The Hausdorff dimension function of the family of conformal iterated function systems of generalized complex continued fractions [*][†]

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Abstract

We consider the family of CIFSs of generalized complex continued fractions with a complex parameter space. This is a new interesting example to which we can apply a general theory of infinite CIFSs and analytic families of infinite CIFSs. We show that the Hausdorff dimension function of the family of the CIFSs of generalized complex continued fractions is continuous in the parameter space and is real-analytic and subharmonic in the interior of the parameter space. As a corollary of these results, we also show that the Hausdorff dimension function has a maximum point and the maximum point belongs to the boundary of the parameter space. [1]

1 Introduction

Iterated function systems arise in many contexts. One of the most famous applications to use the systems is to construct many kinds of fractals. Studies of these fractal sets constructed by the contractive iterated function systems (for short IFS), sometimes called limit sets, have been developed in many directions. Note that general properties of limit sets of systems with finitely many mappings have been well-studied. For example, see Hutchinson [3], Falconer [4], Barnsley [2], Bandt and Graf [1], and Schief [9] and so on.

Around the middle of the 1990's, studies of the limit sets of the conformal IFSs (for short CIFS) with infinitely many mappings were initiated by Mauldin and Urbański [5], [6] and by Moran [7].

Note that Mauldin and Urbański also gave some interesting examples about their general theory in the papers [5] and [6]. In their general theory, they showed deep results to estimate the Hausdorff dimension and the Hausdorff measure of the limit sets.

Moreover, interests in families of CIFSs have emerged. Roy and Urbański especially studied the Hausdorff dimension functions for the families of CIFSs ([8]). They showed that the Hausdorff dimension functions for the families of CIFSs are continuous with respect to the "λ-topology" which they introduced, and if the families are analytic, then the Hausdorff dimension functions for the families of CIFSs are real-analytic and subharmonic.

There exist rich general theories of limit sets of CIFSs for given families of infinite CIFSs. However, the authors do not think we have found sufficiently many examples of families of infinite CIFSs to which we can apply the above general theories. Therefore, the aim of this paper is to present a new interesting family of infinite CIFSs. More precisely, we define a subset of the complex plane as a parameter space and for each point in the parameter space, we introduce a CIFS related to generalized complex continued fractions. The authors found that Mauldin and Urbański's general theories [5], [6] and Roy and Urbański's general theory [8] can apply to this family. The authors also show that the Hausdorff dimension function for the family is continuous in the parameter space and is real-analytic and subharmonic in the interior of the parameter space by applying the general theories of the families of infinite CIFSs. The authors also show that, as a corollary for these results, the Hausdorff dimension function has a maximum point and it belongs to the boundary of the parameter space.

Precise statements are the following. Let

$$A_0 := \{\tau = u + iv \in \mathbb{C} \mid u \geq 0 \text{ and } v \geq 1\}$$

and $X := \{z \in \mathbb{C} \mid |z - 1/2| \leq 1/2\}$. Also, we set $I_\tau := \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{N}\}$ for each $\tau \in A_0$, where $\mathbb{N}$ is the set of the positive integers.

Definition 1.1 (The CIFS of generalized complex continued fractions). For each $\tau \in A_0$, $S_\tau := \{\phi_b \colon X \to X \mid b \in I_\tau\}$ is called the CIFS of generalized complex continued fractions. Here,

$$\phi_b(z) := \frac{1}{z + b} \quad (z \in X).$$

The $\{S_\tau\}_{\tau \in A_0}$ is called the family of CIFSs of generalized complex continued fractions. For each $\tau \in A_0$, let J_τ be the limit set of the CIFS S_τ (see Definition 2.1) and let h_τ be the Hausdorff dimension of the limit set J_τ. Also, we denote by $\text{Int}(A_0)$ the set of interior points of A_0 with respect to the topology in $\mathbb{C}$. We now give the main results of this paper.

Theorem 1.2 (Main result A). Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then $\tau \mapsto h_\tau$ is continuous in A_0. Moreover, for each $\tau \in A_0$, h_τ is equal to the unique zero of the pressure function of S_τ (see Definition 2.2), $1 < h_\tau < 2$ and $h_\tau \to 1$ ($\tau \in A_0, \tau \to \infty$). In particular, the function $\tau \mapsto h_\tau$ is not constant on A_0.

Theorem 1.3 (Main result B). Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then we have that $\tau \mapsto h_\tau$ is real-analytic and subharmonic in $\text{Int}(A_0)$.

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Corollary 1.4 (Main result C)**.** Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, there exists a maximum value of the function $\tau \mapsto h_\tau$ ($\tau \in A_0$) and any maximum point of the function $\tau \mapsto h_\tau$ belongs to the boundary of A_0. In particular, we have that $\max\{h_\tau | \tau \in A_0\} = \max\{h_\tau | \tau \in \partial A_0\}$.

Remark 1.5. It was shown that for each $\tau \in A_0$, $\overline{J_\tau} \setminus J_\tau$ is at most countable and $h_\tau = \dim_{\mathcal{H}}(\overline{J_\tau})$ ([11, Theorem 6.11]). For the readers, we give a proof of this fact in the Appendix of this paper. Also, for each $\tau \in A_0$, since the set of attracting fixed points of elements of the semigroup generated by S_τ is dense in J_τ, Theorem 1.1 of [10] implies that $\overline{J_\tau}$ is equal to the Julia set of the rational semigroup generated by $\{\phi_b^{-1} \mid b \in I_\tau\}$.

The idea and strategies to prove the main results are the following. We first show that for each $\tau \in A_0$, S_τ is a CIFS (see Definition 2.1 and Proposition 3.1). To prove Proposition 3.1, we use some facts in complex analysis (for example, properties of Möbius transformations on the Riemann sphere $\hat{\mathbb{C}}$ and the Koebe distortion theorem). In addition, we show a useful inequality for $\psi_\tau^1(t)$ of the CIFS S_τ (see Definition 2.2 and Lemma 3.3). To prove this inequality, we use the Bounded Distortion Property (BDP) of the CIFS and an appropriate countable partition of $I_\tau \cong \mathbb{N}^2$. Combining the useful inequality, careful observations and Lebesgue's dominated convergence theorem, we show that for each $\tau \in A_0$, S_τ is a hereditarily regular CIFS and thus h_τ is equal to the unique zero of the pressure function (see Definition 2.4) of S_τ with $\theta_\tau = 1$ (see Lemma 3.3). Furthermore, we show that $h_\tau \to 1$ as $\tau \to \infty$ in A_0 (see Lemma 3.4). By using the fact $\theta_\tau = 1$ for each $\tau \in A_0$ and a geometric observation, we show that $1 < h_\tau < 2$ for each $\tau \in A_0$.

Note that since we deal with a family of CIFSs in this paper, we have to consider the family of the pressure functions and zeros of the functions which is parameterized by $\tau \in A_0$. For this reason, we have some difficulties. However, we overcome these difficulties to show this lemma by considering ψ_τ^1 as an infinite series which is parameterized by $\tau \in A_0$ and applying Lebesgue's dominated convergence theorem. This approach is new and important to deal with the family of the pressure functions and zeros of the functions.

