Classical behavior of the integrated density of states for the uniform magnetic field and a randomly perturbed lattice*

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Abstract. For the Schrödinger operators on $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$ with the uniform magnetic field and the scalar potentials located at all sites of a randomly perturbed lattice, the asymptotic behavior of the integrated density of states at the infimum of the spectrum is investigated. The randomly perturbed lattice is the model considered by Fukushima and this describes an intermediate situation between the ordered lattice and the Poisson random field. In this paper the scalar potentials are assumed to decay slowly and its effect to the leading term of the asymptotics are determined explicitly. As the perturbed lattice tends to the Poisson model, the determined leading term tends to that for the Poisson model.

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1. Introduction

Let

$$\mathcal{H} = \mathcal{A}^*\mathcal{A} = \Big(i\frac{\partial}{\partial x_1} - \frac{Bx_2}{2}\Big)^2 + \Big(i\frac{\partial}{\partial x_2} + \frac{Bx_1}{2}\Big)^2 - B \tag{1.1}$$

be the smaller component of the Pauli Hamiltonian $\mathbb{R}^2$ with the uniform magnetic field $B > 0$, where $i = \sqrt{-1}$ and

$$\mathcal{A} = \Big(i\frac{\partial}{\partial x_1} - \frac{Bx_2}{2}\Big) + i\Big(i\frac{\partial}{\partial x_2} + \frac{Bx_1}{2}\Big). \tag{1.2}$$

Let

$$V_\xi(x) = \sum_{q\in\mathbb{Z}^2} u(x - q - \xi_q) \tag{1.3}$$

be a random potential on $\mathbb{R}^2$, where $\xi = (\xi_q)_{q\in\mathbb{Z}^2}$ is a collection of independently and identically distributed $\mathbb{R}^2$-valued random variables with the distribution

$$P_\theta(\xi_q \in dx) = \exp(-|x|^\theta)dx/Z(\theta), \tag{1.4}$$

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$|x|=\sqrt{x_1^2+x_2^2}$, $\theta\in(0,\infty)$, $Z(\theta)$ is the normalizing constant, and u is a nonnegative function belonging to the Kato class K_2 (cf. [2] p-53). Let H_{ξ,Λ_R} be the self-adjoint restriction of the random Schrödinger operator

$$H_\xi=\mathcal{H}+V_\xi, \tag{1.5}$$

to the L^2 space on the cube $\Lambda_R:=(-R/2,R/2)^2$ with the Dirichlet boundary condition and $N_{\xi,R}(\lambda)$ be its number of eigenvalues not exceeding λ. It is well-known that there exists a deterministic increasing function N such that

$$R^{-2}N_{\xi,R}(\lambda)\longrightarrow N(\lambda)\qquad \text{as } R\to\infty \tag{1.6}$$

for any point of continuity of N and almost all ξ under the condition $\sup|x|^\alpha u(x)<\infty$ for some $\alpha>2$ (cf. [2], [7], [10]). This function $(N(\lambda))_{\lambda\geq 0}$ is called as the integrated density of states for the random Schrödinger operator (1.5). For this function, we show the following:

Theorem 1. (i) *If*

$$\operatorname*{essinf}_{|x|\leq R} u(x)>0 \tag{1.7}$$

holds for any $R\geq 1$ *and*

$$u(x)=C_0|x|^{-\alpha}(1+o(1)) \tag{1.8}$$

as $|x|\to\infty$ *for some* $C_0\in(0,\infty)$ *and* $\alpha\in(2,\infty)$*, then we have*

$$\lim_{\lambda\downarrow 0}\lambda^\kappa\log N(\lambda)=\frac{-\kappa^\kappa}{(\kappa+1)^{\kappa+1}}\left\{\int_{\mathbb{R}^2}dq\inf_{y\in\mathbb{R}^2}\Big(\frac{C_0}{|q+y|^\alpha}+|y|^\theta\Big)\right\}^{\kappa+1}, \tag{1.9}$$

where $\kappa=(2+\theta)/(\alpha-2)$.

(ii) *If* (1.7) *holds for any* $R\geq 1$ *and*

$$u(x)=\exp\Big(\frac{-|x|^\alpha}{C_0}(1+o(1))\Big) \tag{1.10}$$

as $|x|\to\infty$ *for some* $C_0\in(0,\infty)$ *and* $\alpha\in(0,2)$*, then we have*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-(2+\theta)/\alpha}\log N(\lambda)=\frac{-2\pi C_0^{(2+\theta)/\alpha}}{(\theta+1)(\theta+2)}. \tag{1.11}$$

We also consider the 3-dimensional problem in the following formulation referring to [8] and [17]. Let

$$\boldsymbol{\mathcal{H}}=\Big(i\frac{\partial}{\partial x_1}-\frac{Bx_2}{2}\Big)^2+\Big(i\frac{\partial}{\partial x_2}+\frac{Bx_1}{2}\Big)^2-B-\frac{\partial^2}{\partial x_3^2} \tag{1.12}$$

be the smaller component of the Pauli Hamiltonian on $\mathbb{R}^3$ with the constant magnetic field B parallel to the x_3-axis. We write any element $\boldsymbol{x}$ of $\mathbb{R}^3$ as $(x_\perp, x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and set

$$\|\boldsymbol{x}\|_p^{\boldsymbol{\theta}} := \begin{cases} \||x_\perp|^{\theta_\perp}, |x_3|^{\theta_3}\|_p = (|x_\perp|^{\theta_\perp p} + |x_3|^{\theta_3 p})^{1/p} \text{ if } p \in [1,\infty), \\ |x_\perp|^{\theta_\perp} \vee |x_3|^{\theta_3} \text{ if } p = \infty, \end{cases} \tag{1.13}$$

for arbitrarily fixed $\boldsymbol{\theta} = (\theta_\perp, \theta_3) \in (0,\infty)^2$ and $p \in [1,\infty]$. Let $\boldsymbol{\xi} = (\boldsymbol{\xi}_{\boldsymbol{q}})_{\boldsymbol{q}\in\mathbb{Z}^3}$ be a collection of independently and identically distributed $\mathbb{R}^3$-valued random variables with the distribution

$$P_{\boldsymbol{\theta}}(\boldsymbol{\xi}_{\boldsymbol{q}} \in d\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_p^{\boldsymbol{\theta}}) d\boldsymbol{x}/Z(\boldsymbol{\theta}, p), \tag{1.14}$$

where $Z(\boldsymbol{\theta}, p)$ is the normalizing constant. Let $\boldsymbol{u}$ be a nonnegative function belonging to the Kato class K_3 (cf. [2] p-53) and satisfying

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{C_0}{\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}}}(1 + o(1)) \tag{1.15}$$

as $|\boldsymbol{x}| \to \infty$ for some $C_0 \in (0,\infty)$, $\tilde{p} \in [1,\infty]$ and $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0,\infty)^2$ such that

$$\frac{2}{\alpha_\perp} + \frac{1}{\alpha_3} < 1. \tag{1.16}$$

Since

$$\int_{\boldsymbol{q}\in\mathbb{R}^3 : |\boldsymbol{q}|\geq 1} \frac{d\boldsymbol{q}}{\|\boldsymbol{q}\|_p^{\boldsymbol{\alpha}}} < \infty \tag{1.17}$$

under the condition (1.16), we can show

$$\boldsymbol{V}_\xi(\boldsymbol{x}) = \sum_{\boldsymbol{q}\in\mathbb{Z}^3} \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}_{\boldsymbol{q}}) \tag{1.18}$$

belong to the local Kato class $K_{3,loc}$ by the same method in Lemma 7.1 in [7]. As in the 2-dimensional case we will consider the integrated density of states $(\boldsymbol{N}(\lambda))_{\lambda\geq 0}$ of the random Schrödinger operators

$$\boldsymbol{H}_{\boldsymbol{\xi}} = \boldsymbol{\mathcal{H}} + \boldsymbol{V}_{\boldsymbol{\xi}} : \tag{1.19}$$

it is a deterministic increasing function satisfying

$$R^{-3}\boldsymbol{N}_{\boldsymbol{\xi},R}(\lambda) \longrightarrow \boldsymbol{N}(\lambda) \qquad \text{as } R \to \infty \tag{1.20}$$

for any point of continuity of $\boldsymbol{N}$ and almost all $\boldsymbol{\xi}$, where $\boldsymbol{N}_{\boldsymbol{\xi},R}(\lambda)$ is the number of eigenvalues not exceeding λ of the self-adjoint operator $\boldsymbol{H}_{\boldsymbol{\xi},\boldsymbol{\Lambda}_R}$ on the L^2 space on the cube $\boldsymbol{\Lambda}_R := (-R/2, R/2)^3$ with the Dirichlet boundary condition (cf. [2], [7], [10]). Our result is the following:

Theorem 2. *We assume*

$$\operatorname*{essinf}_{|\boldsymbol{x}|\le R} \boldsymbol{u}(\boldsymbol{x}) > 0 \tag{1.21}$$

holds for any $R \ge 1$ *and* (1.15) *as* $|\boldsymbol{x}| \to \infty$ *for some* $C_0 \in (0,\infty)$ *and*

$$\frac{2}{\alpha_\perp} + \frac{3}{\alpha_3} > 1. \tag{1.22}$$

We set

$$\kappa(\boldsymbol{\alpha},\boldsymbol{\theta}) = \frac{\dfrac{\theta_\perp}{\alpha_\perp} \vee \dfrac{\theta_3}{\alpha_3} + \dfrac{2}{\alpha_\perp} + \dfrac{1}{\alpha_3}}{1 - \dfrac{2}{\alpha_\perp} - \dfrac{1}{\alpha_3}}, \tag{1.23}$$

and

$$C(\boldsymbol{\alpha},\boldsymbol{\theta},C_0) = \int_{\mathbb{R}^3} d\boldsymbol{q} \inf_{\boldsymbol{y}=(y_\perp,y_3)\in\mathbb{R}^3} \Big(\frac{C_0}{\|\boldsymbol{q}+\boldsymbol{y}\|_{\tilde{p}}^{\boldsymbol{\alpha}}} + \Big\| 1_{\frac{\theta_\perp}{\alpha_\perp} \ge \frac{\theta_3}{\alpha_3}} |y_\perp|^{\theta_\perp}, 1_{\frac{\theta_\perp}{\alpha_\perp} \le \frac{\theta_3}{\alpha_3}} |y_3|^{\theta_3} \Big\|_p \Big) \tag{1.24}$$

(cf. (1.17) *). Then we have*

$$\lim_{\lambda\downarrow 0} \lambda^{\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})} \log \boldsymbol{N}(\lambda) = \frac{-\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})^{\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}}{(1+\kappa(\boldsymbol{\alpha},\boldsymbol{\theta}))^{1+\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}} C(\boldsymbol{\alpha},\boldsymbol{\theta},C_0)^{1+\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}. \tag{1.25}$$

These results are extensions of the results in [6] and [7], where the same problem is considered in the case without magnetic fields. As is discussed in [6] and [7], our model describes an intermediate situation between a completely ordered situation and a completely disordered situation since the point process $\{q+\xi_q\}_{q\in\mathbb{Z}^2}$ converges weakly to the Poisson point process with the intensity 1 as $\theta \to 0$ and converges weakly to the complete lattice $\mathbb{Z}^2$ as $\theta \to \infty$ by slightly modifying the definition as

$$P_\theta(\xi_q \in dx) = \exp(-(1+|x|)^\theta)dx/Z(\theta), \tag{1.26}$$

which brings no essential changes for our results. The results in [6] and [7] show that the leading term of the integrated density of states for each case also tends to that for the corresponding Poisson case as $\theta \to 0$ and decays as $\theta \to \infty$ which reflects that the infimum of the spectrum is strictly positive if the perturbations ξ_q of sites are all bounded. In the case with uniform magnetic fields the asymptotics of the integrated density of states has been investigated mainly for the Poisson case. For this topic and the relation with other topics, refer to a recent survey by Kirsch and Metzger [11]. The first result was given by Broderix, Hundertmark, Kirsch and Leschke [1]: they determined the leading term for the case corresponding to Theorem 1 (i) in this paper where the point process $\{q+\xi_q\}_{q\in\mathbb{Z}^2}$ is replaced by the Poisson point process. As in (1.9), this leading term was determined mainly by the classical effect of the scalar potential as in Pastur's case [14] without magnetic fields. In fact these asymptotics coincide with those of the corresponding classical integrated density of states defined by

$$N_c(\lambda) = E_\theta[|\{(x,p) \in \Lambda_R \times \mathbb{R}^2 : H_{\xi,c}(x,p) \le \lambda\}|](2\pi R)^{-2} \tag{1.27}$$

for any $R \in \mathbb{N}$, where $|\cdot|$ is the 4-dimensional Lebesgue measure and

$$\begin{aligned} &H_{\xi,c}(x,p) \\ =&\Big\{\Big(p_1 - \frac{Bx_2}{2}\Big) + i\Big(p_2 + \frac{Bx_1}{2}\Big)\Big\}^*\Big\{\Big(p_1 - \frac{Bx_2}{2}\Big) + i\Big(p_2 + \frac{Bx_1}{2}\Big)\Big\} \\ &+ V_\xi(x) \\ =&\Big(p_1 - \frac{Bx_2}{2}\Big)^2 + \Big(p_2 + \frac{Bx_1}{2}\Big)^2 + V_\xi(x) \end{aligned} \tag{1.28}$$

is the classical Hamiltonian (cf. [9]). Therefore we may say that only the classical effect from the scalar potential determines the leading term. Then Erdös [4] treated the case where the single site potential u is replaced by a function with a compact support and he determined the corresponding leading term of the integrated density of states, which depends only on the magnetic field and the intensity of the point process and is independent of the precise informations on the single site potential as in Nakao's case [13] without magnetic field referring to Donsker and Varadhan's result [3]. This behavior is different from that of the classical integrated density of states. Thus we may say that the quantum effect appears in this behavior. The borderline between the classical and quantum behaviors was determined by Hupfer, Leschke and Warzel [9]. The borderline corresponds to the case of $\alpha = 2$ in Theorem 1 (ii) in this paper. They also determined the leading term for the case of Theorem 1 (ii) in this paper for the Poisson case. The leading term for the borderline case was determined by Erdös [5]. The leading term for the case that only the classical effect appears was determined also in the 3-dimensional case by Hundertmark, Kirsch and Warzel [8]. Results appearing the quantum effect in 3-dimensional cases were obtained by Warzel [17], where general bounds and the leading order for special cases were obtained. Now our problem is to consider the same problem in our setting. As is discussed in [16] we conjecture that the borderline between the classical and quantum behaviors is the case of $\alpha = 2$ in Theorem 1 (ii) and the case of $2/\alpha_\perp + 3/\alpha_3 = 1$ in Theorem 2 as in the Poisson case. In this paper we treat only the classical case and we will treat the quantum case in [16]. Now Theorems 1 and 2 show that our leading terms coincide with those of the corresponding classical integrated densities of states. They also tend to the corresponding leading term for the Poisson case as $\theta \to 0$.

The leading term we obtained coincides with that for the classical case without magnetic fields obtained in [7]. The classical case without magnetic fields is that of (1.8) with $2 < \alpha < 4$ if the dimension is 2. Theorem 1 shows that this condition for only the classical effect to appear is weakened by the magnetic field, since the effect of the Landau Hamiltonian to the asymptotics of the integrated density of states is weaker than that of the usual Laplacian. The 3-dimensional classical case without magnetic fields is that of (1.15) with $4/\alpha_\perp + 1/\alpha_3 > 1$ and (1.16). This result was obtained for the Poisson case by Kirsch and Warzel [12], where the results for the single site potential with an anisotropic decay are summarized. Theorem 2 shows that this condition for only the classical effect to appear is weakened in the direction perpendicular to the magnetic field and is strengthen in the direction parallel to the magnetic field.

