Negativity of Lyapunov Exponents and Convergence of Generic Random Polynomial Dynamical Systems and Random Relaxed Newton's Methods *

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Abstract

We investigate i.i.d. random complex dynamical systems generated by probability measures on finite unions of the loci of holomorphic families of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. We show that under certain conditions on the families, for a generic system, (especially, for a generic random polynomial dynamical system,) for all but countably many initial values $z \in \hat{\mathbb{C}}$, for almost every sequence of maps $\gamma = (\gamma_1, \gamma_2, \ldots)$, the Lyapunov exponent of γ at z is negative. Also, we show that for a generic system, for every initial value $z \in \hat{\mathbb{C}}$, the orbit of the Dirac measure at z under the iteration of the dual map of the transition operator tends to a periodic cycle of measures in the space of probability measures on $\hat{\mathbb{C}}$. Note that these are new phenomena in random complex dynamics which cannot hold in deterministic complex dynamical systems. We apply the above theory and results of random complex dynamical systems to finding roots of any polynomial by random relaxed Newton's methods and we show that for any polynomial g, for any initial value $z \in \mathbb{C}$ which is not a root of g', the random orbit starting with z tends to a root of g almost surely, which is the virtue of the effect of randomness.

1 Introduction and the main results

In this paper, we investigate the independent and identically-distributed (i.i.d.) random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of rational semigroups (i.e., semigroups of non-constant rational maps where the semigroup operation is functional composition) on $\hat{\mathbb{C}}$.

One motivation for research in (complex) dynamical systems is to describe some mathematical models in various fields to study nature and science. Since nature and any other environments have a lot of random terms, it is very natural and important not only to consider the dynamics of iteration, but also to consider random dynamics. Another motivation for research in complex dynamics is Newton's method to find roots of a complex polynomial, which often is expressed as the dynamics of a rational map g on $\hat{\mathbb{C}}$ with $\deg(g) \geq 2$, where $\deg(g)$ denotes the degree of g. In various fields, we have many mathematical models which are described by the dynamical systems associated with polynomial or rational maps. For each model, it is natural and important to consider a randomized model, since we always have some kind of noise or random terms. Regarding random (complex) dynamics, many researchers in various fields (mathematics, physics, chemistry, etc.) have found and investigated many kinds of new phenomena in random (complex) dynamics which cannot hold in deterministic dynamics. These phenomena arise from the effect of randomness

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and they are called **randomness-induced phenomena** or **noise-induced phenomena** ([20]). In fact, recently these topics are getting more and more attention in many fields.

The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([10]). For research on random complex dynamics of quadratic polynomials, see [3]–[7], [11]. For recent research on random complex dynamics and the various randomness-induced phenomena, see the author's works [34]–[39].

In order to investigate random complex dynamics, it is very natural to study the dynamics of associated rational semigroups. In fact, it is a very powerful tool to investigate random complex dynamics, since random complex dynamics and the dynamics of rational semigroups are related to each other very deeply. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([13]), who were interested in the role of the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([12]), who studied such semigroups from the perspective of random dynamical systems. For recent work on the dynamics of rational semigroups, see the author's papers [30]–[39], and [28, 40, 41].

To introduce the main idea of this paper, we let G be a rational semigroup and denote by $F(G)$ the **Fatou set of** G, which is defined to be the maximal open subset of $\hat{\mathbb{C}}$ where G is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$. We call $J(G) := \hat{\mathbb{C}} \setminus F(G)$ the **Julia set of** G. The Julia set is backward invariant under each element $h \in G$, but might not be forward invariant. This is a difficulty of the theory of rational semigroups. Nevertheless, we utilize this as follows. The key to investigating random complex dynamics is to consider the following **kernel Julia set of** G, which is defined by $J_{\ker}(G) = \bigcap_{g \in G} g^{-1}(J(G))$. This is the largest forward invariant subset of $J(G)$ under the action of G. Note that if G is a group or if G is a commutative semigroup, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup G generated by a family of rational maps h with $\deg(h) \geq 2$, it may happen that $\emptyset = J_{\ker}(G) \neq J(G)$.

Let **Rat** be the space of all non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$, endowed with the distance κ which is defined by $\kappa(f,g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Let $\mathbf{Rat}_+$ be the space of all rational maps g with $\deg(g) \geq 2$. Let $\mathcal{P}$ be the space of all polynomial maps g with $\deg(g) \geq 2$. Let τ be a Borel probability measure on Rat with compact support. We consider the **i.i.d. random dynamics** on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \mathrm{Rat}$ according to τ. Thus this determines a Markov process on the state space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset A of $\hat{\mathbb{C}}$, the **transition probability** $p(x, A)$ from x to A is defined as $p(x, A) = \tau(\{g \in \mathrm{Rat} \mid g(x) \in A\})$. Let G_τ be the rational semigroup generated by the support of τ, i.e., $G_\tau = \{h_1 \circ \cdots \circ h_n \mid n \in \mathbb{N}, h_j \in \operatorname{supp} \tau \text{ for all } j\}$. Let $C(\hat{\mathbb{C}})$ be the space of all complex-valued continuous functions on $\hat{\mathbb{C}}$ endowed with the supremum norm $\|\cdot\|_\infty$. Let M_τ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) = \int \varphi(g(z)) d\tau(g)$. This M_τ is called the **transition operator** of the Markov process induced by τ. For a metric space X, let $\mathfrak{M}_1(X)$ be the space of all Borel probability measures on X endowed with the topology induced by weak convergence (thus $\mu_n \to \mu$ in $\mathfrak{M}_1(X)$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{R}$). Note that if X is a compact metric space, then $\mathfrak{M}_1(X)$ is compact and metrizable. For each $\tau \in \mathfrak{M}_1(X)$, we denote by $\operatorname{supp} \tau$ the topological support of τ. Let $\mathfrak{M}_{1,c}(X)$ be the space of all Borel probability measures τ on X such that $\operatorname{supp} \tau$ is compact. Let $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of M_τ. This M_τ^* can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$ (see Remark 3.9). We define the "pointwise Fatou set" $F_{pt}^0(\tau)$ of the dynamics of M_τ^* as the set of all elements $y \in \hat{\mathbb{C}}$ satisfying that $\{(M_\tau^*)^n \circ \Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous at the one point $y \in \hat{\mathbb{C}}$, where $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$ is the embedding map defined by $\Phi(y) = \delta_y$ (see Definition 3.10). Also, we set $J_{pt}^0(\tau) := \hat{\mathbb{C}} \setminus F_{pt}^0(\tau)$. Moreover, $J_{\ker}(G_\tau)$ is called the **kernel Julia set of** τ.

For each sequence $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$, and for each $n, m \in \mathbb{N}$ with $n \geq m$, we set $\gamma_{n,m} = \gamma_n \circ \cdots \circ \gamma_m$ and we denote by F_γ the set of points $z \in \hat{\mathbb{C}}$ satisfying that there exists an open neighborhood of z on which the sequence $\{\gamma_{n,1}\}_{n=1}^\infty$ is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$. This F_γ is called the **Fatou set of the sequence** γ. Also, we set $J_\gamma := \hat{\mathbb{C}} \setminus F_\gamma$ and this J_γ is called the **Julia set of** γ. Let $\tilde{\tau} := \otimes_{n=1}^{\infty} \tau \in \mathfrak{M}_1((\mathrm{Rat})^{\mathbb{N}})$. For a metric space X, we denote by $\mathrm{Cpt}(X)$ the space of all non-empty compact subsets of X endowed with the Hausdorff metric. For a rational semigroup G, we say that a non-empty compact subset L of $\hat{\mathbb{C}}$ is a **minimal set for** $(G, \hat{\mathbb{C}})$ if $L = \overline{\cup_{h \in G}\{h(z)\}}$. Moreover, we denote by $\mathrm{Min}(G, \hat{\mathbb{C}})$ the sets of all minimal sets for $(G, \hat{\mathbb{C}})$. For any $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, for any $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and for any $z \in \hat{\mathbb{C}}$, we set $T_{L,\tau}(z) = \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}} \mid d(\gamma_{n,1}(z), L) \to 0 \text{ as } n \to \infty\})$. If $L = \{x\}$, then we set $T_{L,\tau} = T_{x,\tau}$. For a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, let $\Gamma_\tau := \operatorname{supp} \tau (\subset \mathrm{Rat})$. Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. We say that a minimal set $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is **attracting for** τ if there exist two open subsets A, B of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus A) \geq 3$ and an $n \in \mathbb{N}$ such that $\overline{B} \subset A$ and such that for each $(\gamma_1, \ldots, \gamma_n) \in \Gamma_\tau^n$, we have $\gamma_n \circ \cdots \circ \gamma_1(A) \subset B$. In this case, we say that L is an **attracting minimal set of** τ. Let $\mathcal{Y}$ be a subset of Rat endowed with the relative topology from Rat. We say that $\mathcal{Y}$ is **mild** if for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$, there exists an attracting minimal set for τ. For example, any non-empty open subset of $\mathcal{P}$ is a mild subset of Rat.

Let $\mathcal{Y}$ be a closed subset of an open subset of Rat, i.e., there exist an open subset $\mathcal{V}$ of Rat and a closed subset $\mathcal{C}$ of Rat such that $\mathcal{Y} = \mathcal{V} \cap \mathcal{C}$. Let $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ be a **holomorphic family of rational maps** (see Definition 3.38) such that Λ is a connected complex manifold and $\lambda \mapsto f_\lambda \in \mathrm{Rat}$ is not constant. We say that $\mathcal{Y}$ is **weakly nice** with respect to $\mathcal{W}$ if $\mathcal{Y} = \{f_\lambda \in \mathrm{Rat} \mid \lambda \in \Lambda\}$ (for more general definition, see 3.41). In this case, for each $n \in \mathbb{N}$, we denote by $S_n(\mathcal{W})$ the set of points $z \in \hat{\mathbb{C}}$ satisfying that $(\lambda_1, \ldots, \lambda_n) \in \Lambda^n \mapsto f_{\lambda_1} \circ \cdots \circ f_{\lambda_n}(z)$ is constant on Λ^n. Also, we set $S(\mathcal{W}) = \cap_{n=1}^{\infty} S_n(\mathcal{W})$. This $S(\mathcal{W})$ is called the **singular set** of $\mathcal{W}$. Note that $\sharp S_1(\mathcal{W}) < \infty$ and $\sharp S(\mathcal{W}) < \infty$. We say that $\mathcal{Y}$ is **nice** with respect to $\mathcal{W}$ if $\mathcal{Y}$ is weakly nice with respect to $\mathcal{W}$ and for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset S(\mathcal{W})$ and for each $z \in L$, either (a) the map $\lambda \mapsto D(f_\lambda)_z$ is non-constant on Λ or (b) $D(f_\lambda)_z = 0$ for all $\lambda \in \Lambda$.

For any closed subset $\mathcal{Y}$ of an open subset of Rat, let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that the sequence $\{\tau_n\}_{n=1}^{\infty}$ in $\mathfrak{M}_{1,c}(\mathcal{Y})$ tends to an element $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ with respect to the topology $\mathcal{O}$ if and only if (a) for each bounded continuous function $\varphi : \mathcal{Y} \to \mathbb{C}$, $\int \varphi \, d\tau_n \to \int \varphi \, d\tau_n$ as $n \to \infty$, and (b) $\Gamma_{\tau_n} \to \Gamma_\tau$ as $n \to \infty$ in $\mathrm{Cpt}(\mathcal{Y})$ with respect to the Hausdorff metric.

We now present the first main result of this paper.

Theorem 1.1 (For the detailed and more general version, see Theorems 3.76, 3.65)**.** *Let $\mathcal{Y}$ be a mild subset of* Rat_+ *and suppose that $\mathcal{Y}$ is nice with respect to a holomorphic family $\mathcal{W}$ of rational maps. Then there exists an open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, the following* (I)–(III) *hold.*

(I) *We have $J_{\ker}(G_\tau) \subset S(\mathcal{W})$, $\sharp J_{\ker}(G_\tau) < \infty$ and $\sharp \mathrm{Min}(G_\tau) < \infty$. Moreover, each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$ is attracting for τ.*

(II) *There exist numbers $l, r \in \mathbb{N}$, probability measures $\eta_1, \ldots, \eta_r \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and functions $\alpha_1, \ldots, \alpha_r : \hat{\mathbb{C}} \to [0,1]$ such that for each $y \in \hat{\mathbb{C}}$ and for each $\varphi \in C(\hat{\mathbb{C}})$, we have*

$$M_\tau^{nl}(\varphi)(y) \to \sum_{i=1}^{r} \alpha_i(y) \int \varphi \, d\eta_i \text{ as } n \to \infty \text{ (pointwise convergence)}, \qquad (1)$$

i.e., we have $(M_\tau^*)^{nl}(\delta_y) \to \sum_{i=1}^{r} \alpha_i(y)\eta_i$ as $n \to \infty$ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ with respect to the weak convergence topology. Also, we have $(M_\tau^*)^l(\sum_i^r \alpha_i(y)\eta_i) = \sum_i^r \alpha_i(y)\eta_i$. Moreover, for each $i = 1, \ldots, r$, supp η_i is included in an element $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and $\cup_{i=1}^{r} \operatorname{supp} \eta_i = \cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L$. Moreover, these functions $\alpha_1, \ldots, \alpha_r$ are locally constant on $F(G_\tau)$. Furthermore, for each $i = 1, \ldots, r$ and for each $y \in F_{pt}^0(\tau)$, we have $\lim_{w \in \hat{\mathbb{C}}, w \to y} \alpha_i(w) = \alpha_i(y)$. Also, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $y \in F_{pt}^0(\tau)$, we have $\lim_{w \in \hat{\mathbb{C}}, w \to y} T_{L,\tau}(w) = T_{L,\tau}(y)$.

(III) *For each $y \in \hat{\mathbb{C}}$, there exists a Borel subset $B_{\tau,y}$ of $(\mathrm{Rat}_+)^{\mathbb{N}}$ with $\tilde{\tau}(B_{\tau,y}) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \ldots,) \in B_{\tau,y}$, we have $d(\gamma_n \circ \cdots \circ \gamma_1(y), \cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L) \to 0$ as $n \to \infty$.*

We remark that statements (I)–(III) in Theorem 1.1 cannot hold for deterministic iteration dynamics of a single $f \in \mathrm{Rat}_+$, since the dynamics of $f : J(f) \to J(f)$, where $J(f)$ denotes the Julia set of f, is chaotic. Thus Theorem 1.1 deals with a randomness-induced phenomenon.

To present the second main theorem, for each $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $\sharp L < \infty$, we define the **Lyapunov exponent** of (τ, L) and denote it by $\chi(\tau, L)$ (see Definition 3.29). Also, if $\mathcal{Y}$ is a weakly nice subset of Rat with respect to a holomorphic family $\mathcal{W}$ of rational maps, we say that $\mathcal{Y}$ is **exceptional with respect to** $\mathcal{W}$ if there exists a non-empty subset L of $S(\mathcal{W})$ such that for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$, we have $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and $\chi(\tau, L) = 0$. We say that $\mathcal{Y}$ is **non-exceptional with respect to** $\mathcal{W}$ if $\mathcal{Y}$ is not exceptional with respect to $\mathcal{W}$ (For the definition in more general setting, see Definition 3.54). For any $g \in \mathrm{Rat}$ and $z \in \hat{\mathbb{C}}$, we denote by $\|Dg_z\|_s$ the norm of the derivative of g at z with respect to the spherical metric. Also, Leb_2 denotes the 2-dimensional Lebesgue measure on $\hat{\mathbb{C}}$ and for any set $B \subset \hat{\mathbb{C}}$, we set $\mathrm{diam}(B) = \sup_{x,y\in B} d(x, y)$.

We now present the second main theorem of this paper.

Theorem 1.2 (For the detailed and more general version, see Theorem 3.81)**.** *Let $\mathcal{Y}$ be a mild subset of* Rat_+ *and suppose that $\mathcal{Y}$ is nice and non-exceptional with respect to a holomorphic family $\mathcal{W} = \{f_\lambda\}_{\lambda\in\Lambda}$ of rational maps. Then there exists an open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all of the following statements* (I) *and* (II) *hold.*

(I) *Let $H_{+,\tau} = \{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) \mid L \subset J_{\ker}(G_\tau), \chi(\tau, L) > 0\}$ and let Ω_τ be the set of points $y \in \hat{\mathbb{C}}$ for which $\tilde{\tau}(\{\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}} \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) \in \cup_{L\in H_{+,\tau}} L\}) = 0$. Then we have $\Omega_\tau = F^0_{pt}(\tau)$, $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \le \aleph_0$ and for each $z \in \Omega_\tau$, $\tilde{\tau}(\{\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}} \mid z \in J_\gamma\}) = 0$. Moreover, $\mathrm{Leb}_2(J_\gamma) = 0$ for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$. Also, $\cup_{L\in H_{+,\tau}} L \subset J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$ and $\sharp J^0_{pt}(\tau) \le \aleph_0$.*

(II) *Let Ω_τ be as in* (I)*. Then $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \le \aleph_0$ and there exist a constant $c_\tau < 0$ and a constant $\rho_\tau \in (0, 1)$ such that for each $z \in \Omega_\tau$, there exists a Borel subset $C_{\tau,z}$ of $(\mathrm{Rat}_+)^{\mathbb{N}}$ with $\tilde{\tau}(C_{\tau,z}) = 1$ satisfying that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in C_{\tau,z}$ and for each $m \in \mathbb{N} \cup \{0\}$, we have the following* (a) *and* (b)*.*

(a)

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(z)}\|_s \le c_\tau < 0.$$

(b) *There exist a constant $\delta = \delta(\tau, z, \gamma, m) > 0$, a constant $\zeta = \zeta(\tau, z, \gamma, m) > 0$ and an attracting minimal set $L = L(\tau, z, \gamma)$ of τ such that*

$$\mathrm{diam}(\gamma_{n+m,1+m}(B(\gamma_{m,1}(z), \delta))) \le \zeta\rho_\tau^n \quad \text{for all } n \in \mathbb{N},$$

and

$$d(\gamma_{n+m,1+m}(\gamma_{m,1}(z)),\ L) \le \zeta\rho_\tau^n \quad \text{for all } n \in \mathbb{N}.$$

Remark 1.3. In Theorems 3.76, 3.81, we show more generalized results in which we deal with random dynamical systems of $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ such that $\mathrm{supp}\,\tau$ is included in a finite union of loci of holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps, and $\mathrm{supp}\,\tau$ meets the locus of each $\mathcal{W}_j$.

We remark that statements (I), (II) in Theorem 1.2 cannot hold for deterministic iteration dynamics of a single $f \in \mathrm{Rat}_+$, since the dynamics of $f : J(f) \to J(f)$ is chaotic, and we have Mañé's result $\dim_H(\{z \in \hat{\mathbb{C}} \mid \liminf_{n\to\infty} \frac{1}{n} \log \|D(f^n)_z\|_s > 0\}) > 0$, where $\dim_H$ denotes the Hausdorff dimension with respect to the spherical distance on $\hat{\mathbb{C}}$ (see [21]). Thus Theorem 1.2 deals with a randomness-induced phenomenon. As we see in Theorems 1.1, 1.2, under the assumptions of Theorems 1.1, 1.2, **regarding generic random complex dynamical systems (in particular, regarding generic random polynomial dynamical systems), the chaoticity is much weaker than that of deterministic complex dynamical systems.** This arises from the effect of randomness and Theorems 1.1, 1.2 deal with randomness-induced phenomena. Note that **the statements in Theorems 1.1, 1.2 are a kind of analogues of the conjecture of density of hyperbolic maps** ([23]) in deterministic complex dynamics.

We remark that in [10] and [36, 37], regarding random complex dynamical systems, results on disappearance of chaos were shown. In [10], it was assumed that $S(\mathcal{W}) = \emptyset$ and the noise is very small, which implies that the systems in the paper have empty kernel Julia sets $J_{\ker}(G_\tau)$ of corresponding rational semigroups. In [37], it was also assumed that $S(\mathcal{W}) = \emptyset$ (for a holomorphic family $\mathcal{W}$ of polynomials, it was assumed $S(\mathcal{W}) \setminus \{\infty\} = \emptyset$) but the range of the noise could be big, and it was shown that the generic systems have empty kernel Julia sets, which implies that the chaoticity of the systems is much weaker than that of deterministic complex dynamical systems. In this paper, it is important that **in Theorems 1.1 and 1.2, the set $\mathcal{A}$ contains many τ such that $J_{\ker}(G_\tau) \neq \emptyset$. Once we have non-empty kernel Julia set, the analysis of the system is much more difficult than the cases with empty kernel Julia sets**, even if the kernel Julia set is finite. We need a new framework and more technical arguments to study such systems.

We apply the results and the methods in the above to finding roots of **any polynomial** by random relaxed Newton's method as we explained below. Let $g \in \mathcal{P}$. Let $\Lambda := \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$ and let $f_\lambda(z) = z - \lambda \frac{g(z)}{g'(z)}$ for each $\lambda \in \Lambda$. Let $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$. Let $\mathcal{Y} := \{f_\lambda \in \mathrm{Rat} \mid \lambda \in \Lambda\}$. Then $\mathcal{Y}$ is called the **random relaxed Newton's method set for** g and $\mathcal{W}$ is called the **random relaxed Newton's method family for** g. Also, $(\mathcal{Y}, \mathcal{W})$ is called the **random relaxed Newton's method scheme for** g. Moreover, for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$, the random dynamical system on $\hat{\mathbb{C}}$ generated by τ is called a **random relaxed Newton's method (or random relaxed Newton's method system) of** (g, τ). Furthermore, let $Q_g := \{z_0 \in \mathbb{C} \mid g(z_0) = 0\}$.

We now present the third main theorem of this paper.

Theorem 1.4 (For the details, see Theorem 4.4)**.** *Let $g \in \mathcal{P}$. Let $(\mathcal{Y}, \mathcal{W})$ be the random relaxed Newton's method scheme for g. Then $\mathcal{Y}$ is a mild subset of Rat_+, the set $\mathcal{Y}$ is nice and non-exceptional with respect to $\mathcal{W}$ and $(\mathcal{Y}, \mathcal{W})$ satisfies the assumptions of Theorems 1.1, 1.2. Moreover, there exists an open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y})$ such that all of the following hold.*

(I) *Let $\tau \in \mathcal{A}$. Then all statements* (I)–(III) *in Theorem 1.1 and statements* (I)–(III) *in Theorem 1.2 hold for τ. Moreover, $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is equal to the union of $\{\{x\} \mid x \in Q_g\} \cup \{\{\infty\}\}$ and $\{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) \mid L \subset \mathbb{C} \setminus Q_g, L \text{ is attracting for } \tau\}$. Also, for each $x \in Q_g$, the minimal set $\{x\}$ is attracting for τ.*

(II) *Let $\tau \in \mathcal{A}$. Let Ω_τ be the set defined in Theorem 1.2. Then $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ and*

$$\Omega_\tau = \{y \in \mathbb{C} \mid \tilde{\tau}(\{\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}} \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) = \infty\}) = 0\}.$$

(III) *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ and suppose that $\mathrm{int}(\Gamma_\tau) \supset \{f_\lambda \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\}$ with respect to the topology in $\mathcal{Y} \cong \Lambda := \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$ and τ is absolutely continuous with respect to the 2-dimensional Lebesgue measure on $\mathcal{Y} \cong \Lambda$ (e.g., let τ be the normalized 2-dimensional Lebesgue measure on $\{f_\lambda \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq r\}$ where $\frac{1}{2} < r < 1$, under the identification $\mathcal{Y} \cong \Lambda$). Then $\tau \in \mathcal{A}$ and all statements* (I)–(III) *in Theorem 1.1 and statements* (I)–(III) *in Theorem 1.2 hold for τ. Moreover, we have*

$$\Omega_\tau = \mathbb{C} \setminus \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\} \text{ and } \sharp(\mathbb{C} \setminus \Omega_\tau) \leq \deg(g) - 1.$$

Moreover, we have $\max_{x \in Q_g} e^{\chi(\tau, \{x\})} < 1$. Moreover, for each $\alpha \in (\max_{x \in Q_g} e^{\chi(\tau, \{x\})}, 1)$, and for each $z \in F^0_{pt}(\tau) = \Omega_\tau$, there exists a Borel subset $C_{\tau,z,\alpha}$ of $(\mathrm{Rat}_+)^{\mathbb{N}}$ with $\tilde{\tau}(C_{\tau,z,\alpha}) = 1$ satisfying that for each $\gamma \in C_{\tau,z,\alpha}$, there exist an element $x = x(\tau, z, \alpha, \gamma) \in Q_g$ and a constant $\xi = \xi(\tau, z, \alpha, \gamma) > 0$ such that

$$d(\gamma_{n,1}(z), x) \leq \xi \alpha^n \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \gamma_{n,1}(z) \to x \text{ as } n \to \infty. \tag{2}$$

Also, for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$, we have $\mathrm{Leb}_2(J_\gamma) = 0$ and for each $z \in F_\gamma$, there exists an element $x = x(\tau, \gamma, z) \in Q_g$ such that

$$\gamma_{n,1}(z) \to x \text{ as } n \to \infty. \tag{3}$$

Moreover, for each $x \in Q_g$ and for each $z \in \Omega_\tau$, we have $\lim_{w\in\hat{\mathbb{C}}, w\to z} T_{x,\tau}(w) = T_{x,\tau}(z)$. Moreover, for any subset B of $\mathbb{C}$ with $\sharp B \geq \deg(g)$, there exists an element $z \in B$ such that

$$\sum_{x\in Q_g} T_{x,\tau}(z) = 1. \tag{4}$$

Furthermore, for each $\varphi \in C(\hat{\mathbb{C}})$ and for each $z \in \Omega_\tau$, we have $M_\tau^n(\varphi)(z) \to \sum_{x\in Q_g} T_{x,\tau}(z)\varphi(x)$ as $n \to \infty$ and this convergence is uniform on any compact subsets of Ω_τ.

Also, there exists an neighborhood $\mathcal{U}$ of τ in $\mathcal{A}$ such that for each $\eta \in \mathcal{U}$, we have $\sharp(\hat{\mathbb{C}}\setminus\Omega_\eta) \leq \aleph_0$ and for each $z \in \Omega_\eta$, there exists a Borel subset $C_{\eta,z}$ of $(\mathrm{Rat}_+)^{\mathbb{N}}$ with $\tilde{\eta}(C_{\eta,z}) = 1$ such that for each $\gamma \in C_{\eta,z}$, there exists an element $x = x(\eta, z, \gamma) \in Q_g$ such that $\gamma_{n,1}(z) \to x$ as $n \to \infty$.

We say that a non-constant polynomial g is **normalized** if $\{z_0 \in \mathbb{C} \mid g(z_0) = 0\}$ is included in $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. For a given polynomial g, sometimes it is not difficult for us to find an element $a \in \mathbb{R}$ with $a > 0$ such that $g(az)$ is a normalized polynomial of z. It is well-known that if $g \in \mathcal{P}$ is a normalized polynomial, then so is g' (see [1]). Thus, we obtain the following corollary.

Corollary 1.5. *Let $g \in \mathcal{P}$ be a normalized polynomial. Let $\Lambda = \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$. Let $z_0 \in \mathbb{C} \setminus \mathbb{D}$. Let $\eta \in \mathfrak{M}_{1,c}(\Lambda)$ be an element such that int$(\mathrm{supp}\,\eta) \supset \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq \frac{1}{2}\}$ and η is absolutely continuous with respect to the 2-dimensional Lebesgue measure on Λ. Under the identification $\mathcal{Y} \cong \Lambda$, we regard η as an element of $\mathfrak{M}_{1,c}(\mathcal{Y}_1)$. Then for $\tilde{\eta}$-a.e. $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$, $\gamma_{n,1}(z_0)$ tends to a root $x = x(\gamma)$ of g. Moreover, if, in addition to the assumptions of our theorem, we know the coefficients of g explicitly, then by the following algorithm in which we consider d-random orbits of z_0 under d-different random relaxed Newton's methods, we can find all roots of g almost surely with arbitrarily small errors.*

(1) *We first consider the random relaxed Newton's method scheme $(\mathcal{Y}_1, \mathcal{W}_1)$ for $g_1 = g$. By Theorem 1.4, for $\tilde{\eta}$-a.e. $\gamma \in \mathrm{Rat}^{\mathbb{N}}$, $\gamma_{n,1}(z_0)$ tends to a root $x = x(\gamma)$ of g. Let x_1 be one of such $x(\gamma)$ (with arbitrarily small error).*

(2) *Let $g_2(z) = g(z)/(z - x_1)$. By using synthetic division, we regard g_2 as a polynomial which divides g_1 (with arbitrarily small error). Note that g_2 is also a normalized polynomial. We consider the random relaxed Newton's method scheme $(\mathcal{Y}_2, \mathcal{W}_2)$ for g_2. As in the first step (replacing g_1 by g_2), we find a root x_2 of g_2, which is also a root of g (with arbitrarily small error).*

(3) *Let $g_3(z) = g_2(z)/(z - x_2)$ and as in the above, we find a root x_3 of g with arbitrarily small error. Continue this method.*

We remark that in Theorem 1.4 and Corollary 1.5, any system has non-empty kernel Julia set of the corresponding rational semigroup (in fact, $\{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\} \cup \{\infty\}$ is included in the kernel Julia set), and in order to analyze such systems we need a new framework and much more technical arguments than those of [36], [37]. See the second remark after Remark 1.3.

Remark 1.6. (I) Regarding the original Newton's method, M. Hurley showed in [15] that for any $k \in \mathbb{N}$ with $k \geq 3$, there exists a polynomial g of $\deg(g) = k$ such that the Newton's method map $N_g(z) = z - g(z)/g'(z)$ for g has $2k - 2$ different attracting cycles. Thus this N_g has $k - 2$ attracting cycles which are not zeros of g. Since attracting cycles are stable under perturbations, it follows that for any $k \geq 3$, the set of elements g for which the Newton's map has attracting cycles which are not zeros of g is a non-empty open subset of $\mathcal{P}_k := \{g \in \mathcal{P} \mid \deg(g) = k\}$.

(II) C. McMullen showed in [22] that for any $k \in \mathbb{N}, k \geq 4$ and for any $l \in \mathbb{N}$, there exists no rational map $\tilde{N} : \mathcal{P}_k \to \mathrm{Rat}_l := \{f \in \mathrm{Rat} \mid \deg(f) = l\}$ such that for any g in an open dense subset of $\mathcal{P}_k$, for any z in an open dense subset of $\hat{\mathbb{C}}$, $\tilde{N}(g)^n(z)$ tends to a root of g as $n \to \infty$.

(III) The essential assumptions on τ in Theorem 1.4 (III) and Corollary 1.5 of this paper do not depend on $g \in \mathcal{P}$. By (I)(II), it follows that **the statements of Theorem 1.4 and Corollary 1.5 cannot hold in the deterministic relaxed Newton's method and any other deterministic complex analytic iterative schemes to find roots of polynomials. Thus Theorem 1.4 and Corollary 1.5 deal with randomness-induced phenomena**.

(IV) J. Hubbard, D. Schleicher and S. Sutherland showed in [16] that for each $d \in \mathbb{N}$, there exists a finite subset B of $\mathbb{C}$ with $\sharp B \leq 1.1d(\log d)^2$ such that for any normalized polynomial g with $\deg(g) = d$ and for every root of g, at least one of the points in B converges to this root under the iteration of the same Newton's method map N_g for g.

Note that this is the first paper to investigate random relaxed Newton's method systematically. It is important that in Theorem 1.4 (III) and Corollary 1.5, the size of the noise is big which enables the system to make the minimal set with period greater than 1 collapse. However, since the size of the noise is big, it is not enough for us to consider the arguments which are similar to those of deterministic dynamics of one map, thus we have to develop the theory of random complex dynamical systems with noise or randomness of any size as in Theorems 1.1, 1.2.

As we see before, in Theorems 1.1 and 1.2, the chaoticity of random complex dynamical systems is much weaker than that of deterministic dynamical systems. However, the random systems may have still a kind of complexity or chaoticity. For example, when we consider the function $T_{L,\tau}$ of probability of tending to one $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, then under certain conditions, this function is continuous on $\hat{\mathbb{C}}$ and even more, this is α-Hölder continuous on $\hat{\mathbb{C}}$ for some $\alpha \in (0,1)$ but there exists an element $\beta \in (0,1)$ such that $T_{L,\tau}$ cannot β-Hölder continuous on $\hat{\mathbb{C}}$. This implies that the system generated by τ does not act mildly (i.e., the transition operator M_τ of τ does not act mildly) on the Banach space $C^\beta(\hat{\mathbb{C}})$ of β-Hölder continuous functions on $\hat{\mathbb{C}}$ endowed with β-Hölder norm $\|\cdot\|_\beta$ (e.g., there exists a $\varphi \in C^\beta(\hat{\mathbb{C}})$ such that $\|M_\tau^n(\varphi)\|_\beta \to \infty$ as $n \to \infty$). Thus regarding the random (complex) dynamical systems, we have the **gradations between chaos and order** (see [36, 37, 17, 18, 39]).

In Theorems 3.79 and 3.82, we show the results on random dynamical systems generated by measures τ on $\mathcal{Y}$ without assuming $\mathcal{Y}$ is mild. We show that considering the mild part $\mathfrak{M}_{1,c,mild}(\mathcal{Y})$ (the set of elements τ which has an attractor, see Definition 3.77), there exists an open and dense subset $\mathcal{A}$ of $\mathfrak{M}_{1,c,mild}(\mathcal{Y})$ such that for each $\tau \in \mathcal{A}$, statements (I)–(III) in Theorem 1.1 and statements (I)(II) in 1.2 hold. Also, denoting by $\mathfrak{M}_{1,c,JF}(\mathcal{Y})$ the set of elements $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ for which $J(G_\tau) = \hat{\mathbb{C}}$ and either $\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ or $\cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L \subset S(\mathcal{W})$, we show that the union of $\mathcal{A}$ and $\mathfrak{M}_{1,c,JF}(\mathcal{Y})$ is dense in $\mathfrak{M}_{1,c}(\mathcal{Y})$ (Theorems 3.79 and 3.82).

Example 1.7. We give some examples of $\mathcal{Y}$ satisfying the assumptions in Theorem 1.2 or the generalized version Theorem 3.81. For the details, see Section 5. In the following, $\mathcal{A}$ denotes the open and dense subset of $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ or $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ (for the notation, see Definition 3.41) in Theorem 1.2 or Theorem 3.81. As mentioned before, **if $J_{\ker}(G_\tau) \neq \emptyset$, it is much more difficult to show the statements on convergence of measures and negativity of Lyapunov exponents in Theorems 1.1, 1.2, 3.76, 3.81 than the cases with $J_{\ker}(G_\tau) = \emptyset$.**

(i) For each $q \in \mathbb{N}$ with $q \geq 2$, let $\mathcal{P}_q := \{f \in \mathcal{P} \mid \deg(f) = q\}$. Let $(q_1, \ldots, q_m) \in \mathbb{N}^m$ with $q_1 < q_2 < \cdots < q_m$ and let $\mathcal{W}_j = \{f\}_{f \in \mathcal{P}_{q_j}}, j = 1, \ldots, m$ and let $\mathcal{Y} = \cup_{j=1}^m \mathcal{P}_{q_j}$. In this case, $S(\mathcal{W}_j) \setminus \{\infty\} = \emptyset$ for each j and the set Ω_τ in Theorem 3.81 is equal to $\hat{\mathbb{C}}$. (Note that this result has been already obtained in [37].)

(ii) Let $q \in \mathbb{N}$ with $q \geq 2$ and let $\mathcal{W} = \{z^q + c\}_{c \in \mathbb{C}}$. Let $\mathcal{Y} = \{z^q + c \mid c \in \mathbb{C}\}$. In this case, $S(\mathcal{W}) \setminus \{\infty\} = \emptyset$ and the set Ω_τ in Theorem 1.2 is equal to $\hat{\mathbb{C}}$. (Note that this result has been already obtained in [37].)

(iii) Let $\mathcal{W} = \{\lambda z(1-z)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ and let $\mathcal{Y} = \{\lambda z(1-z) \in \mathcal{P}_2 \mid \lambda \in \mathbb{C} \setminus \{0\}\}$. In this case, $S(\mathcal{W}) = \{0,1\} \cup \{\infty\}$ and $S(\mathcal{W}) \setminus \{\infty\} \neq \emptyset$. There exists a non-empty open subset $\mathcal{A}'$ of $\mathcal{A}$ such that for each $\tau \in \mathcal{A}'$, we have $F_{pt}^0(\tau) = \Omega_\tau \neq \hat{\mathbb{C}}$ and $J_{\ker}(G_\tau) \neq \emptyset$.

(iv) Let $f \in \mathcal{P}$ such that if $z_0 \in \mathbb{C}$ and $f(z_0) = 0$, then $f'(z_0) \neq 0$. Let $\mathcal{W} = \{z + \lambda f(z)\}_{\lambda \in \mathbb{C}\setminus\{0\}}$ and let $\mathcal{Y} = \{z + \lambda f(z) \in \mathcal{P} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$. In this case, $S(\mathcal{W}) = \{z_0 \in \mathbb{C} \mid f(z_0) = 0\} \cup \{\infty\}$ and $S(\mathcal{W}) \setminus \{\infty\} \neq \emptyset$. Then there exists a non-empty open subset $\mathcal{A}'$ of $\mathcal{A}$ such that for each $\tau \in \mathcal{A}'$, we have $F^0_{pt}(\tau) = \Omega_\tau \neq \hat{\mathbb{C}}$ and $J_{\ker}(G_\tau) \neq \emptyset$.

