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Semihyperbolic transcendental semigroups

By

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Abstract

This paper deals with semihyperbolic semigroups which are generated by entire (possibly transcendental) functions. In particular, a criterion is given assuring that a given entire semigroup is semihyperbolic. Note that a semihyperbolic semigroup G admits holomorphic scaling, that is to say, the branches of local inverses of functions $f \in G$ are of bounded degree and that the preimages shrink to zero in diameter.

1. Introduction

For long years the notion 'complex dynamics' has been associated with the dynamics of groups of Möbius transformations or with the iteration of rational or transcendental functions. Few years ago, Hinkkanen and Martin have initiated the study of rational semigroups, that is to say, semigroups generated by rational functions or polynomials. In particular, each rational semigroup G is a subset of $\mathrm{End}(\mathbf{P}_1)$, the set of holomorphic endomorphisms of the complex sphere $\mathbf{P}_1$. To each rational semigroup G one can attach its Julia set, $\mathcal{J}(G)$, and its Fatou set, $\mathcal{F}(G)$. We refer the reader to the second chapter for the precise definitions. The purpose of the present paper is to give a sufficient condition which assures a given entire semigroup, that is to say, a semigroup $G \subset \mathrm{End}(\mathbf{C})$, to have the property known as 'holomorphic scaling'. Roughly speaking, this means that the branches of local inverses of functions $f \in G$ are of bounded degree, and that the preimages shrink to zero in diameter. We refer the reader to the Main Theorem, cf. Theorem 2 in §2, for the precise statement. This paper is organized as follows. In the next chapter we recall some basic notions and notations. There we will also state the Main Theorem. In the third chapter we give the proof of this main result. Finally, in the last chapter we give examples of entire transcendental semigroup which are semihyperbolic. This paper is based on earlier work of the authors: [10, 14].

2. Notations and basic facts

First, we set up some notations. For basic facts on iteration of rational (and, more generally, meromorphic) functions and the definitions of Julia sets and Fatou

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sets we refer the reader to the monographs [1, 2, 13] and to the lecture notes [12]. Semigroups of rational functions or polynomials have been first studied by Aimo Hinkkanen and Gaven Martin, for further details we refer to their fundamental work [8, 7, 6, 9, 5]. The dynamics of semigroups of meromorphic functions have been studied in [4].

Let $\chi(\cdot,\cdot)$ denote the *chordal metric* on the Riemann sphere $\mathbf{P}_1$, and let $T \subset \mathbf{P}_1$. We write

$$\chi(z, T) = \chi(T, z) := \inf\{\chi(z, w) \mid w \in T\},$$

where $z \in \mathbf{P}_1$, and define

$$\mathrm{diam}(T) := \sup\{\chi(z, w) \mid z, w \in T\}.$$

For some number $\eta > 0$ let $D_\eta(z) := \{\zeta \in \mathbf{P}_1 \mid \chi(z, \zeta) < \eta\}$ and $U_\eta(T) := \bigcup_{z \in T} D_\eta(z) = \{z \in \mathbf{P}_1 \mid \chi(z, T) < \eta\}$. The terms 'closure of T', 'boundary of T', and 'interior of T' refer to the closure, boundary, and interior of T with respect to the topology induced by χ. We write $\bar{T}$ or $cl(T)$ for the closure of T, ∂T for the boundary of T, and $\mathrm{int}(T)$ for the interior of T.

Throughout this paper, f denotes an entire function, in other words, f is a holomorphic mapping from the complex plane $\mathbf{C}$ into itself. Clearly, f is a polynomial or an entire transcendental mapping: $f \in \mathrm{End}(\mathbf{C})$. For a moment we fix an entire function $f : \mathbf{C} \to \mathbf{C}$. Throughout this paper, $CV(f)$ denotes the set of critical values of f and $AV(f)$ the set of asymptotic values of f. All important for dynamics of f is the set $SV(f)$ of the so-called *singular values* of f, that is the set of values where at least one branch of the inverse of f is not well defined as a holomorphic function. Note that a singular value of a transcendental function is either an asymptotic value or a critical value, in short: $SV(f) = CV(f) \cup AV(f)$. Clearly, each critical value is the image of some critical point, that is to say, for each critical value $v \in CV(f)$ there exists some preimage $c \in C(f) := \{c \in \mathbf{C} \mid f'(c) = 0\}$. Throughout this paper ϕ denotes a certain branch of the inverse of f and we write $f^{-1}(T) := \{z \in \mathbf{C} \mid f(z) \in T\}$, where $T \subset \mathbf{C}$. Recall that polynomials do not have asymptotic values. In this paper we deal with subsets of and semigroups in $\mathrm{End}(\mathbf{C})$.