The organization of this paper is as follows. We first prove Theorem 1 in Sections 2 and 3: we give the upper estimate in Section 2 and the lower estimate in Section 3. We next prove Theorem 2 in Sections 4 and 5: we give the upper estimate in Section 4 and the lower estimate in Section 5. In Section 6 we give the leading terms of the low energy asymptotics of the classical integrated densities of states defined in a general setting.

2. 2-dimensional upper estimate

In this section we prove upper estimates necessary to prove Theorem 1. We denote the Laplace-Stieltjes transform of the integrated density of states $N(\lambda)$ by $\widetilde{N}(t)$:

$$\widetilde{N}(t) = \int_0^\infty e^{-t\lambda} dN(\lambda).$$

Then the results are the following: :

Proposition 2.1. (i) *Under the conditions of Theorem 1* (i) *we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{t^{(2+\theta)/(\alpha+\theta)}} \le -\int_{\mathbb{R}^2} dq \inf_{y\in\mathbb{R}^2} \Big(\frac{C_0}{|q+y|^\alpha} + |y|^\theta\Big). \tag{2.1}$$

(ii) *Under the conditions of Theorem 1* (ii) *we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{(\log t)^{(2+\theta)/\alpha}} \le \frac{-2\pi C_0^{(2+\theta)/\alpha}}{(\theta+1)(\theta+2)}. \tag{2.2}$$

(i) is proven by the following proposition and the same proof of Proposition 4 in [7]. The following proposition is an extension of the basic inequality (3.6) in [1] for a $\mathbb{R}^2$-stationary random potential to a $\mathbb{Z}^2$-stationary random potential, which is proven by applying the basic inequality in [1] to the $\mathbb{R}^2$-stationary random potential

$$V_{\xi,\zeta}(x) = \sum_{q\in\mathbb{Z}^2} u(x - q - \xi_q - \zeta_q), \tag{2.3}$$

where $\zeta = (\zeta_q)_{q\in\mathbb{Z}^2}$ is an independent family of random vectors uniformly distributed on Λ_1 :

Proposition 2.2.

$$\widetilde{N}(t) \le \frac{B}{2\pi(1-e^{-2Bt})}\widetilde{N}_1(t),$$

where

$$\widetilde{N}_1(t) = \int_{\Lambda_1} dx E_\theta[\exp(-tV_\xi(x))]. \tag{2.4}$$

Proof of Proposition 2.1 (ii). By replacing the summation by the integration, we have

$$\log \widetilde{N}_1(t) \le \int_{\mathbb{R}^2} dq \log E_\theta\Big[\exp\Big(-t\inf_{x\in\Lambda_2} u(x-q-\xi_0)\Big)\Big].$$

We restrict the integration to $|q| \le \mathcal{L}$ for some finite $\mathcal{L}$. For any $\varepsilon_1 > 0$, there exists R_1 such that $u(x) \ge \exp(-(1+\varepsilon_1)|x|^\alpha/C_0)$ for any $|x|_\infty \ge R_1$, where $|x|_\infty = |x_1| \vee |x_2|$. Thus the right hand side is dominated by

$$\int_{|q|\le\mathcal{L}} dq \log\Bigg\{\int_{|q+y|_\infty\ge R_1+1}\frac{dy}{Z(\theta)}\exp\Big(-t\inf_{x\in\Lambda_2}\exp\Big(-(1+\varepsilon_1)\frac{|x-q-y|^\alpha}{C_0}\Big)-|y|^\theta\Big)$$
$$+\exp\Big(-t\inf_{\Lambda_{2R_1+4}} u\Big)\Bigg\}.$$

By changing the variables, this equals

$$\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{2/\alpha}\int_{|q|\le L} dq\log\Big\{\widetilde{N}_2(t,q)+\exp\Big(-t\inf_{\Lambda_{2R_1+4}} u\Big)\Big\},$$

where $L = \mathcal{L}((1+\varepsilon_1)/(C_0\log t))^{1/\alpha}$,

$$\widetilde{N}_2(t,q) = \Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{2/\alpha}\int_{|q+y|_\infty\ge(R_1+1)((1+\varepsilon_1)/(C_0\log t))^{1/\alpha}}\frac{dy}{Z(\theta)}$$
$$\times\exp\Big(-t^{1-s(t,q,y)}-\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{\theta/\alpha}|y|^\theta\Big),$$

and

$$s(t,q,y) = \sup_{x\in\Lambda_{2((1+\varepsilon_1)/(C_0\log t))^{1/\alpha}}}|x-q-y|^\alpha.$$

We take L as an arbitrary positive constant less than 1 and independent of t. Taking $\varepsilon_2 \in (0,1)$ arbitrarily, we dominate $\widetilde{N}_2(t,q)$ by $\exp(-\widetilde{N}_3(t,q))\varepsilon_2^{-2/\theta}$, where

$$\widetilde{N}_3(t,q) = \inf\Big\{t^{1-s(t,q,y)}+(1-\varepsilon_2)\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{\theta/\alpha}|y|^\theta : y\in\mathbb{R}^2\Big\}.$$

We use the polar coordinate to express as

$$\widetilde{N}_3(t,q) = \inf\Big\{t^{1-(s(t,q)+r)^\alpha}+(1-\varepsilon_2)\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{\theta/\alpha}r^\theta : r>0\Big\}, \tag{2.5}$$

where

$$s(t,q)=\sup_{x\in\Lambda_{2((1+\varepsilon_1)/(C_0\log t))^{1/\alpha}}}|x-q|.$$

We take small $\varepsilon_3>0$ and estimate the infimum in (2.5) as

$$\begin{aligned}\widetilde{N}_3(t,q)\geq&\inf\Big\{(1-\varepsilon_2)\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{\theta/\alpha}r^\theta : r>1-s(t,q)-\varepsilon_3\Big\}\\&\wedge\inf\Big\{t^{1-(s(t,q)+r)^\alpha}:0<r\leq 1-s(t,q)-\varepsilon_3\Big\}\\=&\Big\{(1-\varepsilon_2)\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{\theta/\alpha}(1-s(t,q)-\varepsilon_3)^\theta\Big\}\wedge t^{1-(1-\varepsilon_3)^\alpha}\end{aligned}$$

The right hand side is bounded from below by

$$(1-\varepsilon_2)\Big(\frac{C_0\log t}{1+\varepsilon_1}\Big)^{\theta/\alpha}(1-|q|-2\varepsilon_3)^\theta$$

for sufficiently large t. By taking L as $1-3\varepsilon_3$ and using also the positivity assumption (1.7), we obtain

$$\overline{\lim_{t\uparrow\infty}}\frac{\log\widetilde{N}_1(t)}{(\log t)^{(2+\theta)/\alpha}}\leq-(1-\varepsilon_2)\Big(\frac{C_0}{1+\varepsilon_1}\Big)^{(2+\theta)/\alpha}\int_{|q|\leq 1-3\varepsilon_3}(1-|q|-2\varepsilon_3)^\theta dq.$$

Since $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are arbitrary, we have

$$\overline{\lim_{t\uparrow\infty}}\frac{\log\widetilde{N}_1(t)}{(\log t)^{(2+\theta)/\alpha}}\leq-C_0^{(2+\theta)/\alpha}\int_{|q|\leq 1}(1-|q|)^\theta dq.$$

□

3. 2-dimensional lower estimates

In this section we prove the following lower estimates necessary to prove Theorem 1:

Proposition 3.1. (i) *If* (1.8) *holds, then we have*

$$\underline{\lim_{t\uparrow\infty}}\frac{\log\widetilde{N}(t)}{t^{(2+\theta)/(\alpha+\theta)}}\geq-\int_{\mathbb{R}^2}dq\inf_{y\in\mathbb{R}^2}\Big(\frac{C_0}{|q+y|^\alpha}+|y|^\theta\Big).\tag{3.1}$$

