Packing measure and dimension of the limit sets of IFSs of generalized complex continued fractions*

Kanji INUI†
Center for Mathematics, School of Fundamental Science and Technology,
Faculty of Science and Technology, Keio University
3-14-1, Hiyoshi, Kohoku-ku, Yokohama, 223-8522, JAPAN
E-mail: k_inui@keio.jp

Hiroki SUMI
Course of Mathematical Science, Department of Human Coexistence,
Graduate School of Human and Environmental Studies, Kyoto University
Yoshida-nihonmatsu-cho, Sakyo-ku, Kyoto, 606-8501, JAPAN
E-mail: sumi@math.h.kyoto-u.ac.jp
Homepage: http://www.math.h.kyoto-u.ac.jp/~sumi/index.html

April 9, 2022

Abstract

We consider a family of conformal iterated function systems (for short, CIFSs) of generalized complex continued fractions which is a generalization of the CIFS of complex continued fractions. We show the packing dimension and the Hausdorff dimension of the limit set of each CIFS in the family are equal and the packing measure of the limit set with respect to the packing dimension of the limit set is finite in order to present new and interesting examples of infinite CIFSs. Note that the Hausdorff measure of the limit set with respect to the Hausdorff dimension is zero. To prove the above results, we consider three cases (essentially two cases) and define a 'nice' subset of the index set of the CIFS in each case. In addition, we estimate the cardinality of the 'nice' subsets and the conformal measure of the CIFSs. [1]

1 Introduction and the main results

Fractal geometry has been developed in order to study the geometrical properties of fractals. One of the major studies in Fractal geometry is the study of estimating the dimensions and measures of fractals. By estimating the dimensions and measures of fractals, it is possible to explain phenomena that appear in fractals which are different from the ones that appear in 'usual figures' (see [4]). For this reason, the study of estimating the dimensions and measures of fractals has been the major topic since Fractal geometry began to attract attention not only in mathematics but also in many other fields.

Mathematically speaking, iterated function systems are a powerful method to construct fractals (more precisely, limit set) and in many papers the study of estimating the dimensions and measures of the limit sets has been studied. For example, Mauldin's and Urbański's paper [5] presents the general theory of estimating the dimensions and measures of the limit set constructed by conformal iterated function systems with finitely many mappings (for short, finite CIFSs). Note that by the formula and results in [5], we obtain some estimations of the dimensions and measures of the limit sets. Indeed, by the formula on the Hausdorff dimension of the limit set constructed by finite CIFSs in [5] we obtain the estimation of the Hausdorff dimension of the limit sets, and by the theorem in [5] we deduce the positiveness and finiteness of the Hausdorff measure and the packing measure of the limit sets with respect to the Hausdorff dimension of the limit sets. By these results, we obtain that the packing dimension of the limit set constructed by finite CIFSs equals the Hausdorff dimension of the limit set, we obtain the positiveness and finiteness of the Hausdorff

*2020 Mathematics Subject Classification: 28A78, 28A80
†corresponding author
[1]Keywords: infinite conformal iterated function systems, fractal geometry, packing measures, packing dimension, generalized complex continued fractions.

measure of the limit set constructed by finite CIFSs with respect to the Hausdorff dimensions of the limit set, and we obtain the positiveness and finiteness of the packing measure of the limit set with respect to the packing dimension of the limit set.

In addition, Mauldin's and Urbański's paper [5] presents the general theory of estimating the dimensions and measures of the fractals (more precisely, limit sets too) constructed by conformal iterated function systems with infinitely many mappings (for short, infinite CIFSs), and now, there are many results of the limit set constructed by infinite CIFSs (for example, see [6], [8], [3], [7], [9], [10] and so on). Note that they generalized the above formula and theorems, and by the generalized theorems in their paper we may obtain non-positiveness of the Hausdorff measure of the limit set constructed by infinite CIFSs with respect to the Hausdorff dimension of the limit sets. This theorem indicates we may find a new phenomenon of infinite CIFSs which cannot hold in finite CIFSs.

Moreover, in Mauldin's and Urbański's paper [5], they constructed an interesting example of an infinite CIFS and the limit set which is related to the complex continued fractions. The precise construction of the example is the following. Let $X := \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/2\}$. We call $\hat{S} := \{\hat{\phi}_{(m,n)}(z) \colon X \to X \mid (m,n) \in \mathbb{Z} \times \mathbb{N}\}$ the CIFS of complex continued fractions, where $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers and

$$\hat{\phi}_{(m,n)}(z) := \frac{1}{z + m + ni} \quad (z \in X).$$

Let $\hat{J}$ be the limit set of $\hat{S}$ (see Definition 2.1) and $\hat{h}$ be the Hausdorff dimension of $\hat{J}$. For each $s \geq 0$, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and denoted by $\mathcal{P}^s$ the s-dimensional packing measure. For this example, Mauldin and Urbański showed the following theorem.

Theorem 1.1 (D. Mauldin, M. Urbanski (1996))**.** Let $\hat{S}$ be the CIFS of complex continued fractions as above. Then, we have that $\mathcal{H}^{\hat{h}}(\hat{J}) = 0$ and $0 < \mathcal{P}^{\hat{h}}(\hat{J}) < \infty$.

Note that they obtained an example of infinite CIFS for which the Hausdorff measure of the limit set with respect to the Hausdorff dimension of the limit set is zero and the packing measure of the limit set with respect to the packing dimension of the limit set is positive and finite. That is, they found a new phenomenon of infinite CIFSs which cannot hold in finite CIFSs.

In our previous papers [1] and [2], we considered a family of CIFSs of generalized complex continued fractions which is a generalization of the CIFS $\hat{S}$ of complex continued fractions, in order to present new and interesting examples of infinite CIFSs. We estimated the Hausdorff dimension of the limit set of each CIFS of the generalized complex continued fractions and showed non-positiveness of the Hausdorff measure and positiveness of the packing measure of the limit set with respect to the Hausdorff dimension of the limit set. Note that the family of the CIFSs introduced in the papers [1] and [2] has uncountably many elements. On the other hand, we did not obtain results on the relationship between the Hausdorff dimension and the packing dimension and the finiteness of the packing measure of the limit set with respect to the packing dimensions of the limit set in [1] and [2], and we have been interested in the relationship and the finiteness of the packing measure of the limit set. The aim of this paper is to show the relationship between the Hausdorff dimension and the packing dimension of each limit set and the finiteness of the packing measure of each limit set with respect to the packing dimensions in the family of the CIFSs of the generalized complex continued fractions, in order to find new, interesting and uncountably infinite examples of infinite CIFSs with the phenomenon which cannot hold in finite CIFSs.

The precise statement is the following. Let

$$A_0 := \{\tau = u + iv \in \mathbb{C} \mid u \geq 0 \text{ and } v \geq 1\} \quad \text{and} \quad X := \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/2\},$$

and we set $I_\tau := \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{N}\}$ for each $\tau \in A_0$, where $\mathbb{N}$ is the set of the positive integers.

Definition 1.2 (The CIFS of generalized complex continued fractions)**.** Let $\tau \in A_0$. Then, we say that $S_\tau := \{\phi_b \colon X \to X \mid b \in I_\tau\}$ is the CIFS of generalized complex continued fractions. Here, for each $\tau \in A_0$ and $b \in I_\tau$, ϕ_b is defined by

$$\phi_b(z) := \frac{1}{z + b} \quad (z \in X).$$

We call $\{S_\tau\}_{\tau \in A_0}$ the family of CIFSs of generalized complex continued fractions. For each $\tau \in A_0$, let J_τ be the limit set of the CIFS S_τ (see Definition 2.1) and let h_τ be the Hausdorff dimension of the limit set

J_τ. We remark that this family of the CIFSs is a generalization of $\hat{S}$ in some sense. Indeed, S_τ is related to 'generalized' complex continued fractions since each point of the limit set J_τ of S_τ is of the form

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\cdots}}}$$

for some sequence $(b_1, b_2, b_3, \ldots)$ in I_τ (See Definition 2.1). Note that there are many kinds of general theories for continued fractions and related iterated function systems ([3], [5], [6], [8]). In [2], we showed the following theorem.