Definition 1. A semigroup $G \subset \mathrm{End}(\mathbf{C})$ is called entire semigroup. It is called *finitely generated* if and only if it has a representation $G = \langle f_1, \dots, f_n \rangle$ for a finite nonempty subset $\{f_1, \dots, f_n\} \subset \mathrm{End}(\mathbf{C})$.

The set $\{f_1, \dots, f_n\}$ is called *set of generators* of this group G. We define $AV(G) := \bigcup_{f \in G} AV(f)$ and $SV(G) := \bigcup_{f \in G} SV(f)$.

Remark. Throughout this paper we assume each set of generators to be minimal.

Let G be an entire semigroup generated by $\{f_\lambda\}_{\lambda \in \Lambda}$ with some finite or infinite index set Λ. For some word $w_n = \{w_{n,1}, \dots, w_{n,n}\} \in \Lambda^n$, we define $f_{w_n} := f_{w_{n,1}} \circ \cdots \circ f_{w_{n,n}}$.

Let $\mathscr{S} \subset \operatorname{End}(\mathbf{C})$. In the sequel, $\mathscr{S}$ might be the sequence $\{f^{\circ n}\}_{n \in \mathbf{N}}$ of iterates $f^{\circ n}$ of a fixed function $f \in \operatorname{End}(\mathbf{C})$, a semigroup $G \subset \operatorname{End}(\mathbf{C})$, or an (infinite) subset of $\operatorname{End}(\mathbf{C})$. The *Fatou set* $\mathscr{F}(\mathscr{S})$ is defined as the set of all points $z \in \mathbf{C}$ which have an open neighborhood U such that the family $\{f|_U \mid f \in \mathscr{S}\}$ is normal. As usual, the complement $\mathscr{J}(\mathscr{S}) := \mathbf{C} \backslash \mathscr{F}(\mathscr{S})$ is called the *Julia set* of $\mathscr{S}$. For some set $T \subset \mathbf{C}$ we write $\mathscr{S}(T) := \{f(z) \mid f \in \mathscr{S}, z \in T\}$ and $\mathscr{S}^{-1}(T) := \{z \in \mathbf{C} \mid f(z) \in T$ for some $f \in \mathscr{S}\}$. Note that if $\mathscr{S}$ is a semigroup, then for each $f \in \mathscr{S}$ we have $\{f^{\circ n}\}_{n \in \mathbf{N}} \subset \mathscr{S}$, and therefore $\mathscr{F}(\mathscr{S}) \subset \mathscr{F}(\{f^{\circ n}\}_{n \in \mathbf{N}})$ and $\mathscr{J}(\{f^{\circ n}\}_{n \in \mathbf{N}}) \subset \mathscr{J}(\mathscr{S})$. Throughout this paper, let $\omega_{\mathscr{S}}(z)$ denote the ω-limit set with respect to the semigroup $\mathscr{S}$ for some point $z \in \mathbf{C}$.

Let $\Omega := \overline{\bigcup_{f \in \mathscr{S}} SV(f)}$, i.e. the closure of the postsingular set. It is of great importance whether or not the singular points in the Julia set of f are recurrent. For example, an irrationally indifferent cycle of a single rational or entire transcendental function f requires at least one singular point in the Julia set of f to be recurrent, cf. [11, 10]. The following lemma concerning the critical values can be readily proved.

Lemma 1. *Let $G \subset \operatorname{End}(\mathbf{C})$ be a (not necessarily finitely generated) semigroup and $\{f_\lambda\}_{\lambda \in \Lambda}$ a set of generators of G, that is to say, $G = \langle \{f_\lambda \mid \lambda \in \Lambda\} \rangle$. Then we have*

$$\bigcup_{f \in G} CV(f) = \bigcup_{\lambda \in \Lambda} (CV(f_\lambda) \cup G(\{CV(f_\lambda)\})).$$

Remark. Note that the semigroup G does not have to contain the identity, thus the set of critical values corresponding to some critical point c of some $f \in G$ consists of $f(c)$ and the G-orbit of $f(c)$. For convenience, we define $G^* := G \cup \{id\}$. Then the equation in the above lemma reads as follows:

$$\bigcup_{f \in G} CV(f) = \bigcup_{\lambda \in \Lambda} G^*(\{CV(f_\lambda)\}).$$

Definition 2 (SH_N)**.** Let $G \subset \operatorname{End}(\mathbf{C})$ be an entire semigroup and $N \in \mathbf{N}$. We define SH_N to be the set of all points $z_0 \in \mathbf{C}$ such that there exists a neighborhood U of z_0 satisfying the following condition:

For every $f \in G$ and every component K of $f^{-1}(U)$ the mapping $f|_K : K \to U$ is a proper mapping with $\deg(f|_K : K \to U) \leq N$.