(v) Let $n \in \mathbb{N}$ with $n \geq 2$ and let $w = e^{2\pi i/n} \in \mathbb{C}$. Let $\mathcal{W}_i = \{w^i(z + \lambda(z^n - 1))\}_{\lambda \in \mathbb{C}\setminus\{0\}}$ for each $i = 1, \ldots, n$. Let $i_1, \ldots, i_m \in \{1, \ldots, n\}$ with $i_1 < i_2 \cdots < i_m$. For these $i_1, \ldots, i_m$, let $\mathcal{Y} = \cup_{j=1}^m \{w^{i_j}(z + \lambda(z^n - 1)) \in \mathcal{P} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$. Then there exists a non-empty open subset $\mathcal{A}'$ of $\mathcal{A}$ such that for each $\tau \in \mathcal{A}'$, we have $F^0_{pt}(\tau) = \Omega_\tau \neq \hat{\mathbb{C}}$ and $J_{\ker}(G_\tau) \neq \emptyset$.

The strategy to prove Theorems 1.1, 1.2, 3.76, 3.81 is as follows. Let $\mathcal{Y}$ be a mild subset of Rat_+ and suppose that $\mathcal{Y}$ is nice with respect to a holomorphic family $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ of rational maps. Let $\tau_0 \in \mathfrak{M}_{1,c}(\mathcal{Y})$. Then there exists an element τ which is arbitrarily close to τ_0 and $\mathrm{int}(\Gamma_\tau) \neq \emptyset$. We show that for such τ, we have $J_{\ker}(G_\tau) \subset S(\mathcal{W})$ and hence $\sharp J(G_\tau) < \infty$, by using Montel's theorem (Lemmas 3.44, 3.45). In Proposition 3.63, we develop a theory on **the systems with finite kernel Julia sets** based on careful observations on **limit functions on the Fatou sets** by using the **hyperbolic metrics on the Fatou components** (Lemma 3.60), and we obtain that for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \cap F(G_\tau) \neq \emptyset$, the dynamics in Fatou components which meet L are locally contracting and $\sharp\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$. Also, we develop a theory on **bifurcation of minimal sets** under perturbation which was initiated by the author of this paper in [37] in Lemma 3.71, and applying it and enlarging the support of τ a little bit, we obtain that any $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \cap F(G_\tau) \neq \emptyset$ is attracting for τ. By the theory of finite Markov chains ([9]), we see that for such τ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset J_{\ker}(G_\tau)$, there exists a canonical invariant measure on L (Lemmas 3.25, 3.26, Definition 3.29). It is very important and useful to show that for any $y \in \hat{\mathbb{C}}$, letting $E_y := \{\gamma \in (\mathrm{Rat})^{\mathbb{N}} \mid y \in \cap_{n=1}^\infty \gamma_{n,1}^{-1}(J(G_\tau))\}$,

$$\text{for } \tilde{\tau}\text{ -a.e. } \gamma \in E_y, \text{ we have } d(\gamma_{n,1}(y), J_{\ker}(G_\tau)) \to 0 \text{ as } n \to \infty,$$

by using careful observations on random dynamical systems on general compact metric spaces (Lemma 3.15).

We next observe the local dynamics of G_τ at each point of $S(\mathcal{W})$. By enlarging the support of τ a little bit, by some careful arguments, it turns out that we may assume that each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset S(\mathcal{W})$ satisfies one of the following conditions **(I)–(IV)**. **(I)** "Uniformly expanding". **(II)** "Attracting". **(III)** "There exist a point $z_1 \in L$ and elements $g_1, g_2, g_3 \in G_\tau$ such that $g_1(z_1) = z_1, \|D(g_1)_{z_1}\|_s > 1$, $g_2(z_1) = z_1, \|D(g_2)_{z_1}\|_s < 1$, $g_3(z_1) = z_1$, and g_3 has a Siegel disk with center z_1". **(IV)** "There exists a point $z_1 \in L$ such that for each $\lambda \in \Lambda$, we have $D(f_\lambda)_{z_1} = 0$. Moreover, there exist a point $z_2 \in L$ and an element $g \in G_\tau$ such that $g(z_2) = z_2$ and $\|Dg_{z_2}\|_s > 1$". By using some results on rational semigroups from [13], it turns out that if L is of type (III) or (IV), then $L \subset \mathrm{int}(J(G_\tau))$. In particular, for each $z \in F(G_\tau)$, we have $\overline{G(z)} \cap L = \emptyset$. It turns out that for each $z \in F(G_\tau)$, if $\overline{G(z)}$ does not meet any attracting minimal set of τ, then $\overline{G(z)}$ meets a minimal set L which is uniformly expanding. Thus $\overline{G(z)}$ meets a backward image of L under some element of G_τ, which is included in a compact subset of $J(G_\tau) \setminus S(\mathcal{W})$. By enlarging the support of τ a little bit again, we obtain that for each $z \in F(G_\tau)$, $\overline{G(z)}$ meets an attracting minimal set of τ. From these arguments, we can show that this τ is **weakly mean stable**, i.e., there exist a positive integer n and two non-empty open subsets $V_{1,\tau}, V_{2,\tau}$ of $\hat{\mathbb{C}}$ with $\overline{V_{1,\tau}} \subset V_{2,\tau}$ and $\sharp(\hat{\mathbb{C}} \setminus V_{2,\tau}) \geq 3$ such that (a) for each $(\gamma_1, \ldots, , \gamma_n) \in \Gamma_\tau^n$, we have $\gamma_n \cdots \gamma_1(V_{2,\tau}) \subset V_{1,\tau}$, (b) we have $\sharp D_\tau < \infty$, where $D_\tau := \cap_{g \in G_\tau} g^{-1}(\hat{\mathbb{C}} \setminus V_{2,\tau})$, and (c) for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset D_\tau$, there exist an element $z \in L$ and an element $g_z \in G_\tau$ such that z is a repelling fixed point of g_z. From this fact, we can prove the existence of an open and dense subset $\mathcal{A}$ in Theorems 1.1, 3.76. If we assume further that $\mathcal{Y}$ is non-exceptional with respect to $\mathcal{W}$, then we can show that there exists an open and dense subset $\mathcal{A}'$ of $\mathcal{A}$ such that for each $\tau \in \mathcal{A}'$, (1) for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset S(\mathcal{W})$, we have $\chi(\tau, L) \neq 0$, and (2) for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset S(\mathcal{W})$, if $\chi(\tau, L) > 0$, then for each $z \in L$ and for each $g \in \Gamma_\tau$, we have $Dg_z \neq 0$. Combining this fact and the observations on the local behavior of the systems around the minimal sets with non-zero Lyapunov exponents (Lemmas 3.30–3.36), we can prove that each element of $\tau \in \mathcal{A}'$ satisfies statements (I)–(III) in Theorem 1.2.

By the above arguments, we obtain the following.

Corollary 1.8 (For more generalized result, see Theorem 3.76)**.** *Under the assumptions of Theorem 1.1, the set of weakly mean stable elements* $\tau \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ *is open and dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$*,*

Note that weak mean stability is a new concept introduced by the author of this paper, and it is crucial to consider the density of weakly mean stable elements to prove Theorems 1.1, 1.2, 3.76, 3.81. We emphasize that weakly mean stability implies many interesting properties (Lemma 3.73, Theorem 3.80). We remark that in [36], the notion **mean stability** (i.e., every minimal set is attracting) was introduced by the author of this paper and it was proved in [37] that the set of mean stable elements of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ is open and dense in $\mathfrak{M}_{1,c}(\mathcal{P})$. Mean stability implies weak mean stability, but **the converse is not true**. In fact, there are many examples of mild and nice sets $\mathcal{Y}$ for which there exists a non-empty open subset $\mathcal{A}''$ of $\mathcal{A}$ (where $\mathcal{A}$ is the set in Theorems 1.1, 1.2) such that each $\tau \in \mathcal{A}''$ is **not mean stable (but is weakly mean stable)**. For such examples, see Theorems 1.4, 4.4, Example 1.7 (iii)–(v) and Examples 5.4–5.7.

In Section 2, we give some fundamental notations and definitions, and present some basic facts on rational semigroups. In Section 3, we develope the theory of random complex dynamical systems and prove several theorems including Theorems 1.1, 1.2 and the detailed versions Theorems 3.76, 3.81 of them. In Section 4, we apply Theorems 1.1, 1.2, 3.76, 3.81 and the other results in Section 3 to random relaxed Newton's methods in which we find roots of given polynomials, and we show Theorem 1.4 and the detailed version Theorem 4.4. In Section 5, we give some examples to which we can apply the main theorems.

2 Preliminaries

In this section, we give some fundamental notations and definitions.
Notation. Let (X, d) be a metric space, A a subset of X, and $r > 0$. We set $B(A, r) := \{z \in X \mid d(z, A) < r\}$. Moreover, for a subset C of $\mathbb{C}$, we set $D(C, r) := \{z \in \mathbb{C} \mid \inf_{a \in C} |z - a| < r\}$. Moreover, for any topological space Y and for any subset A of Y, we denote by $\mathrm{int}(A)$ the set of all interior points of A. We denote by $\mathrm{Con}(A)$ the set of all connected components of A.

Definition 2.1. Let Y be a metric space. We set $\mathrm{CM}(Y) := \{f : Y \to Y \mid f \text{ is continuous}\}$ endowed with the compact-open topology. Also, we set $\mathrm{OCM}(Y) := \{f \in \mathrm{CM}(Y) \mid f \text{ is an open map}\}$ endowed with the relative topology from $\mathrm{CM}(Y)$. Moreover, we denote by $C(Y)$ the space of all continuous functions $\varphi : Y \to \mathbb{C}$. When Y is compact, we endow $C(Y)$ with the supremum norm $\|\cdot\|_\infty$.

Remark 2.2. $\mathrm{CM}(Y)$ and $\mathrm{OCM}(Y)$, are semigroups with the semigroup operation being functional composition. If Y is a compact metric space, then $\mathrm{CM}(Y)$ is a complete separable metric space.

Definition 2.3. A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on $\hat{\mathbb{C}}$ with the semigroup operation being functional composition([13, 12]). A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps. We set $\mathrm{Rat} : = \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Moreover, we set $\mathrm{Rat}_+ := \{h \in \mathrm{Rat} \mid \deg(h) \geq 2\}$ endowed with the relative topology from Rat. Also, we set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Remark 2.4. ([2, Theorem 2.8.2, Corollary 2.8.3]) Let $\mathrm{Rat}_m := \{g \in \mathrm{Rat} \mid \deg(g) = m\}$ for each $m \in \mathbb{N}$ and let $\mathcal{P}_m := \{g \in \mathcal{P} \mid \deg(g) = m\}$ for each $m \in \mathbb{N}$ with $m \geq 2$. Then for each m, Rat_m

(resp. $\mathcal{P}_m$) is a connected component of Rat (resp. $\mathcal{P}$). Moreover, Rat_m (resp. $\mathcal{P}_m$) is open and closed in Rat (resp. $\mathcal{P}$) and is a finite dimensional complex manifold. Also, $h_n \to h$ in $\mathcal{P}$ if and only if $\deg(h_n) = \deg(h)$ for each large n and the coefficients of h_n tend to the coefficients of h appropriately as $n \to \infty$.

Definition 2.5. Let Y be a compact metric space and let G be a subsemigroup of $\text{CM}(Y)$. The **Fatou set of** G is defined to be

$$F(G) := \{z \in Y \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t.} \{g|_U : U \to \hat{\mathbb{C}}\}_{g \in G} \text{ is equicontinuous on } U\}.$$

(For the definition of equicontinuity, see [2].) The **Julia set of** G is defined to be $J(G) := \hat{\mathbb{C}} \setminus F(G)$. If G is generated by $\{g_i\}_{i=1}^m$ (i.e., $G = \{g_{i_1} \circ \cdots \circ g_{i_n} \mid n \in \mathbb{N}, i_1, \ldots, i_n \in \{1, \ldots, m\}\}$) , then we write $G = \langle g_1, g_2, \ldots g_m \rangle$. If G is generated by a subset Γ of $\text{CM}(Y)$ (i.e., G is equal to the set $\{h_1 \circ \cdots \circ h_n \mid n \in \mathbb{N}, h_1, \ldots, h_n \in \Lambda\}$), then we write $G = \langle \Gamma \rangle$. For a subset A of Y, we set $G(A) := \bigcup_{g \in G} g(A)$ and $G^{-1}(A) := \bigcup_{g \in G} g^{-1}(A)$. We set $G^* := G \cup \{\text{Id}\}$, where Id denotes the identity map.

Lemma 2.6 ([13, 12])**.** *Let Y be a compact metric space and let G be a subsemigroup of* $\text{OCM}(Y)$*. Then, for each $h \in G$, $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that the equality does not hold in general.*

Regarding the dynamics of rational semigroups G, we have the following. $F(G)$ is G-forward invariant and $J(G)$ is G-backward invariant. Here, we say that a set $A \subset \hat{\mathbb{C}}$ is G-backward invariant, if $g^{-1}(A) \subset A$ for each $g \in G$, and we say that A is G-forward invariant, if $g(A) \subset A$, for each $g \in G$. If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set and $\sharp(E(G)) \leq 2$, where $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp G^{-1}(z) < \infty\}$. ($E(G)$ is called exceptional set of G.) Moreover, if $\sharp J(G) \geq 3$ and if $z \in \hat{\mathbb{C}} \setminus E(G)$, then $J(G) \subset \overline{G^{-1}(z)}$. In particular, if $\sharp J(G) \geq 3$ and $z \in J(G) \setminus E(G)$, then $\overline{G^{-1}(z)} = J(G)$. Also, if $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest closed subset of $\hat{\mathbb{C}}$ containing at least three points which is G-backward invariant. Furthermore, if $\sharp(J(G)) \geq 3$, then we have $J(G) = \overline{\{z \in \hat{\mathbb{C}} \mid z \text{ is a repelling fixed point of some } g \in G\}} = \overline{\cup_{g \in G} J(g)}$. For the proofs of these results, see [13] and [27]. We remark that [27] is a very nice introductory article of rational semigroups.

The following is the key to investigating random complex dynamics.

Definition 2.7. Let Y be a compact metric space and let G be a subsemigroup of $\text{CM}(Y)$. We set $J_{\ker}(G) := \bigcap_{g \in G} g^{-1}(J(G))$. This is called the **kernel Julia set of** G.

Remark 2.8. Let Y be a compact metric space and let G be a subsemigroup of $\text{CM}(Y)$. (1) $J_{\ker}(G)$ is a compact subset of $J(G)$. (2) For each $h \in G$, $h(J_{\ker}(G)) \subset J_{\ker}(G)$. (3) If G is a rational semigroup and if $F(G) \neq \emptyset$, then $\text{int}(J_{\ker}(G)) = \emptyset$. (4) If G is generated by a single map or if G is a group, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup G, it may happen that $\emptyset = J_{\ker}(G) \neq J(G)$ (see [36]).

In the rest of this paper we sometimes need some results of random complex dynamical systems from [36, 37].

3 Random complex dynamical systems

In this section, we develope the theory of random complex dynamical systems and prove several theorems including Theorems 1.1, 1.2 and the detailed versions Theorems 3.76, 3.81 of them.

3.1 Random dynamical systems on general compact metric spaces

In this subsection we show some results on random dynamical systems on general compact metric spaces. It is sometimes important to investigate the dynamics of sequences of maps.

Definition 3.1. Let Y be a compact metric space. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{CM}(Y))^{\mathbb{N}}$ and each $m, n \in \mathbb{N}$ with $m \geq n$, we set $\gamma_{m,n} = \gamma_m \circ \cdots \circ \gamma_n$ and we set $\gamma_{0,1} = \mathrm{Id}$. Also, we set

$$F_{\gamma,0} := \{z \in Y \mid \{\gamma_{n,1}\}_{n=1}^{\infty} \text{ is equicontinuous at the one point } z\},$$
$$F_{\gamma} := \{z \in Y \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{\gamma_{n,1}\}_{n\in\mathbb{N}} \text{ is equicontinuous on } U\},$$

$J_{\gamma,0} := Y \setminus F_{\gamma,0}$ and $J_\gamma := Y \setminus F_\gamma$. The set F_γ is called the **Fatou set of the sequence** γ and the set J_γ is called the **Julia set of the sequence** γ. Moreover, we set $F^{\gamma,0} := \{\gamma\} \times F_{\gamma,0} (\subset (\mathrm{CM}(Y))^{\mathbb{N}} \times Y)$, $F^\gamma := \{\gamma\} \times F_\gamma$ $(\subset (\mathrm{CM}(Y))^{\mathbb{N}} \times Y)$, $J^{\gamma,0} := \{\gamma\} \times J_{\gamma,0} (\subset (\mathrm{CM}(Y))^{\mathbb{N}} \times Y)$ and $J^\gamma := \{\gamma\} \times J_\gamma$ $(\subset (\mathrm{CM}(Y))^{\mathbb{N}} \times Y)$.

Remark 3.2. Let $\gamma \in (\mathrm{Rat})^{\mathbb{N}}$. Then by Montel's theorem, $J_{\gamma,0} = J_\gamma$. Also, if $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$, then by [2, Theorem 2.8.2], $J_\gamma \neq \emptyset$. Moreover, if Γ is a non-empty compact subset of Rat_+ and $\gamma \in \Gamma^{\mathbb{N}}$, then by [31], J_γ is a perfect set and J_γ has uncountably many points.

Lemma 3.3. *Let Y be a compact metric space. Let Γ be a non-empty closed subset of an open subset of* $\mathrm{CM}(Y)$. *Then $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} F^{\gamma,0}$, $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} F^{\gamma}$, $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} J^{\gamma,0}$ and $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} J^{\gamma}$ are Borel measurable subsets of $\Gamma^{\mathbb{N}} \times Y$ and*

$$\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} F^{\gamma,0} = \{(\gamma, y) \in \Gamma^{\mathbb{N}} \times Y \mid \lim_{m\to\infty} \sup_{n\geq 1} \mathrm{diam}\gamma_{n,1}(B(y, \frac{1}{m})) = 0\}, \tag{5}$$

$$\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} F^{\gamma} = \bigcup_{p\in\mathbb{N}} \{(\gamma, y) \in \Gamma^{\mathbb{N}} \times Y \mid \lim_{m\to\infty} \sup_{n\geq 1} \sup_{y'\in B(y,\frac{1}{p})} \mathrm{diam}\gamma_{n,1}(B(y', \frac{1}{m})) = 0\}. \tag{6}$$

Proof. By the definition of F^γ, we obtain (5) and (6). From (5) and (6), it follows that $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} F^{\gamma,0}$ and $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} F^{\gamma}$ are Borel subsets of $\Gamma^{\mathbb{N}} \times Y$. Thus $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} J^{\gamma,0}$ and $\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} J^{\gamma}$ are also Borel subsets of $\Gamma^{\mathbb{N}} \times Y$. □

We now give some notations on random dynamics.

Definition 3.4. For a metric space Y, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on Y endowed with the topology such that $\mu_n \to \mu$ in $\mathfrak{M}_1(Y)$ if and only if for each bounded continuous function $\varphi : Y \to \mathbb{C}$, $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$. Note that if Y is a compact metric space, then $\mathfrak{M}_1(Y)$ is a compact metric space with the metric $d_0(\mu_1, \mu_2) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}{1+|\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}$, where $\{\phi_j\}_{j\in\mathbb{N}}$ is a dense subset of $C(Y)$. Furthermore, for each $\tau \in \mathfrak{M}_1(Y)$, the topological support $\mathrm{supp}\,\tau$ of τ is defined as $\mathrm{supp}\,\tau := \{z \in Y \mid \forall \text{ neighborhood } U \text{ of } z,\ \tau(U) > 0\}$. Note that $\mathrm{supp}\,\tau$ is a closed subset of Y. Furthermore, we set $\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \mathrm{supp}\,\tau \text{ is a compact subset of } Y\}$.

For a complex Banach space $\mathcal{B}$, we denote by $\mathcal{B}^*$ the space of all continuous complex linear functionals $\rho : \mathcal{B} \to \mathbb{C}$, endowed with the weak* topology.

For any $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$, we will consider the i.i.d. random dynamics on Y such that at every step we choose a map $g \in \mathrm{CM}(Y)$ according to τ (thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the state space Y such that for each $x \in Y$ and each Borel measurable subset A of Y, the transition probability $p(x, A)$ from x to A is defined as $p(x, A) = \tau(\{g \in \mathrm{CM}(Y) \mid g(x) \in A\})$).

Definition 3.5. Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$.

1. We set $\Gamma_\tau := \mathrm{supp}\,\tau$ (thus Γ_τ is a closed subset of $\mathrm{CM}(Y)$). Moreover, we set $X_\tau := (\Gamma_\tau)^{\mathbb{N}}$ $(= \{\gamma = (\gamma_1, \gamma_2, \ldots) \mid \gamma_j \in \Gamma_\tau\ (\forall j)\})$ endowed with the product topology. Furthermore, we set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$. This is the unique Borel probability measure on X_τ such that for each cylinder set $A = A_1 \times \cdots \times A_n \times \Gamma_\tau \times \Gamma_\tau \times \cdots$ in X_τ, $\tilde{\tau}(A) = \prod_{j=1}^{n} \tau(A_j)$. We denote by G_τ the subsemigroup of $\mathrm{CM}(Y)$ generated by the subset Γ_τ of $\mathrm{CM}(Y)$.

2. Let M_τ be the operator on $C(Y)$ defined by $M_\tau(\varphi)(z) := \int_{\Gamma_\tau} \varphi(g(z))\ d\tau(g)$. M_τ is called the **transition operator** of the Markov process induced by τ. Moreover, let $M_\tau^* : C(Y)^* \to C(Y)^*$ be the dual of M_τ, which is defined as $M_\tau^*(\mu)(\varphi) = \mu(M_\tau(\varphi))$ for each $\mu \in C(Y)^*$ and each $\varphi \in C(Y)$. Remark: we have $M_\tau^*(\mathfrak{M}_1(Y)) \subset \mathfrak{M}_1(Y)$ and for each $\mu \in \mathfrak{M}_1(Y)$ and each open subset V of Y, we have $M_\tau^*(\mu)(V) = \int_{\Gamma_\tau} \mu(g^{-1}(V))\ d\tau(g)$.

3. We denote by $F_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(Y)$ satisfying that there exists a neighborhood B of μ in $\mathfrak{M}_1(Y)$ such that the sequence $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous on B. We set $J_{meas}(\tau) := \mathfrak{M}_1(Y) \setminus F_{meas}(\tau)$.

4. We denote by $F^0_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(Y)$ satisfying that $\{(M_\tau^*)^n : \mathfrak{M}_1(Y) \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous at the one point μ. Note that $F_{meas}(\tau) \subset F^0_{meas}(\tau)$.

5. We set $J^0_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F^0_{meas}(\tau)$.

Remark 3.6. We have $F_{meas}(\tau) \subset F^0_{meas}(\tau)$ and $J^0_{meas}(\tau) \subset J_{meas}(\tau)$.

Remark 3.7. Let Γ be a closed subset of an open subset $\mathcal{U}$ of Rat. Then there exists a $\tau \in \mathfrak{M}_1(\mathcal{U})$ such that $\operatorname{supp}\tau$ (in the sense of Definition 3.4) is equal to Γ. By using this fact, we sometimes apply the results on random complex dynamics to the study of the dynamics of rational semigroups.

Definition 3.8. Let Y be a compact metric space. Let $\Phi : Y \to \mathfrak{M}_1(Y)$ be the topological embedding defined by: $\Phi(z) := \delta_z$, where δ_z denotes the Dirac measure at z. Using this topological embedding $\Phi : Y \to \mathfrak{M}_1(Y)$, we regard Y as a compact subset of $\mathfrak{M}_1(Y)$.

Remark 3.9. If $h \in \text{Rat}$ and $\tau = \delta_h$, then we have $M_\tau^* \circ \Phi = \Phi \circ h$ on $\hat{\mathbb{C}}$. Moreover, for a general $\tau \in \mathfrak{M}_1(\text{Rat})$, $M_\tau^*(\mu) = \int h_*(\mu) d\tau(h)$ for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Therefore, for a general $\tau \in \mathfrak{M}_1(\text{Rat})$, the map $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$.

Definition 3.10. Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_1(\text{CM}(Y))$. Regarding Y as a compact subset of $\mathfrak{M}_1(Y)$ as in Definition 3.8, we use the following notation.

1. We denote by $F_{pt}(\tau)$ the set of $z \in Y$ satisfying that there exists a neighborhood B of z in Y such that the sequence$\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous on B. We set $J_{pt}(\tau) := Y \setminus F_{pt}(\tau)$.

2. Similarly, we denote by $F^0_{pt}(\tau)$ the set of $z \in Y$ such that the sequence $\{(M_\tau^*)^n|_Y : Y \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous at the one point $z \in Y$. We set $J^0_{pt}(\tau) := Y \setminus F^0_{pt}(\tau)$.

Also, the set $J_{\ker}(G_\tau)$ is called the **kernel Julia set of** τ.

Remark 3.11. We have $F_{pt}(\tau) \subset F^0_{pt}(\tau)$ and $J^0_{pt}(\tau) \subset J_{pt}(\tau) \cap J^0_{meas}(\tau)$ (regarding Y as a compact subset of $\mathfrak{M}_1(Y)$ by using the topological embedding $\Phi : Y \to \mathfrak{M}_1(Y)$).

Remark 3.12. If $\tau = \delta_h \in \mathfrak{M}_1(\text{Rat}_+)$ with $h \in \text{Rat}_+$, then $J^0_{pt}(\tau)$ and $J_{meas}(\tau)$ are uncountable. In fact, we have $\emptyset \neq J(h) \subset J^0_{pt}(\tau)$ and $J(h)$ is uncountable.

Lemma 3.13. *Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_1(\text{CM}(Y))$. Let $y \in Y$. Suppose $\tilde{\tau}(\{\gamma \in (\text{CM}(Y))^{\mathbb{N}} \mid y \in J_{\gamma,0}\}) = 0$. Then $y \in F^0_{pt}(\tau)$.*

Proof. By (6) in Lemma 3.3 and the assumption of our lemma, we obtain that for $\tilde{\tau}$-a.e.$\gamma \in (\text{CM}(Y))^{\mathbb{N}}$, $\lim_{m\to\infty} \sup_{n\geq 1} \operatorname{diam}(\gamma_{n,1}(B(y, \frac{1}{m}))) = 0$. Let $\epsilon > 0$. By Egoroff's theorem, there exists a Borel subset A_1 of X_τ with $\tilde{\tau}(X_\tau \setminus A_1) < \epsilon$ such that

$$\sup_{n\geq 1} \operatorname{diam}(\gamma_{n,1}(B(y, \frac{1}{m}))) \to 0 \tag{7}$$

as $m \to \infty$ uniformly on A_1. Let $\varphi \in C(Y)$. Then there exists a $\delta_1 > 0$ such that if $d(z_1, z_2) < \delta_1$ then $|\varphi(z_1) - \varphi(z_2)| < \epsilon$. By (7), there exists a $\delta_2 > 0$ such that for each $z \in Y$ with $d(z, y) < \delta_2$, for each $\gamma \in A_1$, and for each $n \in \mathbb{N}$, we have $d(\gamma_{n,1}(z), \gamma_{n,1}(y)) < \delta_1$. Therefore for each $z \in Y$ with $d(z, y) < \delta_2$, we have

$$\begin{aligned}
|M_\tau^n(\varphi)(z) - M_\tau^n(\varphi)(y)| &\leq \int_{A_1} |\varphi(\gamma_{n,1}(z) - \varphi(\gamma_{n,1}(y))| d\tilde{\tau}(\gamma) + \int_{X_\tau \setminus A_1} |\varphi(\gamma_{n,1}(z)) - \varphi(\gamma_{n,1}(y))| d\tilde{\tau}(\gamma) \\
&\leq \tilde{\tau}(A_1) \cdot \epsilon + 2\epsilon \cdot \sup_{a \in \hat{\mathbb{C}}} |\varphi(a)| \\
&\leq \epsilon(1 + 2\|\varphi\|_\infty).
\end{aligned}$$

It follows that $y \in F_{pt}^0(\tau)$. Thus we have proved our lemma. $\square$

For a smooth Riemannian real manifold Y with $\dim Y = p$, we denote by Leb_p the (p-dimensional) Lebesgue measure on Y.

Corollary 3.14. *Let Y be a compact smooth manifold with $\dim(Y) = p$ and let $\tau \in \mathfrak{M}_1(\text{CM}(Y))$. Suppose that for $\tilde{\tau}$-a.e.$\gamma \in (\text{CM}(Y))^{\mathbb{N}}$, $Leb_p(J_{\gamma,0}) = 0$. Then $Leb_p(J_{pt}^0(\tau)) = 0$.*

Proof. Under the assumptions of our corollary, Lemma 3.3 and Fubini's theorem imply that for Leb_p-a.e.$y \in Y$, we have $\tilde{\tau}(\{\gamma \in (\text{CM}(Y))^{\mathbb{N}} \mid y \in J_{\gamma,0}\}) = 0$. By Lemma 3.13, it follows that for Leb_p-a.e.$y \in Y$, $y \in F_{pt}^0(\tau)$. Thus we have proved our corollary. $\square$

The following lemma is very important and useful to prove many results.

Lemma 3.15. *Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_1(\text{CM}(Y))$. Let V be a non-empty open subset of Y. Suppose that for each $g \in \Gamma_\tau$, $g(V) \subset V$. Let $L_{\ker} := \cap_{g \in G_\tau} g^{-1}(Y \setminus V)$. Let $y \in Y$ and let $E := \{\gamma \in X_\tau \mid y \in \cap_{j=1}^\infty \gamma_{j,1}^{-1}(Y \setminus V)\}$. Then for $\tilde{\tau}$-a.e.$\gamma \in E$, we have $d(\gamma_{n,1}(y), L_{\ker}) \to 0$ as $n \to \infty$.*

Proof. For each $\delta > 0$ and $n \in \mathbb{N}$, let $A(\delta, n) := \{\gamma \in E \mid \gamma_{n,1}(y) \in (Y \setminus V) \setminus B(L_{\ker}, \delta)\}$ and $C(\delta) := \{\gamma \in E \mid \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0, \gamma_{n,1}(y) \in B(L_{\ker}, \delta)\}$. In order to prove our lemma, it suffices to show that

$$\tilde{\tau}(E \setminus C(\delta)) = 0 \text{ for each } \delta > 0. \tag{8}$$

Since $E \setminus C(\delta) = \cap_{N=1}^\infty \cup_{n=N}^\infty A(\delta, n)$, we have

$$\tilde{\tau}(E \setminus C(\delta)) = \lim_{N \to \infty} \tilde{\tau}(\cup_{n=N}^\infty A(\delta, n)) \leq \lim_{N \to \infty} \sum_{n=N}^\infty \tilde{\tau}(A(\delta, n)).$$

Thus, in order to show (8), it suffices to prove that

$$\sum_{n=1}^\infty \tilde{\tau}(A(\delta, n)) < \infty \text{ for each } \delta > 0. \tag{9}$$

In order to prove (9), let $\delta > 0$. Then for each $z \in (Y \setminus V) \setminus B(L_{\ker}, \delta)$, there exists an element $g_z \in G$ and a neighborhood U_z of z in Y such that $g_z(\overline{U_z}) \subset V$. Since $H := (Y \setminus V) \setminus B(L_{\ker}, \delta)$ is compact, there exist finitely many points $z_1, \ldots, z_r \in Y$ such that $H \subset \cup_{j=1}^r U_{z_j}$. Since $g(V) \subset V$ for each $g \in \Gamma_\tau$, we may assume that there exists an $l \in \mathbb{N}$ such that for each $j = 1, \ldots, r$, there exists an element $\gamma^j = (\gamma_1^j, \ldots, \gamma_l^j) \in \Gamma_\tau^l$ with $g_{z_j} = \gamma_l^j \circ \cdots \circ \gamma_1^j$. Then for each $j = 1, \ldots, r$, there exists a neighborhood W_j of γ^j in Γ_τ^l such that for each $\alpha = (\alpha_1, \ldots, \alpha_l) \in W_j$, $\alpha_l \circ \cdots \circ \alpha_1(\overline{U_{z_j}}) \subset V$. Let $\delta_0 := \min_{j=1}^r \tau^l(W_j)$, where $\tau^l = \otimes_{n=1}^l \tau \in \mathfrak{M}_1((\text{CM}(Y))^l)$. For each $i = 0, 1, \ldots, l-1$ and for each $n \in \mathbb{N}$, let

$$H(\delta, i, n) := \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid \gamma_{i+nl,1}(y) \in (Y \setminus V) \setminus B(L_{\ker}, \delta), \gamma_{i+(n+1)l,1}(y) \in V\}$$

and

$$I(\delta, i, n) := \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid \gamma_{i+nl,1}(y) \in (Y \setminus V) \setminus B(L_{\ker}, \delta)\}.$$

Note that if $n \neq m$ then $H(\delta, i, n) \cap H(\delta, i, m) = \emptyset$. Let $Q_1, \ldots, Q_s$ be mutually disjoint Borel subsets of $(Y \setminus V) \setminus B(L_{\ker}, \delta)$ such that $(Y \setminus V) \setminus B(L_{\ker}, \delta) = \cup_{p=1}^{s} Q_p$ and such that for each $p = 1, \ldots s$ there exists a $j(p) \in \{1, \ldots, r\}$ with $Q_p \subset U_{z_{j(p)}}$. Then for each $i = 0, 1 \ldots, l-1$, we have

$$\begin{aligned}
\tilde{\tau}(H(\delta, i, n)) &= \tau^{i+(n+1)l}\Big(\coprod_{p=1}^{s}\{\gamma \in \Gamma_\tau^{i+(n+1)l} \mid \gamma_{i+nl,1}(y) \in Q_p, \gamma_{i+(n+1)l,1}(y) \in V\}\Big) \\
&= \sum_{p=1}^{s} \tau^{i+(n+1)l}(\{\gamma \in \Gamma_\tau^{i+(n+1)l} \mid \gamma_{i+nl,1}(y) \in Q_p, \gamma_{i+(n+1)l,1}(y) \in V\}) \\
&\geq \sum_{p=1}^{s} \tau^{i+(n+1)l}(\{\gamma \in \Gamma_\tau^{i+(n+1)l} \mid \gamma_{i+nl,1}(y) \in Q_p, (\gamma_{i+nl+1} \ldots, \gamma_{i+(n+1)l}) \in W_{j(p)}\}) \\
&= \sum_{p=1}^{s} \tau^{i+nl}(\{\gamma \in \Gamma_\tau^{i+nl} \mid \gamma_{i+nl,1}(y) \in Q_p\}) \cdot \tau^l(W_{j(p)}) \\
&\geq \delta_0 \tilde{\tau}(I(\delta, i, n)),
\end{aligned}$$

where $\coprod$ denotes the disjoint union. Therefore

$$\begin{aligned}
1 &\geq \tilde{\tau}\Big(\bigcup_{n \in \mathbb{N}} \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid \gamma_{n,1}(y) \in V\}\Big) \\
&\geq \tilde{\tau}\Big(\bigcup_{n=1}^{\infty} H(\delta, i, n)\Big) = \sum_{n=1}^{\infty} \tilde{\tau}(H(\delta, i, n)) \geq \sum_{n=1}^{\infty} \delta_0 \tilde{\tau}(I(\delta, i, n)).
\end{aligned}$$

Thus $\sum_{n=1}^{\infty} \tilde{\tau}(I(\delta, i, n)) < \infty$ for each $i = 0, 1, \ldots, l-1$. Hence

$$\sum_{n=1}^{\infty} \tilde{\tau}(A(\delta, n)) = \sum_{i=0}^{l-1} \sum_{n=1}^{\infty} \tilde{\tau}(I(\delta, i, n)) < \infty.$$

Therefore (9) holds. Thus we have proved our lemma. □

3.2 Systems with hyperbolic kernel Julia sets

In this subsection, we show a result on random complex dynamical systems with hyperbolic kernel Julia sets.

For a holomorphic map $\varphi : U \to \hat{\mathbb{C}}$ defined on an open subset U of $\hat{\mathbb{C}}$ and for any $z \in U$, we denote by $D\varphi_z : T_z U \to T_{\varphi(z)} \hat{\mathbb{C}}$ the complex differential map of φ at z, where $T_z U$ denotes the complex tangential space of U at z and $T_{\varphi(z)} \hat{\mathbb{C}}$ denotes the complex tangential space of $\hat{\mathbb{C}}$ at $\varphi(z)$. Also, we denote by $\|D\varphi_z\|_s$ the norm of $D\varphi_z$ with respect to the spherical metric on $\hat{\mathbb{C}}$.

Definition 3.16. Let Y be a compact metric space and let Γ be a non-empty subset of $\mathrm{CM}(Y)$. We endow Γ with the relative topology from $\mathrm{CM}(Y)$. We define a map $f : \Gamma^{\mathbb{N}} \times Y \to \Gamma^{\mathbb{N}} \times Y$ as follows: For a point $(\gamma, y) \in \Gamma^{\mathbb{N}} \times Y$ where $\gamma = (\gamma_1, \gamma_2, \ldots)$, we set $f(\gamma, y) := (\sigma(\gamma), \gamma_1(y))$, where $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map, that is, $\sigma(\gamma_1, \gamma_2, \ldots) = (\gamma_2, \gamma_3, \ldots)$. The map $f : \Gamma^{\mathbb{N}} \times Y \to \Gamma^{\mathbb{N}} \times Y$ is called the **skew product associated with the generator system** Γ. Moreover, we use the following notation.

