Dynamics of postcritically bounded polynomial semigroups

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Abstract

We investigate the dynamics of semigroups generated by polynomial maps on the Riemann sphere such that the postcritical set in the complex plane is bounded. Moreover, we investigate the associated random dynamics of polynomials. We show that for such a polynomial semigroup, if A and B are two connected components of the Julia set, then one of A and B surrounds the other. A criterion for the Julia set to be connected is given. Moreover, we show that for any $n \in \mathbb{N} \cup \{\aleph_0\}$, there exists a finitely generated polynomial semigroup with bounded planar postcritical set such that the cardinality of the set of all connected components of the Julia set is equal to n. Furthermore, we investigate the fiberwise dynamics of skew products related to polynomial semigroups with bounded planar postcritical set. Using uniform fiberwise quasiconformal surgery on a fiber bundle, we show that if the Julia set of such a semigroup is disconnected, then there exist families of uncountably many mutually disjoint quasicircles with uniform dilatation which are parameterized by the Cantor set, densely inside the Julia set of the semigroup. Moreover, we show that under a certain condition, a random Julia set is almost surely a Jordan curve, but not a quasicircle. Furthermore, we give a classification of polynomial semigroups G such that G is generated by a compact family, the planar postcritical set of G is bounded, and G is (semi-) hyperbolic. Many new phenomena of polynomial semigroups and random dynamics of polynomials that do not occur in the usual dynamics of polynomials are found and systematically investigated.

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Contents

1 Introduction

The theory of complex dynamical systems, which has its origin in the important work of Fatou and Julia in the 1910s, has been investigated by many people and discussed in depth. In particular, since D. Sullivan showed the famous "no wandering domain theorem" using Teichmüller theory in the 1980s, this subject has attracted many researchers from a wide area. For a general reference on complex dynamical systems, see Milnor's textbook [19].

There are several areas in which we deal with generalized notions of classical iteration theory of rational functions. One of them is the theory of dynamics of rational semigroups (semigroups generated by holomorphic maps on the Riemann sphere $\hat{\mathbb{C}}$), and another one is the theory of random dynamics of holomorphic maps on the Riemann sphere.

In this paper, we will discuss these subjects. A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition ([14]). A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([14, 15]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group([45, 13]), who studied such semigroups from the perspective of random dynamical systems. Moreover, the research on rational semigroups is related to that on "iterated function systems" in fractal geometry. In fact, the Julia set of a rational semigroup generated by a compact family has " backward self-similarity" (cf. Lemma 3.1-2). For other research on rational semigroups, see [25, 26, 27, 44, 28, 29, 42, 43], and [32]–[41].

The research on the dynamics of rational semigroups is also directly related to that on the random dynamics of holomorphic maps. The first study in this direction was by Fornaess and Sibony ([11]), and much research has followed. (See [3, 5, 6, 4, 12].)

We remark that the complex dynamical systems can be used to describe some mathematical models. For example, the behavior of the population of a certain species can be described as the dynamical system of a polynomial $f(z) = az(1-z)$ such that f preserves the unit interval and the postcritical set in the plane is bounded (cf. [9]). From this point of view, it is very important to consider the random dynamics of such polynomials (see also Example 1.4). For the random dynamics of polynomials on the unit interval, see [31].

We shall give some definitions for the dynamics of rational semigroups:

Definition 1.1 ([14, 13]). Let G be a rational semigroup. We set

$$F(G) = \{z \in \hat{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\},\ J(G) = \hat{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** of G and $J(G)$ is called the **Julia set** of G. We let $\langle h_1, h_2, \ldots \rangle$ denote the rational semigroup generated by the family $\{h_i\}$. The Julia set of the semigroup generated by a single map g is denoted by $J(g)$.

Definition 1.2.

1. For each rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we set $CV(g) :=$ {all critical values of $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$}. Moreover, for each polynomial map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we set $CV^*(g) := CV(g) \setminus \{\infty\}$.

2. Let G be a rational semigroup. We set

$$P(G) := \overline{\bigcup_{g \in G} CV(g)}\ (\subset \hat{\mathbb{C}}).$$

This is called the **postcritical set** of G. Furthermore, for a polynomial semigroup G, we set $P^*(G) := P(G) \setminus \{\infty\}$. This is called the **planar postcritical set** (or **finite postcritical set**) of G. We say that a polynomial semigroup G is **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$.

Remark 1.3. Let G be a rational semigroup generated by a family Λ of rational maps. Then, we have that $P(G) = \overline{\cup_{g \in G \cup \{Id\}} g(\cup_{h \in \Lambda} CV(h))}$, where Id denotes the identity map on $\hat{\mathbb{C}}$, and that $g(P(G)) \subset P(G)$ for each $g \in G$. From this formula, one can figure out how the set $P(G)$ (resp. $P^*(G)$) spreads in $\hat{\mathbb{C}}$ (resp. $\mathbb{C}$). In fact, in Section 2.8, using the above formula, we present a way to construct examples of postcritically bounded polynomial semigroups (with some additional properties). Moreover, from the above formula, one may, in the finitely generated case, use a computer to see if a polynomial semigroup G is postcritically bounded much in the same way as one verifies the boundedness of the critical orbit for the maps $f_c(z) = z^2 + c$.

Example 1.4. Let $\Lambda := \{h(z) = cz^a(1-z)^b \mid a, b \in \mathbb{N},\ c > 0,\ c(\frac{a}{a+b})^a(\frac{b}{a+b})^b \leq 1\}$ and let G be the polynomial semigroup generated by Λ. Since for each $h \in \Lambda$, $h([0,1]) \subset [0,1]$ and $CV^*(h) \subset [0,1]$, it follows that each subsemigroup H of G is postcritically bounded.

Remark 1.5. It is well-known that for a polynomial g with $\deg(g) \geq 2$, $P^*(\langle g \rangle)$ is bounded in $\mathbb{C}$ if and only if $J(g)$ is connected ([19, Theorem 9.5]).

As mentioned in Remark 1.5, the planar postcritical set is one piece of important information regarding the dynamics of polynomials. Concerning the theory of iteration of quadratic polynomials, we have been investigating the famous "Mandelbrot set".

When investigating the dynamics of polynomial semigroups, it is natural for us to discuss the relationship between the planar postcritical set and the figure of the Julia set. The first question in this regard is:

Question 1.6. Let G be a polynomial semigroup such that each element $g \in G$ is of degree at least two. Is $J(G)$ necessarily connected when $P^*(G)$ is bounded in $\mathbb{C}$?

The answer is **NO.**

Example 1.7 ([44])**.** Let $G = \langle z^3, \frac{z^2}{4} \rangle$. Then $P^*(G) = \{0\}$ (which is bounded in $\mathbb{C}$) and $J(G)$ is disconnected ($J(G)$ is a Cantor set of round circles). Furthermore, according to [36, Theorem 2.4.1], it can be shown that a small perturbation H of G still satisfies that $P^*(H)$ is bounded in $\mathbb{C}$ and that $J(H)$ is disconnected. ($J(H)$ is a Cantor set of quasi-circles with uniform dilatation.)

Question 1.8. What happens if $P^*(G)$ is bounded in $\mathbb{C}$ and $J(G)$ is disconnected?

Problem 1.9. Classify postcritically bounded polynomial semigroups.

In this paper, we show that if G is a postcritically bounded polynomial semigroup with disconnected Julia set, then $\infty \in F(G)$ (cf. Theorem 2.19-1), and for any two connected components of $J(G)$, one of them surrounds the other. This implies that there exists an intrinsic total order " $\leq$ " (called the "surrounding order") in the space $\mathcal{J}_G$ of connected components of $J(G)$, and that every connected component of $F(G)$ is either simply or doubly connected (cf. Theorem 2.7). Moreover, for such a semigroup G, we show that the interior of "the smallest filled-in Julia set" $\hat{K}(G)$ is not empty, and that there exists a maximal element and a minimal element in the space $\mathcal{J}_G$ endowed with the order $\leq$ (cf. Theorem 2.19). From these results, we obtain the result that for a postcritically bounded polynomial semigroup G, the Julia set $J(G)$ is uniformly perfect, even if G is not generated by a compact family of polynomials (cf. Theorem 2.21).

Moreover, we utilize Green's functions with pole at infinity to show that for a postcritically bounded polynomial semigroup G, the cardinality of the set of all connected components of $J(G)$ is less than or equal to that of

$J(H)$, where H is the "real affine semigroup" associated with G (cf. Theorem 2.12). From this result, we obtain a sufficient condition for the Julia set of a postcritically bounded polynomial semigroup to be connected (cf. Theorem 2.14). In particular, we show that if a postcritically bounded polynomial semigroup G is generated by a family of quadratic polynomials, then $J(G)$ is connected (cf. Theorem 2.15). The proofs of the results in this and the previous paragraphs are not straightforward. In fact, we first prove (1) that for any two connected components of $J(G)$ that are included in $\mathbb{C}$, one of them surrounds the other; next, using (1) and the theory of Green's functions, we prove (2) that the cardinality of the set of all connected components of $J(G)$ is less than or equal to that of $J(H)$, where H is the associated real affine semigroup; and finally, using (2) and (1), we prove (3) that $\infty \in F(G)$, $\text{int}(\hat{K}(G)) \neq \emptyset$, and other results in the previous paragraph.

Moreover, we show that for any $n \in \mathbb{N} \cup \{\aleph_0\}$, there exists a finitely generated, postcritically bounded, polynomial semigroup G such that the cardinality of the set of all connected components of $J(G)$ is equal to n (cf. Proposition 2.25, Proposition 2.27 and Proposition 2.28). A sufficient condition for the cardinality of the set of all connected components of a Julia set to be equal to $\aleph_0$ is also given (cf. Theorem 2.26). To obtain these results, we use the fact that the map induced by any element of a semigroup on the space of connected components of the Julia set preserves the order $\leq$ (cf. Theorem 2.7). Note that this is in contrast to the dynamics of a single rational map h or a non-elementary Kleinian group, where it is known that either the Julia set is connected, or the Julia set has uncountably many connected components.

Applying the previous results, we investigate the dynamics of every sequence, or fiberwise dynamics of the skew product associated with the generator system (cf. Section 2.5). Moreover, we investigate the random dynamics of polynomials acting on the Riemann sphere. Let us consider a polynomial semigroup G generated by a compact family Γ of polynomials. For each sequence $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots) \in \Gamma^{\mathbb{N}}$, we examine the dynamics along the sequence γ, that is, the dynamics of the family of maps $\{\gamma_n \circ \cdots \circ \gamma_1\}_{n=1}^{\infty}$. We note that this corresponds to the fiberwise dynamics of the skew product (see Section 2.5) associated with the generator system Γ. We show that if G is postcritically bounded, $J(G)$ is disconnected, and G is generated by a compact family Γ of polynomials; then, for almost every sequence $\gamma \in \Gamma^{\mathbb{N}}$, there exists exactly one bounded component U_γ of the Fatou set of γ, the Julia set of γ has Lebesgue measure zero, there exists no non-constant limit function in U_γ for the sequence γ, and for any point $z \in U_\gamma$, the orbit along γ tends to the interior of the smallest filled-in Julia set $\hat{K}(G)$ of G (cf. Theorem 2.40-

2, Corollary 2.51). Moreover, using the uniform fiberwise quasiconformal surgery (cf. Theorem 4.21), we find sub-skew products $\overline{f}$ such that $\overline{f}$ is hyperbolic and such that every fiberwise Julia set of $\overline{f}$ is a K-quasicircle, where K is a constant not depending on the fibers (cf. Theorem 2.40-3). Reusing the uniform fiberwise quasiconformal surgery, we show that if G is a postcritically bounded polynomial semigroup with disconnected Julia set, then for any non-empty open subset V of $J(G)$, there exists a 2-generator subsemigroup H of G such that $J(H)$ is the disjoint union of "Cantor family of quasicircles" (a family of quasicircles parameterized by a Cantor set) with uniform distortion, and such that $J(H) \cap V \neq \emptyset$ (cf. Theorem 2.45). Note that the uniform fiberwise quasiconformal surgery is based on solving uncountably many Beltrami equations (a kind of partial differential equations).

We also investigate (semi-)hyperbolic, postcritically bounded, polynomial semigroups generated by a compact family Γ of polynomials. We show that if G is such a semigroup with disconnected Julia set, and if there exists an element $g \in G$ such that $J(g)$ is not a Jordan curve, then, for almost every sequence $\gamma \in \Gamma^{\mathbb{N}}$, the Julia set of γ is a Jordan curve but not a quasicircle, the basin of infinity A_γ is a John domain, and the bounded component U_γ of the Fatou set is not a John domain (cf. Theorem 2.48). Moreover, we classify the semi-hyperbolic, postcritically bounded, polynomial semigroups generated by a compact family Γ of polynomials. We show that such a semigroup G satisfies either (I) every fiberwise Julia set is a quasicircle with uniform distortion, or (II) for almost every sequence $\gamma \in \Gamma^{\mathbb{N}}$, the Julia set J_γ is a Jordan curve but not a quasicircle, the basin of infinity A_γ is a John domain, and the bounded component U_γ of the Fatou set is not a John domain, or (III) for every $\alpha, \beta \in \Gamma^{\mathbb{N}}$, the intersection of the Julia sets J_α and J_β is not empty, and $J(G)$ is arcwise connected (cf. Theorem 2.52). Furthermore, we also classify the hyperbolic, postcritically bounded, polynomial semigroups generated by a compact family Γ of polynomials. We show that such a semigroup G satisfies either (I) above, or (II) above, or (III)' for every $\alpha, \beta \in \Gamma^{\mathbb{N}}$, the intersection of the Julia sets J_α and J_β is not empty, $J(G)$ is arcwise connected, and for every sequence $\gamma \in \Gamma^{\mathbb{N}}$, there exist infinitely many bounded components of F_γ (cf. Theorem 2.54). We give some examples of situation (II) above (cf. Example 2.49, Example 2.55 and Section 2.8). Note that situation (II) above is a special and new phenomenon of random dynamics of polynomials that does not occur in the usual dynamics of polynomials.

The key to investigating the dynamics of postcritically bounded polynomial semigroups is the density of repelling fixed points in the Julia set (cf. Theorem 3.2), which can be shown by an application of the Ahlfors five island theorem, and the lower semi-continuity of $\gamma \mapsto J_\gamma$ (Lemma 3.4-2), which is a consequence of potential theory. Moreover, one of the keys to investigating the fiberwise dynamics of skew products is, the observation of non-constant limit functions (cf. Lemma 4.17 and [32]). The key to investigating the dynamics of semi-hyperbolic polynomial semigroups is, the continuity of the map $\gamma \mapsto J_\gamma$ (this is highly nontrivial; see [32]) and the Johnness of the basin A_γ of infinity (cf. [35]). Note that the continuity of the map $\gamma \mapsto J_\gamma$ does not hold in general, if we do not assume semi-hyperbolicity. Moreover, one of the original aspects of this paper is the idea of "combining both the theory of rational semigroups and that of random complex dynamics". It is quite natural to investigate both fields simultaneously. However, no study thus far has done so.

Furthermore, in Section 2.8 and Section 2.4, we provide a way of constructing examples of postcritically bounded polynomial semigroups with some additional properties (disconnectedness of Julia set, semi-hyperbolicity, hyperbolicity, etc.) (cf. Proposition 2.58, Theorem 2.61, Theorem 2.63). For example, by Proposition 2.58, there exists a 2-generator postcritically bounded polynomial semigroup $G = \langle h_1, h_2 \rangle$ with disconnected Julia set such that h_1 has a Siegel disk.

As wee see in Example 1.4 and Section 2.8, it is not difficult to construct many examples, it is not difficult to verify the hypothesis "postcritically bounded", and the class of postcritically bounded polynomial semigroups is very wide.

Throughout the paper, we will see many new phenomena in polynomial semigroups or random dynamics of polynomials that do not occur in the usual dynamics of polynomials. Moreover, these new phenomena are systematically investigated.

In Section 2, we present the main results of this paper. We give some tools in Section 3. The proofs of the main results are given in Section 4.

There are many applications of the results of postcritically bounded polynomial semigroups in many directions. In the sequel [39], we will investigate Markov process on $\hat{\mathbb{C}}$ associated with the random dynamics of polynomials and we will consider the probability $T_\infty(z)$ of tending to $\infty \in \hat{\mathbb{C}}$ starting with the initial value $z \in \hat{\mathbb{C}}$. Applying many results of this paper, it will be shown in [39] that if the associated polynomial semigroup G is postcritically bounded and the Julia set is disconnected, then the function T_∞ defined on $\hat{\mathbb{C}}$ has many interesting properties which are similar to those of the Cantor function. Such a kind of "singular functions in the complex plane" appear very naturally in random dynamics of polynomials and the results of this paper (for example, the results on the space of all connected components of a Julia set) are the keys to investigating that. (The above results have been announced in [40, 41].)

Moreover, as illustrated before, it is very important for us to recall that the complex dynamics can be applied to describe some mathematical models. For example, the behavior of the population of a certain species can be described as the dynamical systems of a polynomial h such that h preserves the unit interval and the postcritical set in the plane is bounded. When one considers such a model, it is very natural to consider the random dynamics of polynomial with bounded postcritical set in the plane (see Example 1.4).

In the sequel [29], we will give some further results on postcritically bounded polynomial semigroups, based on this paper. Moreover, in the sequel [38], we will define a new kind of cohomology theory, in order to investigate the action of finitely generated semigroups, and we will apply it to the study of the dynamics of postcritically bounded polynomial semigroups.

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2 Main results

In this section we present the statements of the main results. Throughout this paper, we deal with semigroups G that might not be generated by a compact family of polynomials. The proofs are given in Section 4.

2.1 Space of connected components of a Julia set, surrounding order

We present some results concerning the connected components of the Julia set of a postcritically bounded polynomial semigroup. The proofs are given in Section 4.1.

Theorem 2.1. *Let G be a rational semigroup generated by a family $\{h_\lambda\}_{\lambda\in\Lambda}$. Suppose that there exists a connected component A of $J(G)$ such that $\sharp A > 1$ and $\cup_{\lambda\in\Lambda} J(h_\lambda) \subset A$. Moreover, suppose that for any $\lambda \in \Lambda$ such that h_λ is a Möbius transformation of finite order, we have $h_\lambda^{-1}(A) \subset A$. Then, $J(G)$ is connected.*

Definition 2.2. We set $\text{Rat} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the topology induced by uniform convergence on $\hat{\mathbb{C}}$ with respect to the spherical distance. We set $\text{Poly} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant polynomial}\}$ endowed with the relative topology from Rat. Moreover, we set $\text{Poly}_{\deg\geq 2} := \{g \in \text{Poly} \mid \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Remark 2.3. Let $d \geq 1$, $\{p_n\}_{n\in\mathbb{N}}$ a sequence of polynomials of degree d, and p a polynomial. Then, $p_n \to p$ in Poly if and only if the coefficients converge appropriately and p is of degree d.

Definition 2.4. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- each element of G is of degree at least two, and
- $P^*(G)$ is bounded in $\mathbb{C}$, i.e., G is postcritically bounded.

Furthermore, we set $\mathcal{G}_{con} = \{G \in \mathcal{G} \mid J(G) \text{ is connected}\}$ and $\mathcal{G}_{dis} = \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$.

Notation: For a polynomial semigroup G, we denote by $\mathcal{J} = \mathcal{J}_G$ the set of all connected components J of $J(G)$ such that $J \subset \mathbb{C}$. Moreover, we denote by $\hat{\mathcal{J}} = \hat{\mathcal{J}}_G$ the set of all connected components of $J(G)$.

Remark 2.5. If a polynomial semigroup G is generated by a compact set in $\text{Poly}_{\deg\geq 2}$, then $\infty \in F(G)$ and thus $\mathcal{J} = \hat{\mathcal{J}}$.

Definition 2.6. For any connected sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" indicates that $K_1 = K_2$, or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Furthermore, "$K_1 < K_2$" indicates $K_1 \leq K_2$ and $K_1 \neq K_2$. Note that "$\leq$" is a partial order in the space of all non-empty compact connected sets in $\mathbb{C}$. This "$\leq$" is called the **surrounding order.**

Theorem 2.7. *Let $G \in \mathcal{G}$ (possibly generated by a non-compact family). Then we have all of the following.*

1. *$(\mathcal{J}, \leq)$ is totally ordered.*
2. *Each connected component of $F(G)$ is either simply or doubly connected.*
3. *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

For the figures of the Julia sets of semigroups $G \in \mathcal{G}_{dis}$, see figure 1 and figure 2.

2.2 Upper estimates of $\sharp(\hat{\mathcal{J}})$

Next, we present some results on the space $\hat{\mathcal{J}}$ and some results on upper estimates of $\sharp(\hat{\mathcal{J}})$. The proofs are given in Section 4.2 and Section 4.3.

Definition 2.8.

1. For a polynomial g, we denote by $a(g) \in \mathbb{C}$ the coefficient of the highest degree term of g.

2. We set $\mathrm{RA} := \{ax + b \in \mathbb{R}[x] \mid a, b \in \mathbb{R}, a \neq 0\}$ endowed with the topology such that, $a_n x + b_n \to ax + b$ if and only if $a_n \to a$ and $b_n \to b$. The space RA is a semigroup with the semigroup operation being functional composition. Any subsemigroup of RA will be called a *real affine semigroup*. We define a map $\Psi : \mathrm{Poly} \to \mathrm{RA}$ as follows: For a polynomial $g \in \mathrm{Poly}$, we set $\Psi(g)(x) := \deg(g)x + \log|a(g)|$.

   Moreover, for a polynomial semigroup G, we set $\Psi(G) := \{\Psi(g) \mid g \in G\}$ $(\subset RA)$.

3. We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ endowed with the topology such that $\{(r, +\infty]\}_{r\in\mathbb{R}}$ makes a fundamental neighborhood system of $+\infty$, and such that $\{[-\infty, r)\}_{r\in\mathbb{R}}$ makes a fundamental neighborhood system of $-\infty$. For a real affine semigroup H, we set

   $$M(H) := \overline{\{x \in \mathbb{R} \mid \exists h \in H, h(x) = x, |h'(x)| > 1\}} \ (\subset \hat{\mathbb{R}}),$$

   where the closure is taken in the space $\hat{\mathbb{R}}$. Moreover, we denote by $\mathcal{M}_H$ the set of all connected components of $M(H)$.

4. We denote by $\eta : \mathrm{RA} \to \mathrm{Poly}$ the natural embedding defined by $\eta(x \mapsto ax + b) = (z \mapsto az + b)$, where $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

5. We define a map $\Theta : \mathrm{Poly} \to \mathrm{Poly}$ as follows. For a polynomial g, we set $\Theta(g)(z) = a(g)z^{\deg(g)}$. Moreover, for a polynomial semigroup G, we set $\Theta(G) := \{\Theta(g) \mid g \in G\}$.

Remark 2.9.

1. The map $\Psi : \mathrm{Poly} \to \mathrm{RA}$ is a semigroup homomorphism. That is, we have $\Psi(g \circ h) = \Psi(g) \circ \Psi(h)$. Hence, for a polynomial semigroup G, the image $\Psi(G)$ is a real affine semigroup. Similarly, the map $\Theta : \mathrm{Poly} \to \mathrm{Poly}$ is a semigroup homomorphism. Hence, for a polynomial semigroup G, the image $\Theta(G)$ is a polynomial semigroup.

2. The maps $\Psi : \text{Poly} \to \text{RA}$, $\eta : \text{RA} \to \text{Poly}$, and $\Theta : \text{Poly} \to \text{Poly}$ are continuous.

Definition 2.10. For any connected sets M_1 and M_2 in $\hat{\mathbb{R}}$, "$M_1 \leq_r M_2$" indicates that $M_1 = M_2$, or each $(x, y) \in M_1 \times M_2$ satisfies $x < y$. Furthermore, "$M_1 <_r M_2$" indicates $M_1 \leq_r M_2$ and $M_1 \neq M_2$.

Remark 2.11. The above "$\leq_r$" is a partial order in the space of non-empty connected subsets of $\hat{\mathbb{R}}$. Moreover, for each real affine semigroup H, $(\mathcal{M}_H, \leq_r)$ is totally ordered.

Theorem 2.12.

1. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, we have $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\mathcal{M}_{\Psi(G)})$. More precisely, there exists an injective map $\tilde{\Psi} : \hat{\mathcal{J}}_G \to \mathcal{M}_{\Psi(G)}$ such that if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Psi}(J_1) <_r \tilde{\Psi}(J_2)$.*

2. *If $G \in \mathcal{G}_{dis}$, then we have that $M(\Psi(G)) \subset \mathbb{R}$ and $M(\Psi(G)) = J(\eta(\Psi(G)))$.*

3. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\hat{\mathcal{J}}_{\eta(\Psi(G))})$.*

Corollary 2.13. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, we have $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\hat{\mathcal{J}}_{\Theta(G)})$. More precisely, there exists an injective map $\tilde{\Theta} : \hat{\mathcal{J}}_G \to \hat{\mathcal{J}}_{\Theta(G)}$ such that if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Theta}(J_1) \in \mathcal{J}_{\Theta(G)}$, $\tilde{\Theta}(J_2) \in \mathcal{J}_{\Theta(G)}$, and $\tilde{\Theta}(J_1) < \tilde{\Theta}(J_2)$.*

Theorem 2.14. *Let $G = \langle h_1, \ldots, h_m \rangle$ be a finitely generated polynomial semigroup in $\mathcal{G}$. For each $j = 1, \ldots, m$, let a_j be the coefficient of the highest degree term of polynomial h_j. Let $\alpha := \min_{j=1,\ldots,m}\{\frac{-1}{\deg(h_j)-1} \log |a_j|\}$ and $\beta := \max_{j=1,\ldots,m}\{\frac{-1}{\deg(h_j)-1} \log |a_j|\}$. We set $[\alpha, \beta] := \{x \in \mathbb{R} \mid \alpha \leq x \leq \beta\}$. If $[\alpha, \beta] \subset \cup_{j=1}^m \Psi(h_j)^{-1}([\alpha, \beta])$, then $J(G)$ is connected.*

Theorem 2.15. *Let G be a polynomial semigroup in $\mathcal{G}$ generated by a (possibly non-compact) family of polynomials of degree two. Then, $J(G)$ is connected.*

Theorem 2.16. *Let G be a polynomial semigroup in $\mathcal{G}$ generated by a (possibly non-compact) family $\{h_\lambda\}_{\lambda \in \Lambda}$ of polynomials. Let a_λ be the coefficient of the highest degree term of the polynomial h_λ. Suppose that for any $\lambda, \xi \in \Lambda$, we have $(\deg(h_\xi) - 1) \log |a_\lambda| = (\deg(h_\lambda) - 1) \log |a_\xi|$. Then, $J(G)$ is connected.*

2.3 Properties of $\mathcal{J}$

In this section, we present some results on $\mathcal{J}$. The proofs are given in Section 4.3.

Definition 2.17. For a polynomial semigroup G, we set

$$\hat{K}(G) := \{z \in \mathbb{C} \mid \bigcup_{g \in G} \{g(z)\} \text{ is bounded in } \mathbb{C}\}$$

and call $\hat{K}(G)$ the **smallest filled-in Julia set** of G. For a polynomial g, we set $K(g) := \hat{K}(\langle g \rangle)$.

Notation: For a set $A \subset \hat{\mathbb{C}}$, we denote by $\text{int}(A)$ the set of all interior points of A.

Proposition 2.18. *Let $G \in \mathcal{G}$. If U is a connected component of $F(G)$ such that $U \cap \hat{K}(G) \neq \emptyset$, then $U \subset \text{int}(\hat{K}(G))$ and U is simply connected. Furthermore, we have $\hat{K}(G) \cap F(G) = \text{int}(\hat{K}(G))$.*

Notation: For a polynomial semigroup G with $\infty \in F(G)$, we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. Moreover, for a polynomial g with $\deg(g) \geq 2$, we set $F_\infty(g) := F_\infty(\langle g \rangle)$.

The following theorem is the key to obtaining further results of post-critically bounded polynomial semigroups and related random dynamics of polynomials.

Theorem 2.19. *Let $G \in \mathcal{G}_{dis}$ (possibly generated by a non-compact family). Then, under the above notation, we have the following.*

1. *We have that $\infty \in F(G)$ and the connected component $F_\infty(G)$ of $F(G)$ containing ∞ is simply connected. Furthermore, the element $J_{\max} = J_{\max}(G) \in \mathcal{J}$ containing $\partial F_\infty(G)$ is the unique element of $\mathcal{J}$ satisfying that $J \leq J_{\max}$ for each $J \in \mathcal{J}$.*

2. *There exists a unique element $J_{\min} = J_{\min}(G) \in \mathcal{J}$ such that $J_{\min} \leq J$ for each element $J \in \mathcal{J}$. Furthermore, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Then, $P^*(G) \subset \hat{K}(G) \subset \mathbb{C} \setminus D$ and $\partial \hat{K}(G) \subset J_{\min}$.*

3. *If G is generated by a family $\{h_\lambda\}_{\lambda \in \Lambda}$, then there exist two elements λ_1 and λ_2 of Λ satisfying:*

   (a) *there exist two elements J_1 and J_2 of $\mathcal{J}$ with $J_1 \neq J_2$ such that $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$;*

(b) $J(h_{\lambda_1}) \cap J_{\min} = \emptyset$;

(c) for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ and $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$; and

(d) h_{λ_1} has an attracting fixed point z_1 in $\mathbb{C}$, $\text{int}(K(h_{\lambda_1}))$ consists of only one immediate attracting basin for z_1, and $K(h_{\lambda_2}) \subset \text{int}(K(h_{\lambda_1}))$. Furthermore, $z_1 \in \text{int}(K(h_{\lambda_2}))$.

4. For each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have that g has an attracting fixed point z_g in $\mathbb{C}$, $\text{int}(K(g))$ consists of only one immediate attracting basin for z_g, and $J_{\min} \subset \text{int}(K(g))$. Note that it is not necessarily true that $z_g = z_f$ when $g, f \in G$ are such that $J(g) \cap J_{\min} = \emptyset$ and $J(f) \cap J_{\min} = \emptyset$ (see Proposition 2.25).

5. We have that $\text{int}(\hat{K}(G)) \neq \emptyset$. Moreover,

   (a) $\mathbb{C} \setminus J_{\min}$ is disconnected, $\sharp J \geq 2$ for each $J \in \hat{\mathcal{J}}$, and

   (b) for each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have that $J_{\min} < g^*(J_{\min})$, $g^{-1}(J(G)) \cap J_{\min} = \emptyset$, $g(\hat{K}(G) \cup J_{\min}) \subset \text{int}(\hat{K}(G))$, and the unique attracting fixed point z_g of g in $\mathbb{C}$ belongs to $\text{int}(\hat{K}(G))$.

6. Let $\mathcal{A}$ be the set of all doubly connected components of $F(G)$. Then, $\cup_{A \in \mathcal{A}} A \subset \mathbb{C}$ and $(\mathcal{A}, \leq)$ is totally ordered.

We present a result on uniform perfectness of the Julia sets of semigroups in $\mathcal{G}$.

Definition 2.20. A compact set K in $\hat{\mathbb{C}}$ is said to be uniformly perfect if $\sharp K \geq 2$ and there exists a constant $C > 0$ such that each annulus A that separates K satisfies that mod $A < C$, where mod A denotes the modulus of A (See the definition in [17]).

Theorem 2.21.

1. Let G be a polynomial semigroup in $\mathcal{G}$. Then, $J(G)$ is uniformly perfect. Moreover, if $z_0 \in J(G)$ is a superattracting fixed point of an element of G, then $z_0 \in \text{int}(J(G))$.

2. If $G \in \mathcal{G}$ and $\infty \in J(G)$, then $G \in \mathcal{G}_{con}$ and $\infty \in \text{int}(J(G))$.

3. Suppose that $G \in \mathcal{G}_{dis}$. Let $z_1 \in J(G) \cap \mathbb{C}$ be a superattracting fixed point of $g \in G$. Then $z_1 \in \text{int}(J_{\min})$ and $J(g) \subset J_{\min}$.

We remark that in [15], it was shown that there exists a rational semigroup G such that $J(G)$ is not uniformly perfect.

We now present results on the Julia sets of subsemigroups of an element of $\mathcal{G}_{dis}$.

Proposition 2.22. *Let $G \in \mathcal{G}_{dis}$ and let $J_1, J_2 \in \mathcal{J} = \mathcal{J}_G$ with $J_1 \leq J_2$. Let A_i be the unbounded component of $\mathbb{C} \setminus J_i$ for each $i = 1, 2$. Then, we have the following.*

1. *Let $Q_1 = \{g \in G \mid \exists J \in \mathcal{J} \text{ with } J_1 \leq J,\ J(g) \subset J\}$ and let H_1 be the subsemigroup of G generated by Q_1. Then $J(H_1) \subset J_1 \cup A_1$.*

2. *Let $Q_2 = \{g \in G \mid \exists J \in \mathcal{J} \text{ with } J \leq J_2,\ J(g) \subset J\}$ and let H_2 be the subsemigroup of G generated by Q_2. Then $J(H_2) \subset \mathbb{C} \setminus A_2$.*

3. *Let $Q = \{g \in G \mid \exists J \in \mathcal{J} \text{ with } J_1 \leq J \leq J_2,\ J(g) \subset J\}$ and let H be the subsemigroup of G generated by Q. Then $J(H) \subset J_1 \cup (A_1 \setminus A_2)$.*

Proposition 2.23. *Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg \geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. Then, there exists an element $h_1 \in \Gamma$ with $J(h_1) \subset J_{\max}$ and there exists an element $h_2 \in \Gamma$ with $J(h_2) \subset J_{\min}$.*

2.4 Finitely generated polynomial semigroups $G \in \mathcal{G}_{dis}$ such that $2 \leq \sharp(\hat{\mathcal{J}}_G) \leq \aleph_0$

In this section, we present some results on various finitely generated polynomial semigroups $G \in \mathcal{G}_{dis}$ such that $2 \leq \sharp(\hat{\mathcal{J}}_G) \leq \aleph_0$. The proofs are given in Section 4.4.

It is well-known that for a rational map g with $\deg(g) \geq 2$, if $J(g)$ is disconnected, then $J(g)$ has uncountably many connected components (See [19]). Moreover, if G is a non-elementary Kleinian group with disconnected Julia set (limit set), then $J(G)$ has uncountably many connected components. However, for general rational semigroups, we have the following examples.

Theorem 2.24. *Let G be a polynomial semigroup in $\mathcal{G}$ generated by a (possibly non-compact) family Γ in $\text{Poly}_{\deg \geq 2}$. Suppose that there exist mutually distinct elements $J_1, \ldots, J_n \in \hat{\mathcal{J}}_G$ such that for each $h \in \Gamma$ and each $j \in \{1, \ldots, n\}$, there exists an element $k \in \{1, \ldots, n\}$ with $h^{-1}(J_j) \cap J_k \neq \emptyset$. Then, we have $\sharp(\hat{\mathcal{J}}_G) = n$.*

Proposition 2.25. *For any $n \in \mathbb{N}$ with $n > 1$, there exists a finitely generated polynomial semigroup $G_n = \langle h_1, \ldots, h_{2n} \rangle$ in $\mathcal{G}$ satisfying $\sharp(\hat{\mathcal{J}}_{G_n}) = n$.*

In fact, let $0 < \epsilon < \frac{1}{2}$ *and we set for each* $j = 1, \ldots, n$, $a_j(z) := \frac{1}{j}z^2$ *and* $\beta_j(z) := \frac{1}{j}(z-\epsilon)^2 + \epsilon$. *Then, for any sufficiently large* $l \in \mathbb{N}$, *there exists an open neighborhood* V *of* $(\alpha_1^l, \ldots, \alpha_n^l, \beta_1^l, \ldots, \beta_n^l)$ *in* $(\text{Poly})^{2n}$ *such that for any* $(h_1, \ldots, h_{2n}) \in V$, *the semigroup* $G = \langle h_1, \ldots, h_{2n} \rangle$ *satisfies that* $G \in \mathcal{G}$ *and* $\sharp(\hat{\mathcal{J}}_G) = n$.

Theorem 2.26. *Let* $G = \langle h_1, \ldots, h_m \rangle \in \mathcal{G}_{dis}$ *be a polynomial semigroup with* $m \geq 3$. *Suppose that there exists an element* $J_0 \in \hat{\mathcal{J}}$ *such that* $\cup_{j=1}^{m-1} J(h_j) \subset J_0$, *and such that for each* $j = 1, \ldots, m-1$, *we have* $h_j^{-1}(J(h_m)) \cap J_0 \neq \emptyset$. *Then, we have all of the following.*

1. $\sharp(\hat{\mathcal{J}}) = \aleph_0$.
2. $J_0 = J_{\min}$, *or* $J_0 = J_{\max}$.
3. *If* $J_0 = J_{\min}$, *then* $J_{\max} = J(h_m)$, $J(G) = J_{\max} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} (h_m)^{-n}(J_{\min})$, *and for any* $J \in \hat{\mathcal{J}}$ *with* $J \neq J_{\max}$, *there exists no sequence* $\{C_j\}_{j \in \mathbb{N}}$ *of mutually distinct elements of* $\hat{\mathcal{J}}$ *such that* $\min_{z \in C_j} d(z, J) \to 0$ *as* $j \to \infty$.
4. *If* $J_0 = J_{\max}$, *then* $J_{\min} = J(h_m)$, $J(G) = J_{\min} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} (h_m)^{-n}(J_{\max})$, *and for any* $J \in \hat{\mathcal{J}}$ *with* $J \neq J_{\min}$, *there exists no sequence* $\{C_j\}_{j \in \mathbb{N}}$ *of mutually distinct elements of* $\hat{\mathcal{J}}$ *such that* $\min_{z \in C_j} d(z, J) \to 0$ *as* $j \to \infty$.

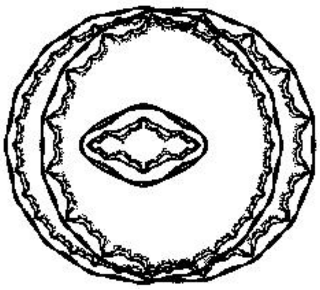
Proposition 2.27. *There exists an open set* V *in* $(\text{Poly}_{\deg \geq 2})^3$ *such that for any* $(h_1, h_2, h_3) \in V$, $G = \langle h_1, h_2, h_3 \rangle$ *satisfies that* $G \in \mathcal{G}_{dis}$, $\cup_{j=1}^2 J(h_j) \subset J_{\min}(G)$, $J_{\max}(G) = J(h_3)$, $h_j^{-1}(J(h_3)) \cap J_{\min}(G) \neq \emptyset$ *for each* $j = 1, 2$, *and* $\sharp(\hat{\mathcal{J}}_G) = \aleph_0$.

Proposition 2.28. *There exists a 3-generator polynomial semigroup* $G = \langle h_1, h_2, h_3 \rangle$ *in* $\mathcal{G}_{dis}$ *such that* $\cup_{j=1}^2 (h_j)^{-1}(J_{\max}(G)) \subset J_{\min}(G)$, $J_{\max}(G) = J(h_3)$, $\sharp(\hat{\mathcal{J}}_G) = \aleph_0$, *there exists a superattracting fixed point* z_0 *of some element of* G *with* $z_0 \in J(G)$, *and* $\text{int}(J_{\min}(G)) \neq \emptyset$.

As mentioned before, these results illustrate new phenomena which can hold in the rational semigroups, but cannot hold in the dynamics of a single rational map or Kleinian groups.

For the figure of the Julia set of a 3-generator polynomial semigroup $G \in \mathcal{G}_{dis}$ with $\sharp \hat{\mathcal{J}}_G = \aleph_0$, see figure 1.

