Non-i.i.d. random holomorphic dynamical systems and the probability of tending to infinity *

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Abstract

We consider random holomorphic dynamical systems on the Riemann sphere whose choices of maps are related to Markov chains. Our motivation is to generalize the facts which hold in i.i.d. random holomorphic dynamical systems. In particular, we focus on the function $\mathbb{T}_{\infty,\tau}$ which represents the probability of tending to infinity. We show some sufficient conditions which make $\mathbb{T}_{\infty,\tau}$ continuous on the whole space and we characterize the Julia sets in terms of the function $\mathbb{T}_{\infty,\tau}$ under certain assumptions.

1 Introduction

1.1 Background

We consider discrete-time random dynamical systems which are **not i.i.d.** The theory of random dynamics is rapidly growing both theoretically and experimentally. We focus in this paper on random **holomorphic** dynamical systems on the Riemann sphere $\widehat{\mathbb{C}}$ from the mathematical viewpoint. Using complex analysis, we can investigate the systems deeply.

The first study of random holomorphic dynamics was given by Fornaess and Sibony [5]. They investigated independent and identically-distributed (i.i.d.) random dynamical systems on $\widehat{\mathbb{C}}$ constructed by small perturbations $\{f_c\}_{c\in B(c_0,\delta)}$ of a rational map $f_{c_0}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which depend holomorphically on the parameter c, where $B(c_0,\delta)$ denotes the open ball with center c_0 and radius δ endowed with the normalized Lebesgue measure. They showed that, if δ is small and f_{c_0} has $k \geq 1$ attractive cycles $\gamma_1,\dots,\gamma_k$, then there exist k continuous functions $T_{\gamma_j}\colon \widehat{\mathbb{C}} \to [0,1]\,(j = 1,\dots,k)$ with the following properties: (i) $\sum_{j=1}^{k} T_{\gamma_j}(z) = 1$ for all $z \in \widehat{\mathbb{C}}$, and (ii) for all $z \in \widehat{\mathbb{C}}$, the random orbit $f_{c_N}\circ\cdots\circ f_{c_2}\circ f_{c_1}(z)$

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tends to the attractive basin of γ_j as $N \to \infty$ with probability $T_{\gamma_j}(z)$. (See [5, Theorem 0.1].)

In [15] the first author generalized this theorem to the case where noise is not small and deeply analyzed the function T_A which represents the probability of tending to an attracting minimal set A. These results are called the Cooperation Principles. His strategy is to consider both random dynamics of rational maps and dynamics of rational semigroups, which are semigroups of non-constant rational maps on $\widehat{\mathbb{C}}$ where the semigroup operation is functional composition. For details on rational semigroups, see [6], [13], [14].

The first author introduced the random relaxed Newton method in [17] and suggested that the random relaxed Newton method might be a more useful method to compute the roots of polynomials than the classical deterministic Newton method. The key is that sufficiently large noise collapses bad attractors and makes the system more stable.

These works find new phenomena which cannot hold in deterministic dynamics. The phenomena are called **noise-induced phenomena** or **randomness-induced phenomena**, which are of great interest from the mathematical viewpoint. For more research on random holomorphic dynamical systems and related fields, see [2], [4], [7], [8], [10], [11], [14], [15], [17], [18].

However, most of the previous studies concern i.i.d. random dynamical systems. It is very natural to generalize the settings and consider non-i.i.d random dynamical systems. In this paper, we especially treat **random dynamical systems with "Markovian rules" whose noise depends on the past**.

We extend the theory of i.i.d. random dynamical systems and **we find new (noise-induced) phenomena which cannot hold in i.i.d. random dynamical systems**. Moreover, our studies may be applied to the skew products whose base dynamical systems have Markov partitions.

We believe that this research will contribute not only toward mathematics but also toward applications to the real world. One motivation for studying dynamical systems is to analyze mathematical models used in the natural or social sciences. Since the environment changes randomly, it is natural to investigate random dynamical systems which describe the time evolution of systems with probabilistic terms. In this sense, it is quite important to understand "Markovian" noise because there are a lot of systems whose noise depends on the past.

Therefore the study of Markov random dynamical systems is natural and meaningful from both the pure and applied mathematical viewpoint. In this paper we aim to generalize the theory of i.i.d. random holomorphic dynamical systems and the theory of rational semigroups simultaneously to the setting of random dynamical systems with Markovian rules and the associated set-valued dynamical systems.

It is essentially new to consider the set-valued dynamical systems with Markovian rules itself, which we call graph directed Markov systems. Although our concept is similar to that of [9], these are completely different. In [9] Mauldin and Urbański are concerned with the limit sets of systems of contracting maps, but in this paper we discuss the Julia sets of general continuous maps and clarify the relationship between the Julia sets of rational semigroups and that of graph directed Markov systems.

1.2 Main results

We now introduce our rigorous settings and present our main results. Let Rat be the space of non-constant holomorphic maps on $\widehat{\mathbb{C}}$ and let $m \in \mathbb{N}$. We endow Rat with the distance κ defined by $\kappa(f, g) := \sup_{z \in \widehat{\mathbb{C}}} d(f(z), g(z))$ where d denotes the spherical distance on $\widehat{\mathbb{C}}$. Suppose that m^2 Borel measures $(\tau_{ij})_{i,j=1,\dots,m}$ on Rat satisfy $\sum_{j=1}^m \tau_{ij}(\mathrm{Rat}) = 1$ for all $i = 1, \dots, m$. For the given $\tau = (\tau_{ij})_{i,j=1,\dots,m}$, we consider the Markov chain on $\widehat{\mathbb{C}} \times \{1, \dots, m\}$ whose transition probability from $(z, i) \in \widehat{\mathbb{C}} \times \{1, \dots, m\}$ to $B \times \{j\}$ is

$$\mathbb{P}\left((z, i), B \times \{j\}\right) = \tau_{ij}(\{f \in \mathrm{Rat}; f(z) \in B\})$$

where B is a Borel subset of $\widehat{\mathbb{C}}$ and $j \in \{1, \dots, m\}$. This system is called the *rational Markov random dynamical system* (rational MRDS for short) induced by τ . For the rest of this subsection, we consider such systems.

Roughly speaking, the MRDS induced by $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ describes the following random dynamical system on the phase space $\hat{\mathbb{C}}$. Fix an initial point $z_0 \in \widehat{\mathbb{C}}$ and choose a vertex $i = 1, \dots, m$ (with some probability if we like). We choose a vertex $i_1 = 1, \dots, m$ with probability $\tau_{ii_1}(\mathrm{Rat})$ and choose a map f_1 according to the probability distribution $\tau_{ii_1}/\tau_{ii_1}(\mathrm{Rat})$. Repeating this, we randomly choose a vertex i_n and a map f_n for each n-th step. We in this paper investigate the asymptotic behavior of random orbits of the form $f_n \circ \cdots \circ f_2 \circ f_1(z_0)$.

In particular, we can apply functional analytical method by extending the phase space from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}} \times \{1, \dots, m\}$. More precisely, we consider iterations of a single transition operator, and it enables us to analyze the MRDS and the above random dynamical system deeply (see Section 3).

Define the vertex set as $V := \{1, 2, \dots, m\}$ and the directed edge set as

$$E := \{(i, j) \in V \times V;\ \tau_{ij}(\mathrm{Rat}) > 0\}.$$

We set $S_\tau := (V, E, (\operatorname{supp} \tau_e)_{e \in E})$ and utilize the terminology of directed graphs following [9]. We define $i : E \to V$ (resp. $t : E \to V$) as the projection to the first (resp. second) coordinate and we call $i(e)$ (resp. $t(e)$) the initial (resp. terminal) vertex of $e \in E$.

A word $e = (e_1, e_2, \dots, e_N) \in E^N$ with length $N \in \mathbb{N}$ is said to be *admissible* if $t(e_n) = i(e_{n+1})$ for all $n = 1, 2, \dots, N-1$. For this word e, we call $i(e_1)$ (resp. $t(e_N)$) the initial (resp. terminal) vertex of e and we denote it by $i(e)$ (resp. $t(e)$). For each $i, j \in V$, we define the following sets.

$$\begin{aligned}
H_i^j(S) &:= \{f_N \circ \cdots \circ f_1;\ \exists N \in \mathbb{N}, \exists e = (e_1, \dots, e_N) \in E^N \text{ s.t.} \\
&\qquad f_n \in \operatorname{supp} \tau_{e_n} (\forall n = 1, \dots, N) \text{ and } e \text{ is admissible with } i(e) = i, t(e) = j\}, \\
J_i(S_\tau) &:= \{z \in \widehat{\mathbb{C}};\ \bigcup_{j \in V} H_i^j(S_\tau) \text{ is not equicontinuous on any neighborhood of } z\}, \\
J_{\ker,i}(S_\tau) &:= \bigcap_{j \in V : H_i^j(S_\tau) \neq \emptyset} \ \bigcap_{h \in H_i^j(S)} h^{-1}(J_j(S)).
\end{aligned}$$

The compact set $J_i(S_\tau)$ is called the Julia set at $i \in V$, which is the set of all initial points where the dynamical system sensitively depends on initial conditions. The subset $J_{\ker,i}(S_\tau)$ is called the kernel Julia set at $i \in V$.

To present our main results, we introduce the following Borel probability measures $\tilde{\tau}_i$ on $(\mathrm{Rat} \times E)^{\mathbb{N}}$.

Definition 1.1. We define Borel probability measures $\tilde{\tau}_i$ $(i = 1, \dots, m)$ on $(\mathrm{Rat} \times E)^{\mathbb{N}}$ by

$$\begin{aligned}
&\tilde{\tau}_i \left(A'_1 \times \cdots \times A'_N \times \prod_{N+1}^{\infty} (\mathrm{Rat} \times E) \right) \\
&= \begin{cases} \tau_{e_1}(A_1) \cdots \tau_{e_N}(A_N), & \text{if } (e_1, \dots, e_N) \text{ is admissible with } i(e_1) = i \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

for N Borel sets A_n ($n = 1, \dots, N$) of Rat and for $(e_1, \dots, e_N) \in E^N$ where $A'_n = A_n \times \{e_n\}$.

For each element $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}}$ of $\operatorname{supp}\tilde{\tau}_i$ we can naturally consider the non-autonomous dynamics of ξ and define the Julia set J_ξ as the set of non-equicontinuity of $\{\gamma_N \circ \cdots \circ \gamma_1\}_{N\in\mathbb{N}}$. The following is a partial generalization of the Cooperation Principle.

Main Result A (Proposition 3.11)**.** If $J_{\ker,j}(S_\tau) = \emptyset$ for all $j \in V$, then the "averaged system" is stable and the non-autonomous Julia set J_ξ is of (Lebesgue) measure-zero for $\tilde{\tau}_i$-almost every ξ.

We say that the system S_τ is *irreducible* if the directed graph (V, E) is strongly connected. Using the theory of rational semigroups, we have the following result.

Main Result B (Corollary 4.3)**.** If S_τ is irreducible and $\#J_j(S_\tau) \geq 3$ for some $j = 1, \dots, m$, then

$$J_i(S_\tau) = \overline{\bigcup_{h\in H_i^i(S_\tau)} \{\text{repelling fixed points of } h\}} = \overline{\bigcup_{\xi\in\operatorname{supp}\tilde{\tau}_i} J_\xi}$$

for all $i = 1, \dots, m$. Here $\#A$ denotes the cardinality of a set A.

We next focus on systems of polynomial maps on $\widehat{\mathbb{C}}$ and the functions which represent the probability of tending to infinity. For the rest of this subsection, suppose that S_τ is irreducible and $\operatorname{supp}\tau_e$ is a compact subset of the space Poly of all polynomial maps on $\widehat{\mathbb{C}}$ of degree 2 or more for each $e \in E$.

Definition 1.2. We define the function $\mathbb{T}_{\infty,\tau} \colon \widehat{\mathbb{C}} \times V \to [0, 1]$ by

$$\mathbb{T}_{\infty,\tau}(z, i) := \tilde{\tau}_i(\{\xi = (\gamma_n, e_n)_{n\in\mathbb{N}};\, d(\gamma_n \circ \cdots \circ \gamma_1(z), \infty) \to 0\, (n \to \infty)\})$$

for any point $(z, i) \in \widehat{\mathbb{C}} \times V$.

We have the following results regarding the relation between the kernel Julia sets $J_{\ker,i}(S_\tau)$ and the continuity of $\mathbb{T}_{\infty,\tau}$.

Main Result C (Proposition 4.24)**.** If $J_{\ker,j}(S_\tau) = \emptyset$ for some $j \in V$, then $\mathbb{T}_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}} \times V$.

Main Result D (Corollary 4.14 (ii))**.** Suppose that there exists $e \in E$ such that

$$\operatorname{supp}\tau_e \supset \{f + c; |c - c_0| < \varepsilon\}$$

for some $f \in$ Poly, $c_0 \in \mathbb{C}$ and $\varepsilon > 0$. Then $J_{\ker,j}(S_\tau) = \emptyset$ for some $j \in V$ and hence $\mathbb{T}_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}} \times V$.

Roughly speaking, if there are sufficiently many maps in one system, then the maps cooperate with one another and thereby eliminate the chaos on average. Consequently the function is continuous on the whole space. This phenomenon cannot hold in deterministic dynamical systems since $\mathbb{T}_{\infty,\tau}$ takes the value 0 on the filled-in Julia set and the value 1 outside of it.

Let us consider systems with finite maps. In this case, we need certain conditions which make $\mathbb{T}_{\infty,\tau}$ continuous.

Definition 1.3. We say that a system S_τ satisfies the *backward separating condition* if $f_1^{-1}(J_{t(e_1)}(S)) \cap f_2^{-1}(J_{t(e_2)}(S)) = \emptyset$ for every $e_1, e_2 \in E$ with the same initial vertex and for every $f_1 \in \operatorname{supp}\tau_{e_1}$, $f_2 \in \operatorname{supp}\tau_{e_2}$, except the case $e_1 = e_2$ and $f_1 = f_2$.

We say that S_τ is *essentially non-deterministic* if there exist $e_1, e_2 \in E$ with $i(e_1) = i(e_2)$ and there exist $f_1 \in \operatorname{supp} \tau_{e_1}, f_2 \in \operatorname{supp} \tau_{e_2}$ such that either $e_1 \neq e_2$ or $f_1 \neq f_2$.

We now present a result for systems with finite maps regarding the continuity of $\mathbb{T}_{\infty,\tau}$ and the set of points where $\mathbb{T}_{\infty,\tau}$ is not locally constant.

Main Result E (Lemma 2.23, Proposition 4.10, Theorem 4.29)**.** Suppose that the polynomial system S_τ satisfies the backward separating condition. If $\operatorname{supp} \tau_e$ is finite for each $e \in E$, then $J_i(S_\tau)$ has no interior points for each $i \in V$ and we have either $\mathbb{T}_{\infty,\tau} \equiv 1$ or

$$J_i(S_\tau) = \{z \in \mathbb{C}; \mathbb{T}_{\infty,\tau}(\cdot, i) \text{ is not constant on any neighborhood of } z\}$$

for each $i \in V$. Moreover, if additionally S_τ is essentially non-deterministic, then $\mathbb{T}_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}} \times V$.