Theorem 1.3 ([2, Theorem 1.3])**.** Let $\{S_\tau\}_{\tau\in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, for each $\tau \in A_0$, we have $\mathcal{H}^{h_\tau}(J_\tau) = 0$ and $0 < \mathcal{P}^{h_\tau}(J_\tau)$.

We now present the main theorem in this paper.

Theorem 1.4 (the main theorem)**.** Let $\{S_\tau\}_{\tau\in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, for each $\tau \in A_0$, we have $\mathcal{P}^{h_\tau}(J_\tau) < \infty$.

Combining Theorem 1.3 and Theorem1.4, we obtain the following corollary.

Corollary 1.5. Let $\{S_\tau\}_{\tau\in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, for each $\tau \in A_0$, we have $0 < \mathcal{P}^{h_\tau}(J_\tau) < \infty$. In particular, for each $\tau \in A_0$, the packing dimension of the limit set J_τ equals the Hausdorff dimension h_τ of the limit set J_τ.

Remark 1.6. By the general theory of finite CIFSs, the Hausdorff measure of the limit set of each finite CIFS with respect to the Hausdorff dimension of the limit set and the packing measure of the limit set with respect to the Hausdorff dimension of the limit set is positive and finite. However, Corollary 1.5 indicates that for each S_τ of the family of CIFSs of generalized complex continued fractions which consists of uncountably many elements, the packing dimension of the limit set equals the Hausdorff dimension of the limit set and the Hausdorff measure of the limit set with respect to the Hausdorff dimension of the limit set is zero and the packing measure of the limit set with respect to the Hausdorff dimension (which equals the packing dimension of the limit set) is positive and finite. This is also a new phenomenon which cannot hold in the finite CIFSs.

Remark 1.7. It was shown that for each $\tau \in A_0$, $\overline{J_\tau} \setminus J_\tau$ is at most countable and $h_\tau = \dim_{\mathcal{H}}(\overline{J_\tau})$ ([1]). Thus, for each $\tau \in A_0$, we have $0 < \mathcal{P}^{h_\tau}(\overline{J_\tau}) = \mathcal{P}^{h_\tau}(J_\tau) < \infty$. Also, for each $\tau \in A_0$, since the set of attracting fixed points of elements of the semigroup generated by S_τ is dense in J_τ, Theorem 1.1 of [11] implies that $\overline{J_\tau}$ is equal to the Julia set of the rational semigroup generated by $\{\phi_b^{-1} \mid b \in I_\tau\}$.

The ideas and strategies to prove the main theorem are the following. To prove the finiteness of the packing measure of the limit set J_τ, we apply Lemma 4.10 in the paper [5] to S_τ for each $\tau \in A_0$. That is, it suffices to show that for each $r > 0$ (which is sufficiently small) and $b \in I_\tau$ with $\mathrm{diam}\phi_b(X)/r \ll 1$, we have

$$m_\tau(B(x,r)) \gtrsim r^{h_\tau}, \tag{1.0.1}$$

where $x := 1/b$, $B(x,r)$ is the open ball with the center x and the radius r with respect to the Euclidean distance in $\mathbb{C}$, m_τ is h_τ-conformal measure of S_τ (see Theorem 2.8) and '$f(r) \gtrsim g(r)$' means that there exists a 'small' constant $c > 0$ such that $f(r) \geq cg(r)$ for all $r > 0$. Note that there is a useful inequality for the conformal measure m_τ :

$$m_\tau\left(\bigcup_{a\in I}\phi_a(X)\right) = \sum_{a\in I} m_\tau\left(\phi_a(X)\right) = \sum_{a\in I}\int_X |\phi_a'(y)|^{h_\tau} m_\tau(\mathrm{d}y) \gtrsim \sum_{a\in I} |a|^{-2h_\tau} m_\tau(X) = \sum_{a\in I} |a|^{-2h_\tau} \tag{1.0.2}$$

for each $I \subset I_\tau$, where we use the property on the S_τ (see Lemma 3.2). To prove the above inequality (1.0.1), we essentially consider the following two cases:

1. $r \ll |x|$ and
2. $r \gg |x|$.

In the first case, by the assumptions, we deduce that $|x|^2 \le r$ and $r < |x|$. We next define $I_{\tau,1} \subset I_\tau$ and show the following inclusions and inequality:

$$B(x,r) \supset \bigcup_{a \in I_{\tau,1}} \phi_a(X), \quad |I_{\tau,1}| \gtrsim \left(\frac{r}{|x|^2}\right)^2 \quad \text{and} \quad |a| \lesssim |x|^{-1}$$

for each $a \in I_{\tau,1}$. Here, for any set A, we denote by $|A|$ the cardinality of A. To prove the above inclusion and inequalities, we prove some additional lemmas. Therefore, by the above inclusion, the inequality (1.0.2) and the above inequalities, we have

$$\begin{aligned} m_\tau(B(x,r)) &\ge m_\tau\left(\bigcup_{a \in I_{\tau,1}} \phi_a(X)\right) \gtrsim \sum_{a \in I_{\tau,1}} |a|^{-2h_\tau} \gtrsim |I_{\tau,1}| \cdot |x|^{2h_\tau} \\ &\gtrsim r^2 \cdot |x|^{2h_\tau - 4} \gtrsim r^2 \cdot r^{h_\tau - 2} = r^{h_\tau}, \end{aligned}$$

where we use the inequality $|x|^2 \le r$ and $h_\tau < 2$.

In the second case, since $r \gg |x|$, we have $B(0,\tilde{r}) \subset B(x,r)$, where $\tilde{r} = cr$ and $c > 0$ is a 'small' positive number. Next, we define $I_\tau(\tilde{r}) \subset I_\tau$ and show the following inclusion and inequalities:

$$B(0,\tilde{r}) \supset \bigcup_{a \in I_\tau(\tilde{r})} \phi_a(X), \quad |I_\tau(\tilde{r})| \gtrsim r^{-2} \quad \text{and} \quad |a| \lesssim r^{-1}$$

for each $a \in I_\tau(\tilde{r})$ (see Lemma 3.5). Therefore, by the above inclusion, the inequality (1.0.2) and the above inequalities, we have

$$\begin{aligned} m_\tau(B(x,r)) &\ge m_\tau(B(0,\tilde{r})) \ge m_\tau\left(\bigcup_{a \in I_\tau(\tilde{r})} \phi_a(X)\right) \gtrsim \sum_{a \in I_\tau(\tilde{r})} |a|^{-2h_\tau} \\ &\gtrsim |I_\tau(\tilde{r})| \cdot r^{2h_\tau} \gtrsim r^{2h_\tau - 2} > r^{h_\tau}, \end{aligned}$$

where we use the inequalities $r^{2h_\tau - 2} > r^{h_\tau}$ since r is sufficiently small and $h_\tau < 2$.

The rest of the paper is organized as follows. In Section 2, we summarize the general theory of the CIFSs and recall some definitions and theorems in the theory. In Section 3, we present some results for the CIFSs of generalized complex continued fractions in the paper [1] and [2]. Also, we prove a slight modification of lemmas in the paper [1] and [2] to prove Theorem 1.4. In Section 4, we prove the main theorem (Theorem 1.4). In this section, we first show some additional lemmas to prove Theorem 1.4 and next present the setting for the proof of the main theorem. Then, we finally show Theorem 1.4. To prove Theorem 1.4, we consider three cases.

2 Conformal iterated function systems

In this section, we summarize the general theory of CIFSs ([1], [2], [5], [6]). We first recall the definition of CIFSs and the limit set of the CIFSs.

Definition 2.1 (Conformal iterated function system)**.** Let $X \subset \mathbb{R}^d$ be a non-empty compact and connected set with the Euclidean norm $|\cdot|$ and let I be a finite set or bijective to $\mathbb{N}$. Suppose that I has at least two elements. We say that $S := \{\phi_i \colon X \to X \mid i \in I\}$ is a conformal iterated function system (for short, CIFS) if S satisfies the following conditions.