(ii) *If* (1.10) *holds with some* $\alpha\in(0,2)$, *then we have*

$$\underline{\lim_{t\uparrow\infty}}\frac{\log\widetilde{N}(t)}{(\log t)^{(2+\theta)/\alpha}}\geq\frac{-2\pi C_0^{(2+\theta)/\alpha}}{(\theta+1)(\theta+2)}.\tag{3.2}$$

Proof. We use the bound

$$\widetilde{N}(t)\geq(2R)^{-2}\exp(-t\lambda_1(\mathcal{H}_{B(R)}))\widetilde{N}_1(t),\tag{3.3}$$

for any $R\in\mathbb{N}$, where

$$\widetilde{N}_1(t)=E_\theta\Big[\exp\Big(-t\int dx|\psi_R(x)|^2V_\xi(x)\Big)\Big],$$

and ψ_R is a normalized eigenfunction for the lowest eigenvalue $\lambda_1(\mathcal{H}_{B(R)})$ of the operator $\mathcal{H}$ restricted to the disk $B(R) = \{x \in \mathbb{R}^2 : |x| < R\}$ by the Dirichlet boundary condition. This is proven by the same method as for the corresponding bound in Theorem (9.6) in [15] for the $\mathbb{R}^2$-stationary random field.

$$\lambda_1(\mathcal{H}_{B(R)}) \le \exp\Big(-\frac{B}{2}R^2(1-\varepsilon_1)\Big) \tag{3.4}$$

is proven by Erdös [4] for sufficiently large R for an arbitrarily taken $\varepsilon_1 > 0$.

If (1.8) holds, then we can show (3.1) by the same proof of Proposition 5 in [7], where R is taken as $\varepsilon_0 t^{1/(\alpha+\theta)}$ with an arbitrarily small $\varepsilon_0 > 0$.

We now assume (1.10).

$$\log \widetilde{N}_1(t) = \sum_{q\in\mathbb{Z}^2} \widetilde{N}_2(t,q),$$

where

$$\widetilde{N}_2(t,q) = \log E_\theta\Big[\exp\Big(-t\int dx|\psi_R(x)|^2 u(x-q-\xi_0)\Big)\Big].$$

If $|q|$ is sufficiently large and ξ_0 is less than $|q|/2$, then we have $|x-q-\xi_0| \ge |q|/2 - R$ and

$$\widetilde{N}_2(t,q) \ge -t\exp\Big(-\frac{1}{C_0}\Big(\frac{|q|}{2}-R\Big)^\alpha(1-\varepsilon_2)\Big) + \log P_\theta(|\xi_0| \le |q|/2),$$

where ε_2 is an arbitrarily small positive constant. By a simple estimate using $\log(1-X) \ge -2X$ for $0 \le X \le 1/2$, we have

$$\sum_{q\in\mathbb{Z}^2\setminus B(R^{1+\varepsilon_3})} \widetilde{N}_2(t,q) \ge -c_1 t\exp(-c_2 R^{(1+\varepsilon_3)\alpha}) - c_3\exp(-c_4 R^{(1+\varepsilon_3)\theta}) \tag{3.5}$$

for large enough R, where ε_3 is an arbitrarily small positive constant, and $c_1,\dots,c_4$ are positive finite constants. We dominate the other part as

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}^2\cap B(R^{1+\varepsilon_3})} \widetilde{N}_2(t,q)\\
\ge&\sum_{q\in\mathbb{Z}^2\cap B(R^{1+\varepsilon_3})} \log\sup_{z\in\mathbb{R}^2}\int_{|y|\le\varepsilon_4}\frac{dy}{Z(\theta)}\\
&\times\exp\Big(-t\int dx|\psi_R(x)|^2u(x-q-y-z)-|y+z|^\theta\Big)\\
\ge& c_5R^{2(1+\varepsilon_3)}\log\frac{\pi\varepsilon_4^2}{Z(\theta)}\\
&-\sum_{q\in\mathbb{Z}^2\cap B(R^{1+\varepsilon_3})}\inf_{z\in\mathbb{R}^2}\sup_{|y|\le\varepsilon_4}\Big(t\int dx|\psi_R(x)|^2u(x-q-y-z)\\
&\qquad+|y+z|^\theta\Big),
\end{aligned}\tag{3.6}$$

where c_5 is a positive finite constant and ε_4 is an arbitrarily small positive constant. We take $z = 0$ for $q \in \mathbb{Z}^2$ satisfying $|q| \geq (1+s)R + \varepsilon_4$ and $z = (-1 + ((1+s)R + \varepsilon_4)/|q|)q$ for other $q \in \mathbb{Z}^2$, where s is a finite positive number specified later. Then the second term of the right hand side of (3.6) is bounded from below by

$$
\begin{aligned}
&- \sum_{q \in \mathbb{Z}^2 \cap (B(R^{1+\varepsilon_3}) \setminus B((1+s)R + \varepsilon_4))} \Big\{ t \exp\Big(- \frac{1-\varepsilon_2}{C_0} (|q| - R - \varepsilon_4)^\alpha \Big) + \varepsilon_4^\theta \Big\} \\
&- \sum_{q \in \mathbb{Z}^2 \cap B((1+s)R + \varepsilon_4)} \Big\{ t \exp\Big(- \frac{1-\varepsilon_2}{C_0} (sR)^\alpha \Big) + ((1+s)R + 2\varepsilon_4 - |q|)^\theta \Big\} \\
\geq & - c_6 (1+s)^2 R^2 t \exp\Big(- \frac{1-\varepsilon_2}{C_0} (sR)^\alpha \Big) - c_7 \varepsilon_4^\theta R^{2(1+\varepsilon_3)} \\
&- (1+\varepsilon_5) \int_{q \in \mathbb{R}^2 : |q| \leq (1+s)R + \varepsilon_4} ((1+s)R + 2\varepsilon_4 - |q|)^\theta dq
\end{aligned}
$$

for large enough R, where ε_5 is an arbitrarily small positive constant, and c_6 and c_7 are positive finite constants. By changing the variables, the last term is bounded from below by

$$
-(1+\varepsilon_5)((1+s)R + 2\varepsilon_4)^{2+\theta} \frac{2\pi}{(\theta+1)(\theta+2)}
$$

for large enough R. The above estimate is summarized as follows:

$$
\begin{aligned}
&\log \widetilde{N}(t) \\
\geq & - t \exp\Big(- \frac{B}{2} R^2 (1 - \varepsilon_1) \Big) \\
&- c_6 (1+s)^2 R^2 t \exp\Big(- \frac{1-\varepsilon_2}{C_0} (sR)^\alpha \Big) - 2 \log(2R) \\
&- c_1 t \exp(-c_2 R^{(1+\varepsilon_3)\alpha}) - c_3 \exp(-c_4 R^{(1+\varepsilon_3)\theta}) \\
&+ c_5 R^{2(1+\varepsilon_3)} \log \frac{\pi \varepsilon_4^2}{Z(\theta)} - c_7 \varepsilon_4^\theta R^{2(1+\varepsilon_3)} \\
&- (1+\varepsilon_5)((1+s)R + 2\varepsilon_4)^{2+\theta} \frac{2\pi}{(\theta+1)(\theta+2)}
\end{aligned}
\tag{3.7}
$$

for large enough R. The first term and the second term change the role according to the value of α.