Remarks. 1. It is all important, that the neighborhood U does *not* depend on the function $f \in G$.

2. Note that the above definition implies that U does *not* contain any asymptotic value of any element $f \in G$, since otherwise $g|_K : K \to U$ would not be a proper mapping.

Definition 3 (semihyperbolic semigroup). An entire semigroup $G \subset \operatorname{End}(\mathbf{C})$ is called *semihyperbolic* if and only if $\mathscr{J}(G) \subset \bigcup_{n \in \mathbf{N}} SH_n$.

This definition assures that each point $z_0 \in \mathscr{J}(G)$ belongs to some SH_n with some $n \in \mathbf{N}$. In other words, the degrees of the local branches of the inverse functions are bounded. Note that in many examples one can prove $\mathcal{J}(G) \subset SH_N$ for some fixed integer N. If, for example $G \subset \mathrm{End}(\mathbf{C})$ is a semihyperbolic semigroup generated by a *finite* number of polynomials, then $\mathcal{J}(G) \subset \mathbf{C}$ is a compact set and, consequently, $\mathcal{J}(G) \subset SH_N$ for some $N \in \mathbf{N}$. However, there are many examples of semigroups which are *not* semihyperbolic. For example, if a polynomial f of degree at least two has an irrationally indifferent fixed point, then it has at least one recurrent critical point in its Julia set. Consequently, every semigroup containing such a polynomial cannot be semihyperbolic.

The main result of this paper gives a sufficient condition for a semigroup of entire functions to be semihyperbolic.

Theorem 2 (Main Theorem). *Let an entire semigroup $G \subset \mathrm{End}(\mathbf{C})$ be generated by $\{f_\lambda\}_{\lambda \in \Lambda}$ with some (not necessarily finite) index set Λ and a point $z_0 \in \mathcal{J}(G)$ satisfy all of the following conditions:*

A.1 *There exists some neighborhood U_1 of z_0 such that for each infinite sequence $\mathcal{S} \subset G$ and each component W of $\mathcal{F}(\mathcal{S})$ there exists some point $\zeta \in W$ such that $\omega_{\mathcal{S}}(\zeta) \cap U_1 = \emptyset$,*

A.2 *there exists some neighborhood U_2 of z_0 and some positive real number $\tilde{\varepsilon}$ such that the set $\mathcal{T} := \{c \in C(f_\lambda) \mid \lambda \in \Lambda, G^*(f_\lambda(c)) \cap U_2 \neq \emptyset\}$ is finite and for each $c \in \mathcal{T}$ we have that $c \in C(f_\lambda)$ implies $\chi(c, \overline{G^*(f_\lambda(c))} \backslash \{c\}) \geq \tilde{\varepsilon} > 0$,*

A.3 $\overline{AV(G)} \subset \mathcal{F}(G)$ *and* $\mathcal{F}(G) \neq \emptyset$.

Then $z_0 \in SH_N$ for some $N \in \mathbf{N}$.

Remark. In this theorem, $\mathcal{T} := \{c \in C(f_\lambda) \mid \lambda \in \Lambda, G^*(f_\lambda(c)) \cap U_2 \neq \emptyset\}$ is regarded as a subset of the plane, and, consequently, $\#\mathcal{T}$ is the number of critical points—counted *without* multiplicity—whose G^*-orbit intersect U_2.

Furthermore, we shall prove the following statement.

Corollary 3. *Let an entire semigroup $G \subset \mathrm{End}(\mathbf{C})$ and some point $z_0 \in \mathcal{J}(G)$ satisfy the assumption of the above theorem. Let $\{w_n\}_{n \in \mathbf{N}}$ be a sequence of words w_n of strictly increasing length. Then z_0 has a neighbourhood U with the following property: For each $n \in \mathbf{N}$, let K_n be a connected component of $f_{w_n}^{-1}(U)$. Then* $\lim_{n \to \infty} \mathrm{diam}(K_n) = 0$.

Clearly, the theorem yields:

Corollary 4. *If a semigroup G satisfies the conditions* **A.1**–**A.3** *for each $z_0 \in \mathcal{J}(G)$, then G is semihyperbolic.*

With the Main Theorem at hand one readily proves the following corollary, compare [14, Lemma 1.14] and [10, Chapter 3].

Corollary 5. *An entire semigroup $G \subset \mathrm{End}(\mathbf{C})$ satisfying the conditions* **A.1**–**A.3** *has no indifferent cycles.*

By the maximum principle, a semigroup $G \subset \mathrm{End}(\mathbf{C})$ cannot have Herman rings. Thus, a semihyperbolic semigroup $G \subset \mathrm{End}(\mathbf{C})$ has no parabolic cycle, Cremer points, Siegel discs, and Herman rings. Note that this does not imply, that $\mathscr{F}(G)$ is the union of attracting basins. In fact, $\mathscr{F}(G)$ might have wandering domains. Recall that this can even happen to polynomial semigroups.

