

# Wegner estimate for Gaussian random magnetic fields

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ABSTRACT. – For the Schrödinger operator on  $L^2(\mathbb{R}^2)$  with the magnetic field which is a sample path of a stationary Gaussian random field, a Wegner type estimate applicable for the proof of the Anderson localization is proven by referring a recent method by Erdős and Hasler, and the theory of the Malliavin calculus.

## 1. INTRODUCTION

For any  $L \geq 1$  and  $\omega$  in a probability space, we consider the self-adjoint operator

$$(1.1) \quad H_L^\omega := \sum_{\iota=1}^2 (i\partial_\iota + A_{L,\iota}^\omega(x))^2$$

with the Dirichlet boundary condition on the open square  $\Lambda_L = (-L/2, L/2)^2$  with the side length  $L$  and 0 as its center, where  $i = \sqrt{-1}$  and  $A_L^\omega$  is a  $C^1$ -map from  $\Lambda_L$  to  $\mathbb{R}^2$  satisfying  $\nabla \times A_L^\omega := \partial_1 A_{L,2}^\omega - \partial_2 A_{L,1}^\omega = B^\omega$ . Its spectrum depends only on  $B^\omega$  and is independent of the choice of the vector potential  $A_L^\omega$ . This is the Schrödinger operator with the magnetic field  $B^\omega$ .

As the magnetic field  $B^\omega$ , we take a Gaussian random field on  $\mathbb{R}^2$ . We assume  $B^\omega(x)$  is stationary with respect to the shift in the space variable  $x \in \mathbb{R}^2$ : the random fields  $B^\omega(\cdot)$  and  $B^\omega(x + \cdot)$  have a same law. Moreover we assume its covariance function  $V(x - y) = \text{Cov}(B^\omega(x), B^\omega(y))$  is

$$(1.2) \quad V(x) = \int \tilde{\sigma}(x - y) \tilde{\sigma}(y) dy,$$

where

$$(1.3) \quad \tilde{\sigma}(x) = \mathcal{P}(\Delta)\sigma(x),$$

$$(1.4) \quad \sigma(x) = (\bar{\sigma}^2 - |x|^2)_+^\nu,$$

$a_+ = \max\{a, 0\}$  is the positive part,  $\bar{\sigma} \in (0, \infty)$ ,  $\nu \in (3/2, \infty)$ ,  $\Delta = \partial_1^2 + \partial_2^2$  and  $\mathcal{P}$  is a non-zero polynomial of the degree less than  $(\nu - 3/2)/2$ . This special form of the covariance makes possible to apply the theory on the Bessel functions. The condition  $\nu > 3/2$  guarantees that the sample path of  $B^\omega$

belongs to the local Sobolev space  $W_{loc}^{2,p}(\mathbb{R}^2)$  of the functions whose derivatives of order  $\leq 2$  are locally  $p$ -th power integrable for any  $p \in [1, \infty)$ . Thus  $B^\omega(x)$  is  $C^1$  in  $x$  by the Sobolev imbedding theorem (see e.g. [1] Theorem 4.12 Part I Case C).

In this paper, we prove the following:

**Theorem 1.** *Under the above assumptions, there exist positive finite constants  $C_0, C_1$  and  $C_2$  such that*

$$(1.5) \quad \mathbb{E}[\text{Tr}[\chi_{[E-\eta, E+\eta]}(H_L^\omega)]] \leq C_0 R^2 \eta L^{C_1}$$

for any  $R \in [1, \infty)$ ,  $L \geq \sqrt{R} \vee C_2$  and  $E, \eta > 0$  satisfying  $E + \eta \leq R$ .

By this theorem and the Lifschitz behavior shown by Theorem 4.3 in [21], the multi scale analysis works well and we obtain the following (cf. [8, 20]):

**Corollary .** *Let  $F^\omega(x)$  be a stationary Gaussian random field with the covariance*

$$\text{Cov}(F^\omega(x), F^\omega(x')) = \int \sigma(x-y)\sigma(x'-y)dy,$$

where  $\sigma$  is the function defined in (1.4) with  $\nu > 7/2$ . Then the operator

$$H^\omega := \sum_{i=1}^2 (i\partial_i + A_i^\omega(x))^2$$

with a  $C^1$  vector potential  $A^\omega$  on  $\mathbb{R}^2$  such that  $\nabla \times A^\omega = -\Delta F^\omega$  exhibits the Anderson localization in the low energies as follows: there exists a positive finite constant  $\varepsilon_0$  such that  $[0, \varepsilon_0]$  is included in the pure point spectrum of  $H^\omega$ , the corresponding eigenfunctions decay exponentially, and

$$\mathbb{E} \left[ \sup_t \left\| |x|^p e^{-itH^\omega} 1_I(H^\omega) 1_K \right\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \right] < \infty$$

for any  $p \in (0, \infty)$ ,  $I \subset [0, \varepsilon_0]$  and any compact set  $K$  in  $\mathbb{R}^2$ , where  $\| \cdot \|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)}$  is the operator norm of bounded operators on  $L^2(\mathbb{R}^2)$ .

In the original Wegner's estimate [23] for the Anderson model, the motivation was to bound the density of states and the idea was to use the monotone dependence of the eigenvalues with respect to the random elements. To obtain the bound of the density of states, we need the linear dependence in the volume:  $C_1$  in (1.5) should be 2. On the other hand the above weak type estimates are known to be sufficient for the proof of the Anderson localization by the multi scale analysis initiated by Fröhlich and Spencer [7] and these estimates have been extended to many models [3, 17, 20]. However the main property of the model to prove the estimates has been the monotone dependence of the eigenvalues with respect to the random

elements. The Schrödinger operator with random magnetic fields never have such a monotonicity. Until recent, the magnetic case had been treated mainly by Klopp's method [9, 11, 13, 14, 22]. His method uses the homogeneity with respect to the random element of the eigenvalues of the corresponding Birman-Schwinger operator. However to obtain the homogeneity in the magnetic case, the corresponding random vector potential had been assumed to be small. Very recently, Erdős and Hasler [4, 5, 6] gave a new method to obtain the Wegner type estimate by posing conditions only on the magnetic fields. This is more preferable since the gauge invariance implies that the spectral structure of the magnetic Schrödinger operator depends only on the magnetic field. Their method use the non-degeneracy of the gradient of eigenvalues with respect to the random elements. To obtain the non-degeneracy they assume that the random magnetic field has fine alloy type structures and is dominated from above and below by positive finite constants. In this paper we extend their theory to the above simple Gaussian random fields. Now Gaussian random fields are not bounded and not positive. The unboundedness brings no serious problem since Gaussian random fields decay exponentially at infinity. Now the non-positivity brings the essential problem. Instead of the positivity, we show a non-degeneracy estimate in Section 4 below for our special Gaussian random fields. The key point of this estimate is the existence of a bound which is a quadratic forms of the white noise with infinite rank. Since the rank is infinity, we can show the probability of the decay of the random field is small enough. This is a same situation with that where the non-degeneracy of the Malliavin covariance is proven [12, 15, 16, 19]. Another key point for the Wegner estimate is the integration by parts on the probability space. This is also the key point of the Malliavin calculus. Then we use the same notation used in the Malliavin calculus: as in Nualart [16]

$$B^\omega(x) = B + \widetilde{B}^\omega(x), B \in \mathbb{R} \text{ and } \widetilde{B}^\omega(x) = \omega(\tilde{\sigma}(x - \cdot)),$$

where  $\omega$  is the isonormal Gaussian process  $(\omega(h))_{h \in L^2(\mathbb{R}^2)}$ : for any  $h \in L^2(\mathbb{R}^2)$ ,  $\omega(h)$  is a Gaussian random variable such that

$$\mathbb{E}[\omega(h)] = 0 \text{ and } \mathbb{E}[\omega(h)\omega(h')] = (h, h')_{L^2(\mathbb{R}^2)}.$$

This  $\omega$  is also called as the white noise and the notation

$$\widetilde{B}^\omega(x) = \int_{\mathbb{R}^2} \tilde{\sigma}(x - y)\omega(dy)$$

is also used (cf. [16] p.8). The  $\sigma$ -field of the probability space is given by that generated by  $\{\omega(h) : h \in L^2(\mathbb{R}^2)\}$  (cf. [16] p.5). Then the measurability of the operator  $H_L^\omega$  is obtained as in [3] Chapter 5. To

reduce the proof of the theorem to the estimates on the non-degeneracy, we apply also the theory on the Bessel functions.

In the following, we mainly consider the case that

$$(1.6) \quad \tilde{\sigma} = \sigma$$

for simplicity. The extension to the case of (1.3) is explained in Remarks 2.1 and 4.1 below. All the estimates in this paper are given as systematic simpler estimates rather than as sharp estimates. As the vector potential, we take as

$$(1.7) \quad A_{L,1}^\omega(x) = (\partial_2 F_L^\omega)(x) \text{ and } A_{L,2}^\omega(x) = -(\partial_1 F_L^\omega)(x)$$

on  $\Lambda_L$ , where

$$(1.8) \quad F_L^\omega(x) = \sum_{\mathbf{n} \in \mathbb{N}^2} \frac{\Phi_{\mathbf{n},L}(x)}{E_{\mathbf{n},L}} \int_{\Lambda_L} \Phi_{\mathbf{n},L}(y) B^\omega(y) dy,$$

and

$$E_{\mathbf{n},L} = \left(\frac{\pi|\mathbf{n}|}{L}\right)^2 \text{ and } \Phi_{\mathbf{n},L}(x) = \frac{2}{L} \prod_{i=1}^2 \sin\left(n_i \pi \left(\frac{x_i}{L} + \frac{1}{2}\right)\right)$$

for  $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ .  $\{E_{\mathbf{n},L}, \Phi_{\mathbf{n},L}\}_{\mathbf{n} \in \mathbb{N}^2}$  is the eigenvalues and a complete orthonormal system consisting of the eigenfunctions of the negative Dirichlet Laplacian  $-\Delta_{\Lambda_L}^D$  (cf. [18], p-266).

The organization of this paper is as follows. In Section 2, we dominate the norm of the gradient of the eigenvalue from below in terms of the magnetic field. In Section 3, we modify the theory by Erdős and Hasler to prove the estimates of the current used in Section 2. In Section 4, we prove the necessary estimate on the non-degeneracy of the Gaussian random field. In Section 5, we modify the theory by Erdős and Hasler to prove Theorem 1.

## 2. A LOWER BOUND OF THE NORM OF THE GRADIENT OF THE EIGENVALUE

Let  $\lambda_\ell(H_L^\omega)$  be the  $\ell$ -th eigenvalue of the operator  $H_L^\omega$ , which is a functional of the isonormal Gaussian process  $\omega = (\omega(\cdot))_{\cdot \in L^2(\mathbb{R}^2)}$ . In Definition 2.6 in [19], the notion of the  $H$ -differentiability of a functional  $\mathcal{F}(\omega)$  at a sample path  $\omega_0$  is defined as the existence of  $D\mathcal{F}(\omega_0) \in L^2(\mathbb{R}^2)$  such that

$$\lim_{\varepsilon \rightarrow 0} \{\mathcal{F}((\omega_0(\cdot) + \varepsilon(\Phi, \cdot))_{\cdot \in L^2(\mathbb{R}^2)}) - \mathcal{F}(\omega_0)\} / \varepsilon = (D\mathcal{F}(\omega_0), \Phi)_{L^2(\mathbb{R}^2)}$$

for any  $\Phi \in L^2(\mathbb{R}^2)$ .  $\lambda_\ell(H_L^\omega)$  is  $H$ -differentiable everywhere in this sense since  $\varepsilon \mapsto \lambda_\ell(H_L^{\omega + \varepsilon\Phi})$  is complex analytic for any  $\Phi \in L^2(\mathbb{R}^2)$  by the regular perturbation theory (cf. [18]§XII.2).