It is worth pointing out that for each $\tau \in A_0$, we obtain inequalities (4), (5) and (6) for the elements of S_τ (Lemmas 4.1, 4.2, 4.3). These inequalities follow from direct calculations. By using these inequalities, we show that $\tau \mapsto S_\tau$ is continuous with respect to the λ-topology. By applying the theory of continuity of the Hausdorff dimension function for a family of CIFSs from [8], we prove the continuity of $\tau \mapsto h_\tau$ in Theorem 1.2. Moreover, by using the above inequalities and the result that S_τ is strongly regular for each $\tau \in A_0$, we can show that for each $\tau \in \text{Int}(A_0)$, there exists an open neighborhood U of τ such that $\{S_\tau\}_{\tau \in U}$ is regularly plane-analytic in the sense of [8]. Combining this with the general theory of real-analyticity of the Hausdorff dimension functions for regularly plane-analytic families of CIFSs from [8], we prove Theorem 1.3.

Finally, by using Theorem 1.2, we obtain that there exists a maximum point of the function $\tau \mapsto h_\tau$. From this fact and Theorem 1.3, we obtain Corollary

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1.4.

The rest of this paper is organized as follows. In section 2, we summarize without proofs the theory of CIFSs and families of the CIFSs. In section 3, we prove some properties about the CIFS of the complex continued fractions. In section 4, we prove the main results.

2 Conformal iterated function systems

In this section, we recall general settings of CIFSs and families of CIFSs ([5], [6], [8]).

Definition 2.1 (Conformal iterated function system). Let $X \subset \mathbb{R}^d$ be a nonempty compact and connected set and let I be a finite set or bijective to $\mathbb{N}$. Suppose that I has at least two elements. We say that $S := \{\phi_i \colon X \to X \mid i \in I\}$ is a conformal iterated function system (for short, CIFS) if S satisfies the following conditions.

1. Injectivity: For all $i \in I$, $\phi_i \colon X \to X$ is injective.

2. Uniform Contractivity: There exists $c \in (0,1)$ such that, for all $i \in I$ and $x, y \in X$, the following inequality holds.

$$|\phi_i(x) - \phi_i(y)| \le c|x - y|.$$

3. Conformality: There exists a positive number ϵ and an open and connected subset $V \subset \mathbb{R}^d$ with $X \subset V$ such that for all $i \in I$, ϕ_i extends to a $C^{1+\epsilon}$diffeomorphism on V and ϕ_i is conformal on V.

4. Open Set Condition(OSC): For all $i, j \in I$ ($i \ne j$), $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ and $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$. Here, $\mathrm{Int}(X)$ denotes the set of interior points of X with respect to the topology in $\mathbb{R}^d$.

5. Bounded Distortion Property(BDP): There exists $K \ge 1$ such that for all $x, y \in V$ and for all $w \in I^* := \bigcup_{n=1}^{\infty} I^n$, the following inequality holds.

$$|\phi'_w(x)| \le K \cdot |\phi'_w(y)|.$$

Here, for each $n \in \mathbb{N}$ and $w = w_1 w_2 \cdots w_n \in I^n$, we set $\phi_w := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ and $|\phi'_w(x)|$ denotes the norm of the derivative of ϕ_w at $x \in X$ with respect to the Euclidean metric on $\mathbb{R}^d$.

6. Cone Condition: For all $x \in \partial X$, there exists an open cone $\mathrm{Con}(x, u, \alpha)$ with a vertex x, a direction u, an altitude $|u|$ and an angle α such that $\mathrm{Con}(x, u, \alpha)$ is a subset of $\mathrm{Int}(X)$.

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I is called an alphabet. We endow I with the discrete topology, and endow $I^\infty := I^\mathbb{N}$ with the product topology. Note that I^∞ is a Polish space. In addition, if I is a finite set, then I^∞ is a compact metrizable space.

Let S be a CIFS. For each $w = w_1 w_2 w_3 \cdots \in I^\infty$, we set $w|_n := w_1 w_2 \cdots w_n \in I^n$ and $\phi_{w|_n} := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$. Then, we have $\bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X)$ is a singleton. We denoted it by $\{x_w\}$. the coding map $\pi \colon I^\infty \to X$ of S is defined by $w \mapsto x_w$. Note that $\pi \colon I^\infty \to X$ is continuous. A limit set of S is defined by

$$J_S := \pi(I^\infty) = \bigcup_{w \in I^\infty} \bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X).$$

For each IFS S, we set $h_S := \dim_\mathcal{H} J_S$, where $\dim_\mathcal{H}$ denote the Hausdorff dimension. For any CIFS S, we define the pressure function of S as follows.

Definition 2.2 (Pressure function). For each $n \in \mathbb{N}$, $[0, \infty]$-valued function ψ_S^n is defined by

$$\psi_S^n(t) := \sum_{w \in I^n} \left(\sup_{z \in X} |\phi_w'(z)| \right)^t \quad (t \geq 0).$$

We set

$$P_S(t) := \lim_{n \to \infty} \frac{1}{n} \log \psi_S^n(t) \in (-\infty, \infty].$$

The function $P_S \colon [0, \infty) \to (-\infty, \infty]$ is called the pressure function of S.

Note that for all $t \geq 0$, $P_S(t)$ exists because of the following proposition.

Proposition 2.3. For all $m, n \in \mathbb{N}$ and $t \geq 0$, we have $\psi_S^{m+n}(t) \leq \psi_S^m(t)\psi_S^n(t)$. In particular, for all $t \geq 0$, $\log \psi_S^n(t)$ is subadditive with respect to $n \in \mathbb{N}$.

We set $\theta_S := \inf\{t \geq 0| \ \psi_S^1(t) < \infty\}$. By using the pressure function, we define properties of CIFSs.

Definition 2.4 (Regular, Strongly regular, Hereditarily regular). Let S be a CIFS. We say that S is regular if there exists $t \geq 0$ such that $P_S(t) = 0$. We say that S is strongly regular if there exists $t \geq 0$ such that $P_S(t) \in (0, \infty)$. We say that S is hereditarily regular if ,for all $I' \subset I$ with $|I \setminus I'| < \infty$, $S' := \{\phi_i \colon X \to X \mid i \in I'\}$ is regular. Here, for any set A, we denote by $|A|$ cardinality of A.

Note that if a CIFS S is hereditarily regular, then S is strong regular and if S is strong regular, then S is regular. We set $F(I) := \{F \subset I| \ |F| < \infty\}$. For each $F \in F(I)$, we set $S_F := \{\phi_i \colon X \to X| \ i \in F\}$. Mauldin and Urbański showed the following results.

Theorem 2.5 ([5] Theorem 3.15). Let S be a CIFS. Then we have

$$h_S = \inf\{t \geq 0 \mid P_S(t) < 0\} = \sup\{h_{S_F}| \ F \in F(I)\} \geq \theta_S.$$

Moreover, if there exists $t \geq 0$ such that $P_S(t) = 0$, then t is the unique zero of the pressure function P_S and we have $t = h_S$.

Theorem 2.6 ([5] Theorem 3.20). Let I be infinite and let S be a CIFS. Then, the following conditions are equivalent:

1. S is hereditarily regular.

2. $\psi_S^1(\theta_S) = \infty$.

Especially, if S is hereditarily regular, we have $\theta_S < h_S$.