Finally, with the same method of proof as used in order to establish Theorem 1.34 and Theorem 1.35 in [14], we obtain:

Theorem 6. *Let G be a semihyperbolic entire semigroup without any elliptic Möbius transformations and not containing the Identity. If $SV(G) \cap \mathscr{F}(G)$ is compact and non empty, and, in addition, if there is no point $z \in \mathscr{F}(G)$ satisfying $\infty \in \overline{G(z)}$, then $\overline{G(z_0)} \subset \mathscr{F}(G)$ for each $z_0 \in \mathscr{F}(G)$. Moreover, if G is generated by $\{f_\lambda\}_{\lambda \in \Lambda}$ with some finite index set Λ, then*

$$\lim_{n \to \infty} \sup \{\chi(f_{w_n}(z_0), SV(G) \cap \mathscr{F}(G)) \mid w_n \in \Lambda^n\} = 0.$$

Note that this statement means, that each attractor of the dynamical system defined by G is contained in $SV(G) \cap \mathscr{F}(G)$. This phenomenon is known in the theory of iterating single rational functions.

Outline of the proof of Theorem 6: Suppose that there exists a point $z_0 \in \mathscr{F}(G)$ and a sequence $\{g_n\}_{n \in \mathbf{N}} \subset G$ satisfying

$$\overline{\bigcup_{n \in \mathbf{N}} g_n(z_0)} \cap \mathscr{J}(G) \neq \emptyset.$$

We may and will assume that there exists a point $y \in \mathscr{J}(G)$ such that $\lim_{n \to \infty} \cdot g_n(z_0) = y$. From the distortion lemma for proper mappings, cf. Lemma 8, and its generalization, Lemma 1.10 in [14], and the backward invariance of $\mathscr{J}(G)$ we derive a contradiction. Hence we have $\overline{G(z_0)} \subset \mathscr{F}(G)$ for every $z_0 \in \mathscr{F}(G)$. From this result we can conclude the second statement of the theorem. For the details, the reader is referred to the paper [14].

3. Limits for inverse mappings

In this section we shall prove the main result, Theorem 2. Both, the result and the proof are motivated by the work of Mañé, cf. [11], and Carleson, Jones and Yoccoz, cf. [3, Theorem 2.1, (D) $\Rightarrow$ (B)]. In their proofs they made use of the fact that Julia sets of rational functions or polynomials are compact subsets of the complex sphere or plane, respectively. But in our setting we have to 'localize' the arguments. We adopt Mañé's idea of 'admissible squares', cf. [11], but several modifications are required by the new setting, that is to say, by the fact that we are dealing with entire semigroups but not with the iteration of a single function.

Proof:

(a) Admissible square. Let $G \subset \mathrm{End}(\mathbf{C})$ be an entire semigroup, $z_0 \in \mathscr{J}(G)$, and a neighbourhood U of z_0 such that the assumptions **A.1–A.3** as given in the Main Theorem are satisfied. In addition, we assume $\mathrm{diam}(U) < \tilde{\varepsilon}$. Without loss of generality we may and will assume $U \subset U_1 \cap U_2$. Choose $\sigma > 0$ such that U contains an open square Q' of side length 4σ and center z_0 as an relatively compact subset. There is a closed square $Q'' \subset Q'$ of side length 2σ and center z_0. In the sequel we shall assume all squares to have sides parallel to the sides of Q'' and to be closed. Let $\xi \in Q''$ and $Q_{1,1}(\xi)$ the unique square of side length σ and center ξ. By construction, $Q_{1,1}(\xi) \subset Q' \subset U$. We call $Q_{1,1}(\xi)$ an *admissible square at level* 1 (centered at ξ).

Now we proceed by induction. Let Q be an admissible square at level m (for some $m \in \mathbf{N}$ and centered at some $\xi \in Q''$) of side length $a = \sigma \cdot 2^{1-m}$. Then Q is covered by 16 closed squares of side length $a/4$. Furthermore, there are 20 closed squares of side length $a/4$ adjacent to Q. We call all these 36 squares *admissible* at level $m+1$; they will be denoted by $Q_{\mu,m+1}$, where $\mu = 1, \dots, 36$. The union of these squares form a new square $\tilde{Q}$, which we call the *square attached to* Q. Clearly, the diameter of admissible or attached squares tends to zero as the level tends to infinity. Each admissible and each attached square is a closed subset of U. Note that for each level m there is an admissible square Q at level m with ζ as an interior point (in fact, there are uncountably many).