The former part of Main Result E is a generalization of the classical fact that the Julia set of polynomial f of degree 2 or more is the boundary of the filled-in Julia set of f. However, the latter part of Main Result E indicates a kind of randomness-induced phenomenon and is a generalization of the fact known in i.i.d. cases [15, Lemma 3.75].

We next present the applications of the main results. For a fixed $m \in \mathbb{N}$, given $f_1, \dots, f_m \in \text{Poly}$ and a given irreducible stochastic matrix $P = (p_{ij})_{i,j=1,\dots,m}$, we define τ_{ij} as the measure $p_{ij}\delta_{f_i}$, where δ_{f_i} denotes the Dirac measure at f_i. We consider the polynomial MRDS induced by $\tau = (\tau_{ij})$. Let $p = (p_1, \dots, p_m)$ be the positive vector such that $\sum_{i=1}^m p_i = 1$ and $pP = p$. Set $T_{\infty,\tau} \colon \widehat{\mathbb{C}} \to [0,1]$ as $T_{\infty,\tau}(z) := \sum_{i=1}^m p_i \mathbb{T}_{\infty,\tau}(z, i)$. In other words, we consider the random dynamical system whose choice of maps is as follows: we choose a map f_{i_1} with probability p_{i_1} at the first step, and after choosing a map f_{i_N} we choose the next map $f_{i_{N+1}}$ with probability $p_{i_N i_{N+1}}$ at the $(N+1)$-st step, where $i_1, \dots, i_N, i_{N+1} \in \{1, \dots, m\}$.

Corollary 1.4 (Corollary 4.30)**.** Suppose that $T_{\infty,\tau} \not\equiv 1$ and $J_i(S_\tau) \cap J_j(S_\tau) = \emptyset$ for all $i, j \in V$ with $i \neq j$. Then $\bigcup_{i \in V} J_i(S_\tau)$ has no interior points and

$$\bigcup_{i \in V} J_i(S_\tau) = \{z \in \mathbb{C}; T_{\infty,\tau} \text{ is not constant on any neighborhood of } z\}.$$

Moreover, if there exist $i, j, k \in \{1, \dots, m\}$ such that $p_{ij} > 0$ and $p_{ik} > 0$ in addition to the assumption above, then $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$.

There are new phenomena in Markov random dynamical systems which cannot hold in i.i.d. random dynamical systems. More precisely, in the i.i.d. case, if the function $T_{\infty,\tau}$ is not identically 1, then there exists $z_0 \in \mathbb{C}$ such that $T_{\infty,\tau}(z_0) = 0$ (see [15] or Lemma 4.22). However, we show the following result. See also Figure 1 and Figure 2.

Main Result F (Proposition 4.23 and Example 4.25)**.** There exists $\tau = (\tau_{ij})_{i,j}$ with $\operatorname{supp} \tau_{ij} \subset \text{Poly}$ such that $T_{\infty,\tau}$ is continuous, $T_{\infty,\tau} \not\equiv 1$, $T_{\infty,\tau}(z) > 0$ for each $z \in \widehat{\mathbb{C}}$ and S_τ is irreducible.

1.3 Organization

This paper is organized as follows. In Section 2, we introduce graph directed Markov systems on a compact metric space (Y, d) and discuss some basic concepts. Although we are most interested in holomorphic dynamics, we first treat such systems on general compact metric spaces in order to show more generality. The concept of graph directed Markov systems is similar to rational semigroups in the theory of i.i.d. random holomorphic

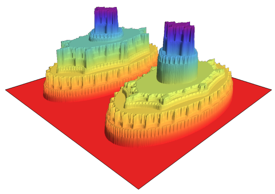

Figure 1: The graph of $1-T_{\infty,\tau}$ with a new phenomenon which cannot hold in i.i.d. random dynamical systems of polynomials.

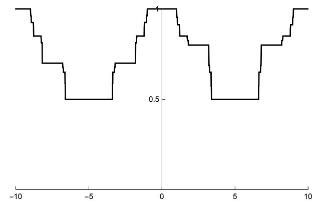

Figure 2: The graph of the function $T_{\infty,\tau}$ on the real line

dynamical systems. We define some kinds of Julia sets and show fundamental properties. We also discuss the dynamics of Markov operators following [15]. In Section 3, we consider Markov random dynamical systems that are induced by given families τ of measures; we define probability measures on the space of infinite product of $\mathrm{CM}(Y)$ and define a Markov operator M_τ induced by τ. Furthermore, we prove Main Result A. In Section 4, we focus on holomorphic dynamical systems on the Riemann sphere $\widehat{\mathbb{C}}$. We consider *rational* graph directed Markov systems in subsection 4.1, and prove Main Result B and other fundamental properties. In subsection 4.2, we investigate *polynomial* Markov random dynamical systems and prove Main Results C, D and E.

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2 Preliminaries

In this section, we introduce graph directed Markov systems on a compact metric space (Y, d) and discuss some basic concepts. These systems are similar to rational semigroups in the theory of i.i.d. random holomorphic dynamical systems. In subsection 2.1, we define some kinds of Julia sets and show fundamental properties of them. In subsection 2.2, we give the definition of skew product maps associated with graph directed Markov systems and consider its dynamics. In subsection 2.3, we discuss dynamics of Markov operators following [15].

2.1 Julia sets of graph directed Markov systems

Notation 2.1. We denote by $\mathrm{CM}(Y)$ the set of all continuous maps from Y to itself and we define a metric κ on $\mathrm{CM}(Y)$ by $\kappa(f,g) := \sup_{y\in Y} d(f(y), g(y))$. The space $\mathrm{CM}(Y)$ is a separable complete metric space since Y is compact. We denote by $\mathrm{OCM}(Y)$ the set of all open continuous maps from Y to itself.

Definition 2.2. Let (V, E) be a directed graph with finite vertices and finite edges, and let Γ_e be a non-empty subset of $\mathrm{CM}(Y)$ indexed by a directed edge $e \in E$. We call $S = (V, E, (\Gamma_e)_{e\in E})$ a graph directed Markov system (GDMS for short) on Y. The symbol $i(e)$ (resp. $t(e)$) denotes the initial (resp. terminal) vertex of each directed edge $e \in E$.

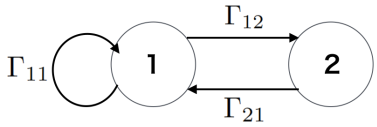

Figure 3: A schematic diagram of a GDMS

In the following, $S = (V, E, (\Gamma_e)_{e\in E})$ denotes a GDMS on Y.

Definition 2.3. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a GDMS.

(i) A word $e = (e_1, e_2, \ldots, e_N) \in E^N$ with length $N \in \mathbb{N}$ is said to be admissible if $t(e_n) = i(e_{n+1})$ for all $n = 1, 2, \ldots, N-1$. For this word e, we call $i(e_1)$ (resp. $t(e_N)$) the initial (resp. terminal) vertex of e and we denote it by $i(e)$ (resp. $t(e)$).

(ii) We set

$$\begin{aligned} H(S) &:= \{f_N \circ \cdots \circ f_2 \circ f_1; \\ &\quad N \in \mathbb{N}, f_n \in \Gamma_{e_n}, t(e_n) = i(e_{n+1})(\forall n = 1, \ldots, N-1)\}, \\ H_i(S) &:= \{f_N \circ \cdots \circ f_2 \circ f_1 \in H(S); \\ &\quad N \in \mathbb{N}, f_n \in \Gamma_{e_n}, t(e_n) = i(e_{n+1})(\forall n = 1, \ldots, N-1), i = i(e_1)\}, \\ H_i^j(S) &:= \{f_N \circ \cdots \circ f_2 \circ f_1 \in H(S); \\ &\quad N \in \mathbb{N}, f_n \in \Gamma_{e_n}, t(e_n) = i(e_{n+1})(\forall n = 1, \ldots, N-1), i = i(e_1), t(e_N) = j\}. \end{aligned}$$

Now we define the Fatou sets and Julia sets of GDMSs. Recall that a subset $\mathcal{F} \subset \mathrm{CM}(Y)$ is said to be equicontinuous at a point $y \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f \in \mathcal{F}$ and for every $z \in Y$ with $d(y, z) < \delta$, we have $d(f(y), f(z)) < \varepsilon$. A subset $\mathcal{F} \subset \mathrm{CM}(Y)$ is said to be equicontinuous on a subset $U \subset Y$ if $\mathcal{F}$ is equicontinuous at every point of U. See [1].

Definition 2.4. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a GDMS.

(i) We denote by $F(S)$ the set of all points $y \in Y$ for which there exists a neighborhood U in Y such that the family $H(S)$ is equicontinuous on U. $F(S)$ is called the Fatou set of S and the complement $J(S) := Y \setminus F(S)$ is called the Julia set of S.

(ii) For each $i \in V$, we denote by $F_i(S)$ the set of all points $y \in Y$ for which there exists a neighborhood U in Y such that the family $H_i(S)$ is equicontinuous on U. $F_i(S)$ is called the Fatou set of S at the vertex i and the complement $J_i(S) := Y \setminus F_i(S)$ is called the Julia set of S at the vertex i.

(iii) Set $\mathbb{F}(S) := \bigcup_{i\in V} F_i(S) \times \{i\}, \mathbb{J}(S) := \bigcup_{i\in V} J_i(S) \times \{i\}$.

Remark 2.5. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a GDMS with just one vertex and just one edge, say $V = \{1\}, E = \{(1,1)\}$. Then $H(S) = H_1(S)$ coincides with the semigroup generated by $\Gamma_{(1,1)}$, where the product is the composition of maps, and the Fatou (resp. Julia) set of the GDMS S coincides with the Fatou (resp. Julia) set of the semigroup $H(S)$ of continuous maps on Y. By definition, the Fatou set of a semigroup G of continuous maps on Y is the set of all points $y \in Y$ for which there exists a neighborhood U in Y such that the family G is equicontinuous on U. The Fatou sets of semigroups are related to i.i.d. random dynamical systems. See [13], [15].

Remark 2.6. The Fatou sets $F(S)$ and $F_i(S)$ are open subsets of Y and the Julia sets $J(S)$ and $J_i(S)$ are compact subsets of Y. Moreover, we have $F(S) = \bigcap_{i \in V} F_i(S)$ and $J(S) = \bigcup_{i \in V} J_i(S)$.

Notation 2.7. For families $\mathcal{F}_1, \mathcal{F}_2 \subset \mathrm{CM}(Y)$ of maps, define

$$\mathcal{F}_2 \circ \mathcal{F}_1 := \{f_2 \circ f_1 \, ; \, f_1 \in \mathcal{F}_1 \text{ and } f_2 \in \mathcal{F}_2\}.$$

Notation 2.8. $B_d(y, r)$ denotes the ball of radius r and center y with respect to the metric d in the space Y. It is also denoted briefly by $B(y, r)$.

Lemma 2.9. Suppose that Y is locally connected, that $\mathcal{F} \subset \mathrm{CM}(Y)$ is not equicontinuous at a point $y_0 \in Y$ and that $h \in \mathrm{CM}(Y)$ satisfies the following condition.

> $\sup\{\operatorname{diam} C; \, C \text{ is a connected component of } h^{-1}(B(y, \varepsilon))\} \to 0$ as $\varepsilon \to 0$ for any point $y \in Y$.

Then $h \circ \mathcal{F}$ is not equicontinuous at the point y_0.

Proof. We first note that the following. Since $\mathcal{F}$ is not equicontinuous at a point $y_0 \in Y$, there exists a positive real number ϵ_0 such that for any $\delta_0 > 0$ there exists $y \in B(y_0, \delta_0)$ and $f \in \mathcal{F}$ such that $d(f(y), f(y_0)) \geq \epsilon_0$. By the assumption, for each $z \in Y$, there exists $\varepsilon > 0$ such that the diameter of each connected component of $h^{-1}(B(z, \varepsilon))$ is less than ϵ_0. Since Y is compact, we may assume that ε does not depend on z.

The proof is by contradiction. Suppose that $h \circ \mathcal{F}$ is equicontinuous at y_0. Then there exists $\delta > 0$ such that for any $y \in B(y_0, \delta)$ and for any $f \in \mathcal{F}$ such that $d(h \circ f(y), h \circ f(y_0)) < \varepsilon$. Since Y is locally connected, we may assume that $B(y_0, \delta)$ is connected. By the definition of ϵ_0, there exists $y_1 \in B(y_0, \delta)$ and $g \in \mathcal{F}$ such that $d(g(y_1), g(y_0)) \geq \epsilon_0$. Let C be the connected component of $h^{-1}(B(h \circ g(y_0), \varepsilon))$ which contains $g(y_0)$, whose diameter is necessarily less than ϵ_0. Since $B(y_0, \delta)$ is connected and $h \circ g(B(y_0, \delta)) \subset B(h \circ g(y_0), \varepsilon)$, we have $g(B(y_0, \delta)) \subset C$, and hence $g(y_1) \in C$. However, this contradicts the fact that $d(g(y_1), g(y_0)) \geq \epsilon_0$ and $\operatorname{diam} C < \epsilon_0$. □

Definition 2.10. A GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ is said to be *irreducible* if the directed graph of S is strongly connected, that is, for any $(i, j) \in V \times V$, there exists an admissible word e such that $i = i(e)$ and $t(e) = j$.

Lemma 2.11. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS such that every $h \in H(S)$ satisfies the condition mentioned in Lemma 2.9:

$$\sup\{\operatorname{diam} C; \, C \text{ is a connected component of } h^{-1}(B(y, \varepsilon))\} \to 0$$

as $\varepsilon \to 0$ for any point $y \in Y$. Then $J_i(S) = J(H_i^i(S))$ for any $i \in V$, where $J(H_i^i(S))$ is the Julia set of the semigroup $H_i^i(S)$.

Proof. Since $J_i(S) \supset J(H_i^i(S))$ trivially, we show $J_i(S) \subset J(H_i^i(S))$. For any $y_0 \in J_i(S)$, there exists a vertex $j \in V$ such that $H_i^j(S)$ is not equicontinuous on any neighborhood of y_0 since $\#V < \infty$. We fix $h \in H_j^i(S)$, which exists by the irreducibility of S. According to Lemma 2.9, $h \circ H_i^j(S)$ is not equicontinuous on any neighborhood of y_0. Thus $H_i^i(S)$ is not equicontinuous on any neighborhood of y_0, and hence $y_0 \in J(H_i^i(S))$. □

Remark 2.12. If $Y = \widehat{\mathbb{C}}$ and $\Gamma_e \subset \mathrm{Rat}$ for all $e \in E$, then every $g \in H(S)$ satisfies the condition mentioned in Lemma 2.9. Thus Lemma 2.11 holds in this case.

Notation 2.13. For a family $\mathcal{F} \subset \mathrm{CM}(Y)$ and a set $X \subset Y$, we set

$$\mathcal{F}(X) := \bigcup_{f \in \mathcal{F}} f(X),\ \mathcal{F}^{-1}(X) := \bigcup_{f \in \mathcal{F}} f^{-1}(X).$$

If $\mathcal{F} = \emptyset$, then we set $\mathcal{F}(X) := \emptyset$, $\mathcal{F}^{-1}(X) := \emptyset$.

Definition 2.14. Let L_i be a subset of Y for each $i \in V$. We consider the familiy $(L_i)_{i \in V}$ indexed by $i \in V$.

(i) $(L_i)_{i \in V}$ is said to be *forward S-invariant* if $\Gamma_e(L_{i(e)}) \subset L_{t(e)}$ for all $e \in E$.