Theorem 2.7 ([5] Proposition 4.4). Let S be a regular CIFS. Then if $\lambda_d(\mathrm{Int}(X) \setminus X_1) > 0$, then $h_S < d$. Here, λ_d is the d-dimensional Lebesgue measure and $X_1 := \cup_{i \in I} \phi_i(X)$.

We now consider families of CIFSs. Let $\mathrm{CIFS}(X, I)$ be the family of all CIFSs with $X \subset \mathbb{C}$ and an infinite alphabet I. For each $S \in \mathrm{CIFS}(X,I)$, let $\pi_S \colon I^\infty \to X$ be the coding map of S. In this paper, for any sequence $\{S^n\}_{n \in \mathbb{N}}$ in $\mathrm{CIFS}(X,I)$ and $S \in \mathrm{CIFS}(X,I)$, we write $\lambda(\{S^n\}_{n \in \mathbb{N}}) = S$ if the following conditions are satisfied.

(L1) For every $i \in I$, $\lim_{n \to \infty}(||\phi_i^n - \phi_i|| + ||(\phi_i^n)' - (\phi_i)'||) = 0$.

(L2) There exist $C > 0$, $M \in \mathbb{N}$ and a finite set $F \subset I$ such that for all $i \in I \setminus F$ and $n \geq M$, $|\ \log ||(\phi_i^n)'|| - \log ||\phi_i'||\ | \leq C$.

Here, we write S^n as $\{\phi_i^n\}_{i \in I}$ and S as $\{\phi_i\}_{i \in I}$. If a sequence $\{S^n\}_{n \in \mathbb{N}}$ in $\mathrm{CIFS}(X,I)$ does not admit any $S \in \mathrm{CIFS}(X,I)$ for which the above conditions are fulfilled, we declare that $\lambda(\{S^n\}_{n \in \mathbb{N}}) = \emptyset$. A sequence $\{S^n\}_{n \in \mathbb{N}} \in \mathrm{CIFS}(X, I)^{\mathbb{N}}$ is called λ-converging if $\lambda(\{S^n\}_{n \in \mathbb{N}}) \in \mathrm{CIFS}(X, I)$. We endow $\mathrm{CIFS}(X,I)$ with the λ-topology ([8]).

Definition 2.8. Let Λ be an open and connected subset of $\mathbb{C}$. Let $\{S^\mu\}_{\mu \in \Lambda}$ be a family of elements of $\mathrm{CIFS}(X, I)$. We write S^μ as $\{\phi_i^\mu\}_{i \in I}$. We say that $\{S^\mu\}_{\mu \in \Lambda}$ is plane-analytic if for all $x \in X$ and $i \in I$, $\mu \mapsto \phi_i^\mu(x)$ is holomorphic in Λ.

Moreover, we say that plane-analytic $\{S^\mu\}_{\mu \in \Lambda}$ is regularly plane-analytic if there exists $\mu_0 \in \Lambda$ such that the following conditions are satisfied.

1. S^{μ_0} is strongly regular.

2. There exists $\eta \in (0,1)$ such that for all $w \in I^\infty$ and $\mu \in \Lambda$, $|\kappa_w^{\mu_0}(\mu) - 1| \leq \eta$. Here, for each $\mu_0 \in \Lambda$ and $w = w_1 w_2 \cdots \in I^\infty$,

$$\pi_\mu := \pi_{S_\mu}, \kappa_w^{\mu_0}(\mu) := \frac{(\phi_{w_1}^\mu)'(\pi_\mu(\sigma w))}{(\phi_{w_1}^{\mu_0})'(\pi_{\mu_0}(\sigma w))} \quad (\mu \in \Lambda).$$

Roy and Urbański showed the following results [8].

Theorem 2.9 ([8] Theorem 5.10). The Hausdorff dimension function $h \colon \mathrm{CIFS}(X, I) \to [0, \infty)$, $S \mapsto h_S$, is continuous when $\mathrm{CIFS}(X, I)$ is endowed with the λ-topology.

Theorem 2.10 ([8] Theorem 6.1). *Let Λ be an open and connected subset of $\mathbb{C}$. Let $\{S^{\mu}\}_{\mu \in \Lambda}$ be a family of elements of $\mathrm{CIFS}(X, I)$. If $\{S^{\mu}\}_{\mu \in \Lambda}$ is regularly plane-analytic, then $\mu \mapsto h_{S^{\mu}}$ is real-analytic in Λ.*

Theorem 2.11 ([8] Theorem 6.3). *Let Λ be an open and connected subset of $\mathbb{C}$. Let $\{S^{\mu}\}_{\mu \in \Lambda}$ be a family of elements of $\mathrm{CIFS}(X, I)$. If $\{S^{\mu}\}_{\mu \in \Lambda}$ is plane-analytic, then $\mu \mapsto 1/h_{S^{\mu}}$ is superharmonic in Λ.*

3 CIFSs of generalized complex continued fractions

In this section, we prove some properties of the CIFSs of generalized complex continued fractions [12]. Note that they are important and interesting examples of infinite CIFSs. We introduce some additional notations. For each $\tau \in A_0$, we set $\pi_{\tau} := \pi_{S_{\tau}}$, $\theta_{\tau} := \theta_{S_{\tau}}$, $\psi_{\tau}^n(t) := \psi_{S_{\tau}}^n(t)$ $(t \geq 0, n \in \mathbb{N})$ and $P_{\tau}(t) := P_{S_{\tau}}(t)$ $(t \geq 0)$.

Proposition 3.1. *For all $\tau \in A_0$, S_{τ} is a CIFS.*

Proof. Let $\tau \in A_0$. Firstly, we show that for all $b \in I_{\tau}$, $\phi_b(X) \subset X$. Let $Y := \{z \in \mathbb{C} | \Re z \geq 1\}$ and let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Möbius transformation defined by $f(z) := 1/z$. Since $f(0) = \infty$, $f(1) = 1$, $f(1/2 + i/2) = 2/(1+i) = (1-i)$, we have $f(\partial X) = \partial Y \cup \{\infty\}$. Moreover, since f(1/2) = 2, we have $f(X) = Y \cup \{\infty\}$. Thus, $f : X \to Y \cup \{\infty\}$ is a homeomorphism. Let $g_b : X \to Y$ be the map defined by $g_b(z) := z + b$. We deduce that $\phi_b = f^{-1} \circ g_b$ and $\phi_b(X) \subset f^{-1}(Y) \subset X$. Therefore, we have proved $\phi_b(X) \subset X$.

We next show that for each $\tau \in A_0$, S_{τ} satisfies the conditions of Definition 2.1.

1. Injectivity.

Since each ϕ_b is a Möbius transformation, each ϕ_b is injective.

2. Uniform Contractivity.

Let $b = m + n\tau(= m + nu + inv)$ be an element of I_{τ} and let $z = x + iy$ and $z' = x' + iy'$ be elements of X. We have

$$|z + b|^2 = |x + m + nu + i(y + nv)|^2$$

$$= (x + m + nu)^2 + (y + nv)^2 \geq (0 + 1 + 0)^2 + (-1/2 + 1)^2 = \frac{5}{4}.$$

Therefore, we deduce that $|z + b| \geq \sqrt{5/4}$. We also deduce that $|z' + b| \geq \sqrt{5/4}$. Finally, we obtain that

$$|\phi_b(z) - \phi_b(z')| = \left| \frac{1}{z + b} - \frac{1}{z' + b} \right|$$

$$= \frac{|z - z'|}{|z + b||z' + b|} \leq \left(\sqrt{\frac{4}{5}} \right)^2 |z - z'| = \frac{4}{5}|z - z'|.$$

Therefore, S_τ is uniformly contractive on X.