(b) Idea of the proof of Theorem 2. The next Main Lemma states that if some f-preimage S_f (for some $f \in G$) of some attached square is 'large' in diameter then the degree of $f|_{S_f}$ is 'large'. Then we shall prove that if $\deg(f|_{S_f})$ is 'large' then some image of S_f is 'large', cf Lemma 9. By induction one can prove that the attached square has to be 'large', too. In other words: All preimages of an admissible square are 'small' if the level is sufficiently high, that is to say, if the admissible square is sufficiently small. Again using Lemma 9 we finally prove that the degree of f restricted to some preimage S_f is 'small'.

By assumption **A.3** and after choosing U small enough, we may assume that $\overline{G(AV(G))} \cap U = \emptyset$. Then $f|_V : V \to W$ is a covering for each simply connected domain $W \subset U$, each $f \in G$, and each connected component V of $f^{-1}(W)$. In general, this covering will be a branched covering. In particular, the degree of the mapping $f|_V : V \to W$ is always well defined and finite.

(c) Main Lemma

Lemma 7. *For given $\varepsilon > 0$ and $N \in \mathbf{N}$ there is some $m_0 \in \mathbf{N}$ such that the following holds: If Q is an admissible square at some level $m \geq m_0$ with some center in Q'', $\tilde{Q}$ the corresponding attached square, S_f a connected component of $f^{-1}(\tilde{Q})$ for some $f \in G$, and $\deg(f|_{S_f} : S_f \to \tilde{Q}) \leq N$, then $\mathrm{diam}(K) \leq \varepsilon$ for each connected component K of $f^{-1}(Q) \cap S_f$.*

Proof: Fix $\varepsilon > 0$ and $N \in \mathbf{N}$. If the lemma is false then there exists a sequence $\{m_k\}_{k \in \mathbf{N}}$ converging to ∞, admissible squares Q_{μ_k, m_k} and functions $f_k \in G$ such that $\mathrm{diam}(K_k) \geq \varepsilon > 0$ and $\deg(f_k|_{S_{f_k}}) \leq N$ for some connected component S_{f_k} of $f_k^{-1}(\tilde{Q}_{\mu_k, m_k})$ and some component K_k of $f_k^{-1}(Q_{\mu_k, m_k}) \cap S_{f_k}$. In particular, $\mathrm{diam}(S_{f_k}) \geq \varepsilon > 0$. Suppose S_{f_k} contains a disc D_k of some fixed positive spherical radius r. After transition to a suitable subsequence, we may and will assume $D_k \to D$ with some disk D. Recall that $f_k(D) \subset U$ for every but finitely many $k \in \mathbf{N}$, hence the sequence $\{f_k|_D\}_{k \in \mathbf{N}}$ is bounded. The latter yields $D \subset \mathscr{F}(\{f_k\}_{k \in \mathbf{N}})$. Let W be the component of $\mathscr{F}(\{f_k\}_{k \in \mathbf{N}})$ containing D. After transition to a subsequence, we may and will assume the admissible squares Q_{μ_k, m_k} to converge to some point $\xi \in U$. Actually, ξ is the limit of the centers of Q_{μ_k, m_k} as k tends to ∞. By the definition of the Fatou set this yields the sequence of functions $f_k|_W$ to converge to the constant limit function $g \equiv \xi$ uniformly on compact subsets of W. This is a contradiction to assumption **A.1**.

Hence, $\lim_{k\to\infty} \operatorname{diam}(D_k) = 0$, where D_k denotes the maximal disc contained in S_{f_k}, and the following distortion lemma for proper mappings yields $f_k^{-1} \to$ *constant* on Q_{μ_k, m_k} as k tends to ∞.

Lemma 8 (Lemma 2.2 of [3]). *For every positive integer N and every real number $r \in]0,1[$ there exists a constant C depending on N and r, such that for every proper selfmapping f of the unit disc $\mathbf{D}$ of degree N and every $z_0 \in \mathbf{D}$*

$$H(f(z_0), C) \subset f(H(z_0, r)) \subset H(f(z_0), r)$$

holds.

Here, $H(z, r)$ denotes the hyperbolic ball of radius r centered at z_0.

(d) Proof. Now we are prepared for the crucial part of the proof. Let d be the cardinality of $\mathscr{T}$, which by assumption **A.2** is finite. Choose $N = 2^d$ and $\varepsilon < \frac{\tilde{\varepsilon}}{36N}$. For these two numbers, Lemma 7 gives an integer $m_0 \in \mathbf{N}$.