1. Injectivity: $\phi_i \colon X \to X$ is injective for each $i \in I$.

2. Uniform Contractivity: There exists $c \in (0,1)$ such that, for all $i \in I$ and $x, y \in X$, the following inequality holds:

$$|\phi_i(x) - \phi_i(y)| \le c|x - y|.$$

3. Conformality: There exists $\epsilon > 0$ and an open and connected subset $V \subset \mathbb{R}^d$ with $X \subset V$ such that for all $i \in I$, ϕ_i extends to a $C^{1+\epsilon}$diffeomorphism on V and ϕ_i is conformal on V i.e. for each $x \in V$ and $i \in I$, there exists $C_i(x) > 0$ such that for each $u, v \in \mathbb{R}^d$,

$$|\phi_i'(x)u - \phi_i'(x)v| = C_i(x)|u - v|.$$

Here, $\phi_i(x)$ denotes the derivative of ϕ_i at $x \in V$.

4. Open Set Condition (OSC): For all $i, j \in I$ $(i \neq j)$, $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ and $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$. Here, $\mathrm{Int}(X)$ denotes the set of interior points of X with respect to the topology in $\mathbb{R}^d$.

5. Bounded Distortion Property(BDP): There exists $\tilde{K} \geq 1$ such that for all $x, y \in V$ and for all $w \in I^* := \bigcup_{n=1}^{\infty} I^n$, the following inequality holds:

$$|\phi'_w(x)| \leq \tilde{K} \cdot |\phi'_w(y)|.$$

   Here, for each $n \in \mathbb{N}$ and $w = w_1 w_2 \cdots w_n \in I^n$, we set $\phi_w := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ and $|\phi'_w(x)|$ denotes the norm of the derivative of ϕ_w at $x \in X$ with respect to the Euclidean norm on $\mathbb{R}^d$.

6. Cone Condition: For all $x \in \partial X$, there exists an open cone $\mathrm{Cone}(x, u, \alpha)$ with a vertex x, a direction u, an altitude $|u|$ and an angle α such that $\mathrm{Cone}(x, u, \alpha)$ is a subset of $\mathrm{Int}(X)$.

We endow I with the discrete topology and endow $I^\infty := I^{\mathbb{N}}$ with the product topology. Note that I^∞ is Polish in general and I^∞ is a compact metrizable space if I is a finite set.

Let S be a CIFS and we set $w|_n := w_1 w_2 \cdots w_n \in I^n$ and $\phi_{w|_n} := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ for each $w = w_1 w_2 w_3 \cdots \in I^\infty$. Note that $\bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X)$ is a singleton (denoted by $\{x_w\}$) and the coding map $\pi_S \colon I^\infty \to X$ of S defined by $\pi_S(w) := x_w$ is well-defined. Then, the limit set J_S of S is defined by

$$J_S := \pi(I^\infty) = \bigcup_{w \in I^\infty} \bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X) (\subset X \subset \mathbb{R}^d).$$

We set $h_S := \dim_{\mathcal{H}} J_S$, where we denote by $\dim_{\mathcal{H}} A$ the Hausdorff dimension of a set $A \subset \mathbb{R}^d$ with respect to the Euclidean distance.

We next recall the pressure function of CIFS S as follows in order to define the regularity of CIFSs.

Definition 2.2. For each $n \in \mathbb{N}$, $[0, \infty]$-valued function ψ_S^n is defined by

$$\psi_S^n(t) := \sum_{w \in I^n} ||\phi'_w||_X^t \quad (t \geq 0).$$

Here, for a C^1 map $f \colon Y \to \mathbb{R}^d$ $(Y \subset \mathbb{R}^d)$, we set

$$|f'(y)| := \sup\{|f'(y)u| \mid u \in \mathbb{R}^d, |u| = 1\} \ (y \in Y) \quad \text{and} \quad ||f'||_Y := \sup\{|f'(y)| \mid y \in Y\}.$$

We set $\theta_S := \inf\{t \geq 0 | \ \psi_S^1(t) < \infty\}$ and $F(S) := \{t \geq 0 \mid \psi_S^1(t) < \infty\}$. Note that by the following lemma, we deduce that $F(S) = (\theta_S, \infty)$ or $F(S) = [\theta_S, \infty)$.

Lemma 2.3 ([5])**.** Let S be a CIFS. Then, $\psi_S^1(t)$ is non-increasing on $[0, \infty)$, and decreasing and convex on $F(S)$. In addition, we have $\psi_S^1(d) \leq \tilde{K}^d$. In particular, $\theta_S \leq d$.

In addition, by the following proposition, we deduce the basic properties of ψ_S^n.

Proposition 2.4 ([5])**.** Let S be a CIFS. For all $m, n \in \mathbb{N}$ and $t \geq 0$, we have

$$\tilde{K}^{-2t} \psi_S^k(t) \psi_S^n(t) \leq \psi_S^{m+n}(t) \leq \psi_S^m(t) \psi_S^n(t).$$

In particular, $\psi_S^n(t) < \infty$ for each $n \in \mathbb{N}$ if and only if $\psi_S^n(t) < \infty$ for some $n \in \mathbb{N}$ (or $n = 1$), and the function $n \mapsto \log \psi_S^n(t)$ is subadditive for all $t \geq 0$. By the subadditivity of $\log \psi_S^n(t)$, we now define the pressure function of S as follows.

Definition 2.5 (Pressure function)**.** The pressure function of S is the function $P_S \colon [0, \infty) \to (-\infty, \infty]$ defined by

$$P_S(t) := \lim_{n \to \infty} \frac{1}{n} \log \psi_S^n(t) \in (-\infty, \infty] \quad (t \geq 0).$$

Proposition 2.6 ([5])**.** Let S be a CIFS and P_S be the pressure function of S. Then, $P_S(t) < \infty$ if and only if $\psi_S^1(t) < \infty$ for each $t \geq 0$. In particular, $\theta_S = \inf\{t \geq 0 \mid P_S(t) < \infty\}$. In addition, P_S is non-increasing on $[0, \infty)$, and decreasing and convex on $F(S)$.

Note that $P_S(0) = \infty$ if and only if I is infinite. By using the pressure function in Definition 2.5, we define the regularity of CIFSs.

Definition 2.7 (Regular, Strongly regular, Hereditarily regular)**.** Let S be a CIFS. We say that

- S is regular if there exists $t \geq 0$ such that $P_S(t) = 0$,
- S is strongly regular if there exists $t \geq 0$ such that $P_S(t) \in (0, \infty)$ and
- S is hereditarily regular if, for all $I' \subset I$ with $|I \setminus I'| < \infty$, $S' := \{\phi_i \colon X \to X \mid i \in I'\}$ is regular.

Here, for any set A, we denote by $|A|$ the cardinality of A.

Note that if a CIFS S is hereditarily regular then S is strong regular, and if S is strong regular then S is regular.

We finally recall the h_S-conformal measure of S. If a CIFS S is regular, there is the following 'nice' probability measure m_S (h_S-conformal measure of S) on J_S. Indeed, we often use m_S in order to estimate the packing measure of the limit set of CIFSs.

Theorem 2.8 ([5] Lemma 3.13)**.** Let S be a CIFS. If S is regular, then there exists the unique Borel probability measure m_S on X such that the following properties hold.

1. $m_S(J_S) = 1$.
2. For all Borel subset A on X and $i \in I$, $m_S(\phi_i(A)) = \int_A |\phi_i'(y)|^{h_S} m_S(\mathrm{d}y)$.
3. For all $i, j \in I$ with $i \neq j$, $m_S(\phi_i(X) \cap \phi_j(X)) = 0$.

We call m_S the h_S-conformal measure of S. As we mentioned above, by the existence of the conformal measure of CIFSs, we estimate the packing measure and obtain the following key theorem to prove Theorem 1.4.