In this section, we dominate the norm of the derivative restricted to a finite dimensional subspace from below in terms of the magnetic field. The object is (2.8) below. The derivative is represented as

$$(D\lambda_\ell(H_L^\omega), \Phi)_{L^2(\mathbb{R}^2)} = \int_{\Lambda_L} j^\omega(x) \cdot (DA_L^\omega(x), \Phi)_{L^2(\mathbb{R}^2)} dx,$$

$$(DA_{L,1}^\omega(x), \Phi)_{L^2(\mathbb{R}^2)} = \sum_{\mathbf{n} \in \mathbb{N}^2} \frac{(\partial_2 \Phi_{\mathbf{n},L})(x)}{E_{\mathbf{n},L}} \int_{\Lambda_L} dy \Phi_{\mathbf{n},L}(y) \int_{\mathbb{R}^2} \sigma(y-z) \Phi(z) dz$$

and

$$(DA_{L,2}^\omega(x), \Phi)_{L^2(\mathbb{R}^2)} = - \sum_{\mathbf{n} \in \mathbb{N}^2} \frac{\partial_1 \Phi_{\mathbf{n},L}(x)}{E_{\mathbf{n},L}} \int_{\Lambda_L} dy \Phi_{\mathbf{n},L}(y) \int_{\mathbb{R}^2} \sigma(y-z) \Phi(z) dz,$$

where  $j^\omega(x) = (j_1^\omega(x), j_2^\omega(x))$  is the current of the eigenfunction  $\psi_\ell$  of the eigenvalue  $\lambda_\ell(H_L^\omega)$  defined by

$$(2.1) \quad j_\ell^\omega(x) = 2 \operatorname{Re} \overline{\psi_\ell} (i\partial_\ell + A_{L,\ell}^\omega(x)) \psi_\ell.$$

As the direction  $\Phi$  of the derivative, we take  $\tilde{\Phi}_{\xi,L}(x)$  defined by 0 on  $\Lambda_{3L}^c$  and

$$\frac{2}{L} \prod_{\iota=1}^2 \sin\left(\xi_\iota \pi \left(\frac{x_\iota}{L} + \frac{1}{2}\right)\right)$$

on  $\Lambda_{3L}$  for  $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ . Since  $\operatorname{supp} \sigma(y - \cdot) \subset \Lambda_{3L}$  for any  $y \in \Lambda_L$  if  $L \geq \bar{\sigma}$ , we have

$$\int_{\Lambda_{3L}} \sigma(y-z) \tilde{\Phi}_{\xi,L}(z) dz = \hat{\sigma}\left(\frac{\xi}{2L}\right) \tilde{\Phi}_{\xi,L}(y),$$

where

$$\hat{\sigma}(\xi) = \int_{\mathbb{R}^2} \exp(-2\pi i \xi \cdot x) \sigma(x) dx$$

is the Fourier transform. In a special form of (1.4), the transform is written as

$$\hat{\sigma}(\xi) = \frac{\bar{\sigma}^{\nu+1} \Gamma(\nu+1)}{\pi^\nu |\xi|^{\nu+1}} J_{\nu+1}(2\pi \bar{\sigma} |\xi|),$$

where

$$J_{\nu+1}(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (t/2)^{2m+\nu+1}}{m! \Gamma(m+\nu+2)}$$

is the Bessel function of the order  $\nu+1$  (cf. [2] 9.1.18 and 11.4.10). The function  $t^{-\nu-1} J_{\nu+1}(t)$  is even, and is known that the zero points  $\{j_{\nu+1,s}\}_{s \in \mathbb{N}}$  on the interval  $(0, \infty)$  are simple and satisfy the asymptotics

$$(2.2) \quad j_{\nu+1,s} = \left(s + \frac{\nu}{2} + \frac{1}{4}\right) \pi + O\left(\frac{1}{s}\right)$$

(cf. [2] 9.5.12). We should take  $\xi$  so that  $2\pi \bar{\sigma} |\xi| / (2L)$  is apart from the zero points  $\{j_{\nu+1,s}\}_s$ . On the other hand, we should take the set of  $\xi$  so that this set includes sufficiently various elements to obtain a

positive lower bound of the norm of the derivatives. One candidate is  $\mathbb{N}^2$  since  $\{\Phi_{\mathbf{n},L}\}_{\mathbf{n} \in \mathbb{N}^2}$  is complete in  $L^2(\mathbb{R}^2)$ . Now we modify  $\mathbb{N}^2$  as  $\{(\mathbf{n}; \varepsilon, L)\}_{\mathbf{n} \in \mathbb{N}^2} \subset (0, \infty)^2$ , where

$$(2.3) \quad (\mathbf{n}; \varepsilon, L) := \begin{cases} \left\{1 + \frac{\varepsilon \bar{j} L}{8\pi \bar{\sigma} |\mathbf{n}|}\right\} \mathbf{n} & \text{if } |\mathbf{n}| \in \left[L \frac{j_{\nu+1,s}}{\pi \bar{\sigma}}, L \frac{j_{\nu+1,s} + \bar{j}/8}{\pi \bar{\sigma}}\right) \\ & \text{for some } s \in \mathbb{N}, \\ \left\{1 - \frac{\varepsilon \bar{j} L}{8\pi \bar{\sigma} |\mathbf{n}|}\right\} \mathbf{n} & \text{if } |\mathbf{n}| \in \left(L \frac{j_{\nu+1,s} - \bar{j}/8}{\pi \bar{\sigma}}, L \frac{j_{\nu+1,s}}{\pi \bar{\sigma}}\right) \\ & \text{for some } s \in \mathbb{N}, \\ \mathbf{n} & \text{otherwise,} \end{cases}$$

$\bar{j} := \inf_{s \in \mathbb{N}} (j_{\nu+1,s+1} - j_{\nu+1,s}) \wedge j_{\nu+1,1}$ , and  $\varepsilon \in (0, 1)$  is specified later. Then we have

$$\pi \bar{\sigma} |(\mathbf{n}; \varepsilon, L)| / L \in [0, \infty) \setminus \bigcup_{s \in \mathbb{N}} \left(j_{\nu+1,s} - \frac{\varepsilon \bar{j}}{8}, j_{\nu+1,s} + \frac{\varepsilon \bar{j}}{8}\right) =: \mathcal{G}_\varepsilon.$$

The asymptotics of the Bessel function itself is known as

$$(2.4) \quad J_{\nu+1}(t) = \sqrt{\frac{2}{\pi t}} \left\{ \cos\left(t - \frac{2\nu+3}{4}\pi\right) + O\left(\frac{1}{t}\right) \right\}$$

(cf. [2] 9.2.1). Thus we can show that

$$(2.5) \quad |\sqrt{t} J_{\nu+1}(t)| \geq c_1 \varepsilon \text{ on } \mathcal{G}_\varepsilon \cap [\bar{j}/2, \infty)$$

for some positive constant  $c_1$ . Indeed we can take  $T_\varepsilon \in (0, \infty)$  such that  $|j_{\nu+1,s} - (s + \nu/2 + 1/4)\pi| < \varepsilon \bar{j}/(16)$  and  $|\sqrt{\pi t/2} J_{\nu+1}(t) - \cos(t - (\nu+1)\pi/2 - \pi/4)| \leq \varepsilon \bar{j}/(16\pi)$  if  $(s + \nu/2 + 1/4)\pi, t \geq T_\varepsilon$  by (2.2) and (2.4). Thus we have  $|\cos(t - (\nu+1)\pi/2 - \pi/4)| \geq \varepsilon \bar{j}/(8\pi)$  and  $|\sqrt{\pi t/2} J_{\nu+1}(t)| \geq \varepsilon \bar{j}/(16\pi)$  on  $\mathcal{G}_\varepsilon \cap [T_\varepsilon + \pi/2, \infty)$ . By the compactness of  $[\bar{j}/2, T_\varepsilon + \pi/2]$  and the simplicity of  $j_{\nu+1,s}$ , we obtain (2.5) (cf. [2] 9.5). By noting also  $\inf_{[0, \bar{j}/2]} J_{\nu+1}(t)/t^{\nu+1} > 0$ , we have

$$\left| \hat{\sigma}\left(\frac{(\mathbf{n}; \varepsilon, L)}{2L}\right) \right| \geq c_2 \left( \left(\varepsilon \left(\frac{L}{\|\mathbf{n}\|}\right)^{\nu+1/2}\right) \wedge 1 \right).$$

Thus, as the direction of the derivatives, we take  $\{\tilde{\Phi}_{(\mathbf{n}; \varepsilon, L), L}(x)\}_{\mathbf{n} \in \mathbb{N}^2: |\mathbf{n}| \leq \mathcal{R}}$ , where the restricting positive number  $\mathcal{R}$  is for the estimates in Section 5 below. Then we have

$$\begin{aligned} |(D\lambda_\ell(H_L^\omega), \tilde{\Phi}_{(\mathbf{n}; \varepsilon, L), L})_{L^2(\mathbb{R}^2)}| &\geq c_2 \left( \left(\varepsilon \left(\frac{L}{\mathcal{R}}\right)^{\nu+1/2}\right) \wedge 1 \right) \left| \int_{\Lambda_L} dx (\nabla \times j^\omega)(x) \right. \\ &\quad \left. \times \sum_{\mathbf{m} \in \mathbb{N}^2} \frac{\Phi_{\mathbf{m}, L}(x)}{E_{\mathbf{m}, L}} \int_{\Lambda_L} dy \Phi_{\mathbf{m}, L}(y) \tilde{\Phi}_{(\mathbf{n}; \varepsilon, L), L}(y) \right|. \end{aligned}$$

By  $\|\tilde{\Phi}_{(\mathbf{n};\varepsilon,L),L} - \tilde{\Phi}_{\mathbf{n},L}\| \leq \varepsilon \bar{j}L/(4\bar{\sigma})$ ,  $E_{\mathbf{m},L} \geq (\pi/L)^2$  and  $E_{\mathbf{n},L} \leq (\pi\mathcal{R}/L)^2$ , we have

$$\begin{aligned} & \sum_{\mathbf{n} \in \mathbb{N}^2: |\mathbf{n}| \leq \mathcal{R}} (D\lambda_\ell(H_L^\omega), \tilde{\Phi}_{(\mathbf{n};\varepsilon,L),L})_{L^2(\mathbb{R}^2)}^2 \\ & \geq c_2^2 \left( \left( \varepsilon^2 \left( \frac{L}{\mathcal{R}} \right)^{2\nu+1} \right) \wedge 1 \right) \left\{ \frac{1}{2\pi^4} \left( \frac{L}{\mathcal{R}} \right)^4 \sum_{\mathbf{n} \in \mathbb{N}^2: |\mathbf{n}| \leq \mathcal{R}} \left( \int_{\Lambda_L} dx (\nabla \times j^\omega)(x) \Phi_{\mathbf{n},L}(x) \right)^2 \right. \\ & \quad \left. - \frac{\bar{j}^2 \varepsilon^2 L^6 \mathcal{R}^2}{16\pi^4 \bar{\sigma}^2} \|\nabla \times j^\omega\|_{L^2(\Lambda_L)}^2 \right\}. \end{aligned}$$

By (3.2) below, we have

$$\begin{aligned} & c_3 L^{22} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^6 \geq \|\nabla(\nabla \times j^\omega)\|_{L^2(\Lambda_L)}^2 \\ & = \sum_{\iota=1}^2 \sum_{\mathbf{n} \in \mathbb{N}^2} (\partial_\iota(\nabla \times j^\omega), \Phi_{\mathbf{n},L}^{(\iota)})_{L^2(\Lambda_L)}^2 = \sum_{\mathbf{n} \in \mathbb{N}^2} \left( \frac{\pi|\mathbf{n}|}{L} \right)^2 (\nabla \times j^\omega, \Phi_{\mathbf{n},L})_{L^2(\Lambda_L)}^2 \\ & \geq \left( \frac{\pi\mathcal{R}}{L} \right)^2 \sum_{\mathbf{n} \in \mathbb{N}^2: |\mathbf{n}| > \mathcal{R}} (\nabla \times j^\omega, \Phi_{\mathbf{n},L})_{L^2(\Lambda_L)}^2, \end{aligned}$$

where, for each  $\iota \in \{1, 2\}$ ,

$$\left\{ \Phi_{\mathbf{n},L}^{(\iota)} = \frac{2}{L} \cos\left(n_\iota \pi \left( \frac{x_\iota}{L} + \frac{1}{2} \right)\right) \sin\left(n_{\widehat{\iota}} \pi \left( \frac{x_{\widehat{\iota}}}{L} + \frac{1}{2} \right)\right) \right\}_{\mathbf{n} \in \mathbb{N}^2}$$

is a complete orthonormal system of the orthogonal complement of  $\{\phi \in L^2(\Lambda_L) : \phi \text{ is independent of } x_\iota\}$  in  $L^2(\Lambda_L)$ , and  $\widehat{1} = 2$  and  $\widehat{2} = 1$ . By using also (3.1) below, we have

$$\begin{aligned} & \sum_{\mathbf{n} \in \mathbb{N}^2: |\mathbf{n}| \leq \mathcal{R}} (D\lambda_\ell(H_L^\omega), \tilde{\Phi}_{(\mathbf{n};\varepsilon,L),L})_{L^2(\mathbb{R}^2)}^2 \\ & \geq c_4 \left( \left( \varepsilon^2 \left( \frac{L}{\mathcal{R}} \right)^{2\nu+1} \right) \wedge 1 \right) \left\{ \left( \frac{L}{\mathcal{R}} \right)^4 \left( \|\nabla \times j^\omega\|_{L^2(\Lambda_L)}^2 - c_3 \frac{L^{24} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^6}{\pi^2 \mathcal{R}^2} \right) \right. \\ & \quad \left. - c_5 \varepsilon^2 L^{24} \mathcal{R}^2 (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^4 \right\}. \end{aligned}$$