1. Let $\pi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_Y : \Gamma^{\mathbb{N}} \times Y \to Y$ be the canonical projections. For each $\gamma \in \Gamma^{\mathbb{N}}$ and $n \in \mathbb{N}$, we set $f_\gamma^n := f^n|_{\pi^{-1}\{\gamma\}} : \pi^{-1}\{\gamma\} \to \pi^{-1}\{\sigma^n(\gamma)\}$. Moreover, we set $f_{\gamma,n} := \gamma_n \circ \cdots \circ \gamma_1$.

2. We set $\tilde{J}(f) := \overline{\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} J^{\gamma}}$, where the closure is taken in the product space $\Gamma^{\mathbb{N}}\times Y$. Furthermore, we set $\tilde{F}(f) := (\Gamma^{\mathbb{N}} \times Y) \setminus \tilde{J}(f)$.

3. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $\hat{J}^{\gamma,\Gamma} := \pi^{-1}\{\gamma\} \cap \tilde{J}(f)$, $\hat{F}^{\gamma,\Gamma} := \pi^{-1}(\{\gamma\}) \setminus \hat{J}^{\gamma,\Gamma}$, $\hat{J}_{\gamma,\Gamma} := \pi_Y(\hat{J}^{\gamma,\Gamma})$, and $\hat{F}_{\gamma,\Gamma} := Y \setminus \hat{J}_{\gamma,\Gamma}$. Note that $J_\gamma \subset \hat{J}_{\gamma,\Gamma}$.

4. When $\Gamma \subset \mathrm{Rat}$, for each $z = (\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$, we set $Df_z := D(\gamma_1)_y$.

Remark 3.17. Under the above notation, let $G = \langle\Gamma\rangle$. Then $\pi_Y(\tilde{J}(f)) \subset J(G)$ and $\pi \circ f = \sigma \circ \pi$ on $\Gamma^{\mathbb{N}} \times Y$. Moreover, for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_1(J_\gamma) \subset J_{\sigma(\gamma)}$, $\gamma_1(\hat{J}_{\gamma,\Gamma}) \subset \hat{J}_{\sigma(\gamma),\Gamma}$, and $f(\tilde{J}(f)) \subset \tilde{J}(f)$. Furthermore, if $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$, then for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_1(J_\gamma) = J_{\sigma(\gamma)}$, $\gamma_1^{-1}(J_{\sigma(\gamma)}) = J_\gamma$, $\gamma_1(\hat{J}_{\gamma,\Gamma}) = \hat{J}_{\sigma(\gamma),\Gamma}$, $\gamma_1^{-1}(\hat{J}_{\sigma(\gamma),\Gamma}) = \hat{J}_{\gamma,\Gamma}$, $f(\tilde{J}(f)) = \tilde{J}(f) = f^{-1}(\tilde{J}(f))$, and $f(\tilde{F}(f)) = \tilde{F}(f) = f^{-1}(\tilde{F}(f))$ (see [31, Lemma 2.4]).

Definition 3.18. Let Γ be a subset of Rat. Let $G = \langle\Gamma\rangle$. We say that a subset A of $J(G)$ is a **hyperbolic set for** Γ if there are constants $C > 0$ and $\lambda > 1$ such that for each $n \in \mathbb{N}$, for each $z \in A$, and for each $\gamma \in \Gamma^{\mathbb{N}}$, we have $\|D(\gamma_{n,1})_z\|_s \geq C\lambda^n$.

We now show a result on the case that $J_{\ker}(G_\tau)$ is a hyperbolic set for Γ_τ.

Proposition 3.19. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $F(G_\tau) \neq \emptyset$ and $J_{\ker}(G_\tau)$ is a hyperbolic set for Γ_τ. Then for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, we have $\mathrm{Leb}_2(\hat{J}_{\gamma,\Gamma_\tau}) = 0$, where Leb_2 denotes the 2-dimensional Lebesgue measure on $\hat{\mathbb{C}}$. Moreover, $\mathrm{Leb}_2(J^0_{pt}(\tau)) = 0$.*

Proof. Suppose that the statement "for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, $\mathrm{Leb}_2(\hat{J}_{\gamma,\Gamma_\tau}) = 0$" is not true. Then since $(\sigma, \tilde{\tau})$ is ergodic, we have for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, $\mathrm{Leb}_2(\hat{J}_{\gamma,\Gamma_\tau}) > 0$. Let $V = F(G_\tau)$. Applying Lemma 3.15, we obtain that

$$(\tilde{\tau} \otimes \mathrm{Leb}_2)(\{(\gamma, y) \in \tilde{J}(f) \mid d(\gamma_{n,1}(y), J_{\ker}(G_\tau)) \not\to 0 \text{ as } n \to \infty\}) = 0.$$

Therefore for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, for Leb_2-a.e.$y \in \hat{J}_{\gamma,\Gamma_\tau}$,we have $d(\gamma_{n,1}(y), J_{\ker}(G_\tau)) \to 0$ as $n \to \infty$. Thus for $\tilde{\tau}$-a.e.γ, there exists a Lebesgue point y of $\hat{J}_{\gamma,\Gamma_\tau}$ such that $d(\gamma_{n,1}(y), J_{\ker}(G_\tau)) \to 0$ as $n \to \infty$. Let (γ, y) be such an element. We may assume that $D(\gamma_{n,1})_y \neq 0$ for each $n \in \mathbb{N}$. Since $J_{\ker}(G_\tau)$ is a hyperbolic set for Γ_τ, we obtain that

$$y \in J_\gamma. \tag{10}$$

Moreover, since $J_{\ker}(G_\tau)$ is a hyperbolic set for Γ_τ and Γ_τ is compact, we have that there exists a constant $\delta > 0$ such that for each $z \in J_{\ker}(G_\tau)$ and for each $g \in \Gamma_\tau$, $g : B(z, 2\delta) \to \hat{\mathbb{C}}$ is injective. Let $n_0 \in \mathbb{N}$ be an element such that

$$\gamma_{n,1}(y) \in B(J_{\ker}(G_\tau), \delta) \quad \text{for each } n \text{ with } n \geq n_0. \tag{11}$$

Combing (10), (11), that y is a Lebesgue point of $\hat{J}_{\gamma,\Gamma_\tau}$, the assumption that $J_{\ker}(G_\tau)$ is a hyperbolic set for Γ_τ and Koebe's distortion theorem, we obtain that there exists an $r > 0$ such that

$$\frac{\mathrm{Leb}_2(\hat{J}_{\sigma^n(\gamma),\Gamma_\tau} \cap B(\gamma_{n,1}(y), r))}{\mathrm{Leb}_2(B(\gamma_{n,1}(y), r))} \to 1 \text{ as } n \to \infty.$$

Therefore there exist a point $z \in J_{\ker}(G_\tau)$ and an element $\alpha \in \Gamma_\tau^{\mathbb{N}}$ such that $B(z, r) \subset \hat{J}_{\alpha,\Gamma_\tau}$. It follows that $\alpha_{n,1}(B(z, r)) \subset J(G_\tau)$ for each $n \in \mathbb{N}$. Since we are assuming $F(G_\tau) \neq \emptyset$, we obtain that $B(z, r) \subset F_\alpha$. However, it contradicts that $J_{\ker}(G_\tau)$ is a hyperbolic set for Γ_τ. Thus we have proved our proposition. $\square$

3.3 Minimal sets with finite cardinality and related lemmas

In this subsection, we show some lemmas regarding random dynamical systems having minimal sets with finite cardinality.

Definition 3.20. For a topological space Y, we denote by $\text{Cpt}(Y)$ the space of all non-empty compact subsets of Y. If Y is a metric space, we endow $\text{Cpt}(Y)$ with the Hausdorff metric.

Definition 3.21. Let G be a rational semigroup. Let $Y \in \text{Cpt}(\hat{\mathbb{C}})$ be such that $G(Y) \subset Y$. Let $K \in \text{Cpt}(\hat{\mathbb{C}})$. We say that K is a **minimal set for** (G, Y) if K is minimal among the space $\{L \in \text{Cpt}(Y) \mid G(L) \subset L\}$ with respect to inclusion. Moreover, we denote by $\text{Min}(G, Y)$ the set of all minimal sets for (G, Y).

Remark 3.22. Let G be a rational semigroup. By Zorn's lemma, it is easy to see that if $K_1 \in \text{Cpt}(\hat{\mathbb{C}})$ and $G(K_1) \subset K_1$, then there exists a $K \in \text{Min}(G, \hat{\mathbb{C}})$ with $K \subset K_1$. Moreover, it is easy to see that for each $K \in \text{Min}(G, \hat{\mathbb{C}})$ and each $z \in K$, $\overline{G(z)} = K$. In particular, if $K_1, K_2 \in \text{Min}(G, \hat{\mathbb{C}})$ with $K_1 \neq K_2$, then $K_1 \cap K_2 = \emptyset$. Moreover, by the formula $\overline{G(z)} = K$, we obtain that for each $K \in \text{Min}(G, \hat{\mathbb{C}})$, either (1) $\sharp K < \infty$ or (2) K is perfect and $\sharp K > \aleph_0$. Furthermore, it is easy to see that if $\Gamma \in \text{Cpt}(\text{Rat}), G = \langle \Gamma \rangle$, and $K \in \text{Min}(G, \hat{\mathbb{C}})$, then $K = \bigcup_{h \in \Gamma} h(K)$.

Remark 3.23. In [36, Remark 3.9], for the statement "for each $K \in \text{Min}(G, Y)$, either (1) $\sharp K < \infty$ or (2) K is perfect", we should assume that each element $g \in G$ is a finite-to-one map.

We now show some lemmas on the minimal sets whose cardinalities are finite (Lemmas 3.25, 3.26).

Definition 3.24. Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_{1,c}(\text{CM}(Y))$. For each $r \in \mathbb{N}$, we set $G_\tau^r := \langle \{g_1 \circ \cdots \circ g_r \mid g_1, \ldots, g_r \in \Gamma_\tau\} \rangle$.

Lemma 3.25. *Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_{1,c}(\text{CM}(Y))$. Let K be a nonempty finite subset of Y. Suppose that $G_\tau(K) \subset K$. Let $\{K_i\}_{i=1}^q = \text{Min}(G_\tau, K)$ where $q = \sharp \text{Min}(G_\tau, K)$. For each $i = 1, \ldots q$, let $p_i \in \mathbb{N}$ be the period of the finite Markov chain with state space K_i induced by τ (i.e. the finite Markov chain with state space K_i whose transition probability $p(x, A)$ from $x \in K_i$ to $A \subset K_i$ satisfies $p(x, A) = \tau(\{g \in \Gamma_\tau \mid g(x) \in A\})$). (For the definition of "period", see [9, p308]). Let $m = \prod_{i=1}^q p_i \in \mathbb{N}$. Let $\{H_j\}_{j=1}^r = \text{Min}(G_\tau^m, K)$ where $r = \sharp(\text{Min}(G_\tau^m, K))$. Then all of the following hold.*

1. *Let $i = 1, \ldots, q$. Then $\sharp(\text{Min}(G_\tau^{p_i}, K_i)) = p_i$. Moreover, there exist $K_{i,1}, \ldots K_{i,p_i} \in \text{Min}(G_\tau^{p_i}, K_i)$ such that $\{K_{i,k}\}_{k=1}^{p_j} = \text{Min}(G_\tau^{p_i}, K_i)$, $K_i = \cup_{k=1}^{p_i} K_{i,k}$ and $h(K_{i,k}) \subset K_{i,k+1}$ for each $h \in \Gamma_\tau$, where $K_{i,p_i+1} := K_{i,1}$. Also, for each $k = 1, \ldots, p_i$ there exists a unique element $\omega_{i,k} \in \mathfrak{M}_1(K_{i,k})$ such that $(M_\tau^*)^{p_i}(\omega_{i,k}) = \omega_{i,k}$. Also, $M_\tau^{np_i}(\varphi) \to (\int \varphi \, d\omega_{i,k}) 1_{K_{i,k}}$ in $C(K_{i,k})$ as $n \to \infty$ for each $\varphi \in C(K_{i,k})$, $\operatorname{supp} \omega_{i,k} = K_{i,k}$ and $M_\tau^* \omega_{i,k} = \omega_{i,k+1}$ in $\mathfrak{M}_1(K_i)$ for each $k = 1, \ldots, p_i$, where $\omega_{i,p_i+1} := \omega_{i,1}$.*

2. *We have $r = \sum_{i=1}^q p_i$ and $\cup_{j=1}^r H_j = \cup_{i=1}^q K_i$. Moreover, we have that $\{H_j \mid j = 1, \ldots, r\} = \{K_{i,k} \mid i = 1, \ldots, q, k = 1, \ldots, p_i\} = \text{Min}(G_\tau^{nm}, K)$ for each $n \in \mathbb{N}$. Moreover, for each $j = 1, \ldots, r$, there exists a unique Borel probability measure η_j on H_j such that $(M_\tau^m)^*(\eta_j) = \eta_j$. Also, $M_\tau^{nm}(\varphi) \to (\int \varphi \, d\eta_j) \cdot 1_{H_j}$ in $C(H_j)$ as $n \to \infty$ for each $\varphi \in C(H_j)$. Also, $\operatorname{supp} \eta_j = H_j$ for each $j = 1, \ldots, r$. Moreover, if $H_j = K_{i,k}$, then $\eta_j = \omega_{i,k}$.*

3. *Let $y \in Y$ and let Ω be a Borel subset of X_τ. Let $A := \{\gamma \in \Omega \mid d(\gamma_{n,1}(y), K) \to 0 \ (n \to \infty)\}$ and $A_j := \{\gamma \in \Omega \mid d(\gamma_{nm,1}(y), H_j) \to 0 \ (n \to \infty)\}$ for each $j = 1, \ldots, r$. Then for each $\varphi \in C(Y)$, we have $\int_A \varphi(\gamma_{nm,1}(y)) d\tilde{\tau}(\gamma) \to \sum_{j=1}^r \tilde{\tau}(A_j) \int \varphi d\eta_j$ as $n \to \infty$.*

Proof. By [9, Theorem 6.6.4 and Lemma 6.7.1], it is easy to see that statements (1)(2) hold. In order to prove statement (3), let $\varphi \in C(Y)$ and let $j \in \{1, \ldots, r\}$. Let $\epsilon > 0$. Then there exists a $\delta_1 > 0$ with $\delta_1 < \frac{1}{2}\min\{d(a,b) \mid a, b \in H_j, a \neq b\}$ such that

$$\text{if } d(z_1, z_2) < \delta_1 \text{ then } |\varphi(z_1) - \varphi(z_2)| < \epsilon. \tag{12}$$

Let $c \in (0,1)$ such that for each $j = 1, \ldots, r$, for each $z \in H_j$, for each $w \in \hat{\mathbb{C}}$ with $d(w,z) < c\delta_1$ and for each $(g_1, \ldots, g_m) \in \Gamma_\tau^m$, we have

$$d(g_1 \circ \cdots \circ g_m(w), g_1 \circ \cdots \circ g_m(z)) < \delta_1. \tag{13}$$

By Egorov's theorem, there exist a Borel measurable subset $A_{j,\epsilon}$ of A_j and a positive integer n_0 such that

$$\tilde{\tau}(A_j \setminus A_{j,\epsilon}) < \epsilon \text{ and for each } \gamma \in A_{j,\epsilon}, \text{ for each } n \text{ with } n \geq n_0, \text{ we have } \gamma_{nm,1}(y) \in B(H_j, c\delta_1). \tag{14}$$

For each $z \in B(H_j, \delta_1)$, let $a(z) \in H_j$ be the point such that $d(z, H_j) = d(z, a(z))$. Since Γ_τ is compact, there exists a compact subset $E_{j,\epsilon}$ of $A_{j,\epsilon}$ such that

$$\tilde{\tau}(A_{j.\epsilon} \setminus E_{j,\epsilon}) < \epsilon. \tag{15}$$

For each $s \in \mathbb{N}$, we set

$$E_{j,\epsilon,s} = \{\gamma \in X_\tau \mid \exists(\alpha_{s+1}, \alpha_{s+2}, \ldots) \in X_\tau \text{ s.t. } (\gamma_1, \ldots, \gamma_s, \alpha_{s+1}, \alpha_{s+2}, \ldots) \in E_{j,\epsilon}\}.$$

Note that denoting by $\pi_s : X_\tau \to \Gamma_\tau^s$ the canonical projection, we have $E_{j,\epsilon,s} = \pi_s^{-1}(\pi_s(E_{j,\epsilon}))$. Moreover, $E_{j,\epsilon,s}$ is a Borel measurable subset of X_τ and we have $E_{j.\epsilon,s} \supset E_{j,\epsilon,s+1}$ for each $s \in \mathbb{N}$. Furthermore, $E_{j,\epsilon} = \cap_{s=1}^{\infty} E_{j,\epsilon,s}$. Hence there exists an $s_0 \in \mathbb{N}$ such that

$$\text{for each } s \in \mathbb{N} \text{ with } s \geq s_0, \text{ we have } \tilde{\tau}(E_{j,\epsilon,s} \setminus E_{j,\epsilon}) < \epsilon. \tag{16}$$

By (14), we have

$$|\int_{A_j} \varphi(\gamma_{nm,1}(y)) d\tilde{\tau}(\gamma) - \int_{A_{j,\epsilon}} \varphi(\gamma_{nm,1}(y)) d\tilde{\tau}(\gamma)| \leq \epsilon \|\varphi\|_\infty \text{ for each } n \geq n_0. \tag{17}$$

Moreover, by (12) and (14), we have

$$|\int_{A_{j,\epsilon}} \varphi(\gamma_{nm,1}(y)) d\tilde{\tau}(\gamma) - \int_{A_{j,\epsilon}} \varphi(a(\gamma_{nm,1}(y))) d\tilde{\tau}(\gamma)| < \epsilon \text{ for each } n \geq n_0. \tag{18}$$

By (13) and (14), for each $\gamma \in A_{j,\epsilon}$ and for each $l \in \mathbb{N}$, we have

$$a(\gamma_{(n_0+l)m,1}(y)) = \gamma_{(n_0+l)m,n_0m+1}(a(\gamma_{n_0m,1}(y))). \tag{19}$$

Let $n_1 := \max\{n_0, s_0\}$. By (15) and (16), we obtain that for each $s \geq n_1$ and $l \in \mathbb{N}$,

$$|\int_{A_{j,\epsilon}} \varphi(\gamma_{(n_1+l)m,n_1m+1}(a(\gamma_{n_1m,1}(y))) d\tilde{\tau}(\gamma) - \int_{E_{j,\epsilon,sm}} \varphi(\gamma_{(n_1+l)m,n_1m+1}(a(\gamma_{n_1m,1}(y))) d\tilde{\tau}(\gamma)| < 2\epsilon\|\varphi\|_\infty. \tag{20}$$

Moreover, for each $l \in \mathbb{N}$, we have

$$\begin{aligned} & \int_{E_{j,\epsilon,n_1m}} \varphi(\gamma_{(n_1+l)m.n_1m+1}(a(\gamma_{n_1m,1}(y)))) d\tilde{\tau}(\gamma) \\ = & \int_{\pi_{n_1m}(E_{j,\epsilon}) \times \Gamma_\tau \times \Gamma_\tau \times \cdots} \varphi(\gamma_{(n_1+l)m.n_1m+1}(a(\gamma_{n_1m,1}(y)))) d\tilde{\tau}(\gamma) \\ = & \int_{\pi_{n_1m}(E_{j,\epsilon})} M_\tau^{lm}(\varphi)(a(\gamma_{n_1m,1}(y))) d\tau^{n_1m}(\gamma). \end{aligned}$$

Since $M_\tau^{lm}(\varphi)(a(\gamma_{n_1m,1}(y))) \to \eta_j(\varphi)$ as $l \to \infty$ for each $\gamma \in \pi_{n_1m}(E_{j,\epsilon})$, it follows that

$$\int_{E_{j,\epsilon,n_1m}} \varphi(\gamma_{(n_1+l)m,n_1m+1}(a(\gamma_{n_1m,1}(y))))d\tilde{\tau}(\gamma) \to \tau^{n_1m}(\pi_{n_1m}(E_{j,\epsilon})) \cdot \eta_j(\varphi) = \tilde{\tau}(E_{j,\epsilon,n_1m}) \cdot \eta_j(\varphi) \tag{21}$$

as $l \to \infty$. Moreover, we have

$$\tilde{\tau}(A \setminus \cup_{j=1}^r A_j) = 0. \tag{22}$$

Combining (14)–(22), we obtain that $\int_A \varphi(\gamma_{nm,1}(y))d\tilde{\tau}(\gamma) \to \sum_{j=1}^r \tilde{\tau}(A_j)\eta_j(\varphi)$ as $n \to \infty$. □

Lemma 3.26. *Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{CM}(Y))$. Let V be a non-empty open subset of Y. Suppose that for each $g \in \Gamma_\tau$, $g(V) \subset V$. Let $L_{\ker} := \cap_{g \in G_\tau} g^{-1}(Y \setminus V)$. Suppose that $1 \le \sharp L_{\ker} < \infty$. Let $\{K_i\}_{i=1}^q = \mathrm{Min}(G_\tau, L_{\ker})$ where $q = \sharp \mathrm{Min}(G_\tau, L_{\ker})$. For each $i = 1, \ldots q$, let $p_i \in \mathbb{N}$ be the period of the finite Markov chain with state space K_i induced by τ. Let $m = \prod_{i=1}^q p_i \in \mathbb{N}$. Let $\{H_j\}_{j=1}^r = \mathrm{Min}(G_\tau^m, L_{\ker})$ where $r = \sharp(\mathrm{Min}(G_\tau^m, L_{\ker}))$. Then all of the following hold.*

(1) *Let $i = 1, \ldots, q$. Then $\sharp(\mathrm{Min}(G_\tau^{p_i}, K_i)) = p_i$. Moreover, there exist $K_{i,1}, \ldots K_{i,p_i} \in \mathrm{Min}(G_\tau^{p_i}, K_i)$ such that $\{K_{i,k}\}_{k=1}^{p_j} = \mathrm{Min}(G_\tau^{p_i}, K_i)$, $K_i = \cup_{k=1}^{p_i} K_{i,k}$ and $h(K_{i,k}) \subset K_{i,k+1}$ for each $h \in \Gamma_\tau$, where $K_{i,p_i+1} := K_{i,1}$. Also, for each $k = 1, \ldots, p_i$ there exists a unique element $\omega_{i,k} \in \mathfrak{M}_1(K_{i,k})$ such that $(M_\tau^*)^{p_i}(\omega_{i,k}) = \omega_{i,k}$. Also, $M_\tau^{np_i}(\varphi) \to (\int \varphi \; d\omega_{i,k})1_{K_{i,k}}$ in $C(K_{i,k})$ as $n \to \infty$ for each $\varphi \in C(K_{i,k})$, $\mathrm{supp}\, \omega_{i,k} = K_{i,k}$ and $M_\tau^* \omega_{i,k} = \omega_{i,k+1}$ in $\mathfrak{M}_1(K_i)$ for each $k = 1, \ldots, p_i$, where $\omega_{i,p_i+1} := \omega_{i,1}$.*

(2) *We have $r = \sum_{i=1}^q p_i$ and $\cup_{j=1}^r H_j = \cup_{i=1}^q K_i$. Moreover, we have that $\{H_j \mid j = 1, \ldots, r\} = \{K_{i,k} \mid i = 1, \ldots, q, k = 1, \ldots, p_i\} = \mathrm{Min}(G_\tau^{nm}, K)$ for each $n \in \mathbb{N}$. Moreover, for each $j = 1, \ldots, r$, there exists a unique Borel probability measure η_j on H_j such that $(M_\tau^m)^*(\eta_j) = \eta_j$. Also, $M_\tau^{nm}(\varphi) \to (\int \varphi \; d\eta_j) \cdot 1_{H_j}$ in $C(H_j)$ as $n \to \infty$ for each $\varphi \in C(H_j)$. Also, $\mathrm{supp}\, \eta_j = H_j$ for each $j = 1, \ldots, r$. Moreover, if $H_j = K_{i,k}$, then $\eta_j = \omega_{i,k}$.*

(3) *Let $y \in Y$. and let Ω be a Borel subset of X_τ. Let $A := \{\gamma \in \Omega \mid y \in \cap_{j=1}^\infty \gamma_{j,1}^{-1}(Y \setminus V)\}$ and $A_j := \{\gamma \in A \mid d(\gamma_{nm,1}(y), H_j) \to 0 \; (n \to \infty)\}$ for each $j = 1, \ldots, r$. Then for each $\varphi \in C(Y)$, we have $\int_A \varphi(\gamma_{nm,1}(y))d\tilde{\tau}(\gamma) \to \sum_{j=1}^r \tilde{\tau}(A_j) \int \varphi d\eta_j$ as $n \to \infty$.*

Proof. By Lemmas 3.15 and 3.25, the statement of our lemma holds. □

3.4 Invariant measures and Lyapunov exponents

In this subsection, we define invariant measures and the Lyapunov exponents for random dynamical systems generated by elements of $\mathfrak{M}_1(\mathrm{Rat})$. Also, We show some results on random complex dynamical systems having minimal sets with non-zero Lyapunov exponents.

We now define τ-invariant measures, τ-ergodic measures and the Lyapunov exponents for $\tau \in \mathfrak{M}_1(\mathrm{Rat})$.

Definition 3.27. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$. Let $\rho \in \mathfrak{M}_1(\hat{\mathbb{C}})$. We say that ρ is **τ-invariant** if $M_\tau^*(\rho) = \rho$. Moreover, we say that a τ-invariant meassusure ρ is **τ-ergodic** if A is a Borel subset of $\hat{\mathbb{C}}$ with $\rho(A) > 0$ and $M_\tau(1_A)(z) = 1_A(z)$ for ρ-a.e.$z \in \hat{\mathbb{C}}$, then $\rho(A) = 1$. For a τ-ergodic measure ρ, we set $\chi(\tau, \rho) := \int \log \|Df_z\|_s d(\tilde{\tau} \otimes \rho)(z)$, where $f : X_\tau \times \hat{\mathbb{C}} \to X_\tau \times \hat{\mathbb{C}}$ denotes the skew product map associated with Γ_τ (see Definition 3.16). This is called the **Lyapunov exponent of** (τ, ρ).

Remark 3.28. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$. Let $\rho \in \mathfrak{M}_1(\hat{\mathbb{C}})$ be a τ-invariant measure. Let $f : X_\tau \times \hat{\mathbb{C}} \to X_\tau \times \hat{\mathbb{C}}$ be the skew product map associated with Γ_τ. Then by [25, Lemma 3.1], the measure $\tilde{\tau} \otimes \rho \in \mathfrak{M}_1(X_\tau \times \hat{\mathbb{C}})$ is f-invariant. Also, by [25, Theorem 4.1], if ρ is τ-ergodic, then $\tilde{\tau} \otimes \rho$ is ergodic with respect to f.

Definition 3.29. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $\sharp L < \infty$. Let $m \in \mathbb{N}$ be the period of the finite Markov chain with state space L induced by τ (see [9, p. 300]). Then by [9, Theorem 6.6.4 and Lemma 6.7.1] we have the following.

- $\sharp\mathrm{Min}(G_\tau^m, L) = m$ and denoting by $\{L_j\}_{j=1}^m = \mathrm{Min}(G_\tau^m, L)$ we have $L = \cup_{j=1}^m L_j$.
- Renumbering $L_1, \ldots, L_m$, for each $j = 1, \ldots, m$ there exists a unique $\omega_{L,j} \in \mathfrak{M}_1(L_j)$ such that $M_\tau^{mn}(\varphi) \to (\omega_{L,j}(\varphi)) \cdot 1_{L_j}$ in $C(L_j)$ as $n \to \infty$ for each $\varphi \in C(L_j)$, $(M_\tau^m)^*(\omega_{L,j}) = \omega_{L,j}, \mathrm{supp}\, \omega_{L,j} = L_j$ and $M_\tau^* \omega_{L,j} = \omega_{L,j+1}$ where $\omega_{L,m+1} := \omega_{L,1}$.
- $\omega_L := \frac{1}{m}\sum_{j=1}^m \omega_{L,j}$ is τ-ergodic.

We call ω_L the **canonical τ-ergodic measure on** L. By [25, Lemma 3.1, Theorem 4.1], $\tilde{\tau} \otimes \omega_L \in \mathfrak{M}_1(X_\tau \times \hat{\mathbb{C}})$ is f-invariant and ergodic with respect to f, where $f : X_\tau \times \hat{\mathbb{C}} \to X_\tau \times \hat{\mathbb{C}}$ is the skew product map associated with Γ_τ. We set $\chi(\tau, L) := \int \log \|Df_z\|_s d(\tilde{\tau} \otimes \omega_L)(z)$. This is called the **Lyapunov exponent of** (τ, L).

We now show a lemma and its corollary on τ-invariant and τ-ergodic measures μ with negative Lyapunov exponents (Lemma 3.30 and Corollary 3.31).

Lemma 3.30. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ be a τ-invariant and τ-ergodic measure. Suppose $\chi(\tau, \mu) < 0$. Then for $(\tilde{\tau} \otimes \mu)$-a.e. $(\gamma, z_0) \in (\mathrm{Rat})^{\mathbb{N}} \times \hat{\mathbb{C}}$, we have $z_0 \in F_\gamma$. Moreover, for μ-a.e. $z_0 \in \hat{\mathbb{C}}$, we have $z_0 \in F_{pt}^0(\tau)$.*

Proof. For each $r \in \mathbb{N}$, let $\psi_r : \Gamma_\tau \times \hat{\mathbb{C}} \to \mathbb{R}$ be the function defined by

$$\psi_r(h, y) = \begin{cases} \log \|Dh_y\|_s & \text{if } \log \|Dh_y\|_s \geq -r \\ -r & \text{if } \log \|Dh_y\|_s < -r. \end{cases}$$

Let $\varphi_r : X_\tau \times \hat{\mathbb{C}} \to \mathbb{R}$ be the function defined by $\varphi_r(\gamma, y) = \psi_r(\gamma_1, y)$. Since $\chi(\tau, \mu) < 0$, there exists an $r \in \mathbb{N}$ such that $\int \varphi_r(z) d(\tilde{\tau} \otimes \mu)(z) < 0$. Let $c_0 = -\int \varphi_r(z) d(\tilde{\tau} \otimes \mu)(z) > 0$. By Birkhoff's ergodic theorem, there exists a Borel subset A of $X_\tau \times \hat{\mathbb{C}}$ with $(\tilde{\tau} \otimes \mu)(A) = 1$ such that for each $(\gamma, z_0) \in A$, $\frac{1}{n}\sum_{j=0}^{n-1} \varphi_r(f^j(\gamma, z_0)) \to -c_0$ as $n \to \infty$. Let $\epsilon_0 \in (0, \frac{1}{4}c_0)$. Let $(\gamma, z_0) \in A$. There exists an $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq n_0$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi_r(\gamma_{j+1}, \gamma_{j,1}(z_0)) = \frac{1}{n}\sum_{j=0}^{n-1} \varphi_r(f^j(\gamma, z_0)) \leq -c_0 + \epsilon_0,$$

where $\gamma_{0,1} = Id$. Let $\epsilon_1 \in \mathbb{R}$ with $0 < \epsilon_1 < \frac{1}{4}c_0$. Since Γ_τ is compact, there exists a $\delta > 0$ such that for each $w \in \hat{\mathbb{C}}$, for each $h \in \Gamma_\tau$ and for each $z \in B(w, \delta)$, we have

$$\log \|Dh_z\|_s \leq \psi_r(h, w) + \epsilon_1, \text{ thus } \|Dh_z\|_s \leq \exp(\psi_r(h, w) + \epsilon_1).$$

There exists a $\delta_1 > 0$ with $\delta_1 < \frac{\delta}{2}$ such that for each $j = 1, \ldots, n_0$, $\gamma_{j,1}(B(z_0, \delta_1)) \subset B(\gamma_{j,1}(z_0), \frac{\delta}{2})$. Therefore we obtain

$$\begin{aligned} \gamma_{n_0,1}(B(z_0, \delta_1)) &\subset B(\gamma_{n_0,1}(z_0), \delta_1 \exp((\sum_{j=0}^{n_0-1} \psi_r(\gamma_{j+1}, \gamma_{j,1}(z_0))) + n_0\epsilon_1)) \\ &\subset B(\gamma_{n_0,1}(z_0), \delta_1 \exp((-c_0 + \epsilon_0 + \epsilon_1)n_0)). \end{aligned}$$

Hence we can show that for each $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \gamma_{n_0+m,1}(B(z_0, \delta_1)) &\subset B(\gamma_{n_0+m,1}(z_0), \delta_1 \exp(\sum_{j=0}^{n_0+m-1} (\psi_r(\gamma_{j+1}, \gamma_{j,1}(z_0)) + (n_0+m)\epsilon_1)) \\ &\subset B(\gamma_{n_0+m,1}(z_0),\ \delta_1 \exp((-c_0 + \epsilon_0 + \epsilon_1)(n_0 + m))) \\ &\subset B(\gamma_{n_0+m,1}(z_0), \frac{\delta}{2}) \end{aligned}$$

by induction on $m \in \mathbb{N} \cup \{0\}$. Therefore $z_0 \in F_\gamma$. Thus for μ-a.e. $z_0 \in \hat{\mathbb{C}}$, we obtain that $\tilde{\tau}(\{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid z_0 \in J_\gamma\}) = 0$. By Lemma 3.13, it follows that for μ-a.e.$z_0 \in \hat{\mathbb{C}}$, $z_0 \in F_{pt}^0(\tau)$. Hence we have proved our lemma. □

Corollary 3.31. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ and let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $\sharp L < \infty$. Suppose $\chi(\tau, L) < 0$. Then for each $z_0 \in L$, for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, we have $z_0 \in F_\gamma$. Moreover, $L \subset F_{pt}^0(\tau)$.*

Proof. Since $\mathrm{supp}\, \omega_L = L$, Lemma 3.30 implies the statement of our corollary. □

We now show some lemmas on minimal sets of G_τ with positive Lyapunov exponents (Lemmas 3.32–3.36).

Lemma 3.32. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $\sharp L < \infty$. Suppose $\chi(\tau, L) > 0$. Suppose also that for each $z_0 \in L$ and for each $g \in \Gamma_\tau$, $Dg_{z_0} \neq 0$. Let $\alpha > 0$. Then there exist $\delta_1 > 0, \delta_2 > 0$ with $\delta_2 < \alpha$, and a Borel subset A of $\Gamma_\tau^{\mathbb{N}}$ with $\tilde{\tau}(A) = 1$, where δ_1 and A do not depend on α, such that for each $z_0 \in L$, for each $z \in B(z_0, \delta_2) \setminus \{z_0\}$ and for each $\gamma \in A$, there exists an $n_1 = n_1(\gamma, z) \in \mathbb{N}$ with $\gamma_{n_1,1}(z) \notin B(L, \delta_1)$. In particular, for each $z_0 \in L$, for $\tilde{\tau}$-a.e.$\gamma \in \Gamma_\tau^{\mathbb{N}}$, we have $z_0 \in J_\gamma$.*

Proof. Since Γ_τ is compact, there exists a $\delta > 0$ such that for each $w_0 \in L$ and for each $g \in \Gamma_\tau$, $g : B(w_0, 5\delta) \to \hat{\mathbb{C}}$ is injective. Let $s := \min\{d(a, b) \mid a, b \in L, a \neq b\} > 0$. Let $0 < \epsilon < \frac{1}{4}\chi(\tau, L)$. Then there exists a $\delta_1 > 0$ with $\delta_1 < \min\{\frac{s}{2}, \delta\}$ such that

(i) for each $w_0 \in L$ and for each $g \in \Gamma_\tau$, we have $g(B(w_0, \delta_1)) \subset B(g(w_0), \frac{s}{2})$, and

(ii) for each $w_0 \in L$ and for each $g \in \Gamma_\tau$, there exists an inverse branch $g_{w_0}^{-1} : B(g(w_0), 2\delta_1) \to B(w_0, \delta)$ of g with $g_{w_0}^{-1}(g(w_0)) = w_0$ such that for each $w \in B(g(w_0), 2\delta_1)$, we have
$$\log \|D(g_{w_0}^{-1})_w\|_s \leq \log \|D(g_{w_0}^{-1})_{g(w_0)}\|_s + \epsilon. \tag{23}$$

By Birkhoff's ergodic theorem, there exists a Borel subset A of X_τ with $\tilde{\tau}(A) = 1$ such that for each $(\gamma, z_0) \in A \times L$, there exists an $n_0 = n_0(\gamma, z_0) \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq n_0$ we have
$$|\frac{1}{n}\sum_{j=0}^{n-1} \log \|D(\gamma_{j+1})_{\gamma_{j,1}(z_0)}\|_s - \chi(\tau, L)| < \epsilon, \text{ thus } e^{n(\chi(\tau,L)-\epsilon)} \leq \|D(\gamma_{n,1})_{z_0}\|_s \leq e^{n(\chi(\tau,L)+\epsilon)}. \tag{24}$$

Let $\delta_2 := \frac{1}{2}\min\{\alpha, \delta_1\} > 0$. Let $z_0 \in L$. Let $z \in B(z_0, \delta_2) \setminus \{z_0\}$. Let $\gamma \in A$. We now prove the following claim.

Claim 1. There exists an $n \in \mathbb{N}$ such that $\gamma_{n,1}(z) \notin B(L, \delta_1)$.