Figure 1: The Julia set of a 3-generator polynomial semigroup $G \in \mathcal{G}_{dis}$ with $\sharp(\hat{\mathcal{J}}_G) = \aleph_0$.



2.5 Fiberwise dynamics and Julia sets

We present some results on the fiberwise dynamics of the skew product related to a postcritically bounded polynomial semigroup with disconnected Julia set. In particular, using the uniform fiberwise quasiconformal surgery on a fiber bundle, we show the existence of a family of quasicircles parameterized by a Cantor set with uniform distortion in the Julia set of such a semigroup. The proofs are given in Section 4.5.

Definition 2.29 ([32, 35]).

1. Let X be a compact metric space, $g : X \to X$ a continuous map, and $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ a continuous map. We say that f is a rational skew product (or fibered rational map on trivial bundle $X \times \hat{\mathbb{C}}$) over $g : X \to X$, if $\pi \circ f = g \circ \pi$ where $\pi : X \times \hat{\mathbb{C}} \to X$ denotes the canonical projection, and if for each $x \in X$, the restriction $f_x := f|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \to \pi^{-1}(\{g(x)\})$ of f is a non-constant rational map, under the canonical identification $\pi^{-1}(\{x'\}) \cong \hat{\mathbb{C}}$ for each $x' \in X$. Let $d(x) = \deg(f_x)$, for each $x \in X$. Let $f_{x,n}$ be the rational map defined by: $f_{x,n}(y) = \pi_{\hat{\mathbb{C}}}(f^n(x,y))$, for each $n \in \mathbb{N}, x \in X$ and $y \in \hat{\mathbb{C}}$, where $\pi_{\hat{\mathbb{C}}} : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the projection map.

   Moreover, if $f_{x,1}$ is a polynomial for each $x \in X$, then we say that $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ is a polynomial skew product over $g : X \to X$.

2. Let Γ be a compact subset of Rat. We set $\Gamma^{\mathbb{N}} := \{\gamma = (\gamma_1, \gamma_2, \ldots) \mid \forall j, \gamma_j \in \Gamma\}$ endowed with the product topology. This is a compact metric space. Let $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ be the shift map, which is defined by $\sigma(\gamma_1, \gamma_2, \ldots) := (\gamma_2, \gamma_3, \ldots)$. Moreover, we define a map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ by: $(\gamma, y) \mapsto (\sigma(\gamma), \gamma_1(y))$, where $\gamma = (\gamma_1, \gamma_2, \ldots)$. This is

called **the skew product associated with the family Γ of rational maps.** Note that $f_{\gamma,n}(y) = \gamma_n \circ \cdots \circ \gamma_1(y)$.

Remark 2.30. Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then, the function $x \mapsto d(x)$ is continuous in X.

Definition 2.31 ([32, 35])**.** Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then, we use the following notation.

1. For each $x \in X$ and $n \in \mathbb{N}$, we set $f_x^n := f^n|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \to \pi^{-1}(\{g^n(x)\}) \subset X \times \hat{\mathbb{C}}$.
2. For each $x \in X$, we denote by $F_x(f)$ the set of points $y \in \hat{\mathbb{C}}$ which has a neighborhood U in $\hat{\mathbb{C}}$ such that $\{f_{x,n} : U \to \hat{\mathbb{C}}\}_{n \in \mathbb{N}}$ is normal. Moreover, we set $F^x(f) := \{x\} \times F_x(f)$ $(\subset X \times \hat{\mathbb{C}})$.
3. For each $x \in X$, we set $J_x(f) := \hat{\mathbb{C}} \setminus F_x(f)$. Moreover, we set $J^x(f) := \{x\} \times J_x(f)$ $(\subset X \times \hat{\mathbb{C}})$. These sets $J^x(f)$ and $J_x(f)$ are called the fiberwise Julia sets.
4. We set $\tilde{J}(f) := \overline{\bigcup_{x \in X} J^x(f)}$, where the closure is taken in the product space $X \times \hat{\mathbb{C}}$.
5. For each $x \in X$, we set $\hat{J}^x(f) := \pi^{-1}(\{x\}) \cap \tilde{J}(f)$. Moreover, we set $\hat{J}_x(f) := \pi_{\hat{\mathbb{C}}}(\hat{J}^x(f))$.
6. We set $\tilde{F}(f) := (X \times \hat{\mathbb{C}}) \setminus \tilde{J}(f)$.

Remark 2.32. We have $\hat{J}^x(f) \supset J^x(f)$ and $\hat{J}_x(f) \supset J_x(f)$. However, strict containment can occur. For example, let h_1 be a polynomial having a Siegel disk with center $z_1 \in \mathbb{C}$. Let h_2 be a polynomial such that z_1 is a repelling fixed point of h_2. Let $\Gamma = \{h_1, h_2\}$. Let $f : \Gamma \times \hat{\mathbb{C}} \to \Gamma \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let $x = (h_1, h_1, h_1, \ldots) \in \Gamma^{\mathbb{N}}$. Then, $(x, z_1) \in \hat{J}^x(f) \setminus J^x(f)$ and $z_1 \in \hat{J}_x(f) \setminus J_x(f)$.

Definition 2.33. Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$. Then for each $x \in X$, we set $K_x(f) := \{y \in \hat{\mathbb{C}} \mid \{f_{x,n}(y)\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{C}\}$, and $A_x(f) := \{y \in \hat{\mathbb{C}} \mid f_{x,n}(y) \to \infty,\ n \to \infty\}$. Moreover, we set $K^x(f) := \{x\} \times K_x(f)$ $(\subset X \times \hat{\mathbb{C}})$ and $A^x(f) := \{x\} \times A_x(f)$ $(\subset X \times \hat{\mathbb{C}})$.

Definition 2.34. Let G be a polynomial semigroup generated by a subset Γ of $\text{Poly}_{\deg \geq 2}$. Suppose $G \in \mathcal{G}_{dis}$. Then we set

$$\Gamma_{\min} := \{h \in \Gamma \mid J(h) \subset J_{\min}\},$$

where $J_{\min}$ denotes the unique minimal element in $(\mathcal{J}, \leq)$ in Theorem 2.19-2. Furthermore, if $\Gamma_{\min} \neq \emptyset$, let $G_{\min,\Gamma}$ be the subsemigroup of G that is generated by $\Gamma_{\min}$.

Remark 2.35. Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose $G \in \mathcal{G}_{dis}$. Then, by Proposition 2.23, we have $\Gamma_{\min} \neq \emptyset$ and $\Gamma \setminus \Gamma_{\min} \neq \emptyset$. Moreover, $\Gamma_{\min}$ is a compact subset of Γ. For, if $\{h_n\}_{n\in\mathbb{N}} \subset \Gamma_{\min}$ and $h_n \to h_\infty$ in Γ, then for a repelling periodic point $z_0 \in J(h_\infty)$ of h_∞, we have that $d(z_0, J(h_n)) \to 0$ as $n \to \infty$, which implies that $z_0 \in J_{\min}$ and thus $h_\infty \in \Gamma_{\min}$.

Notation: Let $\mathcal{F} := \{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of meromorphic functions in a domain V. We say that a meromorphic function ψ is a limit function of $\mathcal{F}$ if there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of positive integers such that $\varphi_{n_j} \to \psi$ locally uniformly on V, as $j \to \infty$.

Definition 2.36. Let G be a rational semigroup.

1. We say that G is hyperbolic if $P(G) \subset F(G)$.

2. We say that G is semi-hyperbolic if there exists a number $\delta > 0$ and a number $N \in \mathbb{N}$ such that for each $y \in J(G)$ and each $g \in G$, we have $\deg(g : V \to B(y, \delta)) \leq N$ for each connected component V of $g^{-1}(B(y, \delta))$, where $B(y, \delta)$ denotes the ball of radius δ with center y with respect to the spherical distance, and $\deg(g : \cdot \to \cdot)$ denotes the degree of finite branched covering. (For background of semi-hyperbolicity, see [32] and [35].)

The following Proposition (2.37-1 and 2.37-2) means that for a polynomial semigroup $G \in \mathcal{G}_{dis}$ generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$, we rarely have the situation that "$\Gamma \setminus \Gamma_{\min}$ is not compact."

Proposition 2.37. *Let G be a polynomial semigroup generated by a compact subset Γ in* $\text{Poly}_{\deg\geq 2}$. *Suppose that $G \in \mathcal{G}_{dis}$ and that $\Gamma \setminus \Gamma_{\min}$ is not compact. Then, all of the following statements 1, 2, 3, and 4 hold.*

1. *Let $h \in \Gamma_{\min}$. Then, $J(h) = J_{\min}(G), K(h) = \hat{K}(G)$, and* $\text{int}(K(h))$ *is a non-empty connected set.*

2. *Either*

   (a) *for each $h \in \Gamma_{\min}$, h is hyperbolic and $J(h)$ is a quasicircle; or*

   (b) *for each $h \in \Gamma_{\min}$,* $\text{int}(K(h))$ *is an immediate parabolic basin of a parabolic fixed point of h.*

3. *For each* $\gamma \in \Gamma^{\mathbb{N}}$*, each limit function of* $\{f_{\gamma,n}\}_{n\in\mathbb{N}}$ *in each connected component of* $F_\gamma(f)$ *is constant.*

4. *Suppose that (a) in statement 2 holds. Then,* $G_{\min,\Gamma}$ *is hyperbolic and* G *is semi-hyperbolic.*

Definition 2.38. Let Γ and S be non-empty subsets of $\mathrm{Poly}_{\deg\geq 2}$ with $S \subset \Gamma$. We set

$$R(\Gamma, S) := \left\{\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}} \mid \sharp(\{n \in \mathbb{N} \mid \gamma_n \in S\}) = \infty\right\}.$$

Definition 2.39. Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. We set

$$C(f) := \{(x, y) \in X \times \hat{\mathbb{C}} \mid y \text{ is a critical point of } f_{x,1}\}.$$

Moreover, we set $P(f) := \overline{\cup_{n\in\mathbb{N}} f^n(C(f))}$, where the closure is taken in the product space $X \times \hat{\mathbb{C}}$. This $P(f)$ is called the **fiber-postcritical set** of f.

We say that f is hyperbolic (along fibers) if $P(f) \subset F(f)$.

We present a result which describes the details of the fiberwise dynamics along γ in $R(\Gamma, \Gamma \setminus \Gamma_{\min})$.

Theorem 2.40. *Let* G *be a polynomial semigroup generated by a compact subset* Γ *of* $\mathrm{Poly}_{\deg\geq 2}$*. Suppose* $G \in \mathcal{G}_{dis}$*. Let* $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ *be the skew product associated with the family* Γ *of polynomials. Then, all of the following statements 1,2, and 3 hold.*

1. *Let* $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$*. Then, each limit function of* $\{f_{\gamma,n}\}_{n\in\mathbb{N}}$ *in each connected component of* $F_\gamma(f)$ *is constant.*

2. *Let* S *be a non-empty compact subset of* $\Gamma \setminus \Gamma_{\min}$*. Then, for each* $\gamma \in R(\Gamma, S)$*, we have the following.*

   (a) *There exists exactly one bounded component* U_γ *of* $F_\gamma(f)$*. Furthermore,* $\partial U_\gamma = \partial A_\gamma(f) = J_\gamma(f)$*.*

   (b) *For each* $y \in U_\gamma$*, there exists a number* $n \in \mathbb{N}$ *such that* $f_{\gamma,n}(y) \in \mathrm{int}(\hat{K}(G))$*.*

   (c) $\hat{J}_\gamma(f) = J_\gamma(f)$*. Moreover, the map* $\omega \mapsto J_\omega(f)$ *defined on* $\Gamma^{\mathbb{N}}$ *is continuous at* γ*, with respect to the Hausdorff topology in the space of non-empty compact subsets of* $\hat{\mathbb{C}}$*.*

   (d) *The 2-dimensional Lebesgue measure of* $\hat{J}_\gamma(f) = J_\gamma(f)$ *is equal to zero.*

3. *Let S be a non-empty compact subset of $\Gamma \setminus \Gamma_{\min}$. For each $p \in \mathbb{N}$, we denote by $W_{S,p}$ the set of elements $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$ such that for each $l \in \mathbb{N}$, at least one of $\gamma_{l+1}, \ldots, \gamma_{l+p}$ belongs to S. Let $\overline{f} := f|_{W_{S,p} \times \hat{\mathbb{C}}} : W_{S,p} \times \hat{\mathbb{C}} \to W_{S,p} \times \hat{\mathbb{C}}$. Then, $\overline{f}$ is a hyperbolic skew product over the shift map $\sigma : W_{S,p} \to W_{S,p}$, and there exists a constant $K_{S,p} \geq 1$ such that for each $\gamma \in W_{S,p}$, $\hat{J}_\gamma(f) = J_\gamma(f) = J_\gamma(\overline{f})$ is a $K_{S,p}$-quasicircle. Here, a Jordan curve ξ in $\hat{\mathbb{C}}$ is said to be a K-quasicircle, if ξ is the image of $S^1 (\subset \mathbb{C})$ under a K-quasiconformal homeomorphism $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. (For the definition of a quasicircle and a quasiconformal homeomorphism, see [17].)*

We now present some results on semi-hyperbolic polynomial semigroups in $\mathcal{G}_{dis}$.

Theorem 2.41. *Let G be a polynomial semigroup generated by a non-empty compact subset Γ of $\text{Poly}_{\deg \geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. If $G_{\min,\Gamma}$ is semi-hyperbolic, then G is semi-hyperbolic.*

Theorem 2.42. *Let G be a polynomial semigroup generated by a non-empty compact subset Γ of $\text{Poly}_{\deg \geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. If $G_{\min,\Gamma}$ is hyperbolic and $(\cup_{h \in \Gamma \setminus \Gamma_{\min}} CV^*(h)) \cap J_{\min}(G) = \emptyset$, then G is hyperbolic.*

Remark 2.43. In [29], it will be shown that in Theorem 2.42, the condition $(\cup_{h \in \Gamma \setminus \Gamma_{\min}} CV^*(h)) \cap J_{\min}(G) = \emptyset$ is necessary.

Theorem 2.44. *Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg \geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Suppose that $G \in \mathcal{G}_{dis}$ and that G is semi-hyperbolic. Let $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ be any element. Then, $\hat{J}_\gamma(f) = J_\gamma(f)$ and $J_\gamma(f)$ is a Jordan curve. Moreover, for each point $y_0 \in \text{int}(K_\gamma(f))$, there exists an $n \in \mathbb{N}$ such that $f_{\gamma,n}(y_0) \in \text{int}(\hat{K}(G))$.*

We next present a result that there exist families of uncountably many mutually disjoint quasicircles with uniform distortion, densely inside the Julia set of a semigroup in $\mathcal{G}_{dis}$.

Theorem 2.45. (Existence of a Cantor family of quasicircles.) *Let $G \in \mathcal{G}_{dis}$ (possibly generated by a non-compact family) and let V be an open subset of $\hat{\mathbb{C}}$ with $V \cap J(G) \neq \emptyset$. Then, there exist elements g_1 and g_2 in G such that all of the following hold.*

1. *$H = \langle g_1, g_2 \rangle$ satisfies that $J(H) \subset J(G)$.*

2. *There exists a non-empty open set U in $\hat{\mathbb{C}}$ such that $g_1^{-1}(\overline{U}) \cup g_2^{-1}(\overline{U}) \subset U$, and such that $g_1^{-1}(\overline{U}) \cap g_2^{-1}(\overline{U}) = \emptyset$.*

3. *$H = \langle g_1, g_2 \rangle$ is a hyperbolic polynomial semigroup.*

4. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family $\Gamma = \{g_1, g_2\}$ of polynomials. Then, we have the following.*

   (a) *$J(H) = \bigcup_{\gamma \in \Gamma^{\mathbb{N}}} J_\gamma(f)$ (disjoint union).*

   (b) *For each connected component J of $J(H)$, there exists an element $\gamma \in \Gamma^{\mathbb{N}}$ such that $J = J_\gamma(f)$.*

   (c) *There exists a constant $K \geq 1$ independent of J such that each connected component J of $J(H)$ is a K-quasicircle.*

   (d) *The map $\gamma \mapsto J_\gamma(f)$, defined for all $\gamma \in \Gamma^{\mathbb{N}}$, is continuous with respect to the Hausdorff topology in the space of non-empty compact subsets of $\hat{\mathbb{C}}$, and injective.*

   (e) *For each element $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f) \cap V \neq \emptyset$.*

   (f) *Let $\omega \in \Gamma^{\mathbb{N}}$ be an element such that $\sharp(\{j \in \mathbb{N} \mid \omega_j = g_1\}) = \infty$ and such that $\sharp(\{j \in \mathbb{N} \mid \omega_j = g_2\}) = \infty$. Then, $J_\omega(f)$ does not meet the boundary of any connected component of $F(G)$.*

2.6 Fiberwise Julia sets that are Jordan curves but not quasicircles

We present a result on a sufficient condition for a fiberwise Julia set $J_x(f)$ to be a Jordan curve but not a quasicircle. The proofs are given in Section 4.6.

Definition 2.46. Let V be a subdomain of $\hat{\mathbb{C}}$ such that $\partial V \subset \mathbb{C}$. We say that V is a John domain if there exists a constant $c > 0$ and a point $z_0 \in V$ ($z_0 = \infty$ when $\infty \in V$) satisfying the following: for all $z_1 \in V$ there exists an arc $\xi \subset V$ connecting z_1 to z_0 such that for any $z \in \xi$, we have $\min\{|z - a| \mid a \in \partial V\} \geq c|z - z_1|$.

Remark 2.47. Let V be a simply connected domain in $\hat{\mathbb{C}}$ such that $\partial V \subset \mathbb{C}$. It is well-known that if V is a John domain, then ∂V is locally connected ([21, page 26]). Moreover, a Jordan curve $\xi \subset \mathbb{C}$ is a quasicircle if and only if both components of $\hat{\mathbb{C}} \setminus \xi$ are John domains ([21, Theorem 9.3]).

Theorem 2.48. *Let G be a polynomial semigroup generated by a compact subset Γ of* $\text{Poly}_{\deg \geq 2}$. *Suppose that $G \in \mathcal{G}_{dis}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let $m \in \mathbb{N}$*

and suppose that there exists an element $(h_1, h_2, \ldots, h_m) \in \Gamma^m$ *such that* $J(h_m \circ \cdots \circ h_1)$ *is not a quasicircle. Let* $\alpha = (\alpha_1, \alpha_2, \ldots) \in \Gamma^{\mathbb{N}}$ *be the element such that for each* $k, l \in \mathbb{N} \cup \{0\}$ *with* $1 \le l \le m$, $\alpha_{km+l} = h_l$*. Then, the following statements 1 and 2 hold.*

1. *Suppose that* G *is hyperbolic. Let* $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ *be an element such that there exists a sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of positive integers satisfying that* $\sigma^{n_k}(\gamma) \to \alpha$ *as* $k \to \infty$*. Then,* $J_\gamma(f)$ *is a Jordan curve but not a quasicircle. Moreover, the unbounded component* $A_\gamma(f)$ *of* $F_\gamma(f)$ *is a John domain, but the unique bounded component* U_γ *of* $F_\gamma(f)$ *is not a John domain.*

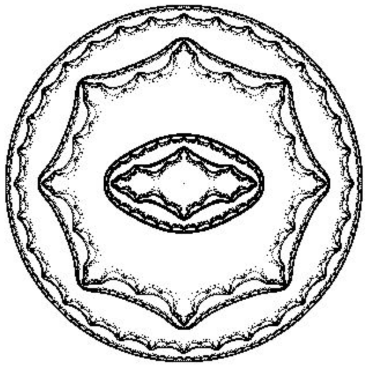
2. *Suppose that* G *is semi-hyperbolic. Let* $\rho_0 \in \Gamma \setminus \Gamma_{\min}$ *be any element and let* $\beta := (\rho_0, \alpha_1, \alpha_2, \ldots) \in \Gamma^{\mathbb{N}}$*. Let* $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ *be an element such that there exists a sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of positive integers satisfying that* $\sigma^{n_k}(\gamma) \to \beta$ *as* $k \to \infty$*. Then,* $J_\gamma(f)$ *is a Jordan curve but not a quasicircle. Moreover, the unbounded component* $A_\gamma(f)$ *of* $F_\gamma(f)$ *is a John domain, but the unique bounded component* U_γ *of* $F_\gamma(f)$ *is not a John domain.*

Example 2.49. Let $g_1(z) := z^2 - 1$ and $g_2(z) := \frac{z^2}{4}$. Let $\Gamma := \{g_1^2, g_2^2\}$. Moreover, let G be the polynomial semigroup generated by Γ. Let $D := \{z \in \mathbb{C} \mid |z| < 0.4\}$. Then, it is easy to see $g_1^2(D) \cup g_2^2(D) \subset D$. Hence, $D \subset F(G)$. Moreover, by Remark 1.3, we have that $P^*(G) = \overline{\cup_{g \in G \cup \{Id\}} g(\{0, -1\})} \subset D \subset F(G)$. Hence, $G \in \mathcal{G}$ and G is hyperbolic. Furthermore, let $K := \{z \in \mathbb{C} \mid 0.4 \le |z| \le 4\}$. Then, it is easy to see that $(g_1^2)^{-1}(K) \cup (g_2^2)^{-1}(K) \subset K$ and $(g_1^2)^{-1}(K) \cap (g_2^2)^{-1}(K) = \emptyset$. Combining it with Lemma 3.1-6 and Lemma 3.1-2, we obtain that $J(G)$ is disconnected. Therefore, $G \in \mathcal{G}_{dis}$. Moreover, it is easy to see that $\Gamma_{\min} = \{g_1^2\}$. Since $J(g_1^2)$ is not a Jordan curve, we can apply Theorem 2.48. Setting $\alpha := (g_1^2, g_1^2, g_1^2, \ldots) \in \Gamma^{\mathbb{N}}$, it follows that for any

$$\gamma \in \mathcal{I} := \{\omega \in R(\Gamma, \Gamma \setminus \Gamma_{\min}) \mid \exists (n_k) \text{ with } \sigma^{n_k}(\omega) \to \alpha\},$$

$J_\gamma(f)$ is a Jordan curve but not a quasicircle, and $A_\gamma(f)$ is a John domain but the bounded component of $F_\gamma(f)$ is not a John domain. (See figure 2: the Julia set of G. In this example, $\hat{\mathcal{J}}_G = \{J_\gamma(f) \mid \gamma \in \Gamma^{\mathbb{N}}\}$ and if $\gamma \neq \omega, J_\gamma(f) \cap J_\omega(f) = \emptyset$.) Note that by Theorem 2.40-3, if $\gamma \notin \mathcal{I}$, then either $J_\gamma(f)$ is not a Jordan curve or $J_\gamma(f)$ is a quasicircle.

Figure 2: The Julia set of $G = \langle g_1^2, g_2^2 \rangle$.



2.7 Random dynamics of polynomials and classification of compactly generated, (semi-)hyperbolic, polynomial semigroups G in $\mathcal{G}$

In this section, we present some results on the random dynamics of polynomials. Moreover, we present some results on classification of compactly generated, (semi-) hyperbolic, polynomial semigroups G in $\mathcal{G}$. The proofs are given in Section 4.7.

Let τ be a Borel probability measure on $\text{Poly}_{\deg\geq 2}$. We consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a polynomial map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ according to the distribution τ. (Hence, this is a kind of Markov process on $\hat{\mathbb{C}}$.)

Notation: For a Borel probability measure τ on $\text{Poly}_{\deg\geq 2}$, we denote by Γ_τ the support of τ on $\text{Poly}_{\deg\geq 2}$. (Hence, Γ_τ is a closed set in $\text{Poly}_{\deg\geq 2}$.) Moreover, we denote by $\tilde{\tau}$ the infinite product measure $\otimes_{j=1}^{\infty}\tau$. This is a Borel probability measure on $\Gamma_\tau^{\mathbb{N}}$. Furthermore, we denote by G_τ the polynomial semigroup generated by Γ_τ.

Definition 2.50. Let X be a complete metric space. A subset A of X is said to be residual if $X \setminus A$ is a countable union of nowhere dense subsets of X. Note that by Baire Category Theorem, a residual set A is dense in X.

Corollary 2.51. *(Corollary of Theorem 2.40-2) Let Γ be a non-empty compact subset of* $\text{Poly}_{\deg\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Suppose $G \in \mathcal{G}_{dis}$. Then, there exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ on* $\text{Poly}_{\deg\geq 2}$ *with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that each $\gamma \in \mathcal{U}$ satisfies all of the following.*

1. *There exists exactly one bounded component U_γ of $F_\gamma(f)$. Furthermore, $\partial U_\gamma = \partial A_\gamma(f) = J_\gamma(f)$.*

2. *Each limit function of $\{f_{\gamma,n}\}_n$ in U_γ is constant. Moreover, for each $y \in U_\gamma$, there exists a number $n \in \mathbb{N}$ such that $f_{\gamma,n}(y) \in \text{int}(\hat{K}(G))$.*

3. *$\hat{J}_\gamma(f) = J_\gamma(f)$. Moreover, the map $\omega \mapsto J_\omega(f)$ defined on $\Gamma^{\mathbb{N}}$ is continuous at γ, with respect to the Hausdorff topology in the space of non-empty compact subsets of $\hat{\mathbb{C}}$.*

4. *The 2-dimensional Lebesgue measure of $\hat{J}_\gamma(f) = J_\gamma(f)$ is equal to zero.*

Next we present a result on compactly generated, semi-hyperbolic, polynomial semigroups in $\mathcal{G}$.

Theorem 2.52. *Let Γ be a non-empty compact subset of $\text{Poly}_{\deg\geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}$ and that G is semi-hyperbolic. Then, exactly one of the following three statements 1, 2, and 3 holds.*

1. *G is hyperbolic. Moreover, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.*

2. *There exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ on $\text{Poly}_{\deg\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component of $F_\gamma(f)$ is not a John domain. Moreover, there exists a dense subset $\mathcal{V}$ of $\Gamma^{\mathbb{N}}$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Furthermore, there exist two elements $\alpha, \beta \in \Gamma^{\mathbb{N}}$ such that $J_\beta(f) < J_\alpha(f)$.*

3. *There exists a dense subset $\mathcal{V}$ of $\Gamma^{\mathbb{N}}$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Moreover, for each $\alpha, \beta \in \Gamma^{\mathbb{N}}$, $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$. Furthermore, $J(G)$ is arcwise connected.*

Corollary 2.53. *Let Γ be a non-empty compact subset of $\text{Poly}_{\deg\geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}_{dis}$ and that G is semi-hyperbolic. Then, either statement 1 or statement 2 in Theorem 2.52 holds. In particular, for any Borel Probability measure τ on $\text{Poly}_{\deg\geq 2}$ with $\Gamma_\tau = \Gamma$, for almost every $\gamma \in \Gamma_\tau^{\mathbb{N}}$ with respect to $\tilde{\tau}$, $J_\gamma(f)$ is a Jordan curve.*

We now classify compactly generated, hyperbolic, polynomial semigroups in $\mathcal{G}$.

Theorem 2.54. *Let Γ be a non-empty compact subset of $\text{Poly}_{\deg\geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}$ and that G is hyperbolic. Then, exactly one of the following three statements 1, 2, 3 holds.*

1. *There exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.*

2. *There exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ on $\text{Poly}_{\deg\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component of $F_\gamma(f)$ is not a John domain. Moreover, there exists a dense subset $\mathcal{V}$ of $\Gamma^{\mathbb{N}}$ such that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is a quasicircle. Furthermore, there exists a dense subset $\mathcal{W}$ of $\Gamma^{\mathbb{N}}$ such that for each $\gamma \in \mathcal{W}$, there are infinitely many bounded connected components of $F_\gamma(f)$.*

3. *For each $\gamma \in \Gamma^{\mathbb{N}}$, there are infinitely many bounded connected components of $F_\gamma(f)$. Moreover, for each $\alpha, \beta \in \Gamma^{\mathbb{N}}$, $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$. Furthermore, $J(G)$ is arcwise connected.*

Example 2.55. Let $h_1(z) := z^2 - 1$ and $h_2(z) := az^2$, where $a \in \mathbb{C}$ with $0 < |a| < 0.1$. Let $\Gamma := \{h_1, h_2\}$. Moreover, let $G := \langle h_1, h_2 \rangle$. Let $U := \{|z| < 0.2\}$. Then, it is easy to see that $h_2(U) \subset U$, $h_2(h_1(U)) \subset U$, and $h_1^2(U) \subset U$. Hence, $U \subset F(G)$. It follows that $P^*(G) \subset \text{int}(\hat{K}(G)) \subset F(G)$. Therefore, $G \in \mathcal{G}$ and G is hyperbolic. Since $J(h_1)$ is not a Jordan curve and $J(h_2)$ is a Jordan curve, Theorem 2.54 implies that there exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each Borel probability measure τ on $\text{Poly}_{\deg\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle. Moreover, for each $\gamma \in \mathcal{U}$, $A_\gamma(f)$ is a John domain, but the bounded component of $F_\gamma(f)$ is not a John domain. Furthermore, by Theorem 2.15, $J(G)$ is connected.

Remark 2.56. Let $h \in \text{Poly}_{\deg\geq 2}$ be a polynomial. Suppose that $J(h)$ is a Jordan curve but not a quasicircle. Then, it is easy to see that there exists a parabolic fixed point of h in $\mathbb{C}$ and the bounded connected component of $F(h)$ is the immediate parabolic basin. Hence, $\langle h \rangle$ is not semi-hyperbolic. Moreover, by [7], $F_\infty(h)$ is not a John domain.

Thus what we see in statement 2 in Theorem 2.52 and statement 2 in Theorem 2.54, as illustrated in Example 2.49 and Example 2.55, is a special and new phenomenon which can hold in the *random* dynamics of a family of polynomials, but cannot hold in the usual iteration dynamics of a single polynomial. Namely, it can hold that for almost every $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a Jordan curve and fails to be a quasicircle all while the basin of infinity $A_\gamma(f)$ is still a John domain. Whereas, if $J(h)$, for some polynomial h, is a Jordan curve which fails to be a quasicircle, then the basin of infinity $F_\infty(h)$ is necessarily **not** a John domain.

Pilgrim and Tan Lei ([22]) showed that there exists a hyperbolic rational map h with *disconnected* Julia set such that "almost every" connected component of $J(h)$ is a Jordan curve but not a quasicircle.

We give a sufficient condition so that statement 1 in Theorem 2.54 holds.

Proposition 2.57. *Let Γ be a non-empty compact subset of $\mathrm{Poly}_{\deg\geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let G be the polynomial semigroup generated by Γ. Suppose that $P^*(G)$ is included in a connected component of $\mathrm{int}(\hat{K}(G))$. Then, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.*

2.8 Construction of examples

We present a way to construct examples of semigroups G in $\mathcal{G}_{dis}$.

Proposition 2.58. *Let G be a polynomial semigroup generated by a compact subset Γ of $\mathrm{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}$ and $\mathrm{int}(\hat{K}(G)) \neq \emptyset$. Let $b \in \mathrm{int}(\hat{K}(G))$. Moreover, let $d \in \mathbb{N}$ be any positive integer such that $d \geq 2$, and such that $(d, \deg(h)) \neq (2, 2)$ for each $h \in \Gamma$. Then, there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, there exists a compact neighborhood V of $g_a(z) = a(z-b)^d + b$ in $\mathrm{Poly}_{\deg\geq 2}$ satisfying that for any non-empty subset V' of V, the polynomial semigroup $H_{\Gamma,V'}$ generated by the family $\Gamma \cup V'$ belongs to $\mathcal{G}_{dis}$, $\hat{K}(H_{\Gamma,V'}) = \hat{K}(G)$ and $(\Gamma \cup V')_{\min} \subset \Gamma$. Moreover, in addition to the assumption above, if G is semi-hyperbolic (resp. hyperbolic), then the above $H_{\Gamma,V'}$ is semi-hyperbolic (resp. hyperbolic).*

Remark 2.59. By Proposition 2.58, there exists a 2-generator polynomial semigroup $G = \langle h_1, h_2 \rangle$ in $\mathcal{G}_{dis}$ such that h_1 has a Siegel disk.

Definition 2.60. Let $d \in \mathbb{N}$ with $d \geq 2$. We set $\mathcal{Y}_d := \{h \in \mathrm{Poly} \mid \deg(h) = d\}$ endowed with the relative topology from Poly.

Theorem 2.61. *Let $m \geq 2$ and let $d_2, \ldots, d_m \in \mathbb{N}$ be such that $d_j \geq 2$ for each $j = 2, \ldots, m$. Let $h_1 \in \mathcal{Y}_{d_1}$ with $\mathrm{int}(K(h_1)) \neq \emptyset$ be such that $\langle h_1 \rangle \in \mathcal{G}$. Let $b_2, b_3, \ldots, b_m \in \mathrm{int}(K(h_1))$. Then, all of the following statements hold.*

1. *Suppose that* $\langle h_1 \rangle$ *is semi-hyperbolic (resp. hyperbolic). Then, there exists a number* $c > 0$ *such that for each* $(a_2, a_3, \dots, a_m) \in \mathbb{C}^{m-1}$ *with* $0 < |a_j| < c$ $(j = 2, \dots, m)$*, setting* $h_j(z) = a_j(z - b_j)^{d_j} + b_j$ $(j = 2, \dots, m)$*, the polynomial semigroup* $G = \langle h_1, \dots, h_m \rangle$ *satisfies that* $G \in \mathcal{G}$*,* $\hat{K}(G) = K(h_1)$ *and* G *is semi-hyperbolic (resp. hyperbolic).*

2. *Suppose that* $\langle h_1 \rangle$ *is semi-hyperbolic (resp. hyperbolic). Suppose also that either (i) there exists a* $j \geq 2$ *with* $d_j \geq 3$*, or (ii)* $\deg(h_1) = 3$*,* $b_2 = \cdots = b_m$*. Then, there exist* $a_2, a_3, \dots, a_m > 0$ *such that setting* $h_j(z) = a_j(z - b_j)^{d_j} + b_j$ $(j = 2, \dots, m)$*, the polynomial semigroup* $G = \langle h_1, h_2, \dots, h_m \rangle$ *satisfies that* $G \in \mathcal{G}_{dis}$*,* $\hat{K}(G) = K(h_1)$ *and* G *is semi-hyperbolic (resp. hyperbolic).*

Definition 2.62. Let $m \in \mathbb{N}$. We set

- $\mathcal{H}_m := \{(h_1, \dots, h_m) \in (\text{Poly}_{\deg \geq 2})^m \mid \langle h_1, \dots, h_m \rangle \text{ is hyperbolic}\}$,
- $\mathcal{B}_m := \{(h_1, \dots, h_m) \in (\text{Poly}_{\deg \geq 2})^m \mid \langle h_1, \dots, h_m \rangle \in \mathcal{G}\}$, and
- $\mathcal{D}_m := \{(h_1, \dots, h_m) \in (\text{Poly}_{\deg \geq 2})^m \mid J(\langle h_1, \dots, h_m \rangle) \text{ is disconnected}\}$.

Moreover, let $\pi_1 : (\text{Poly}_{\deg \geq 2})^m \to \text{Poly}_{\deg \geq 2}$ be the projection defined by $\pi(h_1, \dots, h_m) = h_1$.

Theorem 2.63. *Under the above notation, all of the following statements hold.*

1. $\mathcal{H}_m, \mathcal{H}_m \cap \mathcal{B}_m$*,* $\mathcal{H}_m \cap \mathcal{D}_m$*, and* $\mathcal{H}_m \cap \mathcal{B}_m \cap \mathcal{D}_m$ *are open in* $(\text{Poly}_{\deg \geq 2})^m$.

2. *Let* $d_1, \dots, d_m \in \mathbb{N}$ *be such that* $d_j \geq 2$ *for each* $j = 1, \dots, m$.
   Then, $\pi_1 : \mathcal{H}_m \cap \mathcal{B}_m \cap (\mathcal{Y}_{d_1} \times \cdots \times \mathcal{Y}_{d_m}) \to \mathcal{H}_1 \cap \mathcal{B}_1 \cap \mathcal{Y}_{d_1}$ *is surjective.*

3. *Let* $d_1, \dots, d_m \in \mathbb{N}$ *be such that* $d_j \geq 2$ *for each* $j = 1, \dots, m$ *and such that* $(d_1, \dots, d_m) \neq (2, 2, \dots, 2)$*. Then,* $\pi_1 : \mathcal{H}_m \cap \mathcal{B}_m \cap \mathcal{D}_m \cap (\mathcal{Y}_{d_1} \times \cdots \times \mathcal{Y}_{d_m}) \to \mathcal{H}_1 \cap \mathcal{B}_1 \cap \mathcal{Y}_{d_1}$ *is surjective.*

Remark 2.64. Combining Proposition 2.58, Theorem 2.61, and Theorem 2.63, we can construct many examples of semigroups G in $\mathcal{G}$ (or $\mathcal{G}_{dis}$) with some additional properties (semi-hyperbolicity, hyperbolicity, etc.).

3 Tools

To show the main results, we need some tools in this section.

3.1 Fundamental properties of rational semigroups

Notation: For a rational semigroup G, we set $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp(\cup_{g\in G} g^{-1}(\{z\})) < \infty\}$. This is called the exceptional set of G.

Lemma 3.1 ([14, 13, 32]). *Let G be a rational semigroup.*

1. *For each $h \in G$, we have $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*
2. *If $G = \langle h_1, \ldots, h_m \rangle$, then $J(G) = h_1^{-1}(J(G)) \cup \cdots \cup h_m^{-1}(J(G))$. More generally, if G is generated by a compact subset Γ of Rat, then $J(G) = \cup_{h\in\Gamma} h^{-1}(J(G))$. (We call this property of the Julia set of a compactly generated rational semigroup "backward self-similarity.")*
3. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set.*
4. *If $\sharp(J(G)) \geq 3$, then $\sharp(E(G)) \leq 2$.*
5. *If a point z is not in $E(G)$, then $J(G) \subset \overline{\cup_{g\in G} g^{-1}(\{z\})}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then $\overline{\cup_{g\in G} g^{-1}(\{z\})} = J(G)$.*
6. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

Theorem 3.2 ([14, 13, 32]). *Let G be a rational semigroup. If $\sharp(J(G)) \geq 3$, then $J(G) = \overline{\{z \in \hat{\mathbb{C}} \mid \exists g \in G,\ g(z) = z,\ |g'(z)| > 1\}}$. In particular, $J(G) = \overline{\bigcup_{g\in G} J(g)}$.*

Remark 3.3. If a rational semigroup G contains an element g with $\deg(g) \geq 2$, then $\sharp(J(g)) \geq 3$, which implies that $\sharp(J(G)) \geq 3$.

3.2 Fundamental properties of fibered rational maps

Lemma 3.4. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Then, we have the following.*

1. **([32, Lemma 2.4])** *For each $x \in X$, $(f_{x,1})^{-1}(J_{g(x)}(f)) = J_x(f)$. Furthermore, we have $\hat{J}_x(f) \supset J_x(f)$. Note that* **equality** $\hat{J}_x(f) = J_x(f)$ **does not hold in general.**

   If $g : X \to X$ is a surjective and open map, then $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$, and for each $x \in X$, $(f_{x,1})^{-1}(\hat{J}_{g(x)}(f)) = \hat{J}_x(f)$.