Moreover we use  $\nabla \cdot j^\omega = \partial_1 j_1^\omega + \partial_2 j_2^\omega = 0$  and Lemma 3.2 below to change the bound so that its derivatives in  $\omega$  have simpler representations:

$$\|\nabla \times j^\omega\|_{L^2(\Lambda_L)}^2 = \|\nabla j^\omega\|_{L^2(\Lambda_L)}^2 \geq \left( \frac{\pi}{L} \|j^\omega\|_{L^2(\Lambda_L)} \right)^2 \geq \frac{c_6}{L^6} \mathcal{B}(x_*, \omega),$$

where

$$(2.6) \quad \mathcal{B}(x_*, \omega) := \int_0^{c_7 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2}} \frac{dr}{2\pi r} \times \left| \int_{B(c_7 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2} x_*, r)} B^\omega(x) dx \right|^2$$

and  $x_* \in \mathbb{Z}^2$  such that  $c_7 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2} x_* \in \Lambda_L$ . We now take  $\mathcal{R}$  and  $\varepsilon$  as

$$\mathcal{R}(x_*, \omega) = \left( \frac{2c_3}{\pi^2 c_6} \right)^{1/2} L^{15} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^3 \mathcal{B}(x_*, \omega)^{-1/2}$$

and

$$\begin{aligned}\varepsilon(x_*, \omega) &= \left(\frac{c_6}{4c_5}\right)^{1/2} \mathcal{B}(x_*, \omega)^{1/2} L^{-13} \mathcal{R}(x_*, \omega)^{-3} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2} \\ &= \left(\frac{c_6}{4c_5}\right)^{1/2} \left(\frac{\pi^2 c_6}{2c_3}\right)^{3/2} \mathcal{B}(x_*, \omega)^2 L^{-58} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-11},\end{aligned}$$

respectively. This  $\varepsilon(x_*, \omega)$  is small enough for the definition (2.3) since we deduce

$$\varepsilon(x_*, \omega) \leq c_8 L^{-146} R^{-25}$$

from

$$(2.7) \quad \mathcal{B}(x_*, \omega) \leq \int_0^{c_7 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2}} \frac{\pi r^3}{2} dr \|B^\omega\|_{L^\infty(\Lambda_L)}^2 \leq c_9 L^{-44} R^{-7}.$$

Then we obtain

$$(2.8) \quad \begin{aligned} &\sum_{\mathbf{n} \in \mathbb{N}^2: |\mathbf{n}| \leq \mathcal{R}(x_*, \omega)} (D\lambda_\ell(H_L^\omega), \tilde{\Phi}_{(\mathbf{n}; \varepsilon(x_*, \omega), L), L})_{L^2(\mathbb{R}^2)}^2 \\ &\geq c_{10} \mathcal{B}(x_*, \omega)^{\nu+15/2} L^{-28\nu-192} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-6\nu-37}. \end{aligned}$$

The integral  $\mathcal{B}(x_*, \omega)$  of the magnetic field in the right hand side is dominated from below in Section 4 below.

*Remark 2.1.* To extend the results of this section to the case where  $\sigma$  is replaced by  $\tilde{\sigma}$  defined in (1.3), we have only to avoid not only the set  $\{\xi \in \mathbb{R}^2 : J_{\nu+1}(\pi\tilde{\sigma}|\xi|/L)\}$  but also the set  $\{\xi \in \mathbb{R}^2 : \mathcal{P}(-(\pi|\xi|/L)^2)\}$  in the definition (2.3) of  $(\mathbf{n}; \varepsilon, L)$ .

### 3. THE ESTIMATES OF THE CURRENT

In this section we modify the proof of Lemmas 6.1 and 6.2 in Erdős and Hasler [4] to prove the following:

**Lemma 3.1.** *There exist finite positive constants  $c_1$  and  $c_2$  such that*

$$(3.1) \quad \|\nabla \times j^\omega\|_{L^2(\Lambda_L)} \leq c_1 L^9 (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^2$$

and

$$(3.2) \quad \|\nabla(\nabla \times j^\omega)\|_{L^2(\Lambda_L)} \leq c_2 L^{11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^3$$

for any  $L \geq 2$ ,  $R \geq 1$  and the current  $j^\omega$  of the normalized eigenfunction of the operator  $H_L^\omega$  with the eigenvalue less than  $R$ .

**Lemma 3.2.** *There exist finite positive constants  $c_1$  and  $c_2$ , and  $x_0 \in \Lambda_L$  such that*

$$\begin{aligned} \int_{\Lambda_L} |j^\omega(x)|^2 dx &\geq \int_{B(x_0, c_2 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2})} |j^\omega(x)|^2 dx \\ &\geq \frac{c_1}{L^4} \int_0^{c_2 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2}} \left| \int_{B(x_0, r)} B^\omega(x) dx \right|^2 \frac{dr}{2\pi r} \end{aligned}$$

for any  $x_0 \in B(x_0, c_2 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2})$ ,  $L \geq 2$ ,  $R \geq 1$  and the current  $j^\omega$  of the normalized eigenfunction of the operator  $H_L^\omega$  with the eigenvalue less than  $R$ , where  $B(a, r) = \{x \in \mathbb{R}^2 : |x - a| \leq r\}$  for any  $a \in \mathbb{R}^2$  and  $r \geq 0$ .

Before proving these, we first prepare the following:

**Lemma 3.3.** *There exist finite constants  $c_1$ ,  $c_2$  and  $c_3$  such that*

$$(3.3) \quad \|A_L^\omega\|_{L^\infty(\Lambda_L)} \leq c_1 L \|B^\omega\|_{W^{2,1}(\Lambda_L)} \leq c_2 L^2 \|B^\omega\|_{W^{2,2}(\Lambda_L)}$$

and

$$(3.4) \quad \|\nabla A_L^\omega\|_{L^2(\Lambda_L)} \leq c_3 \|B^\omega\|_{L^2(\Lambda_L)}$$

for any  $L \geq 2$ .

*Proof.* By the integration by parts, we have

$$\begin{aligned} &\int_{\Lambda_L} dy \Phi_{\mathbf{n}, L}(y) B^\omega(y) \\ &= \frac{2L}{n_1 n_2 \pi^2} \left\{ \sum_{\tau_1, \tau_2 \in \{1, -1\}} (-\tau_1)^{n_1} (-\tau_2)^{n_2} B^\omega\left(\tau_1 \frac{L}{2}, \tau_2 \frac{L}{2}\right) \right. \\ &\quad + \int_{-L/2}^{L/2} dy_1 \sum_{\tau \in \{1, -1\}} (-\tau)^{n_2} \partial_1 B^\omega\left(y_1, \tau \frac{L}{2}\right) \cos n_1 \pi \left(\frac{y_1}{L} + \frac{1}{2}\right) \\ &\quad + \int_{-L/2}^{L/2} dy_2 \sum_{\tau \in \{1, -1\}} (-\tau)^{n_1} \partial_2 B^\omega\left(\tau \frac{L}{2}, y_2\right) \cos n_2 \pi \left(\frac{y_2}{L} + \frac{1}{2}\right) \\ &\quad \left. + \int_{\Lambda_L} dy \partial_1 \partial_2 B^\omega(y) \prod_{\iota=1}^2 \cos n_\iota \pi \left(\frac{y_\iota}{L} + \frac{1}{2}\right) \right\}. \end{aligned} \tag{3.5}$$

Thus we have

$$\left| \int_{\Lambda_L} dy \Phi_{\mathbf{n}, L}(y) B^\omega(y) \right| \leq \frac{c_1 L}{n_1 n_2} \|B^\omega\|_{W^{2,1}(\Lambda_L)} \leq \frac{c_2 L^2}{n_1 n_2} \|B^\omega\|_{W^{2,2}(\Lambda_L)}.$$

Since

$$\|\partial_1 \Phi_{\mathbf{n}, L}\|_{L^\infty(\Lambda_L)} = 2n_1 \pi L^{-2} \text{ and } \|\partial_2 \Phi_{\mathbf{n}, L}\|_{L^\infty(\Lambda_L)} = 2n_2 \pi L^{-2},$$

we obtain (3.3). For (3.4), we use the property that  $\{\partial^\alpha \Phi_{\mathbf{n},L}\}_{\mathbf{n} \in \mathbb{N}^2}$  constitutes an orthogonal system in  $L^2(\Lambda_L)$  for any  $\alpha \in \mathbb{Z}_+^2$ :

$$\begin{aligned} \|\nabla A_L^\omega\|_{L^2(\Lambda_L)}^2 &= \|\nabla^{\otimes 2} F_L^\omega\|_{L^2(\Lambda_L)}^2 \\ &= \sum_{\mathbf{n} \in \mathbb{N}^2} \frac{\|\nabla^{\otimes 2} \Phi_{\mathbf{n},L}\|_{L^2(\Lambda_L)}^2}{E_{\mathbf{n},L}^2} \left| \int_{\Lambda_L} \Phi_{\mathbf{n},L}(y) B^\omega(y) dy \right|^2 \\ &\leq c_3 \sum_{\mathbf{n} \in \mathbb{N}^2} \left| \int_{\Lambda_L} \Phi_{\mathbf{n},L}(y) B^\omega(y) dy \right|^2 = c_3 \|B^\omega\|_{L^2(\Lambda_L)}^2. \end{aligned}$$

□

*Proof of Lemma 3.1.* By (2.1), we have

$$|\nabla \times j^\omega| \leq 2|\nabla \times (\overline{\psi_\ell} (i\nabla + A_L^\omega) \psi_\ell)| \leq 2|\nabla \psi_\ell|^2 + 4|A_L^\omega| |\nabla \psi_\ell| |\psi_\ell| + 2|B^\omega| |\psi_\ell|^2$$

and

$$(3.6) \quad \begin{aligned} \|\nabla \times j^\omega\|_{L^2(\Lambda_L)} &\leq c_1 (\|\nabla \psi_\ell\|_{L^4(\Lambda_L)}^2 + \|A_L^\omega\|_{L^\infty(\Lambda_L)} \|\nabla \psi_\ell\|_{L^4(\Lambda_L)} \|\psi_\ell\|_{L^4(\Lambda_L)} \\ &\quad + \|B^\omega\|_{L^\infty(\Lambda_L)} \|\psi_\ell\|_{L^4(\Lambda_L)}^2). \end{aligned}$$

For the derivative, we have

$$\begin{aligned} |\nabla(\nabla \times j^\omega)| &\leq 4|\nabla^{\otimes 2} \psi_\ell| |\psi_\ell| + 4|\nabla A_L^\omega| |\nabla \psi_\ell| |\psi_\ell| + 4|A_L^\omega| |\nabla^{\otimes 2} \psi_\ell| |\psi_\ell| \\ &\quad + 4|A_L^\omega| |\nabla \psi_\ell|^2 + 2|\nabla B^\omega| |\psi_\ell|^2 + 4|B^\omega| |\nabla \psi_\ell| |\psi_\ell| \end{aligned}$$

and

$$(3.7) \quad \begin{aligned} &\|\nabla(\nabla \times j^\omega)\|_{L^2(\Lambda_L)} \\ &\leq c_2 (\|\nabla^{\otimes 2} \psi_\ell\|_{L^4(\Lambda_L)} \|\psi_\ell\|_{L^4(\Lambda_L)} + \|\nabla A_L^\omega\|_{L^2(\Lambda_L)} \|\nabla \psi_\ell\|_{L^\infty(\Lambda_L)} \|\psi_\ell\|_{L^\infty(\Lambda_L)} \\ &\quad + \|A_L^\omega\|_{L^\infty(\Lambda_L)} \|\nabla^{\otimes 2} \psi_\ell\|_{L^4(\Lambda_L)} \|\psi_\ell\|_{L^4(\Lambda_L)} + \|A_L^\omega\|_{L^\infty(\Lambda_L)} \|\nabla \psi_\ell\|_{L^4(\Lambda_L)}^2 \\ &\quad + \|\nabla B^\omega\|_{L^2(\Lambda_L)} \|\psi_\ell\|_{L^\infty(\Lambda_L)}^2 + \|B^\omega\|_{L^\infty(\Lambda_L)} \|\nabla \psi_\ell\|_{L^4(\Lambda_L)} \|\psi_\ell\|_{L^4(\Lambda_L)}). \end{aligned}$$

By the Sobolev inequality, we have

$$(3.8) \quad \|\psi_\ell\|_{L^\infty(\Lambda_L)} \leq c_3 (\|\psi_\ell\|_{L^4(\Lambda_L)} + \|\nabla \psi_\ell\|_{L^4(\Lambda_L)}),$$

$$(3.9) \quad \|\nabla \psi_\ell\|_{L^\infty(\Lambda_L)} \leq c_3 (\|\nabla \psi_\ell\|_{L^4(\Lambda_L)} + \|\nabla^{\otimes 2} \psi_\ell\|_{L^4(\Lambda_L)})$$

and

$$(3.10) \quad \|B^\omega\|_{L^\infty(\Lambda_L)} \leq c_4 \|B^\omega\|_{W^{2,2}(\Lambda_L)}$$