Lemma 9. *Let $f \in G$, in particular, f has a representation of the form $f = f_{\lambda_1} \circ \cdots \circ f_{\lambda_n}$ with $\lambda_1, \ldots, \lambda_n \in \Lambda$. Let $B \subset U$ open, B' some component of $f^{-1}(B)$, and $\deg(f|_{B'} : B' \to B) > N$. Then there exists some $v \in \{1, \ldots, n-1\}$ with $\operatorname{diam}(B_v) \geq \tilde{\varepsilon}$, where $B_v := f_{\lambda_{n-v}} \circ \cdots \circ f_{\lambda_n}(B')$, and $\deg(f_{\lambda_1} \circ \cdots \circ f_{\lambda_{n-v-1}}|_{B_v} : B' \to B) \leq N$.*

Proof: Recall that d is the number of critical points of the generators f_λ whose orbits intersect U. If $\deg(f|_{B'}) > N$ then there exists some v such that $B_v := f_{\lambda_{n-v}} \circ \cdots \circ f_{\lambda_n}(B')$ contains a critical point c of a generator f_λ with $\lambda \in \Lambda$ and an element of the orbit $G^*(f_\lambda(c))$. By the definition of $\tilde{\varepsilon}$ we obtain $\operatorname{diam}(B_v) \geq \tilde{\varepsilon}$. We choose v to be maximal. Recall that we have assumed $\operatorname{diam}(U) < \tilde{\varepsilon}$. Hence $v < n$. Then $\deg(f_{\lambda_1} \circ \cdots \circ f_{\lambda_{n-v-1}}|_{B_v} : B_v \to B) \leq N$.

Now, let n be the smallest integer such that there is some admissible square $Q := Q_{\mu,m}$ at level $m \geq m_0$ with $\operatorname{diam}(K) > \varepsilon$ for some connected component K of $f_w^{-1}(Q)$, some $f_w \in G$, and some n-word $w \in \Lambda^n$. We have $n \geq 1$. Let $\tilde{Q}$ be the square attached to Q. Then Lemma 7 gives $\deg(f_w|_S) > N$ for some connected component S of $f_w^{-1}(\tilde{Q})$. By Lemma 9 there exists some v satisfying $1 \leq v < n$ such that $\operatorname{diam}(f_{\tilde{w}}(S)) > \tilde{\varepsilon}$ for the v-word $\tilde{w} := (w_{n-v+1}, \ldots, w_n) \in \Lambda^v$. We write $\tilde{S} := f_{\tilde{w}}(S)$ and $\hat{w} = (w_1, \ldots, w_{n-v})$. We have $\deg(f_{\hat{w}}|_{\tilde{S}}) \leq N$ and $\tilde{S} \subset \bigcup f_{\hat{w}}^{-1} \cdot (Q_{\tilde{\mu}, m+1})$. Here the union is taken over the $36 \cdot N$ preimages (there are 36 admissible squares $Q_{\tilde{\mu}, m+1}$ at level $m+1 \geq m_0$ forming $\tilde{Q}$, and due to the bound on the degree we have to take into account at most N branches of $f_{\hat{w}}^{-1}$). This yields

$\tilde{\varepsilon} < \mathrm{diam}(f_{\tilde{w}}(S)) \le 36N\varepsilon$. By definition, $36N\varepsilon < \tilde{\varepsilon}$, a contradiction. This proves $n = \infty$.

(e) Summary. For $\varepsilon < \varepsilon_0 := \dfrac{\tilde{\varepsilon}}{36N} < \tilde{\varepsilon}$ arbitrarily small, let m_0 be as in Lemma 7. For each $f \in G$, each admissible square $Q_{\mu,m}$ with $m \ge m_0$, and each connected component K of $f^{-1}(Q_{\mu,m})$ with some $f \in G$, we have proved

$$\mathrm{diam}(K) \le \varepsilon.$$

Lemma 9 yields

$$\deg(f|_K) \le N.$$

In particular, this completes the proof of $\zeta \subset Q \subset SH_N$.

We now turn our attention to Corollary 3. Let $\{w_n\}_{n \in \mathbf{N}}$ be a sequence of words w_n of strictly increasing length. In particular, the length is converging to ∞. Let Q be the admissible square at level m_0 centered at ζ, $D_\delta(\zeta) \subset\subset Q$, and K_n be a connected component of $f_{w_n}^{-1}(D_\delta(\zeta))$. Then as in the proof of the Main lemma one shows that the inner diameter of $f_{w_n}^{-1}(Q)$ converge to zero as n tends to ∞. The distortion lemma for proper mappings, cf 8, yields $\lim_{n \to \infty} \mathrm{diam}(K_n) = 0$. This completes the proof of the corollary.

4. Examples for a semihyperbolic semigroups

In this section we present some examples of semihyperbolic semigroups.