2. ([**16, 32**]) *If $d(x) \geq 2$ for each $x \in X$, then for each $x \in X$, $J_x(f)$ is a non-empty perfect set with $\sharp(J_x(f)) \geq 3$. Furthermore, the map $x \mapsto J_x(f)$ is lower semicontinuous; i.e., for any point $(x, y) \in X \times \hat{\mathbb{C}}$ with $y \in J_x(f)$ and any sequence $\{x^n\}_{n \in \mathbb{N}}$ in X with $x^n \to x$, there exists a sequence $\{y^n\}_{n \in \mathbb{N}}$ in $\hat{\mathbb{C}}$ with $y^n \in J_{x^n}(f)$ for each $n \in \mathbb{N}$ such that $y^n \to y$. However, $x \mapsto J_x(f)$ is* **NOT** *continuous with respect to the Hausdorff topology in general.*

3. *If $d(x) \geq 2$ for each $x \in X$, then $\inf_{x \in X} diam_S J_x(f) > 0$, where $diam_S$ denotes the diameter with respect to the spherical distance.*

4. *If $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ is a polynomial skew product and $d(x) \geq 2$ for each $x \in X$, then we have that there exists a ball B around ∞ such that for each $x \in X$, $B \subset A_x(f) \subset F_x(f)$, and that for each $x \in X$, $J_x(f) = \partial K_x(f) = \partial A_x(f)$. Moreover, for each $x \in X$, $A_x(f)$ is connected.*

5. *If $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ is a polynomial skew product and $d(x) \geq 2$ for each $x \in X$, and if $\omega \in X$ is a point such that $\text{int}(K_w(f))$ is a non-empty set, then $\overline{\text{int}(K_\omega(f))} = K_\omega(f)$ and $\partial(\text{int}(K_\omega(f))) = J_\omega(f)$.*

Proof. For the proof of statement 1, see [32, Lemma 2.4]. For the proof of statement 2, see [16] and [32].

By statement 2, it is easy to see that statement 3 holds. Moreover, it is easy to see that statement 4 holds.

To show statement 5, let $y \in J_\omega(f)$ be a point. Let V be an arbitrary neighborhood of y in $\hat{\mathbb{C}}$. Then, by the self-similarity of Julia sets (see [5]), there exists an $n \in \mathbb{N}$ such that $f_{\omega,n}(V \cap J_\omega(f)) = J_{g^n(\omega)}(f)$. Since $\partial(\text{int}(K_{g^n(\omega)}(f))) \subset J_{g^n(\omega)}(f)$ and $(f_{\omega,n})^{-1}(K_{g^n(\omega)}(f)) = K_\omega(f)$, it follows that $V \cap \partial(\text{int}(K_\omega(f))) \neq \emptyset$. Hence, we obtain $J_\omega(f) = \partial(\text{int}(K_\omega(f))$. Therefore, we have proved statement 5. □

Lemma 3.5. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be a skew product associated with a compact subset Γ of* Rat. *Let G be the rational semigroup generated by Γ. Suppose that $\sharp(J(G)) \geq 3$. Then, we have the following.*

1. *$\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$.*

2. *For each $\gamma = (\gamma_1, \gamma_2, \ldots,) \in \Gamma^{\mathbb{N}}$, $\hat{J}_\gamma(f) = \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$.*

Proof. First, we show statement 1. Since $J_\gamma(f) \subset J(G)$ for each $\gamma \in \Gamma$, we have $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) \subset J(G)$. By Theorem 3.2, we have $J(G) = \overline{\cup_{g \in G} J(g)}$. Since $\cup_{g \in G} J(g) \subset \pi_{\hat{\mathbb{C}}}(\tilde{J}(f))$, we obtain $J(G) \subset \pi_{\hat{\mathbb{C}}}(\tilde{J}(f))$. Therefore, we obtain $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$.

We now show statement 2. Let $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$. By statement 1 in Lemma 3.4, we see that for each $j \in \mathbb{N}$, $\gamma_j \cdots \gamma_1(\hat{J}_\gamma(f)) = \hat{J}_{\sigma^j(\gamma)}(f) \subset J(G)$. Hence, $\hat{J}_\gamma(f) \subset \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$. Suppose that there exists a point $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ such that $y \in \left(\cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))\right) \setminus \hat{J}_\gamma(f)$. Then, we have $(\gamma, y) \in (\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}) \setminus \tilde{J}(f)$. Hence, there exists a neighborhood U of γ in $\Gamma^{\mathbb{N}}$ and a neighborhood V of y in $\hat{\mathbb{C}}$ such that $U \times V \subset \tilde{F}(f)$. Then, there exists an $n \in \mathbb{N}$ such that $\sigma^n(U) = \Gamma^{\mathbb{N}}$. Combining it with Lemma 3.4-1, we obtain $\tilde{F}(f) \supset f^n(U \times V) \supset \Gamma^{\mathbb{N}} \times \{f_{\gamma,n}(y)\}$. Moreover, since we have $f_{\gamma,n}(y) \in J(G) = \pi_{\hat{\mathbb{C}}}(\tilde{J}(f))$, where the last equality holds by statement 1, we get that there exists an element $\gamma' \in \Gamma^{\mathbb{N}}$ such that $(\gamma', f_{\gamma,n}(y)) \in \tilde{J}(f)$. However, it contradicts $(\gamma', f_{\gamma,n}(y)) \in \Gamma^{\mathbb{N}} \times \{f_{\gamma,n}(y)\} \subset \tilde{F}(f)$. Hence, we obtain $\hat{J}_\gamma(f) = \cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$. □

Lemma 3.6. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Then, the following are equivalent.*

1. *$\pi_{\hat{\mathbb{C}}}(P(f)) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.*
2. *For each $x \in X$, $J_x(f)$ is connected.*
3. *For each $x \in X$, $\hat{J}_x(f)$ is connected.*

Proof. First, we show 1 ⇒2. Suppose that 1 holds. Let $R > 0$ be a number such that for each $x \in X$, $B := \{y \in \hat{\mathbb{C}} \mid |y| > R\} \subset A_x(f)$ and $\overline{f_{x,1}(B)} \subset B$. Then, for each $x \in X$, we have $A_x(f) = \cup_{n \in \mathbb{N}} (f_{x,n})^{-1}(B)$ and $(f_{x,n})^{-1}(B) \subset (f_{x,n+1})^{-1}(B)$, for each $n \in \mathbb{N}$. Furthermore, since we assume 1, we see that for each $n \in \mathbb{N}$, $(f_{x,n})^{-1}(B)$ is a simply connected domain, by the Riemann-Hurwitz formula. Hence, for each $x \in X$, $A_x(f)$ is a simply connected domain. Since $\partial A_x(f) = J_x(f)$ for each $x \in X$, we conclude that for each $x \in X$, $J_x(f)$ is connected. Hence, we have shown 1 ⇒ 2.

Next, we show 2 ⇒ 3. Suppose that 2 holds. Let $z_1 \in \hat{J}_x(f)$ and $z_2 \in J_x(f)$ be two points. Let $\{x^n\}_{n \in \mathbb{N}}$ be a sequence such that $x^n \to x$ as $n \to \infty$, and such that $d(z_1, J_{x^n}(f)) \to 0$ as $n \to \infty$. We may assume that there exists a non-empty compact set K in $\hat{\mathbb{C}}$ such that $J_{x^n}(f) \to K$ as $n \to \infty$, with respect to the Hausdorff topology in the space of non-empty compact sets in $\hat{\mathbb{C}}$. Since we assume 2, K is connected. By Lemma 3.4-2, we have $d(z_2, J_{x^n}(f)) \to 0$ as $n \to \infty$. Hence, $z_i \in K$ for each $i = 1, 2$. Therefore, z_1 and z_2 belong to the same connected component of $\hat{J}_x(f)$. Thus, we have shown 2 ⇒ 3.

Next, we show 3 ⇒ 1. Suppose that 3 holds. It is easy to see that $A_x(f) \cap \hat{J}_x(f) = \emptyset$ for each $x \in X$. Hence, $A_x(f)$ is a connected component of

$\hat{\mathbb{C}} \setminus \hat{J}_x(f)$. Since we assume 3, we have that for each $x \in X$, $A_x(f)$ is a simply connected domain. Since $(f_{x,1})^{-1}(A_{g(x)}(f)) = A_x(f)$, the Riemann-Hurwitz formula implies that for each $x \in X$, there exists no critical point of $f_{x,1}$ in $A_x(f) \cap \mathbb{C}$. Therefore, we obtain 1. Thus, we have shown $3 \Rightarrow 1$. □

Corollary 3.7. *Let $G = \langle h_1, h_2 \rangle \in \mathcal{G}$. Then, $h_1^{-1}(J(h_2))$ is connected.*

Proof. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family $\Gamma = \{h_1, h_2\}$. Let $\gamma = (h_1, h_2, h_2, h_2, h_2, \ldots) \in \Gamma^{\mathbb{N}}$. Then, by Lemma 3.4-1, we have $J_\gamma(f) = h_1^{-1}(J(h_2))$. From Lemma 3.6, it follows that $h_1^{-1}(J(h_2))$ is connected. □

Lemma 3.8. *Let G be a polynomial semigroup generated by a compact subset Γ of $\mathrm{Poly}_{\deg \geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Suppose that $G \in \mathcal{G}$. Then for each $\gamma = (\gamma_1, \gamma_2, \ldots,) \in \Gamma^{\mathbb{N}}$, the sets $J_\gamma(f)$, $\hat{J}_\gamma(f)$, and $\cap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$ are connected.*

Proof. From Lemma 3.5-2 and Lemma 3.6, the lemma follows. □

Lemma 3.9. *Under the same assumption as that in Lemma 3.8, let $\gamma, \rho \in \Gamma^{\mathbb{N}}$ be two elements with $J_\gamma(f) \cap J_\rho(f) = \emptyset$. Then, either $J_\gamma(f) < J_\rho(f)$ or $J_\rho(f) < J_\gamma(f)$.*

Proof. Let $\gamma, \rho \in \Gamma^{\mathbb{N}}$ with $J_\gamma(f) \cap J_\rho(f) = \emptyset$. Suppose that the statement "either $J_\gamma(f) < J_\rho(f)$ or $J_\rho(f) < J_\gamma(f)$" is not true. Then, Lemma 3.6 implies that $J_\gamma(f)$ is included in the unbounded component of $\mathbb{C} \setminus J_\rho(f)$, and that $J_\rho(f)$ is included in the unbounded component of $\mathbb{C} \setminus J_\gamma(f)$. From Lemma 3.4-4, it follows that $K_\rho(f)$ is included in the unbounded component $A_\gamma(f) \setminus \{\infty\}$ of $\mathbb{C} \setminus J_\gamma(f)$. However, it causes a contradiction, since $\emptyset \neq P^*(G) \subset \hat{K}(G) \subset K_\rho(f) \cap K_\gamma(f)$. □

Definition 3.10. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$. Let $p \in \mathbb{C}$ and $\epsilon > 0$. We set
$\mathcal{F}_{f,p,\epsilon} := \{\alpha : D(p, \epsilon) \to \mathbb{C} \mid \alpha \text{ is a well-defined inverse branch of } (f_{x,n})^{-1}, x \in X, n \in \mathbb{N}\}$.

Lemma 3.11. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $R > 0, \epsilon > 0$, and $\mathcal{F} := \{\alpha \circ \beta : D(0,1) \to \mathbb{C} \mid \beta : D(0,1) \cong D(p, \epsilon),\ \alpha : D(p, \epsilon) \to \mathbb{C},\ \alpha \in \mathcal{F}_{f,p,\epsilon},\ p \in D(0, R)\}$. Then, $\mathcal{F}$ is normal on $D(0,1)$.*

Proof. Since $d(x) \geq 2$ for each $x \in X$, there exists a ball B around ∞ with $B \subset \hat{\mathbb{C}} \setminus D(0, R + \epsilon)$ such that for each $x \in X$, $f_{x,1}(B) \subset B$. Let $p \in D(0, R)$. Then, for each $\alpha \in \mathcal{F}_{f,p,\epsilon}$, $\alpha(D(p, \epsilon)) \subset \hat{\mathbb{C}} \setminus B$. Hence, $\mathcal{F}$ is normal in $D(0,1)$. □

3.3 A lemma from general topology

Lemma 3.12 ([20])**.** *Let X be a compact metric space and let $f : X \to X$ be a continuous open map. Let A be a compact connected subset of X. Then for each connected component B of $f^{-1}(A)$, we have $f(B) = A$.*

4 Proofs of the main results

In this section, we demonstrate the main results.

4.1 Proofs of results in 2.1

Proof of Theorem 2.1: First, we show the following:
Claim: For any $\lambda \in \Lambda$, $h_\lambda^{-1}(A) \subset A$.

To show the claim, let $\lambda \in \Lambda$ with $J(h_\lambda) \neq \emptyset$ and let B be a connected component of $h_\lambda^{-1}(A)$. Then by Lemma 3.12, $h_\lambda(B) = A$. Combining this with $h_\lambda^{-1}(J(h_\lambda)) = J(h_\lambda)$, we obtain $B \cap J(h_\lambda) \neq \emptyset$. Hence $B \subset A$. This means that $h_\lambda^{-1}(A) \subset A$ for each $\lambda \in \Lambda$ with $J(h_\lambda) \neq \emptyset$. Next, let $\lambda \in \Lambda$ with $J(h_\lambda) = \emptyset$. Then h_λ is either identity or an elliptic Möbius transformation. By hypothesis and Lemma 3.1-1, we obtain $h_\lambda^{-1}(A) \subset A$. Hence, we have shown the claim.

Combining the above claim with $\sharp A \geq 3$, by Lemma 3.1-6 we obtain $J(G) \subset A$. Hence $J(G) = A$ and $J(G)$ is connected. □

Notation: We denote by d the spherical distance on $\hat{\mathbb{C}}$. Given $A \subset \hat{\mathbb{C}}$ and $z \in \hat{\mathbb{C}}$, we set $d(z, A) := \inf\{d(z, w) \mid w \in A\}$. Given $A \subset \hat{\mathbb{C}}$ and $\epsilon > 0$, we set $B(A, \epsilon) := \{a \in \hat{\mathbb{C}} \mid d(a, A) < \epsilon\}$. Furthermore, given $A \subset \mathbb{C}$, $z \in \mathbb{C}$, and $\epsilon > 0$, we set $d_e(z, A) := \inf\{|z - w| \mid w \in A\}$ and $D(A, \epsilon) := \{a \in \mathbb{C} \mid d_e(a, A) < \epsilon\}$.

We need the following lemmas to prove the main results.

Lemma 4.1. *Let $G \in \mathcal{G}$ and let J be a connected component of $J(G)$, $z_0 \in J$ a point, and $\{g_n\}_{n \in \mathbb{N}}$ a sequence in G such that $d(z_0, J(g_n)) \to 0$ as $n \to \infty$. Then $\sup_{z \in J(g_n)} d(z, J) \to 0$ as $n \to \infty$.*

Proof. Suppose there exists a connected component J' of $J(G)$ with $J' \neq J$ and a subsequence $\{g_{n_j}\}_{j \in \mathbb{N}}$ of $\{g_n\}_{n \in \mathbb{N}}$ such that $\min_{z \in J(g_{n_j})} d(z, J') \to 0$ as $j \to \infty$. Since $J(g_{n_j})$ is compact and connected for each j, we may assume, passing to a subsequence, that there exists a non-empty compact connected subset K of $\hat{\mathbb{C}}$ such that $J(g_{n_j}) \to K$ as $j \to \infty$, with respect to the Hausdorff topology. Then $K \cap J \neq \emptyset$ and $K \cap J' \neq \emptyset$. Since $K \subset J(G)$ and K is connected, it contradicts $J' \neq J$. □

Lemma 4.2. *Let $G \in \mathcal{G}$. Then given $J \in \mathcal{J}$ and $\epsilon > 0$, there exists an element $g \in G$ such that $J(g) \subset B(J, \epsilon)$.*

Proof. We take a point $z \in J$. Then, by Theorem 3.2, there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ in G such that $d(z, J(g_n)) \to 0$ as $n \to \infty$. By Lemma 4.1, we conclude that there exists an $n \in \mathbb{N}$ such that $J(g_n) \subset B(J, \epsilon)$. □

Lemma 4.3. *Let G be a polynomial semigroup. Suppose that $J(G)$ is disconnected, and $\infty \in J(G)$. Then, the connected component A of $J(G)$ containing ∞ is equal to $\{\infty\}$.*

Proof. By Lemma 3.12, we obtain $g^{-1}(A) \subset A$ for each $g \in G$. Hence, if $\sharp A \geq 3$, then $J(G) \subset A$, by Lemma 3.1-6. Then $J(G) = A$ and it causes a contradiction, since $J(G)$ is disconnected. □

We now demonstrate Theorem 2.7.

Proof of Theorem 2.7: First, we show statement 1. Suppose the statement is false. Then, there exist elements $J_1, J_2 \in \mathcal{J}$ such that J_2 is included in the unbounded component A_1 of $\mathbb{C} \setminus J_1$, and such that J_1 is included in the unbounded component A_2 of $\mathbb{C} \setminus J_2$. Then we can find an $\epsilon > 0$ such that $\overline{B(J_2, \epsilon)}$ is included in the unbounded component of $\mathbb{C} \setminus \overline{B(J_1, \epsilon)}$, and such that $\overline{B(J_1, \epsilon)}$ is included in the unbounded component of $\mathbb{C} \setminus \overline{B(J_2, \epsilon)}$. By Lemma 4.2, for each $i = 1, 2$, there exists an element $g_i \in G$ such that $J(g_i) \subset B(J_i, \epsilon)$. This implies that $J(g_1) \subset A_2'$ and $J(g_2) \subset A_1'$, where A_i' denotes the unbounded component of $\mathbb{C} \setminus J(g_i)$. Hence we obtain $K(g_2) \subset A_1'$. Let v be a critical value of g_2 in $\mathbb{C}$. Since $P^*(G)$ is bounded in $\mathbb{C}$, we have $v \in K(g_2)$. It implies $v \in A_1'$. Hence $g_1^l(v) \to \infty$ as $l \to \infty$. However, this implies a contradiction since $P^*(G)$ is bounded in $\mathbb{C}$. Hence we have shown statement 1.

Next, we show statement 2. Let F_1 be a connected component of $F(G)$. Suppose that there exist three connected components J_1, J_2 and J_3 of $J(G)$ such that they are mutually disjoint and such that $\partial F_1 \cap J_i \neq \emptyset$ for each $i = 1, 2, 3$. Then, by statement 1, we may assume that we have either (1): $J_i \in \mathcal{J}$ for each $i = 1, 2, 3$ and $J_1 < J_2 < J_3$, or (2): $J_1, J_2 \in \mathcal{J},\ J_1 < J_2$, and $\infty \in J_3$. Each of these cases implies that J_1 is included in a bounded component of $\mathbb{C} \setminus J_2$ and J_3 is included in the unbounded component of $\hat{\mathbb{C}} \setminus J_2$. However, it causes a contradiction, since $\partial F_1 \cap J_i \neq \emptyset$ for each $i = 1, 2, 3$. Hence, we have shown that we have either

Case I: $\sharp\{J : \text{component of } J(G) \mid \partial F_1 \cap J \neq \emptyset\} = 1$ or
Case II: $\sharp\{J : \text{component of } J(G) \mid \partial F_1 \cap J \neq \emptyset\} = 2$.

Suppose that we have Case I. Let J_1 be the connected component of $J(G)$ such that $\partial F_1 \subset J_1$. Let D_1 be the connected component of $\hat{\mathbb{C}} \setminus J_1$ containing F_1. Since $\partial F_1 \subset J_1$, we have $\partial F_1 \cap D_1 = \emptyset$. Hence, we have $F_1 = D_1$. Therefore, F_1 is simply connected.

Suppose that we have Case II. Let J_1 and J_2 be the two connected components of $J(G)$ such that $J_1 \neq J_2$ and $\partial F_1 \subset J_1 \cup J_2$. Let D be the connected component of $\hat{\mathbb{C}} \setminus (J_1 \cup J_2)$ containing F_1. Since $\partial F_1 \subset J_1 \cup J_2$, we have $\partial F_1 \cap D = \emptyset$. Hence, we have $F_1 = D$. Therefore, F_1 is doubly connected. Thus, we have shown statement 2.

We now show statement 3. Let $g \in G$ be an element and J a connected component of $J(G)$. Suppose that $g^{-1}(J)$ is disconnected. Then, by Lemma 3.12, there exist at most finitely many connected components $C_1, \ldots, C_r$ of $g^{-1}(J)$ with $r \geq 2$. Then there exists a positive number ϵ such that denoting by B_j the connected component of $g^{-1}(B(J, \epsilon))$ containing C_j for each $j = 1, \ldots, r$, $\{B_j\}$ are mutually disjoint. By Lemma 3.12, we see that, for each connected component B of $g^{-1}(B(J, \epsilon))$, $g(B) = B(J, \epsilon)$ and $B \cap C_j \neq \emptyset$ for some j. Hence we get that $g^{-1}(B(J, \epsilon)) = \bigcup_{j=1}^{r} B_j$ (disjoint union) and $g(B_j) = B(J, \epsilon)$ for each j. By Lemma 4.2, there exists an element $h \in G$ such that $J(h) \subset B(J, \epsilon)$. Then it follows that $g^{-1}(J(h)) \cap B_j \neq \emptyset$ for each $j = 1, \ldots, r$. Moreover, we have $g^{-1}(J(h)) \subset g^{-1}(B(J, \epsilon)) = \bigcup_{j=1}^{r} B_j$. On the other hand, by Corollary 3.7, we have that $g^{-1}(J(h))$ is connected. This is a contradiction. Hence, we have shown that, for each $g \in G$ and each connected component J of $J(G)$, $g^{-1}(J)$ is connected.

By Lemma 4.3, we get that if $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. Let J_1 and J_2 be two elements of $\mathcal{J}$ such that $J_1 \leq J_2$. Let U_i be the unbounded component of $\mathbb{C} \setminus J_i$, for each $i = 1, 2$. Then $U_2 \subset U_1$. Let $g \in G$ be an element. Then $g^{-1}(U_2) \subset g^{-1}(U_1)$. Since $g^{-1}(U_i)$ is the unbounded connected component of $\mathbb{C} \setminus g^{-1}(J_i)$ for each $i = 1, 2$, it follows that $g^{-1}(J_1) \leq g^{-1}(J_2)$. Hence $g^*(J_1) \leq g^*(J_2)$, otherwise $g^*(J_2) < g^*(J_1)$, and it contradicts $g^{-1}(J_1) \leq g^{-1}(J_2)$. $\square$

4.2 Proofs of results in 2.2

In this section, we prove results in Section 2.2, except Theorem 2.12-2 and Theorem 2.12-3, which will be proved in Section 4.3.

To demonstrate Theorem 2.12, we need the following lemmas.

Lemma 4.4. *Let G be a polynomial semigroup in $\mathcal{G}_{dis}$. Let $J_1, J_2 \in \hat{\mathcal{J}}$ be two elements with $J_1 \neq J_2$. Then, we have the following.*

1. *If $J_1, J_2 \in \mathcal{J}$ and $J_1 < J_2$, then there exists a doubly connected component A of $F(G)$ such that $J_1 < A < J_2$.*

2. *If $\infty \in J_2$, then there exists a doubly connected component A of $F(G)$ such that $J_1 < A$.*

Proof. First, we show statement 1. Suppose that $J_1, J_2 \in \mathcal{J}$ and $J_1 < J_2$. We set $B = \cup_{J \in \mathcal{J}, J_1 \leq J \leq J_2} J$. Then, B is a closed disconnected set. Hence, there exists a multiply connected component A' of $\hat{\mathbb{C}} \setminus B$. Since A' is multiply connected, we have that A' is included in the unbounded component of $\hat{\mathbb{C}} \setminus J_1$, and that A' is included in a bounded component of $\hat{\mathbb{C}} \setminus J_2$. This implies that $A' \cap J(G) = \emptyset$. Let A be the connected component of $F(G)$ such that $A' \subset A$. Since $B \subset J(G)$, we have $F(G) \subset \hat{\mathbb{C}} \setminus B$. Hence, A must be equal to A'. Since A' is multiply connected, Theorem 2.7-2 implies that $A = A'$ is doubly connected. Let J be the connected component $J(G)$ such that $J < A$ and $J \cap \partial A \neq \emptyset$. Then, since $A' = A$ is included in the unbounded component of $\hat{\mathbb{C}} \setminus J_1$, we have that J does not meet any bounded component of $\mathbb{C} \setminus J_1$. Hence, we obtain $J_1 \leq J$, which implies that $J_1 \leq J < A$. Therefore, A is a doubly connected component of $F(G)$ such that $J_1 < A < J_2$. Hence, we have shown statement 1.

Next, we show statement 2. Suppose that $\infty \in J_2$. We set $B = (\cup_{J \in \mathcal{J}, J_1 \leq J} J) \cup J_2$. Then, B is a disconnected closed set. Hence, there exists a multiply connected component A' of $\hat{\mathbb{C}} \setminus B$. By the same method as that of proof of statement 1, we see that A' is equal to a doubly connected component A of $F(G)$ such that $J_1 < A$. Hence, we have shown statement 2. □

Lemma 4.5. *Let H_0 be a real affine semigroup generated by a compact set C in* RA. *Suppose that each element $h \in C$ is of the form $h(x) = b_1(h)x + b_2(h)$, where $b_1(h), b_2(h) \in \mathbb{R}$, $|b_1(h)| > 1$. Then, for any subsemigroup H of H_0, we have $M(H) = J(\eta(H)) \subset \mathbb{R}$.*

Proof. From the assumption, there exists a number $R > 0$ such that for each $h \in C$, $\eta(h)(B(\infty, R)) \subset B(\infty, R)$. Hence, we have $B(\infty, R) \subset F(\eta(H))$, which implies that $J(\eta(H))$ is a bounded subset of $\mathbb{C}$. We consider the following cases:

Case 1: $\sharp(J(\eta(H))) \geq 3$.

Case 2: $\sharp(J(\eta(H))) \leq 2$.

Suppose that we have case 1. Then, from Theorem 3.2, it follows that $M(H) = J(\eta(H)) \subset \mathbb{R}$.

Suppose that we have case 2. Let $b(h)$ be the unique fixed point of $h \in H$ in $\mathbb{R}$. From the hypothesis, we have that for each $h \in H$, $b(h) \in J(\eta(H))$. Since we assume $\sharp(J(\eta(H))) \leq 2$, Lemma 3.1-1 implies that there exists a point $b \in \mathbb{R}$ such that for each $h \in H$, we have $b(h) = b$. Then any element $h \in H$ is of the form $h(x) = c_1(h)(x - b) + c_2(h)$, where $c_1(h), c_2(h) \in \mathbb{R}, |c_1(h)| > 1$. Hence, $M(H) = \{b\} \subset J(\eta(H))$. Suppose that there exists a point c in $J(\eta(H)) \setminus \{b\}$. Since $J(\eta(H))$ is a bounded set of $\mathbb{C}$, and since we have $h^{-1}(J(\eta(H))) \subset J(\eta(H))$ for each $h \in H$ (Lemma 3.1-1), we get that $h^{-1}(c) \in J(\eta(H)) \setminus (\{b\} \cup \{c\})$, for each element $h \in H$. This implies that $\sharp(J(\eta(H))) \geq 3$, which is a contradiction. Hence, we must have that $J(\eta(H)) = \{b\} = M(H)$. □

We need the notion of Green's functions, in order to demonstrate Theorem 2.12.

Definition 4.6. Let D be a domain in $\hat{\mathbb{C}}$ with $\infty \in D$. We denote by $\varphi(D, z)$ Green's function on D with pole at ∞. By definition, this is the unique function on $D \cap \mathbb{C}$ with the properties:

1. $\varphi(D, z)$ is harmonic and positive in $D \cap \mathbb{C}$;
2. $\varphi(D, z) - \log |z|$ is bounded in a neighborhood of ∞; and
3. $\varphi(D, z) \to 0$ as $z \to \partial D$.

Remark 4.7.

1. The limit $\lim_{z\to\infty}(\varphi(D, z) - \log |z|)$ exists and this is called *Robin's constant* of D.
2. If D is a simply connected domain with $\infty \in D$, then we have $\varphi(D, z) = -\log |\psi(z)|$, where $\psi : D \to \{z \in \mathbb{C} \mid |z| < 1\}$ denotes a biholomorphic map with $\psi(\infty) = 0$.
3. It is well-known that for any $g \in \text{Poly}_{\deg\geq 2}$,

$$\varphi(F_\infty(g), z) = \log |z| + \frac{1}{\deg(g) - 1} \log |a(g)| + o(1) \quad \text{as } z \to \infty. \quad (1)$$

   (See [30, p147].) Note that the point $-\frac{1}{\deg(g)-1} \log |a(g)| \in \mathbb{R}$ is the unique fixed point of $\Psi(g)$ in $\mathbb{R}$.

Lemma 4.8. *Let K_1 and K_2 be two non-empty connected compact sets in $\mathbb{C}$ such that $K_1 < K_2$. Let A_i denote the unbounded component of $\hat{\mathbb{C}} \setminus K_i$, for each $i = 1, 2$. Then, we have* $\lim_{z\to\infty}(\log |z| - \varphi(A_1, z)) < \lim_{z\to\infty}(\log |z| - \varphi(A_2, z))$.

Proof. The function $\phi(z) := \varphi(A_2, z) - \varphi(A_1, z) = (\log|z| - \varphi(A_1, z)) - (\log|z| - \varphi(A_2, z))$ is harmonic on $A_2 \cap \mathbb{C}$. This ϕ is bounded around ∞. Hence ϕ extends to a harmonic function on A_2. Moreover, since $K_1 < K_2$, we have $\limsup_{z \to \partial A_2} \phi(z) < 0$. From the maximum principle, it follows that $\phi(\infty) < 0$. Therefore, the statement of our lemma holds. □

In order to demonstrate Theorem 2.12-1, we will prove the following lemma. (Theorem 2.12-2 and Theorem 2.12-3 will be proved in Section 4.3.)

Lemma 4.9. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, there exists an injective map $\tilde{\Psi} : \hat{\mathcal{J}}_G \to \mathcal{M}_{\Psi(G)}$ such that:*

1. *if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Psi}(J_1) <_r \tilde{\Psi}(J_2)$;*
2. *if $J \in \hat{\mathcal{J}}_G$ and $\infty \in J$, then $+\infty \in \tilde{\Psi}(J)$; and*
3. *if $J \in \mathcal{J}_G$, then $\tilde{\Psi}(J) \subset \hat{\mathbb{R}} \setminus \{+\infty\}$.*

Proof. We first show the following claim.
Claim 1: In addition to the assumption of Lemma 4.9, if we have $\infty \in F(G)$, then $M(\Psi(G)) \subset \hat{\mathbb{R}} \setminus \{+\infty\}$.

To show this claim, let $R > 0$ be a number such that $J(G) \subset D(0, R)$. Then, for any $g \in G$, we have $K(g) < \partial D(0, R)$. By Lemma 4.8, we get that there exists a constant $C > 0$ such that for each $g \in G$, $\frac{-1}{\deg(g)-1} \log|a(g)| \le C$. Hence, it follows that $M(\Psi(G)) \subset [-\infty, C]$. Therefore, we have shown Claim 1.

We now prove the statement of the lemma in the case $G \in \mathcal{G}_{con}$. If $\infty \in F(G)$, then claim 1 implies that $M(\Phi(G)) \subset \hat{\mathbb{R}} \setminus \{+\infty\}$ and the statement of the lemma holds. We now suppose $\infty \in J(G)$. We put $L_g := \max_{z \in J(g)} |z|$ for each $g \in G$. Moreover, for each non-empty compact subset E of $\mathbb{C}$, we denote by Cap (E) the logarithmic capacity of E. We remark that $\mathrm{Cap}(E) = \exp(\lim_{z\to\infty}(\log|z| - \varphi(D_E, z)))$, where D_E denotes the connected component of $\hat{\mathbb{C}} \setminus E$ containing ∞. We may assume that $0 \in P^*(G)$. Then, by [1], we have $\mathrm{Cap}(J(g)) \ge \mathrm{Cap}\ ([0, L_g]) \ge L_g/4$ for each $g \in G$. Combining this with $\infty \in J(G)$ and Theorem 3.2, we obtain $+\infty \in M_{\Psi(G)}$ and the statement of the lemma holds.

We now prove the statement of the lemma in the case $G \in \mathcal{G}_{dis}$. Let $\{J_\lambda\}_{\lambda \in \Lambda}$ be the set $\hat{\mathcal{J}}_G$ of all connected components of $J(G)$. By Lemma 4.2, for each $\lambda \in \Lambda$ and each $n \in \mathbb{N}$, there exists an element $g_{\lambda,n} \in G$ such that

$$J(g_{\lambda,n}) \subset B(J_\lambda, \frac{1}{n}). \tag{2}$$

We have that the fixed point of $\Psi(g_{\lambda,n})$ in $\mathbb{R}$ is equal to $\frac{-1}{\deg(g_{\lambda,n})-1}\log|a(g_{\lambda,n})|$. We may assume that $\frac{-1}{\deg(g_{\lambda,n})-1}\log|a(g_{\lambda,n})| \to \alpha_\lambda$ as $n \to \infty$, where α_λ is an element of $\hat{\mathbb{R}}$. For each $\lambda \in \Lambda$, let $B_\lambda \in \mathcal{M}_{\Psi(G)}$ be the element with $\alpha_\lambda \in B_\lambda$. We will show the following claim.

Claim 2: If λ, ξ are two elements in Λ with $\lambda \neq \xi$, then $B_\lambda \neq B_\xi$. Moreover, if $J_\lambda, J_\xi \in \mathcal{J}_G$ and $J_\lambda < J_\xi$, then $B_\lambda <_r B_\xi$. Furthermore, if $J_\xi \in \hat{\mathcal{J}}_G$ with $\infty \in J_\xi$, then $+\infty \in B_\xi$.

To show this claim, let λ and ξ be two elements in Λ with $\lambda \neq \xi$. We have the following two cases:

Case 1: $J_\lambda, J_\xi \in \mathcal{J}_G$ and $J_\lambda < J_\xi$.

Case 2: $J_\lambda \in \mathcal{J}_G$ and $\infty \in J_\xi$.

Suppose that we have case 1. By Lemma 4.4, there exists a doubly connected component A of $F(G)$ such that

$$J_\lambda < A < J_\xi. \tag{3}$$

Let ζ_1 and ζ_2 be two Jordan curves in A such that they are not null-homotopic in A, and such that $\zeta_1 < \zeta_2$. For each $i = 1, 2$, let A_i be the unbounded component of $\hat{\mathbb{C}} \setminus \zeta_i$. Moreover, we set $\beta_i := \lim_{z\to\infty}(\log|z| - \varphi(A_i, z))$, for each $i = 1, 2$. By Lemma 4.8, we have $\beta_1 < \beta_2$. Let $g \in G$ be any element. By (2) and (3), there exists an $m \in \mathbb{N}$ such that $J(g_{\lambda,m}) < \zeta_1$. Since $P^*(G) \subset K(g_{\lambda,m})$, it follows that $P^*(G)$ is included in the bounded component of $\mathbb{C}\setminus\zeta_1$. Hence, we see that

$$\text{either } J(g) < \zeta_1, \text{ or } \zeta_2 < J(g). \tag{4}$$

From Lemma 4.8, it follows that either $\frac{-1}{\deg(g)-1}\log|a(g)| < \beta_1$, or $\beta_2 < \frac{-1}{\deg(g)-1}\log|a(g)|$. This implies that

$$M(\Psi(G)) \subset \hat{\mathbb{R}} \setminus (\beta_1, \beta_2), \tag{5}$$

where $(\beta_1, \beta_2) := \{x \in \mathbb{R} \mid \beta_1 < x < \beta_2\}$. Moreover, combining (2), (3), and (4), we get that there exists a number $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $J(g_{\lambda,n}) < \zeta_1 < \zeta_2 < J(g_{\xi,n})$. From Lemma 4.8, it follows that

$$\frac{-1}{\deg(g_{\lambda,n})-1}\log|a(g_{\lambda,n})| < \beta_1 < \beta_2 < \frac{-1}{\deg(g_{\xi,n})-1}\log|a(g_{\xi,n})|, \tag{6}$$

for each $n \geq n_0$. By (5) and (6), we obtain $B_\lambda <_r B_\xi$.

We now suppose that we have case 2. Then, by Lemma 4.4, there exists a doubly connected component A of $F(G)$ such that $J_\lambda < A$. Continuing the same argument as that of case 1, we obtain $B_\lambda \neq B_\xi$. In order to show $+\infty \in$

B_ξ, let R be any number such that $P^*(G) \subset D(0,R)$. Since $P^*(G) \subset K(g)$ for each $g \in G$, combining it with (2) and Lemma 4.3, we see that there exists an $n_0 = n_0(R)$ such that for each $n \geq n_0$, $D(0,R) < J(g_{\xi,n})$. From Lemma 4.8, it follows that $\frac{-1}{\deg(g_{\xi,n})-1} \log |a(g_{\xi,n})| \to +\infty$. Hence, $+\infty \in B_\xi$. Therefore, we have shown Claim 2.

Combining Claims 1 and 2, the statement of the lemma follows.

Therefore, we have proved Lemma 4.9. $\square$

We now demonstrate Theorem 2.12-1.

Proof of Theorem 2.12-1: From Lemma 4.9, Theorem 2.12-1 follows. $\square$

We now demonstrate Corollary 2.13.

Proof of Corollary 2.13: By Theorem 3.2, we have $J(\Theta(G)) = \overline{\cup_{h\in\Theta(G)} J(h)} = \overline{\cup_{g\in G} J(\Theta(g))}$, where the closure is taken in $\hat{\mathbb{C}}$. Since $J(\Theta(g)) = \{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}$, we obtain

$$J(\Theta(G)) = \overline{\cup_{g\in G}\{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}}, \tag{7}$$

where the closure is taken in $\hat{\mathbb{C}}$. Hence, we see that $\sharp(\hat{\mathcal{J}}_{\Theta(G)})$ is equal to the cardinality of the set of all connected components of $J(\Theta(G)) \cap [0,+\infty]$. Moreover, let $\psi : [0,+\infty] \to \hat{\mathbb{R}}$ be the homeomorphism defined by $\psi(x) := \log(x)$ for $x \in (0,+\infty)$, $\psi(0) := -\infty$, and $\psi(+\infty) = +\infty$. Then, (7) implies that, the map $\psi : [0,\infty] \to \hat{\mathbb{R}}$, maps $J(\Theta(G)) \cap [0,+\infty]$ onto $M(\Psi(\Theta(G)))$. For any $J \in \hat{\mathcal{J}}_{\Theta(G)}$, let $\tilde{\psi}(J) \in \mathcal{M}_{\Psi(\Theta(G))} = \mathcal{M}_{\Psi(G)}$ be the element such that $\psi(J \cap [0,+\infty]) = \tilde{\psi}(J)$. Then, the map $\tilde{\psi} : \hat{\mathcal{J}}_{\Theta(G)} \to \mathcal{M}_{\Psi(\Theta(G))} = \mathcal{M}_{\Psi(G)}$ is a bijection, and moreover, for any $J_1, J_2 \in \mathcal{J}_{\Theta(G)}$, we have that $J_1 < J_2$ if and only if $\tilde{\psi}(J_1) <_r \tilde{\psi}(J_2)$. Furthermore, for any $J \in \hat{\mathcal{J}}_{\Theta(G)}$, $\infty \in J$ if and only if $+\infty \in \tilde{\psi}(J)$. Let $\tilde{\Theta} : \hat{\mathcal{J}}_G \to \hat{\mathcal{J}}_{\Theta(G)}$ be the map defined by $\tilde{\Theta} = \tilde{\psi}^{-1} \circ \tilde{\Psi}$, where $\tilde{\Psi} : \hat{\mathcal{J}}_G \to \mathcal{M}_{\Psi(G)}$ is the map in Lemma 4.9. Then, by Lemma 4.9, $\tilde{\Theta} : \hat{\mathcal{J}}_G \to \hat{\mathcal{J}}_{\Theta(G)}$ is injective, and moreover, if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Theta}(J_1) \in \mathcal{J}_{\Theta(G)}$, $\tilde{\Theta}(J_2) \in \mathcal{J}_{\Theta(G)}$, and $\tilde{\Theta}(J_1) < \tilde{\Theta}(J_2)$.