    3. Conformality.

Let $\tau \in A_0$ and let $b \in I_\tau$. Since ϕ_b is holomorphic on $\mathbb{C} \setminus \{-b\}$, ϕ_b is C^2 and conformal on V.

    4. Open Set Condition.

Note that $\mathrm{Int}(X) = \{z \in \mathbb{C} |\ |z - 1/2| < 1/2\}$. Let $\tau \in A_0$ and let $b \in I_\tau$. Since $f(\partial X) = \partial Y \cup \{\infty\}$, we deduce that for all $b \in I_\tau$,

$$g_b(\mathrm{Int}(X)) \subset \{z = x + iy \in \mathbb{C} |\ x > 1\} = f(\mathrm{Int}(X)).$$

Moreover, if b and b' are distinct elements, then $g_b(\mathrm{Int}(X))$ and $g_{b'}(\mathrm{Int}(X))$ are disjoint. Therefore, we have that for all $b \in I_\tau$,

$$\phi_b(\mathrm{Int}(X)) = f^{-1} \circ g_b(\mathrm{Int}(X)) \subset f^{-1} \circ f(\mathrm{Int}(X)) = \mathrm{Int}(X).$$

And if b and b' is distinct elements,

$$\phi_b(\mathrm{Int}(X)) \cap \phi_{b'}(\mathrm{Int}(X)) = f^{-1}(g_b(X) \cap g_{b'}(X)) = \emptyset.$$

Therefore, S_τ satisfies the Open Set Condition of S_τ.

    6. Cone Condition.

Since X is a closed disk, the Cone Condition is satisfied.

    5. Bounded distortion Property.

Let ϵ be a positive real number which is less than $1/12$ and let $V' := B(1/2, 1/2 + \epsilon)$ be the open ball with center $1/2$ and radius $1/2 + \epsilon$. We set $\tau := u + iv$. And for all $(m, n) \in \mathbb{N}^2$ and $z := x + iy \in V'$, we have that

$$\begin{aligned}
|\phi'_{m+n\tau}(z)| &= \frac{1}{|z + m + n\tau|^2} = \frac{1}{(x + m + nu)^2 + (y + nv)^2} \\
&\leq \frac{1}{(-\epsilon + 1 + 0)^2 + (-1/2 - \epsilon + 1)^2} \\
&= \frac{1}{2\epsilon^2 - 3\epsilon + 5/4} = \frac{1}{2(\epsilon - 3/4)^2 + 1/8} \\
&\leq \frac{1}{2(1/12 - 3/4)^2 + 1/8} = \frac{72}{73} < 1
\end{aligned}$$

For each $z \in V'$, we set

$$z' := \begin{cases} (|z - 1/2| - \epsilon)\dfrac{(z - 1/2)}{|z - 1/2|} + 1/2 & (z \notin X) \\ z & (z \in X). \end{cases}$$

Then, we have that $|z - z'| \leq \epsilon$ and $|z' - 1/2| < 1/2$. It implies that $z' \in X$. Thus, we obtain that $|\phi_b(z) - \phi_b(z')| \leq (72/73)|z - z'| < \epsilon$ and

$$\left|\phi_b(z) - \frac{1}{2}\right| \leq |\phi_b(z) - \phi_b(z')| + \left|\phi_b(z') - \frac{1}{2}\right| < \frac{1}{2} + \epsilon.$$

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It follows that for all $b \in I_\tau$, $\phi_b(V') \subset V'$. In addition, ϕ_b is injective on V' and ϕ_b is holomorphic on $V' := B(1/2, 1/2 + \epsilon)$ since ϕ_b is holomorphic on $\mathbb{C} \setminus \{-b\}$.

Let b be an element of I_τ and $r_0 := 1/2 + \epsilon$. Let f_b be the function defined by

$$f_b(z) := \frac{(\phi_b(r_0 z + 1/2) - \phi_b(1/2))}{r_0 \phi_b'(1/2)} \quad (z \in D := \{z \in \mathbb{C} | |z| < 1\}).$$

Note that f_b is holomorphic on D and $f_b(0) = 0$ and $f_b'(0) = 1$. By using the Koebe distortion theorem, we deduce that for all $z \in D$

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f_b(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}.$$

Let $r_1 := (r_0 + 1/2)/2$. we deduce that there exist $C_1 \geq 1$ and $C_2 \leq 1$ such that for all $z \in B(0, r_1/r_0)(\subset D)$,

$$C_2 \leq \frac{1 - |z|}{(1 + |z|)^3} \quad \text{and} \quad \frac{1 + |z|}{(1 - |z|)^3} \leq C_1.$$

Let $C := C_1/C_2$. Then, we have that for all $z, z' \in B(0, r_1/r_0)$

$$\begin{aligned}
\frac{|\phi_b'(r_0 z + 1/2)|}{|\phi_b'(1/2)|} = |f_b'(z)| &\leq \frac{1 + |z|}{(1 - |z|)^3} \\
&\leq C_1 = C C_2 \leq C \frac{1 - |z'|}{(1 + |z'|)^3} \\
&\leq C|f_b'(z')| \leq C \frac{|\phi_b'(r_0 z' + 1/2)|}{|\phi_b'(1/2)|}.
\end{aligned}$$

It follows that for all $z, z' \in B(0, r_1/r_0)$, $|\phi_b'(r_0 z + 1/2)| \leq C|\phi_b'(r_0 z' + 1/2)|$. Finally, let $V := B(1/2, r_1)$ be the open ball with center $1/2$ and radius r_1. Then, V is an open and connected subset of $\mathbb{C}$ with $X \subset V$ and for all $z, z' \in V$,

$$|\phi_b'(z)| \leq C|\phi_b'(z')|.$$

Therefore, S_τ satisfies the Bounded Distortion Property. $\qquad \square$

Lemma 3.2. Let $\tau \in A_0$. Then, there exists $C \geq 1$ such that for all $z \in B(1/2, r_1)$ and $b \in I_\tau$, we have $C^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq C|b|^{-2}$.

Proof. Note that $|\phi_b'(0)| = |b|^{-2}$. By using the BDP, there exists $C \geq 1$ such that for all $z \in B(1/2, r_1)$, we have $C^{-1}|\phi_b'(0)| \leq |\phi_b'(z)| \leq C|\phi_b'(0)|$. We deduce that $C^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq C|b|^{-2}$. $\qquad \square$

Lemma 3.3. For all $\tau \in A_0$, S_τ is a hereditarily regular CIFS with $\theta_\tau = 1$.