(cf.[1] Theorem 4.12 Part I Case C). In (3.8) and (3.9), we may choose arbitrary  $L^p$  norms with  $p > 2$ . However  $p = 4$  is enough for the present purpose. For any  $p \in [1, \infty)$ , we have

$$(3.11) \quad \|\nabla^{\otimes 2} \psi_\ell\|_{L^p(\Lambda_L)} \leq c_p \|\Delta \psi_\ell\|_{L^p(\Lambda_L)}$$

by the Calderon-Zygmund inequality in the form of Corollary 9.10 in [10]. Since we can derive

$$(3.12) \quad \Delta \psi_\ell = 2iA_L^\omega \cdot \nabla \psi_\ell + (|A_L^\omega|^2 - \lambda_\ell(H_L^\omega))\psi_\ell$$

from the eigenequation by  $\nabla \cdot A_L^\omega = 0$ , we have

$$\|\Delta \psi_\ell\|_{L^4(\Lambda_L)} \leq 2\|A_L^\omega\|_{L^\infty(\Lambda_L)} \|\nabla \psi_\ell\|_{L^4(\Lambda_L)} + (\|A_L^\omega\|_{L^\infty(\Lambda_L)}^2 + R) \|\psi_\ell\|_{L^4(\Lambda_L)}.$$

By the Gagliardo-Nirenberg inequality (cf.[1] Theorem 4.31 with  $m = 1$ ) and the Calderon-Zygmund inequality, we have

$$\|\nabla \psi_\ell\|_{L^4(\Lambda_L)} \leq c_5 \|\Delta \psi_\ell\|_{L^{4/3}(\Lambda_L)}.$$

Since (3.12) is rewritten as

$$(3.13) \quad \Delta \psi_\ell = 2A_L^\omega \cdot (i\nabla + A_L^\omega)\psi_\ell - (|A_L^\omega|^2 + \lambda_\ell(H_L^\omega))\psi_\ell,$$

we have

$$(3.14) \quad \begin{aligned} & \|\Delta \psi_\ell\|_{L^{4/3}(\Lambda_L)} \\ & \leq 2\|A_L^\omega\|_{L^\infty(\Lambda_L)} L^{1/2} \|(i\nabla + A_L^\omega)\psi_\ell\|_{L^2(\Lambda_L)} + (\|A_L^\omega\|_{L^\infty(\Lambda_L)}^2 + R) L^{1/2} \\ & \leq c_6 L^{4+1/2} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R) \end{aligned}$$

by using also Lemma 3.3. Thus we have

$$(3.15) \quad \|\nabla \psi_\ell\|_{L^4(\Lambda_L)} \leq c_7 L^{4+1/2} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R).$$

Similarly we have

$$(3.16) \quad \begin{aligned} & \|\nabla \psi_\ell\|_{L^{4/3}(\Lambda_L)} \leq \|(i\nabla + A_L^\omega)\psi_\ell\|_{L^{4/3}(\Lambda_L)} + \|A_L^\omega \psi_\ell\|_{L^{4/3}(\Lambda_L)} \\ & \leq L^{1/2} (\|(i\nabla + A_L^\omega)\psi_\ell\|_{L^2(\Lambda_L)} + \|A_L^\omega\|_{L^\infty(\Lambda_L)}) \\ & \leq c_8 L^{2+1/2} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{1/2} \end{aligned}$$

and

$$(3.17) \quad \|\psi_\ell\|_{L^4(\Lambda_L)} \leq c_9 \|\nabla \psi_\ell\|_{L^{4/3}(\Lambda_L)} \leq c_{10} L^{2+1/2} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{1/2}.$$

Thus we have

$$(3.18) \quad \|\Delta\psi_\ell\|_{L^4(\Lambda_L)} \leq c_{11}L^{6+1/2}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{3/2}.$$

By applying (3.8)–(3.11), (3.15), (3.17) and (3.18) to each factor in the right hand side of (3.6) and (3.7), we can complete the proof.  $\square$

*Proof of Lemma 3.2.* As in [4], we take  $x_0 \in \Lambda_L$  so that  $|\psi(x_0)| = \max_{\Lambda_L} |\psi_\ell|$ , and we set

$$\langle \psi_\ell \rangle := \int_{B(x_0, l)} \psi_\ell(x) \frac{dx}{\pi l^2}, \quad \langle \nabla \psi_\ell \rangle := \int_{B(x_0, l)} \nabla \psi_\ell(x) \frac{dx}{\pi l^2}$$

and

$$f(x) := \psi_\ell(x) - \langle \psi_\ell \rangle - \langle \nabla \psi_\ell \rangle \cdot (x - x_0)$$

for  $x \in B(x_0, l)$ . Then we have

$$\sup_{B(x_0, l)} |f(x)| \leq c_1 l \|\Delta\psi_\ell\|_{L^2(B(x_0, l))}$$

as in [4]. By (3.18) and (3.15), we have

$$\begin{aligned} \|\Delta\psi_\ell\|_{L^2(B(x_0, l))} &\leq c_2 \sqrt{l} \|\Delta\psi_\ell\|_{L^4(B(x_0, l))} \leq c_2 \sqrt{l} \|\Delta\psi_\ell\|_{L^4(\Lambda_L)} \\ &\leq c_3 \sqrt{l} L^{6+1/2} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{3/2} \end{aligned}$$

and

$$|\langle \nabla \psi_\ell \rangle| \leq c_4 \|\nabla \psi_\ell\|_{L^2(\Lambda_L)} / l \leq c_5 L^5 (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R) / l.$$

Thus by putting  $l = L^{-3/5} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-1/5} |x - x_0|^{2/5}$ , we have

$$|\psi_\ell(x) - \langle \psi_\ell \rangle| \leq c_6 L^{5+3/5} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{6/5} |x - x_0|^{3/5}$$

on  $B(x_0, L^{-1} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-1/3})$ . Then we see that

$$|\psi_\ell(x)| \geq \frac{1}{2L} \text{ on } B(x_0, R_*),$$

where  $R_* := c_7 L^{-11} (\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)^{-2}$ . As in (7.18) in [4], we have

$$\int_{B(x_{00}, R_*/2)} |j^\omega(x)|^2 dx \geq \frac{1}{4L^4} \int_{B(x_{00}, R_*/2)} |A_L^\omega(x) - \nabla \theta_\ell(x)|^2 dx$$

for any  $x_{00} \in B(x_0, R_*/2)$ . By the same proof of Lemma 7.2 in [4], we obtain

$$\int_{B(x_{00}, R_*/2)} |A_L^\omega(x) - \nabla \theta_\ell(x)|^2 dx \geq \int_0^{R_*/2} \frac{dr}{2\pi r} \left| \int_{B(x_{00}, r)} B^\omega(x) dx \right|^2.$$

$\square$

#### 4. NON-DEGENERACY OF THE GAUSSIAN RANDOM FIELD

Let

$$X(R) = \int_0^R \left| \int_{B(r)} B^\omega(x) dx \right|^2 \frac{dr}{2\pi r},$$

where  $B(r) = B(0, r) = \{x \in \mathbb{R}^2 : |x| \leq r\}$  for any  $r \geq 0$ .

In this section, we prove the following:

**Lemma 4.1.** *For any  $\bar{R} \in (0, \infty)$ , there exist  $c, c' \in (0, \infty)$  such that*

$$\mathbb{P}(X(R) \leq t) \leq \exp(-cR^{(2\nu+5)/(2\nu+4)} / t^{1/(2\nu+4)})$$

for any  $R \in (0, \bar{R}]$  and  $t \in (0, c'R^{2\nu+5}]$ .

As its corollary we have the following:

**Corollary .** *For any  $p, \bar{R} \in (0, \infty)$ , there exists  $c \in (0, \infty)$  such that*

$$\mathbb{E}(X(R)^{-p}) \leq cR^{-p(2\nu+5)}$$

for any  $R \in (0, \bar{R}]$ .

To obtain Lemma 4.1, it is enough to show the following:

**Lemma 4.2.** *For any  $\bar{R} \in (0, \infty)$ , there exists  $c \in (0, \infty)$  such that*

$$\mathbb{E}(\exp(-sX(R))) \leq \exp(-cRs^{1/(2\nu+5)})$$

for any  $s \in [1, \infty)$  and  $R \in (0, \bar{R}]$  satisfying  $Rs^{1/(2\nu+5)} \geq 1$ .

In the rest of this section, we prove this lemma. The condition  $\nu > 3/2$  can be extended to  $\nu > 1$  in the following proof. For any  $0 < R_1 < R$ , we have

$$\begin{aligned} & X(R) - X(R_1) \\ &= \int_{R_1}^R \frac{dr}{2\pi r} \left| \int_{B(R_1)} dx B^\omega(x) + \int_{B(r) \setminus B(R_1)} dx \left( B \right. \right. \\ & \quad \left. \left. + \int_{B(R_1+\bar{\sigma})} \sigma(x-y)\omega(dy) + \int_{B(R_1+\bar{\sigma})^c} \sigma(x-y)\omega(dy) \right) \right|^2. \end{aligned}$$

The key point is that  $\omega$  on  $B(R_1 + \bar{\sigma})^c$  is independent of  $X(R_1)$ ,  $B^\omega(x)$  on  $B(R_1)$  and  $\omega$  on  $B(R_1 + \bar{\sigma})$ .

To use this property, we proceed as follows:

$$\begin{aligned} & \mathbb{E}(\exp(-sX(R))) \\ &= \mathbb{E} \left[ \exp \left\{ -sX(R_1) + i \int_{R_1}^R dw(r) \sqrt{\frac{s}{\pi r}} \left( \int_{B(R_1)} dx B^\omega(x) \right. \right. \right. \\ & \quad \left. \left. + \int_{B(r) \setminus B(R_1)} dx \left( B + \int_{B(R_1 + \bar{\sigma})} \sigma(x-y) \omega(dy) \right) \right) \right. \\ & \quad \left. \left. + i \int_{R_1}^R dw(r) \sqrt{\frac{s}{\pi r}} \int_{B(r) \setminus B(R_1)} dx \int_{B(R_1 + \bar{\sigma})^c} \sigma(x-y) \omega(dy) \right\} \right], \end{aligned}$$

where  $w(\cdot)$  is a 1-dimensional Wiener process independent of  $\omega$ . By taking the expectation with respect to  $\omega$  on  $B(R_1 + \bar{\sigma})^c$ , we have

$$\begin{aligned} & \mathbb{E}(\exp(-sX(R))) \\ &= \mathbb{E} \left[ \exp \left\{ -sX(R_1) + i \int_{R_1}^R dw(r) \sqrt{\frac{s}{\pi r}} \left( \int_{B(R_1)} dx B^\omega(x) \right. \right. \right. \\ & \quad \left. \left. + \int_{B(r) \setminus B(R_1)} dx \left( B + \int_{B(R_1 + \bar{\sigma})} \sigma(x-y) \omega(dy) \right) \right) \right. \\ & \quad \left. \left. - \frac{1}{2} \mathbb{E}^\omega \left[ \left( \int_{R_1}^R dw(r) \sqrt{\frac{s}{\pi r}} \int_{B(r) \setminus B(R_1)} dx \int_{B(R_1 + \bar{\sigma})^c} \sigma(x-y) \omega(dy) \right)^2 \right] \right\} \right], \end{aligned}$$

where  $\mathbb{E}^\omega$  is the expectation with respect to  $\omega$ . By taking the absolute value, we have

$$\mathbb{E}(\exp(-sX(R))) \leq \mathbb{E}(\exp(-sX(R_1))) F(R, R_1; s),$$

where

$$\begin{aligned} F(R, R_1; s) &:= \mathbb{E} \left[ \exp \left\{ -\frac{1}{2} \mathbb{E}^\omega \left[ \left( \int_{R_1}^R dw(r) \sqrt{\frac{s}{\pi r}} \right. \right. \right. \right. \\ & \quad \left. \left. \left. \times \int_{B(r) \setminus B(R_1)} dx \int_{B(R_1 + \bar{\sigma})^c} \sigma(x-y) \omega(dy) \right)^2 \right] \right\} \right] \\ &= \mathbb{E} \left[ \exp \left( i \int_{R_1}^R dw(r) \sqrt{\frac{s}{\pi r}} \int_{B(r) \setminus B(R_1)} dx \int_{B(R_1 + \bar{\sigma})^c} \sigma(x-y) \omega(dy) \right) \right] \\ &= \mathbb{E} \left[ \exp \left\{ -s \int_{R_1}^R \frac{dr}{2\pi r} \left( \int_{B(r) \setminus B(R_1)} dx \int_{B(R_1 + \bar{\sigma})^c} \sigma(x-y) \omega(dy) \right)^2 \right\} \right]. \end{aligned}$$

For any sequence  $R = R_0 > R_1 > R_2 > \dots > R_n \downarrow 0$ , we have

$$(4.1) \quad \mathbb{E}(\exp(-sX(R))) \leq \prod_{j=1}^{\infty} F(R_{j-1}, R_j; s).$$