Example 1. The first example is a semihyperbolic entire transcendental semigroup which is not subhyperbolic. Recall that a semigroup G is called subhyperbolic provided that the closure Ω of the postsingular set of G satisfies $\#(\Omega \cap \mathscr{J}(G)) < \infty$. We consider the functions

$$f(z) := 2a(e^{(z-a)^3} - e^{-a^3}); \qquad a > 0 \qquad \text{and}$$

$$g(z) := \mu(e^z - 1); \qquad \mu \in]0,1[.$$

We begin with studying the function g. Clearly, the origin is an attracting fixed point of g. Furthermore, it is the only singular value of g. In fact, it is an asymptotic value of g. Note that g is strictly monotonically increasing on the real axis. Furthermore, for each $x \in]-\infty, 0[$ we obtain $-\mu < g(x) < 0$. This proves $]-\infty, 0] \subset A_g^*(0)$. Here $A_g^*(0)$ denotes the immediate basin of attraction of the origin with respect to the iteration of g. By looking at g' one can prove the existence of some $\delta_g > 0$ such that $U := D_{\delta_g}(0) \cup \{z = x + iy \mid -\infty < x < 0, |y| < \delta_g\}$ is forward invariant with respect to g and, consequently, a subset of $A_g^*(0)$. We fix $\mu \in]0,1[$. Then for some sufficiently large $a_g \in \mathbf{R}$ the monotonicity of g gives

$$x \ge a_g \Rightarrow g(x) \ge \frac{3}{2}x. \qquad (1)$$

Note that g is of finite type, in particular, g has neither Baker domains nor wandering domains. Thus the last formula implies $[a_g, \infty[\subset J(g)$.

We now turn our attention to f. Clearly, the origin is a fixed point of f. After choosing $a \in]0, \infty[$ sufficiently large, we obtain

$$f'(0) = 6a^3 \cdot e^{-a^3} < 1.$$

This proves that the origin is an attracting fixed point of f, too. Note that f is strictly monotonically increasing on $\mathbf{R}$. In particular, for each $x \in]-\infty, 0[$ we obtain $-2a \cdot e^{-a^3} < f(x) < 0$. This proves $]-\infty, 0] \subset A_f^*(0)$. Again one can find some $\delta \in]0, \delta_g]$ such that $U := D_\delta(0) \cup \{z = x + iy \mid -\infty < x < 0, |y| < \delta\}$ is also forward invariant with respect to f and, consequently, a subset of $A_f^*(0)$. Note that $-2a \cdot e^{-a^3} \in]-\infty, 0]$ is the only asymptotic value of f, and that a is the only critical point of f. Let $v := f(a)$. For a sufficiently large and $v \geq a_g$ the monotonicity of f gives

$$x \geq v \Rightarrow f(x) \geq \frac{3}{2}x. \tag{2}$$

This proves $\omega_f(x) = \{\infty\}$ for each $x \in [c, \infty[$. Note that f is of finite type, in particular, it has neither Baker domains nor wandering domains. Thus $[a, \infty[\subset \mathscr{J}(f)$.

Let $G := \langle f, g \rangle$. Clearly, G is an entire transcendental semigroup. By construction,

$$\omega_G(AV(G)) \subset]-\infty, 0] \subset U \subset A_G(0) \subset \mathscr{F}(G).$$

Furthermore we have

$$[v, \infty[\subset \mathscr{J}(g) \cup \mathscr{J}(f) \subset \mathscr{J}(G).$$

Let $w \in \{f, g\}^n$ be an n-word. By the equations (1) and (2) we obtain

$$f_w(v) \geq \left(\frac{3}{2}\right)^n v. \tag{3}$$

In particular, this proves $\omega_G(v) = \{\infty\}$.

We summarize. First of all, we have $SV(G) \subset \mathbf{R}$. Consequently,

$$\mathbf{C} \backslash \mathbf{R} \subset SH_1.$$

Furthermore, $O_G(AV(G)) \subset\subset \mathscr{F}(G)$, and $O_G(CV(G))$ is a discrete subset of $[v, \infty[\subset \mathscr{J}(G)$. Combining the latter statement with equation (3) proves $\mathbf{R} \cap \mathscr{J}(G) \subset SH_2$. Altogether we have proved $\mathscr{J}(G) \subset SH_2$, in particular, G is semihyperbolic. But $O_G(V)$ is not finite, in fact, both, $O_f(v)$ and $O_g(v)$ are infinite. Thus G is not subhyperbolic.

Example 2. This is an example for a semihyperbolic entire transcendental semigroup such that its Julia set has empty interior.