Thus, we have proved Corollary 2.13. $\square$

We now demonstrate Theorem 2.14.

Proof of Theorem 2.14: We have that for any $j = 1, \ldots, m$, $(\Psi(h_j))^{-1}(x) = \frac{1}{\deg(h_j)}(x - \log|a_j|) = \frac{1}{\deg(h_j)}(x - \frac{-1}{\deg(h_j)-1}\log|a_j|) + \frac{-1}{\deg(h_j)-1}\log|a_j|$, where $x \in \mathbb{R}$. Hence, it is easy to see that $\cup_{j=1}^m (\Psi(h_j))^{-1}([\alpha,\beta]) \subset [\alpha,\beta]$. From the assumption, it follows that

$$\cup_{j=1}^m (\Psi(h_j))^{-1}([\alpha,\beta]) = [\alpha,\beta]. \tag{8}$$

Moreover, by Lemma 3.1-2, we have

$$\cup_{j=1}^{m}(\eta(\Psi(h_j)))^{-1}(J(\eta(\Psi(G)))) = J(\eta(\Psi(G))). \qquad (9)$$

Furthermore, by Lemma 4.5, $J(\eta(\Psi(G)))$ is a compact subset of $\mathbb{R}$. Applying [10, Theorem 2.6], it follows that $J(\eta(\Psi(G))) = [\alpha, \beta]$. Combined with Lemma 4.5, we obtain $M(\Psi(G)) = [\alpha, \beta]$. Hence, $M(\Psi(G))$ is connected. Therefore, from Theorem 2.12-1, it follows that $J(G)$ is connected. □

We now demonstrate Theorem 2.15.

Proof of Theorem 2.15: Let C be a set of polynomials of degree two such that C generates G. Suppose that $J(G)$ is disconnected. Then, by Theorem 2.1, there exist two elements $h_1, h_2 \in C$ such that the semigroup $H = \langle h_1, h_2 \rangle$ satisfies that $J(H)$ is disconnected. For each $j = 1, 2$, let a_j be the coefficient of the highest degree term of polynomial h_j. Let $\alpha := \min_{j=1,2}\{\frac{-1}{\deg(h_j)-1}\log|a_j|\}$ and $\beta := \max_{j=1,2}\{\frac{-1}{\deg(h_j)-1}\log|a_j|\}$. Then we have that $\alpha = \min_{j=1,2}\{-\log|a_j|\}$ and $\beta = \max_{j=1,2}\{-\log|a_j|\}$. Since $\Psi(h_j)^{-1}(x) = \frac{1}{2}(x - \log|a_j|) = \frac{1}{2}(x - (-\log|a_j|)) + (-\log|a_j|)$ for each $j = 1, 2$, we obtain $[\alpha, \beta] = \cup_{j=1}^{2}(\Psi(h_j))^{-1}([\alpha, \beta])$. Hence, by Theorem 2.14, it must be true that $J(H)$ is connected. However, this is a contradiction. Therefore, $J(G)$ must be connected. □

We now demonstrate Theorem 2.16.

Proof of Theorem 2.16: For each $\lambda \in \Lambda$, let b_λ be the fixed point of $\Psi(h_\lambda)$ in $\mathbb{R}$. It is easy to see that $b_\lambda = \frac{-1}{\deg(h_\lambda)-1}\log|a_\lambda|$, for each $\lambda \in \Lambda$. From the assumption, it follows that there exists a point $b \in \mathbb{R}$ such that for each $\lambda \in \Lambda$, $b_\lambda = b$. This implies that for any element $g \in G$, the fixed point $b(g) \in \mathbb{R}$ of $\Psi(g)$ in $\mathbb{R}$ is equal to b. Hence, we obtain $M(\Psi(G)) = \{b\}$. Therefore, $M(\Psi(G))$ is connected. From Theorem 2.12-1, it follows that $J(G)$ is connected. □

4.3 Proofs of results in 2.3

In this section, we prove results in 2.3, Theorem 2.12-2 and Theorem 2.12-3.

In order to demonstrate Theorem 2.19, Theorem 2.12-2, and Theorem 2.12-3, we need the following lemma.

Lemma 4.10. *If $G \in \mathcal{G}_{dis}$, then $\infty \in F(G)$.*

Proof. Suppose that $G \in \mathcal{G}_{dis}$ and $\infty \in J(G)$. We will deduce a contradiction. By Lemma 4.3, the element $J \in \hat{\mathcal{J}}_G$ with $\infty \in J$ satisfies that $J = \{\infty\}$.

Hence, by Lemma 4.2, for each $n \in \mathbb{N}$, there exists an element $g_n \in G$ such that $J(g_n) \subset B(\infty, \frac{1}{n})$. Let $R > 0$ be any number which is sufficiently large so that $P^*(G) \subset B(0, R)$. Since we have that $P^*(G) \subset K(g)$ for each $g \in G$, it must hold that there exists an number $n_0 = n_0(R) \in \mathbb{N}$ such that for each $n \geq n_0$, $B(0, R) < J(g_n)$. From Lemma 4.8, it follows that $\lim_{z\to\infty}(\log|z| - \varphi(F_\infty(g_n), z)) \to +\infty$ as $n \to \infty$. Hence, we see that $\frac{-1}{\deg(g_n)-1} \log|a(g_n)| \to +\infty$, as $n \to \infty$. This implies that

$$|a(g_n)|^{-\frac{1}{\deg(g_n)-1}} \to \infty, \text{ as } n \to \infty. \tag{10}$$

Furthermore, by Theorem 2.12-1, we must have that $M(\Psi(G))$ is disconnected.

We now consider the polynomial semigroup $H = \{z \mapsto |a(g)|z^{\deg(g)} \mid g \in G\} \in \mathcal{G}$. By Theorem 3.2, we have $J(H) = \overline{\cup_{h\in H} J(h)}$. Since the Julia set of polynomial $|a(g)|z^{\deg(g)}$ is equal to $\{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}$, it follows that

$$J(H) = \overline{\cup_{g\in G}\{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}}, \tag{11}$$

where the closure is taken in $\hat{\mathbb{C}}$. Moreover, $J(\Theta(G)) = J(H)$. Combining it with (10), (11), and Corollary 2.13, we see that

$$\infty \in J(H), \text{ and } J(H) \text{ is disconnected.} \tag{12}$$

Let $\psi : [0, +\infty] \to \hat{\mathbb{R}}$ be the homeomorphism as in the proof of Corollary 2.13. By (11), we have

$$\psi(J(H) \cap [0, +\infty]) = M(\Psi(H)) = M(\Psi(G)). \tag{13}$$

Moreover, by Lemma 3.1-1, we have

$$h(F(H) \cap [0, +\infty]) \subset F(H) \cap [0, +\infty], \text{ for each } h \in H. \tag{14}$$

Furthermore, we have that

$$\psi \circ h = \Psi(h) \circ \psi \text{ on } [0, +\infty], \text{ for each } h \in H. \tag{15}$$

Combining (13), (14), and (15), we see that

$$\Psi(h)(\hat{\mathbb{R}} \setminus M(\Psi(H))) \subset (\hat{\mathbb{R}} \setminus M(\Psi(H))), \text{ for each } h \in H. \tag{16}$$

By Lemma 4.3 and (12), we get that the connected component J of $J(H)$ containing ∞ satisfies that

$$J = \{\infty\}. \tag{17}$$

Combined with Lemma 4.2, we see that for each $n \in \mathbb{N}$, there exists an element $h_n \in H$ such that

$$J(h_n) \subset B(\infty, \frac{1}{n}). \tag{18}$$

Combining (11), (13), (17), and (18), we obtain the following claim.
Claim 1: $+\infty$ is a non-isolated point of $M(\Psi(H))$ and the connected component of $M(\Psi(H))$ containing $+\infty$ is equal to $\{+\infty\}$.

Let $h \in H$ be an element. Conjugating G by some linear transformation, we may assume that h is of the form $h(z) = z^s, s \in \mathbb{N}, s > 1$. Hence $\Psi(h)(x) = sx, s > 1$. Since 0 is a fixed point of $\Psi(h)$, we have that $0 \in M(\Psi(H))$. By Claim 1, there exists $c_1, c_2 \in [0, +\infty)$ with $c_1 < c_2$ such that the open interval $I = (c_1, c_2)$ is a connected component of $\hat{\mathbb{R}} \setminus M(\Psi(H))$. We now show the following claim.
Claim 2: Let $Q = (r_1, r_2) \subset (0, +\infty)$ be any connected open interval in $\hat{\mathbb{R}} \setminus M(\Psi(H))$, where $0 \leq r_1 < r_2 < +\infty$. Then, we have $r_2 \leq sr_1$.

To show this claim, suppose that $sr_1 < r_2$. Then, it implies that $\cup_{n \in \mathbb{N} \cup \{0\}} \Psi(h)^n(Q) = (r_1, +\infty)$. However, by (16), we have $\cup_{n \in \mathbb{N} \cup \{0\}} \Psi(h)^n(Q) \subset \hat{\mathbb{R}} \setminus M(\Psi(H))$, which implies that the connected component Q' of $\hat{\mathbb{R}} \setminus M(\Psi(H))$ containing Q satisfies that $Q' \supset (r_1, +\infty)$. This contradicts Claim 1. Hence, we obtain Claim 2.

By Claim 2, we obtain $c_1 > 0$. Let $c_3 \in (0, c_1)$ be a number so that $c_2 - c_3 > s(c_1 - c_3)$. Since $c_1 \in M(\Psi(H))$, there exists an element $c \in (c_3, c_1]$ and an element $h_1 \in H$ such that $\Psi(h_1)(c) = c$ and $(\Psi(h_1))'(c) > 1$. Since $c_2 - c_3 > s(c_1 - c_3)$, we obtain

$$c_2 - c > s(c_1 - c). \tag{19}$$

Let $t := (\Psi(h_1))'(c) > 1$. Then, for each $n \in \mathbb{N}$, we have $(\Psi(h_1))^n(I) = (t^n(c_1 - c) + c, t^n(c_2 - c) + c)$. From Claim 2, it follows that $t^n(c_2 - c) + c \leq s(t^n(c_1 - c) + c)$, for each $n \in \mathbb{N}$. Dividing both sides by t^n and then letting $n \to \infty$, we obtain $c_2 - c \leq s(c_1 - c)$. However, this contradicts (19). Hence, we must have that $\infty \in F(G)$. Thus, we have proved Lemma 4.10. $\square$

We now demonstrate Proposition 2.18.

Proof of Proposition 2.18: Let U be a connected component of $F(G)$ with $U \cap \hat{K}(G) \neq \emptyset$. Let $g \in G$ be an element. Then we have $\hat{K}(G) \cap F(G) \subset \text{int}(K(g))$. Since $h(F(G)) \subset F(G)$ and $h(\hat{K}(G) \cap F(G)) \subset \hat{K}(G) \cap F(G)$ for each $h \in G$, it follows that $h(U) \subset \text{int}(K(g))$ for each $h \in G$. Hence $U \subset \text{int}(\hat{K}(G))$. From this, it is easy to see $\hat{K}(G) \cap F(G) = \text{int}(\hat{K}(G))$. By the maximum principle, we see that U is simply connected. $\square$

We now demonstrate Theorem 2.19.

Proof of Theorem 2.19:

First, we show statement 1. By Lemma 4.10, we have that $\infty \in F(G)$. Let $F_\infty(G)$ be the connected component of $F(G)$ containing ∞. Let $J \in \mathcal{J}$ be an element such that $\partial F_\infty(G) \cap J \neq \emptyset$. Let D be the unbounded component of $\hat{\mathbb{C}} \setminus J$. Then $F_\infty(G) \subset D$ and D is simply connected. We show $F_\infty(G) = D$. Otherwise, there exists an element $J_1 \in \mathcal{J}$ such that $J_1 \neq J$ and $J_1 \subset D$. By Theorem 2.7-1, we have either $J_1 < J$ or $J < J_1$. Hence, it follows that $J < J_1$ and we have that J is included in a bounded component D_0 of $\mathbb{C} \setminus J_1$. Since $F_\infty(G)$ is included in the unbounded component D_1 of $\hat{\mathbb{C}} \setminus J_1$, it contradicts $\partial F_\infty(G) \cap J \neq \emptyset$. Hence, $F_\infty(G) = D$ and $F_\infty(G)$ is simply connected.

Next, let $J_{\max}$ be the element of $\mathcal{J}$ with $\partial F_\infty(G) \subset J_{\max}$, and suppose that there exists an element $J \in \mathcal{J}$ such that $J_{\max} < J$. Then $J_{\max}$ is included in a bounded component of $\mathbb{C} \setminus J$. On the other hand, $F_\infty(G)$ is included in the unbounded component of $\hat{\mathbb{C}} \setminus J$. Since $\partial F_\infty(G) \subset J_{\max}$, we have a contradiction. Hence, we have shown that $J \leq J_{\max}$ for each $J \in \mathcal{J}$.

Therefore, we have shown statement 1.

Next, we show statement 2. Since $\emptyset \neq P^*(G) \subset \hat{K}(G)$, we have $\hat{K}(G) \neq \emptyset$. By Proposition 2.18, we have $\partial \hat{K}(G) \subset J(G)$. Let J_1 be a connected component of $J(G)$ with $J_1 \cap \partial \hat{K}(G) \neq \emptyset$. By Lemma 4.3, $J_1 \in \mathcal{J}$. Suppose that there exists an element $J \in \mathcal{J}$ such that $J < J_1$. Let $z_0 \in J$ be a point. By Theorem 3.2, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in G such that $d(z_0, J(g_n)) \to 0$ as $n \to \infty$. Then by Lemma 4.1, $\sup_{z\in J(g_n)} d(z, J) \to 0$ as $n \to \infty$. Since J_1 is included in the unbounded component of $\mathbb{C} \setminus J$, it follows that for a large $n \in \mathbb{N}$, J_1 is included in the unbounded component of $\mathbb{C} \setminus J(g_n)$. However, this causes a contradiction, since $J_1 \cap \hat{K}(G) \neq \emptyset$. Hence, by Theorem 2.7-1, it must hold that $J_1 \leq J$ for each $J \in \mathcal{J}$. This argument shows that if J_1 and J_2 are two connected components of $J(G)$ such that $J_i \cap \partial \hat{K}(G) \neq \emptyset$ for each $i = 1, 2$, then $J_1 = J_2$. Hence, we conclude that there exists a unique minimal element $J_{\min}$ in $(\mathcal{J}, \leq)$ and $\partial \hat{K}(G) \subset J_{\min}$.

Next, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Suppose $D \cap \hat{K}(G) \neq \emptyset$. Let $x \in D \cap \hat{K}(G)$ be a point. By Theorem 3.2 and Lemma 4.1, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in G such that $\sup_{z\in J(g_n)} d(z, J_{\min}) \to 0$ as $n \to \infty$. Then, for a large $n \in \mathbb{N}$, x is in the unbounded component of $\mathbb{C} \setminus J(g_n)$. However, this is a contradiction, since $g_n^l(x) \to \infty$ as $l \to \infty$, and $x \in \hat{K}(G)$. Hence, we have shown statement 2.

Next, we show statement 3. By Theorem 2.1, there exist $\lambda_1, \lambda_2 \in \Lambda$

and connected components J_1, J_2 of $J(G)$ such that $J_1 \neq J_2$ and $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$. By Lemma 4.3, we have $J_i \in \mathcal{J}$ for each $i = 1, 2$. Then $J(h_{\lambda_1}) \cap J(h_{\lambda_2}) = \emptyset$. Since $P^*(G)$ is bounded in $\mathbb{C}$, we may assume $J(h_{\lambda_2}) < J(h_{\lambda_1})$. Then we have $K(h_{\lambda_2}) \subset \text{int}(K(h_{\lambda_1}))$ and $J_2 < J_1$. By statement 2, $J_1 \neq J_{\min}$. Hence $J(h_{\lambda_1}) \cap J_{\min} = \emptyset$. Since $P^*(G)$ is bounded in $\mathbb{C}$, we have that $K(h_{\lambda_2})$ is connected. Let U be the connected component of $\text{int}(K(h_{\lambda_1}))$ containing $K(h_{\lambda_2})$. Since $P^*(G) \subset K(h_{\lambda_2})$, it follows that there exists an attracting fixed point z_1 of h_{λ_1} in $K(h_{\lambda_2})$ and U is the immediate attracting basin for z_1 with respect to the dynamics of h_{λ_1}. Furthermore, by Corollary 3.7, $h_{\lambda_1}^{-1}(J(h_{\lambda_2}))$ is connected. Therefore, $h_{\lambda_1}^{-1}(U) = U$. Hence, $\text{int}(K(h_{\lambda_1})) = U$.

Suppose that there exists an $n \in \mathbb{N}$ such that $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) \neq \emptyset$. Then, by Corollary 3.7, $A := \cup_{s \geq 0} h_{\lambda_1}^{-ns}(J(h_{\lambda_2}))$ is connected and its closure $\overline{A}$ contains $J(h_{\lambda_1})$. Hence $J(h_{\lambda_1})$ and $J(h_{\lambda_2})$ are included in the same connected component of $J(G)$. This is a contradiction. Therefore, for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$. Similarly, for each $n \in \mathbb{N}$, we have $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$. Combining $h_{\lambda_1}^{-1}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ with $z_1 \in K(h_{\lambda_2})$, we obtain $z_1 \in \text{int}(K(h_{\lambda_2}))$. Hence, we have proved statement 3.

We now prove statement 4. Let $g \in G$ be an element with $J(g) \cap J_{\min} = \emptyset$. We show the following:
Claim 2: $J_{\min} < J(g)$.

To show the claim, suppose that $J_{\min}$ is included in the unbounded component U of $\mathbb{C} \setminus J(g)$. Since $\emptyset \neq \partial \hat{K}(G) \subset J_{\min}$, it follows that $\hat{K}(G) \cap U \neq \emptyset$. However, this is a contradiction. Hence, we have shown Claim 2.

Combining Claim 2, Theorem 3.2 and Lemma 4.1, we get that there exists an element $h_1 \in G$ such that $J(h_1) < J(g)$. From an argument which we have used in the proof of statement 3, it follows that g has an attracting fixed point z_g in $\mathbb{C}$ and $\text{int}(K(g))$ consists of only one immediate attracting basin for z_g. Hence, we have shown statement 4.

Next, we show statement 5. Suppose that $\text{int}(\hat{K}(G)) = \emptyset$. We will deduce a contradiction. If $\text{int}(\hat{K}(G)) = \emptyset$, then by Proposition 2.18, we obtain $F(G) \cap \hat{K}(G) = \emptyset$. By statement 3, there exist two elements g_1 and g_2 of G and two elements J_1 and J_2 of $\mathcal{J}$ such that $J_1 \neq J_2$, such that $J(g_i) \subset J_i$ for each $i = 1, 2$, such that g_1 has an attracting fixed point z_0 in $\text{int}(K(g_2))$, and such that $K(g_2) \subset \text{int}(K(g_1))$. Since we assume $F(G) \cap \hat{K}(G) = \emptyset$, we have $z_0 \in P^*(G) \subset \hat{K}(G) \subset J(G)$. Let J be the connected component of $J(G)$ containing z_0. We now show $J = \{z_0\}$. Suppose $\sharp J \geq 2$. Then $J(g_1) \subset \overline{\cup_{n \geq 0} g_1^{-n}(J)}$. Moreover, by Theorem 2.7-3, $g_1^{-n}J$ is connected for each $n \in \mathbb{N}$. Since $g_1^{-n}(J) \cap J \neq \emptyset$ for each $n \in \mathbb{N}$, we see that $\overline{\cup_{n \geq 0} g_1^{-n}(J)}$ is connected. Combining this with $z_0 \in \text{int}(K(g_2))$, $K(g_2) \subset \text{int}(K(g_1))$, $z_0 \in J$ and $J(g_1) \subset \overline{\cup_{n\geq 0} g_1^{-n}(J)}$, we obtain $\overline{\cup_{n\geq 0} g_1^{-n}(J)} \cap J(g_2) \neq \emptyset$. Then it follows that $J(g_1)$ and $J(g_2)$ are included in the same connected component of $J(G)$. This is a contradiction. Hence, we have shown $J = \{z_0\}$. By statement 2, we obtain $\{z_0\} = J_{\min} = P^*(G)$. Let $\varphi(z) := \frac{1}{z-z_0}$ and let $\tilde{G} := \{\varphi g \varphi^{-1} \mid g \in G\}$. Then $\tilde{G} \in \mathcal{G}_{dis}$. Moreover, since $z_0 \in J(G)$, we have that $\infty \in J(\tilde{G})$. This contradicts Lemma 4.10. Therefore, we must have that $\text{int}(\hat{K}(G)) \neq \emptyset$.

Since $\partial \hat{K}(G) \subset J_{\min}$ (statement 2) and $\hat{K}(G)$ is bounded, it follows that $\mathbb{C} \setminus J_{\min}$ is disconnected and $\sharp J_{\min} \geq 2$. Hence, $\sharp J \geq 2$ for each $J \in \mathcal{J} = \hat{\mathcal{J}}$. Now, let $g \in G$ be an element with $J(g) \cap J_{\min} = \emptyset$. we show $J_{\min} \neq g^*(J_{\min})$. If $J_{\min} = g^*(J_{\min})$, then $g^{-1}(J_{\min}) \subset J_{\min}$. Since $\sharp J_{\min} \geq 3$, it follows that $J(g) \subset J_{\min}$, which is a contradiction. Hence, $J_{\min} \neq g^*(J_{\min})$, and so $J_{\min} < g^*(J_{\min})$. Combined with Theorem 2.7-3, we obtain $g^{-1}(J(G)) \cap J_{\min} = \emptyset$. Since $g(\hat{K}(G)) \subset \hat{K}(G)$, we have $g(\text{int}(\hat{K}(G))) \subset \text{int}(\hat{K}(G))$. Suppose $g(\partial \hat{K}(G)) \cap \partial \hat{K}(G) \neq \emptyset$. Then, since $\partial \hat{K}(G) \subset J_{\min}$ (statement 2), we obtain $g(J_{\min}) \cap J_{\min} \neq \emptyset$. This implies $g^{-1}(J_{\min}) \cap J_{\min} \neq \emptyset$, which contradicts $g^{-1}(J(G)) \cap J_{\min} = \emptyset$. Hence, it must hold $g(\partial \hat{K}(G)) \subset \text{int}(\hat{K}(G))$, and so $g(\hat{K}(G)) \subset \text{int}(\hat{K}(G))$. Moreover, since $g^{-1}(J(G)) \cap J_{\min} = \emptyset$, we have that $g(J_{\min})$ is a connected subset of $F(G)$. Since $\partial \hat{K}(G) \subset J_{\min}$ and $g(\partial \hat{K}(G)) \subset \text{int}(\hat{K}(G))$, Proposition 2.18 implies that $g(J_{\min})$ must be contained in $\text{int}(\hat{K}(G))$.

By statement 4, g has a unique attracting fixed point z_g in $\mathbb{C}$. Then, $z_g \in P^*(G) \subset \hat{K}(G)$. Hence, $z_g = g(z_g) \in g(\hat{K}(G)) \subset \text{int}(\hat{K}(G))$. Hence, we have shown statement 5.

We now show statement 6. Since $F_\infty(G)$ is simply connected (statement 1), we have $\cup_{A \in \mathcal{A}} A \subset \mathbb{C}$. Suppose that there exist two distinct elements A_1 and A_2 in $\mathcal{A}$ such that A_1 is included in the unbounded component of $\mathbb{C} \setminus A_2$, and such that A_2 is included in the unbounded component of $\mathbb{C} \setminus A_1$. For each $i = 1, 2$, Let $J_i \in \mathcal{J}$ be the element that intersects the bounded component of $\mathbb{C} \setminus A_i$. Then, $J_1 \neq J_2$. Since $(\mathcal{J}, \leq)$ is totally ordered (Theorem 2.7-1), we may assume that $J_1 < J_2$. Then, it implies that $A_1 < J_2 < A_2$, which is a contradiction. Hence, $(\mathcal{A}, \leq)$ is totally ordered. Therefore, we have proved statement 6.

Thus, we have proved Theorem 2.19. □

We now demonstrate Theorem 2.21.

Proof of Theorem 2.21: First, we show Theorem 2.21-1. If $G \in \mathcal{G}_{con}$, then $J(G)$ is uniformly perfect.

We now suppose that $G \in \mathcal{G}_{dis}$. Let A be an annulus separating $J(G)$. Then A separates $J_{\min}$ and $J_{\max}$. Let D be the unbounded component of $\mathbb{C} \setminus$

$J_{\min}$ and let U be the connected component of $\mathbb{C} \setminus J_{\max}$ containing $J_{\min}$. Then it follows that $A \subset U \cap D$. Since $\sharp J_{\min} > 1$ and $\infty \in F(G)$ (Theorem 2.19), we get that the doubly connected domain $U \cap D$ satisfies mod $(U \cap D) < \infty$. Hence, we obtain mod $A \leq$ mod $(U \cap D) < \infty$. Therefore, $J(G)$ is uniformly perfect.

If a point $z_0 \in J(G)$ is a superattracting fixed point of an element $g \in G$, then, combining uniform perfectness of $J(G)$ and [15, Theorem 4.1], it follows that $z_0 \in \text{int}(J(G))$. Thus, we have shown Theorem 2.21-1.

Next, we show Theorem 2.21-2. If $G \in \mathcal{G}$ and $\infty \in J(G)$, then by Lemma 4.10, we obtain $G \in \mathcal{G}_{con}$. Moreover, Theorem 2.21-1 implies that $\infty \in \text{int}(J(G))$. Therefore, we have shown Theorem 2.21-2.

We now show Theorem 2.21-3. Suppose that $G \in \mathcal{G}_{dis}$. Let $g \in G$ and let $z_1 \in J(G) \cap \mathbb{C}$ with $g(z_1) = z_1$ and $g'(z_1) = 0$. Then, $z_1 \in P^*(G) \subset \hat{K}(G)$. By Theorem 2.19-2, we obtain $z_1 \in J_{\min}$. Moreover, Theorem 2.21-1 implies that $z_1 \in \text{int}(J(G))$. Combining this and $z_1 \in J_{\min}$, we obtain $z_1 \in \text{int}(J_{\min})$. By Theorem 2.19- 5b, we obtain $J(g) \subset J_{\min}$.

Hence, we have shown Theorem 2.21. □

We now demonstrate Theorem 2.12-2.

Proof of Theorem 2.12-2: Suppose $G \in \mathcal{G}_{dis}$. Then, by Lemma 4.10, we obtain $\infty \in F(G)$. Hence, there exists a number $R > 0$ such that for each $g \in G$, $J(g) < \partial B(0, R)$. From Lemma 4.8, it follows that there exists a constant $C_1 > 0$ such that for each $g \in G$, $\frac{-1}{\deg(g)-1} \log |a(g)| < C_1$. This implies that there exists a constant $C_2 \in \mathbb{R}$ such that

$$M(\Psi(G)) \subset [-\infty, C_1]. \tag{20}$$

Moreover, by Theorem 2.19-5, we have that $\text{int}(\hat{K}(G)) \neq \emptyset$. Let B be a closed disc in $\text{int}(\hat{K}(G))$. Then it must hold that for each $g \in G$, $B < J(g)$. Hence, by Lemma 4.8, there exists a constant $C_3 \in \mathbb{R}$ such that for each $g \in G$, $C_3 \leq \frac{-1}{\deg(g)-1} \log |a(g)|$. Therefore, we obtain

$$M(\Psi(G)) \subset [C_3, +\infty]. \tag{21}$$

Combining (20) and (21), we obtain $M(\Psi(G)) \subset \mathbb{R}$. Let C_4 be a large number so that $M(\Psi(G)) \subset D(0, C_4)$. Since for each $g \in G$, the repelling fixed point $-\frac{1}{\deg(g)-1} \log |a(g)|$ of $\eta(\Psi(g))$ belongs to $D(0, C_4) \cap \mathbb{R}$, we see that for each $z \in \mathbb{C} \setminus D(0, C_4)$, $|\eta(\Psi(g))(z)| = |\deg(g)(z - \frac{-1}{\deg(g)-1} \log |a(g)|) + \frac{-1}{\deg(g)-1} \log |a(g)|| \geq \deg(g) C_4 - (\deg(g) - 1) C_4 = C_4$. It follows that $\infty \in F(\eta(\Psi(G)))$. Combining this and Theorem 3.2, we obtain $M(\Psi(G)) = J(\eta(\Psi(G)))$, if $\sharp(J(\eta(\Psi(G)))) \geq 3$.

Suppose that $\sharp(J(\eta(\Psi(G)))) = 2$. Let $g \in G$ be an element and let $b \in \mathbb{R}$ be the unique fixed point of $\Psi(g)$ in $\mathbb{R}$. Then, since $\infty \in F(\eta(\Psi(G)))$, there exists a point $c \in (J(\eta(\Psi(G))) \cap \mathbb{C}) \setminus \{b\}$. By Lemma 3.1-1, $(\eta(\Psi(g)))^{-1}(c) \in J(\eta(\Psi(G))) \setminus \{b, c\}$. This contradicts $\sharp(J(\eta(\Psi(G)))) = 2$. Hence it must hold that $\sharp(J(\eta(\Psi(G)))) \neq 2$.

Suppose that $\sharp(J(\eta(\Psi(G)))) = 1$. Since $M(\Psi(G)) \subset \mathbb{R}$ and $M(\Psi(G)) \cap \mathbb{R} \subset J(\eta(\Psi(G)))$, it follows that $M(\Psi(G)) = J(\eta(\Psi(G)))$.

Therefore, we always have that $M(\Psi(G)) = J(\eta(\Psi(G)))$. Thus, we have proved Theorem 2.12-2. □

We now demonstrate Theorem 2.12-3.

Proof of Theorem 2.12-3: By Theorem 2.12-1 and Theorem 2.12-2, the statement holds. □

We now demonstrate Proposition 2.22.

Proof of Proposition 2.22: First, we show statement 1. Let $g \in Q_1$. We show the following:

Claim 1: For any element $J_3 \in \mathcal{J}$ with $J_1 \leq J_3$, we have $J_1 \leq g^*(J_3)$.

To show this claim, let $J \in \mathcal{J}$ be an element with $J(g) \subset J$. We consider the following two cases;

Case 1: $J \leq J_3$, and

Case 2: $J_1 \leq J_3 \leq J$.

Suppose that we have Case 1. Then, $J_1 \leq J = g^*(J) \leq g^*(J_3)$. Hence, the statement of Claim 1 is true.

Suppose that we have Case 2. If we have $g^*(J_3) < J_3$, then, we have $(g^n)^*(J_3) \leq g^*(J_3) < J_3 \leq J$ for each $n \in \mathbb{N}$. Hence, $\inf\{d(z, J) \mid z \in g^{-n}(J_3), n \in \mathbb{N}\} > 0$. However, since $J(g) \subset J$ and $\sharp J_3 \geq 3$, we obtain a contradiction. Hence, we must have $J_3 \leq g^*(J_3)$, which implies $J_1 \leq J_3 \leq g^*(J_3)$. Hence, we conclude that Claim 1 holds.

Now, let $K_1 := J(G) \cap (J_1 \cup A_1)$. Then, by Claim 1, we obtain $g^{-1}(K_1) \subset K_1$, for each $g \in Q_1$. From Lemma 3.1-6, it follows that $J(H_1) \subset K_1$. Hence, we have shown statement 1.

Next, we show statement 2. Let $g \in Q_2$. Then, by the same method as that of the proof of Claim 1, we obtain the following.

Claim 2: For any element $J_4 \in \mathcal{J}$ with $J_4 \leq J_2$, we have $g^*(J_4) \leq J_2$.

Now, let $K_2 := J(G) \cap (\mathbb{C} \setminus A_2)$. Then, by Claim 2, we obtain $g^{-1}(K_2) \subset K_2$, for each $g \in Q_2$. From Lemma 3.1-6, it follows that $J(H_2) \subset K_2$. Hence, we have shown statement 2.

Next, we show statement 3. By statements 1 and 2, we obtain $J(H) \subset J(H_1) \cap J(H_2) \subset K_1 \cap K_2 \subset (\mathbb{C} \setminus A_2) \cap (J_1 \cup A_1) \subset J_1 \cup (A_1 \setminus A_2)$.

Hence, we have proved Proposition 2.22. □

We now demonstrate Proposition 2.23.

Proof of Proposition 2.23: By Theorem 2.7 and Theorem 2.19, $(\mathcal{J}, \leq)$ is totally ordered and there exists a maximal element $J_{\max}$ and a minimal element $J_{\min}$. Suppose that for any $h \in \Gamma$, $J(h) \cap J_{\max} = \emptyset$. Then, since $\sharp J_{\max} \geq 3$ (Theorem 2.19-5a), we get that for any $h \in \Gamma$, $h^{-1}(J_{\max}) \cap J_{\max} = \emptyset$. Combining it with Theorem 2.7-3, it follows that for any $h \in \Gamma$, $h^{-1}(J(G)) \cap J_{\max} = \emptyset$. However, since $J(G) = \cup_{h \in \Gamma} h^{-1}(J(G))$ (Lemma 3.1-2), it causes a contradiction. Hence, there must be an element $h_1 \in \Gamma$ such that $J(h_1) \subset J_{\max}$.

By the same method as above, we can show that there exists an element $h_2 \in \Gamma$ such that $J(h_2) \subset J_{\min}$. □

4.4 Proofs of results in 2.4

In this section, we prove results in 2.4.

We now prove Theorem 2.24.

Proof of Theorem 2.24: Combining the assumption and Theorem 2.7-3, we get that for each $h \in \Gamma$ and each $j \in \{1, \ldots, n\}$, there exists a $k \in \{1, \ldots, n\}$ with $h^{-1}(J_j) \subset J_k$. Hence,

$$h^{-1}(\cup_{j=1}^n J_j) \subset \cup_{j=1}^n J_j, \text{ for each } h \in \Gamma. \tag{22}$$

Moreover, by Theorem 2.19-5a, we obtain

$$\sharp(\cup_{j=1}^n J_j) \geq 3. \tag{23}$$

Combining (22), (23), and Lemma 3.1-6, it follows that $J(G) \subset \bigcup_{j=1}^n J_j$. Hence, $J(G) = \cup_{j=1}^n J_j$. Therefore, we have proved Theorem 2.24.

□

We now prove Proposition 2.25.

Proof of Proposition 2.25: Let $n \in \mathbb{N}$ with $n > 1$ and let ϵ be a number with $0 < \epsilon < \frac{1}{2}$. For each $j = 1, \ldots, n$, let $\alpha_j(z) = \frac{1}{j} z^2$ and let $\beta_j(z) = \frac{1}{j}(z - \epsilon)^2 + \epsilon$.

For any large $l \in \mathbb{N}$, there exists an open neighborhood U of $\{0, \epsilon\}$ with $U \subset \{z \mid |z| < 1\}$ and a open neighborhood V of $(\alpha_1^l, \ldots, \alpha_n^l, \beta_1^l, \ldots, \beta_n^l)$ in $(\text{Poly})^{2n}$ such that for each $(h_1, \ldots, h_{2n}) \in V$, we have $\cup_{j=1}^{2n} h_j(U) \subset U$ and $\cup_{j=1}^m C(h_j) \cap \mathbb{C} \subset U$, where $C(h_j)$ denotes the set of all critical points of h_j. Then, by Remark 1.3, for each $(h_1, \ldots, h_m) \in V$, $\langle h_1, \ldots, h_{2n} \rangle \in \mathcal{G}$. If l is large enough and V is so small, then, for each $(h_1, \ldots, h_{2n}) \in V$, the set $I_j := J(h_j) \cup J(h_{j+n})$ is connected, for each $j = 1, \ldots, n$, and we have:

$$(h_i)^{-1}(I_j) \cap I_i \neq \emptyset, (h_{i+n})^{-1}(I_j) \cap I_i \neq \emptyset, \tag{24}$$

for each (i,j). Furthermore, for a closed annulus $A = \{z \mid \frac{1}{2} \leq |z| \leq n+1\}$, if $l \in \mathbb{N}$ is large enough and V is so small, then for each $(h_1, \ldots, h_m) \in V$, $\cup_{j=1}^{2n}(h_j)^{-1}(A) \subset \text{int}(A)$ and $\{(h_j)^{-1}(A) \cup (h_{j+n})^{-1}(A)\}_{j=1}^{n}$ are mutually disjoint. Combining it with Lemma 3.1-6 and Lemma 3.1-2, we get that for each $(h_1, \ldots, h_{2n}) \in V$, $J(\langle h_1, \ldots, h_{2n}\rangle) \subset A$ and $\{J_j\}_{j=1}^{n}$ are mutually disjoint, where J_j denotes the connected component of $J(\langle h_1, \ldots, h_{2n}\rangle)$ containing $I_j = J(h_j) \cup J(h_{j+n})$. Combining it with (24) and Theorem 2.24, it follows that for each $(h_1, \ldots, h_{2n}) \in V$, the polynomial semigroup $G = \langle h_1, \ldots, h_{2n}\rangle$ satisfies that $\sharp(\hat{\mathcal{J}}_G) = n$. □

To prove Theorem 2.26, we need the following notation.

Definition 4.11.

1. Let X be a metric space. Let $h_j : X \to X$ $(j = 1, \ldots, m)$ be a continuous map. Let $G = \langle h_1, \ldots, h_m\rangle$ be the semigroup generated by $\{h_j\}$. A non-empty compact subset L of X is said to be a **backward self-similar set with respect to** $\{h_1, \ldots, h_m\}$ if

   (a) $L = \bigcup_{j=1}^{m} h_j^{-1}(L)$ and

   (b) $g^{-1}(z) \neq \emptyset$ for each $z \in L$ and $g \in G$.

   For example, if $G = \langle h_1, \ldots, h_m\rangle$ is a finitely generated rational semigroup, then the Julia set $J(G)$ is a backward self-similar set with respect to $\{h_1, \ldots, h_m\}$. (See Lemma 3.1-2.)

2. We set $\Sigma_m := \{1, \ldots, m\}^{\mathbb{N}}$. For each $x = (x_1, x_2, \ldots,) \in \Sigma_m$, we set $L_x := \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(L)$ $(\neq \emptyset)$.

3. For a finite word $w = (w_1, \ldots, w_k) \in \{1 \ldots, m\}^k$, we set $h_w := h_{w_k} \circ \cdots \circ h_{w_1}$.

4. Under the notation of [24, page 110–page 115], for any $k \in \mathbb{N}$, let $\Omega_k = \Omega_k(L, \{h_1, \ldots, h_m\})$ be the graph (one-dimensional simplicial complex) whose vertex set is $\{1, \ldots, m\}^k$ and that satisfies that mutually different $w^1, w^2 \in \{1, \ldots, m\}^k$ makes a 1-simplex if and only if $\bigcap_{j=1}^{2} h_{w^j}^{-1}(L) \neq \emptyset$.

   Let $\varphi_k : \Omega_{k+1} \to \Omega_k$ be the simplicial map defined by:
   $(w_1, \ldots, w_{k+1}) \mapsto (w_1, \ldots, w_k)$ for each $(w_1, \ldots, w_{k+1}) \in \{1, \ldots, m\}^{k+1}$. Then $\{\varphi_k : \Omega_{k+1} \to \Omega_k\}_{k \in \mathbb{N}}$ makes an inverse system of simplicial maps.