To prove this claim, suppose that for each $n \in \mathbb{N}$, $\gamma_{n,1}(z) \in B(L, \delta_1)$. Let $n_0 = n_0(\gamma, z_0)$ be the number defined above. Let $m \in \mathbb{N}$ with $m \geq n_0$. Then we have $\gamma_m(\gamma_{m-1,1}(z)) = \gamma_{m,1}(z)$, $\gamma_m((\gamma_m)^{-1}_{\gamma_{m-1,1}(z_0)}(\gamma_{m,1}(z))) = \gamma_{m,1}(z)$, $\gamma_{m-1,1}(z) \in B(\gamma_{m-1,1}(z_0), 5\delta)$, $(\gamma_m)^{-1}_{\gamma_{m-1,1}(z_0)}(\gamma_{m,1}(z)) \in B(\gamma_{m-1,1}(z_0), 5\delta)$, and $\gamma_m : B(\gamma_{m-1,1}(z_0), 5\delta) \to \hat{\mathbb{C}}$ is injective. Hence
$$\gamma_{m-1,1}(z) = (\gamma_m)^{-1}_{\gamma_{m-1,1}(z_0)}(\gamma_{m,1}(z)).$$

Similarly, it is easy to see that for each $j = 1, \ldots, m$,
$$\gamma_{m-j,1}(z) = (\gamma_{m-j+1})^{-1}_{\gamma_{m-j,1}(z_0)}(\gamma_{m-j+1,1}(z)). \tag{25}$$

Combining (23), (24), (25), we obtain that
$$\begin{array}{rcl} d(z, z_0) & \leq & \delta_1 \exp(-\sum_{j=1}^{m} \log \|D(\gamma_j)_{\gamma_{j-1,1}(z_0)}\|_s + m\epsilon_1) \\ & = & \delta_1 \|D(\gamma_{m,1})_{z_0}\|_s^{-1} \cdot e^{m\epsilon} \\ & \leq & \delta_1 e^{-m(\chi(\tau,L)-\epsilon)} \cdot e^{m\epsilon} \\ & = & \delta_1 e^{-m(\chi(\tau,L)-2\epsilon)}. \end{array}$$

Since the above inequality holds for any $m \in \mathbb{N}$, it follows that $z = z_0$. However, this is a contradiction. Therefore Claim 1 holds.

By Claim 1, the statement of our lemma holds. $\square$

Lemma 3.33. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $\sharp L < \infty$. Suppose $\chi(\tau, L) > 0$. Suppose also that for each $x \in L$ and for each $g \in \Gamma_\tau$, we have $Dg_x \neq 0$. Let $y \in \hat{\mathbb{C}}$. Let $B = \{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), L) \to 0$ as $n \to \infty\}$. Then for $\tilde{\tau}$-a.e. $\gamma \in B$, there exists a number $n_0 = n_0(\gamma, y) \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq n_0$, we have $\gamma_{n,1}(y) \in L$.*

Proof. Suppose that there exists a Borel subset B_0 of B with $\tilde{\tau}(B_0) > 0$ such that for each $\gamma \in B_0$ and for each $n \in \mathbb{N}$, $\gamma_{n,1}(y) \notin L$. Since $\tilde{\tau}$ is invariant under the shift map $\sigma : X_\tau \to X_\tau$, Lemma 3.32 implies that for $\tilde{\tau}$-a.e. $\gamma \in B_0$, $\limsup_{n\to\infty} d(\gamma_{n,1}(y), L) > 0$. However, this is a contradiction. Hence the statement of our lemma holds. $\square$

Lemma 3.34. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $y \in \hat{\mathbb{C}}$. Then there exists a subset A of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus A) \leq \aleph_0$ such that for each $x \in A$, $\tau(\{g \in \mathrm{Rat} \mid g(x) = y\}) = 0$.*

Proof. For each finite subset $F = \{x_1, \ldots, x_n\}$ of $\hat{\mathbb{C}}$ such that $x_1, \ldots, x_n$ are mutually distinct, let $B_F := \{g \in \mathrm{Rat} \mid g(x_i) = y,$ for each $i = 1, \ldots, n\}$. Since $\mathrm{supp}\,\tau$ is compact, there exists an $N \in \mathbb{N}$ such that for each $g \in \Gamma_\tau$, $\deg(g) \leq N$. Hence, if $\sharp F > N$, then $\tau(B_F) = \tau(B_F \cap \Gamma_\tau) = \tau(\emptyset) = 0$. For each $k \in \mathbb{Z}$ with $0 \leq k \leq N$, let $\mathcal{F}_k = \{F \subset \hat{\mathbb{C}} \mid \sharp F = N + 1 - k, \tau(B_F) > 0\}$. We now prove the following claim.
Claim 1. Let $k \in \mathbb{Z}$ with $0 \leq k < N$. If $\sharp \mathcal{F}_k \leq \aleph_0$, then $\sharp \mathcal{F}_{k+1} \leq \aleph_0$.

To prove this claim, let $0 \leq k < N$ and suppose we have that $\sharp \mathcal{F}_k \leq \aleph_0$. Let $\mathcal{H}$ be the set $\{H \in \mathcal{F}_{k+1} \mid \exists F \in \mathcal{F}_k$ such that $H \subset F\}$. Then $\sharp \mathcal{H} \leq \aleph_0$. Moreover, for each $H_1, H_2 \in \mathcal{F}_{k+1} \setminus \mathcal{H}$ with $H_1 \neq H_2$, we have
$$\tau(B_{H_1} \cap B_{H_2}) = 0. \tag{26}$$
For, let $x \in H_2 \setminus H_1$ and let $F = H_1 \cup \{x\}$. Then $\sharp F = N + 1 - k$ and $H_1 \subset F$. Since $H_1 \notin \mathcal{H}$, we have $F \notin \mathcal{F}_k$. Hence $\tau(B_F) = 0$. Since $B_{H_1} \cap B_{H_2} \subset B_F$, (26) holds. By (26), $\sharp(\mathcal{F}_{k+1} \setminus \mathcal{H}) \leq \aleph_0$. Therefore $\sharp \mathcal{F}_{k+1} \leq \aleph_0$. Thus we have prove Claim 1.

By Claim 1, we obtain that $\sharp\{H \subset \hat{\mathbb{C}} \mid \sharp H = 1, \tau(B_H) > 0\} \leq \aleph_0$. Hence the statement of our lemma holds. $\square$

Lemma 3.35. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let C be a non-empty finite subset of $\hat{\mathbb{C}}$. Then there exists a subset A_C of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus A_C) \leq \aleph_0$ such that for each $x \in A_C$,*
$$\tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ such that } \gamma_{n,1}(x) \in C\}) = 0.$$

Proof. Let $D_{y,n} = \{x \in \hat{\mathbb{C}} \mid \tau^n(\{(\gamma_1, \ldots, \gamma_n) \in \Gamma_\tau^n \mid \gamma_n \cdots \gamma_1(x) = y\}) > 0\}$ for each $y \in C$ and each $n \in \mathbb{N}$, where $\tau^n = \otimes_{j=1}^n \tau \in \mathfrak{M}_{1,c}(\Gamma_\tau^n)$. By using the argument in the proof of Lemma 3.34, we can show that $\sharp D_{y,n} \leq \aleph_0$. Let $A_C = \hat{\mathbb{C}} \setminus (\cup_{y \in C, n \in \mathbb{N}} D_{y,n})$. Then $\sharp(\hat{\mathbb{C}} \setminus A_C) \leq \aleph_0$. For each $x \in A_C$, we have
$$\begin{aligned}
& \tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ such that } \gamma_{n,1}(x) \in C\}) \\
\leq\ & \tilde{\tau}(\cup_{n \in \mathbb{N}, y \in C}\{\gamma \in X_\tau \mid \gamma_{n,1}(x) = y\}) \\
\leq\ & \sum_{n \in \mathbb{N}, y \in C} \tilde{\tau}(\{\gamma \in X_\tau \mid \gamma_{n,1}(x) = y\}) \\
=\ & \sum_{n \in \mathbb{N}, y \in C} \tilde{\tau}(\{(\gamma_1, \ldots, \gamma_n) \in \Gamma_\tau^n \mid \gamma_n \cdots \gamma_1(x) = y\} \times \prod_{j=n+1}^{\infty} \Gamma_\tau) \\
=\ & \sum_{n \in \mathbb{N}, y \in C} \tau^n(\{(\gamma_1, \ldots, \gamma_n) \in \Gamma_\tau^n \mid \gamma_n \cdots \gamma_1(x) = y\}) = 0.
\end{aligned}$$

Thus the statement of our lemma holds. $\square$

Lemma 3.36. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $\sharp L < \infty$. Suppose that $\chi(\tau, L) > 0$ and for each $x \in L$ and for each $g \in \Gamma_\tau$, $Dg_x \neq 0$. Then for each $y \in \hat{\mathbb{C}}$. we have*

$$\tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), L) \to 0 \text{ as } n \to \infty\}) = \tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ such that } \gamma_{n,1}(y) \in L\})$$

and

$$\sharp\{y \in \hat{\mathbb{C}} \mid \tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), L) \to 0 \text{ as } n \to \infty\}) > 0\} \leq \aleph_0.$$

Proof. Lemma 3.33 implies that for each $y \in \hat{\mathbb{C}}$,

$$\tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), L) \to 0 \text{ as } n \to \infty\}) = \tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ such that } \gamma_{n,1}(y) \in L\}).$$

Hence

$$\begin{aligned} & \{y \in \hat{\mathbb{C}} \mid \tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), L) \to 0 \text{ as } n \to \infty\}) > 0\} \\ = \ & \{y \in \hat{\mathbb{C}} \mid \tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ such that } \gamma_{n,1}(y) \in L\}) > 0\} \subset \hat{\mathbb{C}} \setminus A_L, \end{aligned}$$

where A_L is the set for L coming from Lemma 3.35. Since $\sharp(\hat{\mathbb{C}} \setminus A_L) \leq \aleph_0$, the statement of our lemma holds. □

3.5 Systems with finite kernel Julia sets

In this subsection, we show a theorem on the random dynamical systems generated by elements $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ with $J_{\ker}(G_\tau) < \infty$.

Theorem 3.37. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose we have all of the following.*

(i) $\sharp J_{\ker}(G_\tau) < \infty$.

(ii) *For each $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$, we have $\chi(\tau, L) \neq 0$.*

(iii) *For each $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$ with $\chi(\tau, L) > 0$, for each $g \in \Gamma_\tau$ and for each $x \in L$, we have $Dg_x \neq 0$.*

Let $H_+ = \{L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau)) \mid \chi(\tau, L) > 0\}$ and we denote by Ω the set of points $y \in \hat{\mathbb{C}}$ for which $\tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) \in \cup_{L \in H_+} L\}) = 0$. Then, $\sharp(\hat{\mathbb{C}} \setminus \Omega) \leq \aleph_0$ and for each $z \in \Omega$, $\tilde{\tau}(\{\gamma \in X_\tau \mid z \in J_\gamma\}) = 0$. Moreover, for $\tilde{\tau}$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, $\mathrm{Leb}_2(J_\gamma) = 0$. Furthermore, $J^0_{pt}(\tau) \subset \hat{\mathbb{C}} \setminus \Omega$ and $\sharp(J^0_{pt}(\tau)) \leq \aleph_0$.

Proof. By Lemma 3.36, we have $\sharp(\hat{\mathbb{C}} \setminus \Omega) \leq \aleph_0$ and for each $y \in \Omega$,

$$\tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), \cup_{L \in H_+} L) \to 0 \text{ as } n \to \infty\}) = 0. \tag{27}$$

Let $z \in \Omega$. Let $C_z = \{\gamma \in X_\tau \mid z \in J_\gamma\}$. Suppose $\tilde{\tau}(C_z) > 0$. Let H_- be the set of all $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$ with $\chi(\tau, L) < 0$. By Lemma 3.15, for $\tilde{\tau}$-a.e.$\gamma \in C_z$, $d(\gamma_{n,1}(z), \cup_{L \in H_+ \cup H_-} L) \to 0$ as $n \to \infty$. Combining this with (27), we obtain that

$$\text{for } \tilde{\tau} \text{ -a.e. } \gamma \in C_z, d(\gamma_{n,1}(z), \cup_{L \in H_-} L) \to 0 \text{ as } n \to \infty. \tag{28}$$

Let $0 < \epsilon < \frac{1}{2}\tilde{\tau}(C_z)$. By Corollary 3.31, for each $z_0 \in \cup_{L \in H_-} L$. for $\tilde{\tau}$-a.e. γ, we have $z_0 \in F_\gamma$. Combining this with the argument in the proof of Lemma 3.13, we obtain that there exist a Borel subset A_1 of X_τ with $\tilde{\tau}(A_1) \geq 1 - \epsilon$ and a $\delta > 0$ such that for each $z_0 \in \cup_{L \in H_-} L$, for each $\gamma \in A_1$, we have $\sup_{n \geq 1} \mathrm{diam}\gamma_{n,1}(B(z_0, \delta)) \leq \frac{1}{10}\mathrm{diam}\hat{\mathbb{C}}$. In particular,

$$\text{for each } z_0 \in \cup_{L \in H_-} L \text{ and for each } \gamma \in A_1, B(z_0, \delta) \subset F_\gamma. \tag{29}$$

By (28) and Egoroff's theorem, there exist a Borel subset A_2 of C_z with $\tilde{\tau}(A_2) \geq \tilde{\tau}(C_z) - \epsilon$ and an $n_0 \in \mathbb{N}$ such that for each $\gamma \in A_2$,

$$\gamma_{n_0,1}(z) \in B(\cup_{L \in H_-} L, \delta). \tag{30}$$

By (29) and (30), we obtain $A_2 \cap \sigma^{-n_0}(A_1) = \emptyset$. Therefore $\tilde{\tau}(A_2) \leq \tilde{\tau}(X_\tau \setminus \sigma^{-n_0}(A_1)) \leq \epsilon$. Combining this with that $\tilde{\tau}(A_2) \geq \tilde{\tau}(C_z) - \epsilon$, we obtain that $\tilde{\tau}(C_z) \leq 2\epsilon$. However, this is a contradiction because $\epsilon < \frac{1}{2}\tilde{\tau}(C_z)$. Thus, we have proved that for each $z \in \Omega$, $\tilde{\tau}(C_z) = 0$. By Fubini's theorem, it follows that for $\tilde{\tau}$-a.e.γ, $\text{Leb}_2(J_\gamma) = 0$. Moreover, by Lemma 3.13, we obtain that $J^0_{pt}(\tau) \subset \hat{\mathbb{C}} \setminus \Omega$ and $\sharp J^0_{pt}(\tau) \leq \aleph_0$. Thus we have proved our theorem. $\square$

3.6 Random dynamical systems generated by measures on weakly nice sets

In this subsection, we show several results (including Theorems 1.1, 1.2 and their detailed and more generalized version Theorems 3.76, 3.81) regarding random complex dynamical systems generated by measures on weakly nice subsets of Rat.

We now consider holomorphic families of rational maps.

Definition 3.38. Let Λ be a complex manifold. Let $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ be a family of rational maps on $\hat{\mathbb{C}}$. We say that $\mathcal{W}$ is a **holomorphic family of rational maps** if $(z, \lambda) \in \hat{\mathbb{C}} \times \Lambda \mapsto f_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$. **Throughout the paper, we always assume that Λ is connected**. If $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of rational maps and each f_λ is a polynomial, then we say that $\mathcal{W}$ is a **holomorphic family of polynomial maps**. We say that a holomorphic family $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ of rational maps is **non-constant** if $\lambda \in \Lambda \mapsto f_\lambda \in \text{Rat}$ is non-constant.

For each $n \in \mathbb{N}$, we set

$$S_n(\mathcal{W}) = \{z \in \hat{\mathbb{C}} \mid (\lambda_1, \ldots, \lambda_n) \in \Lambda^n \mapsto f_{\lambda_1} \circ \cdots \circ f_{\lambda_n}(z) \text{ is constant on } \Lambda^n\}.$$

Moreover, we set $S(\mathcal{W}) := \cap_{n=1}^\infty S_n(\mathcal{W})$. Each point of $S(\mathcal{W})$ is called a **singular point of $\mathcal{W}$** and the set $S(\mathcal{W})$ is called the **singular set of $\mathcal{W}$**.

Lemma 3.39. *Let $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ be a holomorphic family of rational maps. Then $S_{n+1}(\mathcal{W}) = \cap_{\lambda_{n+1} \in \Lambda} f^{-1}_{\lambda_{n+1}}(S_n(\mathcal{W}))$ and $S(\mathcal{W}) = \cap_{n=1}^\infty \cap_{(\lambda_1, \ldots, \lambda_n) \in \Lambda^n} (f_{\lambda_1} \circ \cdots \circ f_{\lambda_n})^{-1}(S_1(\mathcal{W}))$. Moreover, if, in addition to the assumption, $\mathcal{W}$ is non-constant, then $\sharp S_1(\mathcal{W}) < \infty$ and $\sharp S_n(\mathcal{W}) < \infty$ for each $n \in \mathbb{N}$.*

Proof. We may assume that $\mathcal{W}$ is non-constant. We first show that $\sharp S_1(\mathcal{W}) < \infty$. Suppose that $\sharp S_1(\mathcal{W}) = \infty$. Then there exist a sequence $\{z_n\}$ in $S_1(\mathcal{W})$ and a point $z_\infty \in \hat{\mathbb{C}}$ such that $z_n \to z_\infty$ and $z_n \neq z_\infty$ for each $n \in \mathbb{N}$. By conjugating the family $\mathcal{W}$ by an element of $\text{Aut}(\hat{\mathbb{C}})$, we may assume that $z_\infty \in \mathbb{C}$. Let $b \in \Lambda$. Then there exist an open connected neighborhood Λ_0 of b in Λ and an open connected neighborhood U of z_∞ in $\mathbb{C}$ such that $f_\lambda(z) \in \mathbb{C}$ for all $\lambda \in \Lambda_0$ and all $z \in U$. We may suppose that $\Lambda_0 \subset \mathbb{C}^r$ where $r = \dim \Lambda \in \mathbb{N}$. Let $n \in \mathbb{N}$, $(i_1, \ldots, i_n) \in (\{1, \ldots, r\})^n$ and $z \in U$. Let $g(z) = \frac{\partial^n f_\lambda(z)}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}}|_{\lambda = b}$ for each $z \in U$. Then $g : U \to \mathbb{C}$ is holomorphic in U and $g(z_j) = 0$ for each large j. Hence $g(z) = 0$ for all $z \in U$. Therefore for each $z \in U$, the function $\lambda \mapsto f_\lambda(z) \in \mathbb{C}$ is constant on Λ_0. Thus, for each $z \in U$, the function $\lambda \mapsto f_\lambda(z) \in \hat{\mathbb{C}}$ is constant on Λ. Hence $U \subset S_1(\mathcal{W})$. Therefore

$$z_\infty \in \text{int}(S_1(\mathcal{W})). \tag{31}$$

In particular, $\text{int}(S_1(\mathcal{W})) \neq \emptyset$. We now suppose that $\hat{\mathbb{C}} \neq \text{int}(S_1(\mathcal{W}))$. Then $\partial(\text{int}(S_1(\mathcal{W}))) \neq \emptyset$. If we take any $w_0 \in \partial(\text{int}(S_1(\mathcal{W})))$, then by the argument of the proof of (31), we obtain $w_0 \in \text{int}(S_1(\mathcal{W}))$. However, this contradicts $w_0 \in \partial(\text{int}(S_1(\mathcal{W})))$. Therefore, we must have that

$\hat{\mathbb{C}} = \text{int}(S_1(\mathcal{W}))$. Hence, the function $\lambda \mapsto f_\lambda \in \text{Rat}$ is constant on Λ. However, this contradicts to the assumption that $\mathcal{W}$ is non-constant. Thus, we have that $\sharp S_1(\mathcal{W}) < \infty$.

It is easy to see that $S_{n+1}(\mathcal{W}) \subset \cap_{\lambda_{n+1}\in\Lambda} f_{\lambda_{n+1}}^{-1}(S_n(\mathcal{W}))$. Since $\sharp S_1(\mathcal{W}) < \infty$. it follows that $\sharp S_n(\mathcal{W}) < \infty$. We now prove $\cap_{\lambda_{n+1}\in\Lambda} f_{\lambda_{n+1}}^{-1}(S_n(\mathcal{W})) \subset S_{n+1}(\mathcal{W})$. Let $z \in \cap_{\lambda_{n+1}\in\Lambda} f_{\lambda_{n+1}}^{-1}(S_n(\mathcal{W}))$. Then for each $\lambda_{n+1} \in \Lambda$, we have $f_{\lambda_{n+1}}(z) \in S_n(\mathcal{W})$. Since $\sharp S_n(\mathcal{W}) < \infty$ and Λ is connected, it follows that $\sharp\{f_{\lambda_{n+1}}(z) \in S_n(\mathcal{W}) \mid \lambda_{n+1} \in \Lambda\} = 1$. Therefore we obtain that the cardinality of the set $\{f_{\lambda_1} \circ \cdots \circ f_{\lambda_{n+1}}(z) \mid (\lambda_1 \ldots, \lambda_{n+1}) \in \Lambda^{n+1}\}$ is equal to 1. In particular, $z \in S_{n+1}(\mathcal{W})$. Thus we have proved our lemma. □

Corollary 3.40. *Let $\mathcal{W} = \{f_\lambda\}_{\lambda\in\Lambda}$ be a holomorphic family of rational maps. Then $f_\lambda(S(\mathcal{W})) \subset S(\mathcal{W})$ for all $\lambda \in \Lambda$.*

We now define weakly nice subsets of Rat.

Definition 3.41. We say that a subset $\mathcal{Y}$ of Rat is **weakly nice** (with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps) if there exist an open subset $\mathcal{U}$ of Rat and finitely many non-constant holomorphic families $\mathcal{W}_j = \{f_{j,\lambda}\}_{\lambda\in\Lambda_j}, j = 1, \ldots, m$, of rational maps such that for each $j = 1, \ldots, m$, $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ is a closed subset of $\mathcal{U}$ and $\mathcal{Y} = \cup_{j=1}^m \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$.

Moreover, for a weakly nice set $\mathcal{Y}$ with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps, we set

$$\mathcal{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) := \{\tau \in \mathfrak{M}_1(\mathcal{Y}) \mid \text{supp}\,\tau \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \neq \emptyset \ (\forall j = 1, \ldots, m)\}$$

and

$$\mathcal{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) := \mathfrak{M}_{1,c}(\mathcal{Y}) \cap \mathcal{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m).$$

Here, for the notation "supp τ", see Definition 3.4 (setting $Y = \mathcal{Y}$). (Thus supp τ is a closed subset of $\mathcal{Y}$.) Also, each point of $\cap_{j=1}^m S(\mathcal{W}_j)$ is called a **singular point of** $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ and the set $\cap_{j=1}^m S(\mathcal{W}_j)$ is called the **singular set of** $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$.

Definition 3.42. Let $\mathcal{Y}$ be a closed subset of an open subset of Rat. Let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that the sequence $\{\tau_n\}_{n=1}^\infty$ in $\mathfrak{M}_{1,c}(\mathcal{Y})$ tends to an element $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ with respect to the topology $\mathcal{O}$ if and only if (a) for each bounded $\varphi \in C(\mathcal{Y})$, $\int \varphi d\tau_n \to \int \varphi d\tau$ as $n \to \infty$, and (b) supp $\tau_n \to$ supp τ as $n \to \infty$ in $\text{Cpt}(\mathcal{Y})$ with respect to the Hausdorff topology.

By the definition of weakly nice subsets, it is easy to see the following lemma.

Lemma 3.43. *Let $\mathcal{Y}$ be a weakly nice subset of* Rat *with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then $\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ is closed in $\mathfrak{M}_{1,c}(\mathcal{Y})$ with respect to the topology $\mathcal{O}$.*

The following lemma is easy to show but it is one of the keys to proving many results.

Lemma 3.44. *Let $\mathcal{Y}$ be a weakly nice subset of* Rat *with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Let $\tau \in \mathfrak{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Suppose that int(supp τ) $\neq \emptyset$ with respect to the topology in $\mathcal{Y}$ and $F(G_\tau) \neq \emptyset$. Then $J_{\ker}(G_\tau) \subset S(\mathcal{W}_j)$ for some $j = 1, \ldots, m$ and $\sharp J_{\ker}(G_\tau) < \infty$.*

Proof. Let $\mathcal{W}_j = \{f_{j,\lambda}\}_{\lambda\in\Lambda_j}$ for each j. Then there exists an element $j \in \{1, \ldots, m\}$ such that $\text{int}(\text{supp}\,\tau) \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \neq \emptyset$. Suppose $J_{\ker}(G_\tau) \setminus S(\mathcal{W}_j) \neq \emptyset$. Let $z_0 \in J_{\ker}(G_\tau) \setminus S(\mathcal{W}_j)$. Then there exists an element $n \in \mathbb{N}$ such that the map $(\lambda_1, \ldots, \lambda_n) \in \Lambda_j^n \mapsto f_{j,\lambda_1} \circ \cdots \circ f_{j,\lambda_n}(z_0) \in \hat{\mathbb{C}}$ is non-constant on Λ_j^n. It implies that $\text{int}(J_{\ker}(G_\tau)) \neq \emptyset$. However, this contradicts to the assumption $F(G_\tau) \neq \emptyset$ and Montel's theorem. Thus we must have that $J_{\ker}(G_\tau) \subset S(\mathcal{W}_j)$. Since $\sharp(S_n(\mathcal{W}_j)) < \infty$ (see Lemma 3.39), it follows that $\sharp J_{\ker}(G_\tau) < \infty$. □

Lemma 3.45. *Let $\mathcal{Y}$ be a weakly nice subset of* Rat *with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Let $\tau \in \mathfrak{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Suppose that for each $j = 1, \ldots, m$, we have int(supp $\tau \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}) \neq \emptyset$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$, and that $F(G_\tau) \neq \emptyset$. Then $J_{\ker}(G_\tau) \subset \cap_{j=1}^m S(\mathcal{W}_j)$.*

Proof. By using the argument in the proof of Lemma 3.44, it is easy to see that our lemma holds. □

Lemma 3.46. *Let $\mathcal{Y}$ be a weakly nice subset of* Rat *with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Let $\tau \in \mathfrak{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ such that $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$. Then for each $\rho \in \mathfrak{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$, we have $L \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$.*

Proof. Let $z \in L$. Let $\mathcal{W}_j = \{f_{j,\lambda}\}_{\lambda \in \Lambda_j}$ for each j. Let $\rho \in \mathfrak{M}_1(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Let $h \in \mathrm{supp}\,\rho$. Then there exist an $i \in \{1, \ldots, m\}$ and an element $\lambda_0 \in \Lambda_i$ such that $h = f_{i,\lambda_0}$. Since we have $\mathrm{supp}\,\tau \cap \{f_{i,\lambda} \mid \lambda \in \Lambda_i\} \neq \emptyset$, there exists an element $\lambda_1 \in \Lambda_i$ such that $f_{i,\lambda_1} \in \mathrm{supp}\,\tau$. Since $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, we have $f_{i,\lambda_1}(z) \in L$. Moreover, since $L \subset S(\mathcal{W}_i)$, we have that $h(z) = f_{i,\lambda_0}(z) = f_{i,\lambda_1}(z) \in L$. Hence $h(L) \subset L$. Therefore $L \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$. □

Definition 3.47. Let $\mathcal{Y}$ be a weakly nice subset of Rat with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Then we set

$$S_{\min}(\{\mathcal{W}_j\}_{j=1}^m) = \cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), L \subset \cap_{j=1}^m S(\mathcal{W}_j)} L.$$

Note that this definition does not depend on the choice of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ due to Lemma 3.46.

We now give the definition of attracting minimal sets which was introduced by the author in [37].

Definition 3.48. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. We say that a minimal set $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$ is **attracting** (for Γ) if there exist two open subsets A, B of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus A) \geq 3$ and an $n \in \mathbb{N}$ such that $\overline{B} \subset A$ and such that for each $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, we have $\gamma_n \circ \cdots \circ \gamma_1(A) \subset B$. In this case, we say that L is an **attracting minimal set for** Γ. Also, for an element $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, if $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is attracting for Γ_τ then we say that L is **attracting for** τ, that L is an **attracting minimal set of** Γ_τ, and that L is an **attracting minimal set of** τ.

Definition 3.49. Let $\mathcal{Y}$ be a subset of Rat endowed with the relative topology from Rat. We say that $\mathcal{Y}$ is **mild** if for each $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$, there exists an attracting minimal set for Γ.

We give some examples of mild sets.

Example 3.50 (Examples of mild sets)**.**

(a) Any non-empty open subset $\mathcal{U}$ of $\mathcal{P}$ is a mild set. For, for each $\Gamma \in \mathrm{Cpt}(\mathcal{U})$, the set $\{\infty\}$ is an attracting minimal set for Γ. Also, for any $\Lambda \in \mathrm{Cpt}(\mathcal{P})$, there exists an open subset $\mathcal{V}$ of Rat with $\mathcal{V} \supset \Lambda$ such that $\mathcal{V}$ is mild.

(b) Let $\Lambda \in \mathrm{Cpt}(\mathrm{Rat})$ such that Λ has a minimal set for $(\langle \Lambda \rangle, \hat{\mathbb{C}})$ which is attracting for Λ. Then there exists an open subset $\mathcal{U}$ of Rat with $\mathcal{U} \supset \Lambda$ such that $\mathcal{U}$ is mild.

(c) Let $a \in \hat{\mathbb{C}}$ and let $\mathcal{Y} = \{f \in \mathrm{Rat} \mid a \text{ is an attracting fixed point of } f\}$. Then $\mathcal{Y}$ is a mild subset of Rat.

We now give the definition of mean stability which is introduced by the author in [36].

Definition 3.51. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. Let $G = \langle \Gamma \rangle$. We say that Γ is **mean stable** if there exist non-empty open subsets U and V of $F(G)$ and a number $n \in \mathbb{N}$ such that all of the following hold.

(a) $\overline{V} \subset U$ and $\overline{U} \subset F(G)$.

(b) For each $\gamma \in \Gamma^{\mathbb{N}}, \gamma_{n,1}(\overline{U}) \subset V$.

(c) For each $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ such that $g(z) \in U$.

Also, if Γ is mean stable, we say that G is mean stable (this notion does not depend on the choice of $\Gamma \in \text{Cpt}(\text{Rat})$ with $\langle \Gamma \rangle = G$). Moreover, for an element $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$, if Γ_τ is mean stable, then we say that τ is mean stable.

Remark 3.52. If $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ is mean stable, then the random dynamical system generated by τ has many nice properties (e.g. $J_{\ker}(G_\tau) = \emptyset$, stability of the limit state functions under the perturbation, negativity of Lyapunov exponent for any point of $z \in \hat{\mathbb{C}}$ for $\tilde{\tau}$-a.e. γ etc., see [36, 37]).

We now give a result of the density of mean stable elements. Recall that an element $g \in \text{Aut}(\hat{\mathbb{C}})$ is called loxodromic if g has exactly two fixed points $a, b \in \hat{\mathbb{C}}$ and the modulus of multiplier of (g, a) is strictly larger than 1 and the modulus of multiplier of (g, b) is strictly less than 1.

Lemma 3.53. *Let $\mathcal{Y}$ be a mild subset of* Rat *and suppose that $\mathcal{Y}$ is weakly nice with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Suppose that for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ and for each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, we have $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j) \cap J(G_\tau)$. Then $\mathcal{A} := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) \mid \tau \text{ is mean stable}\}$ is open and dense in $\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with respect to the topology $\mathcal{O}$.*

Proof. By [36, Lemma 3.62], $\mathcal{A}$ is open in $\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with respect to the topology $\mathcal{O}$. To prove the density of $\mathcal{A}$, let $\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Then there exists an element $\rho_0 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is arbitrarily close to ρ with respect to $\mathcal{O}$ such that for each $j \in \{1, \ldots, m\}$, $\text{int}(\text{supp}\, \rho_0 \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}) \neq \emptyset$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$, where $\mathcal{W}_j = \{f_{j,\lambda}\}_{\lambda \in \Lambda_j}$. By Lemma 3.45 and the assumption of our lemma, we obtain $J_{\ker}(G_{\rho_0}) = \emptyset$. Since $\mathcal{Y}$ is mild, each $g \in \Gamma_{\rho_0} \cap \text{Aut}(\hat{\mathbb{C}})$ is loxodromic. By [37, Theorem 1.8] and its proof, if we enlarge $\text{supp}\rho_0$ a little bit, and take an element $\rho_1 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is close to ρ_0, then ρ_1 is mean stable. Thus $\mathcal{A}$ is dense in $\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. $\square$

Definition 3.54. Let $\mathcal{Y}$ be a weakly nice subset of Rat with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. We say that $\mathcal{Y}$ is **exceptional with respect to** $\{\mathcal{W}_j\}_{j=1}^m$ if there exists a non-empty subset L of $\cap_{j=1}^m S(\mathcal{W}_j)$ such that for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$, we have that $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ and $\chi(\tau, L) = 0$. We say that $\mathcal{Y}$ is **non-exceptional with respect to** $\{\mathcal{W}_j\}_{j=1}^m$ if $\mathcal{Y}$ is not exceptional with respect to $\{\mathcal{W}_j\}_{j=1}^m$.

Proposition 3.55. *Let $\mathcal{Y}$ be a mild subset of* Rat *and suppose that $\mathcal{Y}$ is weakly nice and non-exceptional with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then there exists a dense subset $\mathcal{A}$ of the topological space $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that all of the following* (a)(b) *hold.*

(a) *For each $\tau \in \mathcal{A}$ and for each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, we have $\chi(\tau, L) \neq 0$.*

(b) *Let $\tau \in \mathcal{A}$. Then $\sharp J_{\ker}(G_\tau) < \infty$ and $J_{\ker}(G_\tau) \subset \cap_{j=1}^m S(\mathcal{W}_j)$. Moreover, setting $H_+ := \{L \in \text{Min}(G_\tau, J_{\ker}(G_\tau)) \mid \chi(\tau, L) > 0\}$ and denoting by Ω the set of points $y \in \hat{\mathbb{C}}$ for which $\tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) \in \cup_{L \in H_+} L\}) = 0$, we have that $\sharp(\hat{\mathbb{C}} \setminus \Omega) \leq \aleph_0$ and for each $z \in \Omega$, $\tilde{\tau}(\{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid z \in J_\gamma\}) = 0$. Moreover, for $\tilde{\tau}$-a.e. $\gamma \in (\text{Rat})^{\mathbb{N}}$, we have $\text{Leb}_2(J_\gamma) = 0$. Furthermore, $J_{pt}^0(\tau) \subset \hat{\mathbb{C}} \setminus \Omega$ and $\sharp J_{pt}^0(\tau) \leq \aleph_0$.*

Proof. By Lemma 3.53, [36, Propositions 4.7, 4.8] and [37, Remark 3.5], we may assume that there exist a $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\})$ and an $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ such that $L \subset \cap_{j=1}^m S(\mathcal{W}_j)) \cap J(G_\tau)$. For such L, Lemma 3.46 implies that for each $\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$, we have $L \in \text{Min}(G_\rho, \hat{\mathbb{C}})$ and $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$. Let

$$\{L_1, \ldots, L_r\} := \{K \subset \cap_{j=1}^m S(\mathcal{W}_j) \mid K \in \text{Min}(G_\rho, \hat{\mathbb{C}}) \text{ for each } \rho \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)\}.$$

Since $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ is non-exceptional with respect to $\{\mathcal{W}_j\}_{j=1}^m$, for each $k = 1, \dots, r$ there exists a $\tau_k \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ such that $\chi(\tau_k, L_k) \neq 0$. Let $\mathcal{W}_j = \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1. \dots, m$. We consider the following two cases.
Case (I). For each $k = 1, \dots, r$, for each $z \in L_k$ and for each $j = 1, \dots, m$, there exists a $\lambda \in \Lambda_j$ such that $D(f_{j,\lambda})_z \neq 0$.
Case (II). There exist a $k \in \{1, \dots, r\}$, a point $z \in L_k$ and an element $j \in \{1, \dots, m\}$ such that for each $\lambda \in \Lambda_j$, $D(f_{j,\lambda})_z = 0$.
Suppose that we have Case (I). We now prove the following claim.
Claim 1. For each k there exists an element $\rho_k \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is arbitrarily close to τ_k such that $\sharp \text{supp} \rho_k < \infty$, such that for each $g \in \text{supp} \rho_k$ and for each $z \in L_k$, we have $Dg_z \neq 0$, and such that $\chi(\rho_k, L_k) \neq 0$.

To prove this claim, for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ and for each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, let $\mu_{\tau,L}$ be the canonical τ-ergodic measure on L (see Definition 3.29). Let $k \in \{1, \dots, r\}$. We now consider the following two cases.
Case (I)(a). $\chi(\tau_k, L_k) \neq -\infty$. Case (I)(b). $\chi(\tau_k, L_k) = -\infty$.
Suppose we have Case (I)(a). Let $B_k := \{g \in \mathcal{Y} \mid Dg_z = 0 \text{ for some } z \in L_k\}$. Since $\chi(\tau_k, L_k) = \int_{L_k} \int_{\mathcal{Y}} \log \|Dg_z\|_s d\tau_k(g) d\mu_{\tau_k, L_k}(z)$, we obtain that $\tau_k(B_k) = 0$. Let $C_{k,n}$ be the set of elements $g \in \mathcal{Y}$ with $\kappa(g, B_k) \geq 1/n$. Then $\int_{L_k} \int_{C_{k,n}} \log \|Dg_z\|_s d\tau_k(g) d\mu_{\tau_k, L_k}(z) \to \chi(L_k, \tau_k)$ as $n \to \infty$ and $\frac{\tau_k|_{C_{k,n}}}{\tau_k(C_{k,n})} \to \tau_k$ as $n \to \infty$ in $\mathfrak{M}_{1.c}(\mathcal{Y}, \mathcal{O})$. Modifying $\frac{\tau_k|_{C_{k,n}}}{\tau_k(C_{k,n})}$, we obtain ρ_k which is arbitrarily close to τ_k such that $\sharp \text{supp} \rho_k < \infty$, such that for each $g \in \text{supp} \rho_k$ and for each $z \in L_k$, we have $Dg_z \neq 0$, and such that $\chi(\rho_k, L_k) \neq 0$.