Proof. Let $\tau \in A_0$. For each non-negative integer p, we define $K'(p) := \{b = m + n\tau \in I_\tau | (m, n) \in \mathbb{N}^2, m < 2^p, n < 2^p\}$ and $K(p) := K'(p) \setminus K'(p - 1)$. Note that for each non-negative integer p, $|K'(p)| = (2^p - 1)^2$. We deduce that

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for each $p \in \mathbb{N}$, $|K(p)| = |K'(p)| - |K'(p-1)| = (2^p - 1)^2 - (2^{p-1} - 1)^2 = 3 \cdot 4^{p-1} - 2 \cdot 2^{p-1} = 2^{p-1}(3 \cdot 2^{p-1} - 2)$ and $4^{p-1} \leq |K(p)| \leq 3 \cdot 4^{p-1}$.

Let $b = m + n\tau = m + n(u + iv) \in K(p)$. We consider the following two cases.

(i) If $m \geq 2^{p-1}$ then we have

$$
\begin{aligned}
|b|^2 &= |m + nu + inv|^2 \\
&= (m + nu)^2 + (nv)^2 \\
&\geq (2^{p-1} + u)^2 + v^2 \\
&\geq (2^{p-1})^2 + |\tau|^2 = 4^{p-1}\left(1 + \frac{|\tau|^2}{4^{p-1}}\right).
\end{aligned}
$$

(ii) If $n \geq 2^{p-1}$ then we have

$$
\begin{aligned}
|b|^2 &= |m + nu + inv|^2 \\
&= (m + nu)^2 + (nv)^2 \\
&\geq n^2(u^2 + v^2) \geq 4^{p-1}|\tau|^2.
\end{aligned}
$$

Then for any $t \geq 0$, we have

$$
\begin{aligned}
\sum_{b \in I_\tau} |b|^{-2t} &= \sum_{p \in \mathbb{N}} \sum_{b \in K(p)} \left\{|b|^2\right\}^{-t} \\
&\leq \sum_{p \in \mathbb{N}} |K(p)| 4^{-t(p-1)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t} \\
&\leq \sum_{p \in \mathbb{N}} 3 \cdot 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t}.
\end{aligned}
$$

Hence, we deduce that

$$
\sum_{b \in I_\tau} |b|^{-2t} \leq 3 \sum_{p \in \mathbb{N}} 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t}. \tag{1}
$$

Moreover, by the inequality $|\tau|^2 \geq 1$ and the inequality $1 + \dfrac{|\tau|^2}{4^{p-1}} \geq 1$, we deduce that for all $p \in \mathbb{N}$,

$$
3 \cdot 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t} \leq 3 \cdot 4^{(p-1)(1-t)}. \tag{2}
$$

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Also, by the inequality $|b| \leq |m| + |n||\tau| \leq 2^p(1 + |\tau|)$, we have

$$\sum_{b \in I_\tau} |b|^{-2t} = \sum_{p \in \mathbb{N}} \sum_{b \in K(p)} \left\{|b|^{-2}\right\}^t$$
$$\geq \sum_{p \in \mathbb{N}} |K(p)| 4^{-pt} (1 + |\tau|)^{-2t}.$$

Thus, we deduce that

$$\sum_{b \in I_\tau} |b|^{-2t} \geq 4^{-1} \sum_{p \in \mathbb{N}} 4^{p(1-t)} (1 + |\tau|)^{-2t}. \tag{3}$$

Finally, from Lemma 3.2, the inequality (1) and the inequality (3), it follows that $\psi_\tau^1(1) = \infty$ and if $t > 1$, then $\psi_\tau^1(t) < \infty$. Therefore, we deduce that $\theta_\tau = 1$ and by Theorem 2.6, we obtain that for all $\tau \in A_0$, S_τ is hereditarily regular. Hence, we have proved our lemma. $\qquad\square$

Lemma 3.4. We have $\lim_{\tau \to \infty, \tau \in A_0} h_\tau = 1$, i.e., for each $\epsilon > 0$, there exists $N > 0$ such that, for all $\tau \in A_0$ with $|\tau| \geq N$, we have $|h_\tau - 1| < \epsilon$.

Proof. Let $\epsilon > 0$ and $t := 1 + \epsilon > 1$. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be any sequence in A_0 such that $|\tau_n| \to \infty$ as $n \to \infty$. Note that for all $p \in \mathbb{N}$, we have $\left\{\min\{1 + \frac{|\tau_n|^2}{4^{p-1}}, |\tau_n|^2\}\right\}^{-t} \to 0$ as $n \to \infty$. By the inequality (1) and the inequality (2), we deduce that

$$f_n(p) = 3 \cdot 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau_n|^2}{4^{p-1}}, |\tau_n|^2\}\right\}^{-t} \quad (p \in \mathbb{N})$$

is dominated by the integrable function $g(p) = 3 \cdot 4^{(p-1)(1-t)}$ $(p \in \mathbb{N})$ with respect to the counting measure on $\mathbb{N}$. Then, by Lebesgue's dominated convergence theorem, we deduce that $\lim_{n \to \infty} \sum_{b \in I_{\tau_n}} |b|^{-2t} = 0$. By Lemma 3.2, we obtain $\lim_{n \to \infty} \psi_{\tau_n}^1(t) = 0$. It follows that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\tau \in A_0$ with $|\tau| \geq N$, we have $\psi_\tau^1(1 + \epsilon) = \psi_\tau^1(t) < 1$.

By Proposition 2.3, we obtain that $\psi_\tau^n(1 + \epsilon) \leq (\psi_\tau^1(1 + \epsilon))^n < 1$. Therefore, we deduce that $P_\tau(1 + \epsilon) \leq 0$. Thus, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\tau \in A_0$ with $|\tau| \geq N$, $h_\tau \leq 1 + \epsilon$.

Moreover, by Theorem 2.6, for all $\tau \in A_0$, we have $1 - \epsilon \leq 1 = \theta_\tau < h_\tau$. Hence, we have proved our lemma. $\qquad\square$

Theorem 3.5. Let $\tau \in A_0$. Then we have $1 < h_\tau < 2$.

Proof. Let $\tau \in A_0$. By Theorem 2.6, we have $1 = \theta_\tau < h_\tau$. We now show that $h_\tau < 2$. We have

$$\bigcup_{b \in I_\tau} g_b(X) \subset \{z \in \mathbb{C}| \Re z \geq 1 \text{ and } \Im z \geq 0\}.$$

Let U_0 be an open ball such that $U_0 \subset \{z \in \mathbb{C}| \ \Re z \geq 1 \text{ and } \Im z < 0\}$. Since $U_0 \subset Y$, we deduce that $f^{-1}(U_0) \subset f^{-1}(Y) = \text{Int}(X)$. We set $X_1 := \cup_{b \in I_\tau} \phi_b(X)$. Since $U_0 \cap \bigcup_{b \in I_\tau} g_b(X) = \emptyset$, we deduce that $f^{-1}(U_0) \cap X_1 = f^{-1}(U_0 \cap \bigcup_{b \in I_\tau} g_b(X)) = \emptyset$. It follows $\text{Int}(X) \setminus X_1 \supset f^{-1}(U_0)$.

Therefore, we deduce that $\lambda_2(\text{Int}(X) \setminus X_1) > 0$ where, λ_2 is the 2-dimensional Lebesgue measure. By Theorem 2.7, we obtain that $h_\tau < 2$. Hence, we have proved $1 < h_\tau < 2$. $\qquad\square$

4 Proof of the main results

4.1 Proof of Main result A

We first show the following lemma.