Theorem 2.9 ([5] Lemma 4.10)**.** Let S be a regular CIFS and m_S be the h_S-conformal measure of S. Suppose that there exist $L > 0$, $\xi > 0$ and $\gamma \geq 1$ such that for all $b \in I$ and $r > 0$ with $\gamma \cdot \mathrm{diam}\phi_b(X) \leq r \leq \xi$, there exists $x \in \phi_b(V)$ such that $m_S(B(x, r)) \geq L r^{h_S}$, $B(x, r)$ is the open ball with the center x and the radius r with respect to the Euclidean distance in $\mathbb{R}^d$. Then, we have $\mathcal{P}^{h_S}(J_S) < \infty$.

3 CIFSs of generalized complex continued fractions

In this section, we present some results on the CIFSs of generalized complex continued fractions introduced in the papers [1] and [2], which are needed to prove the results of this paper. Note that these CIFSs are important and interesting examples of infinite CIFSs. Rest of this paper, we denote by $B(y, r) \subset \mathbb{R}^d (d \in \mathbb{N})$ the open ball with center $y \in \mathbb{R}^d$ and radius $r > 0$, with respect to the d-dimensional Euclidean norm and we identify $\mathbb{C}$ with $\mathbb{R}^2$.

We first present the following lemma shown in [1] and [2] in order to prove Theorem 1.4.

Lemma 3.1 (Lemmas 3.1, 3.3 and 3.4 in [2])**.** For all $\tau \in A_0$, S_τ is a hereditarily regular CIFS . In addition, we have $1 < h_\tau < 2$.

In addition, in order to prove Theorem 1.4, we next prove the following lemma (a slight modification of Lemma 3.2 in [2]). For the readers, we give a proof of Lemma 3.2.

Lemma 3.2. Let $\tau \in A_0$. Then, there exists $K_0 \geq 1$ such that for all $K \geq K_0$ and $a \in I_\tau$, the following properties hold.

1. $\phi_a(X) \subset B(0, K|a|^{-1})$.
2. $K^{-1}|a|^{-2} \leq |\phi_a'(z)| \leq K|a|^{-2}$ for each $z \in X$.
3. $K^{-1}|a|^{-2} \leq \mathrm{diam}\phi_a(X)$.

Proof. Let $\tau \in A_0$. Note that by using the BDP, there exists a constant $C \geq 1$ such that for all $z, w \in X$,

$$|\phi_a'(z)| \leq C \cdot |\phi_a'(w)|. \tag{3.0.1}$$

We set $K_0 := C(\geq 1)$ and let $K \geq K_0$ and $a \in I_\tau$. Then, by the inequality (3.0.1) with $w = 0 \in X$, we have

$$|\phi_a(z)| \cdot |a| = \frac{|a|}{|a+z|} = \sqrt{\frac{|a|^2}{|a+z|^2}} = \sqrt{\frac{|\phi_a'(z)|}{|\phi_a'(0)|}} \leq \sqrt{C} \leq K_0 \leq K.$$

for each $z \in X$. It follows that $\phi_a(X) \subset B(0, K|a|^{-1})$. Also, by the inequality (3.0.1), we have

$$K^{-1}|a|^{-2} = K^{-1}|\phi_a'(0)| \le C^{-1}|\phi_a'(0)| \le |\phi_a'(z)| \quad \text{and} \quad |\phi_a'(z)| \le C|\phi_a'(0)| \le K|\phi_a'(0)| = K|a|^{-2}$$

for each $z \in X$, which deduce that $K^{-1}|a|^{-2} \le |\phi_a'(z)| \le K|a|^{-2}$. Moreover, by the inequality (3.0.1), we have

$$\begin{aligned}\mathrm{diam}\phi_a(X) \cdot |a|^2 \ge |\phi_a(z) - \phi_a(w)| \cdot |a|^2 &= \left| \frac{1}{z+a} - \frac{1}{w+a} \right| \cdot |a|^2 = \frac{|w-z|}{|z+a||w+a|} \cdot |a|^2 \\ &= |w-z| \cdot \sqrt{\frac{|a|^2}{|a+z|^2}} \cdot \sqrt{\frac{|a|^2}{|a+w|^2}} = |w-z| \cdot \sqrt{\frac{|\phi_a'(z)|}{|\phi_a'(0)|}} \cdot \sqrt{\frac{|\phi_a'(w)|}{|\phi_a'(0)|}} \ge |w-z| \cdot C^{-1}\end{aligned}$$

for all $z, w \in X = B(1/2, 1/2)$. Since $\mathrm{diam}X = \sup\{|z-w| \mid z, w \in X\} = 1$, we obtain that $\mathrm{diam}\phi_a(X) \ge C^{-1}|a|^{-2} \ge K^{-1}|a|^{-2}$. Therefore, we have proved our lemma. □

We recall the following notations used in the paper [2]. We identify I_τ with $\{{}^t(s,t) \in \mathbb{R}^2 \mid s + it \in I_\tau\}$ and $\mathbb{N}^2$ with $\{{}^t(m,n) \in \mathbb{R}^2 \mid m, n \in \mathbb{N}\}$, where for any matrix A, we denote by ${}^tA$ the transpose of A. For each $\tau = u + iv \in A_0$, we set

$$E_\tau := \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix} \quad \text{and} \quad F_\tau := {}^tE_\tau E_\tau = \begin{pmatrix} 1 & u \\ u & |\tau|^2 \end{pmatrix}.$$

Note that by direct calculations, $E_\tau \mathbb{N}^2 = I_\tau$, E_τ is invertible and there exist the eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ of F_τ with $\lambda_1 < \lambda_2$. Note that since F_τ is symmetric, there exist an eigenvector $v_1 \in \mathbb{R}^2$ of F_τ with respect to λ_1 and an eigenvector $v_2 \in \mathbb{R}^2$ of F_τ with respect to λ_2 such that $V_\tau := (v_1, v_2)$ is an orthogonal matrix.

For each $\tau \in A_0$, we set $N_\tau := \sqrt{2\lambda_2}/\sqrt{\lambda_1} + 1$ (> 2). In addition, for each $\tau \in A_0$ and $R > 0$, we set

$$\begin{aligned} D_1(\tau, R) &:= \{{}^t(x,y) \in \mathbb{R}^2 \mid R^2/\lambda_1 < x^2 + y^2 \le (N_\tau R)^2/\lambda_2\} \quad \text{and} \\ D_2(\tau, R) &:= \{{}^t(x,y) \in \mathbb{R}^2 \mid R^2 < x^2 + y^2 \le (N_\tau R)^2\}. \end{aligned}$$

Note that $R/\sqrt{\lambda_1} < (N_\tau R)/\sqrt{\lambda_2}$ for each $R > 0$ since $\sqrt{\lambda_2}/\sqrt{\lambda_1} < N_\tau$ and $R < N_\tau R$ for each $R > 0$ since $1 < N_\tau$. By these notations, we now present the following proposition and lemma shown in the paper [2].

Proposition 3.3 (Proposition 4.3 in [2])**.** Let $R > 0$. Then, for each $R \ge 6$,

$$0 < \frac{R^2 - 7R + 7}{2} \le |\{{}^t(m,n) \in \mathbb{N}^2 \mid m^2 + n^2 \le R^2\}| \le R^2.$$

Lemma 3.4 (Lemmas 4.2 and 4.4 in [2])**.** Let $\tau \in A_0$. Then, there exist $\tilde{C}_\tau > 0$ and $\tilde{L}_\tau > 0$ such that for all $R > \tilde{C}_\tau$,

$$|I_\tau \cap D_2(\tau, R)| \ge \tilde{L}_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R.$$

We finally prove the following lemma (a slight modification of Lemma 3.4). For the readers we give a proof of Lemma 3.5.