We now suppose that we have Case (I)(b). Let $\alpha_n(g, z) = \max\{\log \|Dg_z\|_s, -n\}$ for each $n \in \mathbb{N}$. Since $\chi(\tau_k, L_k) = -\infty$, we have $\int_{L_k} \int_{\mathcal{Y}} \alpha_n(g, z) d\tau_k(g) d\mu_{\tau_k, L_k}(z) \to -\infty$ as $n \to \infty$. Hence for each $M < 0$ there exists an $n \in \mathbb{N}$ such that $\int_{L_k} \int_{\mathcal{Y}} \alpha_n(g, z) d\tau_k(g) d\mu_{\tau_k, L_k}(z) < M$. Therefore there exists a $\rho_k \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is arbitrarily close to τ_k such that $\sharp \text{supp} \rho_k < \infty$, such that for each $g \in \text{supp} \rho_k$ and for each $z \in L_k$, we have $Dg_z \neq 0$, and such that $\int_{L_k} \int_{\mathcal{Y}} \alpha_n(g, z) d\rho_k(g) d\mu_{\rho_k, L_k}(z) < \frac{M}{2}$. Hence $\chi(\rho_k, L_k) \leq \int_{L_k} \int_{\mathcal{Y}} \alpha_n(g, z) d\rho_k(g) d\mu_{\rho_k, L_k}(z) < \frac{M}{2}$. Thus we have proved Claim 1.

For each $n \in \mathbb{N}$, let

$$D_{k,n} := \{((\lambda_{ji})_{i=1,\dots,n})_{j=1,\dots,m} \in \prod_{j=1}^m \Lambda_j^n \mid D(f_{j,\lambda_{ji}})_z = 0 \text{ for some } z \in L_k\}.$$

Moreover, let

$$E_{k,n} := \{(p_{ij})_{i=1,\dots,n,j=1,\dots,m} \in (0,1)^{nm} \mid \sum_{i,j} p_{i,j} = 1\} \times ((\prod_{j=1}^m \Lambda_j^n) \setminus D_{k,n})$$

and let $\alpha_{k,n} : E_{k,n} \to \mathbb{R}$ be the function defined by $\alpha_{k,n}((p_{ij}), (\lambda_{ji})) = \chi(\sum_{j=1}^m \sum_{i=1}^n p_{ij} \delta_{f_{j,\lambda_{ji}}}, L_k)$. Then $(\prod_{j=1}^m \Lambda_j^n) \setminus D_{k,n}$ is connected and $\alpha_{k,n} : E_{k,n} \to \mathbb{R}$ is real-analytic. Hence claim 1 implies the following claim.
Claim 2. There exists an $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq n_0$, the function $\alpha_{k,n} : E_{k,n} \to \mathbb{R}$ is not identically equal to zero in any open subset of $E_{k,n}$.

We now let $\zeta \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ be an arbitrary element. Then there exists an element $\zeta_0 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ arbitrarily close to ζ such that $\sharp \text{supp}\, \zeta_0 < \infty$ and such that for each $g \in \text{supp}\, \zeta_0$, for each k, and for each $z \in L_k$, we have $Dg_z \neq 0$. We may assume that for some $n \geq n_0$ there exists an element $(((p_{ij})_{i=1,\dots,n})_{j=1,\dots,m}, ((\lambda_{ji})_{i=1,\dots,n})_{j=1,\dots,m}) \in \cap_{k=1}^r E_{k,n}$ such that $\zeta_0 = \sum_{j=1}^m \sum_{i=1}^n p_{ij} \delta_{f_{j,\lambda_{ji}}}$. By claim 2, there exists a ζ_1 close to ζ_0 such that for each $g \in \text{supp}\, \zeta_1$, for each k, and for each $z \in L_k$, we have $Dg_z \neq 0$, and such that for each k, $\chi(\zeta_1, L_k) \neq 0$. By enlarging the support of ζ_1, we obtain an element $\zeta_2 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is close to ζ_1 such that for each $g \in \text{supp}\, \zeta_2$, for each k, and for each $z \in L_k$, we have $Dg_z \neq 0$, such that for each k, $\chi(\zeta_2, L_k) \neq 0$, and such that for each $j = 1, \ldots, m$, $\mathrm{int}(\mathrm{supp}\, \zeta_2 \cap \{f_{j,\lambda} \mid \lambda \in \mathcal{W}_j\}) \neq \emptyset$ in the space $\{f_{j,\lambda} \mid \lambda \in \mathcal{W}_j\}$. By Lemma 3.45, we obtain that $J_{\ker}(G_{\zeta_2}) \subset \cap_{j=1}^m S(\mathcal{W}_j)$. In particular, $\sharp J_{\ker}(G_{\zeta_2}) < \infty$ by Lemma 3.39. By Theorem 3.37, denoting by H_+ the set of elements $L \in \mathrm{Min}(G_{\zeta_2}, J_{\ker}(G_{\zeta_2}))$ with $\chi(\zeta_2, L) > 0$ and denoting by Ω the set of elements $y \in \hat{\mathbb{C}}$ for which $\tilde{\zeta}_2(\{\gamma \in X_{\zeta_2} \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) \in \cup_{L \in H_+} L\}) = 0$, we have that $\sharp(\hat{\mathbb{C}} \setminus \Omega) \leq \aleph_0$ and for each $z \in \Omega$, $\tilde{\zeta}_2(\{\gamma \in X_{\zeta_2} \mid z \in J_\gamma\}) = 0$. Moreover, for $\tilde{\zeta}_2$-a.e.$\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, $\mathrm{Leb}_2(J_\gamma) = 0$. Furthermore, $J_{pt}^0(\tau) \subset \hat{\mathbb{C}} \setminus \Omega$ and $\sharp J_{pt}^0(\tau) \leq \aleph_0$.

We now suppose that we have Case (II). Let

$$I := \{k \in \{1, \ldots, r\} \mid \exists z \in L_k\ \exists j \in \{1, \ldots, m\} \text{ such that for each } \lambda \in \Lambda_j, D(f_{j,\lambda})_z = 0\}.$$

We modify the argument in Case (I). Namely, we can choose ζ_1 and ζ_2 in the argument of Case (I) so that $\chi(\zeta_1, L_k) = \chi(\zeta_2, L_k) = -\infty$ for any $k \in I$. For any $k \notin I$, we use the same argument in that of Case (I). Thus we have proved our proposition. $\square$

Lemma 3.56. *Under the assumptions of Proposition 3.55, there exists an open dense subset $\mathcal{A}$ of the topological space $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that all of the following hold.*

(i) *For each $\tau \in \mathcal{A}$ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, we have $\chi(\tau, L) \neq 0$.*

(ii) *For each $\tau \in \mathcal{A}$ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, if $\chi(\tau, L) > 0$, then for each $z \in L$ and for each $g \in G_\tau$, we have $Dg_z \neq 0$.*

Proof. Let $\mathcal{W}_j = \{f_{j,\lambda}\}_{\lambda \in \Lambda_j}$ for all j. We use the arguments in the proof of Proposition 3.55. As in the proof of Proposition 3.55, we may assume that there exists a $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ and an $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ such that $L \subset \cap_{j=1}^m S(\mathcal{W}_j) \cap J(G_\tau)$. Let $L_1, \ldots, L_r$ be as in the proof of Proposition 3.55. Let $\zeta \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Let $\zeta_0 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with $\sharp\Gamma_{\zeta_0} < \infty$ which is arbirarily close to ζ. We classify the elements k of $\{1, \ldots, r\}$ into the following two types (I) and (II).

Type (I). There exist an element $i = 1, \ldots, m$ and an element $z_0 \in L_k$ such that $D(f_{i,\lambda})_{z_0} = 0$ for all $\lambda \in \Lambda_i$.

Type (II). Not type (I).

Note that if k is of type (I), then $\chi(\zeta_0, L_k) = -\infty$. Note also that if k is of type (II), then perturbing ζ_0 if necessary, we may assume that for each $g \in \Gamma_{\zeta_0}$ and for each $z \in L_k$, we have $Dg_z \neq 0$. Therefore, by using the arguments in the proof of Proposition 3.55, we can take $\zeta_1 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with $\sharp\Gamma_{\zeta_1} < \infty$ which is arbitrarily close to ζ_0 such that the following hold.

(a) $\chi(\zeta_1, L_k) = -\infty$ for any k of type (I).

(b) For any k of type (II), for any $z \in L_k$ and for any $g \in \Gamma_{\zeta_1}$, we have $Dg_z \neq 0$.

(c) For any k of type (II) and for any $z \in L_k$, we have $\chi(\zeta_1, L_k) \neq 0$.

Hence for any $\zeta_2 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is close enough to ζ_1, we have the following.

(a)' $\chi(\zeta_2, L_k) < 0$ for any k of type (I).

(b)' For any k of type (II), for any $z \in L_k$ and for any $g \in \Gamma_{\zeta_2}$, we have $Dg_z \neq 0$.

(c)' For any k of type (II) and for any $z \in L_k$, we have $\chi(\zeta_2, L_k) \neq 0$.

Thus we have proved our lemma. $\square$

Definition 3.57. For a topological space X, we denote by $\mathrm{Con}(X)$ the set of connected components of X.

Definition 3.58. Let $\tau \in \mathfrak{M}_1(\text{Rat})$. For an element $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$, we denote by $U_{\tau,L}$ the space of all finite linear combinations of unitary eigenfunctions of $M_\tau : C(L) \to C(L)$, where we say that an element $\varphi \in C(L)\setminus\{0\}$ is a unitary eigenfunction of $M_\tau : C(L) \to C(L)$ if there exists an element $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $M_\tau(\varphi) = \alpha\varphi$ in L. Also, we say that an element $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ is a unitary eigenvalue of $M_\tau : C(L) \to C(L)$ if there exists an element $\varphi \in C(L) \setminus \{0\}$ such that $M_\tau(\varphi) = \alpha\varphi$. Moreover, we denote by $U_{\tau,L,*}$ the set of unitary eigenvalues of $M_\tau : C(L) \to C(L)$.

Definition 3.59. Let U be an open subset of $\hat{\mathbb{C}}$ and let $\{\varphi_n : U \to \hat{\mathbb{C}}\}_{n=1}^{\infty}$ be a sequence of holomorphic maps from U to $\hat{\mathbb{C}}$. We say that a map $\psi : U \to \hat{\mathbb{C}}$ is a **limit function of** $\{\varphi_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{\varphi_{n_j}\}_{j=1}^{\infty}$ of $\{\varphi_n\}_{n=1}^{\infty}$ such that $\varphi_{n_j} \to \psi$ as $j \to \infty$ locally uniformly on U.

The following lemma is very important to analyze the random dynamical system generated by $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ with $\sharp J_{\ker}(G_\tau) < \infty$. The proof is based on careful observations of limit functions on Fatou components of G_τ by using the hyperbolic metrics on the Fatou components of G_τ.

Lemma 3.60. *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ and suppose $\sharp J(G_\tau) \geq 3$. Let $L \in \text{Min}(G, \hat{\mathbb{C}})$ with $L \cap F(G_\tau) \neq \emptyset$. Let $\Omega_L := \cup_{U \in \text{Con}(F(G_\tau)), U \cap L \neq \emptyset} U$. Suppose that $\sharp((\partial \Omega_L) \cap J_{\ker}(G_\tau)) < \infty$. Then we have the following* (I)(II)(III)*.*

(I) *There exists a Borel subset $\mathcal{A}$ of X_τ with $\tilde{\tau}(\mathcal{A}) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}$ and for each $z \in \Omega_L$, there exists a $\delta = \delta(z, \gamma) > 0$ satisfying that $d(\gamma_{n,1}(z), L) \to 0$ and $diam(\gamma_{n,1}(B(z,\delta))) \to 0$ as $n \to \infty$.*

(II) *We have $C(L) = U_{\tau,L} \oplus \{\varphi \in C(L) \mid M_\tau^n(\varphi) \to 0$ as $n \to \infty\}$ in the Banach space $C(L)$ endowed with the supremum norm and $\dim_{\mathbb{C}} U_{\tau,L} < \infty$. Moreover, setting $r_L := \dim_{\mathbb{C}} U_{\tau,L}$, we have $\sharp\text{Min}(G_\tau^{r_L}, L) = r_L$. Also, there exist $L_1, \ldots, L_{r_L} \in \text{Min}(G_\tau^{r_L}, L)$ such that $\{L_j\}_{j=1}^{r_L} = \text{Min}(G_\tau^{r_L}, L)$, $L = \cup_{j=1}^{r_L} L_j$ and $h(L_j) = L_{j+1}$ for each $h \in \Gamma_\tau$, where $L_{r_L+1} := L_1$. Moreover, for each $j = 1, \ldots, r_L$, there exists a unique element $\omega_{L,j} \in \mathfrak{M}_1(L_j)$ such that $(M_\tau^{r_L})^*(\omega_{L,j}) = \omega_{L,j}$. Also, for each $j = 1, \ldots, r_L$, we have $M_\tau^{nr_L}(\varphi) \to (\int \varphi \, d\omega_{L,j}) \cdot 1_{L_j}$ in the Banach space $C(L_j)$ endowed with the supremum norm as $n \to \infty$ for each $\varphi \in C(L_j)$, $\text{supp}\,\omega_{L,j} = L_j$ and $M_\tau^*(\omega_{L,j}) = \omega_{L,j+1}$ in $\mathfrak{M}_1(L)$ where $\omega_{L,r_L+1} = \omega_{L,1}$. Also, we have $U_{\tau,L,*} = \{\alpha \in \mathbb{C} \mid \alpha^{r_L} = 1\}$ and for each $\alpha \in U_{\tau,L,*}$, we have $\dim_{\mathbb{C}}\{\varphi \in C(L) \mid M_\tau\varphi = \alpha\varphi\} = 1$.*

(III) *The function $T_{L,\tau} : \hat{\mathbb{C}} \to [0,1]$ of probability of tending to L is locally constant on $F(G_\tau)$.*

Proof. Let $\Omega = \Omega_L$. Let $U \in \text{Con}(\Omega)$. Let $a \in U \cap L$. To prove item (I), it suffices to prove that there exists a Borel subset $\mathcal{A}_a$ of X_τ with $\tilde{\tau}(\mathcal{A}_a) = 1$ such that for each $\gamma \in \mathcal{A}_a$, each limit function of the sequence $\{\gamma_{n,1}\}_{n=1}^{\infty}$ around a is constant. Since $L \cap F(G_\tau) \neq \emptyset$, we have $L \cap J_{\ker}(G_\tau) = \emptyset$. Hence there exists a $\delta > 0$ such that

$$B(J_{\ker}(G_\tau), \delta) \cap \overline{G_\tau(B(a,\delta))} = \emptyset. \tag{32}$$

Since we are assuming $\sharp(J(G_\tau)) \geq 3$, we have that $J(G_\tau)$ is perfect. Since we are assuming $\sharp(\partial\Omega \cap J_{\ker}(G_\tau)) < \infty$, taking δ so small, we may assume that

$$J(G_\tau) \setminus B((\partial\Omega) \cap J_{\ker}(G_\tau), \delta) \neq \emptyset. \tag{33}$$

By (32) and (33), taking δ so small, we may assume that for each $U_0 \in \text{Con}(F(G_\tau))$ with $L \cap U_0 \neq \emptyset$, we have

$$(\partial U_0) \setminus B((\partial\Omega) \cap J_{\ker}(G_\tau), \delta) \neq \emptyset. \tag{34}$$

For each $z \in (\partial\Omega) \setminus B((\partial\Omega) \cap J_{\ker}(G_\tau), \delta)$, there exist an element $g_z \in G_\tau$ and an open disk neighborhood V_z of z in $\hat{\mathbb{C}}$ such that $g_z(\overline{V_z}) \subset F(G_\tau)$. Since $(\partial\Omega) \setminus (B((\partial\Omega) \cap J_{\ker}(G_\tau), \delta)$ is compact, there exists a finite set $\{z_1, \ldots, z_p\}$ in $(\partial\Omega) \setminus B((\partial\Omega) \cap J_{\ker}(G_\tau), \delta)$ such that

$$\cup_{j=1}^{p} V_{z_j} \supset (\partial\Omega) \setminus B((\partial\Omega) \cap J_{\ker}(G_\tau), \delta) \text{ and } g_{z_j}(\overline{V_{z_j}}) \subset F(G_\tau) \text{ for each } j. \tag{35}$$

For each $j = 1, \ldots, p$ there exists an element $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_{k(j)}) \in \Gamma_\tau^{k(j)}$ for some $k(j) \in \mathbb{N}$ such that $g_{z_j} = \alpha^j_{k(j)} \circ \cdots \circ \alpha^j_1$. Since $G_\tau(F(G_\tau)) \subset F(G_\tau)$, we may assume that there exists a $k \in \mathbb{N}$ such that for each $j = 1, \ldots, p$, we have $k(j) = k$. For each $j = 1, \ldots, p$, let W_j be a compact neighborhood of α^j in Γ_τ^k such that for each $\beta = (\beta_1, \ldots, \beta_k) \in W_j$, we have $\beta_k \circ \cdots \circ \beta_1(\overline{V_{z_j}}) \subset F(G_\tau)$. Also, for each $j = 1, \ldots, p$, let $B_j := \cup_{B \in \mathrm{Con}(\Omega), B \cap V_{z_j} \neq \emptyset} B$. Let $n \in \mathbb{N}$ and let $c_q = 1/q$ for each $q \in \mathbb{N}$. Let $(i_1, \ldots, i_l)$ be a finite sequence of positive integers with $i_1 < \cdots < i_l$. Let $q > 0$. We denote by $A_{q,j}(i_1, \ldots, i_l)$ the set of elements $\gamma \in X_\tau$ which satisfies all of the following (a) and (b).

(a) $\gamma_{kt,1}(a) \in (\hat{\mathbb{C}} \setminus B(\partial\Omega, c_q)) \cap B_j$ if $t \in \{i_1, \ldots, i_l\}$.

(b) $\gamma_{kt,1}(a) \notin (\hat{\mathbb{C}} \setminus B(\partial\Omega, c_q)) \cap B_j$ if $t \in \{1, \ldots, i_l\} \setminus \{i_1, \ldots, i_l\}$.

Moreover, when $l \geq n$, we denote by $B_{q,j,n}(i_1, \ldots, i_l)$ the set of elements $\gamma \in X_\tau$ which satisfies items (a) and (b) above and the following (c).

(c) $(\gamma_{ki_s+1}, \ldots, \gamma_{ki_s+k}) \notin W_j$ for each $s = n, n+1, \ldots, l$.

Furthermore, we denote by $C_{q,j,n}(i_1, \ldots, i_l)$ the set of elements $\gamma \in X_\tau$ which satisfies items (a) and (b) above and the following (d).

(d) $(\gamma_{ki_s+1}, \ldots, \gamma_{ki_s+k}) \notin W_j$ for each $s = n, n+1, \ldots, l-1$.

Furthermore, for each q, j, n, l with $l \geq n$, let $B_{q,j,n,l} := \bigcup_{i_1 < \cdots < i_l} B_{q,j,n}(i_1, \ldots, i_l)$. Let $\mathcal{D} := \bigcup_{q=1}^\infty \bigcup_{j=1}^p \bigcup_{n \in \mathbb{N}} \bigcap_{l \geq n} B_{q,j,n,l}$. We show the following claim.

Claim 1. Let $\gamma \in X_\tau$ be such that there exists a non-constant limit function of the sequence $\{\gamma_{n,1}|_U : U \to \hat{\mathbb{C}}\}_{n=1}^\infty$. Then $\gamma \in \mathcal{D}$.

To show this claim, let $\gamma \in X_\tau$ be an element such that there exists a non-constant limit function of $\{\gamma_{n,1}|_U : U \to \hat{\mathbb{C}}\}_{n=1}^\infty$. Then there exists a $q \in \mathbb{N}$, a $j \in \{1, \ldots, p\}$, and a strictly increasing sequence $\{i_l\}_{l=1}^\infty$ in $\mathbb{N}$ such that $\gamma \in \bigcap_{l=1}^\infty A_{q,j}(i_1, \ldots, i_l)$ and any subsequence of $\{\gamma_{ki_l,1}|_U : U \to \hat{\mathbb{C}}\}_{l=1}^\infty$ does not converge to a constant map. Suppose that there exists a strictly increasing sequence $\{l_p\}_{p=1}^\infty$ in $\mathbb{N}$ such that for each $p \in \mathbb{N}$, $(\gamma_{ki_{l_p}+1}, \ldots, \gamma_{ki_{l_p}+k}) \in W_j$. Since $\sharp J(G_\tau) \geq 3$, for each $A \in \mathrm{Con}(F(G_\tau))$, we can take the hyperbolic metric on A. From the definition of W_j and [24, Pick Theorem], we obtain that there exists a constant $0 < \alpha < 1$ such that for each $p \in \mathbb{N}$ and for each a' in a small neighborhood U_a of a, we have $\|(\gamma_{ki_{l_p}+k} \cdots \gamma_{ki_{l_p}+1})'(\gamma_{ki_{l_p},1}(a'))\|_h \leq \alpha$, where for each $g \in G_\tau$ and for each $z \in F(G_\tau)$, $\|g'(z)\|_h$ denotes the norm of the derivative of g at z measured from the hyperbolic metric on the element of $\mathrm{Con}(F(G_\tau))$ containing z to that on the element of $\mathrm{Con}(F(G_\tau))$ containing $g(z)$. Hence, for each $a' \in U_a$, $\|(\gamma_{ki_{l_p},1})'(a')\|_h \to 0$ as $p \to \infty$. However, this is a contradiction, since $\{\gamma_{ki_{l_p},1}|_U\}_{p=1}^\infty$ does not converge to a constant map. Therefore, $\gamma \in \mathcal{D}$. Thus, we have proved claim 1.

Let $\eta := \max_{j=1}^p (\otimes_{s=1}^k \tau)(\Gamma_\tau^k \setminus W_j)$ (< 1). Then we have for each (l, n) with $l \geq n$,

$$\begin{aligned}\tilde{\tau}(B_{q,j,n}(i_1, \ldots, i_{l+1})) &\leq \tilde{\tau}(C_{q,j,n}(i_1, \ldots, i_{l+1}) \cap \{\gamma \in X_\tau \mid (\gamma_{ki_{l+1}+1}, \ldots, \gamma_{ki_{l+1}+k}) \notin W_j\}) \\ &\leq \tilde{\tau}(C_{q,j,n}(i_1, \ldots, i_{l+1})) \cdot \eta.\end{aligned}$$

Hence, for each l with $l \geq n$,

$$\begin{aligned}\tilde{\tau}(B_{q,j,n,l+1}) =& \tilde{\tau}(\bigcup_{i_1 < \cdots < i_{l+1}} B_{q,j,n}(i_1, \ldots, i_{l+1})) = \sum_{i_1 < \cdots < i_{l+1}} \tilde{\tau}(B_{q,j,n}(i_1, \ldots, i_{l+1})) \\ \leq& \sum_{i_1 < \cdots < i_{l+1}} \eta \tilde{\tau}(C_{q,j,n}(i_1, \ldots, i_{l+1})) = \eta \tilde{\tau}(\bigcup_{i_1 < \cdots < i_{l+1}} C_{q,j,n}(i_1, \ldots, i_{l+1})) \leq \eta \tilde{\tau}(B_{q,j,n,l}).\end{aligned}$$

Therefore $\tilde{\tau}(\mathcal{D}) \leq \sum_{q=1}^\infty \sum_{j=1}^p \sum_{n \in \mathbb{N}} \tilde{\tau}(\bigcap_{l \geq n} B_{q,j,n,l}) = 0$. Hence we have proved item (I) of our lemma.

We now prove item (II). Since $L \cap J_{\ker}(G_\tau) = \emptyset$, Lemmas 3.13 and 3.15 imply that $L \subset F^0_{pt}(\tau)$. Thus we obtain that for each $\varphi \in C(L)$, $\{M^n_\tau(\varphi)|_L\}_{n=1}^\infty$ is equicontinuous on L. Combining this with item (I) and the argument in [36, page 83-87], we easily see that item (II) holds.

We now prove (III). Let U be a connected component of $F(G_\tau)$ and let $x_1, x_2 \in U$. Let $\mathcal{H}_i := \{\gamma \in X_\tau \mid d(\gamma_{n,1}(x_i), L) \to 0 \text{ as } n \to \infty\}$ for each $i = 1, 2$. Then $T_{L,\tau}(x_i) = \tilde{\tau}(\mathcal{H}_i)$. Let $\mathcal{I}_i := \{\gamma \in \mathcal{H}_i \mid \exists n \in \mathbb{N}. \text{ such that } \gamma_{n,1}(x_i) \in \Omega\}$. Let $\mathcal{A}$ be as in (I). Since $\tilde{\tau}(\mathcal{A}) = 1$ and $\tilde{\tau}$ is σ-invariant, we have $\tilde{\tau}(\mathcal{I}_i) = \tilde{\tau}(\mathcal{I}_i \cap \cap_{n=1}^\infty \sigma^{-n}(\mathcal{A}))$. By (I), we have $\mathcal{I}_1 \cap \cap_{n=1}^\infty \sigma^{-n}(\mathcal{A}) = \mathcal{I}_2 \cap \cap_{n=1}^\infty \sigma^{-n}(\mathcal{A})$. Hence $\tilde{\tau}(\mathcal{I}_1) = \tilde{\tau}(\mathcal{I}_2)$. Let $\gamma \in \mathcal{H}_1 \setminus \mathcal{I}_1$. Then $d(\gamma_{n,1}(x_1), L \cap J(G_\tau)) \to 0$ as $n \to \infty$ and every limit function of $\gamma_{n,1}$ on U should be constant. Therefore $\gamma \in \mathcal{H}_2 \setminus \mathcal{I}_2$. Thus $\mathcal{H}_1 \setminus \mathcal{I}_1 \subset \mathcal{H}_2 \setminus \mathcal{I}_2$. Similarly, we have $\mathcal{H}_2 \setminus \mathcal{I}_2 \subset \mathcal{H}_1 \setminus \mathcal{I}_1$. Hence $\mathcal{H}_1 \setminus \mathcal{I}_1 = \mathcal{H}_2 \setminus \mathcal{I}_2$. From these arguments, it follows that $T_{L,\tau}(x_1) = T_{L,\tau}(x_2)$. Therefore $T_{L,\tau}$ is locally constant on $F(G_\tau)$.

Thus, we have completed the proof of Lemma 3.60. □

Definition 3.61. Under the assumptions of Lemma 3.60, we call the number r_L the period of (τ, L).

Remark 3.62. The above argument in the proof of item (I) generalizes the argument in the proof of [36, Lemma 5.2]. In the proof of [36, Lemma 5.2], in order to make the argument more precise, "and $\{\gamma_{ki_l,1}|_U : U \to \hat{\mathbb{C}}\}_{l=1}^\infty$ converges to a non-constant map." ([36, page 81,line -5]) should be "and any subsequence of $\{\gamma_{ki_l,1}|_U : U \to \hat{\mathbb{C}}\}_{l=1}^\infty$ does not converge to a constant map." and "converges to a non-constant map." ([36, page 82, line 4]) should be "does not converge to a constant map." Also, in the proof of [36, Lemma 5.3], the definition of $E_{n,m}$ should be "$E_{n,m} := \{\gamma \in \mathcal{A} \mid \gamma_{ik,1}(a_0) \in \cup_{j=1}^p V_{z_j} \cap B(\partial J(G_\tau), b), i = n, \dots, m\}$, where the number b is equal to $\min\{d(u,v) \mid u \in \partial J(G_\tau), v \in \cup_{j=1}^p \cup_{(\gamma_1,\dots,\gamma_k) \in W_j} \gamma_k \cdots \gamma_1(V_{z_j})\} > 0$".

The following is an important result on random dynamical systems generated by $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ with $\sharp J_{\ker}(G_\tau) < \infty$. In the proof we use the no-wandering-domain theorem ([29]) and the Fatou-Shishikura inequality ([26]).

Proposition 3.63. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ with $\sharp J(G_\tau) \geq 3$. Suppose $\sharp J_{\ker}(G_\tau) < \infty$. Then $1 \leq \sharp \mathrm{Min}(G_\tau) < \infty$ and for each $L \in \mathrm{Min}(G_\tau)$ with $L \cap F(G_\tau) \neq \emptyset$, statements* (I)(II)(III) *of Lemma 3.60 hold.*

Proof. By Lemma 3.60, for each $L \in \mathrm{Min}(G_\tau)$ with $L \cap F(G_\tau) \neq \emptyset$, statements (I)(II)(III) of Lemma 3.60 hold. Also, by Lemma 3.60, we obtain that

$$\text{for each } W \in \mathrm{Con}(F(G_\tau)), \sharp\{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) \mid L \cap W \neq \emptyset\} \leq 1. \tag{36}$$

We now suppose that $\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = \infty$. Then, since we are assuming $\sharp J_{\ker}(G_\tau) < \infty$, we obtain that

$$\sharp\{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) \mid L \cap F(G_\tau) \neq \emptyset\} = \infty. \tag{37}$$

We now show the following claim.
Claim 1. Let $\{L_j\}_{j=1}^\infty$ be a sequence in $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ consisting of mutually distinct elements such that for each j, $L_j \cap F(G_\tau) \neq \emptyset$. Moreover, let $\{w_j\}_{j=1}^\infty$ be a sequence in $\hat{\mathbb{C}}$ such that $w_j \in L_j$ for each j and $\{w_j\}$ tends to a point $w_\infty \in \hat{\mathbb{C}}$. Then $w_\infty \in J_{\ker}(G_\tau)$.

To show this claim, suppose that $w_\infty \notin J_{\ker}(G_\tau)$. Then there exists an element $\alpha \in G_\tau$ such that $\alpha(w_\infty) \in F(G_\tau)$. Let $U \in \mathrm{Con}(F(G_\tau))$ with $\alpha(w_\infty) \in U$. Then for each large j, we have $U \cap L_j \neq \emptyset$. However, this contradicts (36). Hence, Claim 1 holds.

Let $\{L_j\}_{j=1}^\infty$ be a sequence in $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ consisting of mutually distinct elements such that for each j, $L_j \cap F(G_\tau) \neq \emptyset$. For each j, let $z_j \in L_j \cap F(G_\tau)$ be a point. Since $\sharp J(G_\tau) \geq 3$, by the density of repelling fixed points in the Julia set ([27]), either there exists a loxodromic element of $\mathrm{Aut}(\hat{\mathbb{C}}) \cap G$ or there exists an element in G of degree two or more.

Suppose that there exists a loxodromic element $g \in \text{Aut}(\hat{\mathbb{C}}) \cap G$. Let a_g be the attracting fixed point of g. Then for each j, we have $g^n(z_j) \to a_g$ as $n \to \infty$. This implies $a_g \in L_j$ for each j. However, this is a contradiction.

Suppose that there exists an element $g \in G$ with $\deg(g) \geq 2$. By the no-wandering-domain theorem ([29]), we have that for each $z \in F(G_\tau)$, $g^n(z)$ tends to one of the following cycles. (I) attracting cycle. (II) parabolic cycle. (III) Siegel disc cycle. (IV) Hermann ring cycle. Moreover, by the Fatou-Shishikura inequality ([26]), the number of those cycles for one element g is finite. Suppose that there exist a subsequence $\{z_{j_k}\}$ of $\{z_j\}$ and a sequence $\{n_k\}$ in $\mathbb{N}$ such that $g^{n_k}(z_{j_k})$ tends to an attracting or parabolic cycle c_g of g. Then $c_g \in L_{j_k}$ for each large $k \in \mathbb{N}$ and this is a contradiction. Therefore, there exist a subsequence $\{z_{j_k}\}$ of $\{z_j\}$ and a sequence $\{n_k\}$ in $\mathbb{N}$ such that $g^{n_k}(z_{j_k})$ belongs to a Siegel disk cycle or Hermann ring cycle of g for each k. By taking a higher iterate of g, we may assume that the period of the cycle is one. Also, by renaming $g^{n_k}(z_{j_k})$ as z_k, we may assume that there exists a $B \in \text{Con}(F(g))$ which is either Siegel disk or Hermann ring of g such that for each j, we have $z_j \in B \cap L_j \cap F(G_\tau)$. Note that each $B \cap L_j \cap F(G_\tau)$ is a union of analytic Jordan curves in B. Let

$$D := \{z \in \hat{\mathbb{C}} \mid \text{ for each } \delta > 0, \sharp\{j \in \mathbb{N} \mid B(z, \delta) \cap B \cap L_j \cap F(G_\tau) \neq \emptyset\} = \infty\}.$$

Then by Claim 1, we have $D \subset J_{\ker}(G_\tau)$. Since we are assuming $\sharp J_{\ker}(G_\tau) < \infty$, Claim 1 again implies that for any connected component A of ∂B, we cannot have that $A \subset D$. Thus it follows that there exists a point $z_\infty \in J_{\ker}(G_\tau)$ such that $D = \{z_\infty\}$. Therefore B is a Siegel disk for g and z_∞ is the center of B, i.e. $z_\infty \in B$ and $g(z_\infty) = z_\infty$. Let $0 < \epsilon < \frac{1}{2}\min\{d(a, b) \mid a, b \in J_{\ker}(G_\tau), a \neq b\}$. For each $j \in \mathbb{N}$, let C_j be the connected component of $\hat{\mathbb{C}} \setminus (B \cap L_j)$ such that $z_\infty \in C_j$. Let $e_0 := \max\{\|Dh_z\|_s \mid h \in \Gamma_\tau, z \in \hat{\mathbb{C}}\}$. We may assume that for each j, $\max_{a \in C_j} d(z_\infty, a) < \frac{1}{2}e_0^{-1}\epsilon$. Since $z_\infty \in J_{\ker}(G_\tau) \subset J(G_\tau)$, it follows that for each j there exists an element $h_j \in G_\tau$ such that

$$\max_{a \in C_j} d(h_j(z_\infty), h_j(a)) \geq \frac{1}{2}e_0^{-1}\epsilon. \tag{38}$$

We may assume that fixing the generator system Γ_τ of G_τ, the word length of h_j is the minimum among the word lengths of elements of G_τ satisfying the same property as that of h_j. Then, by (38) and the minimality of the word length of h_j, it follows that

$$\max_{a \in C_j} d(h_j(z_\infty), h_j(a)) < \epsilon, \text{ for each } j. \tag{39}$$

Let $a_j \in C_j$ be a point such that $d(h_j(z_\infty), h_j(a_j)) = \max_{a \in C_j} d(h_j(z_\infty), h_j(a))$. Then $a_j \in \partial C_j \subset L_j$ for each j. Hence, setting $u_j = h_j(a_j)$, we have

$$u_j \in L_j \text{ for each } j. \tag{40}$$

By (38) and (39), we have

$$\frac{1}{2}e_0^{-1}\epsilon \leq d(h_j(z_\infty), u_j) < \epsilon. \tag{41}$$

Since $h_j(z_\infty) \in J_{\ker}(G_\tau)$ and by the way of the choice of ϵ, (41) implies that $\{u_j\}_{j=1}^\infty$ cannot accumulate in any point of $J_{\ker}(G_\tau)$. Combining this with (40) and Claim 1, we obtain a contradiction. Hence, $\sharp\text{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.

Thus we have proved our proposition. □

The following is an important and interesting object in random dynamics.

Definition 3.64. Let A be a subset of $\hat{\mathbb{C}}$. Let $\tau \in \mathfrak{M}_1(\text{Rat})$. For each $z \in \hat{\mathbb{C}}$, we set $T_{A,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau \mid d(\gamma_{n,1}(z), A) \to 0 \text{ as } n \to \infty\})$. This is the **probability of tending to** A **regarding the random orbits starting with the initial value** $z \in \hat{\mathbb{C}}$. For any $a \in \hat{\mathbb{C}}$, we set $T_{a,\tau} := T_{\{a\},\tau}$.

We now prove the following theorem regarding the systems with finite kernel Julia sets.

Theorem 3.65. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ with $\sharp J(G_\tau) \geq 3$. Suppose that $\sharp J_{\ker}(G_\tau) < \infty$ and for each $z \in F(G_\tau)$, we have $\overline{G_\tau(z)} \cap (\bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), L \not\subset J_{\ker}(G_\tau)} L) \neq \emptyset$. Then we have the following.*

(i) $\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$. *Moreover, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, we have*
$$C(L) = U_{\tau,L} \oplus \{\varphi \in C(L) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$$
in the Banach space $C(L)$ endowed with the supremum norm and $\dim_{\mathbb{C}} U_{\tau,L} < \infty$. Moreover, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, let $r_L = \dim_C(U_{\tau,L})$. Then $\sharp \mathrm{Min}(G_\tau^{r_L}, L) = r_L$. Also, there exist $L_1, \ldots, L_{r_L} \in \mathrm{Min}(G_\tau^{r_L}, L)$ such that $\{L_j\}_{j=1}^{r_L} = \mathrm{Min}(G_\tau^{r_L}, L)$, $L = \cup_{j=1}^{r_L} L_j$ and $h(L_i) \subset L_{i+1}$ for each $h \in \Gamma_\tau$, where $L_{r_L+1} := L_1$.