(ii) $(L_i)_{i \in V}$ is said to be *backward S-invariant* if $\Gamma_e^{-1}(L_{t(e)}) \subset L_{i(e)}$ for all $e \in E$.

It is easy to prove the following lemma and the proof is left to the readers.

Lemma 2.15. (i) If a GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ satisfies $\Gamma_e \subset \mathrm{OCM}(Y)$ for all $e \in E$, then the family $(F_i(S))_{i \in V}$ (resp. $(J_i(S))_{i \in V}$) of Fatou sets (resp. Julia sets) is forward (resp. backward) S-invariant.

(ii) If $(L_i)_{i \in V}$ is forward S-invariant, then $H_i^j(S)(L_i) \subset L_j$ for every $i, j \in V$. If $(L_i)_{i \in V}$ is backward S-invariant, then $(H_i^j(S))^{-1}(L_j) \subset L_i$ for every $i, j \in V$.

(iii) Let S be an irreducible GDMS and let $(L_i)_{i \in V}$ be a forward S-invariant family. Then $L_i = \emptyset$ for *all* $i \in V$ if and only if $L_j = \emptyset$ for *some* $j \in V$.

Proposition 2.16. If Γ_e is a compact subset of $\mathrm{OCM}(Y)$ for all $e \in E$, then

$$\bigcup_{e \in E\colon i(e) = i} \Gamma_e^{-1}(J_{t(e)}(S)) = J_i(S)$$

for all $i \in V$.

Proof. If there is no $e \in E$ that satisfies $i(e) = i$, the statement is trivial. Hence we may assume there exists some $e \in E$ that satisfies $i(e) = i$.

According to Lemma 2.15, $\bigcup_{i(e)=i} \Gamma_e^{-1}(J_{t(e)}(S)) \subset J_i(S)$. Fix any $y \notin \bigcup_{i(e)=i} \Gamma_e^{-1}(J_{t(e)}(S))$. Since E is finite and Γ_e is compact for all $e \in E$, we have $y \notin J_i(S)$. Thus $\bigcup_{i(e)=i} \Gamma_e^{-1}(J_{t(e)}(S)) \supset J_i(S)$. □

Definition 2.17. We define $J_{\ker,i}(S) := \bigcap_{j \in V : H_i^j(S) \neq \emptyset} \bigcap_{h \in H_i^j(S)} h^{-1}(J_j(S))$ and call it the kernel Julia set of S at the vertex $i \in V$. Here, we set $J_{\ker,i}(S) := \emptyset$ if $H_i^j(S) = \emptyset$ for all $j \in V$. Recall that the kernel Julia set of a semigroup $G \subset \mathrm{CM}(Y)$ is defined by $J_{\ker}(G) = \bigcap_{g \in G} g^{-1}(J(G))$, where $J(G)$ denotes the Julia set of G defined in Remark 2.5 (see [15]). Moreover, we set $\mathbb{J}_{\ker}(S) := \bigcup_{i \in V} J_{\ker,i}(S) \times \{i\} \subset Y \times V$.

Corollary 2.18. Suppose that an irreducible GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ satisfies $\Gamma_e \subset \mathrm{OCM}(Y)$ for all $e \in E$ and every $h \in H(S)$ satisfies the condition mentioned in Lemma 2.9:

$$\sup\{\operatorname{diam} C;\ C \text{ is a connected component of } h^{-1}(B(y, \varepsilon))\} \to 0$$

as $\varepsilon \to 0$ for any point $y \in Y$. Then $J_{\ker,i}(S) = J_{\ker}(H_i^i(S))$, where the right hand side is the kernel Julia set of the semigroup $H_i^i(S)$.

Proof. By Lemma 2.11, we have $J_{\ker,i}(S) \subset J_{\ker}(H_i^i(S))$. We now fix $y \in J_{\ker}(H_i^i(S))$ and fix $h \in H_i^j(S)$. Since S is irreducible, there exists $f \in H_j^i(S)$ so that $f \circ h \in H_i^i(S)$ and hence $f \circ h(y) \in J(H_i^i(S))$. Thus, we have $h(y) \in J_j(S)$ by Lemma 2.11 and Lemma 2.15. □

Notation 2.19. Let $(L_i)_{i\in V}, (\tilde{L}_i)_{i\in V}$ be families of subsets of Y indexed by V. We write $(L_i)_{i\in V} \subset (\tilde{L}_i)_{i\in V}$ if $L_i \subset \tilde{L}_i$ for all $i \in V$.

Lemma 2.20. For kernel Julia sets $(J_{\ker,i}(S))_{i\in V}$, the following statements hold.

(i) The family $(J_{\ker,i}(S))_{i\in V}$ is forward S-invariant.

(ii) If a forward S-invariant family $(L_i)_{i\in V}$ satisfies $(L_i)_{i\in V} \subset (J_i(S))_{i\in V}$, then $(L_i)_{i\in V} \subset (J_{\ker,i}(S))_{i\in V}$.

The proof is immediate by using Lemma 2.15. Now we define a condition that plays an important role in section 4.2.

Definition 2.21. We say that a GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ satisfies the *backward separating condition* if $f_1^{-1}(J_{t(e_1)}(S)) \cap f_2^{-1}(J_{t(e_2)}(S)) = \emptyset$ for every $e_1, e_2 \in E$ with the same initial vertex and for every $f_1 \in \Gamma_{e_1}, f_2 \in \Gamma_{e_2}$, except the case $e_1 = e_2$ and $f_1 = f_2$.

Definition 2.22. We say that a GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ is *essentially non-deterministic* if there exist $e_1, e_2 \in E$ with $i(e_1) = i(e_2)$ and exist $f_1 \in \operatorname{supp} \tau_{e_1}, f_2 \in \operatorname{supp} \tau_{e_2}$ such that either $e_1 \neq e_2$ or $f_1 \neq f_2$.

Lemma 2.23. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a GDMS which satisfies the backward separating condition. If S is essentially non-deterministic, then $J_{\ker,j}(S) = \emptyset$ for some $j \in V$. Moreover, if, in addition to the assumption above, S is irreducible, then $\mathbb{J}_{\ker}(S) = \emptyset$.

Proof. Since S is essentially non-deterministic, there exist $e_1, e_2 \in E$ with $i(e_1) = i(e_2) =: j$ and exist $f_1 \in \operatorname{supp} \tau_{e_1}, f_2 \in \operatorname{supp} \tau_{e_2}$ such that either $e_1 \neq e_2$ or $f_1 \neq f_2$. If there exists some $z \in J_{\ker,j}(S)$, then $f_n(z) \in J_{t(e_n)}(S)$ $(n = 1, 2)$ by definition. However, this implies that $f_1^{-1}(J_{t(e_1)}(S))$ and $f_2^{-1}(J_{t(e_2)}(S))$ share a point z, which contradicts the backward separating condition. If S is irreducible, then $\mathbb{J}_{\ker}(S) = \emptyset$ by Lemma 2.15 and Lemma 2.20. □

2.2 Skew product maps

Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a GDMS on Y. We define the skew product map associated with S and investigate its dynamics. We consider only admissible maps as in subsection 2.1.

Definition 2.24. We say that a sequence $\xi = (\gamma_n, e_n)_{n=1}^N \in (\mathrm{CM}(Y) \times E)^N$ is admissible with length N if $e = (e_1, \ldots, e_N)$ is admissible and $\gamma_n \in \Gamma_{e_n}$ for all $n \in \{1, \ldots, N\}$. Also, we say that a sequence $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in (\mathrm{CM}(Y) \times E)^{\mathbb{N}}$ is admissible if $\gamma_n \in \Gamma_{e_n}$ and $t(e_n) = i(e_{n+1})$ for all $n \in \mathbb{N}$. For any admissible sequence $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}}$ and for any $N, M \in \mathbb{N}$ with $N > M$, we set $\gamma_{N,M} := \gamma_N \circ \cdots \circ \gamma_M$ and $\xi_{N,M} := (\gamma_n, e_n)_{n=M}^N$. Let $h = (\gamma_n, e_n)_{n=1}^N$ be an admissible sequence. We regard h as a map from $Y \times \{i(e_1)\}$ to $Y \times \{t(e_N)\}$ by setting $h(y, i(e_1)) := (\gamma_{N,1}(y), t(e_N))$ $(y \in Y)$.

Definition 2.25. We define the set of all admissible infinite sequences by

$$X(S) := \{\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in (\mathrm{CM}(Y) \times E)^{\mathbb{N}}; \quad \gamma_n \in \Gamma_{e_n} \text{ and } t(e_n) = i(e_{n+1}) \text{ for all } n \in \mathbb{N}\}.$$

We denote by $X_i(S)$ the subset of $X(S)$ consisting of all admissible infinite sequences with initial vertex $i \in V$; thus any $\xi \in X_i(S)$ can be written as $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}}$ such that $i(e_1) = i, \gamma_n \in \Gamma_{e_n}$ and $t(e_n) = i(e_{n+1})$ for all $n \in \mathbb{N}$. We endow $(\mathrm{CM}(Y) \times E)^{\mathbb{N}}$ with the product topology, where E has the discrete topology. We endow $X(S)$ and $X_i(S)$ with the relative topology from $(\mathrm{CM}(Y) \times E)^{\mathbb{N}}$. Note that $X(S)$ and $X_i(S)$ are compact if Γ_e is compact for all $e \in E$.

Definition 2.26. For any $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X(S)$, we denote by F_ξ the set of all points $y \in Y$ for which there exists a neighborhood U in Y such that the family of maps $\{\gamma_{N,1} = \gamma_N \circ \cdots \circ \gamma_1;\ N \in \mathbb{N}\}$ is equicontinuous on U. We call F_ξ the Fatou set of ξ and call the complement $J_\xi := Y \setminus F_\xi$ the Julia set of ξ. Set $F^\xi := \{\xi\} \times F_\xi \subset X(S) \times Y$ and $J^\xi := \{\xi\} \times J_\xi \subset X(S) \times Y$.

Lemma 2.27. (i) For any $\xi \in X_i(S)$, we have $J_\xi \subset J_i(S)$.

(ii) We have

$$\bigcup_{\xi\in X(S)} F^\xi \subset \{(\xi, y) \in X(S) \times Y; \lim_{\varepsilon\to 0} \sup_{n\in\mathbb{N}} \mathrm{diam}(\gamma_{n,1} B(y, \varepsilon)) = 0\}.$$

(iii) For any $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S)$, we have $J_\xi \subset \bigcap_{n\in\mathbb{N}} \gamma_{n,1}^{-1}(J_{t(e_n)}(S))$.

Proof. (i) For any $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S)$, we have $\{\gamma_N \circ \cdots \circ \gamma_1;\ N \in \mathbb{N}\} \subset H_i(S)$. Thus $J_\xi \subset J_i(S)$.

(ii) For any $(\xi, y) \in F^\xi$, we have $y \in F_\xi$ and set $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}}$. For any $\eta > 0$, there exists $\delta > 0$ such that $d(\gamma_{n,1}(y), \gamma_{n,1}(y')) < \eta$ for any $y' \in B(y, \delta)$ and any $n \in \mathbb{N}$. Now we take $\varepsilon < \delta$. Then

$$d(\gamma_{n,1}(y_1), \gamma_{n,1}(y_2)) \le d(\gamma_{n,1}(y_1), \gamma_{n,1}(y)) + d(\gamma_{n,1}(y), \gamma_{n,1}(y_2)) < 2\eta$$

for any $y_1, y_2 \in B(y, \varepsilon)$.

(iii) Suppose $\gamma_{m,1}(y) \in F_{t(e_m)}(S)$ for some $y \in J_\xi$ and $m \in \mathbb{N}$. Then $\gamma_{m,1}^{-1}(F_{t(e_m)}(S))$ is a neighborhood of y. Now $\{\gamma_{n,m+1}\}_{n>m} \subset H_{t(e_m)}(S)$ implies that $\{\gamma_{n,1}\}_{n\in\mathbb{N}}$ is equicontinuous on $\gamma_{m,1}^{-1}(F_{t(e_m)}(S))$. This contradicts the hypothesis that $y \in J_\xi$.

□

Remark 2.28. If $Y = \widehat{\mathbb{C}}$ and $\Gamma_e \subset \mathrm{Rat}$ for all $e \in E$, then the equality holds in the statement of Lemma 2.27(ii), where Rat denotes the space of all non-constant rational maps.

Definition 2.29. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a GDMS and let $\sigma\colon X(S) \to X(S)$ be the (left) shift map. Define the skew product map $\tilde{f} : X(S) \times Y \to X(S) \times Y$ associated with S, by $\tilde{f}(\xi, y) = (\sigma(\xi), \gamma_1(y))$ for any $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X(S)$ and any $y \in Y$. Also we set $\tilde{J}(\tilde{f}) := \overline{\bigcup_{\xi\in X(S)} J^\xi}$, where the closure is taken in the product space $X(S) \times Y$, and we call this the skew product Julia set of $\tilde{f}$.

Lemma 2.30. The skew product map $\tilde{f}$ is continuous on $X(S) \times Y$ and $J_\xi \subset \gamma_1^{-1}(J_{\sigma(\xi)})$ holds for any $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X(S)$. If γ_1 is an open map, then $J_\xi = \gamma_1^{-1}(J_{\sigma(\xi)})$.

Lemma 2.31. With above terminology, we have $\tilde{J}(\tilde{f}) \subset \tilde{f}^{-1}(\tilde{J}(\tilde{f}))$. If $\Gamma_e \subset \mathrm{OCM}(Y)$ for all $e \in E$, then $\tilde{f}$ is open and $\tilde{J}(\tilde{f}) = \tilde{f}^{-1}(\tilde{J}(\tilde{f}))$ holds, where $\mathrm{OCM}(Y)$ denotes the set of all open continuous maps $Y \to Y$.

Proof. By Lemma 2.30, we have $\tilde{f}(\tilde{J}(\tilde{f})) \subset \overline{\bigcup_{\xi \in X(S)} \tilde{f}(J^{\xi})} \subset \overline{\bigcup_{\xi \in X(S)} J^{\sigma(\xi)}} \subset \tilde{J}(\tilde{f})$. Thus, $\tilde{J}(\tilde{f}) \subset \tilde{f}^{-1}(\tilde{J}(\tilde{f}))$. If $\Gamma_e \subset \mathrm{OCM}(Y)$ for all $e \in E$, then $\tilde{f}$ is open. Combining this with Lemma 2.30, we have $\tilde{J}(\tilde{f}) \supset \tilde{f}^{-1}(\tilde{J}(\tilde{f}))$. □

2.3 Markov operators

Throughout this subsection, let $\mathbb{Y}$ be a compact metric space. We discuss Markov operators on the space $\mathrm{C}(\mathbb{Y})$ of all continuous complex functions on $\mathbb{Y}$. The space $\mathrm{C}(\mathbb{Y})$ is a Banach space with the supremum norm $\|\cdot\|_{\mathbb{Y}}$ and its normed dual $\mathrm{C}(\mathbb{Y})^*$ can be regarded as the set of all regular complex Borel measures on $\mathbb{Y}$ by the theorem of F. and M. Riesz.