Lemma 3.5. Let $\tau \in A_0$. Then, there exist $Q_\tau > 0$ and $C_\tau > 0$ such that for all $R \ge C_\tau$, we have

$$|I_\tau \cap D_2(\tau, R)| > Q_\tau R^2. \tag{3.0.2}$$

Proof. Let $\tau \in A_0$ and we set $Q_\tau := \tilde{L}_\tau/2 > 0$, where $\tilde{L}_\tau > 0$ is the number in Lemma 3.4. Note that there exists $C_\tau > \tilde{C}_\tau$ such that for all $R \ge C_\tau$, $(\tilde{L}_\tau R^2)/2 - (7N_\tau R)/(2\sqrt{\lambda_2}) > 0$, which is equivalent to

$$\tilde{L}_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R > \frac{\tilde{L}_\tau}{2} R^2, \tag{3.0.3}$$

where $\tilde{C}_\tau > 0$ is the number in Lemma 3.4. By Lemma 3.4 and the inequality (3.0.3), we deduce that

$$|I_\tau \cap D_2(\tau, R)| \ge \tilde{L}_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R > \frac{\tilde{L}_\tau}{2} R^2 = Q_\tau R^2$$

for all $R \ge C_\tau$. Therefore, we have proved our lemma. □

4 Proof of the main theorem

In this section, we prove the main theorem (Theorem 1.4). Rest of this paper, we use the notations in Section 3.

4.1 Lemmas for the proof of the main theorem

In this subsection, we prove the following lemmas to prove Theorem 1.4.

Lemma 4.1. Let $f(z) := 1/z$ $(z \in \mathbb{C} \setminus \{0\})$. Then, for each $B(x, r)(\subset \mathbb{C})$ with $r < |x|$, we have

$$f(B(x, r)) = B\left(\frac{|x|^2}{|x|^2 - r^2} \cdot \frac{1}{x}, \frac{r}{|x|^2 - r^2}\right).$$

Proof. Let $a \in \mathbb{C}$. Then, $|1/a - \overline{x}/(|x|^2 - r^2)| = r/(|x|^2 - r^2)$ if and only if $|r^2 - \overline{x}(x - a)| = r|a|$, which is also equivalent to the following equation:

$$r^4 - r^2(\overline{x}(x - a) + x(\overline{x} - \overline{a}) + a\overline{a}) + x\overline{x}(x - a)(\overline{x} - \overline{a}) = 0. \tag{4.1.1}$$

Since $(\overline{x}(x-a)+x(\overline{x}-\overline{a})+a\overline{a}) = x\overline{x}+(x-a)(\overline{x}-\overline{a})$, the equation (4.1.1) is equivalent to $(r^2-|x-a|^2)(r^2-|x|^2) = 0$. Since $0 < r < |x|$, we deduce that $|1/a - \overline{x}/(|x|^2 - r^2)| = r/(|x|^2 - r^2)$ if and only if $r = |x - a|$. In addition, since $f(x) = 1/x$ and $r < |x|$, we obtain that

$$\left|f(x) - \frac{|x|^2}{|x|^2 - r^2} \cdot \frac{1}{x}\right| = \frac{r}{|x|^2 - r^2} \frac{r}{|x|} < \frac{r}{|x|^2 - r^2}.$$

That is, $f(x) \in B(\bar{x}/(|x|^2 - r^2), r/(|x|^2 - r^2))$). Therefore, we have proved our lemma. $\square$

Lemma 4.2. Let $\tau \in A_0$, $\tilde{x} \in \mathbb{R}^2$ and $\tilde{R} > 0$. Then, we have

$$E_\tau(B(E_\tau^{-1}\tilde{x}, \tilde{R}/\sqrt{\lambda_2})) \subset B(\tilde{x}, \tilde{R}).$$

Proof. Let ${}^t(x, y) \in B(E_\tau^{-1}\tilde{x}, \tilde{R}/\sqrt{\lambda_2})$. We set $\tilde{y} := {}^t(x, y) - E_\tau^{-1}\tilde{x}$. Since V_τ is orthogonal and $|\tilde{y}|^2 = |{}^t(x, y) - E_\tau^{-1}\tilde{x}|^2 < \tilde{R}^2/\lambda_2$, we have

$$\begin{aligned}
|E_\tau {}^t(x, y) - \tilde{x}|^2 &= |E_\tau\tilde{y}|^2 = {}^t\tilde{y}{}^tE_\tau E_\tau\tilde{y} = {}^t\tilde{y}F_\tau\tilde{y} \\
&= {}^t\tilde{y}V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^tV_\tau\tilde{y} = \lambda_1 z_1^2 + \lambda_2 z_2^2 \leq \lambda_2|V_\tau\tilde{y}|^2 = \lambda_2|\tilde{y}|^2 < \tilde{R}^2,
\end{aligned}$$

where ${}^t(z_1, z_2) := {}^tV_\tau\tilde{y}$. Therefore, we have proved our lemma. $\square$

Lemma 4.3. Let $\tau \in A_0$, let $w \in \mathbb{R}^2$ and $\bar{R} > 0$ with $|w| > \bar{R}$. Then, for each $M \geq 2$, we have

$$\begin{aligned}
&E_\tau B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M|w|}w\right), \frac{\bar{R}}{\sqrt{\lambda_2}M}\right) \subset B(0, |w|) \cap B(w, \bar{R}) \quad \text{and} \\
&E_\tau\left(\mathbb{N}^2 \cap B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M|w|}w\right), \frac{\bar{R}}{\sqrt{\lambda_2}M}\right)\right) \subset I_\tau \cap B(0, |w|) \cap B(w, \bar{R}).
\end{aligned}$$

In particular, we have

$$\left|\mathbb{N}^2 \cap B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M|w|}w\right), \frac{\bar{R}}{\sqrt{\lambda_2}M}\right)\right| \leq |I_\tau \cap B(0, |w|) \cap B(w, \bar{R})|.$$

Proof. Let $\tau \in A_0$, $w \in \mathbb{R}^2$, $\bar{R} > 0$ and $M \geq 2$, and assume that $|w| > \bar{R}$. We first show that the following inclusion:

$$B\left(w - \frac{\bar{R}}{M|w|}w, \frac{\bar{R}}{M}\right) \subset B(0, |w|) \cap B(w, \bar{R}).$$

Indeed, Let $z \in B\left(w - (\bar{R}w)/(M|w|), \bar{R}/M\right)$. Since $|w| > \bar{R}$ and $M \geq 2$, we have $1 > \bar{R}/(M|w|)$ and

$$\begin{aligned}
|z| &\leq \left|z - \left(w - \frac{\bar{R}}{M|w|}w\right)\right| + \left|w - \frac{\bar{R}}{M|w|}w\right| \\
&< \frac{\bar{R}}{M} + \left|1 - \frac{\bar{R}}{M|w|}\right||w| = \frac{\bar{R}}{M} + \left(1 - \frac{\bar{R}}{M|w|}\right)|w| = |w|.
\end{aligned}$$

In addition, since $M \geq 2$, we have $(2\bar{R})/M \leq \bar{R}$ and

$$|z-w| \leq \left| z - \left(w - \frac{\bar{R}}{M|w|}w \right) \right| + \left| \frac{\bar{R}}{M}\frac{w}{|w|} \right| < \frac{\bar{R}}{M} + \frac{\bar{R}}{M} = \frac{2\bar{R}}{M} \leq \bar{R}.$$

Hence, we have proved the desired inclusion. By Lemma 4.2 with $\tilde{x} := w - (\bar{R}w)/(M|w|)$ and $\tilde{R} := \bar{R}/M$, we have

$$E_\tau B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M|w|}w \right), \frac{\bar{R}}{\sqrt{\lambda_2}M} \right) \subset B\left(w - \frac{\bar{R}}{M|w|}w, \frac{\bar{R}}{M} \right) \subset B(0,|w|) \cap B(w,\bar{R}).$$

Moreover, since $I_\tau = E_\tau(\mathbb{N}^2)$ it follows that

$$E_\tau(\mathbb{N}^2) \cap E_\tau\left(B\left(E_\tau^{-1}\left(w - \frac{\bar{R}}{M|w|}w \right), \frac{\bar{R}}{\sqrt{\lambda_2}M} \right) \right) \subset I_\tau \cap B(0,|w|) \cap B(w,\bar{R}).$$

Finally, since E_τ is injective, we have proved our lemma. □

We finally show the following lemma.