Lemma 4.1. Let $\tau \in A_0$ and a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in A_0 satisfies $\lim_{n \to \infty} \tau_n = \tau$. Then, there exist $K \in \mathbb{N}$, $C_1 > 0$ and $C_2 > 0$ such that for all $k \geq K$, $(m, n) \in \mathbb{N}^2$ and $z_0, z' \in X$,

$$C_1 \leq \frac{|z' + m + n\tau_k|^2}{|z + m + n\tau|^2} \leq C_2. \tag{4}$$

Proof. We set $\tau = u + iv$ and we set for each $n \in \mathbb{N}$, $\tau_n = u_n + iv_n$. Since $\lim_{n \to \infty} \tau_n = \tau$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $|u - u_k| \leq 1$ and $|v - v_k| \leq v/3$. Then, for all $(m, n) \in \mathbb{N}^2$ and $z, z' \in X$,

$$\frac{|z' + m + n\tau_k|^2}{|z + m + n\tau|^2}$$

$$\leq \frac{(1 + m + nu_k)^2 + (1/2 + nv_k)^2}{(m + nu)^2 + (-1/2 + nv)^2}$$

$$\leq \frac{(1 + m + n(1 + u))^2 + (1/2 + n(4/3)v)^2}{(m + nu)^2 + (-1/2 + nv)^2}$$

$$= \frac{(1 + m + n(1 + u))^2}{(m + nu)^2 + (-1/2 + nv)^2} + \frac{(1/2 + n(4/3)v)^2}{(m + nu)^2 + (-1/2 + nv)^2}$$

$$\leq \max \left\{ \frac{(1 + (1 + u) + 1)^2}{1^2}, \frac{(1 + (1 + u) + 1)^2}{u^2 + (v - 1/2)^2} \right\} + \frac{(1/2n + (4/3)v)^2}{(v - 1/2n)^2}$$

$$\leq \max \left\{ \frac{(1 + (1 + u)u + 1)^2}{1^2}, \frac{(1(1 + u) + 1)^2}{u^2 + (v - 1/2)^2} \right\} + \frac{(1/2 + (4/3)v)^2}{(v - 1/2)^2} < \infty$$

and

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$$\frac{|z' + m + n\tau_k|^2}{|z + m + n\tau|^2} \geq \frac{(m + nu_k)^2 + (-1/2 + nv_k)^2}{(1 + m + nu)^2 + (1/2 + nv)^2}$$

$$\geq \frac{m^2 + (-1/2 + n(2/3)v)^2}{2(1 + m + n\max\{u, v\})^2}$$

$$\geq \frac{1}{2} \left(\min \left\{ \frac{1}{1 + 1 + \max\{u, v\}}, \frac{((2/3)v - 1/2)^2}{1 + 1 + \max\{u, v\}} \right\} \right)^2 > 0.$$

Therefore, we have proved our lemma. □

We now prove Theorem 1.2.

Proof. Firstly, By Lemma 3.3, for each $\tau \in A_0$, the value h_τ is equal to the unique zero of the pressure function of S_τ. Moreover, by Lemma 3.4 and Theorem 3.5, we have that $1 < h_\tau < 2$ for each $\tau \in A_0$ and $h_\tau \to 1$ as $\tau \to \infty$ in A_0.

We next show that if a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in A_0 satisfies $\lim_{n \to \infty} \tau_n = \tau$, then $\lambda(\{S_{\tau_n}\}_{n \in \mathbb{N}}) = S_\tau$. Since for all $(m, n) \in \mathbb{N}^2$, $\phi_{z+m+n\tau}(z) = 1/(z + m + n\tau)$ and $(\phi_{m+n\tau})'(z) = (-1)/(z + m + n\tau)^2$, condition (L1) is satisfied. Since X is compact, there exist $z_0, z_k \in X$ such that

$$\log\left(\sup_{x \in X} |\phi'_{m+n\tau}| / \sup_{x \in X} |\phi'_{m+n\tau_k}| \right) = \log(\phi'_{m+n\tau}(z_0)/\phi'_{m+n\tau_k}(z_k))$$

$$= \log(|z_k + m + n\tau_k|^2 / |z_0 + m + n\tau|^2).$$

By Lemma 4.1, there exist $C > 0$ and $K \in \mathbb{N}$ such that for each $k \geq K$ and $(m, n) \in \mathbb{N}^2$,

$$\left| \log\left(\sup_{z \in X} |\phi'_{m+n\tau}(z)| \right) - \log\left(\sup_{z \in X} |\phi'_{m+n\tau_k}(z)| \right) \right|$$

$$= \left| \log\left(\sup_{z \in X} |\phi'_{m+n\tau}(z)| / \sup_{z \in X} |\phi'_{m+n\tau_k}(z)| \right) \right| \leq C.$$

Therefore, we have proved that if a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in A_0 satisfies $\lim_{n \to \infty} \tau_n = \tau$, then $\lambda(\{S_{\tau_n}\}_{n \in \mathbb{N}}) = S_\tau$.

We next show that $\tau \mapsto h_\tau$ is continuous in A_0. By Theorem 2.9, $S_\tau \mapsto h_\tau$ is continuous with respect to the λ-topology. By Lemma 3.3 of [8], if $\lambda(\{S_{\tau_n}\}_{n \in \mathbb{N}}) = S_\tau$, then $\lim_{n \to \infty} h_{\tau_n} = h_\tau$. Thus, if $\lim_{n \to \infty} \tau_n = \tau$, then $\lim_{n \to \infty} h_{\tau_n} = h_\tau$. Therefore, we have proved that $\tau \mapsto h_\tau$ is continuous in A_0. □

4.2 Proof of Main result B

In order to prove Theorem 1.3, we first show the following two lemmas.

Lemma 4.2. For each $z \in X$, $(m, n) \in \mathbb{N}^2$ and $\tau \in A_0$, $\phi_{m+n\tau'}(z)$ tends to $\phi_{m+n\tau}(z)$ as τ' tends to τ and this convergence is uniform on $z \in X$ and $(m, n) \in \mathbb{N}^2$.

Proof. We set $z := x + iy$, $\tau := u + iv$ and $\tau' := u' + iv'$. Then, we have

$$
\begin{aligned}
&|\phi_{m+n\tau'}(z) - \phi_{m+n\tau}(z)| \\
&= \left| \frac{1}{z + m + n\tau'} - \frac{1}{z + m + n\tau} \right| \\
&= \left| \frac{n(\tau - \tau')}{(x + m + nu' + i(y + nv'))(x + m + nu + i(y + nv))} \right| \\
&= \frac{n|\tau - \tau'|}{\sqrt{(x + m + nu')^2 + (y + nv')^2} \sqrt{(x + m + nu)^2 + (y + nv)^2}} \\
&\leq \frac{n|\tau' - \tau|}{m^2 + (n - 1/2)^2} \leq 4|\tau' - \tau|,
\end{aligned}
\tag{5}
$$

where, to deduce the last inequality, we use the following.

$$
\begin{aligned}
\frac{n}{m^2 + (n - 1/2)^2} &\leq \frac{n^2}{1 + (n - 1/2)^2} \\
&\leq \frac{1}{1/n^2 + (1 - 1/2n)^2} \leq \frac{1}{(1 - 1/2n)^2} \leq 4.
\end{aligned}
$$