Notation 2.32. We denote by $\mathfrak{M}_1(\mathbb{Y})$ the set of all regular Borel probability measures on $\mathbb{Y}$ and we define the weak*-topology on $\mathfrak{M}_1(\mathbb{Y}) \subset \mathrm{C}(\mathbb{Y})^*$. Namely, $\mu_n \to \mu$ if and only if $\mu_n(\phi) \to \mu(\phi)$ for all $\phi \in \mathrm{C}(\mathbb{Y})$, where we write $\mu(\phi) := \int_{\mathbb{Y}} \phi \, \mathrm{d}\mu$ for any $\mu \in \mathfrak{M}_1(\mathbb{Y})$ and any $\phi \in \mathrm{C}(\mathbb{Y})$. The space $\mathfrak{M}_1(\mathbb{Y})$ is compact by the Banach-Alaoglu theorem.

Remark 2.33. We introduce a metric d_0 on $\mathfrak{M}_1(\mathbb{Y})$ as follows. Take a countable dense subset $\{\phi_k\}_{k\in\mathbb{N}}$ of $\mathrm{C}(\mathbb{Y})$ whose existence is due to the compactness of $\mathbb{Y}$. We define the distance between two points $\mu_1, \mu_2 \in \mathfrak{M}_1(\mathbb{Y})$ by

$$d_0(\mu_1, \mu_2) := \sum_{k\in\mathbb{N}} \frac{1}{2^k} \frac{|\mu_1(\phi_k) - \mu_2(\phi_k)|}{1 + |\mu_1(\phi_k) - \mu_2(\phi_k)|}.$$

Definition 2.34. A linear operator $M \colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{Y})$ is called a Markov operator if $M\mathbb{1}_{\mathbb{Y}} = \mathbb{1}_{\mathbb{Y}}$ and $M\phi \geq 0$ for all $\phi \in \mathrm{C}(\mathbb{Y})$ with $\phi \geq 0$, where we write $\psi \geq 0$ if $\psi(y)$ is non-negative real number for all $y \in \mathbb{Y}$.

Lemma 2.35. The operator norm of a Markov operator $M \colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{Y})$ is equal to one. Thus the adjoint $M^* \colon \mathrm{C}(\mathbb{Y})^* \to \mathrm{C}(\mathbb{Y})^*$ satisfies that $M^*(\mathfrak{M}_1(\mathbb{Y})) \subset \mathfrak{M}_1(\mathbb{Y})$, where

$$(M^*\mu)\phi := \mu(M\phi), \quad \mu \in \mathrm{C}(\mathbb{Y})^*, \phi \in \mathrm{C}(\mathbb{Y}).$$

Proof. Since $M\mathbb{1}_{\mathbb{Y}} = \mathbb{1}_{\mathbb{Y}}$, the operator norm $\|M\| \geq 1$. For any $\phi \in \mathrm{C}(\mathbb{Y})$ with $\|\phi\|_{\mathbb{Y}} \leq 1$, we have $0 \leq |\phi| \leq \mathbb{1}$. Fix any $y \in \mathbb{Y}$ and define $A := M(|\phi|^2)(y), B := |M\phi(y)|$. By the above properties of M, we have $A \leq 1$. On the other hand, there exists a complex number α with modulus 1 such that $\alpha M\overline{\phi}(y) = B$. Then for any $t \in \mathbb{R}$, we have

$$0 \leq M(|\phi - t\alpha|^2)(y) \leq 1 - 2B\,t + t^2.$$

It follows that $|M\phi(y)| = B \leq 1$ and $\|M\phi\|_{\mathbb{Y}} \leq 1$. Hence $\|M\| = 1$. □

Remark 2.36. For each $y \in \mathbb{Y}$, let $\Phi(y)$ be the Dirac measure centered at y. Note that $\Phi \colon \mathbb{Y} \to \mathfrak{M}_1(\mathbb{Y})$ is a topological embedding. We regard $\mathbb{Y}$ as a subset of $\mathfrak{M}_1(\mathbb{Y})$ by using Φ.

Definition 2.37. For a Markov operator $M : \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{Y})$, we consider the family $\{(M^*)^n : \mathfrak{M}_1(\mathbb{Y}) \to \mathfrak{M}_1(\mathbb{Y})\}_{n\in\mathbb{N}}$ of iterations of the adjoint map M^*.

(i) We denote by $F_{\mathrm{meas}}(M^*)$ the set of all points $\mu \in \mathfrak{M}_1(\mathbb{Y})$ for which there exists a neighborhood $\mathcal{U}$ in $\mathfrak{M}_1(\mathbb{Y})$ such that the family $\{(M^*)^n : \mathfrak{M}_1(\mathbb{Y}) \to \mathfrak{M}_1(\mathbb{Y})\}_{n\in\mathbb{N}}$ is equicontinuous on $\mathcal{U}$.

(ii) We denote by $F^0_{\mathrm{meas}}(M^*)$ the set of all points $\mu \in \mathfrak{M}_1(\mathbb{Y})$ satisfying that the family $\{(M^*)^n : \mathfrak{M}_1(\mathbb{Y}) \to \mathfrak{M}_1(\mathbb{Y})\}_{n\in\mathbb{N}}$ is equicontinuous at μ.

(iii) We denote by $F_{\mathrm{pt}}(M^*)$ the set of all points $y \in \mathbb{Y}$ for which there exists a neighborhood U in $\mathbb{Y}$ such that the family $\{(M^*)^n|_{\mathbb{Y}} : \mathbb{Y} \to \mathfrak{M}_1(\mathbb{Y})\}_{n\in\mathbb{N}}$ restricted to $\mathbb{Y} \subset \mathfrak{M}_1(\mathbb{Y})$ is equicontinuous on U.

(iv) We denote by $F^0_{\mathrm{pt}}(M^*)$ the set of all points $y \in \mathbb{Y}$ satisfying that the family $\{(M^*)^n|_{\mathbb{Y}} : \mathbb{Y} \to \mathfrak{M}_1(\mathbb{Y})\}_{n\in\mathbb{N}}$ restricted to $\mathbb{Y} \subset \mathfrak{M}_1(\mathbb{Y})$ is equicontinuous at y.

Lemma 2.38. For a Markov operator $M\colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{Y})$, we have that $y_0 \in F^0_{\mathrm{pt}}(M^*)$ if and only if $\{M^n\phi\colon \mathbb{Y} \to \mathbb{C}\}_{n\in\mathbb{N}}$ is equicontinuous at y_0 for all $\phi \in \mathrm{C}(\mathbb{Y})$.

Proof. Fix a countable dense set $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathrm{C}(\mathbb{Y})$ and let d_0 be the metric mentioned in Remark 2.33. Suppose that $\{M^n\phi\}_{n\in\mathbb{N}}$ is equicontinuous at $y_0 \in \mathbb{Y}$ for all $\phi \in \mathrm{C}(\mathbb{Y})$. Take small $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} 2^{-k} < \varepsilon$. By our assumption, there exists $\delta > 0$ such that for any $y \in B(y_0, \delta)$, any $n \in \mathbb{N}$ and any $k = 1, \dots, N$, we have

$$|M^n\phi_k(y) - M^n\phi_k(y_0)| < \frac{\varepsilon/N}{1-\varepsilon/N}.$$

It follows that

$$d_0((M^*)^n(\delta_y), (M^*)^n(\delta_{y_0})) = \sum_{k\in\mathbb{N}} \frac{1}{2^k} \frac{|M^n\phi_k(y) - M^n\phi_k(y_0)|}{1 + |M^n\phi_k(y) - M^n\phi_k(y_0)|} \le \varepsilon + \sum_{k=1}^{N} \varepsilon/N = 2\epsilon,$$

and hence $y_0 \in F^0_{\mathrm{pt}}(M^*)$.

Conversely, suppose that $y_0 \in F^0_{\mathrm{pt}}(M^*)$, and take any $\phi \in \mathrm{C}(\mathbb{Y})$ and $\varepsilon > 0$. Since $\{\phi_k\}_{k\in\mathbb{N}}$ is dense in $\mathrm{C}(\mathbb{Y})$, there exists $k \in \mathbb{N}$ such that $\|\phi_k - \phi\|_{\mathbb{Y}} < \varepsilon$. By $y_0 \in F^0_{\mathrm{pt}}(M^*)$, there exists $\delta > 0$ such that for any $y \in B(y_0, \delta)$ and any $n \in \mathbb{N}$, we have

$$d_0((M^*)^n(\delta_y), (M^*)^n(\delta_{y_0})) < \frac{1}{2^k} \frac{\varepsilon}{1+\varepsilon}.$$

It follows that $|M^n\phi_k(y) - M^n\phi_k(y_0)| < \varepsilon$ and

$$\begin{aligned} &|M^n\phi(y) - M^n\phi(y_0)| \\ &\le |M^n\phi(y) - M^n\phi_k(y)| + |M^n\phi_k(y) - M^n\phi_k(y_0)| + |M^n\phi_k(y_0) - M^n\phi(y_0)| < 3\varepsilon. \end{aligned}$$

Thus $\{M^n\phi\}_{n\in\mathbb{N}}$ is equicontinuous at y_0. □

Lemma 2.39. For a Markov operator $M\colon \mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{Y})$, we have that $F_{\mathrm{meas}}(M^*) = \mathfrak{M}_1(\mathbb{Y})$ if and only if $F^0_{\mathrm{pt}}(M^*) = \mathbb{Y}$.

Proof. It is easy to check that if $F_{\mathrm{meas}}(M^*) = \mathfrak{M}_1(\mathbb{Y})$ then $F^0_{\mathrm{pt}}(M^*) = \mathbb{Y}$. Conversely, suppose that $F^0_{\mathrm{pt}}(M^*) = \mathbb{Y}$. If there exists some $\mu \in \mathfrak{M}_1(\mathbb{Y}) \setminus F_{\mathrm{meas}}(M^*)$, then there exist $\varepsilon > 0$ such that for any $j \in \mathbb{N}$, there exist some $n_j \in \mathbb{N}$ and some $\mu_j \in \mathfrak{M}_1(\mathbb{Y})$ such that $d_0(\mu, \mu_j) \le j^{-1}$ and $d_0((M^*)^{n_j}(\mu), (M^*)^{n_j}(\mu_j)) \ge \epsilon$. Fix some $N \in \mathbb{N}$ so that $\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon/2$ holds and set

$$\eta = \frac{\varepsilon/N}{1-\varepsilon/N}.$$

Then there exists $\phi = \phi_k \in \mathrm{C}(\mathbb{Y})$ such that $|(M^*)^{n_j}(\mu)(\phi) - (M^*)^{n_j}(\mu_j)(\phi)| \ge \eta$ holds for infinitely many $j \in \mathbb{N}$. By Lemma 2.38 and the assumption that $F^0_{\mathrm{pt}}(M^*) = \mathbb{Y}$, the family $\{M^n\phi\}_{n\in\mathbb{N}}$ is equicontinuous on $\mathbb{Y}$. According to the Arzelá-Ascoli theorem, we can assume that $\{M^{n_j}\phi\}_{j\in\mathbb{N}}$ converges to some $\psi \in \mathrm{C}(\mathbb{Y})$ uniformly on $\mathbb{Y}$. Thus, for sufficiently large $j \in \mathbb{N}$, we have

$$\begin{aligned}
&|(M^*)^{n_j}(\mu)(\phi) - (M^*)^{n_j}(\mu_j)(\phi)| \\
&\le |\mu(M^{n_j}\phi) - \mu(\psi)| + |\mu(\psi) - \mu_j(\psi)| + |\mu_j(\psi) - \mu_j(M^{n_j}\phi)| \\
&< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta,
\end{aligned}$$

which leads to a contradiction. □

3 Settings of Markov random dynamical systems

In this section, we consider a GDMS S_τ that is induced by a given family τ of measures; we define probability measures on the space of infinite product of $\mathrm{CM}(Y)$ and define a Markov operator M_τ induced by τ. Furthermore, we show that almost every random Julia set is null set if the kernel Julia set is empty (Proposition 3.11).

Setting 3.1. Let Y be a compact metric space and let $m \in \mathbb{N}$. Suppose that m^2 measures $(\tau_{ij})_{i,j=1,\dots,m}$ on $\mathrm{CM}(Y)$ satisfy $\sum_{j=1}^{m} \tau_{ij}(\mathrm{CM}(Y)) = 1$ for all $i = 1, \dots, m$. For a given $\tau = (\tau_{ij})_{i,j=1,\dots,m}$, we consider the Markov chain on $Y \times \{1, \dots, m\}$ whose transition probability from $(y, i) \in Y \times \{1, \dots, m\}$ to $B \times \{j\}$ is $\tau_{ij}(\{f \in \mathrm{CM}(Y); f(y) \in B\})$, where B is a Borel subset of Y and $j \in \{1, \dots, m\}$. We call this Markov chain the *Markov random dynamical system* (MRDS for short) induced by τ.

Definition 3.2. (I) When a family τ of measures is given as in Setting 3.1, we define the GDMS S_τ in the following way. Define the vertex set as $V := \{1, 2, \dots, m\}$ and the edge set as

$$E := \{(i, j) \in V \times V;\ \tau_{ij}(\mathrm{CM}(Y)) > 0\}.$$

Also, for each $e = (i, j) \in E$, we define $\Gamma_e := \operatorname{supp} \tau_{ij}$. Set $S_\tau := (V, E, (\Gamma_e)_{e\in E})$, which we call the GDMS induced by τ. We define $i : E \to V$ (resp. $t : E \to V$) as the projection to the first (resp. second) coordinate and we call $i(e)$ (resp. $t(e)$) the initial (resp. terminal) vertex of $e \in E$.

(II) We say that $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ is *irreducible* if S_τ is irreducible.

In the following, let τ be a family of measures as in Setting 3.1. Set $\mathbb{Y} := Y \times V$. We can define a metric on $\mathbb{Y}$ using the metric on Y and regard the compact metric space $\mathbb{Y}$ as m copies of Y: $\mathbb{Y} \cong \bigsqcup_V Y$.

Definition 3.3. We define Borel probability measures $\tilde{\tau}_i$ $(i \in V)$ on $X_i(S_\tau)$ as follows. For N Borel sets A_n $(n = 1, \dots, N)$ of $\mathrm{CM}(Y)$ and for $(e_1, \dots, e_N) \in E^N$, set $A'_n = A_n \times \{e_n\}$. We define the measure $\tilde{\tau}_i$ on $(\mathrm{CM}(Y) \times E)^{\mathbb{N}}$ so that

$$\begin{aligned}
&\tilde{\tau}_i\left(A'_1 \times \cdots \times A'_N \times \prod_{N+1}^{\infty} (\mathrm{CM}(Y) \times E)\right) \\
&= \begin{cases} \tau_{e_1}(A_1) \cdots \tau_{e_N}(A_N), & \text{if } (e_1, \dots, e_N) \text{ is admissible with } i(e_1) = i \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

for each $i \in V$. Note that $\operatorname{supp} \tilde{\tau}_i = X_i(S_\tau)$.

Lemma 3.4. We set $p_{ij} = \tau_{ij}(\mathrm{CM}(Y))$ and set $P = (p_{ij})_{i,j=1,\dots,m}$. Then the following statements hold.

(i) A GDMS S_τ is irreducible if and only if the matrix P is irreducible.

(ii) If S_τ is irreducible, then there exists the unique vecter $p = (p_1, \dots, p_m)$ such that $pP = p$, $\sum_{i\in V} p_i = 1$ and $p_i > 0$ for all $i \in V$.