Lemma 4.4. Let $\tau \in A_0$. Then, there exist $C'_\tau > 0$ and $Q'_\tau > 0$ such that for all $w \in \mathbb{R}^2$ and $R' > 0$ with $|w| > R' \geq C'_\tau$, we have

$$|I_\tau \cap B(0,|w|) \cap B(w,R')| > Q'_\tau (R')^2$$

Proof. Let $\tau \in A_0$. Also, let $w \in \mathbb{R}^2$ and $R' > 0$ with $|w| > R' > 2\sqrt{2\lambda_2}$. We set $\xi = {}^t(\xi_1,\xi_2) := E_\tau^{-1}(w - (R'w)/(2|w|)) \in \mathbb{R}^2$ and let ζ_1 and ζ_2 be minimum integers with $\zeta_1 \geq \xi_1$ and $\zeta_2 \geq \xi_2$. We first show that

$$B\left(\zeta, \frac{R'}{2\sqrt{\lambda_2}} - \sqrt{2} \right) \subset B\left(\xi, \frac{R'}{2\sqrt{\lambda_2}} \right) = B\left(E_\tau^{-1}\left(w - \frac{R'}{2|w|}w \right), \frac{R'}{2\sqrt{\lambda_2}} \right), \tag{4.1.2}$$

where $\zeta := {}^t(\zeta_1,\zeta_2) \in \mathbb{N}^2$. Indeed, note that $R' > 2\sqrt{2\lambda_2}$ if and only if $R'/(2\sqrt{\lambda_2}) - \sqrt{2} > 0$. Since $|\zeta_1 - \xi_1| \leq 1$ and $|\zeta_2 - \xi_2| \leq 1$, we have

$$|\zeta - \xi|^2 = |\zeta_1 - \xi_1|^2 + |\zeta_2 - \xi_2|^2 \leq 2.$$

It follows that for each $a \in B\left(\zeta, R'/(2\sqrt{\lambda_2}) - \sqrt{2}\right)$,

$$|a - \xi| \leq |a - \zeta| + |\zeta - \xi| < \frac{R'}{2\sqrt{\lambda_2}} - \sqrt{2} + \sqrt{2} = \frac{R'}{2\sqrt{\lambda_2}}.$$

Therefore, we have proved the inclusion (4.1.2).

We now set $Q'_\tau := 1/(32\lambda_2)$ and let $C'_\tau \geq 34\sqrt{\lambda_2}$ be a number such that

$$\frac{(R' - 2\sqrt{2\lambda_2})^2}{16\lambda_2} - \frac{(R')^2}{32\lambda_2} > 0 \tag{4.1.3}$$

for each $R' \geq C'_\tau$.

Let $R' \geq C'_\tau$. Note that $l := R'/(2\sqrt{\lambda_2}) - \sqrt{2} > 17 - 2 = 15 > 6$ and by using a geometric observation, we see that

$$|\{{}^t(m,n) \in \mathbb{N}^2 \mid (m-\zeta_1)^2 + (n-\zeta_2)^2 \leq l^2\}| = |\{{}^t(m,n) \in \mathbb{N}^2 \mid m^2 + n^2 \leq l^2\}|.$$

By proposition 3.3, we deduce that

$$\begin{aligned} &|\{{}^t(m,n) \in \mathbb{N}^2 \mid (m-\zeta_1)^2 + (n-\zeta_2)^2 \leq l^2\}| \\ &= |\{{}^t(m,n) \in \mathbb{N}^2 \mid m^2 + n^2 \leq l^2\}| \geq \frac{l^2 - 7l + 7}{2} > \frac{l^2 - 7l}{2} > \frac{l^2}{4}, \end{aligned}$$

where we use the following inequality $(l^2 - 7l)/2 - l^2/4 = (l^2 - 14l)/4 = l(l-14)/4 > 0$ for each $l > 15$.

Finally, let $w \in \mathbb{R}^2$ with $|w| > R'$. Note that $R' \geq C'_\tau > 2\sqrt{2\lambda_2}$. By Lemma 4.3 with $\bar{R} := R'$ and $M = 2$, the inclusion (4.1.2) and the definition of Q'_τ and C'_τ, we have

$$\begin{aligned} |I_\tau \cap B(0,|w|) \cap B(w,R')| &\geq \left| \mathbb{N}^2 \cap B\left(E_\tau^{-1}\left(w - \frac{R'}{2|w|}w \right), \frac{R'}{2\sqrt{\lambda_2}} \right) \right| \geq \left| \mathbb{N}^2 \cap B\left(\zeta, \frac{R'}{2\sqrt{\lambda_2}} - \sqrt{2} \right) \right| \\ &= |\{{}^t(m,n) \in \mathbb{N}^2 \mid (m-\zeta_1)^2 + (n-\zeta_2)^2 \leq l^2\}| \\ &> \frac{l^2}{4} = \frac{(R' - 2\sqrt{2\lambda_2})^2}{16\lambda_2} > \frac{(R')^2}{32\lambda_2} = Q'_\tau (R')^2. \end{aligned}$$

Thus, we have proved our lemma. □

4.2 Proof for Theorem 1.4

We now prove Theorem 1.4. Rest of this section, Let $K \geq 1$ be a number which satisfies 1. $\sim$ 3. in Lemma 3.2.

Proof of Theorem 1.4. It suffices to show that S_τ satisfies the assumption of Theorem 2.9 for each $\tau \in A_0$.

Let $\tau \in A_0$ and we set $r_0 := \min\{1/8, KC_\tau^{-1}\}(>0)$, where $C_\tau > 0$ is the number in Lemma 3.5. Note that there exists $R_0 > \max\{C'_\tau, 1\}(>0)$ such that $(R-1)/R \geq 1/2$ for each $R > R_0$, where $C'_\tau > 0$ is the number in Lemma 4.4. Recall that $N_\tau = \sqrt{2\lambda_2}/\sqrt{\lambda_1} + 1(>2)$. We define constants L'_τ, ξ, γ and L_τ as follows:

$$\begin{aligned} L'_\tau &:= \min\{Q'_\tau/4, (R_0+1)^{-2}\} \ (>0), \\ \xi &:= r_0^2(>0), \\ \gamma &:= K(\geq 1) \quad \text{and} \\ L_\tau &:= \min\left\{L'_\tau(8K)^{-h_\tau}, Q_\tau K^{2-3h_\tau} N_\tau^{-2h_\tau} 2^{2-2h_\tau}\right\} (>0). \end{aligned}$$

Here, Q_τ and Q'_τ are the numbers in Lemmas 3.5 and 4.4 respectively.

Let $b := m + n\tau \in I_\tau$ and $r > 0$ with $\gamma \cdot \operatorname{diam}(\phi_b(X)) \leq r \leq \xi$. We set $x := 1/b = \phi_b(0) \in \phi_b(X) \subset \phi_b(V)$. To prove Theorem 1.4, it suffices to show the following claim:

Claim ($\star$).

$$m_\tau(B(x,r)) \geq L_\tau r^{h_\tau}.$$

Rest of this paper, we consider the following three cases.

Case 1. $r \leq |x|/2$

Case 2. $2|x| \geq r > |x|/2$ and

Case 3. $r > 2|x|$.