Therefore, we have proved our lemma. $\qquad\square$

Lemma 4.3. For any $z, z' \in X$ and for all $\tau \in \mathrm{Int}(A_0)$,

$$
\left| \frac{\phi'_{m+n\tau}(z)}{\phi'_{m+n\tau}(z')} \right| \leq 13.
\tag{6}
$$

Proof. We set $z := x + iy$, $z' := x' + iy'$ and $\tau := u + iv$. Then, we have

$$
\begin{aligned}
\left| \frac{\phi'_{m+n\tau}(z)}{\phi'_{m+n\tau}(z')} \right| &= \left| \frac{(z + m + n\tau)^2}{(z' + m + n\tau)^2} \right| = \frac{(m + nu + x')^2 + (y' + nv)^2}{(m + nu + x)^2 + (y + nv)^2} \\
&\leq \frac{(m + nu + 1)^2 + (1/2 + nv)^2}{(m + nu)^2 + (-1/2 + nv)^2} \\
&\leq \frac{(m + nu + 1)^2}{(m + nu)^2} + \frac{(1/2 + nv)^2}{(-1/2 + nv)^2} \\
&= \left(1 + \frac{1}{m + nu} \right)^2 + \left(\frac{1 + 1/(2nv)}{1 - 1/(2nv)} \right)^2 \leq 2^2 + \left(\frac{3/2}{1/2} \right)^2 = 13.
\end{aligned}
$$

Therefore, we have proved our lemma. $\qquad\square$

We now proved Theorem 1.3.

Proof. We first show that $\tau \mapsto h_\tau$ is subharmonic in $\mathrm{Int}(A_0)$. Let $z \in X$ and Let $(m, n) \in \mathbb{N}^2$. Then, we deduce that the function $\tau \mapsto \phi_{m+n\tau}(z) = 1/(z+m+n\tau)$ is holomorphic since the real part of $-(m+z)/n$ is negative. That is, $-(m+z)/n$ is not an element of $\mathrm{Int}(A_0)$. Thus, the map $\tau \mapsto \phi_{m+n\tau}(z) = 1/(z+m+n\tau)$ is holomorphic in $\mathrm{Int}(A_0)$. Hence, $\{S_\tau\}_{\tau \in \mathrm{Int}(A_0)}$ is plane-analytic. Therefore, by using Theorem 2.11, we have proved that $\tau \mapsto h_\tau$ is subharmonic in $\mathrm{Int}(A_0)$.

We next show that $\tau \mapsto h_\tau$ is real-analytic in $\mathrm{Int}(A_0)$. Since for each $\tau \in A_0$ S_τ is a hereditarily regular CIFS, we have that for each $\tau \in \mathrm{Int}(A_0)$, S_τ is a strongly regular CIFS. We now show that for any $\tau_0 \in \mathrm{Int}(A_0)$, there exists an open ball $U \subset \mathrm{Int}(A_0)$ with center τ_0 and $\eta > 0$ such that for all $\tau \in U$ and $w := (m_i, n_i)_{i \in \mathbb{N}} \in (\mathbb{N}^2)^\infty$, $|\kappa_w^{\tau_0}(\tau) - 1| \leq \eta$, where we denote $(\phi'_{m_1+n_1\tau}(\pi_\tau \sigma w))/(\phi'_{m_1+n_1\tau_0}(\pi_{\tau_0}\sigma w))$ by $\kappa_w^{\tau_0}(\tau)$.

We have that

$$\frac{\phi'_{m_1+n_1\tau}(\pi_\tau \sigma w)}{\phi'_{m_1+n_1\tau_0}(\pi_{\tau_0}\sigma w)} = \frac{\phi'_{m_1+n_1\tau}(\pi_\tau \sigma w)}{\phi'_{m_1+n_1\tau_0}(\pi_\tau \sigma w)} \frac{\phi'_{m_1+n_1\tau_0}(\pi_\tau \sigma w)}{\phi'_{m_1+n_1\tau_0}(\pi_{\tau_0}\sigma w)}. \tag{7}$$

By Lemma 4.2, the first term of the right hand side of (7) tends to 1 as τ tends to τ_0. By Lemma 4.3, the second term of the right hand side of (7) is bounded with respect to $\tau \in \mathrm{Int}(A_0)$ and $w \in (\mathbb{N}^2)^\infty$. Therefore, there exists an open ball $U'' \subset \mathrm{Int}(A_0)$ with center τ_0 such that $|K_w^{\tau_0}|$ is bounded in U''. Note that $K_w^{\tau_0}$ is holomorphic in $\mathrm{Int}(A_0)$. By using the Cauchy formula

$$(K_w^{\tau_0})'(\tau) = \frac{1}{2\pi i} \int_{\partial U''} \frac{K_w^{\tau_0}(\xi)}{(\xi - \tau)^2} \mathrm{d}\xi \quad (\tau \in U''),$$

we deduce that there exists $M > 0$ such that for all $\tau \in U'$, $|(K_w^{\tau_0})'(\tau)| \leq M$. Here, U' is an open ball with center τ_0 such that $U' \subset U''$. Then, we have that

$$\begin{aligned}
|K_w^{\tau_0}(\tau) - 1| &= |K_w^{\tau_0}(\tau) - K_w^{\tau_0}(\tau_0)| \\
&= \left| \int_{\tau_0}^{\tau} (K_w^{\tau_0})'(\xi) \mathrm{d}\xi \right| \\
&\leq \int_{\tau_0}^{\tau} |(K_w^{\tau_0})'(\xi)| |\mathrm{d}\xi| \leq M|\tau - \tau_0|.
\end{aligned}$$

It follows that there exists an open ball $U (\subset U')$ with center τ_0 such that for all $\tau \in U$ and $w \in (\mathbb{N}^2)^\infty$, $|K_w^{\tau_0}(\tau) - 1| \leq \eta$.

Thus, for any $\tau_0 \in \mathrm{Int}(A_0)$, there exists an open ball $U \subset \mathrm{Int}(A_0)$ with center τ_0 such that $\{S_\tau\}_{\tau \in U}$ is regularly plane-analytic. By Theorem 2.10, for any $\tau_0 \in \mathrm{Int}(A_0)$, there exists an open ball $U \subset \mathrm{Int}(A_0)$ with center τ_0 such that $\{S_\tau\}_{\tau \in U}$ is real-analytic. Since τ_0 is arbitrary, we deduce that the map $\tau \mapsto h_\tau$ is real-analytic and subharmonic in $\mathrm{Int}(A_0)$. $\square$

4.3 Proof of Main result C

We now prove Corollary 1.4.