5. Let $\mathcal{C}(|\Omega_k)|)$ be the set of all connected components of the realization $|\Omega_k|$ of Ω_k. Let $\{(\varphi_k)_* : \mathcal{C}(|\Omega_{k+1}|) \to \mathcal{C}(|\Omega_k|)\}_{k \in \mathbb{N}}$ be the inverse system induced by $\{\varphi_k\}_k$.

Notation: We fix an $m \in \mathbb{N}$. We set $\mathcal{W}^* := \cup_{k=1}^{\infty}\{1,\ldots,m\}^k$ (disjoint union) and $\tilde{\mathcal{W}} := \mathcal{W}^* \cup \Sigma_m$ (disjoint union). For an element $x \in \tilde{\mathcal{W}}$, we set $|x| = k$ if $x \in \{1,\ldots,m\}^k$, and $|x| = \infty$ if $x \in \Sigma_m$. (This is called the word length of x.) For any $x \in \tilde{\mathcal{W}}$ and any $j \in \mathbb{N}$ with $j \leq |x|$, we set $x|j := (x_1,\ldots x_j) \in \{1,\ldots,m\}^j$. For any $x^1 = (x_1^1,\ldots,x_p^1) \in \mathcal{W}^*$ and any $x^2 = (x_1^2,x_2^2,\ldots) \in \tilde{\mathcal{W}}$, we set $x^1x^2 := (x_1^1,\ldots,x_p^1,x_1^2,x_2^2,\ldots) \in \tilde{\mathcal{W}}$.

To prove Theorem 2.26, we need the following lemmas.

Lemma 4.12. *Let L be a backward self-similar set with respect to $\{h_1,\ldots,h_m\}$. Then, for each $k \in \mathbb{N}$, the map $|\varphi_k| : |\Omega_{k+1}| \to |\Omega_k|$ induced from $\varphi_k : \Omega_{k+1} \to \Omega_k$ is surjective. In particular, $(\varphi_k)_* : \mathcal{C}(|\Omega_{k+1}|) \to \mathcal{C}(|\Omega_k|)$ is surjective.*

Proof. Let $x^1, x^2 \in \{1,\ldots,m\}^k$ and suppose that $\{x^1,x^2\}$ makes a 1-simplex in Ω_k. Then $h_{x^1}^{-1}(L) \cap h_{x^2}^{-1}(L) \neq \emptyset$. Since $L = \cup_{j=1}^m h_j^{-1}(L)$, there exist x_{k+1}^1 and x_{k+1}^2 in $\{1,\ldots,m\}$ such that $h_{x^1}^{-1}h_{x_{k+1}^1}^{-1}(L) \cap h_{x^2}^{-1}h_{x_{k+1}^2}^{-1}(L) \neq \emptyset$. Hence, $\{x^1x_{k+1}^1, x^2x_{k+1}^2\}$ makes a 1-simplex in Ω_{k+1}. Hence the lemma holds. □

Lemma 4.13. *Let $m \geq 2$ and let L be a backward self-similar set with respect to $\{h_1,\ldots,h_m\}$. Suppose that for each j with $j \neq 1$, $h_1^{-1}(L) \cap h_j^{-1}(L) = \emptyset$. For each k, let $C_k \in \mathcal{C}(|\Omega_k|)$ be the element containing $(1,\ldots,1) \in \{1,\ldots,m\}^k$. Then, we have the following.*

1. *For each $k \in \mathbb{N}$, $C_k = \{(1,\ldots,1)\}$.*
2. *For each $k \in \mathbb{N}$, $\sharp(\mathcal{C}(|\Omega_k|)) < \sharp(\mathcal{C}(|\Omega_{k+1}|))$.*
3. *L has infinitely many connected components.*
4. *Let $x := (1,1,1,\ldots) \in \Sigma_m$ and $x' \in \Sigma_m$ an element with $x \neq x'$. Then, for any $y \in L_x$ and $y' \in L_{x'}$, there exists no connected component A of L such that $y \in A$ and $y' \in A$.*

Proof. We show statement 1 by induction on k. We have $C_1 = \{1\}$. Suppose $C_k = \{(1,\ldots,1)\}$. Let $w \in \{1,\ldots,m\}^{k+1} \cap C_{k+1}$ be any element. Since $(\varphi_k)_*(C_{k+1}) = C_k$, we have $\varphi_k(w) = (1,\ldots,1) \in \{1,\ldots,m\}^k$. Hence, $w|k = (1,\ldots,1) \in \{1,\ldots,m\}^k$. Since $h_1^{-1}(L) \cap h_j^{-1}(L) = \emptyset$ for each $j \neq 1$, we obtain $w = (1,\ldots,1) \in \{1,\ldots,m\}^{k+1}$. Hence, the induction is completed. Therefore, we have shown statement 1.

Since both $(1,\ldots 1,1) \in \{1,\ldots,m\}^{k+1}$ and $(1,\ldots,1,2) \in \{1,\ldots,m\}^{k+1}$ are mapped to $(1,\ldots,1) \in \{1,\ldots,m\}^k$ under φ_k, by statement 1 and Lemma 4.12, we obtain statement 2. For each $k \in \mathbb{N}$, we have

$$L = \coprod_{C \in \mathcal{C}(|\Omega_k|)} \bigcup_{w \in \{1,\ldots,m\}^k \cap C} h_w^{-1}(L). \tag{25}$$

Hence, by statement 2, we conclude that L has infinitely many connected components.

We now show statement 4. Let $k_0 := \min\{l \in \mathbb{N} \mid x'_l \neq 1\}$. Then, by (25) and statement 1, we get that there exist compact sets B_1 and B_2 in L such that $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = L$, $L_x \subset (h_1^{k_0})^{-1}(L) \subset B_1$, and $L_{x'} \subset h_{x'_1}^{-1} \cdots h_{x'_{k_0}}^{-1}(L) \subset B_2$. Hence, statement 4 holds. $\square$

We now demonstrate Theorem 2.26.

Proof of Theorem 2.26: By Theorem 2.19-1 or Remark 2.5, we have $\hat{\mathcal{J}} = \mathcal{J}$. Let $J_1 \in \hat{\mathcal{J}}$ be the element containing $J(h_m)$. By Theorem 2.1, we must have $J_0 \neq J_1$. Then, by Theorem 2.7-1, we have the following two possibilities.

Case 1. $J_0 < J_1$.

Case 2. $J_1 < J_0$.

Suppose we have case 1. Then, by Proposition 2.23, we have that $J_0 = J_{\min}$ and $J_1 = J_{\max}$. Combining it with the assumption and Theorem 2.7-3, we obtain

$$\cup_{j=1}^{m-1} h_j^{-1}(J_{\max}) \subset J_{\min}. \tag{26}$$

By (26) and Theorem 2.7-3, we get

$$\cup_{j=1}^{m-1} h_j^{-1}(J(G)) \subset J_{\min}. \tag{27}$$

Moreover, since $J(h_m) \cap J_{\min} = \emptyset$, Theorem 2.19-5b implies that

$$h_m^{-1}(J(G)) \cap J_{\min} = \emptyset. \tag{28}$$

Then, by (27) and (28), we get

$$h_m^{-1}(J(G)) \cap \left(\cup_{j=1}^{m-1} h_j^{-1}(J(G))\right) = \emptyset. \tag{29}$$

We now consider the backward self-similar set $J(G)$ with respect to $\{h_1, \ldots, h_m\}$. By Lemma 3.1-2, we have

$$J(G) = \cup_{w \in \Sigma_m} (J(G))_w. \tag{30}$$

Combining (29), Lemma 4.13, Lemma 3.8, and (30), we obtain

$$J_{\max} = (J(G))_{m^\infty} \supset J(h_m), \tag{31}$$

where we set $m^\infty := (m, m, m, \ldots) \in \Sigma_m$. Furthermore, by (29) and Lemma 4.13, we get

$$\sharp(\hat{\mathcal{J}}) \geq \aleph_0. \tag{32}$$

Let $x = (x_1, x_2, \ldots) \in \Sigma_m$ be any element with $x \neq m^\infty$ and let $l := \min\{s \in \mathbb{N} \mid x_s \neq m\}$. Then, by (27), we have

$$(J(G))_x = \cap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G)) \subset (h_m^{l-1})^{-1}(J_{\min}). \tag{33}$$

Combining (30) with (31) and (33), we obtain

$$J(G) = J_{\max} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} h_m^{-n}(J_{\min}). \tag{34}$$

By (32) and (34), we get $\sharp(\hat{\mathcal{J}}) = \aleph_0$. Moreover, combining (31), (34), Theorem 2.19-4 and Theorem 2.19-5b, we get that for each $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$. Furthermore, combining (31), Theorem 2.19-4 and Theorem 2.19-5b, we obtain $J_{\max} = (J(G))_{m^\infty} = J(h_m)$. Hence, all statements of Theorem 2.26 are true, provided that we have case 1.

We now assume case 2: $J_1 < J_0$. Then, by Proposition 2.23, we have that $J_0 = J_{\max}$ and $J_1 = J_{\min}$. By the same method as that of case 1, we obtain

$$J_{\min} = (J(G))_{m^\infty} \supset J(h_m), \tag{35}$$

$$J(G) = J_{\min} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} h_m^{-n}(J_{\max}), \tag{36}$$

and

$$\sharp(\hat{\mathcal{J}}) = \aleph_0. \tag{37}$$

Since $J(h_j) \subset J_0$, for each $j = 1, \ldots, m-1$, and $J_0 \neq J_{\min}$, Theorem 2.19-5b implies that for each $j = 1, \ldots, m-1$, $h_j(J(h_m)) \subset \text{int}(K(h_m))$. Hence, for each $j = 1, \ldots, m$, $h_j(K(h_m)) \subset K(h_m)$. Therefore, we have

$$\text{int}(K(h_m)) \subset F(G). \tag{38}$$

By (38) and (35), we obtain $J_{\min} = (J(G))_{m^\infty} = J(h_m)$. Moreover, by (35) and (36), we get that for each $J \in \hat{\mathcal{J}}$ with $J \neq J_{\min}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$. Hence, we have shown Theorem 2.26. □

We now demonstrate Proposition 2.27.

Proof of Proposition 2.27: Let $0 < \epsilon < \frac{1}{2}$ and let $\alpha_1(z) := z^2, \alpha_2(z) := (z-\epsilon)^2 + \epsilon$, and $\alpha_3(z) := \frac{1}{2}z^2$. If we take a large $l \in \mathbb{N}$, then there exists an open neighborhood U of $\{0, \epsilon\}$ with $U \subset \{|z| < 1\}$ and a neighborhood V of $(\alpha_1^l, \alpha_2^l, \alpha_3^l)$ in $(\text{Poly})^3$ such that for each $(h_1, h_2, h_3) \in V$, we have

$\cup_{j=1}^3 h_j(U) \subset U$ and $\cup_{j=1}^3 C(h_j) \cap \mathbb{C} \subset U$, where $C(h_j)$ denotes the set of all critical points of h_j. Then, by Remark 1.3, for each $(h_1, h_2, h_3) \in V$, $\langle h_1, h_2, h_3 \rangle \in \mathcal{G}$. Moreover, if we take an l large enough and V so small, then for each $(h_1, h_2, h_3) \in V$, we have that:

1. $J(h_1) < J(h_3)$;
2. $J(h_1) \cup J(h_2)$ is connected;
3. $h_i^{-1}(J(h_3)) \cap (J(h_1) \cup J(h_2)) \neq \emptyset$, for each $i = 1, 2$;
4. $\cup_{j=1}^3 h_j^{-1}(A) \subset A$, where $A = \{z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq 3\}$; and
5. $h_3^{-1}(A) \cap (\cup_{j=1}^2 h_i^{-1}(A)) = \emptyset$.

Combining statements 4 and 5 above, Lemma 3.1-6, and Lemma 3.1-2, we get that for each $(h_1, h_2, h_3) \in V$, $J(\langle h_1, h_2, h_3 \rangle) \subset A$ and $J(\langle h_1, h_2, h_3 \rangle)$ is disconnected. Hence, for each $(h_1, h_2, h_3) \in V$, we have $\langle h_1, h_2, h_3 \rangle \in \mathcal{G}_{dis}$. Combining it with statements 2 and 3 above and Theorem 2.26, it follows that $J(h_1) \cup J(h_2) \subset J_0$ for some $J_0 \in \hat{\mathcal{J}}_{\langle h_1, h_2, h_3 \rangle}$, $h_j^{-1}(J(h_3)) \cap J_0 \neq \emptyset$ for each $j = 1, 2$, and $\sharp(\hat{\mathcal{J}}_{\langle h_1, h_2, h_3 \rangle}) = \aleph_0$, for each $(h_1, h_2, h_3) \in V$. Since $J(h_1) < J(h_3)$, Theorem 2.26 implies that the connected component J_0 should be equal to $J_{\min}(\langle h_1, h_2, h_3 \rangle)$, and that $J_{\max}(\langle h_1, h_2, h_3 \rangle) = J(h_3)$.

Thus, we have proved Proposition 2.27. □

We now show Proposition 2.28.

Proof of Proposition 2.28: In fact, we show the following claim:

Claim: There exists a polynomial semigroup $G = \langle h_1, h_2, h_3 \rangle$ in $\mathcal{G}$ such that all of the following hold.

1. $\sharp(\hat{\mathcal{J}}) = \aleph_0$.
2. $J_{\min} \supset J(h_1) \cup J(h_2)$ and there exists a superattracting fixed point z_0 of h_1 with $z_0 \in \text{int}(J_{\min})$.
3. $J_{\max} = J(h_3)$.
4. There exists a sequence $\{n_j\}_{j \in \mathbb{N}}$ of positive integers such that $\hat{\mathcal{J}} = \{J_{\min}\} \cup \{J_j \mid j \in \mathbb{N}\}$, where J_j denotes the element of $\hat{\mathcal{J}}$ with $h_3^{-n_j}(J_{\min}) \subset J_j$.
5. For any $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$.

6. G is sub-hyperbolic: i.e., $\sharp(P(G) \cap J(G)) < \infty$ and $P(G) \cap F(G)$ is compact.

To show the claim, let $g_1(z)$ be the second iterate of $z \mapsto z^2 - 1$. Let g_2 be a polynomial such that $J(g_2) = \{z \mid |z| = 1\}$ and $g_2(-1) = -1$. Then, we have $g_1(\sqrt{-1}) = 3 \in \hat{\mathbb{C}} \setminus K(g_1)$. Take a large, positive integer m_1, and let $a := g_1^{m_1}(\sqrt{-1})$. Then,

$$J(\langle g_1^{m_1}, g_2 \rangle) \subset \{z \mid |z| < a\}. \tag{39}$$

Furthermore, since $a > \frac{1}{2} + \frac{\sqrt{5}}{2}$, we have

$$\overline{(g_1^{m_1})^{-1}(\{z \mid |z| < a\})} \subset \{z \mid |z| < a\}. \tag{40}$$

Let g_3 be a polynomial such that $J(g_3) = \{z \mid |z| = a\}$. Since -1 is a superattracting fixed point of $g_1^{m_1}$ and it belongs to $J(g_2)$, by [15, Theorem 4.1], we see that for any $m \in \mathbb{N}$,

$$-1 \in \operatorname{int}(J(\langle g_1^{m_1}, g_2^m \rangle)). \tag{41}$$

Since $J(g_2) \cap \operatorname{int}(K(g_1^{m_1})) \neq \emptyset$ and $J(g_2) \cap F_\infty(g_1^{m_1}) \neq \emptyset$, we can take an $m_2 \in \mathbb{N}$ such that

$$(g_2^{m_2})^{-1}(\{z \mid |z| = a\}) \cap J(\langle g_1^{m_1}, g_2^{m_2} \rangle) \neq \emptyset \tag{42}$$

and

$$\overline{(g_2^{m_2})^{-1}(\{z \mid |z| < a\})} \subset \{z \mid |z| < a\}. \tag{43}$$

Take a small $r > 0$ such that

$$\text{for each } j = 1, 2, 3, \ \ g_j(\{z \mid |z| \leq r\}) \subset \{z \mid |z| < r\}. \tag{44}$$

Take an m_3 such that

$$(g_3^{m_3})^{-1}(\{z \mid |z| = r\}) \cap (\cup_{j=1}^2 (g_j^{m_j})^{-1}(\{z \mid |z| \leq a\})) = \emptyset \tag{45}$$

and

$$g_3^{m_3}(-1) \in \{z \mid |z| < r\}. \tag{46}$$

Let $K := \{z \mid r \leq |z| \leq a\}$. Then, by (40), (43), (44) and (45), we have

$$(g_j^{m_j})^{-1}(K) \subset K, \text{ for } j = 1, 2, 3, \text{and } (g_3^{m_3})^{-1}(K) \cap (\cup_{j=1}^2 (g_j^{m_j})^{-1}(K)) = \emptyset. \tag{47}$$

Let $h_j := g_j^{m_j}$, for each $j = 1, 2, 3$, and let $G = \langle h_1, h_2, h_3 \rangle$. Then, by (47) and Lemma 3.1-6, we obtain:

$$J(G) \subset K \text{ and } h_3^{-1}(J(G)) \cap (\cup_{j=1}^2 h_j^{-1}(J(G))) = \emptyset. \tag{48}$$

Combining it with Lemma 3.1-2, it follows that $J(G)$ is disconnected. Furthermore, combining (44) and (46), we see $G \in \mathcal{G}$, $P(G) \cap J(G) = \{-1\}$, and that $P(G) \cap F(G)$ is compact. By Proposition 2.23, there exists a $j \in \{1, 2, 3\}$ with $J(h_j) \subset J_{\min}$. Since $J(G) \subset K \subset \{z \mid |z| \leq a\}$ and $J(h_3) = \{z \mid |z| = a\}$, we have

$$J(h_3) \subset J_{\max}. \tag{49}$$

Hence, either $J(h_1) \subset J_{\min}$ or $J(h_2) \subset J_{\min}$. Since $J(h_1) \cup J(h_2)$ is connected, it follows that

$$J(h_1) \cup J(h_2) \subset J_{\min}. \tag{50}$$

Combining this with Theorem 2.7-3, we have $h_j^{-1}(J_{\min}) \subset J_{\min}$, for each $j = 1, 2$. Hence,

$$J(\langle h_1, h_2 \rangle) \subset J_{\min}. \tag{51}$$

Since $\sqrt{-1} \in J(h_2)$ and $h_1(\sqrt{-1}) = a \in J(h_3)$, we obtain

$$h_1^{-1}(J(h_3)) \cap J_{\min} \neq \emptyset. \tag{52}$$

Similarly, by (42) and (51), we obtain

$$h_2^{-1}(J(h_3)) \cap J_{\min} \neq \emptyset. \tag{53}$$

Combining (49), (52), (53), and Theorem 2.26, we obtain $\sharp(\hat{\mathcal{J}}) = \aleph_0$, $J_{\max} = J(h_3)$, $J(G) = J_{\max} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} h_3^{-n}(J_{\min})$, and that for any $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$.

Moreover, by (41) and (51) (or by Theorem 2.21-3), the superattracting fixed point -1 of h_1 belongs to $\text{int}(J_{\min})$.

Hence, we have shown the claim.

Therefore, we have proved Proposition 2.28. □

4.5 Proofs of results in 2.5

In this section, we prove results in section 2.5.

To prove results in 2.5, we need the following notations and lemmas.

Definition 4.14 ([32])**.** Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a rational skew product over $g : X \to X$. Let $N \in \mathbb{N}$. We say that a point $(x_0, y_0) \in X \times \hat{\mathbb{C}}$ belongs to $SH_N(f)$ if there exists a neighborhood U of x_0 in X and a positive number δ such that for any $x \in U$, any $n \in \mathbb{N}$, any $x_n \in g^{-n}(x)$, and any connected component V of $(f_{x_n,n})^{-1}(B(y_0, \delta))$, $\deg(f_{x_n,n} : V \to B(y_0, \delta)) \leq N$. Moreover, we set $UH(f) := (X \times \hat{\mathbb{C}}) \setminus \cup_{N \in \mathbb{N}} SH_N(f)$. We say that f is semi-hyperbolic (along fibers) if $UH(f) \subset \tilde{F}(f)$.

Remark 4.15. Under the above notation, we have $UH(f) \subset P(f)$.

Remark 4.16. Let Γ be a compact subset of Rat and let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let G be the rational semigroup generated by Γ. Then, by Lemma 3.5-1, it is easy to see that f is semi-hyperbolic if and only if G is semi-hyperbolic. Similarly, it is easy to see that f is hyperbolic if and only if G is hyperbolic.

Lemma 4.17. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $\omega \in X$, $d(\omega) \geq 2$. Let $x \in X$ be a point and $y_0 \in F_x(f)$ a point. Suppose that there exists a strictly increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ of positive integers such that the sequence $\{f_{x,n_j}\}_{j \in \mathbb{N}}$ converges to a non-constant map around y_0, and such that $\lim_{j \to \infty} f^{n_j}(x, y_0)$ exists. We set $(x_\infty, y_\infty) := \lim_{j \to \infty} f^{n_j}(x, y_0)$. Then, there exists a non-empty bounded open set V in $\mathbb{C}$ and a number $k \in \mathbb{N}$ such that $\{x_\infty\} \times \partial V \subset \tilde{J}(f) \cap UH(f) \subset \tilde{J}(f) \cap P(f)$, and such that for each j with $j \geq k$, $f_{x,n_j}(y_0) \in V$.*

Proof. We set

$$V := \{y \in \hat{\mathbb{C}} \mid \exists \epsilon > 0, \lim_{i \to \infty} \sup_{j > i} \sup_{d(\xi, y) \leq \epsilon} d(f_{g^{n_i}(x), n_j - n_i}(\xi), \xi) = 0\}.$$

Then, by [32, Lemma 2.13], we have $\{x_\infty\} \times \partial V \subset \tilde{J}(f) \cap UH(f) \subset \tilde{J}(f) \cap P(f)$. Moreover, since for each $x \in X$, $f_{x,1}$ is a polynomial with $d(x) \geq 2$, Lemma 3.4-4 implies that there exists a ball B around ∞ such that $B \subset \hat{\mathbb{C}} \setminus V$.

From the assumption, there exists a number $a > 0$ and a non-constant map $\varphi : D(y_0, a) \to \hat{\mathbb{C}}$ such that $f_{x,n_j} \to \varphi$ as $j \to \infty$, uniformly on $D(y_0, a)$. Hence, $d(f_{x,n_j}(y), f_{x,n_i}(y)) \to 0$ as $i, j \to \infty$, uniformly on $D(y_0, a)$. Moreover, since φ is not constant, there exists a positive number ϵ such that for each large i, $f_{x,n_i}(D(y_0, a)) \supset D(y_\infty, \epsilon)$. Therefore, it follows that $d(f_{g^{n_i}(x), n_j - n_i}(\xi), \xi) \to 0$ as $i, j \to \infty$ uniformly on $D(y_\infty, \epsilon)$. Thus, $y_\infty \in V$. Hence, there exists a number $k \in \mathbb{N}$ such that for each $j \geq k$, $f_{x,n_j}(y_0) \in V$. Therefore, we have proved Lemma 4.17. $\square$

Remark 4.18. In [32, Lemma 2.13] and [35, Theorem 2.6], the sequence (n_j) of positive integers should be strictly increasing.

Lemma 4.19. *Let Γ be a non-empty compact subset of $\text{Poly}_{\deg \geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let G be the polynomial semigroup generated by Γ. Let $\gamma \in \Gamma^{\mathbb{N}}$ be a point. Let $y_0 \in F_\gamma(f)$ and suppose that there exists a strictly increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ of positive integers such that $\{f_{\gamma,n_j}\}_{j \in \mathbb{N}}$ converges to a non-constant map around y_0. Moreover, suppose that $G \in \mathcal{G}$. Then, there exists a number $j \in \mathbb{N}$ such that $f_{\gamma,n_j}(y_0) \in \text{int}(\hat{K}(G))$.*

Proof. By Lemma 4.17, there exists a bounded open set V in $\mathbb{C}$, a point $\gamma_\infty \in \Gamma^{\mathbb{N}}$, and a number $j \in \mathbb{N}$ such that $\{\gamma_\infty\} \times \partial V \subset \tilde{J}(f) \cap P(f)$, and such that $f_{\gamma,n_j}(y_0) \in V$. Then, we have $\partial V \subset P^*(G)$. Since $g(P^*(G)) \subset P^*(G)$ for each $g \in G$, the maximum principle implies that $V \subset \text{int}(\hat{K}(G))$. Hence, $f_{\gamma,n_j}(y_0) \in \text{int}(\hat{K}(G))$. Therefore, we have proved Lemma 4.19. □

Lemma 4.20. *Let G be a polynomial semigroup generated by a compact subset Γ of* $\text{Poly}_{\deg \geq 2}$*. If a sequence $\{g_n\}_{n \in \mathbb{N}}$ of elements of G tends to a constant $w_0 \in \hat{\mathbb{C}}$ locally uniformly on a domain $V \subset \hat{\mathbb{C}}$, then $w_0 \in P(G)$.*

Proof. Since $\infty \in P(G)$, we may assume that $w_0 \in \mathbb{C}$. Suppose $w_0 \in \mathbb{C} \setminus P(G)$. Then, there exists a $\delta > 0$ such that $B(w_0, 2\delta) \subset \mathbb{C} \setminus P(G)$. Let $z_0 \in V$ be a point. Then, for each large $n \in \mathbb{N}$, there exists a well-defined inverse branch α_n of g_n^{-1} on $B(w_0, 2\delta)$ such that $\alpha_n(g_n(z_0)) = z_0$. Let $B := B(w_0, \delta)$. Since Γ is compact, there exists a connected component $F_\infty(G)$ of $F(G)$ containing ∞. Let C be a compact neighborhood of ∞ in $F_\infty(G)$. Then, we must have that there exists a number n_0 such that $\alpha_n(B) \cap C = \emptyset$ for each $n \geq n_0$, since $g_n \to \infty$ uniformly on C as $n \to \infty$, which follows from that $\deg(g_n) \to \infty$ and local degree at ∞ of g_n tends to ∞ as $n \to \infty$. Hence, $\{\alpha_n|_B\}_{n \geq n_0}$ is normal in B. However, for a small ϵ so that $B(z_0, 2\epsilon) \subset V$, we have $g_n(B(z_0, \epsilon)) \to w_0$ as $n \to \infty$, and this is a contradiction. Hence, we must have that $w_0 \in P(G)$. □

We now demonstrate Proposition 2.37-1, 2.37-2, and 2.37-3. (Proposition 2.37-4 will be proved after Theorem 2.41 is proved.)

Proof of Proposition 2.37-1, 2.37-2, and 2.37-3 : Since $\Gamma \setminus \Gamma_{\min}$ is not compact, there exists a sequence $\{h_j\}_{j \in \mathbb{N}}$ in $\Gamma \setminus \Gamma_{\min}$ and an element $h_\infty \in \Gamma_{\min}$ such that $h_j \to h_\infty$ as $j \to \infty$. By Theorem 2.19-5b, for each $j \in \mathbb{N}$, $h_j(K(h_\infty))$ is included in a connected component U_j of $\text{int}(\hat{K}(G))$. Let $z_1 \in \text{int}(\hat{K}(G))$ $(\subset \text{int}(K(h_\infty)))$ be a point. Then, $h_\infty(z_1) \in \text{int}(\hat{K}(G))$ and $h_j(z_1) \to h_\infty(z_1)$ as $j \to \infty$. Hence, we may assume that there exists a connected component U of $\text{int}(\hat{K}(G))$ such that for each $j \in \mathbb{N}$, $h_j(K(h_\infty)) \subset U$. Therefore, $K(h_\infty) = h_\infty(K(h_\infty)) \subset \overline{U}$. Since $\overline{U} \subset K(h_\infty)$, we obtain $K(h_\infty) = \overline{U}$. Since $U \subset \text{int}(K(h_\infty)) \subset \overline{U}$ and U is connected, it follows that $\text{int}(K(h_\infty))$ is connected. Moreover, we have $U \subset \text{int}(K(h_\infty)) \subset \text{int}(\overline{U}) \subset \text{int}(\hat{K}(G))$. Thus,

$$\text{int}(K(h_\infty)) = U. \tag{54}$$

Furthermore, since

$$J(h_\infty) < J(h_j) \text{ for each } j \in \mathbb{N}, \tag{55}$$

and $h_j \to h_\infty$ as $j \to \infty$, we obtain

$$J(h_j) \to J(h_\infty) \text{ as } j \to \infty, \tag{56}$$

with respect to the Hausdorff topology. Combining that $h_j \in \Gamma \setminus \Gamma_{\min}$ for each $j \in \mathbb{N}$ with Theorem 2.19-4, (54), (55), and (56), we see that for each $h \in \Gamma_{\min}$, $K(h) = K(h_\infty)$. Combining it with (54), (55) and (56), it follows that statements 1 and 2 in Proposition 2.37 hold.

We now show that statement 3 holds. Let $\gamma \in \Gamma^{\mathbb{N}}$ and let $y_0 \in \text{int}(K_\gamma(f))$ be a point. Suppose that there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of positive integers such that f_{γ,n_j} tends to a non-constant map as $j \to \infty$ around y_0. Then, by Lemma 4.19, there exists a number $k \in \mathbb{N}$ such that $f_{\gamma,n_k}(y_0) \in \text{int}(\hat{K}(G))$. Hence, the sequence $\{f_{\sigma^{n_k}(\gamma),n_{k+j}-n_k}\}_{j\in\mathbb{N}}$ converges to a non-constant map around $y_1 := f_{\gamma,n_k}(y_0) \in \text{int}(\hat{K}(G))$. However, combining Theorem 2.19-5b and statements 1 and 2 in Proposition 2.37, we have that for each $h \in \Gamma$, $h : \text{int}(\hat{K}(G)) \to \text{int}(\hat{K}(G))$ is a contraction map with respect to the hyperbolic distance on $\text{int}(\hat{K}(G))$. This causes a contradiction. Therefore, statement 3 in Proposition 2.37 holds.

Thus, we have proved Proposition 2.37-1, 2.37-2, and 2.37-3. □

We now demonstrate Theorem 2.40-1 and Theorem 2.40-2.

Proof of Theorem 2.40-1 and Theorem 2.40-2: First, we will show the following claim.

Claim 1. Let $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$. Then, for any point $y_0 \in F_\gamma(f)$, there exists no non-constant limit function of $\{f_{\gamma,n}\}_{n\in\mathbb{N}}$ around y_0.

To show this claim, by Proposition 2.37-3, we may assume that $\Gamma \setminus \Gamma_{\min}$ is compact. Suppose that there exists a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of positive integers such that f_{γ,n_j} tends to a non-constant map as $j \to \infty$ around y_0. By Lemma 4.19, there exists a number $k \in \mathbb{N}$ such that $f_{\gamma,n_k}(y_0) \in \text{int}(\hat{K}(G))$. Hence, we get that the sequence $\{f_{\sigma^{n_k}(\gamma),n_{k+j}-n_k}\}_{j\in\mathbb{N}}$ converges to a non-constant map around the point $y_1 := f_{\gamma,n_k}(y_0) \in \text{int}(\hat{K}(G))$. However, since we are assuming that $\Gamma \setminus \Gamma_{\min}$ is compact, Theorem 2.19-5b implies that $\cup_{h\in\Gamma\setminus\Gamma_{\min}} h(\hat{K}(G))$ is a compact subset of $\text{int}(\hat{K}(G))$, which implies that if we take the hyperbolic metric for each connected component of $\text{int}(\hat{K}(G))$, then there exists a constant $0 < c < 1$ such that for each $z \in \text{int}(\hat{K}(G))$ and each $h \in \Gamma \setminus \Gamma_{\min}$, we have $\|h'(z)\| \leq c$, where $\| \cdot \|$ denotes the norm of the derivative measured from the hyperbolic metric on the connected component W_1 of $\text{int}(\hat{K}(G))$ containing z to that of the connected component W_2 of $\text{int}(\hat{K}(G))$ containing $h(z)$. This causes a contradiction, since we have that $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ and the sequence $\{f_{\sigma^{n_k}(\gamma),n_{k+j}-n_k}\}_{j\in\mathbb{N}}$ converges to a non-constant map around the point $y_1 \in \text{int}(\hat{K}(G))$. Hence, we have shown Claim

1.

Next, let S be a non-empty compact subset of $\Gamma\setminus\Gamma_{\min}$ and let $\gamma \in R(\Gamma, S)$. We show the following claim.

Claim 2. For each point y_0 in each bounded component of $F_\gamma(f)$, there exists a number $n \in \mathbb{N}$ such that $f_{\gamma,n}(y_0) \in \text{int}(\hat{K}(G))$.

To show this claim, suppose that there exists no $n \in \mathbb{N}$ such that $f_{\gamma,n}(y_0) \in \text{int}(\hat{K}(G))$, and we will deduce a contradiction. By Claim 1, $\{f_{\gamma,n}\}_{n\in\mathbb{N}}$ has only constant limit functions around y_0. Moreover, if a point $w_0 \in \mathbb{C}$ is a constant limit function of $\{f_{\gamma,n}\}_{n\in\mathbb{N}}$, then by Lemma 4.20, we must have $w_0 \in P^*(G) \subset \hat{K}(G)$. Since we are assuming that there exists no $n \in \mathbb{N}$ such that $f_{\gamma,n}(y_0) \in \text{int}(\hat{K}(G))$, it follows that $w_0 \in \partial\hat{K}(G)$. Combining it with Theorem 2.19-2, we obtain $w_0 \in \partial\hat{K}(G) \subset J_{\min}$. From this argument, we get that

$$d(f_{\gamma,n}(y_0), J_{\min}) \to 0, \text{ as } n \to \infty. \tag{57}$$

However, since γ belongs to $R(\Gamma, S)$, the above (57) implies that the sequence $\{f_{\gamma,n}(y_0)\}_{n\in\mathbb{N}}$ accumulates in the compact set $\cup_{h\in S} h^{-1}(J_{\min})$, which is apart from $J_{\min}$, by Theorem 2.19-5b. This contradicts (57). Hence, we have shown that Claim 2 holds.

Next, we show the following claim.

Claim 3. There exists exactly one bounded component U_γ of $F_\gamma(f)$.

To show this claim, we take an element $h \in \Gamma_{\min}$ (Note that $\Gamma_{\min} \neq \emptyset$, by Proposition 2.23). We write the element γ as $\gamma = (\gamma_1, \gamma_2, \ldots)$. For any $l \in \mathbb{N}$ with $l \geq 2$, let $s_l \in \mathbb{N}$ be an integer with $s_l > l$ such that $\gamma_{s_l} \in S$. We may assume that for each $l \in \mathbb{N}$, $s_l < s_{l+1}$. For each $l \in \mathbb{N}$, let $\gamma^l := (\gamma_1, \gamma_2, \ldots, \gamma_{s_l-1}, h, h, h, \ldots) \in \Gamma^{\mathbb{N}}$ and $\tilde{\gamma}^l := \sigma^{s_l-1}(\gamma) = (\gamma_{s_l}, \gamma_{s_l+1}, \ldots) \in \Gamma^{\mathbb{N}}$. Moreover, let $\rho := (h, h, h, \ldots) \in \Gamma^{\mathbb{N}}$. Since $h \in \Gamma_{\min}$, we have

$$J_\rho(f) = J(h) \subset J_{\min}. \tag{58}$$

Moreover, since γ_{s_l} does not belong to $\Gamma_{\min}$, combining it with Theorem 2.19-5b we obtain $\gamma_{s_l}^{-1}(J(G)) \cap J_{\min} = \emptyset$. Hence, we have that for each $l \in \mathbb{N}$,

$$J_{\tilde{\gamma}^l}(f) = \gamma_{s_l}^{-1}(J_{\sigma^{s_l}(\gamma)}(f)) \subset \gamma_{s_l}^{-1}(J(G)) \subset \hat{\mathbb{C}} \setminus J_{\min}. \tag{59}$$

Combining (58), (59), and Lemma 3.9, we obtain

$$J_\rho(f) < J_{\tilde{\gamma}^l}(f), \tag{60}$$

which implies

$$J_{\gamma^l}(f) = (f_{\gamma,s_l-1})^{-1}(J_\rho(f)) < (f_{\gamma,s_l-1})^{-1}(J_{\tilde{\gamma}^l}(f)) = J_\gamma(f). \tag{61}$$

From Lemma 3.9 and (61), it follows that there exists a bounded component U_γ of $F_\gamma(f)$ such that for each $l \in \mathbb{N}$ with $l \geq 2$,

$$J_{\gamma^l}(f) \subset U_\gamma. \tag{62}$$

We now suppose that there exists a bounded component V of $F_\gamma(f)$ with $V \neq U_\gamma$, and we will deduce a contradiction. Under the above assumption, we take a point $y \in V$. Then, by Claim 2, we get that there exists a number $l \in \mathbb{N}$ such that $f_{\gamma,l}(y) \in \text{int}(\hat{K}(G))$. Since $s_l > l$, we obtain $f_{\gamma,s_l-1}(y) \in \text{int}(\hat{K}(G)) \subset K(h)$, where, $h \in \Gamma_{\min}$ is the element which we have taken before. By (60), we have that there exists a bounded component B of $F_{\tilde{\gamma}^l}(f)$ containing $K(h)$. Hence, we have $f_{\gamma,s_l-1}(y) \in B$. Since the map $f_{\gamma,s_l-1} : V \to B$ is surjective, it follows that $V \cap ((f_{\gamma,s_l-1})^{-1}(J(h))) \neq \emptyset$. Combined with $(f_{\gamma,s_l-1})^{-1}(J(h)) = (f_{\gamma^l,s_l-1})^{-1}(J(h)) = J_{\gamma^l}(f)$, we obtain $V \cap J_{\gamma^l}(f) \neq \emptyset$. However, this causes a contradiction, since we have (62) and $U_\gamma \cap V = \emptyset$. Hence, we have shown Claim 3.

Next, we show the following claim.

Claim 4. $\partial U_\gamma = \partial A_\gamma(f) = J_\gamma(f)$.

To show this claim, since $U_\gamma = \text{int}(K_\gamma(f))$, Lemma 3.4-5 implies that $\partial U_\gamma = J_\gamma(f)$. Moreover, by Lemma 3.4-4, we have $\partial A_\gamma(f) = J_\gamma(f)$. Thus, we have shown Claim 4.

We now show the following claim.

Claim 5. $\hat{J}_\gamma(f) = J_\gamma(f)$ and the map $\omega \mapsto J_\omega(f)$ is continuous at γ with respect to the Hausdorff topology in the space of non-empty compact subsets of $\hat{\mathbb{C}}$.

To show this claim, suppose that there exists a point z with $z \in \hat{J}_\gamma(f) \setminus J_\gamma(f)$. Since $\hat{J}_\gamma(f) \setminus J_\gamma(f)$ is included in the union of bounded components of $F_\gamma(f)$, combining it with Claim 2, we get that there exists a number $n \in \mathbb{N}$ such that $f_{\gamma,n}(z) \in \text{int}(\hat{K}(G)) \subset F(G)$. However, since $z \in \hat{J}_\gamma(f)$, we must have that $f_{\gamma,n}(z) = \pi_{\hat{\mathbb{C}}}(f_\gamma^n(z)) \in \pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$. This is a contradiction. Hence, we obtain $\hat{J}_\gamma(f) = J_\gamma(f)$. Combining it with Lemma 3.4-2, it follows that $\omega \mapsto J_\omega(f)$ is continuous at γ. Therefore, we have shown Claim 5.

Combining all Claims $1, \ldots, 5$, it follows that statements 1, 2a, 2b, and 2c in Theorem 2.40 hold.