We next estimate each  $F(R_{j-1}, R_j; s)$ : if we set

$$X(R_{j-1}, R_j) := \int_{R_j}^{R_{j-1}} \frac{dr}{2\pi r} \left( \int_{B(r) \setminus B(R_j)} dx \int_{B(R_j + \bar{\sigma})^c} \sigma(x-y) \omega(dy) \right)^2,$$

then

$$\begin{aligned}
& 1 - F(R_{j-1}, R_j; s) \\
&= \mathbb{E}[sX(R_{j-1}, R_j)] \\
&\quad - \mathbb{E}\left[s^2 X(R_{j-1}, R_j)^2 \int_0^1 dt_1 \int_0^{t_1} dt_2 \exp(-t_2 s X(R_{j-1}, R_j))\right] \\
&\geq s \mathbb{E}[X(R_{j-1}, R_j)] - \frac{s^2}{2} \mathbb{E}[X(R_{j-1}, R_j)^2].
\end{aligned}$$

Since the 4-th moment of a centered Gaussian random variable is proportional to the square of the variance of the Gaussian variable, we have

$$\begin{aligned}
& \mathbb{E}[X(R_{j-1}, R_j)^2] \\
&\leq \left( \int_{R_j}^{R_{j-1}} \frac{dr}{2\pi r} \mathbb{E}\left[\left(\int_{B(r)\setminus B(R_j)} dx \int_{B(R_j+\bar{\sigma})^c} \sigma(x-y)\omega(dy)\right)^4\right]^{1/2} \right)^2 \\
&= 3\mathbb{E}[X(R_{j-1}, R_j)]^2.
\end{aligned}$$

By the polar coordinate, we have

$$\begin{aligned}
\mathbb{E}[X(R_{j-1}, R_j)] &= \int_{R_j}^{R_{j-1}} \frac{dr}{2\pi r} \int_{B(R_j+\bar{\sigma})^c} dy \left( \int_{B(r)\setminus B(R_j)} dx \sigma(x-y) \right)^2 \\
&= \int_{R_j}^{R_{j-1}} \frac{dr}{r} \int_{R_j+\bar{\sigma}}^\infty dr_0 r_0 \left( \int_{R_j}^r dr_1 r_1 \int_0^{2\pi} d\theta_1 \left( (\bar{\sigma} - r_1 + r_0)(\bar{\sigma} + r_1 - r_0) \right. \right. \\
&\quad \left. \left. - r_1 r_0 \left( 2 \sin \frac{\theta_1}{2} \right)_+^\nu \right)^2 \right) \\
&= \int_{R_j}^{R_{j-1}} \frac{dr}{r} \left( \prod_{\ell=1}^2 \int_{R_j}^r dr_\ell r_\ell \right) \int_{\bar{\sigma}+R_j}^{\bar{\sigma}+(r_1 \wedge r_2)} dr_0 r_0 \\
&\quad \times \left( \prod_{\ell=1}^2 \int_0^{2\pi} d\theta_\ell \left( (\bar{\sigma} - r_\ell + r_0)(\bar{\sigma} + r_\ell - r_0) - r_\ell r_0 \left( 2 \sin \frac{\theta_\ell}{2} \right)_+^\nu \right) \right).
\end{aligned}$$

By changing the variables, we have

$$\begin{aligned}
& \mathbb{E}[X(R_{j-1}, R_j)] \\
&= \int_0^1 \frac{dr(R_{j-1} - R_j)}{R_j + (R_{j-1} - R_j)r} \left( \prod_{\ell=1}^2 \int_0^r dr_\ell (R_{j-1} - R_j)(R_j + (R_{j-1} - R_j)r_\ell) \right) \\
(4.2) \quad & \times \int_0^{r_1 \wedge r_2} dr_0 (R_{j-1} - R_j)(\bar{\sigma} + R_j + (R_{j-1} - R_j)r_0) \\
& \times \left( \prod_{\ell=1}^2 \frac{\{(2\bar{\sigma} - (R_{j-1} - R_j)(r_\ell - r_0))(R_{j-1} - R_j)(r_\ell - r_0)\}^{\nu+1/2}}{\{(R_j + (R_{j-1} - R_j)r_\ell)(\bar{\sigma} + R_j + (R_{j-1} - R_j)r_0)\}^{1/2}} \right. \\
& \left. \times 2 \int_0^{\pi \mathcal{R}_{\ell,j}} d\theta_\ell \left( 1 - \left( 2 \mathcal{R}_{\ell,j} \sin \frac{\theta_\ell}{2 \mathcal{R}_{\ell,j}} \right)_+^\nu \right) \right),
\end{aligned}$$

where

$$\mathcal{R}_{\iota,j} := \left\{ \frac{(R_j + (R_{j-1} - R_j)r_\iota)(\bar{\sigma} + R_j + (R_{j-1} - R_j)r_0)}{(2\bar{\sigma} - (R_{j-1} - R_j)(r_\iota - r_0))(R_{j-1} - R_j)(r_\iota - r_0)} \right\}^{1/2}.$$

Since  $2\theta/\pi \leq \sin \theta \leq \theta$  for  $0 \leq \theta \leq \pi/2$ , we have

$$\left(\frac{3}{4}\right)^\nu \pi \left(\frac{1}{2\pi} \wedge \mathcal{R}_{\iota,j}\right) \leq \int_0^{\pi \mathcal{R}_{\iota,j}} d\theta_\iota \left(1 - \left(2\mathcal{R}_{\iota,j} \sin \frac{\theta_\iota}{2\mathcal{R}_{\iota,j}}\right)_+^2\right)^\nu \leq \frac{\pi}{2} \int_0^1 d\theta (1 - \theta^2)^\nu.$$

Moreover by  $R_j \leq R_j + (R_{j-1} - R_j)r_\iota \leq R_{j-1}$  for any  $\iota \in \{0, 1, 2\}$ , we obtain

$$c_1(R_{j-1} - R_j)^{2\nu+5} \left(\frac{R_j}{R_{j-1}}\right)^3 \leq \mathbb{E}[X(R_{j-1}, R_j)] \leq c_2(R_{j-1} - R_j)^{2\nu+5} \left(\frac{R_{j-1}}{R_j}\right)^2$$

and

$$\begin{aligned} & F(R_{j-1}, R_j; s) \\ & \leq 1 - c_1 s (R_{j-1} - R_j)^{2\nu+5} \left(\frac{R_j}{R_{j-1}}\right)^3 + \frac{3c_2^2 s^2}{2} (R_{j-1} - R_j)^{2(2\nu+5)} \left(\frac{R_{j-1}}{R_j}\right)^4 \end{aligned}$$

if  $R_{j-1} - R_j \leq \bar{\sigma}$ .

We now take  $\{R_j\}_j$  as follows: taking  $\varepsilon \in (0, (\bar{\sigma} \wedge 1)/2)$  and preparing the sequence

$$b_k := \begin{cases} \frac{1 - 2\varepsilon}{(1 - \varepsilon)\lceil 1/\varepsilon \rceil} & \text{for } k \in [0, \lceil 1/\varepsilon \rceil] \cap \mathbb{N}, \\ \varepsilon^{k - \lceil 1/\varepsilon \rceil} & \text{for } k \in (\lceil 1/\varepsilon \rceil, \infty) \cap \mathbb{N}, \end{cases}$$

whose elements are in  $(0, \varepsilon)$  and whose sum is 1, we set  $R_0 = R$  and

$$R_{j-1} - R_j = \frac{b_k R}{\lfloor R s^{1/(2\nu+5)} \rfloor} \text{ for } j \in ((k-1)\lfloor R s^{1/(2\nu+5)} \rfloor, k\lfloor R s^{1/(2\nu+5)} \rfloor] \cap \mathbb{N},$$

where  $\lfloor a \rfloor = \max\{(-\infty, a] \cap \mathbb{Z}\}$  and  $\lceil a \rceil = \min\{[a, \infty) \cap \mathbb{Z}\}$  for any  $a \in \mathbb{R}$ . Then we have  $R_{j-1} - R_j \leq \bar{\sigma}$

and

$$(4.3) \quad R_{j-1}/R_j \leq 1/\varepsilon.$$

Indeed, we have

$$R_j = \begin{cases} R \left(1 - \frac{j}{\lfloor R s^{1/(2\nu+5)} \rfloor} \frac{1 - 2\varepsilon}{(1 - \varepsilon)\lceil 1/\varepsilon \rceil}\right) & \text{for } j \in [0, \lceil 1/\varepsilon \rceil \lfloor R s^{1/(2\nu+5)} \rfloor] \cap \mathbb{N}, \\ R \varepsilon^{k - \lceil 1/\varepsilon \rceil} \left(\frac{\varepsilon}{1 - \varepsilon} + k - \frac{j}{\lfloor R s^{1/(2\nu+5)} \rfloor}\right) & \\ & \text{for } j \in ((k-1)\lfloor R s^{1/(2\nu+5)} \rfloor, k\lfloor R s^{1/(2\nu+5)} \rfloor] \cap \mathbb{N} \\ & \text{with } k \in (\lceil 1/\varepsilon \rceil, \infty) \cap \mathbb{N}. \end{cases}$$

If  $j \in [0, \lceil 1/\varepsilon \rceil \lfloor Rs^{1/(2\nu+5)} \rfloor] \cap \mathbb{N}$ , then

$$\frac{R_{j-1}}{R_j} = 1 + \frac{1-2\varepsilon}{\lfloor Rs^{1/(2\nu+5)} \rfloor (1-\varepsilon) \lceil 1/\varepsilon \rceil} \bigg/ \left( 1 - \frac{j}{\lfloor Rs^{1/(2\nu+5)} \rfloor} \frac{1-2\varepsilon}{(1-\varepsilon) \lceil 1/\varepsilon \rceil} \right)$$

takes its maximum at  $j = \lceil 1/\varepsilon \rceil \lfloor Rs^{1/(2\nu+5)} \rfloor$  and the maximum is

$$1 + \frac{1-2\varepsilon}{\lfloor Rs^{1/(2\nu+5)} \rfloor \varepsilon \lceil 1/\varepsilon \rceil} \leq 2 \leq \frac{1}{\varepsilon}.$$

If  $j \in [2 + (k-1) \lfloor Rs^{1/(2\nu+5)} \rfloor, k \lfloor Rs^{1/(2\nu+5)} \rfloor] \cap \mathbb{N}$  with  $k \in (\lceil 1/\varepsilon \rceil, \infty) \cap \mathbb{N}$ , then

$$\frac{R_{j-1}}{R_j} = 1 + \frac{1}{\lfloor Rs^{1/(2\nu+5)} \rfloor} \bigg/ \left( \frac{\varepsilon}{1-\varepsilon} + k - \frac{j}{\lfloor Rs^{1/(2\nu+5)} \rfloor} \right)$$

takes its maximum at  $j = k \lfloor Rs^{1/(2\nu+5)} \rfloor$  and the maximum is

$$1 + \frac{1-\varepsilon}{\varepsilon \lfloor Rs^{1/(2\nu+5)} \rfloor} \leq \frac{1}{\varepsilon}.$$

Finally if  $j = 1 + (k-1) \lfloor Rs^{1/(2\nu+5)} \rfloor$  with  $k \in (\lceil 1/\varepsilon \rceil, \infty) \cap \mathbb{N}$ , then

$$R_{j-1} = \frac{R\varepsilon^{k-\lceil 1/\varepsilon \rceil}}{1-\varepsilon},$$

$$R_j = R_{j-1} - (R_{j-1} - R_j) = R\varepsilon^{k-\lceil 1/\varepsilon \rceil} \left( \frac{1}{1-\varepsilon} - \frac{1}{\lfloor Rs^{1/(2\nu+5)} \rfloor} \right)$$

and

$$\frac{R_{j-1}}{R_j} = \frac{1}{1-\varepsilon} \bigg/ \left( \frac{1}{1-\varepsilon} - \frac{1}{\lfloor Rs^{1/(2\nu+5)} \rfloor} \right) \leq \frac{1}{\varepsilon}.$$

Therefore we obtain (4.3).