When $\alpha < 2$, we take $1/s$ as an arbitrarily small positive constant and set $\widehat{R} = sR$. Then the right hand side of (3.7) is bounded from below by

$$
-c_6 t \exp\Big(- \frac{1 - 2\varepsilon_2}{C_0} \widehat{R}^\alpha \Big) - (1 + 2\varepsilon_5) \widehat{R}^{2+\theta} \frac{2\pi}{(\theta+1)(\theta+2)}
$$

for large enough $\widehat{R}$. We now take

$$
\widehat{R} = \Big(\frac{C_0 \log t}{1 - 2\varepsilon_2} \Big)^{1/\alpha}.
$$

Then we obtain

$$\log \widetilde{N}(t) \geq -c_6 - (1+2\varepsilon_5)\Big(\frac{C_0 \log t}{1-2\varepsilon_2}\Big)^{(2+\theta)/\alpha} \frac{2\pi}{(\theta+1)(\theta+2)}$$

for large enough t, from which we obtain (3.2). □

4. **3-dimensional upper estimate**

In this section we prove an upper estimates necessary to prove Theorem 2. We denote the Laplace-Stieltjes transform of the integrated density of states $\boldsymbol{N}(\lambda)$ by $\widetilde{\boldsymbol{N}}(t)$:

$$\widetilde{\boldsymbol{N}}(t) = \int_0^\infty e^{-t\lambda} d\boldsymbol{N}(\lambda).$$

Then the result is the following:

Proposition 4.1. *Under the conditions of Theorem 2, we have*

$$\overline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{\boldsymbol{N}}(t)}{t^{\widetilde{\kappa}(\boldsymbol{\alpha},\boldsymbol{\theta})}} \leq -C(\boldsymbol{\alpha},\boldsymbol{\theta},C_0), \tag{4.1}$$

where

$$\widetilde{\kappa}(\boldsymbol{\alpha},\boldsymbol{\theta}) = \frac{\dfrac{\theta_\perp}{\alpha_\perp} \vee \dfrac{\theta_3}{\alpha_3} + \dfrac{2}{\alpha_\perp} + \dfrac{1}{\alpha_3}}{\dfrac{\theta_\perp}{\alpha_\perp} \vee \dfrac{\theta_3}{\alpha_3} + 1}. \tag{4.2}$$

To prove this proposition, we use the following simple extension of Proposition 2.2:

Proposition 4.2.

$$\widetilde{\boldsymbol{N}}(t) \leq \frac{B}{2\pi(1-e^{-2Bt})\sqrt{4\pi t}} \widetilde{\boldsymbol{N}}_1(t),$$

where

$$\widetilde{\boldsymbol{N}}_1(t) = \int_{\boldsymbol{\Lambda}_1} d\boldsymbol{x} E_\theta[\exp(-t\boldsymbol{V}_{\boldsymbol{\xi}}(\boldsymbol{x}))]. \tag{4.3}$$

Proof of Proposition 4.1. By replacing the summation by the integration, we have

$$\log \widetilde{\boldsymbol{N}}_1(t) \leq \int_{\mathbb{R}^3} d\boldsymbol{q} \log E_\theta \Big[\exp\Big(-t \inf_{\boldsymbol{x}\in\Lambda_2} \boldsymbol{u}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{\xi}_{\boldsymbol{0}})\Big)\Big].$$

We restrict the integration to $|q_\perp|_\infty \leq \mathcal{L}_\perp$ and $|q_3| \leq \mathcal{L}_3$ for some finite $\mathcal{L}_\perp$ and $\mathcal{L}_3$. For any $\varepsilon_1 > 0$, there exists R_1 such that $\boldsymbol{u}(\boldsymbol{x}) \geq C_0(1-\varepsilon_1)/\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}}$ for any $|\boldsymbol{x}|_\infty \geq R_1$, where $|\boldsymbol{x}|_\infty = \max_i |x_i|$. Thus the right hand side is dominated by

$$\int_{|q_\perp|_\infty \leq \mathcal{L}_\perp, |q_3|\leq \mathcal{L}_3} d\boldsymbol{q} \quad \log \Bigg\{ \int_{|\boldsymbol{q}+\boldsymbol{y}|_\infty \geq R_1+2} \frac{d\boldsymbol{y}}{Z(\boldsymbol{\theta},p)} \exp\Big(-t \inf_{\boldsymbol{x}\in\boldsymbol{\Lambda}_2} \frac{C_0(1-\varepsilon_1)}{\|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}\|_{\tilde{p}}^{\boldsymbol{\alpha}}} - \|\boldsymbol{y}\|_p^\theta\Big)$$
$$+ \exp\Big(-t \inf_{\boldsymbol{\Lambda}_{2R_1+6}} \boldsymbol{u}\Big)\Bigg\}.$$

By changing the variables $(q_\perp, q_3)$ to $(t^{\eta_\perp} q_\perp, t^{\eta_3} q_3)$ and $(y_\perp, y_3)$ to $(t^{\eta_\perp} y_\perp, t^{\eta_3} y_3)$ with $\eta_\perp, \eta_3 \in (0,\infty)$ satisfying $\eta_\perp \alpha_\perp = \eta_3 \alpha_3$, we see that this equals

$$t^{2\eta_\perp+\eta_3} \int_{|\boldsymbol{q}|_\infty \le L} d\boldsymbol{q} \log\Big\{ \widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q}) + \exp\Big(-t \inf_{\boldsymbol{\Lambda}_{2R_1+6}} \boldsymbol{u} \Big) \Big\}, \tag{4.4}$$

where

$$\begin{aligned} &\widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q}) \\ =& t^{2\eta_\perp+\eta_3} \int_{\substack{|q_\perp + y_\perp|_\infty \ge (R_1+2)t^{-\eta_\perp} \\ |q_3+y_3| \ge (R_1+2)t^{-\eta_3}}} \frac{d\boldsymbol{y}}{Z(\boldsymbol{\theta},p)} \exp\Big(-t^{1-\eta_\perp\alpha_\perp} \inf_{\substack{|x_\perp|_\infty \le t^{-\eta_\perp} \\ |x_3| \le t^{-\eta_3}}} \frac{C_0(1-\varepsilon_1)}{\|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}\|_{\tilde{p}}^{\alpha}} \\ &- t^{\eta_*\alpha_*} \Big\| |y_*|^{\theta_*}, \frac{|y_{**}|^{\theta_{**}}}{t^{|\eta_\perp\theta_\perp - \eta_3\theta_3|}} \Big\|_p \Big), \end{aligned}$$

we take L as an arbitrary constant independent of t by taking $(\mathcal{L}_\perp, \mathcal{L}_3)$ appropriately, and we take $*$ and $**$ as elements of $\{\perp, 3\}$ such that $\eta_*\theta_* = (\eta_\perp\theta_\perp) \vee (\eta_3\theta_3)$ and $** \ne *$. Moreover we take $\eta_\perp$ and η_3 as $\eta_* = 1/(\theta_* + \alpha_*)$ and $\eta_{**} = \alpha_*/\{\alpha_{**}(\theta_*+\alpha_*)\}$ so that $1-\eta_\perp\alpha_\perp = \eta_*\theta_*$. Then, taking $\varepsilon_2, \varepsilon_3 > 0$ sufficiently small and using the positivity assumption, we can dominate $\widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q})$ by $\exp(-t^{\eta_*\theta_*}\widetilde{\boldsymbol{N}}_3(t,\boldsymbol{q}))\varepsilon_2^{-2/\theta_\perp - 1/\theta_3}$ for large enough t, where

$$\begin{aligned} &\widetilde{\boldsymbol{N}}_3(t,\boldsymbol{q}) \\ &= \inf_{\boldsymbol{x}\in\boldsymbol{\Lambda}_{\varepsilon_3}, \boldsymbol{y}\in\mathbb{R}^3} \Big\{ \frac{C_0(1-\varepsilon_1)}{\|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}\|_{\tilde{p}}^{\alpha}} + (1-\varepsilon_2) \Big\| |y_*|^{\theta_*}, \frac{|y_{**}|^{\theta_{**}}}{t^{|\eta_\perp\theta_\perp-\eta_3\theta_3|}} \Big\|_p \Big\}. \end{aligned} \tag{4.5}$$

Therefore we obtain

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{\boldsymbol{N}}(t)}{t^{2\eta_\perp+\eta_3+\eta_*\theta_*}} \le -\int_{|q|_\infty \le L} \widetilde{\boldsymbol{N}}_4(\boldsymbol{q}) d\boldsymbol{q},$$

where

$$\widetilde{\boldsymbol{N}}_4(\boldsymbol{q}) = \inf_{\boldsymbol{x}\in\boldsymbol{\Lambda}_{\varepsilon_3}, \boldsymbol{y}\in\mathbb{R}^3} \Big\{ \frac{C_0(1-\varepsilon_1)}{\|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}\|_{\tilde{p}}^{\alpha}} + (1-\varepsilon_2) \Big\| |y_*|^{\theta_*}, 1_{\theta_\perp\eta_\perp = \theta_3\eta_3} |y_{**}|^{\theta_{**}} \Big\|_p^{\boldsymbol{\theta}} \Big\}.$$

Since $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and L are arbitrary, we can complete the proof. □

5. 3-dimensional lower estimate

In this section we prove the following lower estimates necessary to prove Theorem 2:

Proposition 5.1. *If* (1.15) *holds with* (1.16) *and* (1.22)*, then we have*

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{\boldsymbol{N}}(t)}{t^{\widetilde{\kappa}(\boldsymbol{\alpha},\boldsymbol{\theta})}} \ge -C(\boldsymbol{\alpha},\boldsymbol{\theta},C_0) \tag{5.1}$$

with (4.2) *and* (1.24).