We now show statement 2d. Let $\gamma \in R(\Gamma, S)$ be an element. Suppose that $m_2(J_\gamma(f)) > 0$, where m_2 denotes the 2-dimensional Lebesgue measure. Then, there exists a Lebesgue density point $b \in J_\gamma(f)$ so that

$$\lim_{s \to 0} \frac{m_2\left(D(b,s) \cap J_\gamma(f)\right)}{m_2(D(b,s))} = 1. \tag{63}$$

Since γ belongs to $R(\Gamma, S)$, there exists an element $\gamma_\infty \in S$ and a sequence $\{n_j\}_{j \in \mathbb{N}}$ of positive integers such that $n_j \to \infty$ and $\gamma_{n_j} \to \gamma_\infty$ as $j \to \infty$, and such that for each $j \in \mathbb{N}$, $\gamma_{n_j} \in S$. We set $b_j := f_{\gamma,n_j-1}(b)$, for each $j \in \mathbb{N}$. We may assume that there exists a point $a \in \mathbb{C}$ such that $b_j \to a$ as $j \to \infty$. Since $\gamma_{n_j}(b_j) = f_{\gamma,n_j}(b) = \pi_{\hat{\mathbb{C}}}(f_\gamma^{n_j}(\gamma, b)) \in \pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$, we obtain $a \in \gamma_\infty^{-1}(J(G))$. Moreover, by Theorem 2.19-5b, we obtain

$$a \in \gamma_\infty^{-1}(J(G)) \subset \mathbb{C} \setminus J_{\min}. \tag{64}$$

Combining it with Theorem 2.19-2, it follows that

$$r := d_e(a, P(G)) > 0. \tag{65}$$

Let ϵ be arbitrary number with $0 < \epsilon < \frac{r}{10}$. We may assume that for each $j \in \mathbb{N}$, we have $b_j \in D(a, \frac{\epsilon}{2})$. For each $j \in \mathbb{N}$, let φ_j be the well-defined inverse branch of $(f_{\gamma,n_j-1})^{-1}$ on $D(a,r)$ such that $\varphi_j(b_j) = b$. Let $V_j := \varphi_j(D(b_j, r-\epsilon))$, for each $j \in \mathbb{N}$. We now show the following claim.
Claim 6. diam $V_j \to 0$, as $\jmath \to \infty$.

To show this claim, suppose that this is not true. Then, there exists a strictly increasing sequence $\{j_k\}_{k\in\mathbb{N}}$ of positive integers and a positive constant κ such that for each $k \in \mathbb{N}$, diam $V_{j_k} \geq \kappa$. From Koebe distortion theorem, it follows that there exists a positive constant c_0 such that for each $k \in \mathbb{N}$, $V_{j_k} \supset D(b, c_0)$. This implies that for each $k \in \mathbb{N}$, $f_{\gamma,v_k}(D(b,c_0)) \subset D(b_{j_k}, r-\epsilon)$, where $v_k := n_{j_k} - 1$. Since $v_k \to \infty$ as $k \to \infty$ and $f_{\gamma',n}|_{F_\infty(G)} \to \infty$ for any $\gamma' \in \Gamma^{\mathbb{N}}$, it follows that for any $n \in \mathbb{N}$, $f_{\gamma,n}(D(b,c_0)) \subset (\hat{\mathbb{C}} \setminus F_\infty(G))$, which implies that $b \in F_\gamma(f)$. However, it contradicts $b \in J_\gamma(f)$. Hence, Claim 6 holds.

Combining Koebe distortion theorem and Claim 6, we see that there exist a constant $K > 0$ and two sequences $\{r_j\}_{j\in\mathbb{N}}$ and $\{R_j\}_{j\in\mathbb{N}}$ of positive numbers such that $K \leq \frac{r_j}{R_j} < 1$ and $D(b, r_j) \subset V_j \subset D(b, R_j)$ for each $j \in \mathbb{N}$, and such that $R_j \to 0$ as $j \to \infty$. From (63), it follows that

$$\lim_{j\to\infty} \frac{m_2\left(V_j \cap F_\gamma(f)\right)}{m_2(V_j)} = 0. \tag{66}$$

For each $j \in \mathbb{N}$, let $\psi_j : D(0,1) \to \varphi_j(D(a,r))$ be a biholomorphic map such that $\psi_j(0) = b$. Then, there exists a constant $0 < c_1 < 1$ such that for each $j \in \mathbb{N}$,

$$\psi_j^{-1}(V_j) \subset D(0, c_1). \tag{67}$$

Combining it with (66) and Koebe distortion theorem, it follows that

$$\lim_{j\to\infty} \frac{m_2\left(\psi_j^{-1}(V_j \cap F_\gamma(f))\right)}{m_2(\psi_j^{-1}(V_j))} = 0. \tag{68}$$

Since $\varphi_j^{-1}(\psi_j(D(0,1)) \subset D(a,r)$ for each $j \in \mathbb{N}$, combining (67) and Cauchy's formula yields that there exists a constant $c_2 > 0$ such that for any $j \in \mathbb{N}$,

$$|(f_{\gamma,n_j-1} \circ \psi_j)'(z)| \leq c_2 \ \text{ on } \psi_j^{-1}(V_j). \tag{69}$$

Combining (68) and (69), we obtain

$$\begin{aligned} \frac{m_2\left(D(b_j, r-\epsilon) \cap F_{\sigma^{n_j-1}(\gamma)}(f)\right)}{m_2(D(b_j, r-\epsilon))} &= \frac{m_2\left((f_{\gamma,n_j-1} \circ \psi_j)(\psi_j^{-1}(V_j \cap F_\gamma(f)))\right)}{m_2(D(b_j, r-\epsilon))} \\ = \frac{\int_{\psi_j^{-1}(V_j \cap F_\gamma(f))} |(f_{\gamma,n_j-1} \circ \psi_j)'(z)|^2 \, dm_2(z)}{m_2(\psi_j^{-1}(V_j))} &\cdot \frac{m_2(\psi_j^{-1}(V_j))}{m_2(D(b_j, r-\epsilon))} \to 0, \end{aligned}$$

as $j \to \infty$. Hence, we obtain

$$\lim_{j\to\infty} \frac{m_2\left(D(b_j, r-\epsilon) \cap J_{\sigma^{n_j-1}(\gamma)}(f)\right)}{m_2(D(b_j, r-\epsilon))} = 1.$$

Since $J_{\sigma^{n_j-1}(\gamma)}(f) \subset J(G)$ for each $j \in \mathbb{N}$, and $b_j \to a$ as $j \to \infty$, it follows that

$$\frac{m_2(D(a, r-\epsilon) \cap J(G))}{m_2(D(a, r-\epsilon))} = 1.$$

This implies that $D(a, r-\epsilon) \subset J(G)$. Since this is valid for any ϵ, we must have that $D(a,r) \subset J(G)$. It follows that the point a belongs to a connected component J of $J(G)$ such that $J \cap P^*(G) \neq \emptyset$. However, Theorem 2.19-2 implies that the component J is equal to $J_{\min}$, which causes a contradiction since we have (64). Hence, we have shown statement 2d in Theorem 2.40-2.

Therefore, we have proved Theorem 2.40-1 and Theorem 2.40-2. □

In order to demonstrate Theorem 2.40-3, we need the following result.

Theorem 4.21. *(***Uniform fiberwise quasiconformal surgery***) Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Suppose that f is hyperbolic and that $\pi_{\hat{\mathbb{C}}}(P(f)) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. Moreover, suppose that for each $x \in X$, $\text{int}(K_x(f))$ is connected. Then, there exists a constant K such that for each $x \in X$, $J_x(f)$ is a K-quasicircle.*

Proof. Step 1: By [32, Theorem 2.14-(4)], the map $x \mapsto J_x(f)$ is continuous with respect to the Hausdorff topology. Hence, there exists a positive constant C_1 such that for each $x \in X$, $\inf\{d(a,b) \mid a \in J^x(f),\ b \in$

$\pi^{-1}(\{x\}) \cap P^*(f)\} > C_1$, where $P^*(f) := P(f) \setminus \pi_{\hat{\mathbb{C}}}^{-1}(\{\infty\})$, and $d(\cdot, \cdot)$ denotes the spherical distance, under the canonical identification $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$. Moreover, from the assumption, we have that for each $x \in X$, $\mathrm{int}(K_x(f)) \neq \emptyset$. Since X is compact, it follows that for each $x \in X$, there exists an analytic Jordan curve ζ_x in $K^x(f) \cap F^x(f)$ such that:

1. $\pi^{-1}(\{x\}) \cap P^*(f)$ is included in the bounded component V_x of $\pi^{-1}(\{x\}) \setminus \zeta_x$;
2. $\inf_{z \in \zeta_x} d(z,\ J^x(f) \cup (\pi^{-1}(\{x\}) \cap P^*(f))) \geq C_2$, where C_2 is a positive constant independent of $x \in X$; and
3. there exist finitely many Jordan curves $\xi_1, \ldots, \xi_k$ in $\mathbb{C}$ such that for each $x \in X$, there exists a j with $\pi_{\hat{\mathbb{C}}}(\zeta_x) = \xi_j$.

Step 2: By [35, Corollary 2.7], there exists an $n \in \mathbb{N}$ such that for each $x \in X$, $W_x := (f_x^n)^{-1}(V_{g^n(x)}) \supset \overline{V_x}$, $\inf\{d(a,b) \mid a \in \partial W_x, b \in \partial V_x, x \in X\} > 0$, and $\mathrm{mod}\ (W_x \setminus \overline{V_x}) \geq C_3$, where C_3 is a positive constant independent of $x \in X$. In order to prove the theorem, since $J_x(f^n) = J_x(f)$ for each $x \in X$, replacing $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ by $f^n : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$, we may assume $n = 1$.
Step 3: For each $x \in X$, let $\varphi_x : \pi^{-1}(\{x\}) \setminus \overline{V_x} \to \pi^{-1}(\{x\}) \setminus \overline{D(0, \frac{1}{2})}$ be a biholomorphic map such that $\varphi_x(x, \infty) = (x, \infty)$, under the canonical identification $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$. We see that φ_x extends analytically over $\partial V_x = \zeta_x$. For each $x \in X$, we define a quasi-regular map $h_x : \pi^{-1}(\{x\}) \cong \hat{\mathbb{C}} \to \pi^{-1}(\{g(x)\}) \cong \hat{\mathbb{C}}$ as follows:

$$h_x(z) := \begin{cases} \varphi_{g(x)} f_x \varphi_x^{-1}(z), & \text{if } z \in \varphi_x(\pi^{-1}(\{x\}) \setminus W_x), \\ z^{d(x)}, & \text{if } z \in \overline{D(0, \frac{1}{2})}, \\ \tilde{h}_x(z), & \text{if } z \in \varphi_x(W_x \setminus \overline{V_x}), \end{cases}$$

where $\tilde{h}_x : \varphi_x(W_x \setminus \overline{V_x}) \to D(0, \frac{1}{2}) \setminus \overline{D(0, (\frac{1}{2})^{d(x)})}$ is a regular covering and a K_0-quasiregular map with dilatation constant K_0 independent of $x \in X$.
Step 4: For each $x \in X$, we define a Beltrami differential $\mu_x(z)\frac{d\overline{z}}{dz}$ on $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$ as follows:

$$\begin{cases} \frac{\partial_{\overline{z}} \tilde{h}_x}{\partial_z \tilde{h}_x} \frac{d\overline{z}}{dz}, & \text{if } z \in \varphi_x(W_x \setminus \overline{V_x}), \\ (h_{g^m(x)} \cdots h_x)^* (\frac{\partial_{\overline{z}} \tilde{h}_{g^m(x)}}{\partial_z \tilde{h}_{g^m(x)}} \frac{d\overline{z}}{dz}), & \text{if } z \in (h_{g^m(x)} \cdots h_x)^{-1}(\varphi_{g^m(x)}(W_{g^m(x)} \setminus \overline{V_{g^m(x)}})), \\ 0, & \text{otherwise.} \end{cases}$$

Then, there exists a constant k with $0 < k < 1$ such that for each $x \in X$, $\|\mu_x\|_\infty \leq k$. By the construction, we have $h_x^*(\mu_{g(x)}\frac{d\overline{z}}{dz}) = \mu_x \frac{d\overline{z}}{dz}$, for each

$x \in X$. By the measurable Riemann mapping theorem ([17, page 194]), for each $x \in X$, there exists a quasiconformal map $\psi_x : \pi^{-1}(\{x\}) \to \pi^{-1}(\{x\})$ such that $\partial_{\overline{z}}\psi_x = \mu_x \partial_z \psi_x$, $\psi_x(0) = 0$, $\psi_x(1) = 1$, and $\psi_x(\infty) = \infty$, under the canonical identification $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$. For each $x \in X$, let $\hat{h}_x := \psi_{g(x)} h_x \psi_x^{-1} : \pi^{-1}(\{x\}) \to \pi^{-1}(\{g(x)\})$. Then, $\hat{h}_x$ is holomorphic on $\pi^{-1}(\{x\})$. By the construction, we see that $\hat{h}_x(z) = c(x) z^{d(x)}$, where $c(x) = \psi_{g(x)} h_x \psi_x^{-1}(1) = \psi_{g(x)} h_x(1)$. Moreover, by the construction again, we see that there exists a positive constant C_4 such that for each $x \in X$, $\frac{1}{C_4} \leq |h_x(1)| \leq C_4$. Furthermore, [17, Theorem 5.1 in page 73] implies that under the canonical identification $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$, the family $\{\psi_x^{-1}\}_{x \in X}$ is normal in $\hat{\mathbb{C}}$. Therefore, it follows that there exists a positive constant C_5 such that for each $x \in X$, $\frac{1}{C_5} \leq |c(x)| \leq C_5$. Let $\tilde{J}_x$ be the set of non-normality of the sequence $\{\hat{h}_{g^m(x)} \cdots \hat{h}_x\}_{m \in \mathbb{N}}$ in $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$. Since $\hat{h}_x(z) = c(x) z^{d(x)}$ and $\frac{1}{C_5} \leq |c(x)| \leq C_5$ for each $x \in X$, we get that for each $x \in X$, $\tilde{J}_x$ is a round circle. Moreover, [17, Theorem 5.1 in page 73] implies that $\{\psi_x\}_{x \in X}$ and $\{\psi_x^{-1}\}_{x \in X}$ are normal in $\hat{\mathbb{C}}$ (under the canonical identification $\pi^{-1}(\{x\}) \cong \hat{\mathbb{C}}$). Combining it with [35, Corollary 2.7], we see that for each $x \in X$, $J^x(f) = \varphi_x^{-1}(\psi_x^{-1}(\tilde{J}_x))$, and it follows that there exists a constant K such that for each $x \in X$, $J_x(f)$ is a K-quasicircle.

Thus, we have proved Theorem 4.21. □

Remark 4.22. Theorem 4.21 generalizes a result in [23, THÉORÈME 5.2], where O. Sester investigated hyperbolic polynomial skew products $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ such that for each $x \in X$, $d(x) = 2$.

We now demonstrate Theorem 2.40-3.

Proof of Theorem 2.40-3: First, we remark that the subset $W_{S,p}$ of $\Gamma^{\mathbb{N}}$ is a σ-invariant compact set. Hence, $\overline{f} : W_{S,p} \times \hat{\mathbb{C}} \to W_{S,p} \times \hat{\mathbb{C}}$ is a polynomial skew product over $\sigma : W_{S,p} \to W_{S,p}$. Suppose that $\tilde{J}(\overline{f}) \cap P(\overline{f}) \neq \emptyset$ and let $(\gamma, y) \in \tilde{J}(\overline{f}) \cap P(\overline{f})$ be a point. Then, since the point $\gamma = (\gamma_1, \gamma_2, \ldots)$ belongs to $W_{S,p}$, there exists a number $j \in \mathbb{N}$ such that $\gamma_j \in S$. Combining it with Theorem 2.19-5b and Theorem 2.19-2, we have $\gamma_j^{-1}(J(G)) \subset \mathbb{C} \setminus \hat{K}(G) \subset \mathbb{C} \setminus P(G)$. Moreover, we have that $\pi_{\hat{\mathbb{C}}}(\overline{f}_\gamma^{j-1}(\gamma, y)) = \pi_{\hat{\mathbb{C}}}(f_\gamma^{j-1}(\gamma, y)) \in J_{\sigma^{j-1}(\gamma)}(f) = \gamma_j^{-1}\left(J_{\sigma^j(\gamma)}(f)\right) \subset \gamma_j^{-1}(J(G))$. Hence, we obtain

$$\pi_{\hat{\mathbb{C}}}(\overline{f}_\gamma^{j-1}(\gamma, y)) \in \mathbb{C} \setminus P(G). \tag{70}$$

However, since $(\gamma, y) \in P(\overline{f})$, we have that $\pi_{\hat{\mathbb{C}}}(\overline{f}_\gamma^{j-1}(\gamma, y)) \in \pi_{\hat{\mathbb{C}}}(P(\overline{f})) \subset P(G)$, which contradicts (70). Hence, we must have that $\tilde{J}(\overline{f}) \cap P(\overline{f}) = \emptyset$. Therefore, $\overline{f} : W_{S,p} \times \hat{\mathbb{C}} \to W_{S,p} \times \hat{\mathbb{C}}$ is a hyperbolic polynomial skew product over the shift map $\sigma : W_{S,p} \to W_{S,p}$.

Combining this with Theorem 2.40-2a and Theorem 4.21, we conclude that there exists a constant $K_{S,p} \geq 1$ such that for each $\gamma \in W_{S,p}$, $J_\gamma(\overline{f})$ is a $K_{S,p}$-quasicircle. Moreover, by Theorem 2.40-2c, we have $J_\gamma(\overline{f}) = J_\gamma(f) = \hat{J}_\gamma(f)$.

Hence, we have shown Theorem 2.40-3. $\square$

To demonstrate Theorem 2.41, we need the following.

Lemma 4.23. *Let G be a polynomial semigroup generated by a non-empty compact set Γ in $\text{Poly}_{\deg \geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. Then, we have $\hat{K}(G_{\min,\Gamma}) = \hat{K}(G)$.*

Proof. Since $G_{\min,\Gamma} \subset G$, we have $\hat{K}(G) \subset \hat{K}(G_{\min,\Gamma})$. Moreover, it is easy to see $\hat{K}(G_{\min,\Gamma}) = \cap_{g \in G_{\min,\Gamma}} K(g)$. Let $g \in G_{\min,\Gamma}$ and $h \in \Gamma \setminus \Gamma_{\min}$. For each $\alpha \in \Gamma_{\min}$, we have $\alpha^{-1}(J_{\min}(G)) \subset J_{\min}(G)$. Since $\sharp(J_{\min}(G)) \geq 3$ (Theorem 2.19-5a), Lemma 3.1-6 implies that $J(g) \subset J_{\min}(G)$. Hence, from Theorem 2.19-5b, it follows that

$$h(J(g)) \subset \text{int}(\hat{K}(G)) \subset \text{int}(\hat{K}(g)). \tag{71}$$

Since $J(g)$ is connected and each connected component of $\text{int}(K(g))$ is simply connected, the above (71) implies that $h(K(g)) \subset K(g)$. Hence, we obtain $h(\hat{K}(G_{\min,\Gamma})) = h(\cap_{g \in G_{\min,\Gamma}} K(g)) \subset \cap_{g \in G_{\min,\Gamma}} K(g) = \hat{K}(G_{\min,\Gamma})$. Combined with that $\alpha(\hat{K}(G_{\min,\Gamma})) \subset \hat{K}(G_{\min,\Gamma})$ for each $\alpha \in \Gamma_{\min}$, it follows that for each $\beta \in G$, $\beta(\hat{K}(G_{\min,\Gamma})) \subset \hat{K}(G_{\min,\Gamma})$. Therefore, we obtain $\hat{K}(G_{\min,\Gamma}) \subset \hat{K}(G)$. Thus, it follows that $\hat{K}(G_{\min,\Gamma}) = \hat{K}(G)$. $\square$

Definition 4.24. Let G be a rational semigroup and N a positive integer. We denote by $SH_N(G)$ the set of points $z \in \hat{\mathbb{C}}$ satisfying that there exists a positive number δ such that for each $g \in G$, $\deg(g : V \to B(z, \delta)) \leq N$, for each connected component V of $g^{-1}(B(z, \delta))$. Moreover, we set $UH(G) := \hat{\mathbb{C}} \setminus \cup_{N \in \mathbb{N}} SH_N(G)$.

Lemma 4.25. *Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg \geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$ and that $\Gamma \setminus \Gamma_{\min}$ is not compact. Moreover, suppose that (a) in Proposition 2.37-2 holds. Then, there exists an open neighborhood $\mathcal{U}$ of $\Gamma_{\min}$ in Γ and an open set U in $\text{int}(\hat{K}(G))$ with $\overline{U} \subset \text{int}(\hat{K}(G))$ such that:*

1. *$\cup_{h \in \mathcal{U}} h(U) \subset U$;*
2. *$\cup_{h \in \mathcal{U}} CV^*(h) \subset U$, and*

3. *denoting by H the polynomial semigroup generated by $\mathcal{U}$, we have that $P^*(H) \subset \text{int}(\hat{K}(G)) \subset F(H)$ and that H is hyperbolic.*

Proof. Let $h_0 \in \Gamma_{\min}$ be an element. Let $\mathcal{E} := \{\psi(z) = az + b \mid a, b \in \mathbb{C}, |a| = 1, \psi(J(h_0)) = J(h_0)\}$. Then, by [2], $\mathcal{E}$ is compact in Poly. Moreover, by [2], we have the following two claims:
Claim 1: If $J(h_0)$ is a round circle with the center b_0 and radius r, then $\mathcal{E} = \{\psi(z) = a(z - b_0) + b_0 \mid |a| = r\}$.
Claim 2: If $J(h_0)$ is not a round circle, then $\sharp\mathcal{E} < \infty$.

Let z_0 be the unique attracting fixed point of h_0 in $\mathbb{C}$. Let $g \in G_{\min,\Gamma}$. By [2], for each $n \in \mathbb{N}$, there exists an $\psi_n \in \mathcal{E}$ such that $h_0^n g = \psi_n g h_0^n$. Hence, for each $n \in \mathbb{N}$, $h_0^n g(z_0) = \psi_n g h_0^n(z_0) = \psi_n g(z_0)$. Combining it with Claim 1 and Claim 2, it follows that there exists an $n \in \mathbb{N}$ such that $h_0^n(g(z_0)) = z_0$. For this n, $g(z_0) = \psi_n^{-1}(h_0^n(g(z_0))) = \psi_n^{-1}(z_0) \in \cup_{\psi\in\mathcal{E}}\psi(z_0)$. Combining it with Claim 1 and Claim 2 again, we see that the set $C := \overline{\cup_{g\in G_{\min,\Gamma}}\{g(z_0)\}}$ is a compact subset of $\text{int}(\hat{K}(G))$. Let d_H be the hyperbolic distance on $\text{int}(\hat{K}(G))$. Let $R > 0$ be a large number such that setting $U := \{z \in \text{int}(\hat{K}(G)) \mid \min_{a\in C} d_H(z, a) < R\}$, we have $\cup_{h\in\Gamma_{\min}} CV^*(h) \subset U$. Then, for each $h \in \Gamma_{\min}$, $\overline{h(U)} \subset U$. Therefore, there exists an open neighborhood $\mathcal{U}$ of $\Gamma_{\min}$ in Γ such that $\cup_{h\in\mathcal{U}} h(U) \subset U$, and such that $\cup_{h\in\mathcal{U}} CV^*(h) \subset U$. Let H be the polynomial semigroup generated by $\mathcal{U}$. From the above argument, we obtain $P^*(H) = \overline{\cup_{g\in H} CV^*(g)} \subset \overline{\cup_{g\in H\cup\{Id\}} g\left(\cup_{h\in\mathcal{U}} CV^*(h)\right)} \subset \overline{\cup_{g\in H\cup\{Id\}} g(U)} \subset \overline{U} \subset \text{int}(\hat{K}(G)) \subset F(H)$. Hence, H is hyperbolic. Thus, we have proved Lemma 4.25. $\square$

We now demonstrate Theorem 2.41.

Proof of Theorem 2.41: Suppose that $G_{\min,\Gamma}$ is semi-hyperbolic. We will consider the following two cases:
Case 1: $\Gamma \setminus \Gamma_{\min}$ is compact.
Case 2: $\Gamma \setminus \Gamma_{\min}$ is not compact.

Suppose that we have Case 1. Since $UH(G_{\min,\Gamma}) \subset P(G_{\min,\Gamma})$, $G_{\min,\Gamma} \in \mathcal{G}$, and $G_{\min,\Gamma}$ is semi-hyperbolic, we obtain $UH(G_{\min,\Gamma}) \cap \mathbb{C} \subset F(G_{\min,\Gamma}) \cap \hat{K}(G_{\min,\Gamma}) = \text{int}(\hat{K}(G_{\min,\Gamma}))$. By Lemma 4.23, we have $\hat{K}(G_{\min,\Gamma}) = \hat{K}(G)$. Hence, we obtain

$$UH(G_{\min,\Gamma}) \cap \mathbb{C} \subset \text{int}(\hat{K}(G)) \subset \mathbb{C} \setminus J_{\min}(G). \tag{72}$$

Therefore, there exists a positive integer N and a positive number δ such that for each $z \in J_{\min}(G)$ and each $h \in G_{\min,\Gamma}$, we have

$$\deg(h : V \to D(z, \delta)) \leq N, \tag{73}$$

for each connected component V of $h^{-1}(D(z,\delta))$. Moreover, combining Theorem 2.19-5b and Theorem 2.19-2, we obtain $\cup_{\alpha\in\Gamma\setminus\Gamma_{\min}}\alpha^{-1}(J_{\min}(G))\cap P^*(G) = \emptyset$. Hence, there exists a number δ_1 such that for each $z \in \cup_{\alpha\in\Gamma\setminus\Gamma_{\min}}\alpha^{-1}(J_{\min}(G))$ and each $\beta \in G\cup\{Id\}$,

$$\deg(\beta : W \to D(z,\delta_1)) = 1, \tag{74}$$

for each connected component W of $\beta^{-1}(D(z,\delta_1))$. For this δ_1, there exists a number $\delta_2 > 0$ such that for each $z \in J_{\min}(G)$ and each $\alpha \in \Gamma\setminus\Gamma_{\min}$,

$$\operatorname{diam} B \le \delta_1,\ \deg(\alpha : B \to D(z,\delta_2)) \le \max\{\deg(\alpha) \mid \alpha \in \Gamma\setminus\Gamma_{\min}\} \tag{75}$$

for each connected component B of $\alpha^{-1}(D(z,\delta_2))$. Furthermore, by [32, Lemma 1.10] (or [33]), we have that there exists a constant $0 < c < 1$ such that for each $z \in J_{\min}(G)$, each $h \in G_{\min,\Gamma}\cup\{Id\}$, and each connected component V of $h^{-1}(D(z,c\delta))$,

$$\operatorname{diam} V \le \delta_2. \tag{76}$$

Let $g \in G$ be any element.

Suppose that $g \in G_{\min,\Gamma}$. Then, by (73), for each $z \in J_{\min}(G)$, we have $\deg(g : V \to D(z,c\delta)) \le N$, for each connected component V of $g^{-1}(D(z,c\delta))$.

Suppose that g is of the form $g = h\circ\alpha\circ g_0$, where $h \in G_{\min,\Gamma}\cup\{Id\}$, $\alpha \in \Gamma\setminus\Gamma_{\min}$, and $g_0 \in G\cup\{Id\}$. Then, combining (74), (75), and (76), we get that for each $z \in J_{\min}(G)$, $\deg(g : W \to D(z,c\delta)) \le N\cdot\max\{\deg(\alpha) \mid \alpha \in \Gamma\setminus\Gamma_{\min}\}$, for each connected component W of $g^{-1}(D(z,c\delta))$.

From the above argument, we see that $J_{\min}(G) \subset SH_{N'}(G)$, where $N' := N\cdot\max\{\deg(\alpha) \mid \alpha \in \Gamma\setminus\Gamma_{\min}\}$. Moreover, by Theorem 2.19-2, we see that for any point $z \in J(G)\setminus J_{\min}(G)$, $z \in SH_1(G)$. Hence, we have shown that $J(G) \subset \hat{\mathbb{C}}\setminus UH(G)$. Therefore, G is semi-hyperbolic, provided that we have Case 1.

We now suppose that we have Case 2. Then, by Proposition 2.37, we have that for each $h \in \Gamma_{\min}$, $K(h) = \hat{K}(G)$ and $\operatorname{int}(K(h))$ is non-empty and connected. Moreover, for each $h \in \Gamma_{\min}$, $\operatorname{int}(K(h))$ is an immediate basin of an attracting fixed point $z_h \in \mathbb{C}$. Let $\mathcal{U}$ be the open neighborhood of $\Gamma_{\min}$ in Γ as in Lemma 4.25. Denoting by H the polynomial semigroup generated by $\mathcal{U}$, we have $P^*(H) \subset \operatorname{int}(\hat{K}(G))$. Therefore, there exists a number $\delta > 0$ such that

$$D(J(G),\delta) \subset \mathbb{C}\setminus P(H). \tag{77}$$

Moreover, combining Theorem 2.19-5b and that $\Gamma\setminus\mathcal{U}$ is compact, we see that there exists a number $\epsilon > 0$ such that

$$\overline{\bigcup_{\alpha\in\Gamma\setminus\mathcal{U}}\alpha^{-1}(D(J_{\min}(G),\epsilon))} \subset A_0, \tag{78}$$

where A_0 denotes the unbounded component of $\mathbb{C} \setminus J_{\min}(G)$. Combining it with Theorem 2.19-2, it follows that there exists a number $\delta_1 > 0$ such that

$$D\left(\bigcup_{\alpha \in \Gamma \setminus \mathcal{U}} \alpha^{-1}(D(J_{\min}(G), \epsilon)),\ \delta_1\right) \subset \mathbb{C} \setminus P(G). \tag{79}$$

For this δ_1, there exists a number $\delta_2 > 0$ such that for each $\alpha \in \Gamma \setminus \mathcal{U}$ and each $x \in D(J_{\min}(G), \epsilon)$,

$$\text{diam } B \leq \delta_1,\ \deg(\alpha : B \to D(x, \delta_2)) \leq \max\{\deg(\beta) \mid \beta \in \Gamma \setminus \mathcal{U}\} \tag{80}$$

for each connected component B of $\alpha^{-1}(D(x, \delta_2))$. By Lemma 3.11 and (77), there exists a constant $c > 0$ such that for each $h \in H$ and each $z \in J_{\min}(G)$,

$$\text{diam } V \leq \min\{\delta_2, \epsilon\}, \tag{81}$$

for each connected component V of $h^{-1}(D(z, c\delta))$. Let $z \in J_{\min}(G)$ and $g \in G$. We will show that $z \in \mathbb{C} \setminus UH(G)$.

Suppose that $g \in H$. Then, (77) implies that for each connected component V of $g^{-1}(D(z, c\delta))$, $\deg(g : V \to D(z, c\delta)) = 1$.

Suppose that g is of the form $g = h \circ \alpha \circ g_0$, where $h \in H \cup \{Id\}, \alpha \in \Gamma \setminus \mathcal{U}, g_0 \in G \cup \{Id\}$. Let W be a connected component of $g^{-1}(D(z, c\delta))$ and let $W_1 := g_0(W)$ and $V := \alpha(W_1)$. Let z_1 be the point such that $\{z_1\} = V \cap h^{-1}(\{z\})$. If $z_1 \in \mathbb{C} \setminus D(J_{\min}(G), \epsilon)$, then, by (81) and Theorem 2.19-2, $V \subset D(z_1, \epsilon) \subset \mathbb{C} \setminus P(G)$. Hence, $\deg(\alpha \circ g_0 : W \to V) = 1$, which implies that $\deg(g : W \to D(z, c\delta)) = 1$. If $z_1 \in D(J_{\min}(G), \epsilon)$, then by (81), $V \subset D(z_1, \delta_2)$. Combining it with (79) and (80), we obtain $\deg(\alpha \circ g_0 : W \to V) = \deg(\alpha : W_1 \to V) \leq \max\{\deg(\beta) \mid \beta \in \Gamma \setminus \mathcal{U}\}$. Therefore, $\deg(g : W \to D(z, c\delta)) \leq \max\{\deg(\beta) \mid \beta \in \Gamma \setminus \mathcal{U}\}$. Thus, $J_{\min}(G) \subset \mathbb{C} \setminus UH(G)$.

Moreover, Theorem 2.19-2 implies that $J(G) \setminus J_{\min}(G) \subset \mathbb{C} \setminus P(G) \subset \mathbb{C} \setminus UH(G)$. Therefore, $J(G) \subset \mathbb{C} \setminus UH(G)$, which implies that G is semi-hyperbolic.

Thus, we have proved Theorem 2.41. □

We now demonstrate Theorem 2.42.

Proof of Theorem 2.42: We use the same argument as that in the proof of Theorem 2.41, but we modify it as follows:

1. In (72), we replace $UH(G_{\min,\Gamma}) \cap \mathbb{C}$ by $P^*(G_{\min,\Gamma})$.
2. In (73), we replace N by 1.

3. We replace (75) by the following (75)' diam $B \le \delta_1$, $\deg(\alpha : B \to D(z, \delta_2)) = 1$.

4. We replace (80) by the following (80)' diam $B \le \delta_1$, $\deg(\alpha : B \to D(x, \delta_2)) = 1$. (We take the number $\epsilon > 0$ so small.)

With these modification, it is easy to see that G is hyperbolic.

Thus, we have proved Theorem 2.42. □

We now demonstrate Proposition 2.37-4.

Proof of Proposition 2.37-4: Suppose that (a) in Proposition 2.37-2 holds. By Lemma 4.25, $G_{\min,\Gamma}$ is hyperbolic. Combining it with Theorem 2.41, it follows that G is semi-hyperbolic. Thus, we have proved Proposition 2.37-4. □

To demonstrate Theorem 2.44, we need the following proposition.

Proposition 4.26. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a semi-hyperbolic polynomial skew product over $g : X \to X$. Suppose that for each $x \in X$, $d(x) \ge 2$, and that $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$. Let $\omega \in X$ be a point. If $\mathrm{int}(K_\omega(f))$ is a non-empty connected set, then $J_\omega(f)$ is a Jordan curve.*

Proof. By [35, Theorem 1.12] and Lemma 3.6, we get that the unbounded component $A_\omega(f)$ of $F_\omega(f)$ is a John domain. Combining it, that $A_\omega(f)$ is simply connected (cf. Lemma 3.6), and [21, page 26], we see that $J_\omega(f) = \partial(A_\omega(f))$ (cf. Lemma 3.4) is locally connected. Moreover, by Lemma 3.4-5, we have $\partial(\mathrm{int}(K_\omega(f))) = J_\omega(f)$. Hence, we see that $\hat{\mathbb{C}} \setminus J_\omega(f)$ has exactly two connected components $A_\gamma(f)$ and $\mathrm{int}(K_\omega(f))$, and that $J_\omega(f)$ is locally connected. From [22, Lemma 5.1], it follows that $J_\gamma(f)$ is a Jordan curve. Thus, we have proved Proposition 4.26. □

We now demonstrate Theorem 2.44.

Proof of Theorem 2.44: Let $\gamma \in R(\Gamma, \Gamma \setminus \Gamma_{\min})$ and $y \in \mathrm{int}(K_\gamma(f))$. Combining Theorem 2.40-1 and [32, Lemma 1.10], we obtain $\liminf_{n\to\infty} d(f_{\gamma,n}(y), J(G)) > 0$. Combining this with Lemma 4.20 and Theorem 2.40-1, we see that there exists a point $a \in P^*(G) \cap F(G)$ such that $\liminf_{n\to\infty} d(f_{\gamma,n}(y), a) = 0$. Since $P^*(G) \cap F(G) \subset \mathrm{int}(\hat{K}(G))$, it follows that there exists a positive integer l such that

$$f_{\gamma,l}(y) \in \mathrm{int}(\hat{K}(G)). \tag{82}$$

Combining (82) and the same method as that in the proof of Claim 3 in the proof of Theorem 2.40-1 and Theorem 2.40-2, we get that there exists exactly one bounded component U_γ of $F_\gamma(f)$. Combining it with Proposition 4.26, it follows that $J_\gamma(f)$ is a Jordan curve. Moreover, by [32, Theorem 2.14-(4)], we have $\hat{J}_\gamma(f) = J_\gamma(f)$.

Thus, we have proved Theorem 2.44. $\square$

We now demonstrate Theorem 2.45.

Proof of Theorem 2.45: Let V be an open set with $J(G) \cap V \neq \emptyset$. We may assume that V is connected. Then, by Theorem 3.2, there exists an element $\alpha_1 \in G$ such that $J(\alpha_1) \cap V \neq \emptyset$. Since we have $G \in \mathcal{G}_{dis}$, Theorem 2.1 implies that there exists an element $\alpha_2 \in G$ such that no connected component J of $J(G)$ satisfies $J(\alpha_1) \cup J(\alpha_2) \subset J$. Hence, we have $\langle \alpha_1, \alpha_2 \rangle \in \mathcal{G}_{dis}$. Since $J(\alpha_1) \cap V \neq \emptyset$, combined with Lemma 3.4-2, we get that there exists an $l_0 \in \mathbb{N}$ such that for each l with $l \geq l_0$, $J(\alpha_2 \alpha_1^l) \cap V \neq \emptyset$. Moreover, since no connected component J of $J(G)$ satisfies $J(\alpha_1) \cup J(\alpha_2) \subset J$, Lemma 3.4-2 implies that there exists an $l_1 \in \mathbb{N}$ such that for each l with $l \geq l_1$, $J(\alpha_2 \alpha_1^l) \cap J(\alpha_1 \alpha_2^l) = \emptyset$. We fix an $l \in \mathbb{N}$ with $l \geq \max\{l_0, l_1\}$. We now show the following claim.

Claim 1. The semigroup $H_0 := \langle \alpha_2 \alpha_1^l, \alpha_1 \alpha_2^l \rangle$ is hyperbolic, and for the skew product $\tilde{f} : \Gamma_0^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma_0^{\mathbb{N}} \times \hat{\mathbb{C}}$ associated with $\Gamma_0 = \{\alpha_2 \alpha_1^l, \alpha_1 \alpha_2^l\}$, there exists a constant $K \geq 1$ such that for any $\gamma \in \Gamma_0^{\mathbb{N}}$, $J_\gamma(\tilde{f})$ is a K-quasicircle.

To show this claim, applying Theorem 2.40-3 with $\Gamma = \{\alpha_1, \alpha_2\}, S = \Gamma \setminus \Gamma_{\min}$, and $p = 2l + 1$, we see that the polynomial skew product $\overline{f} : W_{S,2l+1} \times \hat{\mathbb{C}} \to W_{S,2l+1} \times \hat{\mathbb{C}}$ over $\sigma : W_{S,2l+1} \to W_{S,2l+1}$ is hyperbolic, and that there exists a constant $K \geq 1$ such that for each $\gamma \in W_{S,2l+1}$, $J_\gamma(\overline{f})$ is a K-quasicircle. Moreover, combining the hyperbolicity of $\overline{f}$ above and Remark 4.16, we see that the semigroup H_1 generated by the family $\{\alpha_{j_1} \circ \cdots \circ \alpha_{j_{l+1}} \mid 1 \leq \exists k_1 \leq l+1 \text{ with } j_{k_1} = 1, 1 \leq \exists k_2 \leq l+1 \text{ with } j_{k_2} = 2\}$ is hyperbolic. Hence, the semigroup H_0, which is a subsemigroup of H_1, is hyperbolic. Therefore, Claim 1 holds.

We now show the following claim.

Claim 2. We have either $J(\alpha_2 \alpha_1^l) < J(\alpha_1 \alpha_2^l)$, or $J(\alpha_1 \alpha_2^l) < J(\alpha_2 \alpha_1^l)$.