Let $\mathscr{R} := \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$ and $\mathscr{L} := \{z \in \mathbf{C} \mid \mathrm{Re}(z) < 0\}$ the right and left half plane, respectively. For some number $t \in]0,1[$ we define $f_1(z) = te^z - 1$ and $f_2(z) = te^{z+\pi i} - 1$. Let $G := \langle f_1, f_2 \rangle$. One readily proves

$$f_j(\mathscr{L}) = \{z \in \mathbf{C} \mid |z+1| < t\} =: \mathscr{D} \subset\subset \mathscr{L}$$

and

$$|f_j'(z)| \le t < 1$$

for each $j = 1, 2$ and $z \in \mathscr{L}$. This proves $\mathscr{D} \subset \mathscr{L} \subset \mathscr{F}(G)$. Clearly, $CV(G) = \emptyset$ and $AV(f_j) = \{-1\} \subset \mathscr{D}$. This proves

$$G(SV(G)) \subset\subset \mathscr{L} \subset \mathscr{F}(G),$$

thus $\mathscr{J}(G) \subset SH_1$, in particular, G is semihyperbolic. We now turn our attention to

Claim. $\mathrm{Int}(\mathscr{J}(G)) = \emptyset$.

The main ingredient in the proof is the so-called 'open set condition'.

Definition 4. Let $G = \langle f_1, \dots, f_n \rangle$ be a finitely generated semigroup. It is said to satisfy the open set condition (with respect to the generators $f_1, \dots, f_n$) if there exists some open set $U \subset \mathbf{C}$ such that the sets $f_j^{-1}(U)$ are mutually disjoint and $f_j^{-1}(U) \subset U$ holds for each generator f_j.

We prove that in this example one can choose $U := \mathscr{R}$. We have already shown $f_j(\mathscr{L}) \subset \mathscr{L}$, this yields

$$f_j^{-1}(\mathscr{R}) \subset \mathscr{R}.$$

Let $z := x + iy \in \mathscr{R}$. Then

$$|(\pi - \mathrm{Im}\,(f_2^{-1}(z))) \mod 2\pi| < \frac{\pi}{2}.$$

But

$$\mathrm{Im}\,(f_1^{-1}(z)) \mod 2\pi \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 1\right].$$

Hence, G satisfies the open set condition with respect to the generators f_1 and f_2. Note that $f_j(\mathscr{L}) \subset \mathscr{D} \subset\subset \mathscr{L} \subset \mathscr{F}(G)$. The continuity of the generators f_1 and f_2 gives the existence of some open neighborhood (with respect to the complex plane) of $\overline{\mathscr{L}}$ contained in $\mathscr{F}(G)$. This in turn implies $\mathscr{R} \backslash \mathscr{J}(G) \neq \emptyset$. The claim now follows from [14, Prop. 4.3].

Note that Sumi has proved the proposition for rational semigroups, only. But the reader will immediately see that the proof carries over to entire transcendental semigroups.

Example 3. For some semigroup G and some polynomial f, let $P(G)$ or $P(f)$ denotes the postcritical set of G respectively f. Let $c \in \mathbf{C}$ be a point such that

$c \in \mathcal{J}(f_c)$ and that c is not recurrent (with respect to f_c) but $\omega_{f_c}(c)$ is infinite. In particular, $f_c(z) = z^2 + c$ is semihyperbolic but not subhyperbolic. Then $P(f_c) \subset \mathcal{J}(f_c)$. Let $g(z) = (z-a)^2 + a$ where $a \in \mathbf{C}$ is a point such that $\mathcal{J}(f_c)$ is included in $A_\infty(g)$ which is the connected component of $\mathcal{F}(g)$ containing ∞. Let U be an open disk such that $\mathcal{J}(f_c) \cup \mathcal{J}(g) \subset U$. There exists a number $n \in \mathbf{N}$ such that $\overline{f_c^{-n}(U)} \subset U$, $\overline{g^{-n}(U)} \subset U$, $\overline{f_c^{-n}(U)} \cap \overline{g^{-n}(U)} = \emptyset$, and $\overline{f_c^{-n}(U)} \subset A_\infty(g)$. Let $f_1 = f_c^n$, $f_2 = g^n$, and $G = \langle f_1, f_2 \rangle$. Then G satisfies the open set condition with the open set U. Hence the interior of $\mathcal{J}(G)$ is empty. This shows that the assumption **A.3** in the Main Theorem holds. By construction and the choice of n we have $P(G) \cap \mathcal{J}(G) = P(f)$ and

$$P(G) \cap \mathcal{F}(G) \subset \{a\} \cup (\mathbf{P}_1 \backslash U) \subset\subset \mathcal{F}(G).$$

So the assumption **A.2** holds, too. For any $z_0 \in \mathcal{F}(G)$ we have $\omega_G(z_0) \subset \{a, \infty\}$. This proves that the assumption **A.1** is satisfied. By the Main Theorem we conclude that G is semihyperbolic. Since f_c is not subhyperbolic, G is not subhyperbolic.

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