Since  $\nu > 1$ , by taking  $\varepsilon$  sufficiently small, we have

$$\log F(R_{j-1}, R_j; s) \leq -c_3 b_k^{2\nu+5}$$

for  $j \in ((k-1) \lfloor Rs^{1/(2\nu+5)} \rfloor, k \lfloor Rs^{1/(2\nu+5)} \rfloor] \cap \mathbb{N}$ . By applying this to the right hand side of (4.1), we obtain

$$\log \mathbb{E}(\exp(-sX(R))) \leq -c_3 \sum_{k=1}^{\infty} b_k^{2\nu+5} \lfloor Rs^{1/(2\nu+5)} \rfloor \leq -c_4 Rs^{1/(2\nu+5)}.$$

*Remark 4.1.* The results of this section is extended to the case where  $\sigma$  is replaced by

$$\tilde{\sigma}(x) = \tilde{\mathcal{P}}(x)\sigma(x)$$

and  $\tilde{\mathcal{P}}$  is a bounded function on  $\mathbb{R}^2$  such that  $\tilde{\mathcal{P}}(x_0) \neq 0$  for some  $x_0 \in \mathbb{R}^2$  satisfying  $|x_0| = \bar{\sigma}$  and that  $\tilde{\mathcal{P}}$  is continuous at any points of a neighborhood of  $x_0$ . In this case, the equation (4.2) is changed to

$$\begin{aligned}
& \mathbb{E}[X(R_{j-1}, R_j)] \\
&= \int_0^1 \frac{dr(R_{j-1} - R_j)}{R_j + (R_{j-1} - R_j)r} \left( \prod_{\ell=1}^2 \int_0^r dr_\ell (R_{j-1} - R_j)(R_j + (R_{j-1} - R_j)r_\ell) \right) \\
&\quad \times \int_0^{r_1 \wedge r_2} dr_0 (R_{j-1} - R_j)(\bar{\sigma} + R_j + (R_{j-1} - R_j)r_0) \int_0^{2\pi} \frac{d\theta_0}{2\pi} \\
&\quad \times \left( \prod_{\ell=1}^2 \frac{\{(2\bar{\sigma} - (R_{j-1} - R_j)(r_\ell - r_0))(R_{j-1} - R_j)(r_\ell - r_0)\}^{\nu+1/2}}{\{(R_j + (R_{j-1} - R_j)r_\ell)(\bar{\sigma} + R_j + (R_{j-1} - R_j)r_0)\}^{1/2}} \right) \\
&\quad \times \int_0^{\pi \mathcal{R}_{\ell,j}} d\theta_\ell \left( 1 - \left( 2\mathcal{R}_{\ell,j} \sin \frac{\theta_\ell}{2\mathcal{R}_{\ell,j}} \right)_+^\nu \right) \\
&\quad \times \sum_{\tau \in \{+, -\}} \tilde{\mathcal{P}} \left( (R_j + (R_{j-1} - R_j)r_\ell) \left( \cos \left( \theta_0 + \frac{\tau \theta_\ell}{\mathcal{R}_{\ell,j}} \right), \sin \left( \theta_0 + \frac{\tau \theta_\ell}{\mathcal{R}_{\ell,j}} \right) \right) \right. \\
&\quad \quad \left. - (\bar{\sigma} + R_j + (R_{j-1} - R_j)r_0) (\cos \theta_0, \sin \theta_0) \right).
\end{aligned}$$

The upper estimate of this is obtained by the same method. To obtain the lower estimate, we restrict the integral with respect to  $\theta_0$  to an interval  $I$  such that  $\inf\{\tau \mathcal{P}(x) : |x + \bar{\sigma}(\cos \theta, \sin \theta)| \leq \delta \text{ for some } \theta \in I\} > 0$  for some  $\delta > 0$  and  $\tau \in \{+, -\}$ .

This extension is applicable to the case of (1.3). Indeed if the degree of  $\mathcal{P}$  is  $m$ , then  $\nu > 2m + 3/2$  and  $\tilde{\sigma}(x) = \mathcal{Q}(|x|)(\bar{\sigma}^2 - |x|^2)_+^{\nu-2m}$ , where  $\mathcal{Q}$  is a polynomial of the degree  $2m$ . By the factor theorem, we can write  $\mathcal{Q}(r) = (\bar{\sigma} - r)^h \tilde{\mathcal{Q}}(r)$ , where  $h \in \mathbb{Z} \cap [0, 2m]$  and  $\tilde{\mathcal{Q}}$  is a polynomial of the degree  $2m - h$  such that  $\tilde{\mathcal{Q}}(\bar{\sigma}) \neq 0$ . Then, since  $\tilde{\sigma}(x) = (\bar{\sigma}^2 - |x|^2)_+^{\nu-2m+h} \tilde{\mathcal{Q}}(|x|)(\bar{\sigma} + |x|)^{-h}$ ,  $\tilde{\mathcal{Q}}(|x|)(\bar{\sigma} + |x|)^{-h}$  is bounded on  $B(\bar{\sigma})$  and  $\tilde{\mathcal{Q}}(\bar{\sigma})(2\bar{\sigma})^{-h} \neq 0$ , the results in this section hold for this case if  $\nu$  is replaced by  $\nu - 2m + h$ .

## 5. PROOF OF THEOREM 1

In this section we modify Erdős and Hasler [4] to prove Theorem 1 by applying the results proved in the preceding sections.

We first cut off high energies:

$$(5.1) \quad \text{Tr}[\chi_{[E-\eta, E+\eta]}(H_L^\omega)] \leq \text{Tr}[\chi_{[(t(E)-\eta) \vee t(0), t(E)+\eta]}(t(H_L^\omega))]$$

for any  $E, \eta > 0$  such that  $E + \eta \leq R$ , where  $t(u) := (u + 1)(5R)^3 / (5R + u + 1)^3$ . By (2.8) and  $\inf_{u \in (0, R)} t'(u) > 0$ , the right hand side of (5.1) is less than or equal to

$$\begin{aligned}
& c_1 \sum_{\ell} \chi_{[(t(E) - \eta) \vee t(0), t(E) + \eta]}(t(\lambda_{\ell}(H_L^{\omega}))) L^{28\nu + 192} \\
& \times \sum_{m \in \mathbb{N}} \tilde{\chi}_{[m-1, m]}(\|B^{\omega}\|_{W^{2,2}(\Lambda_L)}^2 + R) m^{6\nu + 37} \\
& \times \sum_{x_* \in (c_2 L^{-11} m^{-2} \mathbb{Z}^2) \cap \Lambda_L} \mathcal{B}(m, x_*, \omega)^{-\nu - 15/2} \\
& \times \sum_{\mathbf{n} \in \mathbb{N}^2} \tilde{\chi}_{[0, \infty)}(\mathcal{R}(m, x_*, \omega) - |\mathbf{n}|)(Dt(\lambda_{\ell}(H_L^{\omega})), \tilde{\Phi}_{(\mathbf{n}; \varepsilon(m, x_*, \omega), L), L})^2,
\end{aligned}$$

where

$$\begin{aligned}
(5.2) \quad \mathcal{B}(m, x_*, \omega) &= \int_0^{c_2 L^{-11} m^{-2}} \left| \int_{B(x_*, r)} B^{\omega}(x) dx \right|^2 \frac{dr}{2\pi r}, \\
\mathcal{R}(m, x_*, \omega) &= c_3 L^{15} m^3 \mathcal{B}(m, x_*, \omega)^{-1/2}, \\
\varepsilon(m, x_*, \omega) &= c_4 \mathcal{B}(m, x_*, \omega)^{1/2} L^{-13} m^{-2} \mathcal{R}(m, x_*, \omega)^{-3} \\
&= c_4 c_3^{-3} \mathcal{B}(m, x_*, \omega)^2 L^{-58} m^{-11},
\end{aligned}$$

and, for each interval  $I$ ,  $\tilde{\chi}_I$  is a  $[0, 1]$ -valued smooth function on  $\mathbb{R}$  such that  $\tilde{\chi}_I = 1$  on  $I$  and  $\tilde{\chi}_I(x) = 0$  if  $\text{dist}(x, I) \geq 1$ . Let  $F$  and  $G$  be functions on  $\mathbb{R}$  such that  $F' = \chi_{[(t(E) - \eta) \vee t(0), t(E) + \eta]}$ ,  $G' = F$ , and  $F = G = 0$  on  $(-\infty, (t(E) - \eta) \vee t(0))$ . Then we have

$$\begin{aligned}
& \chi_{[(t(E) - \eta) \vee t(0), t(E) + \eta]}(t(\lambda_{\ell}(H_L^{\omega}))) (\tilde{D}_{\mathbf{n}} t(\lambda_{\ell}(H_L^{\omega})))^2 \\
& = \tilde{D}_{\mathbf{n}}^2 G(t(\lambda_{\ell}(H_L^{\omega}))) - F(t(\lambda_{\ell}(H_L^{\omega}))) \tilde{D}_{\mathbf{n}}^2 t(\lambda_{\ell}(H_L^{\omega})),
\end{aligned}$$

where  $\tilde{D}_{\mathbf{n}}(\cdot) := (D(\cdot), \tilde{\Phi}_{(\mathbf{n}; \varepsilon(m, x_*, \omega), L), L})_{L^2(\mathbb{R}^2)}$ . As shown in Lemma 5.2 of [4], we have

$$(5.3) \quad \sum_{\ell} F(t(\lambda_{\ell}(H_L^{\omega}))) \tilde{D}_{\mathbf{n}}^2 t(\lambda_{\ell}(H_L^{\omega})) \geq \text{Tr}[F(t(H_L^{\omega})) \tilde{D}_{\mathbf{n}}^2 t(H_L^{\omega})]$$

and

$$(5.4) \quad \sum_{\ell} \tilde{D}_{\mathbf{n}}^2 G(t(\lambda_{\ell}(H_L^{\omega}))) = \tilde{D}_{\mathbf{n}}^2 \text{Tr}[G(t(H_L^{\omega}))].$$

We proceed to estimate the terms including the right hand side of (5.3) under the condition that

$$(5.5) \quad \|B^{\omega}\|_{W^{2,2}(\Lambda_L)}^2 + R \in [m - 2, m + 1].$$

As in [4], we first show

$$\|\tilde{D}_{\mathbf{n}}^2 t(H_L^{\omega})\|_{L^2(\Lambda_L) \rightarrow L^2(\Lambda_L)} \leq c_5 L^4$$

by decomposing as

$$\begin{aligned}
& \tilde{D}_{\mathbf{n}}^2 t(H_L^\omega) \\
&= (5R)^3 (\tilde{D}_{\mathbf{n}}^2 H_L^\omega) (5R + 1 + H_L^\omega)^{-3} \\
&\quad - 2(5R)^3 \sum_{k=1}^3 (\tilde{D}_{\mathbf{n}} H_L^\omega) (5R + 1 + H_L^\omega)^{-k} (\tilde{D}_{\mathbf{n}} H_L^\omega) (5R + 1 + H_L^\omega)^{-4+k} \\
&\quad - (5R)^3 \sum_{k=1}^3 (H_L^\omega + 1) (5R + 1 + H_L^\omega)^{-k} (\tilde{D}_{\mathbf{n}}^2 H_L^\omega) (5R + 1 + H_L^\omega)^{-4+k} \\
&\quad + 2(5R)^3 \sum_{\substack{1 \leq k, k' \leq 4 \\ k+k' \leq 4}} (H_L^\omega + 1) (5R + 1 + H_L^\omega)^{-k} (\tilde{D}_{\mathbf{n}} H_L^\omega) (5R + 1 + H_L^\omega)^{-k'} \\
&\quad \quad \quad \times (\tilde{D}_{\mathbf{n}} H_L^\omega) (5R + 1 + H_L^\omega)^{-5+k+k'}.
\end{aligned}$$

For this, we use

$$\begin{aligned}
\tilde{D}_{\mathbf{n}} H_L^\omega &= 2(i\nabla + A_L^\omega) \cdot (\tilde{D}_{\mathbf{n}} A_L^\omega) = 2(\tilde{D}_{\mathbf{n}} A_L^\omega) \cdot (i\nabla + A_L^\omega), \\
\tilde{D}_{\mathbf{n}}^2 H_L^\omega &= 2|(\tilde{D}_{\mathbf{n}} A_L^\omega)|^2 + 2(i\nabla + A_L^\omega) \cdot (\tilde{D}_{\mathbf{n}}^2 A_L^\omega) \\
&= 2|\tilde{D}_{\mathbf{n}} A_L^\omega|^2 + 2(\tilde{D}_{\mathbf{n}}^2 A_L^\omega) \cdot (i\nabla + A_L^\omega), \\
\|(i\nabla + A_L^\omega)(5R + 1 + H_L^\omega)^{-1/2}\|_{L^2(\Lambda_L) \rightarrow L^2(\Lambda_L) \otimes \mathbb{C}^2} &\leq 1
\end{aligned}$$

and the following estimates obtained from Lemma 3.3:

$$(5.6) \quad \|\tilde{D}_{\mathbf{n}} A_L^\omega\|_{L^\infty(\Lambda_L)} \leq c_6 L^2$$

and

$$(5.7) \quad \|\tilde{D}_{\mathbf{n}}^2 A_L^\omega\|_{L^\infty(\Lambda_L)} \leq \frac{c_7}{L^{144} m^{53/2}}.$$