Proof. We use the bound

$$
\begin{aligned}
&\widetilde{\boldsymbol{N}}(t)\\
\geq&(2R_\perp)^{-2}(2R_3)^{-1}\exp\Big\{-t\lambda_1(\mathcal{H}_{B(R_\perp)})\\
&-t\lambda_1\Big(-\Big(\frac{d^2}{dx^2}\Big)_{(-R_3,R_3)}\Big)\Big\}\widetilde{\boldsymbol{N}}_1(t),
\end{aligned}
\tag{5.2}
$$

where $R_\perp$, $R_3 \in \mathbb{N}$, $\mathcal{H}_{R_\perp}$ is the same symbol used in (3.3), $(d^2/(dx^2))_{(-R_3,R_3)}$ is the restriction to the interval $(-R_3, R_3)$ of the 1-dimensional Laplacian by the Dirichlet boundary condition,

$$
\widetilde{\boldsymbol{N}}_1(t) = E\Bigg[\exp\Big(-t\int d\boldsymbol{x}\psi_{R_\perp}(x_\perp)^2\phi_{R_3}(x_3)^2\boldsymbol{V}_\xi(\boldsymbol{x})\Big)\Bigg],
$$

and $\psi_{R_\perp}$ and ϕ_{R_3} are the normalized ground states of $\mathcal{H}_{B(R_\perp)}$ and $-(d^2/(dx^2))_{(-R_3,R_3)}$, respectively. This can be proven by the same method as for the corresponding bound in (3.3) and Theorem (9.6) in [15] for $\mathbb{R}^d$-stationary random fields. We have Erdös's bound

$$
\lambda_1(\mathcal{H}_{R_\perp}) \leq \exp\Big(-\frac{B}{2}R_\perp^2(1-\varepsilon_1)\Big)
\tag{5.3}
$$

as in (3.4) and

$$
\lambda_1\Big(-\Big(\frac{d^2}{dx^2}\Big)_{(-R_3,R_3)}\Big) = \Big(\frac{\pi}{2R_3}\Big)^2,
\tag{5.4}
$$

where ε_1 is an arbitrarily fixed positive constant. By replacing the summation by integration, we have

$$
\log\widetilde{\boldsymbol{N}}_1(t) \geq \int_{\mathbb{R}^3}\widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q})d\boldsymbol{q},
$$

where

$$
\widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q}) = \log E\Big[\exp\Big(-t\int d\boldsymbol{x}\psi_{R_\perp}(x_\perp)^2\phi_{R_3}(x_3)^2\sup_{\boldsymbol{z}\in\boldsymbol{\Lambda}_1}\boldsymbol{u}(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{z}-\boldsymbol{\xi_0})\Big)\Big].
$$

For any $\varepsilon_2 > 0$, there exists R_2 such that $\boldsymbol{u}(\boldsymbol{x}) \leq C_0(1+\varepsilon_2)/\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}}$ for any $|\boldsymbol{x}| \geq R_2$ by the assumption (1.15). To use this bound in the above right hand side, we need $\inf\{|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{z}-\boldsymbol{\xi_0}| : |x_\perp| < R_\perp, |x_3| < R_3, \boldsymbol{z} \in \boldsymbol{\Lambda}_1\} \geq R_2$. A sufficient condition of this is $|q_\perp| \geq 2(R_2 + R_\perp + 2)$ with $|\xi_{0,\perp}| \leq |q_\perp|/2$ or $|q_3| \geq 2(R_2 + R_3 + 1)$ with $|\xi_{0,3}| \leq |q_3|/2$. We fix $\beta_\perp, \beta_3 > 0$ and take t large enough so that $t^{\beta_\perp} \geq 2(R_2 + R_\perp + 2)$ and $t^{\beta_3} \geq 2(R_2 + R_3 + 1)$. Then, by using also $\log(1-X) \geq -2X$ for $0 \leq X \leq 1/2$

we obtain

$$\begin{aligned}&\int_{|q_\perp|\geq t^{\beta_\perp},|q_3|^{1/\beta_3}\leq|q_\perp|^{1/\beta_\perp}} d\boldsymbol{q}\, \widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q})\\ \geq&\int_{|q_\perp|\geq t^{\beta_\perp},|q_3|^{1/\beta_3}\leq|q_\perp|^{1/\beta_\perp}} d\boldsymbol{q}\,\Big(\frac{-tC_0(1+\varepsilon_2)2^{\alpha_\perp}}{(|q_\perp|-2(R_\perp+2))^{\alpha_\perp}}\\ &+\log P(|\xi_{0,\perp}|\leq|q_\perp|/2)\Big)\\ \geq&-c_1t^{1+\beta_3-\beta_\perp(\alpha_\perp-2)}-c_2\exp(-c_3t^{\beta_\perp\theta_\perp})\end{aligned}\tag{5.5}$$

if

$$\alpha_\perp>2+\beta_3/\beta_\perp,\tag{5.6}$$

and

$$\begin{aligned}&\int_{|q_3|\geq t^{\beta_3},|q_\perp|^{1/\beta_\perp}\leq|q_3|^{1/\beta_3}} d\boldsymbol{q}\, \widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q})\\ \geq&\int_{|q_3|\geq t^{\beta_3},|q_\perp|^{1/\beta_\perp}\leq|q_3|^{1/\beta_3}} d\boldsymbol{q}\,\Big(\frac{-tC_0(1+\varepsilon_2)2^{\alpha_3}}{(|q_3|-2(R_3+1))^{\alpha_\perp}}\\ &+\log P(|\xi_{0,3}|\leq|q_3|/2)\Big)\\ \geq&-c_4t^{1+2\beta_\perp-\beta_3(\alpha_3-1)}-c_5\exp(-c_6t^{\beta_3\theta_3})\end{aligned}\tag{5.7}$$

if

$$\alpha_3>1+2\beta_\perp/\beta_3.\tag{5.8}$$

The other part is estimated as

$$\begin{aligned}&\int_{|q_\perp|\leq t^{\beta_\perp},|q_3|\leq t^{\beta_3}} d\boldsymbol{q}\,\widetilde{\boldsymbol{N}}_2(t,\boldsymbol{q})\\ \geq&\int_{|q_\perp|\leq t^{\beta_\perp},|q_3|\leq t^{\beta_3}} d\boldsymbol{q}\,\log\int_{|q_\perp+y_\perp|\geq R_2+R_\perp+2 \text{ or } |q_3+y_3|\geq R_2+R_3+1}\frac{d\boldsymbol{y}}{Z(\boldsymbol{\theta},p)}\\ &\times\exp\Big(\frac{-tC_0(1+\varepsilon_2)}{\inf_{|x_\perp|<R_\perp,|x_3|<R_3,\boldsymbol{z}\in\boldsymbol{\Lambda}_1}\{\|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{z}-\boldsymbol{y}\|_{\tilde{p}}^{\boldsymbol{\alpha}}\}}-\|\boldsymbol{y}\|_p^{\boldsymbol{\theta}}\Big).\end{aligned}\tag{5.9}$$