We now consider Case 1. Note that by the assumption and Lemma 3.2, we have

$$0 < r \ (\leq |x|/2) < |x| \quad \text{and} \quad |x|^2 = K \cdot K^{-1}|b|^{-2} \leq \gamma \cdot \operatorname{diam}\phi_b(X) \leq r. \tag{4.2.1}$$

We set $f(z) := 1/z$ ($z \in \mathbb{C} \setminus \{0\}$). We set

$$w := \frac{|x|^2}{|x|^2 - r^2} \cdot \frac{1}{x} \quad \text{and} \quad R := \frac{r}{|x|^2 - r^2}$$

for simplicity. Also, we set

$$I_\tau(x,r) := \{a \in I_\tau \mid \phi_a(X) \subset B(x,r)\} = \{a \in I_\tau \mid B(a+1/2, 1/2) \subset f(B(x,r)) = B(w,R)\} \tag{4.2.2}$$

and $I_{\tau,1} := I_\tau(x,r) \cap B(0,|w|)$, where we use Lemma 4.1 in the equation (4.2.2). Note that $|w| = |x|/(|x|^2 - r^2)$.

We next show that $|I_{\tau,1}| > L'_\tau R^2$ for each $R > 0$. To prove this inequality, we consider the following two cases (Cases 1-1, 1-2). Recall that $R_0 > \max\{C'_\tau, 1\}(>0)$ is the positive real number such that $(R-1)/R \geq 1/2$ for each $R > R_0$.

Case 1-1. $R \geq R_0 + 1$.

Note that by the equation (4.2.2), we have

$$\begin{aligned} I_{\tau,1} &= \{a \in I_\tau \mid B(a+1/2, 1/2) \subset B(w,R)\} \cap B(0,|w|) \\ &\supset \{a \in I_\tau \mid a \in B(w, R-1)\} \cap B(0,|w|) = I_\tau \cap B(w,R-1) \cap B(0,|w|). \end{aligned} \tag{4.2.3}$$

Since $|x| > r$, we have $|w| > R$ ($> R_0 > 0$). By the inclusion (4.2.3) and Lemma 4.4 with $R' := R - 1(\geq R_0 \geq C'_\tau)$, we have

$$|I_{\tau,1}| \geq |I_\tau \cap B(w,R-1) \cap B(0,|w|)| > Q'_\tau(R-1)^2 \geq \frac{Q'_\tau}{4} R^2 \geq L'_\tau R^2.$$

Case 1-2. $R_0 + 1 > R$.

We show that $b \in I_{\tau,1}$. Note that since $|b| = 1/|x| \leq |x|/(|x|^2 - r^2) = |w|$, we have $b \in B(0,|w|)$. Therefore, we have only to show that $b \in I_\tau(x,r)$. By using a geometric observation, it suffices to show that $|w - (b+1/2)| \leq$

$R-1/2$. To this end, we set $A := 1-|b|^2r^2$ for simplicity. Note that by the inequality (4.2.1) and $|x| = 1/|b|$, $A = 1-|b|^2r^2 > 0$ and $|b|^2r - 1 \geq 0$. Then, we have

$$\begin{aligned}
&\left|2|b|^2r - A\right|^2 - |2r^2|b|^2b - A|^2 \\
&= 4|b|^4r^2 - 2A\cdot 2|b|^2r + A^2 - \left(2r^2|b|^2b - A\right)\left(2r^2|b|^2\overline{b} - A\right) \\
&= 4|b|^4r^2 - 2A\cdot 2|b|^2r - (2|b|^2r^2)^2\cdot|b|^2 + 2|b|^2r^2A\cdot(b+\overline{b}) \\
&= 4|b|^4r^2 - 4A|b|^2r - 4|b|^6r^4 + 4|b|^2r^2A\cdot\Re(b) \\
&= 4|b|^4r^2(1-|b|^2r^2) + 4A|b|^2r(r\Re(b) - 1) \\
&= 4A|b|^4r^2 + 4A|b|^2r(r\Re(b)-1) = 4A|b|^2r(|b|^2r - 1 + r\Re(b)) \geq 0,
\end{aligned}$$

where $\Re(b) \geq 0$ is the real part of b. In addition, since $|b|^2r - 1 \geq 0$, we have

$$2|b|^2r - A = |b|^2r^2 + |b|^2r + |b|^2r - 1 \geq |b|^2r - 1 \geq 0.$$

Therefore, we deduce that $|2r^2|b|^2b - A| \leq 2|b|^2r - A$ and it follows that

$$\begin{aligned}
\left|w - \left(b + \frac{1}{2}\right)\right| &= \left|\frac{|x|^2}{|x|^2 - r^2}\frac{1}{x} - b - \frac{1}{2}\right| = \left|\frac{b}{1-|b|^2r^2} - b - \frac{1}{2}\right| \\
&= \frac{|2b - 2b(1-|b|^2r^2) - (1-|b|^2r^2)|}{2(1-|b|^2r^2)} = \frac{|2b|b|^2r^2 - A|}{2(1-|b|^2r^2)} \\
&\leq \frac{2|b|^2r - A}{2(1-|b|^2r^2)} = \frac{|b|^2r}{1-|b|^2r^2} - \frac{1-|b|^2r^2}{2(1-|b|^2r^2)} = \frac{r}{|x|^2 - r^2} - \frac{1}{2} = R - \frac{1}{2}.
\end{aligned}$$

Thus, we have proved $B(b+1/2, 1/2) \subset B(w, R)$ and $b \in I_{\tau,1}$. By the assumption $R_0 + 1 > R$, we deduce that

$$|I_{\tau,1}| \geq 1 > \left(\frac{R}{R_0 + 1}\right)^2 \geq L'_\tau R^2.$$

Hence, we have proved $|I_{\tau,1}| > L'_\tau R^2$ for each $R > 0$.

We now show the statement of Claim ($\star$) for Case 1. $r \leq |x|/2$. Rest of this paper, we denote by m_τ the h_τ-conformal measure for S_τ for each $\tau \in A_0$. Note that by Theorem 2.8 and Lemma 3.2, we have

$$\begin{aligned}
m_\tau\left(\bigcup_{a\in I_{\tau,1}} \phi_a(X)\right) &= \sum_{a\in I_{\tau,1}} m_\tau\left(\phi_a(X)\right) = \sum_{a\in I_{\tau,1}} \int_X |\phi'_a(y)|^{h_\tau} m_\tau(\mathrm{d}y) \\
&\geq \sum_{a\in I_{\tau,1}} K^{-h_\tau}|a|^{-2h_\tau} m_\tau(X) = \sum_{a\in I_{\tau,1}} K^{-h_\tau}|a|^{-2h_\tau}.
\end{aligned}$$

Therefore, since $\phi_a(X) \subset B(x,r)$ and $|a| \leq |w|$ for each $a \in I_{\tau,1}$ we deduce that

$$m_\tau(B(x,r)) \geq m_\tau\left(\bigcup_{a\in I}\phi_a(X)\right) = \sum_{a\in I_{\tau,1}} K^{-h_\tau}|a|^{-2h_\tau} \geq \sum_{a\in I_{\tau,1}} K^{-h_\tau}|w|^{-2h_\tau} \geq |I_{\tau,1}|\cdot K^{-h_\tau}|w|^{-2h_\tau}.$$

In addition, since $r \leq |x|/2$, we obtain that $|x|^2 - r^2 \geq 3|x|^2/4$ (i.e. $|w| = |x|/(|x|^2 - r^2) \leq 4/(3|x|)$) and

$$\begin{aligned}
m_\tau(B(x,r)) &\geq L'_\tau R^2 \cdot K^{-h_\tau}\left(\frac{3|x|}{4}\right)^{2h_\tau} \geq L'_\tau\left(\frac{r}{|x|^2}\right)^2\cdot K^{-h_\tau}\left(\frac{9}{16}\right)^{h_\tau}|x|^{2h_\tau} \\
&\geq L'_\tau K^{-h_\tau}2^{-h_\tau}\cdot r^2\,|x|^{2h_\tau - 4} \geq L'_\tau(2K)^{-h_\tau}\cdot r^2 r^{h_\tau - 2} \geq L'_\tau(2K)^{-h_\tau}r^{h_\tau} \\
&\geq L_\tau r^{h_\tau},
\end{aligned}$$

where we use the inequality $|I_{\tau,1}| > L'_\tau R^2$ for each $R > 0$ and the inequality $|x|^{2(h_\tau - 2)} \geq r^{h_\tau - 2}$ since $|x|^2 \leq r$ and $h_\tau < 2$ (see Lemma 3.1). Thus, we have proved the statement of Claim ($\star$) for Case 1. $r \leq |x|/2$.