To show this claim, since $J(\alpha_2 \alpha_1^l) \cap J(\alpha_1 \alpha_2^l) = \emptyset$ and $H_0 \in \mathcal{G}$, combined with Lemma 3.9, we obtain Claim 2.

By Claim 2, we have the following two cases.

Case 1. $J(\alpha_2 \alpha_1^l) < J(\alpha_1 \alpha_2^l)$.

Case 2. $J(\alpha_1 \alpha_2^l) < J(\alpha_2 \alpha_1^l)$.

We may assume that we have Case 1 (when we have Case 2, we can show all statements of our theorem, using the same method as below). Let $A := K(\alpha_1 \alpha_2^l) \setminus \text{int}(K(\alpha_2 \alpha_1^l))$. By Claim 1, we have that $J(\alpha_1 \alpha_2^l)$ and $J(\alpha_2 \alpha_1^l)$ are quasicircles. Moreover, since $H_0 \in \mathcal{G}_{dis}$ and H_0 is hyperbolic, we must have $P^*(H_0) \subset \text{int}(K(\alpha_2 \alpha_1^l))$. Therefore, it follows that if we take a small open neighborhood U of A, then there exists a number $n \in \mathbb{N}$ such that, setting $h_1 := (\alpha_2 \alpha_1^l)^n$ and $h_2 := (\alpha_1 \alpha_2^l)^n$, we have that

$$h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}) \subset U \text{ and } h_1^{-1}(\overline{U}) \cap h_2^{-1}(\overline{U}) = \emptyset. \tag{83}$$

Moreover, combining Lemma 3.4-2 and that $J(h_1) \cap V \neq \emptyset$, we get that there exists a $k \in \mathbb{N}$ such that $J(h_2 h_1^k) \cap V \neq \emptyset$. We set $g_1 := h_1^{k+1}$ and $g_2 := h_2 h_1^k$. Moreover, we set $H := \langle g_1, g_2 \rangle$. Since H is a subsemigroup of H_0 and H_0 is hyperbolic, we have that H is hyperbolic. Moreover, (83) implies that $g_1^{-1}(\overline{U}) \cup g_2^{-1}(\overline{U}) \subset U$ and $g_1^{-1}(\overline{U}) \cap g_2^{-1}(\overline{U}) = \emptyset$. Hence, we have shown that for the semigroup $H = \langle g_1, g_2 \rangle$, statements 1,2, and 3 in Theorem 2.45 hold.

From statement 2 and Lemma 3.1-6, we obtain $J(H) \subset \overline{U}$ and $g_1^{-1}(J(H)) \cap g_2^{-1}(J(H)) = \emptyset$. Combining this with Lemma 3.1-2 and Lemma 3.5-2, it follows that the skew product $f : \Gamma_1^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma_1^{\mathbb{N}} \times \hat{\mathbb{C}}$ associated with $\Gamma_1 = \{g_1, g_2\}$ satisfies that $J(H)$ is equal to the disjoint union of the sets $\{\hat{J}_\gamma(f)\}_{\gamma \in \Gamma_1^{\mathbb{N}}}$. Moreover, since H is hyperbolic, [32, Theorem 2.14-(2)] implies that for each $\gamma \in \Gamma_1^{\mathbb{N}}$, $\hat{J}_\gamma(f) = J_\gamma(f)$. In particular, the map $\gamma \mapsto J_\gamma(f)$ from $\Gamma_1^{\mathbb{N}}$ into the space of non-empty compact sets in $\hat{\mathbb{C}}$, is injective. Since $J_\gamma(f)$ is connected for each $\gamma \in \Gamma_1^{\mathbb{N}}$ (Claim 1), it follows that for each connected component J of $J(H)$, there exists an element $\gamma \in \Gamma_1^{\mathbb{N}}$ such that $J = J_\gamma(f)$. Furthermore, by Claim 1, each connected component J of $J(H)$ is a K-quasicircle, where K is a constant not depending on J. Moreover, by [32, Theorem 2.14-(4)], the map $\gamma \mapsto J_\gamma(f)$ from $\Gamma_1^{\mathbb{N}}$ into the space of non-empty compact sets in $\hat{\mathbb{C}}$, is continuous with respect to the Hausdorff topology. Therefore, we have shown that statements 4a,4b,4c, and 4d hold for $H = \langle g_1, g_2 \rangle$ and $f : \Gamma_1^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma_1^{\mathbb{N}} \times \hat{\mathbb{C}}$.

We now show that statement 4e holds. Since we are assuming Case 1, Proposition 2.23 implies that $\{h_1, h_2\}_{\min} = \{h_1\}$. Hence $J(g_1) < J(g_2)$. Combining it with Proposition 2.23 and statement 4b, we obtain

$$J(g_1) = J_{\min}(H) \text{ and } J(g_2) = J_{\max}(H). \tag{84}$$

Moreover, since $J(g_1) = J(\alpha_2 \alpha_1^l)$, $J(\alpha_2 \alpha_1^l) \cap V \neq \emptyset$, $J(g_2) = J(h_2 h_1^k)$, and $J(h_2 h_1^k) \cap V \neq \emptyset$, it follows that

$$J_{\min}(H) \cap V \neq \emptyset \text{ and } J_{\max}(H) \cap V \neq \emptyset. \tag{85}$$

Let $\gamma \in \Gamma^{\mathbb{N}}$ be an element such that $J_\gamma(f) \cap (J_{\min}(H) \cup J_{\max}(H)) = \emptyset$. By statement 4b, we obtain

$$J_{\min}(H) < J_\gamma(f) < J_{\max}(H). \tag{86}$$

Since we are assuming V is connected, combining (85) and (86), we obtain $J_\gamma(f) \cap V \neq \emptyset$. Therefore, we have proved that statement 4e holds.

We now show that statement 4f holds. To show that, let $\omega = (\omega_1, \omega_2, \ldots) \in \Gamma_1^{\mathbb{N}}$ be an element such that $\sharp(\{j \in \mathbb{N} \mid \omega_j = g_1\}) = \sharp(\{j \in \mathbb{N} \mid \omega_j = g_2\}) = \infty$. For each $r \in \mathbb{N}$, let $\omega^r = (\omega_{r,1}, \omega_{r,2}, \ldots) \in \Gamma_1^{\mathbb{N}}$ be the element such that $\begin{cases} \omega_{r,j} = \omega_j & (1 \leq j \leq r), \\ \omega_{r,j} = g_1 & (j \geq r+1). \end{cases}$ Moreover, let $\rho^r = (\rho_{r,1}, \rho_{r,2}, \ldots) \in \Gamma_1^{\mathbb{N}}$ be the element such that $\begin{cases} \rho_{r,j} = \omega_j & (1 \leq j \leq r), \\ \rho_{r,j} = g_2 & (j \geq r+1). \end{cases}$ Combining (84), statement 4a, and statement 4b, we see that for each $r \in \mathbb{N}$, $J(g_1) < J_{\sigma^r(\omega)}(f) < J(g_2)$. Hence, by Theorem 2.7-3, we get that for each $r \in \mathbb{N}$, $(f_{\omega,r})^{-1}(J(g_1)) < (f_{\omega,r})^{-1}\left(J_{\sigma^r(\omega)}(f)\right) < (f_{\omega,r})^{-1}(J(g_2))$, where $f_{\omega,r}(y) = \pi_{\hat{\mathbb{C}}}(f^r(\omega, y))$. Since we have $(f_{\omega,r})^{-1}(J(g_1)) = J_{\omega^r}(f)$, $(f_{\omega,r})^{-1}\left(J_{\sigma^r(\omega)}(f)\right) = J_\omega(f)$, and $(f_{\omega,r})^{-1}(J(g_2)) = J_{\rho^r}(f)$, it follows that

$$J_{\omega^r}(f) < J_\omega(f) < J_{\rho^r}(f), \tag{87}$$

for each $r \in \mathbb{N}$. Moreover, since $\omega^r \to \omega$ and $\rho^r \to \omega$ in $\Gamma_1^{\mathbb{N}}$ as $r \to \infty$, statement 4d implies that $J_{\omega^r}(f) \to J_\omega(f)$ and $J_{\rho^r}(f) \to J_\omega(f)$ as $r \to \infty$, with respect to the Hausdorff topology. Combined with (87), statement 4b and statement 4c, we get that for any connected component W of $F(H)$, we must have $\partial W \cap J_\omega(f) = \emptyset$. Since $F(G) \subset F(H)$, it follows that for any connected component W' of $F(G)$, $\partial W' \cap J_\omega(f) = \emptyset$. Therefore, we have shown that statement 4f holds.

Thus, we have proved Theorem 2.45. □

4.6 Proofs of results in 2.6

In this section, we demonstrate Theorem 2.48. We need the following notations and lemmas.

Definition 4.27. Let h be a polynomial with $\deg(h) \geq 2$. Suppose that $J(h)$ is connected. Let ψ be a biholomorphic map $\hat{\mathbb{C}} \setminus \overline{D(0,1)} \to F_\infty(h)$ with $\psi(\infty) = \infty$ such that $\psi^{-1} \circ h \circ \psi(z) = z^{\deg(h)}$, for each $z \in \hat{\mathbb{C}} \setminus \overline{D(0,1)}$. (For the existence of the biholomorphic map ψ, see [19, Theorem 9.5].) For each $\theta \in \partial D(0,1)$, we set $T(\theta) := \psi(\{r\theta \mid 1 < r \leq \infty\})$. This is called the external ray (for $K(h)$) with angle θ.

Lemma 4.28. *Let h be a polynomial with* $\deg(h) \geq 2$. *Suppose that $J(h)$ is connected and locally connected and $J(h)$ is not a Jordan curve. Moreover, suppose that there exists an attracting periodic point of h in $K(h)$. Then, for any $\epsilon > 0$, there exist a point $p \in J(h)$ and elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$, such that all of the following hold.*

1. *For each $i = 1, 2$, the external ray $T(\theta_i)$ lands at the point p.*

2. *Let V_1 and V_2 be the two connected components of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$. Then, for each $i = 1, 2$, $V_i \cap J(h) \neq \emptyset$. Moreover, there exists an i such that diam $(V_i \cap K(h)) \leq \epsilon$.*

Proof. Let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to F_\infty(h)$ be a biholomorphic map with $\psi(\infty) = \infty$ such that for each $z \in \hat{\mathbb{C}} \setminus \partial D(0,1)$, $\psi^{-1} \circ h \circ \psi(z) = z^{\deg(h)}$. Since $J(h)$ is locally connected, the map $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to F_\infty(h)$ extends continuously over $\partial D(0,1)$, mapping $\partial D(0,1)$ onto $J(h)$. Moreover, since $J(h)$ is not a Jordan curve, it follows that there exist a point $p_0 \in J(h)$ and two points $t_{0,1}, t_{0,2} \in \partial D(0,1)$ with $t_{0,1} \neq t_{0,2}$ such that two external rays $T(t_{0,1})$ and $T(t_{0,2})$ land at the same point p_0. Considering a higher iterate of h if necessary, we may assume that there exists an attracting fixed point of h in $\mathrm{int}(K(h))$. Let $a \in \mathrm{int}(K(h))$ be an attracting fixed point of h and let U be the connected component of $\mathrm{int}(K(h))$ containing a. Then, there exists a critical point $c \in U$ of h. Let V_0 be the connected component of $\hat{\mathbb{C}} \setminus (T(t_1) \cup T(t_2) \cup \{p_0\})$ containing a. Moreover, for each $n \in \mathbb{N}$, let V_n be the connected component of $(h^n)^{-1}(V_0)$ containing a. Since $c \in U$, we get that for each $n \in \mathbb{N}$, $c \in V_n$. Hence, setting $e_n := \deg(h^n : V_n \to V_0)$, it follows that

$$e_n \to \infty \text{ as } n \to \infty. \tag{88}$$

We fix an $n \in \mathbb{N}$ satisfying $e_n > d$, where $d := \deg(h)$. Since $\deg(h^n : V_n \cap F_\infty(h) \to V_0 \cap F_\infty(h)) = \deg(h^n : V_n \to V_0)$, we have that the number of connected components of $V_n \cap F_\infty(h)$ is equal to e_n. Moreover, every connected component of $V_n \cap F_\infty(h)$ is a connected component of $(h^n)^{-1}(V_0 \cap F_\infty(h))$. Hence, it follows that there exist mutually disjoint arcs $\xi_1, \xi_2, \ldots, \xi_{e_n}$ in $\mathbb{C}$ satisfying all of the following.

1. For each j, $h^n(\xi_j) = (T(t_1) \cup T(t_2) \cup \{p_0\}) \cap \mathbb{C}$.

2. For each j, $\xi_j \cup \{\infty\}$ is the closure of union of two external rays and $\xi_j \cup \{\infty\}$ is a Jordan curve.

3. $\partial V_n = \xi_1 \cup \cdots \cup \xi_{e_n} \cup \{\infty\}$.

For each $j = 1, \ldots, e_n$, let W_j be the connected component of $\hat{\mathbb{C}} \setminus (\xi_j \cup \{\infty\})$ that does not contain V_n. Then, each W_j is a connected component of $\hat{\mathbb{C}} \setminus \overline{V_n}$. Hence, for each (i, j) with $i \neq j$, $W_i \cap W_j = \emptyset$. Since the number of critical values of h in $\mathbb{C}$ is less than or equal to $d - 1$, we have that $\sharp(\{1 \leq j \leq e_n \mid W_j \cap CV(h) = \emptyset\}) \geq e_n - (d-1)$. Therefore, denoting by $u_{1,j}$ the number of well-defined inverse branches of h^{-1} on W_j, we obtain

$$\sum_{j=1}^{e_n} u_{1,j} \geq d(e_n - (d-1)) \geq d.$$

Inductively, denoting by $u_{k,j}$ the number of well-defined inverse branches of $(h^k)^{-1}$ on W_j, we obtain

$$\sum_{j=1}^{e_n} u_{k,j} \geq d(d - (d-1)) \geq d, \text{for each } k \in \mathbb{N}. \tag{89}$$

For each $k \in \mathbb{N}$, we take a well-defined inverse branch ζ_k of $(h^k)^{-1}$ on a domain W_j, and let $B_k := \zeta_k(W_j)$. Then, $h^k : B_k \to W_j$ is biholomorphic. Since ∂B_k is the closure of finite union of external rays and h^{n+k} maps each connected component of $(\partial B_k) \cap \mathbb{C}$ onto $(T(t_1) \cup T(t_2) \cup \{p_0\}) \cap \mathbb{C}$, B_k is a Jordan domain. Hence, $h^k : B_k \to W_j$ induces a homeomorphism $\partial B_k \cong \partial W_j$. Therefore, ∂B_k is the closure of union of two external rays, which implies that $B_k \cap F_\infty(h)$ is a connected component of $(h^k)^{-1}(W_j \cap F_\infty(h))$. Hence, we obtain

$$l\left(\overline{\psi^{-1}(B_k \cap F_\infty(h))} \cap \partial D(0,1)\right) \to 0 \text{ as } k \to \infty, \tag{90}$$

where $l(\cdot)$ denotes the arc length of a subarc of $\partial D(0,1)$. Since $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to F_\infty(h)$ extends continuously over $\partial D(0,1)$, (90) implies that diam $(B_k \cap J(h)) \to 0$ as $k \to \infty$. Hence, there exists a $k \in \mathbb{N}$ such that diam $\overline{(B_k \cap K(h))} \leq \epsilon$. Let $\theta_1, \theta_2 \in \partial D(0,1)$ be two elements such that $\partial B_k = \overline{T(\theta_1) \cup T(\theta_2)}$. Then, there exists a point $p \in J(h)$ such that each $T(\theta_i)$ lands at the point p. By [19, Lemma 17.5], any of two connected components of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ intersects $J(h)$.

Thus, we have proved Lemma 4.28. □

Lemma 4.29. *Let G be a polynomial semigroup generated by a compact subset Γ of $\mathrm{Poly}_{\deg \geq 2}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Suppose $G \in \mathcal{G}_{dis}$. Let $m \in \mathbb{N}$ and suppose that there exists an element $(h_1, \ldots, h_m) \in \Gamma^m$ such that setting $h = h_m \circ \cdots \circ h_1$, $J(h)$ is connected and locally connected, and $J(h)$ is not a Jordan curve. Moreover, suppose that there exists an attracting periodic point of h in $K(h)$. Let $\alpha = (\alpha_1, \alpha_2, \ldots) \in \Gamma^{\mathbb{N}}$ be the element such that for each $k, l \in \mathbb{N} \cup \{0\}$ with $1 \leq l \leq m$, $\alpha_{km+l} = h_l$. Let $\rho_0 \in \Gamma \setminus \Gamma_{\min}$ be an element and let $\beta = (\rho_0, \alpha_1, \alpha_2, \ldots) \in \Gamma^{\mathbb{N}}$. Moreover, let $\psi_\beta : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_\beta(f)$ be a biholomorphic map with $\psi_\beta(\infty) = \infty$. Furthermore, for each $\theta \in \partial D(0,1)$, let $T_\beta(\theta) = \psi_\beta(\{r\theta \mid 1 < r \leq \infty\})$. Then, for any $\epsilon > 0$, there exist a point $p \in J_\beta(f)$ and elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$, such that all of the following statements 1 and 2 hold.*

1. *For each $i = 1, 2$, $T_\beta(\theta_i)$ lands at p.*

2. *Let V_1 and V_2 be the two connected components of $\hat{\mathbb{C}} \setminus (T_\beta(\theta_1) \cup T_\beta(\theta_2) \cup \{p\})$. Then, for each $i = 1, 2$, $V_i \cap J_\beta(f) \neq \emptyset$. Moreover, there exists an i such that diam $(V_i \cap K_\beta(f)) \leq \epsilon$ and such that $V_i \cap J_\beta(f) \subset \rho_0^{-1}(J(G)) \subset \mathbb{C} \setminus P(G)$.*

Proof. We use the notation and argument in the proof of Lemma 4.28. Taking a higher iterate of h, we may assume that $d := \deg(h) > \deg(\rho_0)$. Then, from (89), it follows that for each $k \in \mathbb{N}$, we can take a well-defined inverse branch ζ_k of $(h^k)^{-1}$ on a domain W_j such that setting $B_k := \zeta_k(W_j)$, B_k does not contain any critical value of ρ_0. By (90), there exists a $k \in \mathbb{N}$ such that diam $(B_k \cap J(h)) \leq \epsilon'$, where $\epsilon' > 0$ is a small number. Let B be a connected component of $\rho_0^{-1}(B_k)$. Then, there exist a point $p \in J_\beta(f)$ and elements $\theta_1, \theta_2 \in \partial D(0, 1)$ with $\theta_1 \neq \theta_2$ such that for each $i = 1, 2$, $T_\beta(\theta_i)$ lands at p, and such that B is a connected component of $\hat{\mathbb{C}} \setminus (T_\beta(\theta_1) \cup T_\beta(\theta_2) \cup \{p\})$. Taking ϵ' so small, we obtain diam $(B \cap K_\beta(f)) =$ diam $(B \cap J_\beta(f)) \leq \epsilon$. Moreover, since $\rho_0 \in \Gamma \setminus \Gamma_{\min}$, combining Theorem 2.19-2 and Theorem 2.19-5b, we obtain $J_\beta(f) = \rho_0^{-1}(J(h)) \subset \rho_0^{-1}(J(G)) \subset \mathbb{C} \setminus P(G)$. Hence, $B \cap J_\beta(f) \subset \rho_0^{-1}(J(G)) \subset \mathbb{C} \setminus P(G)$. Therefore, we have proved Lemma 4.29. □

Lemma 4.30. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\gamma \in X$ be a point. Suppose that $J_\gamma(f)$ is a Jordan curve. Then, for each $n \in \mathbb{N}$, $J_{g^n(\gamma)}(f)$ is a Jordan curve. Moreover, for each $n \in \mathbb{N}$, there exists no critical value of $f_{\gamma,n}$ in $J_{g^n(\gamma)}(f)$.*

Proof. Since $(f_{\gamma,1})^{-1}(K_{g(\gamma)}(f)) = K_\gamma(f)$, it follows that $\text{int}(K_{g(\gamma)}(f))$ is a non-empty connected set. Moreover, $J_{g(\gamma)}(f) = f_{\gamma,1}(J_\gamma(f))$ is locally connected. Furthermore, by Lemma 3.4-4 and Lemma 3.4-5, $\partial(\text{int}(K_{g(\gamma)}(f))) = \partial(A_{g(\gamma)}(f)) = J_{g(\gamma)}(f)$. Combining the above arguments and [22, Lemma 5.1], we get that $J_{g(\gamma)}(f)$ is a Jordan curve. Inductively, we conclude that for each $n \in \mathbb{N}$, $J_{g^n(\gamma)}(f)$ is a Jordan curve.

Furthermore, applying the Riemann-Hurwitz formula to the map $f_{\gamma,n} : \text{int}(K_\gamma(f)) \to \text{int}(K_{g^n(\gamma)}(f))$, we obtain $1 + p = \deg(f_{\gamma,n})$, where p denotes the cardinality of the critical points of $f_{\gamma,n} : \text{int}(K_\gamma(f)) \to \text{int}(K_{g^n(\gamma)}(f))$ counting multiplicities. Hence, $p = \deg(f_{\gamma,n}) - 1$. It implies that there exists no critical value of $f_{\gamma,n}$ in $J_{g^n(\gamma)}(f)$. □

Lemma 4.31. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\mu > 0$ be a number. Then, there exists a number $\delta > 0$ such that the following statement holds.*

- *Let $\omega \in X$ be any point and $p \in J_\omega(f)$ any point with* $\min\{|p-b| \mid (\omega, b) \in P(f), b \in \mathbb{C}\} > \mu$. *Suppose that $J_\omega(f)$ is connected. Let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_\omega(f)$ be a biholomorphic map with $\psi(\infty) = \infty$. For each $\theta \in \partial D(0,1)$, let $T(\theta) = \psi(\{r\theta \mid 1 < r \leq \infty\})$. Suppose that there exist two elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$ such that for each $i = 1, 2,$, $T(\theta_i)$ lands at p. Moreover, suppose that a connected component V of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ satisfies that diam $(V \cap K_\omega(f)) \leq \delta$. Furthermore, let $\gamma \in X$ be any point and suppose that there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that $g^{n_k}(\gamma) \to \omega$ as $k \to \infty$. Then, $J_\gamma(f)$ is not a quasicircle.*

Proof. Let $\mu > 0$. Let $R > 0$ with $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) \subset D(0,R)$. Combining Lemma 3.11 and Lemma 3.4-3, we see that there exists a $\delta_0 > 0$ with
$0 < \delta_0 < \frac{1}{20} \min\{\inf_{x \in X} \text{diam } J_x(f), \mu\}$ such that the following statement holds:

- Let $x \in X$ be any point and $n \in \mathbb{N}$ any element. Let $p \in D(0,R)$ be any point with $\min\{|p-b| \mid (g^n(x), b) \in P(f), b \in \mathbb{C}\} > \mu$. Let $\phi : D(p,\mu) \to \mathbb{C}$ be any well-defined inverse branch of $(f_{x,n})^{-1}$ on $D(p,\mu)$. Let A be any subset of $D(p, \frac{\mu}{2})$ with diam $A \leq \delta_0$. Then,
$$\text{diam } \phi(A) \leq \frac{1}{10} \inf_{x \in X} \text{diam } J_x(f). \tag{91}$$

We set $\delta := \frac{1}{10}\delta_0$. Let $\omega \in X$ and $p \in J_\omega(f)$ with $\min\{|p-b| \mid (\omega, b) \in P(f), b \in \mathbb{C}\} > \mu$. Suppose that $J_\omega(f)$ is connected and let $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \to A_\omega(f)$ be a biholomorphic map with $\psi(\infty) = \infty$. Setting $T(\theta) := \psi(\{r\theta \mid 1 < r \leq \infty\})$ for each $\theta \in \partial D(0,1)$, suppose that there exist two elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$ such that for each $i = 1, 2$, $T(\theta_i)$ lands at p. Moreover, suppose that a connected component V of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ satisfies that
$$\text{diam}(V \cap K_\omega(f)) \leq \delta. \tag{92}$$
Furthermore, let $\gamma \in X$ and suppose that there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that $g^{n_k}(\gamma) \to \omega$ as $k \to \infty$. We now suppose that $J_\gamma(f)$ is a quasicircle, and we will deduce a contradiction. Since $g^{n_k}(\gamma) \to \omega$ as $k \to \infty$, we obtain
$$\max\{d_e(b, K_\omega(f)) \mid b \in J_{g^{n_k}(\gamma)}(f)\} \to 0 \text{ as } k \to \infty. \tag{93}$$
We take a point $a \in V \cap J_\omega(f)$ and fix it. By Lemma 3.4-2, there exists a number $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, there exists a point y_k satisfying that
$$y_k \in J_{g^{n_k}(\gamma)}(f) \cap D(a, \frac{|a-p|}{10k}). \tag{94}$$

Let V' be the connected component of $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$ with $V' \neq V$. Then, by [19, Lemma 17.5],

$$V' \cap J_\omega(f) \neq \emptyset. \tag{95}$$

Combining (95) and Lemma 3.4-2, we see that there exists a $k_1 (\geq k_0) \in \mathbb{N}$ such that for each $k \geq k_1$,

$$V' \cap J_{g^{n_k}(\gamma)}(f) \neq \emptyset. \tag{96}$$

By assumption and Lemma 4.30, for each $k \geq k_1$, $J_{g^{n_k}(\gamma)}(f)$ is a Jordan curve. Combining it with (94) and (96), there exists a $k_2 (\geq k_1) \in \mathbb{N}$ satisfying that for each $k \geq k_2$, there exists a smallest closed subarc ξ_k of $J_{g^{n_k}(\gamma)}(f) \cong S^1$ such that $y_k \in \xi_k$, $\xi_k \subset \overline{V}$, $\sharp(\xi_k \cap (T(\theta_1) \cup T(\theta_2) \cup \{p\})) = 2$, and such that $\xi_k \neq J_{g^{n_k}(\gamma)}(f)$. For each $k \geq k_2$, let $y_{k,1}$ and $y_{k,2}$ be the two points such that $\{y_{k,1}, y_{k,2}\} = \xi_k \cap (T(\theta_1) \cup T(\theta_2) \cup \{p\})$. Then, (93) implies that

$$y_{k,i} \to p \text{ as } k \to \infty, \text{ for each } i = 1, 2. \tag{97}$$

Combining that $\xi_k \subset V \cup \{y_{k,1}, y_{k,2}\}$, (93), and (92), we get that there exists a $k_3 (\geq k_2) \in \mathbb{N}$ such that for each $k \geq k_3$,

$$\text{diam } \xi_k \leq \frac{\delta_0}{2}. \tag{98}$$

Moreover, combining (94) and (97), we see that there exists a constant $C > 0$ such that

$$\text{diam } \xi_k > C. \tag{99}$$

Combining (97), (98), and (99), we may assume that there exists a constant $C > 0$ such that for each $k \in \mathbb{N}$,

$$C < \text{ diam } \xi_k \leq \frac{\delta_0}{2} \text{ and } \xi_k \subset D(p, \delta_0). \tag{100}$$

By Lemma 4.30, each connected component v of $(f_{\gamma,n_k})^{-1}(\xi_k)$ is a subarc of $J_\gamma(f) \cong S^1$ and $f_{\gamma,n_k} : v \to \xi_k$ is a homeomorphism. For each $k \in \mathbb{N}$, let λ_k be a connected component of $(f_{\gamma,n_k})^{-1}(\xi_k)$, and let $z_{k,1}, z_{k,2} \in \lambda_k$ be the two endpoints of λ_k such that $f_{\gamma,n_k}(z_{k,1}) = y_{k,1}$ and $f_{\gamma,n_k}(z_{k,2}) = y_{k,2}$. Then, combining (91) and (100), we obtain

$$\text{diam}\lambda_k < \text{ diam } (J_\gamma(f) \setminus \lambda_k), \text{ for each large } k \in \mathbb{N}. \tag{101}$$

Moreover, combining (97), (100), and Koebe distortion theorem, it follows that

$$\frac{\text{diam } \lambda_k}{|z_{k,1} - z_{k,2}|} \to \infty \text{ as } k \to \infty. \tag{102}$$

Combining (101) and (102), we conclude that $J_\gamma(f)$ cannot be a quasicircle, since we have the following well-known fact:
Fact ([17, Chapter 2]): Let ξ be a Jordan curve in $\mathbb{C}$. Then, ξ is a quasicircle if and only if there exists a constant $K > 0$ such that for each $z_1, z_2 \in \xi$ with $z_1 \neq z_2$, we have $\frac{\text{diam } \lambda(z_1,z_2)}{|z_1 - z_2|} \leq K$, where $\lambda(z_1, z_2)$ denotes the smallest closed subarc of ξ such that $z_1, z_2 \in \lambda(z_1, z_2)$ and such that diam $\lambda(z_1, z_2) <$ diam $(\xi \setminus \lambda(z_1, z_2))$.

Hence, we have proved Lemma 4.31. □

We now demonstrate Theorem 2.48-1.

Proof of Theorem 2.48-1: Let γ be as in Theorem 2.48-1. Then, by Theorem 2.44, $J_\gamma(f)$ is a Jordan curve. Moreover, setting $h = h_m \circ \cdots \circ h_1$, since h is hyperbolic and $J(h)$ is not a quasicircle, $J(h)$ is not a Jordan curve. Combining it with Lemma 4.31 and Lemma 4.28, it follows that $J_\gamma(f)$ is not a quasicircle. Moreover, $A_\gamma(f)$ is a John domain (cf. [35, Theorem 1.12]). Combining the above arguments with [21, Theorem 9.3], we conclude that the bounded component U_γ of $F_\gamma(f)$ is not a John domain.

Thus, we have proved Theorem 2.48-1. □

We now demonstrate Theorem 2.48-2.

Proof of Theorem 2.48-2: Let ρ_0, β, γ be as in Theorem 2.48-2. By Theorem 2.44, $J_\gamma(f)$ is a Jordan curve. By Theorem 2.19-5, we have $\emptyset \neq \text{int}(\hat{K}(G)) \subset \text{int}(K(h))$. Moreover, h is semi-hyperbolic. Hence, h has an attracting periodic point in $K(h)$. Combining Lemma 4.31 and Lemma 4.29, we get that $J_\gamma(f)$ is not a quasicircle. Combining it with the argument in the proof of Theorem 2.48-1, it follows that $A_\gamma(f)$ is a John domain, but the bounded component U_γ of $F_\gamma(f)$ is not a John domain.

Thus, we have proved Theorem 2.48-2. □

4.7 Proofs of results in 2.7

In this subsection, we will demonstrate results in Section 2.7.

we now prove Corollary 2.51.

Proof of Corollary 2.51: By Remark 2.35, there exists a compact subset S of $\Gamma \setminus \Gamma_{\min}$ such that the interior of S with respect to the space Γ is not empty. Let $\mathcal{U} := R(\Gamma, S)$. Then, it is easy to see that $\mathcal{U}$ is residual in $\Gamma^{\mathbb{N}}$, and that for each Borel probability measure τ on $\text{Poly}_{\deg \geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$. Moreover, by Theorem 2.40-1 and Theorem 2.40-2, each $\gamma \in \mathcal{U}$ satisfies properties 1,2,3, and 4 in Corollary 2.51. Hence, we have proved Corollary 2.51. □

To demonstrate Theorem 2.52, we need several lemmas.

Lemma 4.32. *Let Γ be a compact set in* $\text{Poly}_{\deg\geq 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ. Let G be the polynomial semigroup generated by Γ. Suppose that $G \in \mathcal{G}$ and that G is semi-hyperbolic. Moreover, suppose that there exist two elements $\alpha, \beta \in \Gamma^{\mathbb{N}}$ such that $J_\beta(f) < J_\alpha(f)$. Let $\gamma \in \Gamma^{\mathbb{N}}$ and suppose that there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $\sigma^{n_k}(\gamma) \to \alpha$ as $k \to \infty$. Then, $J_\gamma(f)$ is a Jordan curve.*

Proof. Since G is semi-hyperbolic, [32, Theorem 2.14-(4)] implies that

$$J_{\sigma^{n_k}(\gamma)}(f) \to J_\alpha(f) \text{ as } k \to \infty, \tag{103}$$

with respect to the Hausdorff topology in the space of non-empty compact subsets of $\hat{\mathbb{C}}$. Combining it with Lemma 3.9, we see that there exists a number $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$,

$$J_\beta(f) < J_{\sigma^{n_k}(\gamma)}(f). \tag{104}$$

We will show the following claim.
Claim: $\text{int}(K_\gamma(f))$ is connected.

To show this claim, suppose that there exist two distinct components U_1 and U_2 of $\text{int}(K_\gamma(f))$. Let $y_i \in U_i$ be a point, for each $i = 1, 2$. Let $\epsilon > 0$ be a number such that $\overline{D(K_\beta(f), \epsilon)}$ is included in a connected component U of $\text{int}(K_\alpha(f))$. Then, combining [32, Theorem 2.14-(5)] and Lemma 4.20, we get that there exists a number $k_1 \in \mathbb{N}$ with $k_1 \geq k_0$ such that for each $k \geq k_1$ and each $i = 1, 2$,

$$f_{\gamma,n_k}(y_i) \in D(P^*(G), \epsilon) \subset \overline{D(K_\beta(f), \epsilon)} \subset U. \tag{105}$$

Combining (105), (103) and (104), we get that there exists a number $k_2 \in \mathbb{N}$ with $k_2 \geq k_1$ such that for each $k \geq k_2$,

$$f_{\gamma,n_k}(U_1) = f_{\gamma,n_k}(U_2) = V_k, \tag{106}$$

where V_k denotes the connected component of $\text{int}(K_{\sigma^{n_k}(\gamma)}(f))$ containing $J_\beta(f)$. From (104) and (106), it follows that

$$(f_{\gamma,n_k})^{-1}(J_\beta(f)) \subset \text{int}(K_\gamma(f)) \text{ and } (f_{\gamma,n_k})^{-1}(J_\beta(f)) \cap U_i \neq \emptyset \ (i = 1, 2), \tag{107}$$

which implies that

$$(f_{\gamma,n_k})^{-1}(J_\beta(f)) \text{ is disconnected.} \tag{108}$$

For each $k \geq k_2$, let $\omega^k := (\gamma_1, \ldots, \gamma_{n_k}, \beta_1, \beta_2, \ldots) \in \Gamma^{\mathbb{N}}$. Then for each $k \geq k_2$,

$$(f_{\gamma,n_k})^{-1}(J_\beta(f)) = J_{\omega^k}(f). \tag{109}$$

Since $G \in \mathcal{G}$, combining (108), (109) and Lemma 3.6 yields a contradiction. Hence, we have proved the claim.

From the above claim and Proposition 4.26, it follows that $J_\gamma(f)$ is a Jordan curve. □

Lemma 4.33. *Let* Γ *be a non-empty compact subset of* $\mathrm{Poly}_{\deg\geq 2}$. *Let* $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ *be the skew product associated with the family* Γ *of polynomials. Let* G *be the polynomial semigroup generated by* Γ. *Let* $\alpha, \rho \in \Gamma^{\mathbb{N}}$ *be two elements. Suppose that* $G \in \mathcal{G}$, *that* G *is semi-hyperbolic, that* α *is a periodic point of* $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$, *that* $J_\alpha(f)$ *is a quasicircle, and that* $J_\rho(f)$ *is not a Jordan curve. Then, for each* $\epsilon > 0$, *there exist* $n \in \mathbb{N}$ *and two elements* $\theta_1, \theta_2 \in \partial D(0,1)$ *with* $\theta_1 \neq \theta_2$ *satisfying all of the following.*

1. *Let* $\omega = (\alpha_1, \ldots, \alpha_n, \rho_1, \rho_2, \ldots) \in \Gamma^{\mathbb{N}}$ *and let* $\psi : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \cong A_\omega(f)$ *be a biholomorphic map with* $\psi(\infty) = \infty$. *Moreover, for each* $i = 1, 2$, *let* $T(\theta_i) := \psi(\{r\theta_i \mid 1 < r \leq \infty\})$. *Then, there exists a point* $p \in J_\omega(f)$ *such that for each* $i = 1, 2$, $T(\theta_i)$ *lands at* p.

2. *Let* V_1 *and* V_2 *be the two connected components of* $\hat{\mathbb{C}} \setminus (T(\theta_1) \cup T(\theta_2) \cup \{p\})$. *Then, for each* $i = 1, 2$, $V_i \cap J_\omega(f) \neq \emptyset$. *Moreover, there exists an* $i \in \{1, 2\}$ *such that* $\mathrm{diam}\,(V_i \cap K_\omega(f)) \leq \epsilon$, *and such that* $V_i \cap J_\omega(f) \subset D(J_\alpha(f), \epsilon)$.

Proof. For each $\gamma \in \Gamma^{\mathbb{N}}$, let $\psi_\gamma : \hat{\mathbb{C}} \setminus \overline{D(0,1)} \cong A_\gamma(f)$ be a biholomorphic map with $\psi_\gamma(\infty) = \infty$. Moreover, for each $\theta \in \partial D(0,1)$, let $T_\gamma(\theta) := \psi_\gamma(\{r\theta \mid 1 < r \leq \infty\})$. Since G is semi-hyperbolic, combining [35, Theorem 1.12], Lemma 3.6, and [21, page 26], we see that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is locally connected. Hence, for each $\gamma \in \Gamma^{\mathbb{N}}$, ψ_γ extends continuously over $\hat{\mathbb{C}} \setminus D(0,1)$ such that $\psi_\gamma(\partial D(0,1)) = J_\gamma(f)$. Moreover, since $G \in \mathcal{G}$, it is easy to see that for each $\gamma \in \Gamma^{\mathbb{N}}$, there exists a number $a_\gamma \in \mathbb{C}$ with $|a_\gamma| = 1$ such that for each $z \in \hat{\mathbb{C}} \setminus \overline{D(0,1)}$, we have $\psi_{\sigma(\gamma)}^{-1} \circ f_{\gamma,1} \circ \psi_\gamma(z) = a_\gamma z^{d(\gamma)}$.

Let $m \in \mathbb{N}$ be an integer such that $\sigma^m(\alpha) = \alpha$ and let $h := \alpha_m \circ \cdots \circ \alpha_1$. Moreover, for each $n \in \mathbb{N}$, we set $\omega^n := (\alpha_1, \ldots, \alpha_{mn}, \rho_1, \rho_2, \ldots) \in \Gamma^{\mathbb{N}}$. Then, $\omega^n \to \alpha$ in $\Gamma^{\mathbb{N}}$ as $n \to \infty$. Combining it with [32, Theorem 2.14-(4)], we obtain

$$J_{\omega^n}(f) \to J_\alpha(f) \text{ as } n \to \infty, \tag{110}$$

with respect to the Hausdorff topology. Let ξ be a Jordan curve in $\mathrm{int}(K(h))$ such that $P^*(\langle h \rangle)$ is included in the bounded component B of $\mathbb{C} \setminus \xi$. By

(110), there exists a $k \in \mathbb{N}$ such that $J_{\omega^k}(f) \cap (\xi \cup B) = \emptyset$. We now show the following claim.
Claim 1: $\xi \subset \text{int}(K_{\omega^k}(f))$.

To show this claim, suppose that ξ is included in $A_{\omega^k}(f) = \hat{\mathbb{C}} \setminus (K_{\omega^k}(f))$. Then, it implies that $f_{\omega^k,u} \to \infty$ on $P^*(\langle h \rangle)$ as $u \to \infty$. However, this is a contradiction, since $G \in \mathcal{G}$. Hence, we have shown Claim 1.

By Claim 1, we see that $P^*(\langle h \rangle)$ is included in a bounded component B_0 of $\text{int}(K_{\omega^k}(f))$. We now show the following claim.
Claim 2: $J_{\omega^k}(f)$ is not a Jordan curve.