(iii) Assume S_τ is irreducible and define the probability measure $\tilde{\tau}$ on $(\mathrm{CM}(Y) \times E)^{\mathbb{N}}$ as $\tilde{\tau} = \sum_{i=1}^m p_i \tilde{\tau}_i$, where the vector p is as above. Then $\operatorname{supp} \tilde{\tau} = X(S_\tau)$ and $\tilde{\tau}$ is an invariant probability measure with respect to the shift map on $X(S_\tau)$.

Proof. We show (iii). For a Borel set $\tilde{A}$ of $(\mathrm{CM}(Y) \times E)^{\mathbb{N}}$, we prove $\tilde{\tau}(\sigma^{-1}(\tilde{A})) = \tilde{\tau}(\tilde{A})$. We may assume

$$\tilde{A} = A'_1 \times \cdots \times A'_N \times \prod_{N+1}^{\infty} (\mathrm{CM}(Y) \times E), \quad A'_n = A_n \times \{e_n\}.$$

If the word $(e_1, \dots, e_N)$ is not admissible, then $\tilde{\tau}(\sigma^{-1}(\tilde{A})) = 0 = \tilde{\tau}(\tilde{A})$. If $(e_1, \dots, e_N)$ is admissible, then

$$\sigma^{-1}(\tilde{A}) = (\mathrm{CM}(Y) \times E) \times \tilde{A} = \bigsqcup_{i\in V} \bigsqcup_{i(e)=i} (\mathrm{CM}(Y) \times \{e\}) \times \tilde{A}$$

and hence

$$\tilde{\tau}(\sigma^{-1}(\tilde{A})) = \sum_{i\in V} p_i p_{ii(e_1)} \tau_{e_1}(A_1) \cdots \tau_{e_N}(A_N) = p_{i(e_1)} \tau_{e_1}(A_1) \cdots \tau_{e_N}(A_N) = \tilde{\tau}(\tilde{A}).$$

□

Definition 3.5. For $\tau = (\tau_{ij})_{i,j\in V}$, we define the transition operator M_τ of τ as follows.

$$M_\tau \phi(y, i) := \sum_{j\in V} \int_{\Gamma_{ij}} \phi(\gamma(y), j)\, \mathrm{d}\tau_{ij}(\gamma), \quad (y, i) \in \mathbb{Y}.$$

Here, ϕ is a complex-valued Borel measurable function on $\mathbb{Y}$.

Remark 3.6. For the transition operator M_τ, the following statements hold.

(i) If $\phi \in \mathrm{C}(\mathbb{Y})$, then $M_\tau \phi \in \mathrm{C}(\mathbb{Y})$.

(ii) The transition operator M_τ is a Markov operator on $\mathrm{C}(\mathbb{Y})$ in the sense of subsection 2.3 (see Definition 2.34).

Lemma 3.7. If $(y, i) \in \mathbb{Y}, n \in \mathbb{N}$ and $\phi \in \mathrm{C}(\mathbb{Y})$, then

$$(M_\tau^n \phi)(y, i) = \int_{X_i(S_\tau)} \phi(\xi_{n,1}(y, i))\, \mathrm{d}\tilde{\tau}_i(\xi).$$

For the meaning of $\xi_{n,1}(y, i)$, see Notation 2.24.

Proof. We use induction on n. If the statement holds for $n = N$, then

$$\begin{aligned}(M_\tau^{N+1} \phi)(y, i) &= \sum_{i_0\in V} \int_{\Gamma_{ii_0}} (M_\tau^N \phi)(\gamma_0(y), i_0)\, \mathrm{d}\tau_{ii_0}(\gamma_0) \\ &= \sum_{i_0\in V} \int_{\Gamma_{ii_0}} \int_{X_{i_0}(S_\tau)} \phi(\xi_{N,1}(\gamma_0(y), i_0))\, \mathrm{d}\tilde{\tau}_{i_0}(\xi) \mathrm{d}\tau_{ii_0}(\gamma_0).\end{aligned}$$

Set $\xi_0 = (\gamma_0, e_0), e_0 = (i\, i_0)$ and $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}}, e = (e_n)_{n=1}^N$. If $\phi = 1\!\!1_{B\times\{j\}}$, then

$$\begin{aligned}
&(M_\tau^{N+1}\phi)(y,i)\\
&= \sum_{i_0\in V} \tau_{i\, i_0} \otimes \tilde{\tau}_{i_0}(\{(\xi_0,\xi)\,;\, i_0 = i(e), t(e) = j \text{ and } \gamma_N \circ\cdots\circ \gamma_0(y) \in B\})\\
&= \sum_{e'} \tau_{e_0}\otimes\tau_{e_1}\otimes\cdots\otimes\tau_{e_N}(\{(\gamma_n)_{n=0}^N \in \mathrm{CM}(Y)^{N+1}\,;\, \gamma_N\circ\cdots\circ\gamma_0(y)\in B\})\\
&= \int_{X_i(S_\tau)} \phi(\xi_{N+1,1}(y,i))\, \mathrm{d}\tilde{\tau}_i(\xi).
\end{aligned}$$

Here, the summation is taken over all admissible words $e' = (e_n)_{n=0}^N$ with initial vertex i, terminal vertex j and length $N+1$. This completes the proof since any continuous function ϕ can be approximated by simple functions. $\square$

Lemma 3.8. If $\tilde{\tau}_i(\{\xi\in X_i(S_\tau); y\in J_\xi\}) = 0$ holds for $(y,i)\in\mathbb{Y}$, then $(y,i)\in F^0_{\mathrm{pt}}(M_\tau^*)$.

Proof. By assumption, we have $y\in F_\xi$ for $\tilde{\tau}_i$-almost every $\xi\in X_i(S_\tau)$. For such $\xi = (\gamma_n,e_n)_{n\in\mathbb{N}}$, we have $\lim_{\eta\to 0}\sup_{n\in\mathbb{N}} \mathrm{diam}(\gamma_{n,1}B(y,\eta)) = 0$ by Lemma 2.27. For any $\phi\in \mathrm{C}(\mathbb{Y})$ and any $\varepsilon > 0$, the function ϕ is uniformly continuous on the compact space $\mathbb{Y}$. Thus there exists $\delta_1 > 0$ such that for any $z_1, z_2 \in \mathbb{Y}$ with $d(z_1,z_2) < \delta_1$, we have $|\phi(z_1) - \phi(z_2)| < \varepsilon$. By Egoroff's theorem, there exists a Borel set $B\subset X_i(S_\tau)$ with $\tilde{\tau}_i(B^c) = \tilde{\tau}_i(X_i(S_\tau)\setminus B) < \varepsilon$ satisfying the following property; there exists $\eta_0 > 0$ such that $\sup_{n\in\mathbb{N}} \mathrm{diam}(\gamma_{n,1}B(y,\eta_0)) < \delta_1$ for any $\xi = (\gamma_n,e_n)_{n\in\mathbb{N}} \in B$. Hence, for any $z_1 = (y_1, i)$ whose distance from $z = (y,i)$ is less than η_0, we have

$$\begin{aligned}
|(M_\tau^n\phi)(z) - (M_\tau^n\phi)(z_1)| &\le \int_{X_i(S_\tau)} |\phi(\xi_{n,1}(z)) - \phi(\xi_{n,1}(z_1))|\, \mathrm{d}\tilde{\tau}_i(\xi)\\
&= \int_B + \int_{B^c} \le \varepsilon\tilde{\tau}_i(B) + 2\|\phi\|\tilde{\tau}_i(B^c) \le \varepsilon(1+2\|\phi\|).
\end{aligned}$$

By Lemma 2.38, we have $z = (y,i)\in F^0_{\mathrm{pt}}(M_\tau^*)$. $\square$

Corollary 3.9. Let λ be a Borel finite measure on $\mathbb{Y}$. If $\lambda(J_\xi) = 0$ for all $i\in V$ and for $\tilde{\tau}_i$ -a.e. $\xi\in X_i(S_\tau)$, then $\lambda(\mathbb{Y}\setminus F^0_{\mathrm{pt}}(M_\tau^*)) = 0$.

Proof. The statement follows easily from Lemma 3.8 and Fubini's theorem. $\square$

Lemma 3.10. Let $(U_j)_{j\in V}$ be a forward S_τ-invariant family such that each U_j is a non-empty open subset of Y. Set $L_{\ker,j} = \bigcap_{k\in V:\, H_j^k(S_\tau)\neq\emptyset} \bigcap_{h\in H_j^k(S_\tau)} h^{-1}(Y\setminus U_k)$ for $j\in V$. Also, for $(y,i)\in\mathbb{Y}$, we set

$$E = \{(\gamma_n,e_n)_{n\in\mathbb{N}} \in X_i(S_\tau); y\in \bigcap_{n\in\mathbb{N}} \gamma_{n,1}^{-1}(Y\setminus U_{t(e_n)})\}.$$

Then $d(\gamma_{n,1}(y), L_{\ker,t(e_n)}) \to 0\, (n\to\infty)$ for $\tilde{\tau}_i$ -a.e. $(\gamma_n,e_n)_{n\in\mathbb{N}}\in E$, where $d(a,\emptyset) := \infty$ $(a\in Y)$.

Proof. Let $z = (y,i)\in\mathbb{Y}$, $\mathbb{U} := \bigcup_{j\in V} U_j\times\{j\}$ and $\mathbb{L}_{\ker} := \bigcup_{j\in V} L_{\ker,j}\times\{j\}$. For any $\varepsilon > 0$ and for any $n\in\mathbb{N}$, we set

$$\begin{aligned}
A(\varepsilon,n) &:= \{\xi\in E\,;\, \xi_{n,1}(z)\notin \mathbb{U}\cup B(\mathbb{L}_{\ker},\varepsilon)\},\\
C(\varepsilon) &:= \{\xi\in E\,;\, \exists N\in\mathbb{N} \text{ such that } \xi_{n,1}(z)\in B(\mathbb{L}_{\ker},\varepsilon) \text{ for any } n\ge N\}.
\end{aligned}$$

Here, $B(\mathbb{L}_{\ker}, \varepsilon) = \{y \in \mathbb{Y};\, d(y, \mathbb{L}_{\ker}) < \varepsilon\}$ and we set $B(\mathbb{L}_{\ker}, \varepsilon) = \emptyset$ if $\mathbb{L}_{\ker} = \emptyset$. We prove that $\tilde{\tau}_i(E \setminus C(\varepsilon)) = 0$ for any $\varepsilon > 0$. For this purpose, fix a small $\varepsilon > 0$. It suffices to show $\sum_{n\in\mathbb{N}} \tilde{\tau}_i(A(\varepsilon, n)) < \infty$. For, since $E \setminus C(\varepsilon) = \limsup_{n\to\infty} A(\varepsilon, n)$, the statement follows by combining these with the Borel-Cantelli lemma.

In order to show $\sum_{n\in\mathbb{N}} \tilde{\tau}_i(A(\varepsilon, n)) < \infty$, we set $\mathbb{K} := \mathbb{Y} \setminus (\mathbb{U} \cup B(\mathbb{L}_{\ker}, \varepsilon))$. Then there exist subsets $K_j \subset \bigcup_{k\in V:\, H_j^k(S_\tau)\neq\emptyset} \bigcup_{h\in H_j^k(S_\tau)} h^{-1}(U_k)$, $j \in V$, such that $\mathbb{K} = \bigcup_{j\in V} K_j \times \{j\}$. Since $\mathbb{K}$ is compact, there exist finitely many open sets W_q in $\mathbb{Y}$ $(q = 1, \ldots, p)$ and finitely many admissible sequences $g_q \in (\mathrm{CM}(Y) \times E)^{l_q}$ $(q = 1, \ldots, p)$ such that $\mathbb{K} \subset \bigcup_{q=1}^{p} W_q$ and $g_q(W_q) \subset \mathbb{U}$. Note that we may assume there exists $l \in \mathbb{N}$ such that $l = l_q$ for all $q = 1, \ldots, p$ since $(U_j)_{j\in V}$ is forward S_τ-invariant. Then, for each $q = 1, \ldots, p$, there exists an open neighborhood $O_q \subset (\mathrm{CM}(Y) \times E)^l$ of g_q such that $g(W_q) \subset \mathbb{U}$ for all $g \in O_q$. We put $\tilde{O}_q := O_q \times \prod_{l+1}^{\infty}(\mathrm{CM}(Y) \times E)$ and $\delta := \min_{q=1,\ldots,p} \tilde{\tau}_{i_q}(\tilde{O}_q) > 0$, where $i_q \in V$ is the initial vertex of g_q.

For each $k \geq 0$ and $r = 0, \ldots, l-1$, we set

$$I(k, r) := \{\xi \in X_i(S_\tau)\,;\, \xi_{kl+r,1}(z) \in \mathbb{K}\} \text{ and } H(k, r) := \{\xi \in I(k, r)\,;\, \xi_{(k+1)l+r,1}(z) \in \mathbb{U}\}.$$

Here, $I(0, 0) := \emptyset$. If $k \neq k'$, then $H(k, r) \cap H(k', r) = \emptyset$. Since $\mathbb{K} \subset \bigcup_{q=1}^{p} W_q$, there exist s Borel sets $B_1, \ldots, B_s$ on $\mathbb{Y}$ for some $s \in \mathbb{N}$ with the following property; $\mathbb{K} = \bigsqcup_{t=1}^{s} B_t$, where $\bigsqcup$ denotes the disjoint union, and for each $t = 1, \ldots, s$ there exists $q(t) \in \{1, \ldots, r\}$ such that $B_t \subset W_{q(t)}$. Then, we have

$$\begin{aligned}
\tilde{\tau}_i(H(k, r)) &= \sum_{t=1}^{s} \tilde{\tau}_i(\{\xi \in X_i(S_\tau)\,;\, \xi_{kl+r,1}(z) \in B_t, \xi_{(k+1)l+r,1}(z) \in \mathbb{U}\}) \\
&\geq \sum_{t=1}^{s} \tilde{\tau}_i(\{\xi \in X_i(S_\tau)\,;\, \xi_{kl+r,1}(z) \in B_t, \xi_{(k+1)l+r,kl+r+1} \in O_{q(t)}\}) \\
&\geq \sum_{t=1}^{s} \tilde{\tau}_i(\{\xi \in X_i(S_\tau)\,;\, \xi_{kl+r,1}(z) \in B_t\})\delta = \tilde{\tau}_i(I(k, r))\delta
\end{aligned}$$

and hence

$$1 \geq \tilde{\tau}_i(\bigcup_{k\geq 0} H(k, r)) = \sum_{k=0}^{\infty} \tilde{\tau}_i(H(k, r)) \geq \delta \sum_{k=0}^{\infty} \tilde{\tau}_i(I(k, r)).$$

It follows that $\sum_{n\in\mathbb{N}} \tilde{\tau}_i(A(\varepsilon, n)) \leq l/\delta < \infty$. □

The following proposition is one of the main results of this paper. The statement means that almost surely the random Julia set is of measure-zero and the averaged system is stable if the kernel Julia set is empty.

Proposition 3.11. Let λ be a Borel finite measure on Y. Suppose that $\mathbb{J}_{\ker}(S_\tau) = \emptyset$ and $\Gamma_e \subset \mathrm{OCM}(Y)$ for all $e \in E$. Then, $F_{\mathrm{meas}}(M_\tau^*) = \mathfrak{M}_1(\mathbb{Y})$ and $\lambda(J_\xi) = 0$ holds for any $i \in V$ and for $\tilde{\tau}_i$ -a.e. $\xi \in X_i(S_\tau)$.