To prove (5.7), we also use

$$\|\tilde{D}_{\mathbf{n}}^2 B^\omega\|_{W^{2,1}(\Lambda_L)} \leq c_8 L^2 \|\tilde{D}_{\mathbf{n}} \tilde{\Phi}_{(\mathbf{n}; \varepsilon(m, x_*, \omega), L), L}\|_{L^\infty(\Lambda_{3L})} \leq \frac{c_9}{L^{145} m^{53/2}},$$

which is proven by the following: since we have

$$(5.8) \quad \mathcal{B}(m, x_*, \omega) \leq \frac{c_{10}}{L^{44} m^7}$$

and

$$\begin{aligned}
& |\tilde{D}_{\mathbf{n}}\mathcal{B}(m, x_*, \omega)| \\
(5.9) \quad &= \left| \int_0^{c_2 L^{-11} m^{-2}} \frac{dr}{\pi r} \int_{B(x_*, r)} dx' B^\omega(x') \right. \\
&\quad \times \left. \int_{B(x_*, r)} dx \int_{\mathbb{R}^2} dy \sigma(x-y) \tilde{\Phi}_{(\mathbf{n}; \varepsilon(m, x_*, \omega), L), L}(y) \right| \\
&\leq \frac{c_{11}}{L^{45} m^{15/2}}
\end{aligned}$$

under the condition (5.5) by the same method for (2.7), we have

$$\begin{aligned}
& |\tilde{D}_{\mathbf{n}}\varepsilon(m, x_*, \omega)| = 2c_4 c_3^{-3} \mathcal{B}(m, x_*, \omega) L^{-58} m^{-11} |\tilde{D}_{\mathbf{n}}\mathcal{B}(m, x_*, \omega)| \leq \frac{c_{12}}{L^{147} m^{53/2}}, \\
& |\tilde{D}_{\mathbf{n}}(\mathbf{n}; \varepsilon(m, x_*, \omega), L)| \\
&= \left| \sum_{\tau \in \{+, -\}, s \in \mathbb{N}} \chi_{[0, \bar{j}/8)}(\tau(\pi\bar{\sigma}|\mathbf{n}|/L - j_{\nu+1, s})) \frac{\tau \bar{j} L \mathbf{n}}{8\pi\bar{\sigma}|\mathbf{n}|} \tilde{D}_{\mathbf{n}}\varepsilon(m, x_*, \omega) \right| \\
&\leq \frac{c_{13}}{L^{146} m^{53/2}},
\end{aligned}$$

and

$$\begin{aligned}
& |\tilde{D}_{\mathbf{n}}\tilde{\Phi}_{(\mathbf{n}; \varepsilon(m, x_*, \omega), L), L}| \\
&= \left| \frac{2}{L} \sum_{\iota=1}^2 \cos\left((\mathbf{n}; \varepsilon(m, x_*, \omega), L)_\iota \pi \left(\frac{x_\iota}{L} + \frac{1}{2}\right)\right) \right. \\
&\quad \times \left. \sin\left((\mathbf{n}; \varepsilon(m, x_*, \omega), L)_{\bar{\iota}} \pi \left(\frac{x_{\bar{\iota}}}{L} + \frac{1}{2}\right)\right) \pi \left(\frac{x_\iota}{L} + \frac{1}{2}\right) \tilde{D}_{\mathbf{n}}(\mathbf{n}; \varepsilon(m, x_*, \omega), L) \right| \\
&\leq \frac{c_{14}}{L^{147} m^{53/2}}.
\end{aligned}$$

Thus we obtain

$$|\mathrm{Tr}[F(t(H_L^\omega)) \tilde{D}_{\mathbf{n}}^2 t(H_L^\omega)]| \leq c_5 L^4 \mathrm{Tr}[F(t(H_L^\omega))].$$

Let  $E^*(E)$  be the root of  $t(E^*) = t(E)$  in  $(5R/2 - 1, \infty)$  for  $E \in (0, 5R/2 - 1)$ . This is solved as

$$\begin{aligned}
& E^*(E) \\
&= \frac{2(5R)^3}{\sqrt{(E+1)^2(E+15R+1)^2 + 4(5R)^3(E+1) + (E+1)(E+15R+1)}} - 1 \\
&\leq (5R)^{3/2}.
\end{aligned}$$

Since  $t'(u) \geq t'(R) > 5^3/7^4 > 1/(20)$  for  $u \in (0, R)$ ,  $F(t(\lambda_\ell(H_L^\omega))) \neq 0$  implies

$$(E - 20\eta)_+ \leq \lambda_\ell(H_L^\omega) \leq E^*((E - 20\eta)_+).$$

By  $F \leq 2\eta$  and applying the Weyl bound as in [4], we have

$$\begin{aligned} \text{Tr}[F(t(H_L^\omega))] &\leq 2\eta \#\{\text{spec}(H_L^\omega) \cap [0, E^*((E - 20\eta)_+)]\} \\ &\leq c_{15}\eta L^2 E^*((E - 20\eta)_+) \leq c_{16}\eta L^2 R^{3/2}. \end{aligned}$$

We next estimate the terms including the right hand side of (5.4). For this we apply the theory of the Malliavin calculus. For any separable Hilbert space  $H$ ,  $p \in (1, \infty)$ ,  $k \in \mathbb{Z}_+$  and any element  $F$  of polynomial functionals

$$\begin{aligned} \mathcal{P}(H) = \{F(\omega) = \sum_{m=1}^M p_m(\omega(\varphi_1), \omega(\varphi_2), \dots, \omega(\varphi_N)) h_m : M, N \in \mathbb{N}, \\ p_1, p_2, \dots : \text{polynomials}, \varphi_1, \varphi_2, \dots \in L^2(\mathbb{R}^2), h_1, h_2, \dots \in H\}, \end{aligned}$$

we define a norm by

$$\|F\|_{\mathbb{D}^{k,p}(H)} := \| \|F\|_H \|_{L^p(\mathbb{P})} + \| \|D^k F\|_{(L^2(\mathbb{R}^2))^{\otimes k} \otimes H} \|_{L^p(\mathbb{P})},$$

and a Banach space  $\mathbb{D}^{k,p}(H)$  by the completion of  $\mathcal{P}(H)$  with respect to this norm. Let  $(\mathbb{D}^{-k,q}(H), \|\cdot\|_{\mathbb{D}^{-k,q}(H)})$  be its dual space, where  $q \in (1, \infty)$  such that  $1/p + 1/q = 1$  (cf. [12] Chapter V §8, [16] §1.2, §1.3, [19] §4.2). We abbreviate as  $\mathbb{D}^{k,p}(\mathbb{R}) = \mathbb{D}^{k,p}$ . The derivative operator  $D$  can be extended to a continuous operator from  $\mathbb{D}^{k,p}(H)$  to  $\mathbb{D}^{k-1,p}(L^2(\mathbb{R}^2) \otimes H)$  for any  $k \in \mathbb{Z}$  (cf. [12] Chapter V Theorem 8.5, [19] Proposition 4.13). Let  $\delta$  be its dual operator:  $\mathbb{E}[(\delta(G), F)_H] = \mathbb{E}[(G, DF)_{L^2(\mathbb{R}^2) \otimes H}]$  for any  $G \in \mathbb{D}^{-k+1,q}(L^2(\mathbb{R}^2) \otimes H)$  and  $F \in \mathbb{D}^{k,p}(H)$ . This is also a continuous operator from  $\mathbb{D}^{k+1,q}(L^2(\mathbb{R}^2) \otimes H)$  to  $\mathbb{D}^{k,q}(H)$  for any  $k \in \mathbb{Z}$ . Now we apply this fact as follows:

$$\begin{aligned} &\mathbb{E}[(D(D \text{Tr}[G(t(H_L^\omega))]), \Phi_{\tilde{\mathbf{n}},L})_{L^2(\mathbb{R}^2)}, \tilde{\Phi}_{\tilde{\mathbf{n}},L})_{L^2(\mathbb{R}^2)} \Psi(\omega)] \\ &= \mathbb{E}[(D \text{Tr}[G(t(H_L^\omega))]), \tilde{\Phi}_{\tilde{\mathbf{n}},L})_{L^2(\mathbb{R}^2)} \delta(\Psi(\omega) \tilde{\Phi}_{\tilde{\mathbf{n}},L})] \\ &= \mathbb{E}[\text{Tr}[G(t(H_L^\omega))] \delta(\delta(\Psi(\omega) \tilde{\Phi}_{\tilde{\mathbf{n}},L}) \tilde{\Phi}_{\tilde{\mathbf{n}},L})] \end{aligned}$$

and

$$\begin{aligned} &\left| \mathbb{E} \left[ \text{Tr}[G(t(H_L^\omega))] \delta(\delta(\Psi(\omega) \tilde{\Phi}_{\tilde{\mathbf{n}},L}) \tilde{\Phi}_{\tilde{\mathbf{n}},L}) \right] \right| \\ &\leq \| \text{Tr}[G(t(H_L^\omega))] \|_{L^{p_1}(\mathbb{P})} \| \delta(\delta(\Psi(\omega) \tilde{\Phi}_{\tilde{\mathbf{n}},L}) \tilde{\Phi}_{\tilde{\mathbf{n}},L}) \|_{L^{p_2}(\mathbb{P})} \\ (5.10) \quad &\leq c_{17} \| \text{Tr}[G(t(H_L^\omega))] \|_{L^{p_1}(\mathbb{P})} \| \delta(\Psi(\omega) \tilde{\Phi}_{\tilde{\mathbf{n}},L}) \tilde{\Phi}_{\tilde{\mathbf{n}},L} \|_{\mathbb{D}^{1,p_2}(L^2(\mathbb{R}^2))} \\ &\leq c_{18} \| \text{Tr}[G(t(H_L^\omega))] \|_{L^{p_1}(\mathbb{P})} \| \Psi(\omega) \tilde{\Phi}_{\tilde{\mathbf{n}},L} \|_{\mathbb{D}^{2,p_3}(L^2(\mathbb{R}^2))} \| \tilde{\Phi}_{\tilde{\mathbf{n}},L} \|_{\mathbb{D}^{1,p_4}(L^2(\mathbb{R}^2))} \\ &\leq c_{19} \| \text{Tr}[G(t(H_L^\omega))] \|_{L^{p_1}(\mathbb{P})} \| \Psi(\omega) \|_{\mathbb{D}^{2,p_5}} \| \tilde{\Phi}_{\tilde{\mathbf{n}},L} \|_{\mathbb{D}^{2,p_4}(L^2(\mathbb{R}^2))}^2, \end{aligned}$$

where  $\tilde{\mathbf{n}} := (\mathbf{n}; \varepsilon(m, x_*, \omega), L)$ ,  $p_1, p_2, p_3, p_4, p_5 \in (1, \infty)$  satisfying  $1/p_1 + 1/p_2 = 1$ ,  $1/p_3 + 1/p_4 = 1/p_2$ ,  $1/p_4 + 1/p_5 = 1/p_3$ , and

$$\Psi(\omega) := \tilde{\chi}_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)\tilde{\chi}_{[0, \infty)}(\mathcal{R}(m, x_*, \omega) - |\mathbf{n}|)\mathcal{B}(m, x_*, \omega)^{-\nu-15/2}.$$

To justify this estimate, we should show  $\text{Tr}[G(t(H_L^\omega))] \in L^{p_1}(\mathbb{P})$ ,  $\Psi(\omega) \in \mathbb{D}^{2, p_5}$  and  $\tilde{\Phi}_{\tilde{\mathbf{n}}, L} \in \mathbb{D}^{2, p_4}(L^2(\mathbb{R}^2))$ .

This is reduced to show the finiteness of the norms by Proposition 4.21 in [19].