(ii) *For each $j = 1, \ldots, r_L$, there exists a unique element $\omega_{L,j} \in \mathfrak{M}_1(L_j)$ such that $(M_\tau^{r_L})^*(\omega_{L,j}) = \omega_{L,j}$. Also, for each $j = 1, \ldots, r_L$, we have $M_\tau^{nr_L}(\varphi) \to (\int \varphi \, d\omega_{L,j}) \cdot 1_{L_j}$ in the Banach space $C(L_j)$ endowed with the supremum norm as $n \to \infty$ for each $\varphi \in C(L_j)$, $\mathrm{supp}\, \omega_{L,j} = L_j$ and $M_\tau^*(\omega_{L,j}) = \omega_{L,j+1}$ in $\mathfrak{M}_1(L)$ where $\omega_{L,r_L+1} = \omega_{L,1}$. Moreover, $U_{\tau,L,*} = \{a \in \mathbb{C} \mid a^{r_L} = 1\}$ and for each $a \in U_{\tau,L,*}$, we have $\dim_{\mathbb{C}}\{\varphi \in C(L) \mid M_\tau \varphi = a\varphi\} = 1$.*

(iii) *Let $l := \prod_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} r_L$. For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, for each $j = 1, \ldots, r_L$ and for each $y \in \hat{\mathbb{C}}$, let $\alpha(L_j, y) = \tilde{\tau}(\{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid d(\gamma_{nl,1}(y), L_j) \to 0 \text{ as } n \to \infty.\})$. Then for each $y \in \hat{\mathbb{C}}$ and for each $\varphi \in C(\hat{\mathbb{C}})$, we have*
$$M_\tau^{nl}(\varphi)(y) \to \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{j=1}^{r_L} \alpha(L_j, y) \int \varphi \, d\omega_{L,j} \text{ as } n \to \infty \text{ (pointwise convergence)}, \quad (42)$$
i.e. we have
$$(M_\tau^*)^{nl}(\delta_y) \to \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{j=1}^{r_L} \alpha(L_j, y) \omega_{L,j} \text{ as } n \to \infty \text{ in } \mathfrak{M}_1(\hat{\mathbb{C}}) \quad (43)$$
with respect to the weak convergence topology. Also,
$$(M_\tau^*)^l(\textstyle\sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{j=1}^{r_L} \alpha(L_j, y)\omega_{L,j}) = \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{j=1}^{r_L} \alpha(L_j, y)\omega_{L,j}.$$

(iv) *For each $z \in \hat{\mathbb{C}}$ there exists a Borel subset A_z of $\Gamma_\tau^{\mathbb{N}}$ with $\tilde{\tau}(A_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in A_z$, we have $d(\gamma_{n,1}(z), \cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L) \to 0$ as $n \to \infty$.*

(v) *There exists a Borel subset $\mathcal{A}$ of $\mathrm{Rat}^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}) = 1$ such that for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$, for each point $z \in \Omega_L := \cup_{U \in \mathrm{Con}(F(G_\tau)): U \cap L \neq \emptyset} U$ and for each $\gamma \in \mathcal{A}$, there exists a $\delta = \delta(z, \gamma) > 0$ such that $\mathrm{diam}(\gamma_{n,1}(B(z, \delta))) \to 0$ and $d(\gamma_{n,1}(z), L) \to 0$ as $n \to \infty$. In particular, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$ and for each $j = 1, \ldots, r_L$, if $y \in \Omega_{L,j} := \cup_{U \in \mathrm{Con}(F(G_\tau)): U \cap L_j \neq \emptyset} U$, then $\alpha(L_j, y) = 1$, and if $y \in \Omega_{L',i}$ with $(L', i) \neq (L, j)$ then $\alpha(L_j, y) = 0$.*

(vi) *Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $j = 1, \ldots, r_L$. Then the functions $T_{L,\tau} : \hat{\mathbb{C}} \to [0.1]$ and $\alpha(L_j, \cdot) : \hat{\mathbb{C}} \to [0, 1]$ are locally constant on $F(G_\tau)$.*

(vii) *Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $j = 1, \ldots, r_L$. Then for each $y \in F_{pt}^0(\tau)$, we have*
$$\lim_{z \in \hat{\mathbb{C}}, z \to y} T_{L,\tau}(z) = T_{L,\tau}(y) \text{ and } \lim_{z \in \hat{\mathbb{C}}, z \to y} \alpha(L_j, z) = \alpha(L_j, y).$$

Proof. Let $\mathcal{L} := \bigcup_{L \in \text{Min}(G_\tau, \hat{\mathbb{C}}), L \not\subset J_{\ker}(G_\tau)} L$. Then, by the assumption of our theorem, we have $\mathcal{L} \neq \emptyset$. Moreover, we have $\mathcal{L} = \bigcup_{L \in \text{Min}(G_\tau, \hat{\mathbb{C}}), L \cap F(G_\tau) \neq \emptyset} L$. Let

$$V := \bigcup \{U \in \text{Con}(F(G_\tau)) \mid \exists L \in \text{Min}(G_\tau, \hat{\mathbb{C}}) \text{ with } L \cap U \neq \emptyset\}.$$

Then $G_\tau(V) \subset V$. Moreover, by the assumptions of our theorem, we obtain $\cap_{g \in G_\tau} g^{-1}(\hat{\mathbb{C}} \setminus V) = J_{\ker}(G_\tau)$. Therefore, the statements (i)–(v) of our theorem follow from Lemma 3.26, Proposition 3.63 and Lemma 3.15.

We now prove statement (vi). Let $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ and $j = 1, \ldots, r_L$. If $L \not\subset J_{\ker}(G_\tau)$, then by Lemma 3.60 (III) and its proof, the functions $T_{L,\tau}$ and $\alpha(L_j, \cdot)$ are locally constant on $F(G_\tau)$. If $L \subset J_{\ker}(G_\tau)$, then for any $U \in \text{Con}(F(G_\tau))$, for any $x, y \in U$ and for any $\gamma \in X_\tau$ with $d(\gamma_{n,1}(x), L) \to 0$ $(n \to \infty)$, we have that any limit function of $\{\gamma_{n,1}\}_{n=1}^\infty$ is constant on U. Hence $d(\gamma_{n,1}(y), L) \to 0$ as $n \to \infty$. This argument implies that $T_{L,\tau}$ is constant on U for any $U \in \text{Con}(F(G_\tau))$. Therefore $T_{L,\tau}$ is locally constant on $F(G_\tau)$. By the same method as above, we can show that $\alpha(L_j, \cdot)$ is locally constant on $F(G_\tau)$.

We now prove (vii). Let $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ and let $j = 1, \ldots, r_L$. Since $\sharp \text{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$, there exists an element $\varphi_L \in C(\hat{\mathbb{C}})$ such that $\varphi_L|_L = 1$ and $\varphi_L|_{L'} = 0$ for any $L' \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ with $L' \neq L$. By statement (iv), we have $T_{L,\tau}(x) = \lim_{n\to} M_\tau^n(\varphi_L)(x)$ for any $x \in \hat{\mathbb{C}}$. Thus for any $y \in F_{pt}^0(\tau)$, we have $\lim_{z \in \hat{\mathbb{C}}, z \to y} T_{L,\tau}(z) = T_{L,\tau}(y)$. Similarly, we can show that for any $y \in F_{pt}^0(\tau)$, $\lim_{z \in \hat{\mathbb{C}}, z \to y} \alpha(L_j, z) = \alpha(L_j, y)$.

Thus we have proved our theorem. □

We now prove the following theorem, which is a generalization of [36, Theorem 3.15].

Theorem 3.66. *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$. Suppose that $\sharp J_{\ker}(G_\tau) < \infty$, $\sharp J(G_\tau) \geq 3$ and that for each $L \in \text{Min}(G_\tau, J_{\ker}(G_\tau))$, $\chi(\tau, L) < 0$. Then we have $F_{pt}^0(\tau) = \hat{\mathbb{C}}, F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$, $\text{Leb}_2(J_\gamma) = 0$ for $\tilde{\tau}$-a.e.$\gamma \in (\text{Rat})^{\mathbb{N}}$, and all statements in* [36, Theorem 3.15 (1)–(3),(4a),(5)(6),(8)–(16), (19)(20)] *hold for τ. Moreover, for each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of X_τ with $\tilde{\tau}(\mathcal{A}_z) = 1$ satisfying that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$ and for each $m \in \mathbb{N} \cup \{0\}$, we have*

$$\lim_{n\to\infty} \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(z)}\|_s = 0.$$

Also, if, in addition to the assumptions of our theorem, each $L \in Min(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$ is attracting for τ, then there exist a constant $c_\tau < 0$ and a constant $\rho_\tau \in (0,1)$ such that for each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of X_τ with $\tilde{\tau}(\mathcal{A}_z) = 1$ such that for each $\gamma \in \mathcal{A}_z$ and for each $m \in \mathbb{N} \cup \{0\}$, we have the following (a) *and* (b).

(a)
$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(z)}\|_s \leq c_\tau < 0.$$

(b) *There exist a constant $\delta = \delta(\tau, z, \gamma, m) > 0$, a constant $\zeta = \zeta(\tau, z, \gamma, m) > 0$ and an attracting minimal set $L = L(\tau, z, \gamma)$ of τ such that*

$$diam(\gamma_{n+m,1+m}(B(\gamma_{m,1}(z), \delta))) \leq \zeta \rho_\tau^n \quad \textit{for all } n \in \mathbb{N},$$

and

$$d(\gamma_{n+m,1+m}(\gamma_{m,1}(z)),\ L) \leq \zeta \rho_\tau^n \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We modify the proof of Theorem 3.37. By the assumption of our theorem, the set Ω in the proof of Theorem 3.37 is equal to $\hat{\mathbb{C}}$. By Theorem 3.37, we see that for each $z \in \hat{\mathbb{C}}$, $\tilde{\tau}(\{\gamma \in X_\tau \mid z \in J_\gamma\}) = 0$ and $F_{pt}^0(\tau) = \hat{\mathbb{C}}$. Therefore by [36, Lemma 4.2], $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.

Let $\delta_1 > 0$ be a small number. Let $\epsilon > 0$ be an arbitrarily small number. Then by the argument in the proof of Lemma 3.13, there exist a $\delta_2 > 0$ with $\delta_2 < \delta_1$ and a Borel subset A_ϵ of X_τ with $\tilde{\tau}(A_\epsilon) \geq 1 - \epsilon$ such that for each $L \in \text{Min}(G_\tau, J_{\ker}(G_\tau))$, for each $z \in L$, and for each $\gamma = (\gamma_1, \gamma_2 \ldots) \in A_\epsilon$, we have $\text{diam}(\gamma_{n,1}(B(z, \delta_2))) \leq \delta_1$. For this δ_2, by the argument in the proof of Lemma 3.13 again, there exist a $\delta_3 > 0$ and a Borel subset B_ϵ of X_τ with $\tilde{\tau}(B_\epsilon) \geq 1 - \epsilon$ such that for each $L \in \text{Min}(G_\tau, J_{\ker}(G_\tau))$, for each $z \in L$, and for each $\gamma = (\gamma_1, \gamma_2 \ldots) \in B_\epsilon$, we have $\text{diam}(\gamma_{n,1}(B(z, \delta_3))) \leq \delta_2$.

Let $I_1 := \{L \in \text{Min}(G_\tau, \hat{\mathbb{C}}) \mid L \subset J_{\ker}(G_\tau)\}$ and $I_2 := \{L \in \text{Min}(G_\tau, \hat{\mathbb{C}}) \mid L \cap F(G_\tau) \neq \emptyset\}$. Note that $I_1 \cup I_2 = \text{Min}(G_\tau, \hat{\mathbb{C}})$. For each $L \in I_2$, let $W_L := \cup_{U \in \text{Con}(F(G_\tau)), U \cap L \neq \emptyset} U$. Then for each $z \in \hat{\mathbb{C}}$, there exists an element $g_z \in G_\tau$ such that $g_z(z) \in B(\cup_{L \in I_1} L, \delta_3) \cup \cup_{L \in I_2} W_L$. Let $\delta_z > 0$ be a number such that $g_z(\overline{B(z, \delta_z)}) \subset B(\cup_{L \in I_1} L, \delta_3) \cup \cup_{L \in I_2} W_L$. Since $\hat{\mathbb{C}}$ is compact, there exist finitely many points $z_1, \ldots, z_n \in \hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} = \cup_{j=1}^n B(z_j, \delta_{z_j})$. Note that $G_\tau(W_L) \subset W_L$ for each $L \in I_2$. Thus if $g_z(z) \in W_L$ for some $L \in I_2$, then for each $g \in G_\tau$, we have $g g_z(z) \in W_L$. Moreover, for each $L \in I_1$, for each $z \in L$, for each $\gamma \in B_\epsilon$ and for each $n \in \mathbb{N}$, we have $\gamma_{n,1}(B(z, \delta_3)) \subset B(\cup_{L \in I_1} L, \delta_2)$. Hence, considering $\alpha_j \circ g_{z_j}$ for some $\alpha_j \in G_\tau$ for each j, we have the following claim.

Claim 1. There exist an $l \in \mathbb{N}$ and n elements $h_{z_1}, \ldots, h_{z_n} \in G_\tau$ such that each h_{z_j} is the composition of l elements of Γ_τ and such that for each $j = 1 \ldots, n$, we have $h_{z_j}(\overline{B(z, \delta_{z_j})}) \subset B(\cup_{L \in I_1} L, \delta_2) \cup \cup_{L \in I_2} W_L$.

For each $j = 1, \ldots, l$, let $(\gamma_1^j, \ldots, \gamma_l^j) \in \Gamma_\tau^l$ be an element such that $h_{z_j} = \gamma_l^j \circ \cdots \circ \gamma_1^j$. For each $j = 1, \ldots, n$, let V_j be a neighborhood of $(\gamma_1^j, \ldots, \gamma_l^j) \in \Gamma_\tau^l$ such that for each $(\alpha_1, \ldots, \alpha_l) \in V_j$, we have $\alpha_l \cdots \alpha_1(\overline{B(z_j, \delta_{z_j})}) \subset B(\cup_{L \in I_1} L, \delta_2) \cup \cup_{L \in I_2} W_L$. Let $\Omega_1, \ldots, \Omega_t$ be the measurable partition of $\hat{\mathbb{C}}$ such that each Ω_i is a finite intersection of elements of $\{B(z_j, \delta_{z_j})\}_{j=1}^n$. For each $i = 1, \ldots, t$, let $\varphi(i) \in \{1, \ldots, n\}$ be an element such that $\Omega_i \subset B(z_{\varphi(i)}, \delta_{z_{\varphi(i)}})$. For each $z \in \hat{\mathbb{C}}$, let $i(z) \in \{1, \ldots, t\}$ be the unique element such that $z \in \Omega_{i(z)}$. Let $j(z) = \varphi(i(z)) \in \{1, \ldots, n\}$. For each $n \in \mathbb{N}$ and each $z \in \hat{\mathbb{C}}$, let $C_{n,z}$ be the set of elements $\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau$ satisfying the following.

- $(\gamma_1, \ldots, \gamma_l) \notin V_{j(z)}, (\gamma_{l+1}, \ldots, \gamma_{2l}) \notin V_{j(\gamma_{l,1}(z))}, \ldots, (\gamma_{(n-2)l+1}, \ldots, \gamma_{(n-1)l}) \notin V_{j(\gamma_{(n-2)l,1}(z))}$, and $(\gamma_{(n-1)l+1}, \ldots, \gamma_{nl}) \in V_{j(\gamma_{(n-1)l,1}(z))}$.

Similarly, let $D_{n,z} := \{\gamma \in C_{n,z} \mid (\gamma_{nl+1}, \gamma_{nl+2}, \ldots) \notin A_\epsilon\}$. Moreover, let E_z be the set of elements $\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau$ satisfying that for each $n \in \mathbb{N}, (\gamma_{(n-1)l+1}, \ldots, \gamma_{nl}) \notin V_{j(\gamma_{(n-1)l,1}(z))}$. Then for each $z \in \hat{\mathbb{C}}$ we have

$$\{\gamma \in X_\tau \mid \gamma_{n,1}(z) \notin B(\cup_{L \in I_1} L, \delta_1) \cup \cup_{L \in I_2} W_L \text{ for infinitely many } n \in \mathbb{N}\} \subset \cup_{n=1}^\infty D_{n,z} \cup E_z.$$

It is easy to see that $\tilde{\tau}(E_z) = 0$. Moreover,

$$\tilde{\tau}(\cup_{n=1}^\infty D_{n,z}) = \sum_{n=1}^\infty \tilde{\tau}(D_{n,z}) = \sum_{n=1}^\infty \tilde{\tau}(C_{n,z}) \cdot \tilde{\tau}(X_\tau \setminus A_\epsilon) = \tilde{\tau}(\cup_{n=1}^\infty C_{n,z}) \cdot \tilde{\tau}(X_\tau \setminus A_\epsilon) \leq \epsilon.$$

Hence

$$\tilde{\tau}(\{\gamma \in X_\tau \mid \gamma_{n,1}(z) \notin B(\cup_{L \in I_1} L, \delta_1) \cup \cup_{L \in I_2} W_L \text{ for infinitely many } n \in \mathbb{N}\}) \leq \epsilon. \quad (44)$$

Since ϵ, δ_1 are arbitrary, combining (44) and Lemma 3.60 implies that for each $z \in \hat{\mathbb{C}}$, there exists a Borel subset Q_z of X_τ with $\tilde{\tau}(Q_z) = 1$ such that for each $\gamma \in Q_z$, we have

$$d(\gamma_{n,1}(z), \cup_{L \in \text{Min}(G_\tau, \hat{\mathbb{C}})} L) \to 0 \text{ as } n \to \infty. \quad (45)$$

By the result $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$, (45), Lemma 3.25 and its proof, Lemma 3.30 and its proof, Lemma 3.60, the proof of [36, Theorem 3.15] and [36, Theorem 3.14], the statement of our theorem holds. $\square$

Remark 3.67. Under the assumptions of Theorem 3.66, suppose that $J_{\ker}(G_\tau) \neq \emptyset$. Then τ is not mean stable. Also, τ does not satisfy the assumptions of [36, Theorem 3.15], although most of the statements of [36, Theorem 3.15] hold for τ. Note that we have many examples $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ with $J_{\ker}(G_\tau) \neq \emptyset$ satisfying the assumptions of Theorem 3.66. See Section 5, Example 5.4.

We now give the definition of nice sets and strongly nice sets of Rat.

Definition 3.68. Let $\mathcal{Y}$ be a weakly nice subset of Rat with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps, where $\mathcal{W}_j = \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$.

- We say that $\mathcal{Y}$ is **nice** (with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps) if for each $z \in S_{\min}(\{\mathcal{W}_j\}_{j=1}^m)$ (see Definition 3.47) and for each $j = 1, \ldots, m$, either (a) the map $\lambda \mapsto D(f_{j,\lambda})_z$ is non-constant on Λ_j or (b) $D(f_{j,\lambda})_z = 0$ for all $\lambda \in \Lambda_j$.
- We say that a finite sequence $\{z_i\}_{i=1}^n$ of points of $\hat{\mathbb{C}}$ is a **peripheral cycle** for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ if there exists a $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$ such that both of the following (a)(b) hold.
  - (a) $\{z_i \mid i = 1, \ldots, n\} \subset (\cup_{j=1}^m S_1(\mathcal{W}_j)) \setminus \cup_{L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}}), L \subset \cup_{j=1}^m S_1(\mathcal{W}_j)} L$.
  - (b) There exists a finite sequence $\{\gamma_i\}_{i=1}^n$ of elements of Γ such that for each $i = 1, \ldots, n$, there exists a number $j_i \in \{1, \ldots, n\}$ satisfying that for each $i = 1, \ldots, n$, we have $\gamma_i \in \{f_{j_i,\lambda} \mid \lambda \in \Lambda_{j_i}\}$, $z_i \in S_1(\mathcal{W}_{j_i})$ and $\gamma_i(z_i) = z_{i+1}$ where $z_{n+1} := z_1$.
- We say that $\mathcal{Y}$ is **strongly nice** with respect to $\{\mathcal{W}_j\}_{j=1}^m$ if $\mathcal{Y}$ is nice with respect to $\{\mathcal{W}_j\}_{j=1}^m$ and there exists no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$.

Definition 3.69. Let $\mathcal{Y}$ be a weakly nice subset of Rat_+ with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps, where $\mathcal{W}_j = \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$. Let $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$ such that $\Gamma \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \neq \emptyset$ for each $j = 1, \ldots, m$. Let $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$ with $L \neq \hat{\mathbb{C}}$. Let $g \in \Gamma$ and $j \in \{1, \ldots, m\}$. We say that g is a **strict bifurcation element for** (Γ, L) **with corresponding suffix** j if one of the following statements (a)(b) holds.

(a) $g \in \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ and there exists a point $z \in (L \cap J(\langle \Gamma \rangle)) \setminus S_1(\mathcal{W}_j)$ such that $g(z) \in L \cap J(\langle \Gamma \rangle)$.

(b) $g \in \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ and there exist an open subset U of $\hat{\mathbb{C}}$ with $(U \cap L) \setminus S_1(\mathcal{W}_j) \neq \emptyset$ and finitely many elements $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ such that $g \circ \gamma_{n-1} \cdots \gamma_1(U) \subset U$ and U is a subset of a Siegel disk or a Hermann ring of $g \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$.

Lemma 3.70. *Let $\mathcal{Y}$ be a weakly nice subset of Rat_+ with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps, where $\mathcal{W}_j = \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$. Suppose there exists no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Let $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$ such that $\Gamma \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \neq \emptyset$ for each $j = 1, \ldots, m$. Let $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$ with $L \neq \hat{\mathbb{C}}$. Suppose that $J_{\ker}(\langle \Gamma \rangle) \subset \cap_{j=1}^m S(\mathcal{W}_j)$ and $\sharp L = \infty$. Suppose also that L is not attracting for Γ. Then there exists an element $(g, j) \in \Gamma \times \{1, \ldots, m\}$ such that g is a strict bifurcation element for (Γ, L) with corresponding suffix j. Moreover, if $(h, i) \in \Gamma \times \{1, \ldots, m\}$ and h is a strict bifurcation element for (Γ, L) with corresponding suffix i, then $h \in \partial(\Gamma \cap \{f_{i,\lambda} \mid \lambda \in \Lambda_i\})$ with respect to the topology in $\{f_{i,\lambda} \mid \lambda \in \Lambda_i\}$.*

Proof. Let $G = \langle \Gamma \rangle$. [37, Lemma 3.8] implies that we have one of the following two situations (I)(II).

(I) There exist an element $(g, j) \in \Gamma \times \{1, \ldots, m\}$ with $g \in \Gamma \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ and a point $z_0 \in L \cap J(G)$ such that $g(z_0) \in L \cap J(G)$.

(II) There exist an open subset U of $\hat{\mathbb{C}}$ with $U \cap L \neq \emptyset$ and finitely many elements $\gamma_1, \ldots, \gamma_r \in \Gamma$ such that $\gamma_r \circ \cdots \gamma_1(U) \subset U$ and U is a subset of a Siegel disk or a Hermann ring of $\gamma_r \circ \cdots \circ \gamma_1$.

Suppose we have case (II). Since $\sharp L = \infty$, by using [36, Remark 3.9] and [37, Remark 2.24] we obtain that $\sharp(U \cap L) = \infty$. Let $j \in \{1, \ldots, m\}$ with $\gamma_r \in \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$. Since $\sharp S_1(\mathcal{W}_j) < \infty$ (Lemma 3.39), it follows that γ_r is a strict bifurcation element with corresponding suffix j.

Suppose we have case (I). Then there exist a sequence $\{\gamma_i\}_{i=1}^{\infty}$ in Γ with $\gamma_i \in \{f_{j_i,\lambda} \mid \lambda \in \Lambda_{j_i}\}$, $j_i \in \{1, \ldots, m\}$ and a point $z_0 \in L \cap J(G)$ such that $\gamma_i \cdots \gamma_1(z_0) \in L \cap J(G)$ for each i. We now consider the following two cases (a)(b).

(a) There exists an $i \in \mathbb{N}$ such that $\gamma_i \cdots \gamma_1(z_0) \notin S_1(\mathcal{W}_{j_{i+1}})$.

(b) For each $i \in \mathbb{N}$, $\gamma_i \cdots \gamma_1(z_0) \in S_1(\mathcal{W}_{j_{i+1}})$.

Suppose we have case (a). Then γ_{i+1} is a strict bifurcation element with corresponding suffix j_{i+1}.

Suppose we have case (b). Since $\sharp L = \infty$ and $\sharp \cup_{j=1}^m S_1(\mathcal{W}_j) < \infty$ (Lemma 3.39), we have that $L \not\subset \cup_{j=1}^m S_1(\mathcal{W}_j)$. Then for each $i \in \mathbb{N}$, we have

$$\gamma_i \cdots \gamma_1(z_0) \in (\cup_{j=1}^m S_1(\mathcal{W}_j)) \setminus \cup_{K \in \mathrm{Min}(G,\hat{\mathbb{C}}), K \subset \cup_{j=1}^m S_1(\mathcal{W}_j)} K.$$

Since we are assuming case (b) and since $\sharp \cup_{j=1}^m S_1(\mathcal{W}_j) < \infty$, there exist two elements $i, j \in \mathbb{N}$ with $j > i$ such that $\gamma_j \cdots \gamma_i \cdots \gamma_1(z_0) = \gamma_i \cdots \gamma_1(z_0)$. This contradicts to the assumption that there exists no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$.

We now suppose that $(h, i) \in \Gamma \times \{1, \ldots, m\}$ is a strict bifurcation element for (Γ, L) with corresponding suffix i. Supoose that $h \in \mathrm{int}(\Gamma \cap \{f_{i,\lambda} \mid \lambda \in \Lambda_i\})$ with respect to the topology in $\{f_{i,\lambda} \mid \lambda \in \Lambda_i\}$. Then for each $z \in \hat{\mathbb{C}} \setminus S_1(\mathcal{W}_i)$, we have that $\mathrm{int}(\langle \Gamma \rangle(z)) \neq \emptyset$. Hence it is easy to see that $\mathrm{int}(L) \cap J(G) \neq \emptyset$. It implies that $L = \hat{\mathbb{C}}$. However, this contradicts the assumption of our lemma. Hence $h \in \partial(\Gamma \cap \{f_{i,\lambda} \mid \lambda \in \Lambda_i\})$.

Thus we have proved our lemma. $\square$

Lemma 3.71. *Let $\mathcal{Y}$ be a weakly nice subset of Rat_+ with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps, where $\mathcal{W}_j = \{f_{j,\lambda}\}_{\lambda \in \Lambda_j}, j = 1, \ldots, m$. Suppose that there exists no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Let $\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ and suppose that the interior of $\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ is not empty with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$. Suppose also that $F(G_\rho) \neq \emptyset$. Then we have the following.*

(i) *$J_{\ker}(G_\rho) \subset \cap_{j=1}^m S(\mathcal{W}_j)$, $\sharp J_{\ker}(G_\rho) < \infty$ and $\sharp \mathrm{Min}(G_\rho) < \infty$.*

(ii) *Let $L \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$. Suppose that L is not attracting for ρ. Then there exists an element $(g, j) \in \Gamma_\rho \times \{1, \ldots, m\}$ such that g is a strict bifurcation element for (Γ_ρ, L) with corresponding suffix j. Moreover, if $(h, i) \in \Gamma_\rho \times \{1, \ldots, m\}$ such that h is a strict bifurcation element for (Γ_ρ, L) with corresponding suffix i, then h belongs to the boundary of $\Gamma_\rho \cap \{f_{i,\lambda} \mid \lambda \in \Lambda_i\}$, where the boundary of $\Gamma_\rho \cap \{f_{i,\lambda} \mid \lambda \in \Lambda_i\}$ is taken with respect to the topology in $\{f_{i,\lambda} \mid \lambda \in \Lambda_i\}$.*

(iii) *Suppose that there exists an element $L_0 \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ which is attracting for ρ. Then there exists an open neighborhood V of ρ in $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\rho_1 \in V$ satisfying that $\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset int(\Gamma_{\rho_1} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$, we have the following.*

   (a) *$\sharp \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}}) =$*
   $\sharp(\{L' \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) \mid L' \subset \cap_{j=1}^m S(\mathcal{W}_j)\})$
   $+\sharp\{L' \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) \mid L' \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ and L' is attracting for $\rho\}$.

   (b) *For each $L \in \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}})$ there exists a unique $L' \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $L' \subset L$ such that either "$L' \subset \cap_{j=1}^m S(\mathcal{W}_j)$" or "$L' \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ and L' is attracting for ρ".*

   (c) *In item (b), if $L' \subset \cap_{j=1}^m S(\mathcal{W}_j)$, then $L = L'$. If $L' \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ and L' is attracting for ρ, then L is attracting for ρ_1.*

(d) *Each* $L \in \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}})$ *with* $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ *is attracting for* ρ_1.

(iv) *Suppose that each element* $L_0 \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ *is not attracting for* ρ. *Let* $\rho_1 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ *be an element such that* $\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset$ *int*$(\Gamma_{\rho_1} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ *with respect to the topology in* $\{f_{j.\lambda} \mid \lambda \in \Lambda_j\}$ *for each* $j = 1, \dots, m$. *Then we have the following.*

(a) *If there exists an element* $L \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ *with* $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, *then* $\mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}}) = \{L \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) \mid L \subset \cap_{j=1}^m S(\mathcal{W}_j)\}$.

(b) *If there exists no* $L \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ *with* $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, *then* $\mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}}) = \hat{\mathbb{C}}$ *and* $J(G_{\rho_1}) = \hat{\mathbb{C}}$.

Proof. By Lemma 3.45, we obtain that $J_{\ker}(G_\rho) \subset \cap_{j=1}^m S(\mathcal{W}_j)$. Thus by Lemma 3.39, $\sharp J_{\ker}(G_\rho) < \infty$. From Proposition 3.63, it follows that $\sharp \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) < \infty$. Thus statement (i) holds.

To prove statement (ii), since $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ and since $\mathrm{int}(\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}) \neq \emptyset$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \dots, m$, we obtain that $\sharp L = \infty$. Moreover, since $\mathrm{int}(\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}) \neq \emptyset$ for each j and since $J(G_\rho) \setminus \cup_{j=1}^m S_1(\mathcal{W}_j) \neq \emptyset$, we have $\mathrm{int}(J(G_\rho)) \neq \emptyset$. Combining this with the assumption $F(G_\rho) \neq \emptyset$, we obtain that $\hat{\mathbb{C}}$ cannot be a minimal set for $(G_\rho, \hat{\mathbb{C}})$. Thus statement (ii) follows from Lemma 3.70.

To prove statement (iii), let V be a small open neighborhood V of ρ in $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ and let $\rho_1 \in V$ such that $\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset \mathrm{int}(\Gamma_{\rho_1} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$. Taking V small enough, we have that for each $\rho' \in V$, $F(G_{\rho'}) \neq \emptyset$. By Zorn's lemma, for each $L \in \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}})$ there exists an element $L' \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $L' \subset L$. If $L' \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ and L' is not attracting for ρ, then statement (ii) (for ρ and ρ_1) implies a contradiction. Hence either $L' \subset \cap_{j=1}^m S(\mathcal{W}_j)$ or L' is attracting for ρ. If $L' \subset \cap_{j=1}^m S(\mathcal{W}_j)$, then Lemma 3.46 implies that $L' = L$. Suppose $L' \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ and L' is attracting for ρ. Then taking V so small, [37, Lemma 5.2] implies that L is attacting for ρ_1 and there is no $L'' \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $L'' \neq L'$ such that $L'' \subset L$. Also, by Lemma 3.46 again, for any $K \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $K \subset \cap_{j=1}^m S(\mathcal{W}_j)$, we have $K \in \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}})$. Moreover, by [37, Lemma 5.2] again, for any $K \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $K \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ which is attracting for ρ, there exists a unique element $\tilde{K} \in \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}})$ close to K, and this $\tilde{K}$ is attracting for ρ_1. From these arguments, statement (iii) follows.

We now prove statement (iv). Suppose that each $L_0 \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ is not attracting for ρ. Let $\rho_1 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ be an element such that $\Gamma_\rho \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset \mathrm{int}(\Gamma_{\rho_1} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ for each $j = 1, \dots, m$. Let $L \in \mathrm{Min}(G_{\rho_1}, \hat{\mathbb{C}})$. Suppose that $L \neq \hat{\mathbb{C}}$ and $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$. Then $\emptyset \neq \mathrm{int}(L)$, $\sharp(\hat{\mathbb{C}} \setminus \mathrm{int}(L)) \geq 3$ and $G_{\rho_1}(\mathrm{int}(L)) \subset \mathrm{int}(L)$. Hence $\emptyset \neq \mathrm{int}(L) \subset F(G_{\rho_1})$. Also, L is not attracting for ρ_1 (otherwise by Zorn's lemma there exists an element $L_0 \in \mathrm{Min}(G_\rho, L)$ which is attracting for ρ). By applying statement (ii) for ρ and ρ_1, we obtain a contradiction. Thus either $L = \hat{\mathbb{C}}$ or $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$. If $L = \hat{\mathbb{C}}$, then since $\mathrm{int}(J(G_{\rho_1})) \neq \emptyset$ (see the argument in the proof of (ii)), we obtain that $F(G_{\rho_1}) = \emptyset$. Hence statements (a) and (b) in (iv) hold.

Thus we have proved our lemma. □

Definition 3.72. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. We say that Γ is **weakly mean stable** if there exist a positive integer n and two non-empty open subsets $V_{1,\Gamma}, V_{2,\Gamma}$ of $\hat{\mathbb{C}}$ with $\overline{V_{1,\Gamma}} \subset V_{2,\Gamma}$ and $\sharp(\hat{\mathbb{C}} \setminus V_{2,\Gamma}) \geq 3$ such that the following three conditions hold.

(a) For each $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$, $\gamma_n \circ \cdots \circ \gamma_1(V_{2,\Gamma}) \subset V_{1,\Gamma}$.

(b) Let $D_\Gamma := \cap_{g \in \langle \Gamma \rangle} g^{-1}(\hat{\mathbb{C}} \setminus V_{2,\Gamma})$. Then $\sharp D_\Gamma < \infty$.

(c) For each $L \in \mathrm{Min}(\langle \Gamma \rangle, D_\Gamma)$ there exist an element $z \in L$ and an element $g_z \in \langle \Gamma \rangle$ such that z is a repelling fixed point of g_z.

Moreover, we say that $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is weakly mean stable if $\operatorname{supp}\tau$ is weakly mean stable. If $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is weakly mean stable, then we set $V_{i,\tau} = V_{i,\Gamma_\tau}$ and $D_\tau = D_{\Gamma_\tau}$.

Lemma 3.73. *Let $\mathcal{A} := \{\Gamma \in \mathrm{Cpt}(\mathrm{Rat}) \mid \Gamma$ is weakly mean stable$\}$. Then $\mathcal{A}$ is open in* Cpt(Rat). *In particular, the set $\mathcal{A}' := \{\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}) \mid \tau$ is weakly mean stable$\}$ is open in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$.*

Proof. Let $\Gamma \in \mathcal{A}$. For this Γ, let $V_{1,\Gamma}, V_{2,\Gamma}, n$ as in Definition 3.72. Let $V'_{1,\Gamma}$ be an open subset of $\hat{\mathbb{C}}$ such that $\overline{V_{1,\Gamma}} \subset V'_{1,\Gamma} \subset \overline{V'_{1,\Gamma}} \subset V_{2,\Gamma}$. Then there exists a neighborhood $\mathcal{U}$ of Γ in Cpt(Rat) such that for each $\Lambda \in \mathcal{U}$ and for each $(\gamma_1, \ldots, \gamma_n) \in \Lambda^n$, we have $\gamma_n \circ \cdots \circ \gamma_1(V_{2,\Gamma}) \subset V'_{1,\Gamma}$.

For each $L \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$, let $z_L \in L$ and $g_L \in \langle\Gamma\rangle$ such that z_L is a repelling fixed point of g_L. Let $\epsilon > 0$ be a small number. By considering linearizing coordinate for g_L at z_L and the fundamental region for g_L near z_L, it is easy to see that for each $L \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$ there exist small simply connected open neighborhoods $H_{L,\Gamma,1}, H_{L,\Gamma,2}$ of z_L with $\overline{H_{L,\Gamma,2}} \subset H_{L,\Gamma,1}$ such that for each $z \in B(z_L, \epsilon) \setminus \{z_L\}$ there exists an element $n \in \mathbb{N}$ such that $g_L^n(z) \in H_{L,\Gamma,1} \setminus H_{L,\Gamma,2}$.

Shrinking $\mathcal{U}$ if necessary, we may assume that for each $\Lambda \in \mathcal{U}$ and for each $L \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$ there exist $z_{L,\Lambda} \in B(z_L, \frac{\epsilon}{2})$ and $g_{L,\Lambda} \in \langle\Lambda\rangle$ such that $z_{L,\Lambda}$ is a repelling fixed point of $g_{L,\Lambda}$ and such that $g_{L,\Lambda} \to g_L$ and $z_{L,\Lambda} \to z_L$ as $\Lambda \to \Gamma$. Since the linearizing coordinate for a repelling fixed point is continuous on Rat, if $\mathcal{U}$ is small enough, then for each $\Lambda \in \mathcal{U}$, for each $L \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$ there exist two small simply connected open neighborhoods $H_{L,\Lambda,1}, H_{L,\Lambda,2}$ of $z_{L,\Lambda}$ with $\overline{H_{L,\Lambda,2}} \subset H_{L,\Lambda,1}$ such that the following hold.

1. For each $z \in B(z_{L,\Lambda}, \epsilon) \setminus \{z_{L,\Lambda}\}$ there exists an element $n \in \mathbb{N}$ with $g^n_{L,\Lambda}(z) \in H_{L,\Lambda,1} \setminus H_{L,\Lambda,2}$.