Proof. Note that the Fatou set $F_j(S_\tau)$ at each $j \in V$ is not empty since $\mathbb{J}_{\ker}(S_\tau) = \emptyset$. By Lemma 2.15 the family $(F_j(S_\tau))_{j\in V}$ of Fatou sets is forward S_τ-invariant. Hence we can apply Lemma 3.10 with $U_j := F_j(S_\tau)$. Therefore, we have

$$\tilde{\tau}_i\{(\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S_\tau); y \in \bigcap_{n\in\mathbb{N}} \gamma_{n,1}^{-1}\left(Y \setminus F_{t(e_n)}(S_\tau)\right)\} = 0$$

for all $(y, i) \in \mathbb{Y}$. By Lemma 2.27, it follows that $\tilde{\tau}_i(\{\xi \in X_i(S_\tau); y \in J_\xi\}) = 0$. By virtue of Fubini's theorem, we have $\lambda(J_\xi) = 0$ for $\tilde{\tau}_i$ -a.e. $\xi \in X_i(S_\tau)$. Furthermore, by Lemma 3.8, we know $(y, i) \in F_{\mathrm{pt}}^0(M_\tau^*)$ for any $(y, i) \in \mathbb{Y}$. Lemma 2.39 implies $F_{\mathrm{meas}}(M_\tau^*) = \mathfrak{M}_1(\mathbb{Y})$. □

4 Rational MRDSs on $\widehat{\mathbb{C}}$

In this section, we focus on holomorphic dynamical systems on the Riemann sphere $\widehat{\mathbb{C}}$. We denote by Rat the space of all non-constant holomorphic maps from $\widehat{\mathbb{C}}$ to itself with the topology of uniform convergence or the compact-open topology. Recall that each element f of Rat can be expressed as the quotient $p(z)/q(z)$ of two polynomials without common zeros and the degree of f is defined by the maximum of the degrees of p and q. We denote by Poly the subspace of Rat consisting of all polynomial maps of degree two or more. We consider *rational* GDMSs or *polynomial* GDMSs as in Definition 4.1.

In subsection 4.1, we discuss the Julia sets of rational GDMSs and show some fundamental properties. These discussions are the generalization of those of rational semigroups (see [13]). Moreover, we show some sufficient conditions for the kernel Julia sets to be empty. In subsection 4.2, we focus on a polynomial MRDS and the function $\mathbb{T}_{\infty,\tau}$ which represents the probability of tending to ∞. We show that the function $\mathbb{T}_{\infty,\tau}$ is continuous on the whole space and varies precisely on the Julia set of associated system under certain conditions.

4.1 Julia sets

Definition 4.1. We say that $S = (V, E, (\Gamma_e)_{e\in E})$ is a rational (resp. polynomial) GDMS on $\widehat{\mathbb{C}}$ if $\Gamma_e \subset \text{Rat}$ (resp. $\Gamma_e \subset \text{Poly}$) for all $e \in E$.

Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a rational GDMS. Recall that the Julia set $J_i(S)$ of S at the vertex $i \in V$ is equal to the Julia set of the rational semigroup $H_i^i(S)$ (see Remark 2.12). It is well known that the Julia set $J(G)$ of a rational semigroup G is equal to the closure of the set of repelling fixed points of elements of G if $J(G)$ contains at least three points. For this reason, we introduce the following important definition.

Definition 4.2. A rational GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ is said to be non-elementary if the Julia set $J_i(S)$ at i contains at least three points for all $i \in V$.

Consequently, we obtain a characterization of the Julia set of a rational GDMS.

Corollary 4.3. If a rational GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ is non-elementary and irreducible, then

$$J_i(S) = \overline{\bigcup_{h\in H_i^i(S)} \{\text{repelling fixed points of } h\}} = \overline{\bigcup_{\xi\in X_i(S)} J_\xi}$$

for all $i \in V$. Here, a fixed point z_0 of h is said to be repelling if the modulus of multiplier of h at z_0 is greater than 1.

Here are some basic properties of the Julia set. Although some claims can be directly proved by using the theory of rational semigroups, we do not use it in order not to require preliminary knowledge.

Lemma 4.4. Let $(L_i)_{i\in V}$ be a family that is backward S-invariant and suppose that each L_i is a compact set which contains at least three points. Then $(J_i(S))_{i\in V} \subset (L_i)_{i\in V}$.

Proof. Since $(L_i)_{i\in V}$ is backward S-invariant, it follows that $H_i^j(S)(\widehat{\mathbb{C}} \setminus L_i) \subset \widehat{\mathbb{C}} \setminus L_j$ for all $i, j \in V$. If $\#L_j \geq 3$ for each $j \in V$, then $\widehat{\mathbb{C}} \setminus L_i \subset F_i(S)$ for each $i \in V$ by Montel's theorem. □

Definition 4.5. A point z is called an *exceptional point* of S at the vertex $i \in V$ if $\#(H_i^i(S))^{-1}(z) < 3$, where $(H_i^i(S))^{-1}(z) = \bigcup_{h \in H_i^i(S)} h^{-1}(z)$. We define $\mathcal{E}_i(S)$ as the set of all exceptional points z of S at $i \in V$.

Lemma 4.6. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be an irreducible rational GDMS. Let $j \in V$. Then we have the following statements.

(i) If $z \notin \mathcal{E}_j(S)$, then $J_i(S) \subset \overline{(H_i^j(S))^{-1}(z)}$ for all $i \in V$.

(ii) If $z \in J_j(S) \setminus \mathcal{E}_j(S)$, then $J_i(S) = \overline{(H_i^j(S))^{-1}(z)}$ for all $i \in V$.

Proof. Set $L_i := \overline{(H_i^j(S))^{-1}(z)}$. Then $(L_i)_{i\in V}$ is backward S-invariant and each L_i contains at least three points since S is irreducible. Lemma 4.4 implies (i). Combining (i) and Lemma 2.15 implies (ii). □

Remark 4.7. Lemma 4.6 suggests an algorithm for computing pictures of the Julia set. Figure 8 is drawn in this manner.

Lemma 4.8. If a rational GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ is non-elementary, then each Julia set $J_i(S)$ is a perfect set.

Proof. Suppose $J_i(S)$ has an isolated point z_0. Then there exists an open neighborhood U of z_0 in $\widehat{\mathbb{C}}$ such that $U \cap J_i(S) = \{z_0\}$. Set $U^* := U \setminus \{z_0\}$. We, therefore, have that $U^* \subset F_i(S)$ and $H_i^j(S)(U^*) \subset F_j(S)$ for all $j \in V$. By assumption, $\widehat{\mathbb{C}} \setminus F_j(S)$ contains at least three points for all $j \in V$. Thus, by the strengthened Montel's theorem [3, p203] it follows that $H_i^j(S)$ is normal on the whole U. This contradicts that $z_0 \in J_i(S)$. □

Lemma 4.9. If a rational GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ is non-elementary and irreducible, then $\#\mathcal{E}_i(S) < 3$ for all $i \in V$.

Proof. The proof is by contradiction. Suppose that there exists $k \in V$ such that $\mathcal{E}_k(S)$ has three distinct points a, b and c. For each $i \in V$, we set $L_i := \overline{(H_i^k(S))^{-1}(\{a, b, c\})}$. Then $(L_i)_{i\in V}$ is backward S-invariant and each L_i contains at least three points. By Lemma 4.4, we have $J_k(S) \subset L_k$. However, this contradicts Lemma 4.8 since L_k is finite. □

Proposition 4.10. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be an irreducible rational GDMS such that Γ_e is a finite set for each $e \in E$. If S satisfies the backward separating condition, then either $\mathrm{int}(J_i(S)) = \emptyset$ for all $i \in V$, or $J_i(S) = \widehat{\mathbb{C}}$ for all $i \in V$.

Proof. We assume that $\mathrm{int}(J_i(S)) \neq \emptyset$ for some $i \in V$ and prove that $J_i(S) = \widehat{\mathbb{C}}$. Let U be a connected open subset of $\mathrm{int}(J_i(S))$. By the backward separating condition and Proposition 2.16, there uniquely exist $e_1 \in E$ and $f_1 \in \Gamma_{e_1}$ such that $i = i(e_1)$ and $U \subset f_1^{-1}(J_{t(e_1)}(S))$. Furthermore, for $e \in E$ with $i(e) = i$ and $f \in \Gamma_e$, if $e \neq e_1$ or $f \neq f_1$, then $U \cap f^{-1}(J_{t(e)}(S)) = \emptyset$. Inductively, there uniquely exist $e_n \in E$ and $f_n \in \Gamma_{e_n}$ such that $t(e_n) = i(e_{n+1})$ and $f_n \circ \cdots \circ f_1(U) \subset J_{t(e_n)}(S)$ for any $n \in \mathbb{N}$.

By Lemma 2.11, we have $U \subset J_i(S) = J(H_i^i(S))$ and hence there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset H_i^i(S)$ that contains no subsequence which converges locally uniformly on U. It follows from Montel's theorem that there exists a subsequence $\{h_{n(k)}\}$ such that $h_{n(k)} \in \{f_n \circ \cdots \circ f_1\}_n$ for all $k \in \mathbb{N}$. Thus we have $h_{n(k)}(U) \subset J_i(S)$ for all $k \in \mathbb{N}$, and hence $J_i(S) = \widehat{\mathbb{C}}$ by Montel's theorem again. □

Now we investigate the kernel Julia sets of rational GDMSs.

Lemma 4.11. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be an irreducible rational GDMS. If $\mathrm{int}(J_{\ker,j}(S)) \neq \emptyset$ for some $j \in V$, then $J_{\ker,i}(S) = \widehat{\mathbb{C}}$ for all $i \in V$.

Proof. The proof is by contradiction. It suffices to show that $J_i(S) = \widehat{\mathbb{C}}$ for all $i \in V$. We assume that there exists $i \in V$ such that $J_{\ker,i}(S) \neq \widehat{\mathbb{C}}$. Then $\#\widehat{\mathbb{C}} \setminus J_{\ker,i}(S) \geq 3$ and $h(\mathrm{int}(J_{\ker,j}(S))) \subset J_{\ker,i}(S)$ for all $h \in H_j^i(S)$ by Lemma 2.20. It consequently follows that $H_j^i(S)$ is normal on $\mathrm{int}(J_{\ker,j}(S))$. Now we fix some $g \in H_j^i$ and hence $g \circ H_j^j(S) \subset H_j^i(S)$. By Lemma 2.9 and the Arzelá-Ascoli theorem, the family $H_j^j(S)$ is equicontinuous on $\mathrm{int}(J_{\ker,j}(S))$, so that $\mathrm{int}(J_{\ker,j}(S)) \subset F_j(S)$. This contradicts the fact that $\emptyset \neq \mathrm{int}(J_{\ker,j}(S)) \subset J_j(S)$. □

Definition 4.12. Let Λ be a connected finite-dimensional complex manifold. Let $\{g_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\}_{\lambda\in\Lambda}$ be a family of non-constant rational maps on $\widehat{\mathbb{C}}$. We say that $\{g_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of rational maps (over Λ) if the associated map $\widehat{\mathbb{C}} \times \Lambda \ni (z, \lambda) \mapsto g_\lambda(z) \in \widehat{\mathbb{C}}$ is holomorphic.

Proposition 4.13. Suppose that an irreducible rational GDMS $S = (V, E, (\Gamma_e)_{e\in E})$ has $e \in E$ with the following property. For all $z \in J_{i(e)}(S)$, there exists a holomorphic family of rational maps $\{g_\lambda\}_{\lambda\in\Lambda} \subset \Gamma_e$ such that the map $\Theta : \Lambda \ni \lambda \mapsto g_\lambda(z) \in \widehat{\mathbb{C}}$ is non-constant.

If, in addition, $F_{i(e)}(S) \neq \emptyset$, then $J_{\ker,i}(S) = \emptyset$ for all $i \in V$.

Proof. The proof is by contradiction. Suppose there exists an element $z \in J_{\ker,i(e)}(S)$. Fix a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda} \subset \Gamma_e$ such that the map $\Theta : \Lambda \ni \lambda \mapsto g_\lambda(z) \in \widehat{\mathbb{C}}$ is non-constant. Then $\Theta(\Lambda)$ is non-empty and open in $\widehat{\mathbb{C}}$ and $\Theta(\Lambda) \subset J_{\ker,t(e)}(S)$ by Lemma 2.20. It follows from Lemma 4.11 that $J_{t(e)}(S) \supset J_{\ker,t(e)}(S) = \widehat{\mathbb{C}}$, and this contradicts the assumption $F_{i(e)}(S) \neq \emptyset$. By Lemma 2.15, $J_{\ker,i}(S) = \emptyset$ for all $i \in V$. □

Corollary 4.14. Let S be an irreducible polynomial GDMS. Suppose that Γ_e is a compact subset of Poly for each $e \in E$.

(i) If there exists $e \in E$ such that $\mathrm{int}(\Gamma_e) \neq \emptyset$, then $J_{\ker,i}(S) = \emptyset$ for all $i \in V$. Here, the symbol int denotes the set of all interior points in Poly.

(ii) If there exists $e \in E$, $f \in$ Poly and a non-empty open set U in $\mathbb{C}$ such that $\{f + c\,;\, c \in U\} \subset \Gamma_e$, then $J_{\ker,i}(S) = \emptyset$ for all $i \in V$.

Proof. Since Γ_e is a compact subset of Poly for each $e \in E$, we have $\infty \in F_i(S)$ for each $i \in V$. Combining this with Proposition 4.13, the statements (i) and (ii) of our corollary hold. □

4.2 Probability tending to ∞

In this subsection, we investigate polynomial MRDS induced by $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ and its associated GDMS $S_\tau = (V, E, (\Gamma_e)_{e\in E})$ such that Γ_e is a compact subset of Poly for each $e \in E$. For the definition of S_τ, see Setting 3.1 and Definition 3.2. Polynomial maps of degree 2 or more have a common attracting fixed point at infinity, and hence some random orbits may tend to infinity. We define the function $\mathbb{T}_{\infty,\tau} \colon \widehat{\mathbb{C}} \times V \to [0, 1]$ which represents the probability of tending to infinity and give some sufficient conditions for $\mathbb{T}_{\infty,\tau}$ to be continuous on the whole space. Moreover, we show that $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$ and varies precisely on the Julia set $\mathbb{J}(S_\tau)$ under certain conditions. Recall that $\mathbb{Y} := \widehat{\mathbb{C}} \times V$.

Definition 4.15. We define the function $\mathbb{T}_{\infty,\tau}\colon \mathbb{Y} \to [0,1]$ by

$$\mathbb{T}_{\infty,\tau}(z,i) := \tilde{\tau}_i(\{\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S_\tau)\,;\, d(\gamma_{n,1}(z), \infty) \to 0\,(n \to \infty)\})$$

for any point $(z,i) \in \widehat{\mathbb{C}} \times V$. If S_τ is irreducible, we fix the vector p of Lemma 3.4 and define $T_{\infty,\tau}\colon \widehat{\mathbb{C}} \to [0,1]$ by

$$T_{\infty,\tau}(z) := \sum_{i=1}^{m} p_i \mathbb{T}_{\infty,\tau}(z,i) = \tilde{\tau}(\{\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X(S_\tau)\,;\, d(\gamma_{n,1}(z), \infty) \to 0\,(n \to \infty)\}).$$

The function $T_{\infty,\tau}$ is associated with the following random dynamical systems. Fix an initial point $z \in \widehat{\mathbb{C}}$. We choose a vertex $i = 1, \dots, m$ with probability p_i. At the first step, we choose a vertex $i_1 = 1, \dots, m$ with probability $\tau_{ii_1}(\mathrm{Poly})$ and choose a map f_1 according to the probability distribution $\tau_{ii_1}/\tau_{ii_1}(\mathrm{Poly})$. Repeating this, we randomly choose a map f_n for each n-th step. Then the random orbit $f_n \circ \cdots \circ f_2 \circ f_1(z)$ tends to the point at infinity with probability $T_{\infty,\tau}(z)$.