We next proceed to estimates of each norm in the right hand side of (5.10). For the first norm, a sufficient uniform estimate

$$\text{Tr}[G(t(H_L^\omega))] \leq c_{20}R^2\eta L^2.$$

is obtained by applying the Feynman-Kac-Itô formula and the diamagnetic inequality as in [4]. By the Hölder inequality, we have

$$\begin{aligned} & \|\Psi(\omega)\|_{\mathbb{D}^{2, p_5}} \\ (5.11) \quad & \leq \|\tilde{\chi}_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)\|_{\mathbb{D}^{2, p_6}} \\ & \quad \times \|\tilde{\chi}_{[0, \infty)}(\mathcal{R}(m, x_*, \omega) - |\mathbf{n}|)\|_{\mathbb{D}^{2, p_7}} \|\mathcal{B}(m, x_*, \omega)^{-\nu-15/2}\|_{\mathbb{D}^{2, p_8}}, \end{aligned}$$

where  $p_6, p_7, p_8 \in (1, \infty)$  satisfying  $1/p_6 + 1/p_7 + 1/p_8 = 1/p_5$ . Since

$$\begin{aligned} & D^2 \tilde{\chi}_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R) \\ & = \tilde{\chi}'_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)(D\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2)^{\otimes 2} \\ & \quad + \tilde{\chi}''_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)D^2\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2, \\ & \quad \|D\|\partial^\alpha B^\omega\|_{L^2(\Lambda_L)}\|_{L^2(\mathbb{R}^2)}^2 \\ & = \int_{\mathbb{R}^2} dy \left| 2 \int_{\Lambda_L} dx (\partial^\alpha B^\omega)(x)(\partial^\alpha \sigma)(x-y) \right|^2 \leq c_{\alpha, 1} m L^2 \end{aligned}$$

and

$$\begin{aligned} & \|D^2\|\partial^\alpha B^\omega\|_{L^2(\Lambda_L)}\|_{L^2(\mathbb{R}^2)}^{\otimes 2} \\ & = \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dy' \left| 2 \int_{\Lambda_L} dx (\partial^\alpha \sigma)(x-y)(\partial^\alpha \sigma)(x-y') \right|^2 \leq c_{\alpha, 2} L^4 \end{aligned}$$

under (5.5) for any  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$  satisfying  $\alpha_1 + \alpha_2 \leq 2$ , we have

$$\begin{aligned} & \|\tilde{\chi}_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)\|_{\mathbb{D}^{2, p_6}} \\ & \leq c_{21} m L^2 \mathbb{P}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R \in [m-2, m+1])^{1/p_6}. \end{aligned}$$

The probability is estimated as

$$\begin{aligned}
& \mathbb{P}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R \in [m-2, m+1]) \\
& \leq L^2 \mathbb{P}((m-R-2)/L^2 \leq \|B^\omega\|_{W^{2,2}(\Lambda_1)}^2) \\
& \leq L^2 \mathbb{E}[\exp(c_{22}(\|B^\omega\|_{W^{2,2}(\Lambda_1)}^2 - (m-R-2)/L^2))] \\
& \leq c_{23} L^2 \exp(-(m-R-2)/L^2)
\end{aligned}$$

by Chebyshev's inequality and Fernique's theorem (cf. [12] p.402). Thus we have

$$\|\tilde{\chi}_{[m-1, m]}(\|B^\omega\|_{W^{2,2}(\Lambda_L)}^2 + R)\|_{\mathbb{D}^{2, p_6}} \leq c_{24} m L^{2+2/p_6} \exp(-(m-R-1)/(L^2 p_6)).$$

Similarly we have

$$\begin{aligned}
& \|\tilde{\chi}_{[0, \infty)}(\mathcal{R}(m, x_*, \omega) - |\mathbf{n}|)\|_{\mathbb{D}^{2, p_7}} \\
& \leq c_{25} \mathbb{P}(\mathcal{R}(m, x_*, \omega) - |\mathbf{n}| \in [-1, \infty))^{1/p_9} (1 + \|\|D\mathcal{R}(m, x_*, \omega)\|_{L^2(\mathbb{R})}\|_{L^{p_{10}}(\mathbb{P})} \\
& \quad + \|\|D^2\mathcal{R}(m, x_*, \omega)\|_{L^2(\mathbb{R})^{\otimes 2}}\|_{L^{p_{10}}(\mathbb{P})})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(\mathcal{R}(m, x_*, \omega) - |\mathbf{n}| \in [-1, \infty))^{1/p_9} & \leq \mathbb{E}\left[\left(\frac{\mathcal{R}(m, x_*, \omega) + 1}{|\mathbf{n}|}\right)^{3p_9}\right]^{1/p_9} \\
& \leq \|\mathcal{R}(m, x_*, \omega) + 1\|_{L^{3p_9}(\mathbb{P})}^3 / |\mathbf{n}|^3,
\end{aligned}$$

where  $p_9, p_{10} \in (1, \infty)$  satisfying  $1/p_9 + 1/p_{10} = 1/p_7$ . Thus the remained estimates for  $\|\Psi(\omega)\|_{\mathbb{D}^{2, p_5}}$  are those of  $\|\|D^k \mathcal{B}(m, x_*, \omega)^{-h}\|_{L^2(\mathbb{R})^{\otimes k}}\|_{L^p(\mathbb{P})}$  with  $(k, h, p) \in \{0, 1, 2\} \times (0, \infty) \times (1, \infty)$ . Since

$$\begin{aligned}
& \|D\mathcal{B}(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)}^2 \\
(5.12) \quad & = \int_{\mathbb{R}^2} dy \left| \int_0^{c_2 L^{-11} m^{-2}} \frac{dr}{\pi r} \int_{B(x_*, r)} dx \sigma(x-y) \int_{B(x_*, r)} dx' B^\omega(x') \right|^2 \\
& \leq \frac{c_{26} \|B^\omega\|_{L^\infty(B(x_*, 1))}^2}{L^{88} m^{16}}
\end{aligned}$$

and

$$\begin{aligned}
& \|D^2\mathcal{B}(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)^{\otimes 2}}^2 \\
(5.13) \quad & = \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dy' \left| \int_0^{c_2 L^{-11} m^{-2}} \frac{dr}{\pi r} \right. \\
& \quad \times \left. \int_{B(x_*, r)} dx \sigma(x-y) \int_{B(x_*, r)} dx' \sigma(x'-y') \right|^2 \\
& \leq \frac{c_{27}}{L^{88} m^{16}},
\end{aligned}$$

are deduced as in (2.7), we have

$$\begin{aligned} & \|\|DB(m, x_*, \omega)^{-h}\|_{L^2(\mathbb{R})^{\otimes k}}\|_{L^p(\mathbb{P})} \\ & \leq \frac{hc_{26}^{1/2}}{L^{44}m^8} \|\mathcal{B}(m, x_*, \omega)^{-h-1}\|_{L^{p'}(\mathbb{P})} \|\|B^\omega\|_{L^\infty(B(x_*, 1))}\|_{L^{p''}(\mathbb{P})} \end{aligned}$$

and

$$\begin{aligned} & \|\|D^2\mathcal{B}(m, x_*, \omega)^{-h}\|_{L^2(\mathbb{R})^{\otimes k}}\|_{L^p(\mathbb{P})} \\ & \leq \frac{h(h+1)c_{26}}{L^{88}m^{16}} \|\mathcal{B}(m, x_*, \omega)^{-h-2}\|_{L^{p'}(\mathbb{P})} \|\|B^\omega\|_{L^\infty(B(x_*, 1))}\|_{L^{2p''}(\mathbb{P})}^2 \\ & \quad + \frac{hc_{27}^{1/2}}{L^{44}m^8} \|\mathcal{B}(m, x_*, \omega)^{-h-1}\|_{L^p(\mathbb{P})}, \end{aligned}$$

where  $p', p'' \in (1, \infty)$  such that  $1/p' + 1/p'' = 1/p$ . We here do not use the condition (5.5) as in (5.8) and (5.9), since we have a simpler estimate

$$(5.14) \quad \|\|B^\omega\|_{L^\infty(B(x_*, 1))}\|_{L^{p''}(\mathbb{P})} \leq c_{p''}.$$

We can now apply the corollary of Lemma 4.1 to obtain

$$(5.15) \quad \|\mathcal{B}(m, x_*, \omega)^{-1}\|_{L^p(\mathbb{P})} \leq c_p(L^{11}m^2)^{2\nu+5}$$

for any  $p \in [1, \infty)$ . Finally, since

$$\begin{aligned} & \|D\varepsilon(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)} = 2c_4c_3^{-3}L^{-58}m^{-11}\mathcal{B}(m, x_*, \omega)\|DB(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)} \\ & \leq c_{28}\|B^\omega\|_{L^\infty(B(x_*, 1))}^3L^{-146}m^{-27} \end{aligned}$$

and

$$\begin{aligned} & \|D^2\varepsilon(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)^{\otimes 2}} \\ & = 2c_4c_3^{-3}L^{-58}m^{-11}(\|DB(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)}^2 + \mathcal{B}(m, x_*, \omega)\|D^2\mathcal{B}(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)^{\otimes 2}}) \\ & \leq c_{28}\|B^\omega\|_{L^\infty(B(x_*, 1))}^2L^{-146}m^{-27} \end{aligned}$$

are deduced from (5.2), (5.12), (5.13) and  $\mathcal{B}(m, x_*, \omega) \leq c_{29}\|B^\omega\|_{L^\infty(B(x_*, 1))}^2L^{-44}m^{-8}$ , we have

$$\begin{aligned} & \|D^2\tilde{\Phi}_{\vec{n}, L}\|_{L^2(\mathbb{R}^2)^{\otimes 3}} \\ & \leq c_{30}(L^2\|D\varepsilon(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)}^2 + L\|D^2\varepsilon(m, x_*, \omega)\|_{L^2(\mathbb{R}^2)^{\otimes 2}}) \\ & \leq c_{31}(\|B^\omega\|_{L^\infty(B(x_*, 1))}^6L^{-290}m^{-54} + \|B^\omega\|_{L^\infty(B(x_*, 1))}^2L^{-145}m^{-27}) \end{aligned}$$

and

$$\|\tilde{\Phi}_{\vec{n}, L}\|_{\mathbb{D}^{2, p_4}(L^2(\mathbb{R}^2))} \leq c_{32}$$

by (5.14).

Thus we obtain

$$\begin{aligned} & \mathbb{E}[\mathrm{Tr}[\chi_{[E-\eta, E+\eta]}(H_L^\omega)]] \\ & \leq c_{33}\eta R^2 L^{c_{34}} \sum_{m \in \mathbb{N}} m^{c_{35}} \exp(-(m - R - 1)/(L^2 p_6)) \sum_{\mathbf{n} \in \mathbb{N}^2} |\mathbf{n}|^{-3} \\ & \leq c_{36}\eta R^2 L^{c_{37}} \end{aligned}$$

for  $L \geq \sqrt{R} \vee c_{38}$ .

## References

1. Adams, R. A. and Fournier, J. J. F., *Sobolev spaces*, Elsevier/Academic Press, Amsterdam, 2003.
2. Abramowitz, M. and Stegun, I. A.(eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., New York. 1992.
3. Carmona, R. and Lacroix, J., *Spectral theory of random Schrödinger operators*, Birkhäuser Boston Inc., Boston, MA, 1990
4. Erdős, L., and Hasler, D., Wegner estimate and Anderson localization for random magnetic fields *Comm. Math. Phys.*, **309** (2012), 507–542.
5. Erdős, L., and Hasler, D., Anderson localization for random magnetic Laplacian on  $\mathbb{Z}^2$ , *arXiv:1101.2139*, to appear in *Annales Henri Poincaré*.
6. Erdős, L., and Hasler, D., Anderson localization at band edges for random magnetic fields, *arXiv:1103.3744*, to appear in *J. Stat. Phys.*
7. Fröhlich, J. and Spencer, T., Absence of diffusion in the Anderson tight binding model for large disorder or low energy, *Comm. Math. Phys.*, **88** (1983), 151–184.
8. Germinet, F. and Klein, A., Bootstrap multiscale analysis and localization in random media, *Comm. Math. Phys.*, **222** (2001), 415–448.
9. Ghribi, F., Hislop, P. D. and Klopp, F., Localization for Schrödinger operators with random vector potentials, in: Germinet, F. and Hislop, P. D. ed., *Adventures in mathematical physics*, Papers from the International Conference on Transport and Spectral Problems in Quantum Mechanics held in honor of Jean-Michel Combes at the Université de Cergy-Pontoise, Cergy-Pontoise, September 4–6, 2006, *Contemp. Math.*, **447**, (American Mathematical Society, Providence, RI, 2007), 123–138.
10. Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order*, Springer, Berlin, 2001.
11. Hislop, P. D. and Klopp, F., The integrated density of states for some random operators with non-sign definite potentials, *J. Funct. Anal.*, **195** (2002), 12–47.
12. Ikeda, N. and Watanabe, S., *Stochastic differential equations and diffusion processes*, North-Holland, Amsterdam, 1989.
13. Klopp, F., Localization for some continuous random Schrödinger operators, *Comm. Math. Phys.*, **167** (1995), 553–569.
14. Klopp, F., Nakamura, S., Nakano, F. and Nomura, Y., Anderson localization for 2D discrete Schrödinger operators with random magnetic fields, *Ann. Henri Poincaré*, **4**, (2003), 795–811.

15. Malliavin, P., Stochastic calculus of variation and hypoelliptic operators, in: Itô, K. ed., *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, (Wiley, New York, 1978), 195–263.
16. Nualart, D., *The Malliavin calculus and related topics*, Springer, New York, 2010.
17. Pastur, L. and Figotin, A., *Spectra of random and almost-periodic operators*, Springer-Verlag, Berlin, 1992.
18. Reed, M. and Simon, B., *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
19. Shigekawa, I., *Stochastic analysis*, American Mathematical Society, Providence, RI, 2004.
20. Stollmann, P., *Caught by disorder, Bound states in random media*, Birkhäuser, Boston, 2001.
21. Ueki, N., Simple examples of Lifschitz tails in Gaussian random magnetic fields, *Ann. Henri Poincaré*, **1** (2000), 473–498.
22. Ueki, N., Wegner estimates and localization for random magnetic fields, *Osaka J. Math.*, **45** (2008), 565–608.
23. Wegner, F., Bounds on the density of states in disordered systems, *Z. Phys. B*, **44** (1981), 9–15.

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