2. There exist two small numbers $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 < \epsilon_2 < \frac{1}{3}\min\{d(a,b) \mid a, b \in D_\Gamma, a \neq b\}$ such that for each $\Lambda \in \mathcal{U}$ and for each $L \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$,

$$B(z_L, \epsilon_1) \subset H_{L,\Lambda,2},\ H_{L,\Lambda,1} \subset B(z_L, \epsilon_2). \tag{46}$$

For each $w \in D_\Gamma$, let $L_w \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$ be an element such that $\langle\Gamma\rangle(w) \cap L_w \neq \emptyset$. Moreover, let $h_w \in \langle\Gamma\rangle$ such that $h_w(w) = z_{L_w}$. Taking $\mathcal{U}$ small enough, there exists a $\delta > 0$ with

$$\delta < \epsilon_1, \delta < \frac{1}{3}\min\{d(a,b) \mid a, b \in D_\Gamma, a \neq b\}. \tag{47}$$

such that for each $\rho \in \mathcal{U}$ and for each $w \in D_\Gamma$, there exists an element $h_{w,\Lambda} \in \langle\Lambda\rangle$ close to h_w such that

$$h_{w,\Lambda}(B(w,\delta)) \subset B(z_{L_w}, \frac{\epsilon}{2}) \subset B(z_{L_w,\Lambda}, \epsilon). \tag{48}$$

Let $K_\delta = \hat{\mathbb{C}} \setminus B(D_\Gamma, \delta)$. Then for each $z \in K_\delta$ there exists an element $\alpha_z \in \langle\Gamma\rangle$ such that $\alpha_z(z) \in V_{2,\Gamma}$. Since K_δ is compact, there exist a finite set $\{z_1, \ldots, z_q\}$ in K_δ, a number $\epsilon_0 > 0$ and elements $\beta_1, \ldots, \beta_q \in \langle\Gamma\rangle$ such that

$$K_\delta \subset \cup_{j=1}^q B(z_j, \epsilon_0) \tag{49}$$

and $\beta_j(\overline{B(z_j, \epsilon_0)}) \subset V_{2,\Gamma}$ for all $j = 1, \ldots, q$. Hence shrinking $\mathcal{U}$ if necessary, we have that for each $\Lambda \in \mathcal{U}$, there exist elements $\beta_{1,\Lambda}, \ldots, \beta_{q,\Lambda} \in \langle\Lambda\rangle$ such that

$$\beta_{j,\Lambda}(B(z_j, \epsilon_0)) \subset V_{2,\Gamma},\ \text{for all } j = 1, \ldots, q. \tag{50}$$

We now let $\Lambda \in \mathcal{U}$ and let $z_0 \in \cap_{g\in\langle\Lambda\rangle} g^{-1}(\hat{\mathbb{C}} \setminus V_{2,\Gamma})$. Then by (49) and (50), we have $z_0 \notin K_\delta$. Thus $z_0 \in B(D_\Gamma, \delta)$. Moreover, by (46) and (47), we have $H_{L,\Lambda,2} \setminus H_{L,\Lambda,1} \subset K_\delta$ for all $L \in \mathrm{Min}(\langle\Gamma\rangle, D_\Gamma)$ and for all $\Lambda \in \mathcal{U}$. Combining this with (48), we obtain that taking an element $w \in D_\Gamma$ with $d(z_0, w) < \delta$, we have $h_{w,\Lambda}(z_0) = z_{L_w,\Lambda}$ for all $\Lambda \in \mathcal{U}$. It follows that

$$\bigcap_{g\in\langle\Lambda\rangle} g^{-1}(\hat{\mathbb{C}} \setminus V_{2,\Gamma}) \subset \bigcup_{L\in\mathrm{Min}(\langle\Gamma\rangle, D_\Gamma), w\in D_\Gamma} h^{-1}_{w,\Lambda}(z_{L,\Lambda}) \quad \text{for all } \Lambda \in \mathcal{U}. \tag{51}$$

Since the right hand side of the above is a finite set, we obtain that $\sharp D_\Lambda < \infty$, where $D_\Lambda := \cap_{g\in\langle\Lambda\rangle} g^{-1}(\hat{\mathbb{C}} \setminus V_{2,\Gamma})$. Moreover, by (51), we have that for each $K \in \mathrm{Min}(\langle\Lambda\rangle, D_\Lambda)$ there exist an element $z \in K$ and an element $\zeta_z \in \langle\Lambda\rangle$ such that z is a repelling fixed point of ζ_z. Thus Λ is wealky mean stable. Hence we have proved our lemma. $\square$

Lemma 3.74. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ be weakly mean stable. Let D_Γ be as in Definition 3.72. Then $D_\Gamma = J_{\ker}(\langle\Gamma\rangle)$, $\sharp(J_{\ker}(\langle\Gamma\rangle)) < \infty$, and for each $z \in F(\langle\Gamma\rangle)$, we have*

$$\overline{\langle\Gamma\rangle(z)} \cap \left(\bigcup_{L\in\mathrm{Min}(\langle\Gamma\rangle,\hat{\mathbb{C}}), L\not\subset J_{\ker}(\langle\Gamma\rangle)} L \right) \neq \emptyset.$$

In particular, if $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is weakly mean stable and $\sharp J(G_\tau) \geq 3$, then statements (i)-(vii) in Theorem 3.65 hold.

Proof. By definition of D_Γ, we have $\langle\Gamma\rangle(D_\Gamma) \subset D_\Gamma$. Also, by condition (c) in Definition 3.72, we have $D_\Gamma \subset J(\langle\Gamma\rangle)$. Thus $D_\Gamma \subset J_{\ker}(\langle\Gamma\rangle)$. Let $V_{2,\Gamma}$ be as in Definition 3.72. Then $V_{2,\Gamma} \subset F(\langle\Gamma\rangle)$. Since $\langle\Gamma\rangle(J_{\ker}(\langle\Gamma\rangle)) \subset J_{\ker}(\langle\Gamma\rangle) \subset J(\langle\Gamma\rangle) \subset \hat{\mathbb{C}} \setminus V_{2,\Gamma}$, we obtain $J_{\ker}(\langle\Gamma\rangle) \subset D_\Gamma$. Hence we have $D_\Gamma = J_{\ker}(\langle\Gamma\rangle)$. By definition of weakly mean stable elements again, we have $\sharp D_\Gamma < \infty$. Thus $\sharp J_{\ker}(\langle\Gamma\rangle) < \infty$. Let $V_{2,\Gamma}$ be as in Definition 3.72 for Γ. Let $z \in F(\langle\Gamma\rangle)$. Since $z \notin J_{\ker}(\langle\Gamma\rangle)$ and $D_\Gamma = J_{\ker}(\langle\Gamma\rangle)$, it follows that $\langle\Gamma\rangle(z) \cap V_{2,\Gamma} \neq \emptyset$. Thus $\overline{\langle\Gamma\rangle(z)} \cap (\bigcup_{L\in\mathrm{Min}(\langle\Gamma\rangle,\hat{\mathbb{C}}), L\not\subset J_{\ker}(\langle\Gamma\rangle)} L) \neq \emptyset$. If $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is weakly mean stable and $\sharp J(G_\tau) \geq 3$, then combining the above argument and Theorem 3.65 implies that statements (i)–(vii) in Theorem 3.65 hold for τ. $\square$

Lemma 3.75. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. Let $G = \langle\Gamma\rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ with $\sharp L < \infty$. Then we have the following.*

- (i) *Suppose that for each $z \in L$ and for each $g \in G$ with $g(z) = z$, we have $\|Dg_z\|_s > 1$. Then there exist a constant $C_1 > 0$ and a constant $\alpha > 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$ and for each $z \in L$, we have $\|D(\gamma_{n,1})_z\|_s \geq C_1\alpha^n$.*
- (ii) *Suppose that for each $z \in L$ and for each $g \in G$ with $g(z) = z$, we have $\|Dg_z\|_s < 1$. Then there exist a constant $C_2 > 0$ and a constant $\beta < 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$ and for each $z \in L$, we have $\|D(\gamma_{n,1})_z\|_s \leq C_2\beta^n$.*

Proof. We first prove statement (i). We show the following claim.
Claim 1. Under the assumptions of our lemma and statement (i), let $k \in \mathbb{N}$ with $1 \leq k \leq \sharp L$. Then there exist a constant $A_k > 0$ and a constant $\alpha_k > 1$ such that for any subset $H \subset L$ with $\sharp H = k$, for any $n \in \mathbb{N}$, for any $z \in H$ and for any $\gamma \in \Gamma^{\mathbb{N}}$, if $\gamma_{j,1}(z) \in H$ for each $j = 1, \ldots, n$, then $\|D(\gamma_{n,1})_z\|_s \geq A_k\alpha_k^n$.

To prove this claim, we use the induction on k. Apparently, the statement of the conclusion of the claim holds for $k = 1$. Suppose that the statement of the conclusion of the claim holds for k, where $1 \leq k < \sharp L$. Let $u \in \mathbb{N}$ with $u \geq 2$ such that for each $u' \in \mathbb{N}$ with $u' \geq u$, we have

$$(\min_{w\in L} \|Dg_w\|_s) A_k \alpha_k^{u'} \geq 2. \tag{52}$$

For this u, let

$$B := \min\{\|D(\rho_r \circ \cdots \circ \rho_1)_w\|_s \mid w \in L, r \leq u, (\rho_1, \ldots, \rho_r) \in \Gamma^r, \rho_r \circ \cdots \circ \rho_1(w) = w\} > 1. \tag{53}$$

Also, let $v \in \mathbb{N}$ be a large number such that

$$(\min_{w\in L} \|Dg_w\|_s) A_k \alpha_k^{uv} \cdot (\min\{\|D(\rho_r \circ \cdots \circ \rho_1)_w\|_s \mid w \in L, r \leq u, (\rho_1, \ldots, \rho_r) \in \Gamma^r\}) > 2. \tag{54}$$

Let $p \in \mathbb{N}$ be a large number such that

$$B^p \cdot \min\{\|D(\rho_r \circ \cdots \circ \rho_1)_w\|_s \mid w \in L, r \leq u, (\rho_1, \ldots, \rho_r) \in \Gamma^r\} > 2. \tag{55}$$

Let $n \in \mathbb{N}$ with $n > puv$. Let $H \subset L$ with $\sharp H = k+1$. Let $\gamma \in \Gamma^{\mathbb{N}}, z \in H$ and suppose that $\gamma_{j,1}(z) \in H$ for each $j = 1, \ldots, n$. Let $j_1, \ldots, j_m \in \mathbb{N}$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ such that $\gamma_{j_i,1}(z) = z$ for each $i = 1, \ldots, m$ and $\gamma_{l,1}(z) \neq z$ for each $l \in \{1, \ldots, n\} \setminus \{j_i \mid i = 1, \ldots, m\}$. Also, let $j_0 := 0$ and $j_{m+1} = n$. (If there is no $j \in \mathbb{N}$ such that $\gamma_{j,1}(z) = z$, then we set $j_0 = 0, m = 0, j_1 = n$.) We now want to show that $\|D(\gamma_{n,1})_z\|_s \geq 2$. In order to do that, we consider the following three cases 1,2,3.

Case 1. $j_{m+1} - j_m > u$. In this case, by the definition of $\{j_i\}$, assumptions of our lemma and (52), we obtain that $\|D(\gamma_{n,1})_z\|_s \geq 2$.

Case 2. $j_{m+1} - j_m \leq u$ and there exists an element $q \in \mathbb{N} \cup \{0\}$ with $0 \leq q \leq m-1$ such that $j_{q+1} - j_q > uv$. In this case, by the definition of $\{j_i\}$, assumptions of our lemma and (54), we obtain that $\|D(\gamma_{n,1})_z\|_s \geq 2$.

Case 3. $j_{m+1} - j_m \leq u$ and for each $i \in \mathbb{N} \cup \{0\}$ with $0 \leq i \leq m-1$, $j_{i+1} - j_i \leq uv$. In this case, we have $puv < n = \sum_{i=0}^{m}(j_{i+1} - j_i) \leq (m+1)uv$. Hence $m \geq p$. Combining this with the definition of $\{j_i\}$ and (55), we obtain that

$$\|D(\gamma_{n,1})_z\|_s \geq B^m \cdot \min\{\|D(\rho_r \circ \cdots \circ \rho_1)_w\|_s \mid r \leq u, (\rho_1, \ldots, \rho_r) \in \Gamma^r\} \geq 2.$$

From these arguments, the induction step for $k+1$ is complete. Thus we have proved Claim 1. By Claim 1, statement (i) of our lemma holds.

By the similar method to the above, we can show that statement (ii) of our lemma holds.

Thus we have proved our lemma. $\square$

We now prove the following theorem, which is one of the main results of this paper.

Theorem 3.76. *Let $\mathcal{Y}$ be a mild subset of* Rat_+ *and suppose that $\mathcal{Y}$ is strongly nice with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then the set*

$$\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) \mid \tau \textit{ is weakly mean stable}\}$$

is open and dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$. Moreover, there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all of the following statements (i)–(v) *hold.*

(i) *τ is weakly mean stable.*

(ii) *Let D_τ be as in Definition 3.72 for τ. Then $\sharp J_{\ker}(G_\tau) < \infty$, $D_\tau = J_{\ker}(G_\tau) \subset \cap_{j=1}^m S(\mathcal{W}_j)$ and $\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.*

(iii) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$, we have that L is attracting for τ.*

(iv) *For each $z \in F(G_\tau)$, we have that $\overline{G_\tau(z)} \cap (\cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), L \not\subset J_{\ker}(G_\tau)} L) \neq \emptyset$.*

(v) *All statements* (i)–(vii) *of Theorem 3.65 hold for τ.*

Proof. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ be an element. There exists an element $\tau_0 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with $\sharp \mathrm{supp}\, \tau_0 < \infty$ arbitrarily close to τ. Since $\mathcal{Y}$ is nice with respect to $\{\mathcal{W}_j\}_{j=1}^m$, we may assume that for each $z \in S_{\min}(\{\mathcal{W}_j\}_{j=1}^m)$ and for each $j \in \{1, \ldots, m\}$, either

- $Dh_z \neq 0$ for all $h \in \Gamma_{\tau_0} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$, or
- $D(f_{j,\lambda})_z = 0$ for all $\lambda \in \Lambda_j$.

By enlarging the support of τ_0 a little bit, we obtain an element $\tau_1 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ arbitrarily close to τ such that $\mathrm{int}(\Gamma_{\tau_1} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}) \neq \emptyset$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$. By enlarging the support of τ_1 a little bit again, Lemma 3.71 implies that, we can obtain an element $\tau_2 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ arbitrarily close to τ such that $\mathrm{int}(\Gamma_{\tau_2} \cap \{f_{j,\lambda} \mid$

$\lambda \in \Lambda_j\}) \neq \emptyset$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \dots, m$, such that $J_{\ker}(G_{\tau_2}) \subset \cap_{j=1}^m S(\mathcal{W}_j)$, such that $\sharp J_{\ker}(G_{\tau_2}) < \infty$, such that $\sharp \mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}}) < \infty$, and such that each $L \in \mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}})$ with $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$ is attracting for τ_2. We now prove the following claim.
Claim 1. There exists an element $\tau_3 \in (\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ arbitrarily close to τ_2 such that the interior of $\Gamma_{\tau_3} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ is not empty with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \dots, m$, and such that for each $L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, exactly one of the following (I)-(IV) holds.

(I) For each $z \in L$ and for each $g \in G_{\tau_3}$ with $g(z) = z$, we have $\|Dg_z\|_s > 1$.

(II) For each $z \in L$ and for each $g \in G_{\tau_3}$ with $g(z) = z$, we have $\|Dg_z\|_s < 1$.

(III) There exist a point $z_1 \in L$ and elements $g_1, g_2, g_3 \in G_{\tau_3}$ such that $g_1(z_1) = z_1, \|D(g_1)_{z_1}\|_s > 1$, $g_2(z_1) = z_1, 0 < \|D(g_2)_{z_1}\| < 1$, $g_3(z_1) = z_1$, and z_1 is the center of a Siegel disk of g_3. Moreover, there exist some elements $\alpha_1, \dots, \alpha_l \in \Gamma_{\tau_3}$ with $\alpha_k \in \mathrm{int}(\Gamma_{\tau_3} \cap \{f_{j_k,\lambda} \mid \lambda \in \Lambda_{j_k}\})$ with respect to the topology in $\{f_{j_k,\lambda} \mid \lambda \in \Lambda_{j_k}\}$, $k = 1, \dots, l$, such that $g_3 = \alpha_1 \circ \cdots \circ \alpha_l$.

(IV) There exist a point $z_1 \in L$ and a $j \in \{1, \dots, m\}$ such that for each $\lambda \in \Lambda_j$, we have $D(f_{j,\lambda})_z = 0$. Moreover, there exist a point $z_2 \in L$ and an element $g \in G_{\tau_3}$ such that $g(z_2) = z_2$ and $\|Dg_{z_2}\|_s > 1$.

To prove this claim, we first remark that regarding the minimal set $L \in \mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$ of type (I), by Lemmas 3.46 and 3.75, if we perturb τ_2 a little bit to τ_3', then $L \in \mathrm{Min}(G_{\tau_3'}, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$ and L is of type (I) for τ_3'. By Lemmas 3.46 and 3.75 again, the similar thing holds for minimal sets $L \in \mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$ of type (II). Let $\tau_3 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ be an element such that τ_3 is close to τ_2 and $\Gamma_{\tau_2} \subset \mathrm{int}(\Gamma_{\tau_3} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$, for each $j = 1, \dots, m$. Regarding the element τ_3, suppose that we do not have (I) or (II). Then there exist a point $z_1 \in L$, an element $g_1 \in G_{\tau_3}$, a point $z_2 \in L$, and an element $h_2 \in G_{\tau_3}$ such that $g_1(z_1) = z_1$, $\|D(g_1)_{z_1}\|_s \geq 1$, $h_2(z_2) = z_2$, and $\|D(h_2)_{z_2}\|_s \leq 1$. Since $\mathcal{Y}$ is nice with respect to $\{\mathcal{W}_j\}_{j=1}^m$, by enlarging the support of τ_3 a little bit, we may assume that $\|D(g_1)_{z_1}\|_s > 1$ and $\|D(h_2)_{z_2}\|_s < 1$. (For, if $g_1 = \gamma_n \circ \cdots \circ \gamma_1$ where $\gamma_k \in \Gamma_{\tau_3} \cap \{f_{j_k,\lambda} \mid \lambda \in \Lambda_{j_k}\}$, $k = 1, \dots, n$, we may assume that $\gamma_n \in \mathrm{int}(\Gamma_{\tau_3} \cap \{f_{j_n,\lambda} \mid \lambda \in \Lambda_{j_n}\}$. Since $\mathcal{Y}$ is nice with respect to $\{\mathcal{W}_j\}_{j=1}^m$, perturbing γ_n a little bit if necessary, we may assume that $\|D(g_1)_{z_1}\| > 1$. Similar argument is valid for h_2.) Let $\alpha, \beta \in \Gamma_{\tau_3}$ such that $\alpha(z_2) = z_1$ and $\beta(z_1) = z_2$. Take a large $n \in \mathbb{N}$ so that $\|D(\alpha h_2^n \beta)_{z_1}\|_s < 1$. Let $g_2 = \alpha h_2^n \beta$. Suppose we do not have (IV). Then we may assume that $0 < \|D(g_2)_{z_1}\|_s$. In order to take an element g_3 as in (III), let $a = \|D(g_1)_{z_1}\|_s > 1$ and $b = \|D(g_2)_{z_1}\|_s \in (0, 1)$. Let

$$\Omega := \{m \log a + n \log b \mid (m, n) \in (\mathbb{N} \cup \{0\})^2 \setminus \{(0, 0)\}\}.$$

We now prove the following subclaim which is needed in the proof of Claim 1.
Subclaim $(*)$. $0 \in \overline{\Omega}$ with respect to the topology in $\mathbb{R}$.

To prove this subclaim, let $\Omega_+ = \Omega \cap \{x \in \mathbb{R} \mid x \geq 0\}$ and $\Omega_- := \{x \in \mathbb{R} \mid x \leq 0\}$. Suppose that $0 \notin \overline{\Omega}$. Then $\inf \Omega_+ > 0$ and $\sup \Omega_- < 0$. Suppose that $\inf \Omega_+ > -\sup \Omega_-$. Then for each $\epsilon > 0$ with $\epsilon < \max\{\inf \Omega_+ + \sup \Omega_-, -\sup \Omega_-\}$, there exist an element $c_1 \in \Omega_+$ with $c_1 < \inf \Omega_+ + \epsilon$ and an element $d_1 \in \Omega_-$ with $d_1 > \sup \Omega_- - \epsilon$. Then $c_1 + d_1 \geq \inf \Omega_+ + \sup \Omega_- - \epsilon > 0$. Hence $c_1 + d_1 \in \Omega_+$. However, $c_1 + d_1 \leq \inf \Omega_+ + \sup \Omega_- + \epsilon < \inf \Omega_+$. This is a contradiction. Thus we must have that $\inf \Omega_+ \leq -\sup \Omega_-$. Similarly, we must have that $\inf \Omega_+ \geq -\sup \Omega_-$. Hence $\inf \Omega_+ = -\sup \Omega_-$. This implies $0 \in \overline{\Omega}$. However, this is a contradiction. Thus we have proved subclaim $(*)$.

Going back to the proof of Claim 1, for each $i = 1, 2$, we write $g_i = \gamma_1^i \circ \cdots \circ \gamma_{p_i}^i$ where $\gamma_k^i \in \Gamma_{\tau_3} \cap \{f_{j_{i,k},\lambda} \mid \lambda \in \Lambda_{j_{i,k}}\}$. By enlarging the support of τ_3 a little bit, we may assume that $\gamma_k^i \in \mathrm{int}(\Gamma_{\tau_3} \cap \{f_{j_{i,k},\lambda} \mid \lambda \in \Lambda_{j_{i,k}}\})$ with respect to the topology in $\{f_{j_{i,k},\lambda} \mid \lambda \in \Lambda_{j_{i,k}}\}$ for each i, k.

Then there exist an $\epsilon > 0$ and a neighborhood $V_{k,i}$ of γ_k^i in $\text{int}(\Gamma_{\tau_3} \cap \{f_{j_{k,i},\lambda} \mid \lambda \in \Lambda_{j_{k,i}}\})$ such that $(\log a - \epsilon, \log a + \epsilon) \subset \{\log \|D(\tilde{\gamma}_1^1 \cdots \tilde{\gamma}_{p_1}^1)_{z_1}\|_s \mid \tilde{\gamma}_k^1 \in V_{k,1}, k = 1\ldots, p_1\}$ and $(\log b - \epsilon, \log b + \epsilon) \subset \{\log \|D(\tilde{\gamma}_1^2 \cdots \tilde{\gamma}_{p_2}^2)_{z_1}\|_s \mid \tilde{\gamma}_k^2 \in V_{k,2}, k = 1\ldots, p_2\}$. We set

$$\tilde{\Omega} := \{m \log \|D(\tilde{\gamma}_1^1 \cdots \tilde{\gamma}_{p_1}^1)_{z_1}\|_s + n \log \|D(\tilde{\gamma}_1^2 \cdots \tilde{\gamma}_{p_2}^2)_{z_1}\|_s \\ \mid (m, n) \in (\mathbb{N} \cup \{0\})^2 \setminus \{(0, 0)\}, \tilde{\gamma}_k^1 \in V_{k,1}, \tilde{\gamma}_k^2 \in V_{k,2}, \forall k\}.$$

Then for each $c \in \Omega$, we have $(c - \epsilon, c + \epsilon) \subset \tilde{\Omega}$. By Subclaim $(*)$, it follows that $0 \in \tilde{\Omega}$. Therefore there exist an element $(m, n) \in (\mathbb{N} \cup \{0\})^2 \setminus \{(0, 0)\}$, p_1-elements $\tilde{\gamma}_k^1 \in V_{k,1}, k = 1, \ldots, p_1$, and p_2-elements $\tilde{\gamma}_k^2 \in V_{k,2}, k = 1, \ldots, p_2$ such that setting $h_3 = (\tilde{\gamma}_1^1 \cdots \tilde{\gamma}_{p_1}^1)^m (\tilde{\gamma}_1^2 \cdots \tilde{\gamma}_{p_2}^2)^n$, we have $\|D(h_3)_{z_1}\|_s = 1$. Perturbing $\tilde{\gamma}_k^i$ a little bit, we obtain an element g_3 which is close to h_3 such that $g_3'(z_1)$ is a Brjuno number (we may assume $z_1 \in \mathbb{C}$ by conjugating G_{τ_3} by an element of $\text{Aut}(\hat{\mathbb{C}})$). Thus g_3 has a Siegel disk whose center is z_1 ([24]). Thus we have proved Claim 1.

By Lemma 3.75, we have the following two claims.

Claim 2 There exists a $k \in \mathbb{N}$ such that for each $L \in \text{Min}(G_{\tau_3}, S_{\min}(\{\mathcal{W}_j\}_{j=1}^m))$ of type (I), for each $z \in L$ and for each $(\gamma_1, \ldots, \gamma_k) \in \Gamma_{\tau_3}^k$, we have $\|D(\gamma_k \circ \cdots \circ \gamma_1)_z\|_s > 2$.

Claim 3. There exists a $k \in \mathbb{N}$ such that for each $L \in \text{Min}(G_{\tau_3}, S_{\min}(\{\mathcal{W}\}_{j=1}^m))$ of type (II), for each $z \in L$ and for each $(\gamma_1, \ldots, \gamma_k) \in \Gamma_{\tau_3}^k$, we have $\|D(\gamma_k \circ \cdots \circ \gamma_1)_z\|_s < \frac{1}{2}$. Moreover, there exists a neighborhood V of L with $\sharp(\hat{\mathbb{C}} \setminus V) \geq 3$ such that for each $(\gamma_1, \ldots, \gamma_k) \in \Gamma_{\tau_3}$, we have $\gamma_k \circ \cdots \circ \gamma_1(V) \subset V$. In particular, L is attracting for G_{τ_3} and $L \subset F(G_{\tau_3})$.

Throughout the rest of the proof, we fix an element $k \in \mathbb{N}$ which satisfies the statements in Claims 2,3.

We now prove the following claim.

Claim 4. Let $L \in \text{Min}(G_{\tau_3}, S_{\min}(\{\mathcal{W}_j\}_{j=1}^m))$ be of type (III). Then $L \subset \text{int}(J(G_{\tau_3}))$. In particular, for each $z \in F(G_{\tau_3})$, we have $\overline{G_{\tau_3}(z)} \cap L = \emptyset$.

To prove Claim 4, let z_1, g_1, g_2, g_3 be as in (III). Since z_1 is a repelling fixed point of g_1, we have $z_1 \in J(G_{\tau_3})$. Since $J(G_{\tau_3})$ is perfect (see [13]), there exists a point $w \in J(G_{\tau_3}) \cap (B \setminus \{z_1\})$, where B denotes the Siegel disk of g_3 whose center is z_1. Therefore there exists a g_3-invariant analytic Jordan curve ζ in $J(G_{\tau_3}) \cap B$ with $w \in \zeta$. If K is a compact subset in $\hat{\mathbb{C}} \setminus S_1(\mathcal{W}_j)$, A is a subset of $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ with $\text{int}(A) \neq \emptyset$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$, and $h_0 \in \text{int}(A)$, then there exists an $\epsilon > 0$ such that for each $z \in K$, $B(h_0(z), \epsilon) \subset \{h(z) \mid h \in A\}$. From this fact and that $\sharp(S_1(\mathcal{W}_j)) < \infty$ for each j, it follows that $\zeta \subset \text{int}(J(G_{\tau_3}))$. Similarly, for each $w' \in B \cap J(G_{\tau_3})$, if we take the g_3-invariant analytic Jordan curve ζ' in B with $w' \in \zeta'$, then $\zeta' \subset \text{int}(J(G_{\tau_3}))$. From this argument, we obtain that $z_1 \in \text{int}(J(G_{\tau_3}))$. Therefore $L \subset \text{int}(J(G_{\tau_3}))$. Thus we have proved Claim 4.

We now prove the following claim.

Claim 5. Let $L \in \text{Min}(G_{\tau_3}, S_{\min}(\{\mathcal{W}_j\}_{j=1}^m))$ be of type (IV). Then $L \subset \text{int}(J(G_{\tau_3}))$. In particular, for each $z \in F(G_{\tau_3})$, $\overline{G_{\tau_3}(z)} \cap L = \emptyset$.

To prove Claim 5, let $j \in \{1, \ldots, m\}$, $z_1, z_2 \in L$ and $g \in G_{\tau_3}$ be as in (IV). Since z_2 is a repelling fixed point of g, we have $z_2 \in J(G_{\tau_3})$. Moreover, let $\lambda \in \Lambda_j$ with $f_{j,\lambda} \in \Gamma_{\tau_3}$ and let $\alpha, \beta \in G_{\tau_3}$ such that $\alpha(z_2) = z_1, \beta(f_{j,\lambda}(z_1)) = z_2$. Then $\beta \circ f_{j,\lambda} \circ \alpha(z_2) = z_2$ and $D(\beta \circ f_{j,\lambda} \circ \alpha)_{z_2} = 0$. By [14, Corollary 4.1], we obtain that $z_2 \in \text{int}(J(\langle g, \beta \circ f_{j,\lambda} \circ \alpha \rangle)) \subset \text{int}(J(G_{\tau_3}))$. Moreover, for each $z \in L$, there exists an element $\gamma \in G_{\tau_3}$ such that $\gamma(z) = z_2$. Thus $L \subset \text{int}(J(G_{\tau_3}))$. Hence we have proved Claim 5.

Let $\mathcal{I} := \{L \in \text{Min}(G_{\tau_3}, \hat{\mathbb{C}}) \mid L \subset \cap_{j=1}^m S(\mathcal{W}_j), L \text{ is of type } (I)\}$. Let

$$C_{\tau_3} := \{w \in \hat{\mathbb{C}} \setminus \cup_{L \in \mathcal{I}} L \mid \exists (\gamma_1, \ldots, \gamma_k) \in \Gamma_{\tau_3}^k \text{ s.t. } \gamma_k \cdots \gamma_1(w) \in \cup_{L \in \mathcal{I}} L\}.$$

Note that $C_{\tau_3} \subset J(G_{\tau_3})$. Moreover, by Claim 2,

$$C_{\tau_3} \cap \cup_{L \in \text{Min}(G_{\tau_3}, \hat{\mathbb{C}})} L = \emptyset \text{ and } C_{\tau_3} \text{ is compact.} \tag{56}$$

We now prove the following claim.
Claim 6. Let $z \in F(G_{\tau_3})$. If $\overline{G_{\tau_3}(z)} \cap (\cup_{L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}}), L \cap F(G_{\tau_3}) \neq \emptyset} L) = \emptyset$, then

$$\overline{G_{\tau_3}(z)} \cap (\cup_{L \in \mathcal{I}, L \subset J(G_{\tau_3})} L) \neq \emptyset \text{ and } \overline{G_{\tau_3}(z)} \cap C_{\tau_3} \neq \emptyset.$$

To prove Claim 6, let $z \in F(G_{\tau_3})$ and suppose $\overline{G_{\tau_3}(z)} \cap (\cup_{L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}}), L \cap F(G_{\tau_3}) \neq \emptyset} L) = \emptyset$. Since $\overline{G_{\tau_3}(z)} \cap \cup_{L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}})} L \neq \emptyset$, Claims 3,4,5 imply that $\overline{G_{\tau_3}(z)} \cap (\cup_{L \in \mathcal{I}, L \subset J(G_{\tau_3})} L) \neq \emptyset$. Let $\delta_1 > 0$ be a number such that for each $(\gamma_1, \ldots, \gamma_k) \in \Gamma_{\tau_3}^k$, for each $L \in \mathcal{I}$ and for each $x \in L$, we have $\gamma_k \cdots \gamma_1|_{B(x, \delta_1)}$ is injective and we can take well-defined inverse branch $\zeta : B(\gamma_k \cdots \gamma_1(x), \delta_1) \to \hat{\mathbb{C}}$ of $\gamma_k \cdots \gamma_1$ such that $\zeta(\gamma_k \cdots \gamma_1(x)) = x$. We may assume

$$\delta_1 < (1/2) \cdot \min\{d(a, b) \mid L \in \mathcal{I}, a, b \in L, a \neq b\}.$$

Let $\delta_2 \in (0, \delta_1)$ be a number such that for each $L \in \mathcal{I}$, for each $x \in L$ and for each $y \in B(x, \delta_2)$, we have $d(\gamma_k \cdots \gamma_1(y), \gamma_k \cdots, \gamma_1(x)) < \delta_1$. Let $\epsilon \in (0, \delta_1)$ be any small number with $\epsilon < d(z, \cup_{L \in \mathcal{I}} L)$. Then there exist an element $\gamma = (\gamma_1, \gamma_2, \ldots) \in X_{\tau_3}$ an element $n \in \mathbb{N}$, and an element $L \in \mathcal{I}$ such that $\gamma_{nk,1}(z) \in B(L, \epsilon)$. We may assume that n is the minimum one. Suppose $\gamma_{(n-1)k,1}(z) \in \cup_{L \in \mathcal{I}} B(L, \delta_2)$. Then there exist an element $L_0 \in \mathcal{I}$ and an element $z_0 \in L_0$ such that $\gamma_{(n-1)k,1}(z) \in B(z_0, \delta_2)$. It implies that $d(\gamma_{nk,1}(z),\ \gamma_{nk} \cdots \gamma_{(n-1)k+1}(z_0)) < \delta_1$. Let $\xi : B(\gamma_{nk} \cdots \gamma_{(n-1)k+1}(z_0), \delta_1) \to \hat{\mathbb{C}}$ be the well-defined inverse branch of $\gamma_{nk} \cdots \gamma_{(n-1)k+1}$ such that $\xi(\gamma_{nk} \cdots \gamma_{(n-1)k+1}(z_0)) = z_0$. By Claim 2, taking δ_1 small enough, we obtain that

$$\xi(B(\gamma_{nk} \cdots \gamma_{(n-1)k+1}(z_0), \epsilon)) \subset B(z_0, \frac{3}{4}\epsilon) \subset B(z_0, \delta_1).$$

Since $\gamma_{nk} \cdots \gamma_{(n-1)k+1}|_{B(z_0, \delta_1)}$ is injective, it follows that

$$\gamma_{(n-1)k,1}(z) \in \xi(B(\gamma_{nk} \cdots \gamma_{(n-1)k+1}(z_0), \epsilon)) \subset B(z_0, \epsilon).$$

However, this contradicts the minimality of n. Therefore we should have that $\gamma_{(n-1)k,1}(z) \notin \cup_{L \in \mathcal{I}} B(L, \delta_2)$. Since the above argument is valid for arbitrarily small $\epsilon > 0$, we obtain that $\overline{G_{\tau_3}(z)} \cap C_{\tau_3} \neq \emptyset$. Thus we have proved Claim 6.

Let $p \in \mathbb{N}$ with $p > \sum_{j=1}^{m} \sharp S_1(\mathcal{W}_j) + 1$ and let $H := \{z \in F(G_{\tau_3}) \mid \overline{G_{\tau_3}(z)} \cap C_{\tau_3} \neq \emptyset\}$. For each $z \in H$ and for each $n \in \mathbb{N}$, there exist an element $(w_{z,n,0}, \ldots, w_{z,n,p}) \in (G_{\tau_3}(z))^{p+1}$ and an element $(\gamma_{z,n,1}, \ldots, \gamma_{z,n,p}) \in \Gamma_{\tau_3}^p$ such that $\gamma_{z,n,i+1}(w_{z,n,i}) = w_{z,n,i+1}$ for each $i = 0, \ldots, p-1$ and such that $d(w_{z,n,p}, C_{\tau_3}) < \frac{1}{n}$. We may assume that for each $i = 0, \ldots, p$, there exists an element $w_{z,\infty,i} \in \overline{G_{\tau_3}(z)}$ such that $w_{z,n,i} \to w_{z,\infty,i}$ as $n \to \infty$. Moreover, we may assume that for each $i = 1, \ldots, p$, there exists an element $\gamma_{z,\infty,i} \in \Gamma_{\tau_3}$ such that $\gamma_{z,n,i} \to \gamma_{z,\infty,i}$ as $n \to \infty$. Then we have that $\gamma_{z,\infty,i+1}(w_{z,\infty,i}) = w_{z,\infty,i+1}$ for each $i = 0, \ldots, p-1$. Since $w_{z,\infty,p} \in C_{\tau_3} \subset J(G_{\tau_3})$, we obtain that $w_{z,\infty,i} \in J(G_{\tau_3})$ for each $i = 0, \ldots, p$. For each $i = 1, \ldots, p$, let $j_{z,i} \in \{1, \ldots, m\}$ be an element such that $\gamma_{z,\infty,i} \in \Gamma_{\tau_3} \cap \{f_{j_{z,i},\lambda} \mid \lambda \in \Lambda_{j_{z,i}}\}$. We now prove the following claim.
Claim 7. There exists a number $\epsilon > 0$ such that for each $z \in H$, there exists an element $i \in \mathbb{N}$ with $1 \leq i \leq p$ such that $d(w_{z,\infty,i-1}, S_1(\mathcal{W}_{j_{z,i}})) > \epsilon$.