We need the following lemma which can be easily shown.

Lemma 4.16. Let Γ be a compact subset of Poly. Then there exists an open neighborhood U of ∞ such that for all $\gamma = (\gamma_1, \gamma_2, \dots) \in \Gamma^{\mathbb{N}}$, we have $\gamma_{n,1} \to \infty$ as $n \to \infty$ locally uniformly on U.

Corollary 4.17. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a polynomial GDMS such that Γ_e is a compact subset of Poly for all $e \in E$. Then the Julia set $J_i(S)$ is a compact subset of $\mathbb{C}$ for all $i \in V$.

Definition 4.18. Let $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X(S)$. We denote by A_ξ the set of all points z such that $\gamma_{n,1}(z) \to \infty$ as $n \to \infty$ and denote by K_ξ the complement $\widehat{\mathbb{C}} \setminus A_\xi$.

Lemma 4.19. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a polynomial GDMS such that Γ_e is a compact subset of Poly for all $e \in E$. Then the set A_ξ is a non-empty open set and $J_\xi = \partial K_\xi = \partial A_\xi$ for each $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X(S)$.

Proof. Set $\Gamma := \bigcup_{e\in E} \Gamma_e$. We apply Lemma 4.16 and fix the open neighborhood U of ∞ in Lemma 4.16. It follows easily that $A_\xi = \bigcup_{n\in\mathbb{N}} \gamma_{n,1}^{-1}(U)$ and hence A_ξ is a non-empty open set. For any open set W of $\widehat{\mathbb{C}}$ which meets $\partial K_\xi = \partial A_\xi$, the family $\{\gamma_{n,1}\}_{n\in\mathbb{N}}$ is not equicontinuous on W. Thus $\partial K_\xi \subset J_\xi$. Conversely, since $\gamma_{n,1}(z) \to \infty$ as $n \to \infty$ locally uniformly on A_ξ, we have $A_\xi \subset F_\xi$. In addition, since $\gamma_{n,1}(K_\xi \setminus \partial K_\xi) \subset \widehat{\mathbb{C}} \setminus U$ for any $n \in \mathbb{N}$, we have $K_\xi \setminus \partial K_\xi \subset F_\xi$ by Montel's theorem. Therefore, $J_\xi \subset \partial K_\xi$. □

Definition 4.20. Let $S = (V, E, (\Gamma_e)_{e\in E})$ be a polynomial GDMS such that Γ_e is a compact subset of Poly for all $e \in E$. We denote by $K_i(S)$ the set of all points $z \in \widehat{\mathbb{C}}$ such that $H_i(S)(z)$ is bounded in $\mathbb{C}$. We call $K_i(S)$ the smallest filled-in Julia set of S at $i \in V$.

For the rest of this subsection, we consider a polynomial MRDS induced by τ and $S_\tau = (V, E, (\Gamma_e)_{e\in E})$ such that Γ_e is a compact subset of Poly for each $e \in E$.

Lemma 4.21. The function $\mathbb{T}_{\infty,\tau}$ is locally constant on $\mathbb{F}(S_\tau)$. If τ is irreducible (i.e., S_τ is irreducible), then $T_{\infty,\tau}$ is locally constant on $F(S_\tau)$.

Proof. Fix any connected component U of the Fatou set $F_i(S_\tau)$ at $i \in V$. For each $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S_\tau)$, it follows by Lemma 2.15 that $\gamma_{n,1}(U)$ is contained in some connected component of $F_{t(e_n)}(S_\tau)$. Thus, for any point $z \in U$, we have $\gamma_{n,1}(z) \to \infty$ if and only if there exists $N \in \mathbb{N}$ such that $\gamma_{N,1}(z)$ is contained in the connected component of $F_{t(e_N)}(S_\tau)$ which contains ∞. Consequently, the function $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is constant on U and hence $\mathbb{T}_{\infty,\tau}$ is locally constant on $\mathbb{F}(S_\tau)$. If S_τ is irreducible, then $T_{\infty,\tau}$ is locally constant on $F(S_\tau) = \bigcap_{i \in V} F_i(S_\tau)$. □

Lemma 4.22. (i) $K_i(S_\tau) = \{z \in \widehat{\mathbb{C}}\,;\, \mathbb{T}_{\infty,\tau}(z,i) = 0\}$ for all $i \in V$.

(ii) The smallest filled-in Julia set $K_i(S_\tau)$ is empty for all $i \in V$ if and only if $\mathbb{T}_{\infty,\tau}(\cdot, i) \equiv 1$ for all $i \in V$.

(iii) If $\mathbb{T}_{\infty,\tau} \equiv 1$, then $\mathbb{J}_{\ker}(S_\tau) = \emptyset$.

Proof. Evidentally, we have $\mathbb{T}_{\infty,\tau}(\cdot, i) \equiv 0$ on $K_i(S_\tau)$. Let $U_{\infty,j}$ be the connected component of $F_j(S_\tau)$ which contains ∞. Then the family $(U_{\infty,j})_{j\in V}$ is forward S_τ-invariant. For any $z \notin K_i(S_\tau)$, there exists $h \in H_i^j(S_\tau)$ such that $h(z) \in U_{\infty,j}$. Thus, there exist a finite admissible word $(e_1, \ldots, e_N) \in E^N$ with initial vertex i and maps $\alpha_n \in \Gamma_{e_n} = \operatorname{supp} \tau_{e_n}$ such that $h = \alpha_N \circ \cdots \circ \alpha_1$. For each $n = 1, \ldots, N$, there exists a neighborhood A_n of α_n in Poly such that $\gamma_N \circ \cdots \circ \gamma_1(z) \in U_{\infty,j}$ for all $\gamma_n \in A_n$ $(n = 1, \ldots, N)$. Now we set

$$\tilde{A} = A'_1 \times \cdots \times A'_N \times \prod_{N+1}^{\infty} (\mathrm{Poly} \times E), \quad A'_n = A_n \times \{e_n\},$$

then $\mathbb{T}_{\infty,\tau}(z,i) \geq \tilde{\tau}_i(\tilde{A}) > 0$. This implies (i). The rest of claims easily follows from (i) and Lemma 3.10 (with $U_j = U_{\infty,j}$ for all $j \in V$). □

If $\#V = 1$ (when the system is i.i.d.), then either $T_{\infty,\tau} \equiv 1$ or there exists $z_0 \in \mathbb{C}$ such that $T_{\infty,\tau}(z_0) = 0$ by Lemma 4.22. However, this is not the case when $\#V > 1$ as the following Proposition 4.23 shows. This fact illustrates the difference between i.i.d. random dynamical systems and Markov case. In other words, we found a new phenomenon which cannot hold in i.i.d. case. For a concrete example of this phenomenon, see Example 4.25 .

Proposition 4.23. Suppose τ is irreducible. If there exist $i, j \in V$ such that $K_i(S_\tau) \neq \emptyset$ and $K_i(S_\tau) \cap K_j(S_\tau) = \emptyset$, then $T_{\infty,\tau} \not\equiv 1$ and $T_{\infty,\tau}(z) > 0$ for all $z \in \widehat{\mathbb{C}}$.

Proof. If $z \notin K_i(S_\tau)$, then $T_{\infty,\tau}(z) \geq p_i \mathbb{T}_{\infty,\tau}(z,i) > 0$ since $p_i > 0$. If $z \in K_i(S_\tau)$, then $T_{\infty,\tau}(z) \leq \sum_{j \neq i} p_j < 1$. Also, we have $z \notin K_j(S_\tau)$, and it follows that $T_{\infty,\tau}(z) \geq p_j \mathbb{T}_{\infty,\tau}(z,j) > 0$ since $p_j > 0$ □

The following proposition claims that $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$ if $\mathbb{J}_{\ker}(S_\tau) = \emptyset$. Combining this proposition with Corollary 4.14 or Lemma 2.23, we obtain some examples of τ satisfying that $\mathbb{T}_{\infty,\tau}$ is continuous.

Proposition 4.24. Let $\phi \in \mathrm{C}(\mathbb{Y})$ be a continuous function with $\phi(\infty, i) = 1$ and $\|\phi\|_{\mathbb{Y}} = 1$. Suppose that the support of $\phi(\cdot, i)$ is contained in the connected component of $F_i(S_\tau)$ which contains ∞ for all $i \in V$. Then the following statements hold.

(i) The sequence $\{M_\tau^n \phi\}_{n\in\mathbb{N}}$ converges pointwise to $\mathbb{T}_{\infty,\tau}$ on $\mathbb{Y}$ as $n \to \infty$.

(ii) The equation $M_\tau \mathbb{T}_{\infty,\tau} = \mathbb{T}_{\infty,\tau}$ holds.

(iii) If $\mathbb{J}_{\ker}(S_\tau) = \emptyset$, then $\{M_\tau^n \phi\}_{n\in\mathbb{N}}$ converges uniformly to $\mathbb{T}_{\infty,\tau}$ on $\mathbb{Y}$ as $n \to \infty$ and $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$. If, in addition to the assumption above, τ is irreducible, then $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$.

Proof. Note that all sequences $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S_\tau)$ converge to ∞ locally uniformly on the connected component of $F_i(S_\tau)$ which contains ∞ for all $i \in V$.

(i) Fix any point z of $\widehat{\mathbb{C}}$ and any $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S_\tau)$. If $z \in A_\xi$, then $\phi(\xi_{n,1}(z,i)) \to 1$; othewise $\phi(\xi_{n,1}(z,i)) = 0$ for all $n \in \mathbb{N}$. Thus, combining Lemma 3.7 with the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}(M_\tau^n\phi)(z,i) &= \lim_{n\to\infty}\int_{X_i(S_\tau)} \phi(\xi_{n,1}(z,i))\,\mathrm{d}\tilde{\tau}_i(\xi)\\ &= \int_{X_i(S_\tau)} \mathbb{1}_{\{\eta;z\in A_\eta\}}(\xi)\,\mathrm{d}\tilde{\tau}_i(\xi) = \mathbb{T}_{\infty,\tau}(z,i).\end{aligned}$$

(ii) It follows immediately from (i).

(iii) If $\mathbb{J}_{\ker}(S_\tau) = \emptyset$, then the sequence $\{M_\tau^n\phi\}_{n\in\mathbb{N}}$ is equicontinuous on $\mathbb{Y}$ by Proposition 3.11, Lemma 2.39 and Lemma 2.38. Moreover, $\{M_\tau^n\phi\}$ is uniformly bounded since $\|M_\tau^n\phi\|_{\mathbb{Y}} \le \|\phi\|_{\mathbb{Y}} = 1$. By the Arzelá-Ascoli theorem, it follows that any subsequence of $\{M_\tau^n\phi\}$ has a subsequence which converges uniformly to $\mathbb{T}_{\infty,\tau}$ on $\mathbb{Y}$. Therefore $\{M_\tau^n\phi\}_{n\in\mathbb{N}}$ converges uniformly to $\mathbb{T}_{\infty,\tau}$ on $\mathbb{Y}$ and the limit $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$.

□

The following example illustrates a new phenomenon which cannot hold in i.i.d. systems. For this example, we can apply Proposition 4.23 and Proposition 4.24.

Example 4.25. Let $g_1(z) = z^2 - 1, g_2(z) = z^2/4$ and set

$$\begin{aligned}&f_1(z) = g_1 \circ g_1(z-5) + 5,\ f_2(z) = g_2 \circ g_2(z-5) + 5,\\ &f_3(z) = g_2 \circ g_2(z+5) - 5,\ f_4(z) = g_1 \circ g_1(z+5) - 5,\\ &h_1(z) = f_1(z+10),\ h_2(z) = f_3(z-10).\end{aligned}$$

We consider the polynomial MRDS induced by

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} & \tau_{14}\\ \tau_{21} & \tau_{22} & \tau_{23} & \tau_{24}\\ \tau_{31} & \tau_{32} & \tau_{33} & \tau_{34}\\ \tau_{41} & \tau_{42} & \tau_{43} & \tau_{44}\end{pmatrix} = \begin{pmatrix} \frac{1}{2}\delta_{f_1} & \frac{1}{2}\delta_{f_1} & & \\ \frac{1}{4}\delta_{f_2} & \frac{1}{4}\delta_{f_2} & \frac{1}{2}\delta_{h_2} & \\ & & \frac{1}{2}\delta_{f_3} & \frac{1}{2}\delta_{f_3}\\ \frac{1}{2}\delta_{h_1} & & \frac{1}{4}\delta_{f_4} & \frac{1}{4}\delta_{f_4}\end{pmatrix}.$$

This system satisfies the assumptions of Proposition 4.23 and of Proposition 4.24 (iii): $K_1(S_\tau) \neq \emptyset$ and $K_1(S_\tau) \cap K_3(S_\tau) = \emptyset$; $\mathbb{J}_{\ker}(S_\tau) = \emptyset$ and τ is irreducible. Therefore, it follows that $T_{\infty,\tau} \not\equiv 1$, $T_{\infty,\tau}(z) > 0$ for all $z \in \widehat{\mathbb{C}}$ and $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$. Figure 5 illustrates the function $1 - T_{\infty,\tau}$ which represents the probability of not tending to infinity.

Moreover, the system in this example is postcritically bounded; i.e. the set

$$\overline{\bigcup_{h\in H(S_\tau)} \{c \in \mathbb{C}; c \text{ is a critical value of } h\} \setminus \{\infty\}}$$

is bounded in $\mathbb{C}$. If the i.i.d. random dynamical system is postcritically bounded, then the connected components of Julia set "surround" one another and $T_{\infty,\tau}$ has the monotonicity with the surrounding order (see [16, Theorem 2.4]). However, as this example illustrates, the "surrounding order" is not totally ordered regarding the set of connected components of the Julia set of S_τ and $T_{\infty,\tau}$ does not have the monotonicity in a general non-i.i.d. irreducible system, even if the system is postcritically bounded.

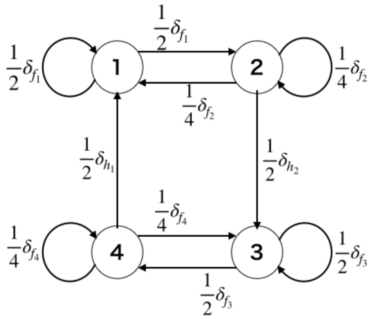

Figure 4: The schematic GDMS of Example 4.25.