We now consider Case 2. We set $\tilde{r} := r/4$. Then, by the assumption, we have $\tilde{r} \leq |x|/2$. In addition, since $|x|^2 = K\cdot K^{-1}|b|^{-2} \leq \gamma\cdot\mathrm{diam}\phi_b(X) \leq r \leq \xi = r_0^2$, we have

$$|x|^2 \leq r_0\cdot|x| \leq \frac{1}{8}\cdot 2r = \tilde{r},$$

where used the definition of $r_0 > 0$. Therefore, the positive real number $\tilde{r}$ satisfies the inequalities $\tilde{r} \le |x|/2$ (the assumption of Case 1.), $\tilde{r} < |x|$ and $|x|^2 \le \tilde{r}$ (the inequalities (4.2.1)) instead of $r > 0$. By the same argument as Case 1. with $\tilde{r} > 0$ instead of $r > 0$, we have

$$m_\tau(B(x,r)) \ge m_\tau(B(x,\tilde{r})) \ge L'_\tau(2K)^{-h_\tau}\tilde{r}^{h_\tau} \ge L'_\tau(2K)^{-h_\tau}4^{-h_\tau}r^{h_\tau} \ge L'_\tau(8K)^{-h_\tau}r^{h_\tau} \ge L_\tau r^{h_\tau}.$$

Thus, we have proved the statement of Claim ($\star$) for Case 2. $2|x| \ge r > |x|/2$.

We now consider Case 3. We set $\bar{r} := r/2$. Note that $B(0,\bar{r}) \subset B(x,r)$ since $|y-x| \le |x|+|y| < r/2+\bar{r} = r$ for each $y \in B(0,\bar{r})$. We set $r_\tau := KC_\tau^{-1}(>0)$ for simplicity, where C_τ is the number in Lemma 3.5. We set

$$I_\tau(\bar{r}) := \{a \in I_\tau \mid \bar{r}/N_\tau \le K|a|^{-1} < \bar{r}\}.$$

We show that

$$|I_\tau(\bar{r})| > Q_\tau K^2\bar{r}^{-2}. \tag{4.2.4}$$

Indeed, note that by the definition of $r > 0$, $\xi > 0$ and $r_0 > 0$, we have $\bar{r} = r/2 < r_0 \le r_\tau$. We set $R := K\bar{r}^{-1}$ for simplicity. Note that $\bar{r} < r_\tau$ if and only if $R > C_\tau$ and

$$I_\tau(\bar{r}) = \{a \in I_\tau \mid K\bar{r}^{-1} < |a| \le N_\tau K\bar{r}^{-1}\} = I_\tau \cap D_2(\tau, K\bar{r}^{-1}) = I_\tau \cap D_2(\tau, R).$$

By Lemma 3.5, we obtain that

$$|I_\tau(\tilde{r})| = |I_\tau \cap D_2(\tau,R)| > Q_\tau R^2 = Q_\tau K^2\bar{r}^{-2}.$$

Therefore, we have proved the inequality (4.2.4). Now, by Theorem 2.8 and Lemma 3.2, we have

$$\begin{aligned} m_\tau\left(\bigcup_{a\in I_\tau(\bar{r})}\phi_a(X)\right) &= \sum_{a\in I_\tau(\bar{r})} m_\tau(\phi_a(X)) = \sum_{a\in I_\tau(\bar{r})}\int_X |\phi'_a(y)|^{h_\tau} m_\tau(\mathrm{d}y) \\ &\ge \sum_{a\in I_\tau(\bar{r})} K^{-h_\tau}|a|^{-2h_\tau}m_\tau(X) = \sum_{a\in I_\tau(\bar{r})} K^{-h_\tau}|a|^{-2h_\tau}. \end{aligned}$$

In addition, since $\phi_a(X) \subset B(0,\bar{r})$ (see Lemma 3.2) and $|a|^{-1} \ge \bar{r}/(N_\tau K)$ for each $a \in I_\tau(\bar{r})$ we deduce that

$$\begin{aligned} m_\tau(B(x,r)) \ge m_\tau(B(0,\bar{r})) &\ge m_\tau\left(\bigcup_{a\in I_\tau(\bar{r})}\phi_a(X)\right) \ge \sum_{a\in I_\tau(\bar{r})} K^{-h_\tau}\left(\frac{\bar{r}}{KN_\tau}\right)^{2h_\tau} \\ &= |I_\tau(\bar{r})| \cdot K^{-3h_\tau}N_\tau^{-2h_\tau}\bar{r}^{2h_\tau} > Q_\tau K^{2-3h_\tau}N_\tau^{-2h_\tau} \cdot \bar{r}^{2h_\tau-2} \\ &= Q_\tau K^{2-3h_\tau}N_\tau^{-2h_\tau}2^{2-2h_\tau} \cdot r^{2h_\tau-2} \ge Q_\tau K^{2-3h_\tau}N_\tau^{-2h_\tau}2^{2-2h_\tau}r^{h_\tau} \ge L_\tau r^{h_\tau}, \end{aligned}$$

where we use the inequalities (4.2.4) and $r^{2h_\tau-2} > r^{h_\tau}$ since $r < r_0 \le 1/8 < 1$ and $h_\tau < 2$ (see Lemma 3.1). Thus, we have proved the statement of Claim ($\star$) for Case 3. $r > 2|x|$.

Hence, by the three cases (Cases 1. $\sim$ 3.), we have proved Theorem 1.4. $\square$

Acknowledgment

The authors would like to thank Mariusz Urbański for helpful comments on [5]. The second author is partially supported by JSPS Grant-in-Aid for Scientific Research (B) Grant Number JP 19H01790.

References

[1] K. Inui, H. Okada, H. Sumi, The Hausdorff dimension function of the family of conformal iterated function systems of generalized complex continued fractions, Discrete and Continuous Dynamical Systems Series A, **40**, no.2, pp.753–766 (2020).

[2] K. Inui, H. Sumi, Hausdorff measures and packing measures of limit sets of CIFSs of generalized complex continued fractions. J. Difference Equ. Appl. **26** , no. 1, pp.104–121 (2020).

[3] M. Iosifescu, C. Kraaikamp, *Metrical theory of continued fractions*, Mathematics and its Applications, **547**, Kluwer Academic Publishers, Dordrecht (2002).

[4] B. Mandelbrot, *The Fractal Geometry of Nature*, Spektrum Akademischer Verlag, (1982)

[5] R. D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proceedings of the London Mathematical Society, **73**, no. 1, pp.105–154 (1996).

[6] R. D. Mauldin, M. Urbański, Conformal iterated function systems with applications to the geometry of continued fractions, Trans. Amer. Math. Soc. **351**, no. 12, pp.4995–5025 (1999).

[7] R. D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Tracts in Mathematics **148**, Cambridge University Press, Cambridge (2003).

[8] E. Mihailescu, M. Urbański, Hausdorff dimension of the limit set of countable conformal iterated function systems with overlaps. Fractal geometry and dynamical systems in pure and applied mathematics. I. Fractals in pure mathematics, pp.273–289, Contemp. Math., **600**, Amer. Math. Soc., Providence, RI, (2013).

[9] E. Mihailescu, M. Urbański, Random countable iterated function systems with overlaps and applications. Adv. Math. **298** , pp.726–758, (2016).

[10] M. Roy, M. Urbański, Regularity properties of Hausdorff dimension in infinite conformal iterated function systems, Ergodic Theory Dynam. Systems, **25**, no. 6, pp.1961–1983 (2005).

[11] R. Stankewitz, Density of repelling fixed points in the Julia set of a rational or entire semigroup, II, Discrete Contin. Dyn. Syst. **32**, no. 7, pp.2583–2589 (2012).