By the same change of variables as in (4.4) and (4.5), we find that the right hand side equals

$$t^{2\eta_\perp+\eta_3}\int_{|q_\perp|\leq t^{\beta_\perp-\eta_\perp},|q_3|\leq t^{\beta_3-\eta_3}} d\boldsymbol{q}\,\log\int_{\substack{|q_\perp+y_\perp|\geq(R_2+R_\perp+2)/t^{\eta_\perp}\\ \text{or } |q_3+y_3|\geq(R_2+R_3+1)/t^{\eta_3}}}\frac{d\boldsymbol{y}t^{2\eta_\perp+\eta_3}}{Z(\boldsymbol{\theta},p)}\exp(-t^{\eta_*\theta_*}\widetilde{\boldsymbol{N}}_3(t,\boldsymbol{y},\boldsymbol{q})),$$

where

$$\begin{aligned}
&\widetilde{\boldsymbol{N}}_3(t,\boldsymbol{q},\boldsymbol{y})\\
=&C_0(1+\varepsilon_2)/\inf\{|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{z}-\boldsymbol{y}|_{\tilde{p}}^{\boldsymbol{\alpha}} : |x_\perp| \le R_\perp/t^{\eta_\perp}, |x_3| \le R_3/t^{\eta_3},\\
&\qquad\qquad |z_\perp|_\infty \le 1/(2t^{\eta_\perp}), |z_3| \le 1/(2t^{\eta_3})\}\\
&+\left\| |y_*|^{\theta_*}, \frac{|y_{**}|^{\theta_{**}}}{t^{|\eta_\perp\theta_\perp-\eta_3\theta_3|}}\right\|_p.
\end{aligned} \tag{5.10}$$

Taking $\gamma_\perp, \gamma_3 \in (0,\infty)$ and $\boldsymbol{w} \in \mathbb{R}^3$, we restrict the integration to $B(w_\perp, t^{-\gamma_\perp}) \times (w_3 - t^{-\gamma_3}, w_3 + t^{-\gamma_3})$. Then we can bound the integrand with respect to q from below by

$$\log \frac{2\pi t^{2(\eta_\perp-\gamma_\perp)+(\eta_3-\gamma_3)}}{Z(\boldsymbol{\theta},p)} - t^{\eta_*\theta_*}\widetilde{\boldsymbol{N}}_4(t,\boldsymbol{q}), \tag{5.11}$$

where

$$\begin{aligned}
&\widetilde{\boldsymbol{N}}_4(t,\boldsymbol{q})\\
=&\inf\Big\{\sup_{\boldsymbol{y}\in B(w_\perp,t^{-\gamma_\perp})\times(w_3-t^{-\gamma_3},w_3+t^{-\gamma_3})} \widetilde{\boldsymbol{N}}_3(t,\boldsymbol{q},\boldsymbol{y})\\
&: w_3 \in \mathbb{R}, w_\perp \in \mathbb{R}^2,\\
&d(B(w_\perp, t^{-\gamma_\perp}), -q_\perp) \ge (R_2+R_\perp+2)t^{-\eta_\perp}\Big\}\\
&\wedge \inf\Big\{\sup_{\boldsymbol{y}\in B(w_\perp,t^{-\gamma_\perp})\times(w_3-t^{-\gamma_3},w_3+t^{-\gamma_3})} \widetilde{\boldsymbol{N}}_3(t,\boldsymbol{q},\boldsymbol{y})\\
&: w_\perp \in \mathbb{R}^2, w_3 \in \mathbb{R},\\
&d((w_3-t^{-\gamma_3}, w_3+t^{-\gamma_3}), -q_3) \ge (R_2+R_3+1)t^{-\eta_3}\Big\}.
\end{aligned} \tag{5.12}$$

We now specify $R_\perp$ and R_3 as the integer parts of $\varepsilon_3 t^{\eta_\perp}$ and $\varepsilon_3 t^{\eta_3}$, respectively, where ε_3 is an arbitrarily fixed positive number. We take $\beta_\perp$ and β_3 as $\beta_\perp = \eta_\perp + \zeta\varepsilon_4$ and $\beta_3 = \eta_3 + \varepsilon_4$, respectively, where ε_4 is also an arbitrarily fixed positive number and ζ is a number satisfying

$$\frac{1}{\alpha_\perp - 2} < \zeta < \frac{\alpha_3 - 1}{2}.$$

By taking ε_4 small, we have $1-2\eta_3, 1+\beta_3-\beta_\perp(\alpha_\perp-2), 1+2\beta_\perp-\beta_3(\alpha_3-1), 2\beta_\perp+\beta_3 < 2\eta_\perp+\eta_3+\eta_*\theta_*$ and (5.6) and (5.8). Thus we obtain

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{\boldsymbol{N}}(t)}{t^{2\eta_\perp+\eta_3+\eta_*\theta_*}} \ge -\varlimsup_{t\uparrow\infty} \int_{|q_\perp|\le t^{\beta_\perp-\eta_\perp}, |q_3|\le t^{\beta_3-\eta_3}} \widetilde{\boldsymbol{N}}_4(t,\boldsymbol{q})d\boldsymbol{q}. \tag{5.13}$$

When $|q_\perp| \le t^{\beta_\perp-\eta_\perp}$ and $|q_3| \le t^{\beta_3-\eta_3}$, we have $t^{-1} \le |q_\perp|^{-1/(\beta_\perp-\eta_\perp)}$ and $t^{-1} \le |q_3|^{-1/(\beta_3-\eta_3)}$. Thus, for large $|q|$, by taking $\boldsymbol{w}$ as $\boldsymbol{0}$, we can dominate $\widetilde{\boldsymbol{N}}_4(t,\boldsymbol{q})$ by

$$(|q_\perp|-1)_+^{-\alpha_\perp} \wedge (|q_3|-1)_+^{-\alpha_3} + |q_\perp|^{-\gamma_\perp\theta_\perp/(\beta_\perp-\eta_\perp)} \wedge |q_3|^{-\gamma_3\theta_3/(\beta_3-\eta_3)}.$$

This is integrable if we take $\gamma_\perp$ and γ_3 large enough so that $\gamma_\perp\theta_\perp/(\beta_\perp-\eta_\perp) > 3$ and $\gamma_3\theta_3/(\beta_3-\eta_3) > 3$. Thus, by the Lebesgue convergence theorem, we have

$$\begin{aligned}
&\lim_{t\uparrow\infty}\int_{|q_\perp|\le t^{\beta_\perp-\eta_\perp},|q_3|\le t^{\beta_3-\eta_3}} \widetilde{\boldsymbol{N}}_4(t,\boldsymbol{q})d\boldsymbol{q}\\
&= \int_{\mathbb{R}^3} d\boldsymbol{q}\inf\Big\{\frac{C_0(1+\varepsilon_2)}{\inf\limits_{|x_\perp|,|x_3|\le\varepsilon_3}\|\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{y}\|_{\tilde{p}}^{\boldsymbol{\alpha}}} + \||y_*|^{\theta_*}, 1_{\eta_3\theta_3=\eta_*\theta_*}|y_{**}|^{\theta_{**}}\|_p\\
&\qquad : |y_\perp+q_\perp| \vee |y_3+q_3| \ge \varepsilon_3\Big\}.
\end{aligned}$$

Since ε_2 and ε_3 are arbitrary, this completes the proof. □

6. Classical integrated density of states

In this section we summarize the results on the leading terms of the low energy asymptotics of the classical integrated densities of states in a general setting.