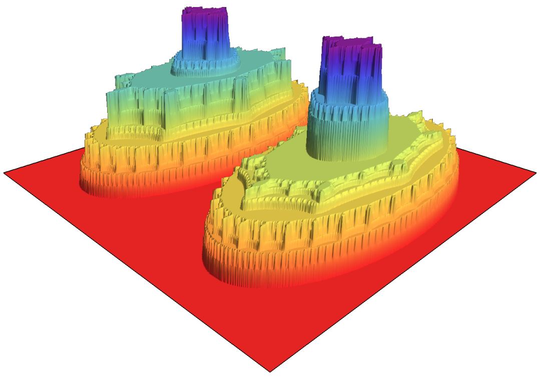

Figure 5: The graph of the function $1 - T_{\infty,\tau}$ with $0 < T_{\infty,\tau} \not\equiv 1$, which cannot hold in i.i.d. random dynamical systems of polynomials.

Corollary 4.26. Suppose that the polynomial GDMS S_τ satisfies the assumption (i) or (ii) of Corollary 4.14. Then $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$ and $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$.

Corollary 4.27. Suppose that the polynomial GDMS S_τ satisfies the assumption of Lemma 2.23. Then $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$. Moreover, if S_τ is irreducible in addition to the assumption above, then $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$.

Corollary 4.26 can be paraphrased by saying that $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$ if some $\Gamma_e = \operatorname{supp} \tau_e$ contains sufficiently many polynomials. In contrast, the backward separating condition (one of the assumptions of Lemma 2.23) seems to be familiar to the GDMS with less polynomials. We focus on the latter case and show more sophisticated results. We now begin with the following easy lemma.

Lemma 4.28. Suppose that τ is irreducible and satisfies the backward separating condition. Then $J_i(S_\tau) \cap \mathcal{E}_i(S_\tau) = \emptyset$ for all $i \in V$.

Proof. We divide the proof into two cases.

Case 1. Suppose that S_τ is essentially non-deterministic. Then there exist two edges $e_1, e_2 \in E$ with the same initial vertex $j \in V$ and two maps $f_1 \in \Gamma_{e_1}, f_2 \in \Gamma_{e_2}$ such that either $e_1 \neq e_2$ or $f_1 \neq f_2$. Fix any $g_n \in H^j_{t(e_n)}(S_\tau)$ and set $h_n := g_n \circ f_n \in H^j_j(S_\tau)$ for each $n = 1, 2$. Then it is easy to see that $h_1^{-1}(J_j(S_\tau)) \cap h_2^{-1}(J_j(S_\tau)) = \emptyset$. Now we have $h_n^{-1}(J_j(S_\tau) \cap \mathcal{E}_j(S_\tau)) \subset J_j(S_\tau) \cap \mathcal{E}_j(S_\tau)$ for each $n = 1, 2$ and $\#(J_j(S_\tau) \cap \mathcal{E}_j(S_\tau)) \leq 2$ by Lemma 4.9. Therefore, we can show $J_j(S_\tau) \cap \mathcal{E}_j(S_\tau) = \emptyset$, and hence $J_i(S_\tau) \cap \mathcal{E}_i(S_\tau) = \emptyset$ for all $i \in V$ since S_τ is irreducible.

Case 2. Suppose that $\# \bigcup_{i(e)=j} \Gamma_e = 1$ for all $j \in V$. In this case, it follows that $H_i^i(S_\tau)$ is a polynomial semigroup generated by a single map h_i. Then the Julia set $J_i(S_\tau)$ is equal to the Julia set $J(h_i)$ of h_i; the exceptional set $\mathcal{E}_i(S_\tau)$ is equal to the exceptional set $\mathcal{E}(h_i)$ of h_i. Here, $J(h_i)$ and $\mathcal{E}(h_i)$ is defined for the iteration of h_i, which is classically well known. By [12, Lemma 4.9], we have $J_i(S_\tau) \cap \mathcal{E}_i(S_\tau) = \emptyset$.

□

The following theorem is one of the main theorems in this paper and gives a sufficient condition that the function $\mathbb{T}_{\infty,\tau}$ which represents the probability of tending to infinity varies precisely on the Julia set $\mathbb{J}(S_\tau)$ and the function $\mathbb{T}_{\infty,\tau}$ is continuous on the whole space.

Theorem 4.29. Suppose that τ is irreducible and the polynomial GDMS S_τ satisfies the backward separating condition and satisfies that $\#\Gamma_e < \infty$ for all $e \in E$. If $K_j(S_\tau) \neq \emptyset$ for some $j \in V$, then the Julia set $J_i(S_\tau)$ at i is equal to the set of all points where $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is not locally constant for all $i \in V$. Moreover, if, in addition to the assumption above, S_τ is essentially non-deterministic, then $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$ and $\mathbb{T}_{\infty,\tau}(J_i(S_\tau) \times \{i\}) = [0, 1]$ for all $i \in V$, and hence $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$.

Proof. First consider the case where $\# \bigcup_{i(e)=j} \Gamma_e = 1$ for all $j \in V$. Then it follows that $H_i^i(S_\tau)$ is a polynomial semigroup generated by a single map h_i, and hence the smallest filled-in Julia set $K_i(S_\tau)$ is equal to the filled-in Julia set $K(h_i)$ of h_i, which is classically well known. By definition, the function $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is 0 on $K(h_i)$ and 1 outside of $K(h_i)$. Thus $\partial K(h_i) = J(h_i) = J_i(S_\tau)$ is equal to the set of all points where $\mathbb{T}_{\infty,\tau}$ is not locally constant. For the classical iteration theory, see [12, §9]. (Remark: We denote by $K(h)$ the set of all $z \in \mathbb{C}$ for which the orbit of z under h is bounded. This set is called *filled* Julia set in [12].)

We next consider the case where there exist two edges $e_1, e_2 \in E$ with the same initial vertex and two maps $f_1 \in \Gamma_{e_1}, f_2 \in \Gamma_{e_2}$ such that either $e_1 \neq e_2$ or $f_1 \neq f_2$. The proof is by contradiction. Let $i \in V$ and suppose that $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is constant on a neighborhood U_0 of $z_0 \in J_i(S_\tau)$ in $\widehat{\mathbb{C}}$. Fix any $z \in J_i(S_\tau)$. By Lemma 4.28 and Lemma 4.6, we have $J_i(S_\tau) = \overline{(H_i^i(S_\tau))^{-1}(z)}$. Thus there exists $z' \in U_0 \cap (H_i^i(S_\tau))^{-1}(z)$, and hence there exists $h \in H_i^i(S_\tau)$ such that $h(z') = z$. This h can be written as $h = \alpha_N \circ \cdots \circ \alpha_1$, where $(\alpha_n, e_n)_{n=1}^N$ is an admissible finite sequence. Set $\tilde{A} := \prod_{n=1}^N (\{\alpha_n\} \times \{e_n\}) \times \prod_{N+1}^\infty (\text{Poly} \times E)$. Since $\#\Gamma_e < \infty$, it follows that $\tilde{\tau}_i(\tilde{A}) > 0$. Now, for all $\xi = (\gamma_n, e_n)_{n\in\mathbb{N}} \in X_i(S_\tau) \setminus \tilde{A} = \tilde{A}^c$, we have $\xi_{N,1}(z', i) \in \mathbb{F}(S_\tau)$ by the backward separating condition. By Proposition 4.24 and Lemma 3.7,

$$\begin{aligned}
\mathbb{T}_{\infty,\tau}(z', i) &= \int_{X_i(S_\tau)} \mathbb{T}_{\infty,\tau}(\xi_{N,1}(z', i)) \, \mathrm{d}\tilde{\tau}_i \\
&= \int_{\tilde{A}} \mathbb{T}_{\infty,\tau}(\xi_{N,1}(z', i)) \, \mathrm{d}\tilde{\tau}_i + \int_{\tilde{A}^c} \mathbb{T}_{\infty,\tau}(\xi_{N,1}(z', i)) \, \mathrm{d}\tilde{\tau}_i \\
&= \mathbb{T}_{\infty,\tau}(z, i)\tilde{\tau}_i(\tilde{A}) + \int_{\tilde{A}^c} \mathbb{T}_{\infty,\tau}(\xi_{N,1}(z', i)) \, \mathrm{d}\tilde{\tau}_i.
\end{aligned}$$

We take a small neighborhood $U' \subset U_0$ of z' in $\widehat{\mathbb{C}}$ so that $\xi(z', i)$ and $\xi(z_1', i)$ are contained in the same connected component of the Fatou set $\mathbb{F}(S_\tau)$ for all $z_1' \in U'$ and for all admissible sequences $\xi \neq (\alpha_n, e_n)_{n=1}^N$ with lengh N. Then $h(U')$ is a neighborhood of $h(z') = z$. Take any $z_1 \in h(U')$ and take $z_1' \in U'$ so that $h(z_1') = z_1$. By our assumption that $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is constant on U_0, we have $\mathbb{T}_{\infty,\tau}(z', i) = \mathbb{T}_{\infty}(z_1', i)$. Since $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is constant on each connected component of $F_i(S_\tau)$ by Lemma 4.21, it follows that

$$\begin{aligned}\mathbb{T}_{\infty,\tau}(z,i) &= (\tilde{\tau}_i(\tilde{A}))^{-1}\left(\mathbb{T}_{\infty,\tau}(z',i) - \int_{\tilde{A}^c} \mathbb{T}_{\infty,\tau}(\xi_{N,1}(z',i))\,\mathrm{d}\tilde{\tau}_i\right) \\ &= (\tilde{\tau}_i(\tilde{A}))^{-1}\left(\mathbb{T}_{\infty,\tau}(z_1',i) - \int_{\tilde{A}^c} \mathbb{T}_{\infty,\tau}(\xi_{N,1}(z_1',i))\,\mathrm{d}\tilde{\tau}_i\right) \\ &= \mathbb{T}_{\infty,\tau}(z_1,i).\end{aligned}$$

Namely, $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is constant on the neighborhood $h(U')$ of $z \in J_i(S_\tau)$. It follows that $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is locally constant at any point of $J_i(S_\tau)$. However, combining this with Lemma 4.21, we obtain that $\mathbb{T}_{\infty,\tau}(\cdot, i)$ is constant on $\hat{\mathbb{C}}$, which contradicts Lemma 4.22, irreducibility of τ and the assumption that $K_j(S_\tau) \neq \emptyset$. Thus $J_i(S_\tau)$ is equal to the set of all points where $\mathbb{T}_{\infty,\tau}$ is not locally constant.

Moreover, the function $\mathbb{T}_{\infty,\tau}$ is continuous on $\mathbb{Y}$ and $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$ by Corollary 4.27. Since $\mathbb{T}_{\infty,\tau}|_{K_i(S_\tau)\times\{i\}} = 0$ and $\mathbb{T}_{\infty,\tau}|_{\{\infty\}\times\{i\}} = 1$, it follows that $\mathbb{T}_{\infty,\tau}(\hat{\mathbb{C}} \times \{i\}) = [0, 1]$. Further, since $\mathbb{T}_{\infty,\tau}$ is locally constant on $F_i(S_\tau) \times \{i\}$, thus it follows that $\mathbb{T}_{\infty,\tau}(J_i(S_\tau) \times \{i\}) = [0, 1]$. □

Now we apply the results to the following random dynamical systems. This is an immediate consequence of Theorem 4.29, Corollary 4.17 and Proposition 4.10.

Corollary 4.30. For a given $m \in \mathbb{N}$, given $f_1, \dots, f_m \in$ Poly and a given irreducible stochastic matrix $P = (p_{ij})_{i,j=1,\dots,m}$, we define τ_{ij} as the measure $p_{ij}\delta_{f_i}$, where δ_{f_i} denotes the Dirac measure at f_i. Suppose that the polynomial GDMS S_τ induced by $\tau = (\tau_{ij})$ satisfies $K_i(S_\tau) \neq \emptyset$ for some $i \in V$ and $J_i(S_\tau) \cap J_j(S_\tau) = \emptyset$ for all $i, j \in V$ with $i \neq j$. Then $\mathrm{int}(J(S_\tau)) = \emptyset$ and the Julia set $J(S_\tau)$ is equal to the set of all points where $T_{\infty,\tau}$ is not locally constant. If, in addition to the assumption above, there exist $i, j, k \in \{1, \dots, m\}$ with $j \neq k$ such that $p_{ij} > 0$ and $p_{ik} > 0$, then $T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$.

Example 4.31. Let $g_1(z) = z^2 - 1, g_2(z) = z^2/4$ and set

$$m = 2,\ f_i = g_i \circ g_i\ (i = 1, 2) \text{ and } P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

Define $\tau_{ij} = p_{ij}\delta_{f_i}$ and $\tau = (\tau_{ij})$. The polynomial GDMS S_τ satisfies all the assumption of Corollary 4.30.

Figure 6 illustrates the grapf of $1 - T_{\infty,\tau}$, which represents the probability of NOT tending to ∞. The function $1 - T_{\infty,\tau}$ is continuous on $\widehat{\mathbb{C}}$ and varies on the Julia set $J(S_\tau)$. Figure 7 illustrates the image of Figure 6 viewed from the top. The Julia set $J(S_\tau)$ is illustrated as the set of all points where the color varies.

Figure 8 illustrates the Julia set $J(S_\tau)$. The Julia set contains neither isolated points nor interior points.

It follows from [15, Example 6.2] that the Hausdorff dimension of the Julia set $J(S_\tau)$ is strictly less than 2 for this example.

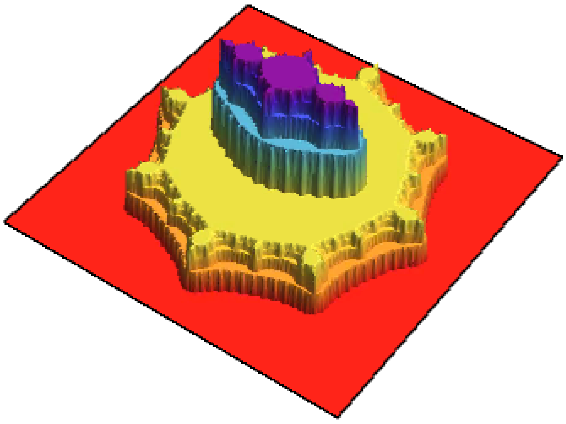

Figure 6: The graph of $1 - T_{\infty,\tau}$.

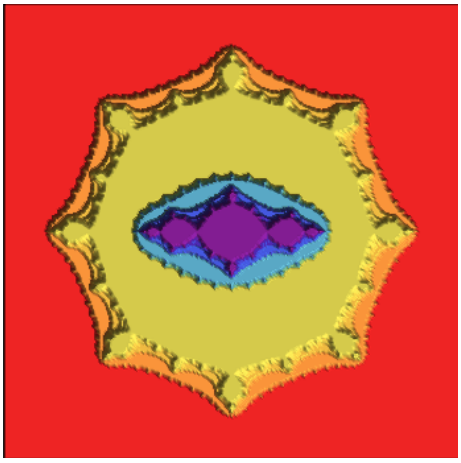

Figure 7: The graph of $1 - T_{\infty,\tau}$ viewed from above.

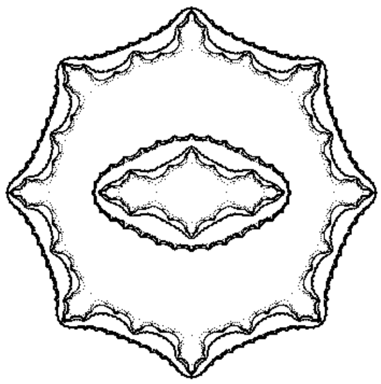

Figure 8: The Julia set $J(S_\tau)$ for Example 4.31.

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