To show this claim, suppose that $J_{\omega^k}(f)$ is a Jordan curve. Then, Lemma 4.30 implies that $J_\rho(f)$ is a Jordan curve. However, this is a contradiction. Hence, we have shown Claim 2.

By Claim 2, there exist two distinct elements $t_1, t_2 \in \partial D(0,1)$ and a point $p_0 \in J_{\omega^k}(f)$ such that for each $i = 1, 2$, $T_\rho(t_i)$ lands at the point p_0. Let W_0 be the connected component of $\hat{\mathbb{C}} \setminus (T_\rho(t_1) \cup T_\rho(t_2) \cup \{p_0\})$ such that W_0 does not contain B_0. Then, we have

$$\overline{W_0} \cap P^*(\langle h \rangle) = \emptyset. \tag{111}$$

For each $j \in \mathbb{N}$, we take a connected component W_j of $(h^j)^{-1}(W_0)$. Then, $h^j : W_j \to W_0$ is biholomorphic. We set $\zeta_j := (h^j|_{W_j})^{-1}$ on W_0. By (111), there exists a number $R > 0$ and a number $a > 0$ such that for each j, ζ_j is analytically continued to a univalent function $\tilde{\zeta}_j : B(\overline{W_0 \cap D(0,R)}, a) \to \hat{\mathbb{C}}$ and $W_j \cap (J_{\omega^{k+j}}(f)) \subset \tilde{\zeta}_j(W_0 \cap D(0,R))$. Hence, we obtain

$$\text{diam } (W_j \cap K_{\omega^{k+j}}(f)) = \text{diam } (W_j \cap J_{\omega^{k+j}}(f)) \to 0 \text{ as } j \to \infty. \tag{112}$$

Combining (110) and (112), there exists an $s \in \mathbb{N}$ such that $\text{diam } (W_s \cap K_{\omega^{k+s}}(f)) \le \epsilon$, and such that $W_s \cap J_{\omega^{k+s}}(f) \subset D(J_\alpha(f), \epsilon)$.

Each connected component of $(\partial W_s) \cap \mathbb{C}$ is a connected component of $(h^s)^{-1}((T_{\omega^k}(t_1) \cup \underline{T_{\omega^k}(t_2) \cup \{p_0\}) \cap \mathbb{C})}$, and there are some $u_1, \ldots, u_v \in \partial D(0,1)$ such that $\partial W_s = \cup_{i=1}^v T_{\omega^{k+s}}(u_i)$. Hence, W_s is a Jordan domain. Therefore, $h^s : \overline{W_s} \to \overline{W_0}$ is a homeomorphism. Thus, $h^s : (\partial W_s) \cap \mathbb{C} \to (\partial W_0) \cap \mathbb{C}$ is a homeomorphism. Hence, $(\partial W_s) \cap \mathbb{C}$ is connected. It follows that there exist two elements $\theta_1, \theta_2 \in \partial D(0,1)$ with $\theta_1 \neq \theta_2$ and a point $p \in J_{\omega^{k+s}}(f)$ such that $\partial W_s = T_{\omega^{k+s}}(\theta_1) \cup T_{\omega^{k+s}}(\theta_2) \cup \{p\}$, and such that for each $i = 1, 2$, $T_{\omega^{k+s}}(\theta_i)$ lands at the point p. By [19, Lemma 17.5], each of two connected components of $\hat{\mathbb{C}} \setminus (T_{\omega^{k+s}}(\theta_1) \cup T_{\omega^{k+s}}(\theta_2) \cup \{p\})$ intersects $J_{\omega^{k+s}}(f)$.

Hence, we have proved Lemma 4.33. □

Lemma 4.34. *Let Γ be a non-empty compact subset of* $\text{Poly}_{\deg \ge 2}$. *Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with the family Γ of polynomials. Let G be the polynomial semigroup generated by Γ. Let $\alpha, \beta, \rho \in \Gamma^{\mathbb{N}}$*

be three elements. Suppose that $G \in \mathcal{G}$, that G is semi-hyperbolic, that α is a periodic point of $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$, that $J_\beta(f) < J_\alpha(f)$, and that $J_\rho(f)$ is not a Jordan curve. Then, there exists an $n \in \mathbb{N}$ such that setting $\omega := (\alpha_1, \ldots, \alpha_n, \rho_1, \rho_2, \ldots) \in \Gamma^{\mathbb{N}}$ and $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists \{m_j\}_{j\in\mathbb{N}}, \exists \{n_k\}_{k\in\mathbb{N}}, \sigma^{m_j}(\gamma) \to \alpha,\ \sigma^{n_k}(\gamma) \to \omega\}$, we have that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain.

Proof. Let $p \in \mathbb{N}$ be a number such that $\sigma^p(\alpha) = \alpha$ and let $u := \alpha_p \circ \cdots \circ \alpha_1$. We show the following claim.
Claim 1: $J(u)$ is a quasicircle.

To show this claim, by assumption, we have $J_\beta(f) < J(u)$. Let $\zeta := (\alpha_1, \ldots, \alpha_p, \beta_1, \beta_2, \ldots) \in \Gamma^{\mathbb{N}}$. Then, we have $J_\zeta(f) = u^{-1}(J_\beta(f))$. Moreover, since $G \in \mathcal{G}$, we have that $J_\zeta(f)$ is connected. Hence, it follows that $u^{-1}(J_\beta(f))$ is connected. Let U be a connected component of $\text{int}(K(u))$ containing $J_\beta(f)$ and V a connected component of $\text{int}(K(u))$ containing $u^{-1}(J_\beta(f))$. By Lemma 3.9, it must hold that $U = V$. Therefore, we obtain $u^{-1}(U) = U$. Thus, $\text{int}(K(u)) = U$. Since G is semi-hyperbolic, it follows that $J(u)$ is a quasicircle. Hence, we have proved Claim 1.

Let $\mu := \frac{1}{3}\min\{|b - c| \mid b \in J_\alpha(f), c \in P^*(G)\}$. Since $J_\beta(f) < J_\alpha(f)$, we have $P^*(G) \subset K_\beta(f)$. Hence, $\mu > 0$. Applying Lemma 4.31 to the above (f, μ), let δ be the number in the statement of Lemma 4.31. We set $\epsilon := \min\{\delta, \mu\}(> 0)$. Applying Lemma 4.33 to the above $(\Gamma, \alpha, \rho, \epsilon)$, let $(n, \theta_1, \theta_2, \omega)$ be the element in the statement of Lemma 4.33. We set $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists \{m_j\}_{j\in\mathbb{N}}, \exists \{n_k\}_{k\in\mathbb{N}}, \sigma^{m_j}(\gamma) \to \alpha, \sigma^{n_k}(\gamma) \to \omega\}$. Then, combining the statement Lemma 4.31 and that of Lemma 4.33, it follows that for any $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is not a quasicircle. Moreover, by Lemma 4.32, we see that for any $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve. Furthermore, combining the above argument, [35, Theorem 1.12], Lemma 3.6, and [21, Theorem 9.3], we see that for any $\gamma \in \mathcal{U}$, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain. Therefore, we have proved Lemma 4.34. □

We now demonstrate Theorem 2.52.
Proof of Theorem 2.52: We suppose the assumption of Theorem 2.52. We will consider several cases. First, we show the following claim.
Claim 1: If $J_\gamma(f)$ is a Jordan curve for each $\gamma \in \Gamma^{\mathbb{N}}$, then statement 1 in Theorem 2.52 holds.

To show this claim, Lemma 4.30 implies that for each $\gamma \in X$, any critical point $v \in \pi^{-1}(\{\gamma\})$ of $f_\gamma : \pi^{-1}(\{\gamma\}) \to \pi^{-1}(\{\sigma(\gamma)\})$ (under the canonical identification $\pi^{-1}(\{\gamma\}) \cong \pi^{-1}(\{\sigma(\gamma)\}) \cong \hat{\mathbb{C}}$) belongs to $F^\gamma(f)$. Moreover, by [32, Theorem 2.14-(2)], $\tilde{J}(f) = \cup_{\gamma\in\Gamma^{\mathbb{N}}} J^\gamma(f)$. Hence, it follows that

$C(f) \subset \tilde{F}(f)$. Therefore, $C(f)$ is a compact subset of $\tilde{F}(f)$. Since f is semi-hyperbolic, [32, Theorem 2.14-(5)] implies that $P(f) = \overline{\bigcup_{n\in\mathbb{N}} f^n(C(f))} \subset \tilde{F}(f)$. Hence, $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is hyperbolic. Combining it with Remark 4.16, we conclude that G is hyperbolic. Moreover, Theorem 4.21 implies that there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle. Hence, we have proved Claim 1.

Next, we will show the following claim.

Claim 2: If $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$ for each $(\alpha, \beta) \in \Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$, then $J(G)$ is arcwise connected.

To show this claim, since G is semi-hyperbolic, combining [35, Theorem 1.12], Lemma 3.6, and [21, page 26], we get that for each $\gamma \in \Gamma^{\mathbb{N}}$, $A_\gamma(f)$ is a John domain and $J_\gamma(f)$ is locally connected. In particular, for each $\gamma \in \Gamma^{\mathbb{N}}$,

$$J_\gamma(f) \text{ is arcwise connected.} \tag{113}$$

Moreover, by [32, Theorem 2.14-(2)], we have

$$\tilde{J}(f) = \cup_{\gamma\in\Gamma^{\mathbb{N}}} J^\gamma(f). \tag{114}$$

Combining (113), (114) and Lemma 3.5-1, we conclude that $J(G)$ is arcwise connected. Hence, we have proved Claim 2.

Next, we will show the following claim.

Claim 3: If $J_\alpha(f) \cap J_\beta(f) \neq \emptyset$ for each $(\alpha, \beta) \in \Gamma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$, and if there exists an element $\rho \in \Gamma^{\mathbb{N}}$ such that $J_\rho(f)$ is not a Jordan curve, then statement 3 in Theorem 2.52 holds.

To show this claim, let $\mathcal{V} := \cup_{n\in\mathbb{N}} (\sigma^n)^{-1}(\{\rho\})$. Then, $\mathcal{V}$ is a dense subset of $\Gamma^{\mathbb{N}}$. From Lemma 4.30, it follows that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Combining this result with Claim 2, we conclude that statement 3 in Theorem 2.52 holds. Hence, we have proved Claim 3.

We now show the following claim.

Claim 4: If there exist two elements $\alpha, \beta \in \Gamma^{\mathbb{N}}$ such that $J_\alpha(f) \cap J_\beta(f) = \emptyset$, and if there exists an element $\rho \in \Gamma^{\mathbb{N}}$ such that $J_\rho(f)$ is not a Jordan curve, then statement 2 in Theorem 2.52 holds.

To show this claim, using Lemma 3.9, We may assume that $J_\beta(f) < J_\alpha(f)$. Combining this, Lemma 3.9, [32, Theorem 2.14-(4)], and that the set of all periodic points of σ in $\Gamma^{\mathbb{N}}$ is dense in $\Gamma^{\mathbb{N}}$, we may assume further that α is a periodic point of σ. Applying Lemma 4.34 to $(\Gamma, \alpha, \beta, \rho)$ above, let $n \in \mathbb{N}$ be the element in the statement of Lemma 4.34, and we set $\omega = (\alpha_1, \ldots, \alpha_n, \rho_1, \rho_2, \ldots) \in \Gamma^{\mathbb{N}}$ and $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists (m_j), \exists (n_k), \sigma^{m_j}(\gamma) \to \alpha, \sigma^{n_k}(\gamma) \to \omega\}$. Then, by the statement of Lemma 4.34, we have that for each $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve but not a quasicircle, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain. Moreover, $\mathcal{U}$

is residual in $\Gamma^{\mathbb{N}}$, and for any Borel probability measure τ on $\text{Poly}_{\deg\geq 2}$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$. Furthermore, let $\mathcal{V} := \cup_{n\in\mathbb{N}}(\sigma^n)^{-1}(\{\rho\})$. Then, $\mathcal{V}$ is a dense subset of $\Gamma^{\mathbb{N}}$, and the argument in the proof of Claim 3 implies that for each $\gamma \in \mathcal{V}$, $J_\gamma(f)$ is not a Jordan curve. Hence, we have proved Claim 4.

Combining Claims 1,2,3 and 4, Theorem 2.52 follows. $\square$

We now demonstrate Corollary 2.53.

Proof of Corollary 2.53: From Theorem 2.52, Corollary 2.53 immediately follows. $\square$

To demonstrate Theorem 2.54, we need several lemmas.

Notation: For a subset A of $\hat{\mathbb{C}}$, we denote by $\mathcal{C}(A)$ the set of all connected components of A.

Lemma 4.35. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\alpha \in X$ be a point. Suppose that $2 \leq \sharp\left(\mathcal{C}(\text{int}(K_\alpha(f)))\right) < \infty$. Then, $\sharp\left(\mathcal{C}(\text{int}(K_{g(\alpha)}(f)))\right) < \sharp\left(\mathcal{C}(\text{int}(K_\alpha(f)))\right)$. In particular, there exists an $n \in \mathbb{N}$ such that $\text{int}(K_{g^n(\alpha)}(f))$ is a non-empty connected set.*

Proof. Suppose that $2 \leq \sharp(\mathcal{C}(\text{int}(K_{g(\alpha)}(f)))) = \sharp(\mathcal{C}(\text{int}(K_\alpha(f)))) < \infty$. We will deduce a contradiction. Let $\{V_j\}_{j=1}^r = \mathcal{C}(\text{int}(K_{g(\alpha)}(f)))$, where $2 \leq r < \infty$. Then, by the assumption above, we have that $\mathcal{C}(\text{int}(K_{g(\alpha)}(f))) = \{f_{\alpha,1}(V_j)\}_{j=1}^r$. For each $j = 1, \ldots, r$, let p_j be the number of critical points of $f_{\alpha,1} : V_j \to f_{\alpha,1}(V_j)$ counting multiplicities. Then, by the Riemann-Hurwitz formula, we have that for each $j = 1, \ldots, r$, $\chi(V_j) + p_j = d\chi(f_{\alpha,1}(V_j))$, where $\chi(\cdot)$ denotes the Euler number and $d := \deg(f_{\alpha,1})$. Since $\chi(V_j) = \chi(f_{\alpha,1}(V_j)) = 1$ for each j, we obtain $r + \sum_{j=1}^r p_j = rd$. Since $\sum_{j=1}^r p_j \leq d-1$, it follows that $rd - r \leq d - 1$. Therefore, we obtain $r \leq 1$, which is a contradiction. Thus, we have proved Lemma 4.35. $\square$

Lemma 4.36. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$ such that for each $x \in X$, $d(x) \geq 2$. Let $\omega \in X$ be a point. Suppose that f is hyperbolic, that $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$, and that $\text{int}(K_\omega(f))$ is not connected. Then, there exist infinitely many connected components of $\text{int}(K_\omega(f))$.*

Proof. Suppose that $2 \leq \sharp(\mathcal{C}(\text{int}(K_\omega(f)))) < \infty$. Then, by Lemma 4.35, there exists an $n \in \mathbb{N}$ such that $\text{int}(K_{g^n(\omega)}(f))$ is connected. We set $U := \text{int}(K_{g^n(\omega)}(f))$. Let $\{V_j\}_{j=1}^r$ be the set of all connected components of $(f_{\omega,n})^{-1}(U)$. Since $\text{int}(K_\omega(f))$ is not connected, we have $r \geq 2$. For each $j = 1, \ldots, r$, we set $d_j := \deg(f_{\omega,n} : V_j \to U)$. Moreover, we denote by p_j the number of critical points of $f_{\omega,n} : V_j \to U$ counting multiplicities. Then, by the Riemann-Hurwitz formula, we see that for each $j = 1, \ldots, r$, $\chi(V_j) + p_j = d_j \chi(U)$. Since $\chi(V_j) = \chi(U) = 1$ for each $j = 1, \ldots, r$, it follows that

$$r + \sum_{j=1}^{r} p_j = d, \tag{115}$$

where $d := \deg(f_{\omega,n})$. Since f is hyperbolic and $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$, we have $\sum_{j=1}^{r} p_j = d - 1$. Combining it with (115), we obtain $r = 1$, which is a contradiction. Hence, we have proved Lemma 4.36. □

Lemma 4.37. *Let $f : X \times \hat{\mathbb{C}} \to X \times \hat{\mathbb{C}}$ be a polynomial skew product over $g : X \to X$. Let $\alpha \in X$ be an element. Suppose that $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$, that f is hyperbolic, and that $\mathrm{int}(K_\alpha(f)))$ is connected. Then, there exists a neighborhood $\mathcal{U}_0$ of α in X satisfying the following.*

- *Let $\gamma \in X$ and suppose that there exists a sequence $\{m_j\}_{j \in \mathbb{N}} \subset \mathbb{N}, m_j \to \infty$ such that for each $j \in \mathbb{N}$, $g^{m_j}(\gamma) \in \mathcal{U}_0$. Then, $J_\gamma(f)$ is a Jordan curve.*

Proof. Let $P^*(f) := P(f) \setminus \pi_{\hat{\mathbb{C}}}^{-1}(\{\infty\})$. By assumption, we have $\pi_{\hat{\mathbb{C}}}(P^*(f) \cap \pi^{-1}(\{\alpha\})) \subset \mathrm{int}(K_\alpha(f))$. Since $\mathrm{int}(K_\alpha(f))$ is simply connected, there exists a Jordan curve ξ in $\mathrm{int}(K_\alpha(f))$ such that $\pi_{\hat{\mathbb{C}}}(P^*(f) \cap \pi^{-1}(\{\alpha\}))$ is included in the bounded component B of $\mathbb{C} \setminus \xi$. Since f is hyperbolic, [32, Theorem 2.14-(4)] implies that the map $x \mapsto J_x(f)$ is continuous with respect to the Hausdorff topology. Hence, there exists a neighborhood $\mathcal{U}_0$ of α in X such that for each $\beta \in \mathcal{U}_0$, $J_\beta(f) \cap (\xi \cup B) = \emptyset$. Moreover, since $P(f)$ is compact, shrinking $\mathcal{U}_0$ if necessary, we may assume that for each $\beta \in \mathcal{U}_0$, $\pi_{\hat{\mathbb{C}}}(P^*(f) \cap \pi^{-1}(\{\beta\})) \subset B$. Since $\pi_{\hat{\mathbb{C}}}(P(f)) \cap \mathbb{C}$ is bounded in $\mathbb{C}$, it follows that for each $\beta \in \mathcal{U}_0$, $\xi < J_\beta(f)$. Hence, for each $\beta \in \mathcal{U}_0$, there exists a connected component V_β of $\mathrm{int}(K_\beta(f))$ such that

$$\pi_{\hat{\mathbb{C}}}(P^*(f) \cap \pi^{-1}(\{\beta\})) \subset V_\beta. \tag{116}$$

Let $\gamma \in X$ be an element and suppose that there exists a sequence $\{m_j\}_{j \in \mathbb{N}} \subset \mathbb{N}, m_j \to \infty$ such that for each $j \in \mathbb{N}$, $g^{m_j}(\gamma) \in \mathcal{U}_0$. We will show that $\mathrm{int}(K_\gamma(f))$ is connected. Suppose that there exist two distinct connected components W_1 and W_2 of $\mathrm{int}(K_\gamma(f))$. Then, combining [35, Corollary 2.7] and (116), we get that there exists a $j \in \mathbb{N}$ such that

$$\pi_{\hat{\mathbb{C}}}(P^*(f) \cap \pi^{-1}(\{\beta\})) \subset f_{\gamma,m_j}(W_1) = f_{\gamma,m_j}(W_2). \tag{117}$$

We set $W = f_{\gamma,m_j}(W_1) = f_{\gamma,m_j}(W_2)$. Let $\{V_i\}_{i=1}^r$ be the set of all connected components of $(f_{\gamma,m_j})^{-1}(W)$. Since $W_1 \neq W_2$, we have $r \geq 2$. For each $i = 1, \ldots, r$, we denote by p_i the number of critical points of $f_{\gamma,m_j} : V_i \to W$ counting multiplicities. Moreover, we set $d_i := \deg(f_{\gamma,m_j} : V_i \to W)$. Then, by the Riemann-Hurwitz formula, we see that for each $i = 1, \ldots, r$, $\chi(V_i) + p_i = d_i \chi(W)$. Since $\chi(V_i) = \chi(W) = 1$, it follows that

$$r + \sum_{i=1}^{r} p_i = d, \text{ where } d := \deg(f_{\gamma,m_j}). \tag{118}$$

By (117), we have $\sum_{i=1}^{r} p_i = d-1$. Hence, (118) implies $r = 1$, which is a contradiction. Therefore, $\text{int}(K_\gamma(f))$ is a non-empty connected set. Combining it with Proposition 4.26, we conclude that $J_\gamma(f)$ is a Jordan curve.

Thus, we have proved Lemma 4.37. □

We now demonstrate Theorem 2.54.

Proof of Theorem 2.54: We suppose the assumption of Theorem 2.54. We consider the following three cases.
Case 1: For each $\gamma \in \Gamma^{\mathbb{N}}$, $\text{int}(K_\gamma(f))$ is connected.
Case 2: For each $\gamma \in \Gamma^{\mathbb{N}}$, $\text{int}(K_\gamma(f))$ is disconnected.
Case 3: There exist two elements $\alpha \in \Gamma^{\mathbb{N}}$ and $\beta \in \Gamma^{\mathbb{N}}$ such that $\text{int}(K_\alpha(f))$ is connected and such that $\text{int}(K_\beta(f))$ is disconnected.

Suppose that we have Case 1. Then, by Theorem 4.21, there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.

Suppose that we have Case 2. Then, by Lemma 4.36, we get that for each $\gamma \in \Gamma^{\mathbb{N}}$, there exist infinitely many connected components of $\text{int}(K_\gamma(f))$. Moreover, by Theorem 2.52, we see that statement 3 in Theorem 2.52 holds. Hence, statement 3 in Theorem 2.54 holds.

Suppose that we have Case 3. By Lemma 4.36, there exist infinitely many connected components of $\text{int}(K_\beta(f))$. Let $\mathcal{W} := \cup_{n\in\mathbb{N}}(\sigma^n)^{-1}(\{\beta\})$. Then, for each $\gamma \in \mathcal{W}$, there exist infinitely many connected components of $\text{int}(K_\gamma(f))$. Moreover, $\mathcal{W}$ is dense in $\Gamma^{\mathbb{N}}$.

Next, combining Lemma 4.37 and that the set of all periodic points of $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is dense in $\Gamma^{\mathbb{N}}$, we may assume that the above α is a periodic point of σ. Then, $J_\alpha(f)$ is a quasicircle. We set $\mathcal{V} := \cup_{n\in\mathbb{N}}(\sigma^n)^{-1}(\{\alpha\})$. Then $\mathcal{V}$ is dense in $\Gamma^{\mathbb{N}}$. Let $\gamma \in \mathcal{V}$ be an element. Then there exists an $n \in \mathbb{N}$ such that $\sigma^n(\gamma) = \alpha$. Since $(f_{\gamma,n})^{-1}(K_\alpha(f)) = K_\gamma(f)$, it follows that $\sharp(\mathcal{C}(\text{int}(K_\gamma(f)))) < \infty$. Combining it with Lemma 4.36 and Proposition 4.26, we get that $J_\gamma(f)$ is a Jordan curve. Combining it with that $J_\alpha(f)$ is a quasicircle, it follows that $J_\gamma(f)$ is a quasicircle.

Next, let $\mu := \frac{1}{3}\min\{|b - c| \mid b \in J(G),\ c \in P^*(G)\}(> 0)$. Applying Lemma 4.31 to (f, μ) above, let δ be the number in the statement of

Lemma 4.31. We set $\epsilon := \min\{\delta, \mu\}$ and $\rho := \beta$. Applying Lemma 4.33 to $(\Gamma, \alpha, \rho, \epsilon)$ above, let $(n, \theta_1, \theta_2, \omega)$ be the element in the statement of Lemma 4.33. Let $\mathcal{U} := \{\gamma \in \Gamma^{\mathbb{N}} \mid \exists \{m_j\}_{j\in\mathbb{N}}, \exists \{n_k\}_{k\in\mathbb{N}}, \sigma^{m_j}(\gamma) \to \alpha, \sigma^{n_k}(\gamma) \to \omega\}$. Then, combining the statement of Lemma 4.31 and that of Lemma 4.33, it follows that for any $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is not a quasicircle. Moreover, by Lemma 4.37, we get that for any $\gamma \in \mathcal{U}$, $J_\gamma(f)$ is a Jordan curve. Combining the above argument, [35, Theorem 1.12], Lemma 3.6, and [21, Theorem 9.3], we see that for any $\gamma \in \mathcal{U}$, $A_\gamma(f)$ is a John domain, and the bounded component U_γ of $F_\gamma(f)$ is not a John domain. Furthermore, it is easy to see that $\mathcal{U}$ is residual in $\Gamma^{\mathbb{N}}$, and that for any Borel probability measure τ on $\text{Poly}_{\deg\geq 2}$ with $\Gamma_\tau = \Gamma$, $\tilde{\tau}(\mathcal{U}) = 1$. Thus, we have proved Theorem 2.54. □

Remark 4.38. Using the above method (especially, using Lemma 4.28, Lemma 4.31 and Lemma 4.37), we can also construct an example of a polynomial skew product $f : \mathbb{C}^2 \to \mathbb{C}^2, f(x, y) = (p(x), q_x(y))$, where $p : \mathbb{C} \to \mathbb{C}$ is a polynomial with $\deg(p) \geq 2$, $q_x : \mathbb{C} \to \mathbb{C}$ is a monic polynomial with $\deg(q_x) \geq 2$ for each $x \in \mathbb{C}$, and $(x, y) \to q_x(y)$ is a polynomial of (x, y), such that all of the following hold:

- f satisfies the Axiom A; and
- for almost every $x \in J(p)$ with respect to the maximal entropy measure of $p : \mathbb{C} \to \mathbb{C}$, the fiberwise Julia set $J_x(f)$ is a Jordan curve but not a quasicircle, the fiberwise basin $A_x(f)$ of ∞ is a John domain, and the bounded component of $F_x(f)$ is not a John domain.

For the related topics of Axiom A polynomial skew products on $\mathbb{C}^2$, see [8].

We now demonstrate Proposition 2.57.

Proof of Proposition 2.57: Since $P^*(G) \subset \text{int}(\hat{K}(G)) \subset F(G)$, G is hyperbolic. Let $\gamma \in \Gamma^{\mathbb{N}}$ be any element. We will show the following claim.

Claim: $\text{int}(K_\gamma(f))$ is a non-empty connected set.

To show this claim, since G is hyperbolic, $\text{int}(K_\gamma(f))$ is non-empty. Suppose that there exist two distinct connected components W_1 and W_2 of $\text{int}(K_\gamma(f))$. Since $P^*(G)$ is included in a connected component U of $\text{int}(\hat{K}(G)) \subset F(G)$, [35, Corollary 2.7] implies that there exists an $n \in \mathbb{N}$ such that $P^*(G) \subset f_{\gamma,n}(W_1) = f_{\gamma,n}(W_2)$. Let $W := f_{\gamma,n}(W_1) = f_{\gamma,n}(W_2)$. Then, any critical value of $f_{\gamma,n}$ in $\mathbb{C}$ is included in W. Using the method in the proof of Lemma 4.37, we see that $(f_{\gamma,n})^{-1}(W)$ is connected. However, this is a contradiction, since $W_1 \neq W_2$. Hence, we have proved the above claim.

From Claim above and Theorem 4.21, it follows that there exists a constant $K \geq 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma(f)$ is a K-quasicircle.

Hence, we have proved Proposition 2.57. □

4.8 Proofs of results in 2.8

We now demonstrate Proposition 2.58.

Proof of Proposition 2.58: Conjugating G by $z \mapsto z+b$, we may assume that $b=0$. For each $h \in \Gamma$, we set $a_h := a(h)$ and $d_h := \deg(h)$. Let $r>0$ be a number such that $\overline{D(0,r)} \subset \text{int}(\hat{K}(G))$.

Let $h \in \Gamma$ and let $\alpha > 0$ be a number. Since $d \geq 2$ and $(d, d_h) \neq (2,2)$, it is easy to see that $(\frac{r}{\alpha})^{\frac{1}{d}} > 2\left(\frac{2}{|a_h|}(\frac{1}{\alpha})^{\frac{1}{d-1}}\right)^{\frac{1}{d_h}}$ if and only if

$$\log \alpha < \frac{d(d-1)d_h}{d+d_h-d_h d}(\log 2 - \frac{1}{d_h}\log\frac{|a_h|}{2} - \frac{1}{d}\log r). \tag{119}$$

We set

$$c_0 := \min_{h\in\Gamma} \exp\left(\frac{d(d-1)d_h}{d+d_h-d_h d}(\log 2 - \frac{1}{d_h}\log\frac{|a_h|}{2} - \frac{1}{d}\log r)\right) \in (0,\infty). \tag{120}$$

Let $0<c<c_0$ be a small number and let $a \in \mathbb{C}$ be a number with $0<|a|<c$. Let $g_a(z) = az^d$. Then, we obtain $K(g_a) = \{z \in \mathbb{C} \mid |z| \leq (\frac{1}{|a|})^{\frac{1}{d-1}}\}$ and $g_a^{-1}(\{z \in \mathbb{C} \mid |z| = r\}) = \{z \in \mathbb{C} \mid |z| = (\frac{r}{|a|})^{\frac{1}{d}}\}$. Let $D_a := \overline{D(0, 2(\frac{1}{|a|})^{\frac{1}{d-1}})}$. Since $h(z) = a_h z^{d_h}(1+o(1))$ $(z \to \infty)$ uniformly on Γ, it follows that if c is small enough, then for any $a \in \mathbb{C}$ with $0<|a|<c$ and for any $h \in \Gamma$, $h^{-1}(D_a) \subset \left\{z \in \mathbb{C} \mid |z| \leq 2\left(\frac{2}{|a_h|}(\frac{1}{|a|})^{d-1}\right)^{\frac{1}{d_h}}\right\}$. This implies that for each $h \in \Gamma$,

$$h^{-1}(D_a) \subset g_a^{-1}(\{z \in \mathbb{C} \mid |z| < r\}). \tag{121}$$

Moreover, if c is small enough, then for any $a \in \mathbb{C}$ with $0<|a|<c$ and any $h \in \Gamma$,

$$\hat{K}(G) \subset g_a^{-1}(\{z \in \mathbb{C} \mid |z| < r\}),\ \overline{h(\hat{\mathbb{C}} \setminus D_a)} \subset \hat{\mathbb{C}} \setminus D_a. \tag{122}$$

Let $a \in \mathbb{C}$ with $0<|a|<c$. By (121) and (122), there exists a compact neighborhood V of g_a in $\text{Poly}_{\deg\geq 2}$, such that

$$\hat{K}(G) \cup \bigcup_{h\in\Gamma} h^{-1}(D_a) \subset \text{int}\left(\cap_{g\in V} g^{-1}(\{z \in \mathbb{C} \mid |z| < r\})\right), \text{ and} \tag{123}$$

$$\bigcup_{h\in\Gamma\cup V} \overline{h(\hat{\mathbb{C}} \setminus D_a)} \subset \hat{\mathbb{C}} \setminus D_a, \tag{124}$$

which implies that

$$\text{int}(\hat{K}(G)) \cup (\hat{\mathbb{C}} \setminus D_a) \subset F(H_{\Gamma,V}), \tag{125}$$

where $H_{\Gamma,V}$ denotes the polynomial semigroup generated by the family $\Gamma \cup V$.

By (123), we obtain that for any non-empty subset V' of V,

$$\hat{K}(G) = \hat{K}(H_{\Gamma,V'}), \tag{126}$$

where $H_{\Gamma,V'}$ denotes the polynomial semigroup generated by the family $\Gamma \cup V'$. If the compact neighborhood V of g_a is so small, then

$$\bigcup_{g \in V} CV^*(g) \subset \text{int}(\hat{K}(G)). \tag{127}$$

Since $P^*(G) \subset \hat{K}(G)$, combining it with (126) and (127), we get that for any non-empty subset V' of V, $P^*(H_{\Gamma,V'}) \subset \hat{K}(H_{\Gamma,V'})$. Therefore, for any non-empty subset V' of V, $H_{\Gamma,V'} \in \mathcal{G}$.

We now show that for any non-empty subset V' of V, $J(H_{\Gamma,V'})$ is disconnected and $(\Gamma \cup V')_{\min} \subset \Gamma$. Let

$$U := \left(\text{int}(\bigcap_{g \in V} g^{-1}(\{z \in \mathbb{C} \mid |z| < r\}))\right) \setminus \bigcup_{h \in \Gamma} h^{-1}(D_a).$$

Then, for any $h \in \Gamma$,

$$h(U) \subset \hat{\mathbb{C}} \setminus D_a. \tag{128}$$

Moreover, for any $g \in V$, $g(U) \subset \text{int}(\hat{K}(G))$. Combining it with (125), (128), and Lemma 3.1-2, it follows that $U \subset F(H_{\Gamma,V})$. If the neighborhood V of g_a is so small, then there exists an annulus A in U such that for any $g \in V$, A separates $J(g)$ and $\cup_{h \in \Gamma} h^{-1}(J(g))$. Hence, it follows that for any non-empty subset V' of V, the polynomial semigroup $H_{\Gamma,V'}$ generated by the family $\Gamma \cup V'$ satisfies that $J(H_{\Gamma,V'})$ is disconnected and $(\Gamma \cup V')_{\min} \subset \Gamma$.

We now suppose that in addition to the assumption, G is semi-hyperbolic. Let V' be any non-empty subset of V. Since $(\Gamma \cup \overline{V'})_{\min} \subset \Gamma$, Theorem 2.41 implies that the above $H_{\Gamma,V'}$ is semi-hyperbolic.

We now suppose that in addition to the assumption, G is hyperbolic. Let V' be any non-empty subset of V. By (126) and (127), we have

$$\bigcup_{g \in \Gamma \cup \overline{V'}} CV^*(g) \subset \text{int}(\hat{K}(H_{\Gamma,\overline{V'}})). \tag{129}$$

Since $(\Gamma \cup \overline{V'})_{\min} \subset \Gamma$, combining it with (129) and Theorem 2.42, we obtain that $H_{\Gamma,V'}$ is hyperbolic.

Thus, we have proved Proposition 2.58. $\square$

We now demonstrate Theorem 2.61.

Proof of Theorem 2.61: First, we show 1. Let $r > 0$ be a number such that $D(b_j, 2r) \subset \text{int}(K(h_1))$ for each $j = 1, \dots, m$. If we take $c > 0$ so small, then for each $(a_2, \dots, a_m) \in \mathbb{C}^{m-1}$ such that $0 < |a_j| < c$ for each $j = 2, \dots, m$, setting $h_j(z) = a_j(z - b_j)^{d_j} + b_j$ $(j = 2, \dots, m)$, we have

$$h_j(K(h_1)) \subset D(b_j, r) \subset \text{int}(K(h_1)) \ (j = 2, \dots, m). \tag{130}$$

Hence, $K(h_1) = \hat{K}(G)$, where $G = \langle h_1, \dots, h_m \rangle$. Moreover, by (130), we have $P^*(G) \subset K(h_1)$. Hence, $G \in \mathcal{G}$.

If $\langle h_1 \rangle$ is semi-hyperbolic, then using the same method as that of Case 1 in the proof of Theorem 2.41, we obtain that G is semi-hyperbolic.

We now suppose that $\langle h_1 \rangle$ is hyperbolic. By (130), we have $\cup_{j=2}^m CV^*(h_j) \subset \text{int}(\hat{K}(G))$. Combining it with the same method as that in the proof of Theorem 2.42, we obtain that G is hyperbolic. Hence, we have proved statement 1.

We now show statement 2. Suppose we have case (i). We may assume $d_m \geq 3$. Then, by statement 1, there exists an element $a > 0$ such that setting $h_j(z) = a(z - b_j)^{d_j} + b_j$ $(j = 2, \dots, m-1)$, $G_0 = \langle h_1, \dots, h_{m-1} \rangle$ satisfies that $G_0 \in \mathcal{G}$ and $\hat{K}(G_0) = K(h_1)$ and if $\langle h_1 \rangle$ is semi-hyperbolic (resp. hyperbolic), then G_0 is semi-hyperbolic (resp. hyperbolic). Combining it with Proposition 2.58, it follows that there exists an $a_m > 0$ such that setting $h_m(z) = a_m(z - b_m)^{d_m} + b_m$, $G = \langle h_1, \dots, h_m \rangle$ satisfies that $G \in \mathcal{G}_{dis}$ and $\hat{K}(G) = \hat{K}(G_0) = K(h_1)$ and if G_0 is semi-hyperbolic (resp. hyperbolic), then G is semi-hyperbolic (resp. hyperbolic).

Suppose now we have case (ii). Then by Proposition 2.58, there exists an $a_2 > 0$ such that setting $h_j(z) = a_2(z - b_j)^2 + b_j$ $(j = 2, \dots, m)$, $G = \langle h_1, \dots, h_m \rangle = \langle h_1, h_2 \rangle$ satisfies that $G \in \mathcal{G}_{dis}$ and $\hat{K}(G) = K(h_1)$ and if $\langle h_1 \rangle$ is semi-hyperbolic (resp. hyperbolic), then G is semi-hyperbolic (resp. hyperbolic).

Thus, we have proved Theorem 2.61. □

We now demonstrate Theorem 2.63.

Proof of Theorem 2.63: Statements 2 and 3 follow from Theorem 2.61.

We now show statement 1. By [36, Theorem 2.4.1], $\mathcal{H}_m$ and $\mathcal{H}_m \cap \mathcal{D}_m$ are open.

We now show that $\mathcal{H}_m \cap \mathcal{B}_m$ is open. In order to do that, let $(h_1, \dots, h_m) \in \mathcal{H}_m \cap \mathcal{B}_m$. Let $\epsilon > 0$ such that $D(P^*(\langle h_1, \dots, h_m \rangle),\ 3\epsilon) \subset F(\langle h_1, \dots, h_m \rangle)$. By [32, Theorem 1.35], there exists an $n \in \mathbb{N}$ such that for each $(i_1, \dots, i_n) \in \{1, \dots, m\}^n$,

$$h_{i_n} \cdots h_{i_1}(D(P^*(\langle h_1, \dots, h_m \rangle),\ 2\epsilon)) \subset D(P^*(\langle h_1, \dots, h_m \rangle),\ \epsilon/2).$$

Hence, there exists a neighborhood U of $(h_1, \ldots, h_m)$ in $(\text{Poly}_{\deg \geq 2})^m$ such that for each $(g_1, \ldots, g_m) \in U$ and each $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$,

$$g_{i_n} \cdots g_{i_1}(D(P^*(\langle h_1, \ldots, h_m \rangle),\ 2\epsilon)) \subset D(P^*(\langle h_1, \ldots, h_m \rangle),\ \epsilon).$$

If U is small, then for each $(g_1, \ldots, g_m) \in U$, $\cup_{j=1}^m CV^*(g_j) \subset D(P^*(\langle h_1, \ldots, h_m \rangle),\ \epsilon)$. Hence, if U is small enough, then for each $(g_1, \ldots, g_m) \in U$, $P^*(\langle g_1, \ldots, g_m \rangle) \subset D(P^*(\langle h_1, \ldots, h_m \rangle), \epsilon)$. Hence, for each $(g_1, \ldots, g_m) \in U$, $\langle g_1, \ldots, g_m \rangle \in \mathcal{G}$. Therefore, $\mathcal{H}_m \cap \mathcal{B}_m$ is open.

Thus, we have proved Theorem 2.63. □

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