The 2-dimensional results are the following:

Theorem 3. (i) *If* (1.7) *holds for any* $R \ge 1$ *and* (1.8) *as* $|x| \to \infty$ *for some* $C_0 \in (0,\infty)$ *and* $\alpha \in (2,\infty)$, *then the classical integrated density of states defined by* (1.27) *has the same leading term with* (1.9)*:*

$$\lim_{\lambda\downarrow 0}\lambda^\kappa \log N_c(\lambda) = \frac{-\kappa^\kappa}{(\kappa+1)^{\kappa+1}}\left\{\int_{\mathbb{R}^2} dq \inf_{y\in\mathbb{R}^2}\left(\frac{C_0}{|q+y|^\alpha}+|y|^\theta\right)\right\}^{\kappa+1}. \tag{6.1}$$

(ii) *If* (1.7) *holds for any* $R \ge 1$ *and* (1.10) *as* $|x| \to \infty$ *for some* $C_0 \in (0,\infty)$ *and* $\alpha \in (0,2+\theta)$, *then the classical integrated density of states defined by* (1.27) *has the same leading term with* (1.11)*:*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-(2+\theta)/\alpha}\log N_c(\lambda) = \frac{-2\pi C_0^{(2+\theta)/\alpha}}{(\theta+1)(\theta+2)}. \tag{6.2}$$

(iii) *If* (1.7) *holds for any* $R \ge 1$ *and* (1.10) *as* $|x| \to \infty$ *for some* $C_0 \in (0,\infty)$ *and* $\alpha \in (2+\theta,\infty)$, *then we have*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-1}\log N_c(\lambda) = -1. \tag{6.3}$$

(iv) *If* (1.7) *holds for any* $R \ge 1$ *and* (1.10) *as* $|x| \to \infty$ *for some* $C_0 \in (0,\infty)$ *and* $\alpha = 2+\theta$, *then we have*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-1}\log N_c(\lambda) = -1 - \frac{2\pi C_0}{(\theta+1)(\theta+2)}. \tag{6.4}$$

(v) *If* $\operatorname{supp} u$ *is compact, then the classical integrated density of states defined by* (1.27) *satisfies the following:*

$$N_c(\lambda) = \frac{K_c \lambda}{4\pi}(1 + o(1)) \tag{6.5}$$

as $\lambda \downarrow 0$, *where*

$$K_c = \int_{\Lambda_1} dx \prod_{q \in \mathbb{Z}^2} P_\theta(x - q - \xi_q \notin \operatorname{supp} u). \tag{6.6}$$

(i)-(iv) are proven by the same methods in Sections 2 and 3, and (v) is proven by the Lebesgue convergence theorem.

Similarly by the methods in Sections 4 and 5, and the Lebesgue convergence theorem, we have the following 3-dimensional results:

Theorem 4. *Let* $\boldsymbol{N}_c(\lambda)$ *be the classical integrated density of states of our 3-dimensional setting defined similarly as in* (1.27).

(i) *If* (1.21) *holds for any* $R \geq 1$ *and* (1.15) *as* $|\boldsymbol{x}| \to \infty$ *for some* $C_0 \in (0, \infty)$, $\tilde{p} \in [1, \infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0, \infty)^2$ *satisfying* (1.16), *then* $\boldsymbol{N}_c(\lambda)$ *has the same leading term with* (1.25)*:*

$$\lim_{\lambda \downarrow 0} \lambda^{\kappa(\boldsymbol{\alpha}, \boldsymbol{\theta})} \log \boldsymbol{N}_c(\lambda) = \frac{-\kappa(\boldsymbol{\alpha}, \boldsymbol{\theta})^{\kappa(\boldsymbol{\alpha}, \boldsymbol{\theta})}}{(1 + \kappa(\boldsymbol{\alpha}, \boldsymbol{\theta}))^{1+\kappa(\boldsymbol{\alpha}, \boldsymbol{\theta})}} C(\boldsymbol{\alpha}, \boldsymbol{\theta}, C_0)^{1+\kappa(\boldsymbol{\alpha}, \boldsymbol{\theta})}. \tag{6.7}$$

(ii) *If* (1.21) *holds for any* $R \geq 1$ *and*

$$\boldsymbol{u}(\boldsymbol{x}) = \exp\Big(\frac{-\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}}}{C_0}(1 + o(1)) \Big) \tag{6.8}$$

as $|\boldsymbol{x}| \to \infty$ *for some* $C_0 \in (0, \infty)$, $\tilde{p} \in [1, \infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0, \infty)^2$ *satisfying*

$$\chi(\boldsymbol{\alpha}, \boldsymbol{\theta}) > 1, \tag{6.9}$$

then $\boldsymbol{N}_c(\lambda)$ *satisfies*

$$\lim_{\lambda \downarrow 0} (\log(1/\lambda))^{-\chi(\boldsymbol{\alpha}, \boldsymbol{\theta})} \log \boldsymbol{N}_c(\lambda) = -C_0^{\chi(\boldsymbol{\alpha}, \boldsymbol{\theta})} D(\boldsymbol{\alpha}, \boldsymbol{\theta}, p, \tilde{p}), \tag{6.10}$$

where

$$\chi(\boldsymbol{\alpha}, \boldsymbol{\theta}) := \frac{2}{\alpha_\perp} + \frac{1}{\alpha_3} + \Big(\frac{\theta_\perp}{\alpha_\perp} \wedge \frac{\theta_3}{\alpha_3} \Big) \tag{6.11}$$

and

$$D(\boldsymbol{\alpha}, \boldsymbol{\theta}, p, \tilde{p}) := \int_{\|\boldsymbol{q}\|_{\tilde{p}}^{\boldsymbol{\alpha}} \leq 1} d\boldsymbol{q} \inf_{\boldsymbol{y}: \|\boldsymbol{q}+\boldsymbol{y}\|_{\tilde{p}}^{\boldsymbol{\alpha}} \geq 1} \Big\| 1_{\frac{\theta_\perp}{\alpha_\perp} \leq \frac{\theta_3}{\alpha_3}} |y_\perp|^{\theta_\perp}, 1_{\frac{\theta_\perp}{\alpha_\perp} \geq \frac{\theta_3}{\alpha_3}} |y_3|^{\theta_3} \Big\|_p. \tag{6.12}$$

(iii) *If* (1.21) *holds for any* $R \geq 1$ *and* (6.8) *as* $|\boldsymbol{x}| \to \infty$ *for some* $C_0 \in (0,\infty)$, $\tilde{p} \in [1,\infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0,\infty)^2$ *satisfying*

$$\chi(\boldsymbol{\alpha}, \boldsymbol{\theta}) < 1, \tag{6.13}$$

then $\boldsymbol{N}_c(\lambda)$ *satisfies*

$$\lim_{\lambda \downarrow 0} (\log(1/\lambda))^{-\chi(\boldsymbol{\alpha},\boldsymbol{\theta})} \log \boldsymbol{N}_c(\lambda) = -\frac{3}{2}. \tag{6.14}$$

(iv) *If* (1.21) *holds for any* $R \geq 1$ *and* (6.8) *as* $|\boldsymbol{x}| \to \infty$ *for some* $C_0 \in (0,\infty)$, $\tilde{p} \in [1,\infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0,\infty)^2$ *satisfying*

$$\chi(\boldsymbol{\alpha}, \boldsymbol{\theta}) = 1, \tag{6.15}$$

then $\boldsymbol{N}_c(\lambda)$ *satisfies*

$$\lim_{\lambda \downarrow 0} (\log(1/\lambda))^{-1} \log \boldsymbol{N}_c(\lambda) = -\frac{3}{2} - C_0 D(\boldsymbol{\alpha}, \boldsymbol{\theta}, p, \tilde{p}). \tag{6.16}$$

(v) *If* $\operatorname{supp} \boldsymbol{u}$ *is compact, then* $\boldsymbol{N}_c(\lambda)$ *satisfies the following:*

$$\boldsymbol{N}_c(\lambda) = \frac{\boldsymbol{K}_c \lambda^{3/2}}{6\pi^2}(1 + o(1)) \tag{6.17}$$

as $\lambda \downarrow 0$, *where*

$$\boldsymbol{K}_c = \int_{\boldsymbol{\Lambda}_1} d\boldsymbol{x} \prod_{\boldsymbol{q} \in \mathbb{Z}^3} P_{\boldsymbol{\theta}}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}_{\boldsymbol{q}} \notin \operatorname{supp} \boldsymbol{u}). \tag{6.18}$$

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