Proof. For each $n \in \mathbb{N}$, let $B_n := A_0 \cap \{z \in \mathbb{C} | \ |\Re z| \le n \text{ and } |\Im z| \le n\}$. Note that for all $n \in \mathbb{N}$, the map $\tau \mapsto h_\tau$ is subharmonic in $\text{Int}(B_n)$ by Theorem 1.3. Let $\epsilon := (h_i - 1)/2$, where $i = \sqrt{-1}$. By Lemma 3.4, we deduce that there exists $N \in \mathbb{N}$ such that for all $\tau \in A_0 \setminus B_N$, $|h_\tau - 1| < \epsilon$. It follows that $(h_i - 1)/2 > h_\tau - 1$. Then, we obtain that for all $\tau \in A_0 \setminus B_N$,

$$h_i > 2h_\tau - 1 = h_\tau + (h_\tau - 1) > h_\tau.$$

Since the function $\tau \mapsto h_\tau$ is continuous in B_N, there exists a maximum point of the function $\tau \mapsto h_\tau$ in A_0 and

$$\max\{h_\tau | \ \tau \in A_0\} = \max\{h_\tau | \ \tau \in B_N\}.$$

Since the function $\tau \mapsto h_\tau$ is subharmonic in $\text{Int}(A_0)$, there exists no maximum point of the function $\tau \mapsto h_\tau$ in $\text{Int}(A_0)$. Thus, we have proved Corollary 1.4. $\square$

5 Appendix: the proof of the fact $\overline{J_\tau} \setminus J_\tau$ is at most countable

In this section, for the readers, we give the proof of the fact for each $\tau \in A_0$, $\overline{J_\tau} \setminus J_\tau$ is at most countable and $h_\tau = \dim_{\mathcal{H}}(\overline{J_\tau})$ ([11, Theorem 6.1]). We introduce some additional notations.

Definition 5.1. Let S be a CIFS. We write S as $\{\phi_i\}_{i \in I}$. Suppose that I is a countable infinite set. We set $I^* = \bigcup_{n \in \mathbb{N}} I^n$. Let $z \in X$ and $\{z_i\}_{i \in I'} \subset X$ with $I' \subset I$ and $|I'| = \infty$. We say that $\lim_{i \in I'} z_i = z$ if for each $\epsilon > 0$, there exists $F' \subset I'$ with $|F'| < \infty$ such that if $i \in I' \setminus F'$, then $|z_i - z| < \epsilon$.

We set

$$X_S(\infty) := \{\lim_{i \in I'} z_i \in X | \ ^\exists I' \subset I, ^\exists \{z_i\}_{i \in I'} \text{ s.t. } |I'| = \infty, z_i \in \phi_i(X) \ (i \in I')\}.$$

Mauldin and Urbański showed the following results. ([5])

Lemma 5.2. [5, Lemma 2.5 and Lemma 2.1] Let S be a CIFS. We write S as $\{\phi_i\}_{i \in I}$. Suppose that I is a countable infinite set. Then we have that

$$\overline{J_S} = J_S \cup \bigcup_{w \in I^*} \phi_w(X_S(\infty)) \cup X_S(\infty).$$

Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. We set $X_\tau(\infty) := X_{S_\tau}(\infty)$. Sugita showed the following result ([11]).

Theorem 5.3. Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, we have that for all $\tau \in A_0$, $X_\tau(\infty) = \{0\}$. In particular, for each $\tau \in A_0$,

$$\overline{J_\tau} = J_\tau \cup \bigcup_{w \in I^*} \phi_w(\{0\}) \cup \{0\} \quad \text{and} \quad \overline{J_\tau} \setminus J_\tau \subset \bigcup_{w \in I^*} \phi_w(\{0\}) \cup \{0\}.$$

Proof. We first show that for all $\tau \in A_0$, $0 \in X_\tau(\infty)$. We set $I'_\tau := \{m + \tau \in I_\tau \mid m \in \mathbb{N}\} \subset I_\tau$ and $b_m := m + \tau \in I'_\tau$. Then, we have that $|I'_\tau| = \infty$ and since $0 \in X$, $\phi_{b_m}(0) \in \phi_{b_m}(X)$. Let $\epsilon > 0$. Then, there exists $M \in \mathbb{N}$ such that $M > 1/\epsilon$. Let $F_\tau := \{m + \tau \in I_\tau \mid m \in \mathbb{N}, m \leq M\} \subset I'_\tau$. We obtain that $|F_\tau| < \infty$ and if $b_m \in I'_\tau \setminus F_\tau$, then $\phi_{b_m}(0) \in \phi_{b_m}(X)$ and

$$|\phi_{b_m}(0)| = \left| \frac{1}{m + \tau} \right| < \frac{1}{m} < \frac{1}{M} < \epsilon.$$

We next show that for each $\tau \in A_0$, $a \in X_\tau(\infty)$ implies $a = 0$. Suppose that there exists $a \in X_\tau(\infty)$ such that $a \neq 0$. Then, there exist $I'_\tau \subset I_\tau$ and $\{z'_b\}_{b \in I'_\tau}$ such that $|I'_\tau| = \infty$, $z'_b \in \phi_b(X)$ $(b \in I'_\tau)$ and $\lim_{b \in I'_\tau} z'_b = a$. Let $\delta := |a|/2 > 0$. Then, there exists $F'_\tau \subset I'_\tau$ such that $|F'_\tau| < \infty$ and for all $b \in I'_\tau \setminus F'_\tau$, $|z'_b - a| < \delta$. In particular, for all $b \in I'_\tau \setminus F'_\tau$,

$$|z'_b| \geq |a| - |z'_b - a| > \delta. \tag{*}$$

Moreover, for each $z \in X$, $\tau \in A_0$ and $b \in I_\tau$, we write $z := x + yi$, $\tau := u + iv$ and $b := m + n\tau$. Note that

$$
\begin{aligned}
|z + b|^2 &= |x + m + nu + i(y + nv)|^2 \\
&= (x + m + nu)^2 + (y + nv)^2 \\
&\geq (0 + m + nu)^2 + (-1/2 + nv)^2 \geq m^2 + (n - 1/2)^2.
\end{aligned}
$$

Let $M := 1/\delta$. By using the above inequality, there exists $N_\delta \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $x \in X$, if $m \geq N_\delta$ or $n \geq N_\delta$, then $|z + b| > M = 1/\delta$. In particular, $b \in I_\tau \setminus F_\tau(N_\delta)$ implies that for all $z \in X$, $|\phi_b(z)| < \delta$. Here, $F_\tau(N_\delta) := \{b := m + n\tau \in I_\tau \mid n \leq N_\delta, m \leq N_\delta\}$.

By the inequality (*) and $|F_\tau(N_\delta)| < \infty$, this contradicts that there exist $b \in I'_\tau \setminus (F'_\tau \cup F_\tau(N_\delta))$ and $z'_b \in \phi_b(X)$ such that $|z'_b| > \delta$. Therefore, we have proved that for all $\tau \in A_0$, $X_\tau(\infty) = \{0\}$. $\qquad\square$

Corollary 5.4. Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, we have that for all $\tau \in A_0$, $\dim_{\mathcal{H}}(\overline{J_\tau}) = h_\tau$.

Proof. By Theorem 5.3, we obtain that $\overline{J_\tau} \setminus J_\tau$ is at most countable. Note that if A is at most countable, then $\dim_{\mathcal{H}} A = 0$. Thus,

$$\dim_{\mathcal{H}}(\overline{J_\tau}) = \max\{\dim_{\mathcal{H}} J_\tau, \dim_{\mathcal{H}}(\overline{J_\tau} \setminus J_\tau)\} = \max\{\dim_{\mathcal{H}} J_\tau, 0\} = h_\tau$$

Therefore, we have proved Corollary 5.4. $\qquad\square$

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