To prove Claim 7, suppose that the statement of Claim 7 does not hold. Then for each $r \in \mathbb{N}$ there exists a $z_r \in H$ such that for each $i \in \mathbb{N}$ with $1 \leq i \leq p$, we have $d(w_{z_r,\infty,i-1}, S_1(\mathcal{W}_{j_{z_r,i}})) < \frac{1}{r}$. We may assume that for each $i = 1, \ldots, p$, there exist an element $a_{i-1} \in C_{\tau_3}$, an element $j_i \in \{1, \ldots, m\}$, and an element $\gamma_i \in \Gamma_{\tau_3} \cap \{f_{j_i,\lambda} \mid \lambda \in \Lambda_{j_i}\}$ such that $j_{z_r,i} = j_i$ for each r, such that $w_{z_r,\infty,i-1} \to a_{i-1}$ as $r \to \infty$, and such that $\gamma_{z_r,\infty,i} \to \gamma_i$ as $r \to \infty$. Also we may assume that there exists an element $a_p \in C_{\tau_3}$ such that $w_{z_r,\infty,p} \to a_p$ as $r \to \infty$. Then we have that $a_{i-1} \in S_1(\mathcal{W}_{j_i})$ and $\gamma_i(a_{i-1}) = a_i$ for each $i = 1, \ldots, p$ and thus $a_{i-1} \notin \cup_{L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}})} L$ for each $i = 1, \ldots, p$ (by (56) and the fact $a_p \in C_{\tau_3}$). However, this contradicts to the assumption that there exists no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Thus we have proved Claim 7.

Since $w_{z,\infty,i} \in J(G_{\tau_3})$ for each $z \in H$ and $i = 1, \ldots, p$, Claim 7 implies that if $\tau_4 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ is an element such that $\Gamma_{\tau_3} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset \mathrm{int}(\Gamma_{\tau_4} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ with respect to the topology in $\{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \ldots, m$, then for each $z \in H$ there exists an element $g_z \in \Gamma_{\tau_4}$ such that

$$\overline{G_{\tau_3}(g_z(z))} \cap \cup_{L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}}),\ L \text{ is attracting for } \tau_3} L \neq \emptyset.$$

Combining this with Lemma 3.71 and Claim 6, we easily see that if we assume further that τ_4 is close enough to τ_3, then for each $L \in \mathrm{Min}(G_{\tau_4}, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_{\tau_4})$, we have that L is attracting for τ_4 and

$$\text{for each } z \in F(G_{\tau_4}), \text{ we have that } \overline{G_{\tau_4}(z)} \cap \cup_{L \in \mathrm{Min}(G_{\tau_4}, \hat{\mathbb{C}}),\ L \text{ is attracting for } \tau_4} L \neq \emptyset. \tag{57}$$

Moreover, Lemma 3.71 implies that there exist two non-empty open neighborhoods $V_{1,\tau_4}, V_{2,\tau_4}$ of the union of attracting minimal sets for $(G_{\tau_4}, \hat{\mathbb{C}})$ and an element $n \in \mathbb{N}$ such that $\overline{V_{1,\tau_4}} \subset V_{2,\tau_4}$, $\sharp(\hat{\mathbb{C}} \setminus V_{2,\tau_4}) \geq 3$ and for each $(\gamma_1, \ldots, \gamma_n) \in \Gamma_{\tau_4}^n$, we have $\gamma_n \circ \cdots \circ \gamma_1(V_{2,\tau_4}) \subset V_{1,\tau_4}$. By (57) and Lemma 3.71 (i), we have

$$D_{\tau_4} := \bigcap_{g \in G_{\tau_4}} g^{-1}(\hat{\mathbb{C}} \setminus V_{2,\tau_4}) = J_{\ker}(G_{\tau_4}) \subset \cap_{j=1}^m S(\mathcal{W}_j). \tag{58}$$

Furthermore, for each $L \in \mathrm{Min}(G_{\tau_4}, \hat{\mathbb{C}})$ with $L \subset \cap_{j=1}^m S_1(\mathcal{W}_j)$, L satisfies exactly one of (I)–(IV) in Claim 1. Therefore τ_4 is weakly mean stable. By (58), Lemma 3.46, Lemma 3.73 and its proof, and Lemma 3.74, we see that there exists an neighborhood V of τ_4 in $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\tau_5 \in V$, we have that statements (i)(ii)(iii)(iv)(v) in our theorem hold for τ_5. Thus we have proved Theorem 3.76. $\square$

Definition 3.77. Let $\mathcal{Y}$ be a weakly nice subset of Rat with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. We set

$$\mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) \mid \exists L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) \text{ which is attracting for } \tau\}.$$

Also, we denote by $\mathfrak{M}_{1,c,JF}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ the set of elements $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ satisfying that $J(G_\tau) = \hat{\mathbb{C}}$ and either $\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ or $\cup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L \subset \cap_{j=1}^m S(\mathcal{W}_j)$.

Remark 3.78. Let $\mathcal{Y}$ be a weakly nice subset of Rat with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then it is easy to see that $\mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ is an open subset of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$.

We now prove a theorem in which we do not assume that $\mathcal{Y}$ is mild with $\{\mathcal{W}_j\}_{j=1}^m$.

Theorem 3.79. *Let $\mathcal{Y}$ be a strongly nice subset of* Rat_+ *with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then the set*

$$\{\tau \in \mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) \mid \tau \textit{ is weakly mean stable}\}$$

is open and dense in $(\mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$. Moreover, there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all statements (i)–(v) *in Theorem 3.76 hold. Furthermore, we have*

$$\overline{\mathcal{A} \cup \mathfrak{M}_{1,c,JF}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)} = \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$$

with respect to the topology $\mathcal{O}$.

Proof. By using the argument in the proof of Theorem 3.76, we obtain that the set of mean stable elements $\tau \in \mathfrak{M}_{1,c,mild}$ is open and dense in $(\mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$, and there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c,mild}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all statements *(i)–(v)* in Theorem 3.76 hold. To prove the last statement of the theorem, let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ and suppose that there exists no element in $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ which is attracting for τ. We want to find an element in $\mathfrak{M}_{1,c,JF}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ which is arbitrarily close to τ, by using the arguments in the proof of Theorem 3.76 with some modifications. We take τ_1 close to τ as in the proof of Theorem 3.76. We may assume that there exists no element in $\mathrm{Min}(G_{\tau_1}, \hat{\mathbb{C}})$ which is attracting for τ_1. We now consider the following two cases.
Case 1. $F(G_{\tau_1}) = \emptyset$. Case 2. $F(G_{\tau_1}) \neq \emptyset$.
Suppose we have Case 1. Let $L \in \mathrm{Min}(G_{\tau_1}, \hat{\mathbb{C}})$ and suppose $L \neq \hat{\mathbb{C}}$ and $L \not\subset \cap_{j=1}^m S(\mathcal{W}_j)$. Then $\emptyset \neq \mathrm{int}(L)$, $\sharp(\hat{\mathbb{C}} \setminus (\mathrm{int}(L))) \geq 3$ and $G_{\tau_1}(\mathrm{int}(L)) \subset \mathrm{int}(L)$. Hence by Montel's theorem, we obtain $\emptyset \neq \mathrm{int}(L) \subset F(G_{\tau_1})$. However, this is a contradiction. Thus $\tau_2 \in \mathfrak{M}_{1,c,JF}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$.

Suppose that we have Case 2. Let $\tau_2 \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ such that $\Gamma_{\tau_1} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset \mathrm{int}(\Gamma_{\tau_2} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ for each $j = 1, \dots, m$, and such that τ_2 is close to τ_1. Then by Lemma 3.71 (iv), we have that either $\mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ or $\cup_{L \in \mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}})} L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, and if $\mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ then $J(G_{\tau_2}) = \hat{\mathbb{C}}$. Thus we may assume that $\cup_{L \in \mathrm{Min}(G_{\tau_2}, \hat{\mathbb{C}})} L \subset \cap_{j=1}^m S(\mathcal{W}_j)$. Under this condition, if $F(G_{\tau_2}) = \emptyset$, then $\tau_2 \in \mathfrak{M}_{1,c,JF}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Thus we may assume $F(G_{\tau_2}) \neq \emptyset$. By the argument in the proof of Claim 1 in the proof of Theorem 3.76, there exists an element τ_3 close to τ_2 such that $\Gamma_{\tau_2} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset \mathrm{int}(\Gamma_{\tau_3} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\}$ for each $j = 1, \dots, m$, and such that the statement in Claim 1 in the proof of Theorem 3.76 holds for τ_3. By Lemma 3.46, we have

$$\cup_{L \in \mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}})} L \subset \cap_{j=1}^m S(\mathcal{W}_j). \tag{59}$$

Also, since $\Gamma_{\tau_2} \subset \Gamma_{\tau_3}$, there exists no element in $\mathrm{Min}(G_{\tau_3}, \hat{\mathbb{C}})$ which is attracting for τ_3. As before, we may assume that $F(G_{\tau_3}) \neq \emptyset$. There exists a $k \in \mathbb{N}$ for which the statement of Claim 2 in the proof of Theorem 3.76 holds. We fix such an element. It is easy to see that statements in Claims 4,5 hold for τ_3 even under our assumptions. Let $\mathcal{I}, C_{\tau_3}$ be as in the proof of Theorem 3.76. Then the statement of Claim 6 in the proof of Theorem 3.76 holds for τ_3. More precisely, we have that

$$\text{if } z \in F(G_{\tau_3}), \text{ then } \overline{G_{\tau_3}(z)} \cap (\cup_{L \in \mathcal{I}} L) \neq \emptyset \text{ and } \overline{G_{\tau_3}(z)} \cap C_{\tau_3} \neq \emptyset. \tag{60}$$

As in the proof of Theorem 3.76, let $p > \sum_{j=1}^m \sharp S(\mathcal{W}_j) + 1$ and let

$$H := \{z \in F(G_{\tau_3}) \mid \overline{G_{\tau_3}(z)} \cap C_{\tau_3} \neq \emptyset\}. \tag{61}$$

Then we have that

$$\text{the statement of Claim 7 in the proof of Theorem 3.76 holds for our } \tau_3. \tag{62}$$

Let τ_4 be an element close to τ_3 such that $\Gamma_{\tau_3} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\} \subset \mathrm{int}(\Gamma_{\tau_4} \cap \{f_{j,\lambda} \mid \lambda \in \Lambda_j\})$ for each $j = 1, \dots, m$. Then by (59) and Lemma 3.46, we have that

$$\cup_{L \in \mathrm{Min}(G_{\tau_4}, \hat{\mathbb{C}})} L \subset \cap_{j=1}^m S(\mathcal{W}_j). \tag{63}$$

Moreover, by (62), we see that for each $z \in H$ there exists an element $g_z \in \Gamma_{\tau_4}$ such that $\overline{G_{\tau_3}(g_z(z))} \cap \mathrm{int}(J(G_{\tau_3})) \neq \emptyset$. In particular, $H \subset J(G_{\tau_4})$. Combining this with (60) and (61), it follows that $F(G_{\tau_3}) \subset J(G_{\tau_4})$. Hence $J(G_{\tau_4}) = \hat{\mathbb{C}}$. Therefore we obtain that $\tau_4 \in \mathfrak{M}_{1,c,JF}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Thus we have proved our theorem. □

We now prove the following theorem on the systems generated by weakly mean stable elements.

Theorem 3.80. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be weakly mean stable. Suppose $\sharp J(G_\tau) \geq 3$. Suppose that for each $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$, we have $\chi(L,\tau) \neq 0$. Suppose also that for each $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$, if $\chi(L,\tau) > 0$ then for each $z \in L$ and for each $g \in \Gamma_\tau$, we have $Dg_z \neq 0$. Then all of the following hold.*

(i) $\sharp J_{\ker}(G_\tau) < \infty$.

(ii) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$, we have that L is attracting for G_τ.*

(iii) *For each $z \in F(G_\tau)$, we have that $\overline{G_\tau(z)} \cap ((\cup_{L \in \mathrm{Min}(G_\tau,\hat{\mathbb{C}}), L \not\subset J_{\ker}(G_\tau)} L) \neq \emptyset$.*

(iv) *All statements (i) –(vii) in Theorem 3.65 hold for τ.*

(v) *Let $H_{+,\tau} = \{L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau)) \mid \chi(\tau, L) > 0\}$ and let Ω_τ be the set of points $y \in \hat{\mathbb{C}}$ for which $\tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N}$ s.t. $\gamma_{n,1}(y) \in \cup_{L \in H_{+,\tau}} L\}) = 0$. Then we have $\Omega_\tau = F^0_{pt}(\tau)$, $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ and for each $z \in \Omega_\tau$, $\tilde{\tau}(\{\gamma \in X_\tau \mid z \in J_\gamma\}) = 0$. Moreover, for $\tilde{\tau}$-a.e.$\gamma \in X_\tau$, we have $\mathrm{Leb}_2(J_\gamma) = 0$. Moreover, $\cup_{L \in H_{+,\tau}} L \subset J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$ and $\sharp J^0_{pt}(\tau) \leq \aleph_0$.*

(vi) *Let Ω_τ be as in (v). Then $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ and there exist a constant $c_\tau < 0$ and a constant $\rho_\tau \in (0,1)$ such that for each $z \in \Omega_\tau$, there exists a Borel subset $C_{\tau,z}$ of X_τ with $\tilde{\tau}(C_{\tau,z}) = 1$ satisfying that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in C_{\tau,z}$ and for each $m \in \mathbb{N} \cup \{0\}$, we have the following* (a) *and* (b).

   (a)
$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(z)}\|_s \leq c_\tau < 0.$$

   (b) *There exist a constant $\delta = \delta(\tau, z, \gamma, m) > 0$, a constant $\zeta = \zeta(\tau, z, \gamma, m) > 0$ and an attracting minimal set $L = L(\tau, z, \gamma)$ of τ such that*
$$\mathrm{diam}(\gamma_{n+m,1+m}(B(\gamma_{m,1}(z), \delta))) \leq \zeta \rho_\tau^n \quad \textit{for all } n \in \mathbb{N},$$
and
$$d(\gamma_{n+m,1+m}(\gamma_{m,1}(z)),\ L) \leq \zeta \rho_\tau^n \quad \textit{for all } n \in \mathbb{N}.$$

(vii) *For $\tilde{\tau}$-a.e. $\gamma \in X_\tau$, for Leb_2-a.e. $z \in \hat{\mathbb{C}}$. there exists an attracting minimal set $L = L(\tau, \gamma, z)$ such that $d(\gamma_{n,1}(z), L) \to \infty$. Also, for $\tilde{\tau}$-a.e. $\gamma \in X_\tau$, for each $z \in F_\gamma$, there exists an attracting minimal set $L = L(\tau, \gamma, z)$ such that $d(\gamma_{n,1}(z), L) \to \infty$.*

(viii) *Let Ω_τ be as in* (v). *Then we have $\Omega_\tau = F^0_{pt}(\tau)$, $\sharp(\hat{\mathbb{C}} \setminus F^0_{pt}(\tau)) \leq \aleph_0$ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, for each $j = 1, \ldots, r_L$, where $r_L = \dim_{\mathbb{C}} U_{\tau,L}$, and for each $y \in F^0_{pt}(\tau)$, we have that $\lim_{z\in\hat{\mathbb{C}}, z\to y} T_{L,\tau}(z) = T_{L,\tau}(y)$ and $\lim_{z\in\hat{\mathbb{C}}, z\to y} \alpha(L_j, z) = \alpha(L_j, y)$, where $\alpha(L_j, \cdot)$ is the function coming from Theorem 3.65 (iii).*

(ix) *Let $H_{+,\tau}$ and Ω_τ be as in* (v). *Let $y \in J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$. Then there exist an element $L \in H_{+,\tau}$ and $j \in \{1, \ldots, r_L\}$ such that all of the following hold.*

   (a) *$T_{L,\tau}(z)$ does not tend to $T_{L,\tau}(y)$ as $z \to y$.*

   (b) *$\alpha(L_j, z)$ does not tend to $\alpha(L_j, y)$ as $z \to y$. Here, for the notation L_j, see Theorem 3.65 (i).*

   (c) *Let $\varphi_L \in C(\hat{\mathbb{C}})$ be any element such that $\varphi_L|_L = 1$ and $\varphi_L|_{L'} = 0$ for any $L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L' \neq L$. Then the convergence in (42) in Theorem 3.65 for $\varphi = \varphi_L$ is not uniform in any neighborhood of y.*

(d) *There exists a Borel subset $E_{\tau,y}$ of X_τ with $\tilde{\tau}(E_{\tau,y}) = T_{L,\tau}(y) > 0$ such that for each $\gamma \in E_{\tau,y}$, there exists an element $m \in \mathbb{N}$ such that $\gamma_{m,1}(y) \in L$ and $\lim_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(y)}\|_s = \chi(\tau, L) > 0$.*

Proof. By Lemma 3.74, statements (i)–(iv) hold. By Theorem 3.37, we have $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \le \aleph_0$ and for each $z \in \Omega_\tau$, $\tilde{\tau}(\{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid z \in J_\gamma\}) = 0$. Moreover, for $\tilde{\tau}$-a.e.$\gamma \in X_\tau$, we have $\mathrm{Leb}_2(J_\gamma) = 0$. Moreover, $J^0_{pt}(\tau) \subset \hat{\mathbb{C}} \setminus \Omega_\tau$ and $\sharp J^0_{pt}(\tau) \le \aleph_0$. In order to prove $\Omega_\tau = F^0_{pt}(\tau)$, let $y \in \hat{\mathbb{C}} \setminus \Omega_\tau$. Then by Lemma 3.36,

$$\tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(y), \cup_{L\in H_{+,\tau}} L) \to 0 \text{ as } n \to \infty\}) > 0. \tag{64}$$

Since $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \le \aleph_0$, there exists a sequence $\{x_m\}_{m=1}^\infty$ in Ω_τ such that $x_m \to y$ as $m \to \infty$. Then by Lemma 3.36 again, we have $\tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(z_m), \cup_{L\in H_{+,\tau}} L) \to 0 \text{ as } n \to \infty\}) = 0$ for each $m \in \mathbb{N}$. Combining this with (iv) and Theorem 3.65 (iv), we obtain that

$$\tilde{\tau}(\{\gamma \in X_\tau \mid d(\gamma_{n,1}(z_m), \cup_{L\in \mathrm{Min}(G_\tau,\hat{\mathbb{C}})\setminus H_{+,\tau}} L) \to 0 \text{ as } n \to \infty\}) = 1 \text{ for each } m \in \mathbb{N}. \tag{65}$$

By (64) and (65), it follows that $y \in J^0_{pt}(\tau)$. Hence $\Omega_\tau = F^0_{pt}(\tau)$. Also, by the definition of Ω_τ, we have $\cup_{L\in H_{+,\tau}} L \subset \hat{\mathbb{C}} \setminus \Omega_\tau$. Thus statement (v) holds. Moreover, by using the above argument, we can show that there exist an element $L \in H_{+,\tau}$ and an element $j \in \{1, \dots, r_L\}$ such that (a) and (b) in (ix) hold. By statement (a) in (ix) and the fact $\sharp\mathrm{Min}(G_\tau) < \infty$, statement (c) in (ix) holds. Statement (d) in (ix) follows from the definition of Ω_τ, Lemma 3.36 and Birkhoff's ergodic theorem. Hence statement (ix) holds.

We now prove statement (vi). By Theorem 3.65 (iv) and Lemma 3.36, it follows that for each $z \in \Omega_\tau$, there exists a Borel subset $D_{\tau,z}$ of X_τ with $\tilde{\tau}(D_{\tau,z}) = 1$ satisfying that for each $\gamma = (\gamma_1, \gamma_2, \dots) \in D_{\tau,z}$, we have

$$d\left(\gamma_{n,1}(z), \bigcup_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}}), L \text{ is attracting}} L \cup \bigcup_{L\in\mathrm{Min}(G_\tau,J_{\ker}(G_\tau)) \text{ and } \chi(L,\tau)<0} L\right) \to 0 \text{ as } n \to \infty. \tag{66}$$

There exist a constant $\lambda_\tau \in (0,1)$ and a constant $C_\tau > 0$ such that for each $\gamma = (\gamma_1, \gamma_2 \dots,) \in X_\tau$, for each $z \in \cup_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}}), L \text{ is attracting}} L$ and for each $n \in \mathbb{N}$, we have $\|D(\gamma_{n,1})_z\|_s \le C_\tau \lambda_\tau^n$. Let $c_\tau := \max\{\log \lambda_\tau, \max_{L\in\mathrm{Min}(G_\tau,J_{\ker}(G_\tau)), \chi(L,\tau)<0} \chi(L,\tau)\} < 0$ (if there exists no $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$ with $\chi(L,\tau) < 0$, then we set $c_\tau = \log \lambda_\tau$). Then for each $z \in \Omega_\tau$, there exists a Borel subset $C_{\tau,z}$ of $D_{\tau,z}$ with $\tilde{\tau}(C_{\tau,z}) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots) \in C_{\tau,z}$ and for each $m \in \mathbb{N} \cup \{0\}$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(z)}\|_s \le c_\tau < 0.$$

Also, by (66) and Lemma 3.30 and its proof, there exists an element $\rho_\tau \in (0,1)$ such that we can arrange $C_{\tau,z}$ so that for any $\gamma \in C_{\tau,z}$ and for any $m \in \mathbb{N} \cup \{0\}$, statement (vi)(b) holds. Hence statement (vi) holds for τ.

By statements (vi) and (v), statement (vii) holds.

By (iv)(v) and Theorem 3.65 (vii), statement (viii) holds. Thus we have proved our theorem. □

We now prove the following theorem, which is one of the main results of this paper.

Theorem 3.81. *Let $\mathcal{Y}$ be a mild subset of Rat_+ and suppose that $\mathcal{Y}$ is non-exceptional and strongly nice with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all of the following hold.*

(i) *τ is weakly mean stable.*

(ii) *For each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *with* $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, *we have* $\chi(L,\tau) \neq 0$. *Moreover, for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *with* $L \subset \cap_{j=1}^m S(\mathcal{W}_j)$, *if* $\chi(L,\tau) > 0$, *then for each* $z \in L$ *and for each* $g \in \Gamma_\tau$, *we have* $Dg_z \neq 0$.

(iii) $\sharp J_{\ker}(G_\tau) < \infty$ *and* $J_{\ker}(G_\tau) \subset \cap_{j=1}^m S(\mathcal{W}_j)$.

(iv) *For each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *with* $L \not\subset J_{\ker}(G_\tau)$, *we have that* L *is attracting for* τ.

(v) *For each* $z \in F(G_\tau)$, *we have that* $\overline{G_\tau(z)} \cap ((\cup_{L \in \mathrm{Min}(G_\tau,\hat{\mathbb{C}}), L \not\subset J_{\ker}(G_\tau)} L) \neq \emptyset$.

(vi) *All statements (i) –(vii) in Theorem 3.65 hold for* τ.

(vii) *Let* $H_{+,\tau} = \{L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau)) \mid \chi(\tau, L) > 0\}$ *and let* Ω_τ *be the set of points* $y \in \hat{\mathbb{C}}$ *for which* $\tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) \in \cup_{L \in H_{+,\tau}} L\}) = 0$. *Then we have* $\Omega_\tau = F^0_{pt}(\tau)$, $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ *and for each* $z \in \Omega_\tau$, $\tilde{\tau}(\{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid z \in J_\gamma\}) = 0$. *Moreover, for* $\tilde{\tau}$*-a.e.*$\gamma \in X_\tau$, *we have* $\mathrm{Leb}_2(J_\gamma) = 0$. *Moreover,* $\cup_{L \in H_{+,\tau}} L \subset J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$ *and* $\sharp J^0_{pt}(\tau) \leq \aleph_0$.

(viii) *Let* Ω_τ *be as in* (vii). *Then* $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ *and there exist a constant* $c_\tau < 0$ *and a constant* $\rho_\tau \in (0,1)$ *such that for each* $z \in \Omega_\tau$, *there exists a Borel subset* $C_{\tau,z}$ *of* X_τ *with* $\tilde{\tau}(C_{\tau,z}) = 1$ *satisfying that for each* $\gamma = (\gamma_1, \gamma_2, \ldots) \in C_{\tau,z}$ *and for each* $m \in \mathbb{N} \cup \{0\}$, *we have the following* (a) *and* (b).

   (a)
$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(z)}\|_s \leq c_\tau < 0.$$

   (b) *There exist a constant* $\delta = \delta(\tau, z, \gamma, m) > 0$, *a constant* $\zeta = \zeta(\tau, z, \gamma, m) > 0$ *and an attracting minimal set* $L = L(\tau, z, \gamma)$ *of* τ *such that*
$$\mathrm{diam}(\gamma_{n+m,1+m}(B(\gamma_{m,1}(z), \delta))) \leq \zeta \rho_\tau^n \quad \textit{for all } n \in \mathbb{N},$$
   and
$$d(\gamma_{n+m,1+m}(\gamma_{m,1}(z)),\ L) \leq \zeta \rho_\tau^n \quad \textit{for all } n \in \mathbb{N}.$$

(ix) *For* $\tilde{\tau}$*-a.e.* $\gamma \in X_\tau$, *for* Leb_2*-a.e.* $z \in \hat{\mathbb{C}}$. *there exists an attracting minimal set* $L = L(\tau, \gamma, z)$ *such that* $d(\gamma_{n,1}(z), L) \to \infty$. *Also, for* $\tilde{\tau}$*-a.e.* $\gamma \in X_\tau$, *for each* $z \in F_\gamma$, *there exists an attracting minimal set* $L = L(\tau, \gamma, z)$ *such that* $d(\gamma_{n,1}(z), L) \to \infty$.

(x) *Let* Ω_τ *be as in* (vii). *Then we have* $\Omega_\tau = F^0_{pt}(\tau)$, $\sharp(\hat{\mathbb{C}} \setminus F^0_{pt}(\tau)) \leq \aleph_0$ *and for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, *for each* $j = 1, \ldots, r_L$, *where* $r_L = \dim_{\mathbb{C}} U_{\tau,L}$, *and for each* $y \in F^0_{pt}(\tau)$, *we have that* $\lim_{z \in \hat{\mathbb{C}}, z \to y} T_{L,\tau}(z) = T_{L,\tau}(y)$ *and* $\lim_{z \in \hat{\mathbb{C}}, z \to y} \alpha(L_j, z) = \alpha(L_j, y)$, *where* $\alpha(L_j, \cdot)$ *is the function coming from Theorem 3.65 (iii).*

(xi) *Let* $H_{+,\tau}$ *and* Ω_τ *be as in* (vii). *Let* $y \in J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$. *Then there exist an element* $L \in H_{+,\tau}$ *and an element* $j \in \{1, \ldots, r_L\}$ *such that all of the following hold.*

   (a) $T_{L,\tau}(z)$ *does not tend to* $T_{L,\tau}(y)$ *as* $z \to y$.

   (b) $\alpha(L_j, z)$ *does not tend to* $\alpha(L_j, y)$ *as* $z \to y$. *Here, for the notation* L_j, *see Theorem 3.65 (i).*

   (c) *Let* $\varphi_L \in C(\hat{\mathbb{C}})$ *be any element such that* $\varphi_L|_L = 1$ *and* $\varphi_L|_{L'} = 0$ *for any* $L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *with* $L' \neq L$. *Then the convergence in (42) in Theorem 3.65 for* $\varphi = \varphi_L$ *is not uniform in any neighborhood of* y.

   (d) *There exists a Borel subset* $E_{\tau,y}$ *of* X_τ *with* $\tilde{\tau}(E_{\tau,y}) = T_{L,\tau}(y) > 0$ *such that for each* $\gamma \in E_{\tau,y}$, *there exists an element* $m \in \mathbb{N}$ *such that* $\gamma_{m,1}(y) \in L$ *and* $\lim_{n\to\infty} \frac{1}{n} \log \|D(\gamma_{n+m,1+m})_{\gamma_{m,1}(y)}\|_s = \chi(\tau, L) > 0$.

Proof. By Theorem 3.76 and Lemma 3.56, there exists an open and dense subset $\mathcal{A}$ of the space $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$ such that for each $\tau\in\mathcal{A}$, statements (i), (ii) and (iii) hold. By Theorem 3.80, for each $\tau\in\mathcal{A}$, statements (iv)–(xi) hold. Thus we have proved our theorem. $\square$

We now prove a theorem in which we do not assume that $\mathcal{Y}$ is mild with $\{\mathcal{W}_j\}_{j=1}^m$.

Theorem 3.82. *Let $\mathcal{Y}$ be a non-exceptional and strongly nice subset of Rat_+ with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c,mild}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$ such that for each $\tau\in\mathcal{A}$, all statements* (i)–(xi) *in Theorem 3.81 hold. Furthermore, we have*

$$\overline{\mathcal{A}\cup\mathfrak{M}_{1,c,JF}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m)}=\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m)$$

with respect to the topology $\mathcal{O}$.

Proof. By the arguments in the proof of Theorem 3.81 and Theorem 3.79, it is easy to see that the statements of our theorem hold. $\square$

We now give corollaries of Theorems 3.76 and 3.81.

Corollary 3.83. *Let $\mathcal{Y}$ be a mild subset of Rat_+ and suppose that $\mathcal{Y}$ is strongly nice with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Then the set*

$$\{\tau\in\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m)\mid \tau \textit{ is weakly mean stable and } \sharp\, supp\,\tau<\infty\}$$

is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$. Moreover, there exists a dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$ such that for each $\tau\in\mathcal{A}$, we have $\sharp\, supp\,\tau<\infty$ and all statements (i)–(v) *of Theorem 3.76 hold for τ.*

Corollary 3.84. *Let $\mathcal{Y}$ be a mild supset of Rat_+ and suppose that $\mathcal{Y}$ is non-exceptional and strongly nice with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of rational maps. Let $\mathcal{A}$ be the largest open and dense subset of $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$ given in Theorem 3.81. Let $\mathcal{A}^f:=\{\tau\in\mathcal{A}\mid\sharp\Gamma_\tau<\infty\}$. Then $\mathcal{A}^f$ is a dense subset of $\mathcal{A}$ and is a dense subset of $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$ such that for each $\tau\in\mathcal{A}^f$, we have that $\sharp\mathrm{supp}\,\tau<\infty$ and all statements* (i)–(xi) *in Theorem 3.81 hold for τ. Also, let $\mathcal{A}_+:=\{\tau\in\mathcal{A}\mid\exists L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}})$ s.t. $\chi(\tau,L)>0\}$ and let $\mathcal{A}_+^f:=\mathcal{A}_+\cap\mathcal{A}^f$. Then $\mathcal{A}_+$ is an open subset of $\mathcal{A}$ (hence an open subset of $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$) and $\mathcal{A}_+^f$ is a dense subset of $\mathcal{A}_+$. Moreover, for each $\tau\in\mathcal{A}_+^f$, we have $\overline{J_{pt}^0(\tau)}=J(G_\tau)$ which is a perfect set.*

Proof. It is easy to show that $\mathcal{A}^f$ is dense in $\mathcal{A}$. Thus $\mathcal{A}^f$ is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y},\{\mathcal{W}_j\}_{j=1}^m),\mathcal{O})$. Also, by statement (ii) in Theorem 3.81, it is easy to show that $\mathcal{A}_+$ is open in $\mathcal{A}$. In order to prove the last statement, suppose $\tau\in\mathcal{A}_+^f$. Since $\cup_{L\in H_{+,\tau}}L\subset J_{pt}^0$, we have $J_{pt}^0\neq\emptyset$. Moreover, since $\Gamma_\tau\subset\mathrm{Rat}_+$, we have $J_{pt}^0\subset J(G_\tau)\subset\hat{\mathbb{C}}\setminus E(G_\tau)$ (recall that $E(G_\tau)$ denotes the exceptional set of G_τ). Hence $\overline{G_\tau^{-1}(J_{pt}^0(\tau))}\supset J(G_\tau)$ (see [13, Lemma 3.2]). Also, by the definition of Ω_τ, since Γ_τ is finite, we have $\hat{\mathbb{C}}\setminus\Omega=(G_\tau\cup\{Id\})^{-1}(\cup_{L\in H_{+,\tau}}L)$ and $G_\tau^{-1}(\hat{\mathbb{C}}\setminus\Omega_\tau)\subset\hat{\mathbb{C}}\setminus\Omega_\tau$. Furthermore, by Theorem 3.81 (vii), we have $J_{pt}^0(\tau)=\hat{\mathbb{C}}\setminus\Omega_\tau$. It follows that $\overline{G_\tau^{-1}(J_{pt}^0(\tau))}\subset\overline{J_{pt}^0(\tau)}\subset J(G_\tau)$. Therefore $\overline{J_{pt}^0(\tau)}=\overline{G_\tau^{-1}(J_{pt}^0(\tau))}=J(G_\tau)$. Finally, by [13, Lemma 3.1], $J(G_\tau)$ is perfect. $\square$

4 Random relaxed Newton's method

In this section we apply Theorems 3.76, 3.81 and the other results in the previous sections to random relaxed Newton's methods in which we find roots of given any polynomial.

Definition 4.1. Let $g \in \mathcal{P}$. Let $\Lambda := \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$ and let $f_\lambda(z) = z - \lambda \frac{g(z)}{g'(z)}$ for each $\lambda \in \Lambda$. Let $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$. Let $\mathcal{Y} := \{f_\lambda \in \text{Rat} \mid \lambda \in \Lambda\}$. Then $\mathcal{Y}$ is called the random relaxed Newton's method set for g and $\mathcal{W}$ is called the **random relaxed Newton's method family for** g. Also, $(\mathcal{Y}, \mathcal{W})$ is called the **random relaxed Newton's method scheme for** g. Moreover, for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$, the random dynamical system on $\hat{\mathbb{C}}$ generated by τ is called a **random relaxed Newton's method (or random relaxed Newton's method system) of** (g, τ). Also, let $Q_g := \{z_0 \in \mathbb{C} \mid g(z_0) = 0\}$.

We need the following lemma to investigate random relaxed Newton's methods and other examples to which we can apply Theorems 3.76 and 3.81. The proof is easy and it is left to the reader.

Lemma 4.2. *Let $\mathcal{Y}$ be a nice subset of* Rat *with respect to a holomorphic family $\mathcal{W} = \{f_\lambda\}_{\lambda \in \Lambda}$ of rational maps. Then $\mathcal{Y}$ is strongly nice with respect to $\mathcal{W}$ and $\mathcal{Y}$ satisfies the assumptions in Theorem 3.76. Moreover, if, in addition to the assumption of our lemma, $\mathcal{Y}$ satisfies that for each $\Gamma \in \text{Cpt}(\{f_\lambda \mid \lambda \in \Lambda\})$ and for each $L \in \text{Min}(\langle \Gamma \rangle, S(\mathcal{W}))$, we have $\sharp L = 1$, then $\mathcal{Y}$ is non-exceptional and strongly nice with respect to $\mathcal{W}$.*

We now show that we can apply Theorem 3.81 to random relaxed Newton's methods.

Lemma 4.3. *Let $g \in \mathcal{P}$ and let $(\mathcal{Y}, \mathcal{W})$ be the random relaxed Newton's method scheme for g. Then $\mathcal{Y}$ is a mild subset of* Rat *and $\mathcal{Y}$ is non-exceptional and strongly nice with respect to $\mathcal{W}$. Also, for each $x \in Q_g$ and $\lambda \in \Lambda$, we have that $f_\lambda(x) = x$ and $f'_\lambda(x) = 1 - \frac{\lambda}{m_x}$, where m_x denotes the order of g at the zero x, and $|f'_\lambda(x)| < 1$. Moreover, for each $\lambda \in \Lambda$, we have $f_\lambda(\infty) = \infty$, the multiplier of f_λ at ∞ is equal to $(1 - \frac{\lambda}{\deg(g)})^{-1}$, and $\|D(f_\lambda)_\infty\|_s = |1 - \frac{\lambda}{\deg(g)}|^{-1} > 1$. Moreover, we have $S(\mathcal{W}) = Q_g \amalg \{\infty\} \amalg \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\}$. Moreover, for each $\Gamma \in \text{Cpt}(\mathcal{Y})$, we have $\text{Min}(\langle \Gamma \rangle, S(\mathcal{W})) = \{\{x\} \mid x \in Q_g\} \cup \{\{\infty\}\}$.*

Proof. It is easy to see that $S(\mathcal{W}) = Q_g \amalg \{\infty\} \amalg \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\}$ and for each $\Gamma \in \text{Cpt}(\mathcal{Y})$, we have $\text{Min}(\langle \Gamma \rangle, S(\mathcal{W})) = \{\{x\} \mid x \in Q_g\} \cup \{\{\infty\}\}$.

It is easy to see that $f_\lambda(x) = x$ and $f'_\lambda(x) = 1 - \frac{\lambda}{m_x}$ for each $x \in Q_g$ and $\lambda \in \Lambda$. Since $\{1 - \frac{\lambda}{m_x} \mid \lambda \in \Lambda\} = \{z \in \mathbb{C} \mid |z - (1 - \frac{1}{m_x})| < \frac{1}{m_x}\}$, we have $|f'_\lambda(x)| < 1$ for all $x \in Q_g, \lambda \in \Lambda$. Similarly, it is easy to see that for each $\lambda \in \Lambda$, we have $f_\lambda(\infty) = \infty$, the multiplier of f_λ at ∞ is equal to $(1 - \frac{\lambda}{\deg(g)})^{-1}$, and $\|D(f_\lambda)_\infty\|_s = |1 - \frac{\lambda}{\deg(g)}|^{-1} > 1$. From the above arguments, we obtain that $\mathcal{Y}$ is a mild subset of Rat and $\mathcal{Y}$ is non-exceptional and nice with respect to $\mathcal{W}$. By Lemma 4.2, it follows that $\mathcal{Y}$ is strongly nice with respect to $\mathcal{W}$. $\square$

We now prove the following theorem on random relaxed Newton's methods.

Theorem 4.4. *Let $g \in \mathcal{P}$. Let $(\mathcal{Y}, \mathcal{W})$ be the random relaxed Newton's method scheme for g. Then we have the following.*

(i) *There exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W}), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all statements* (i)–(xi) *in Theorem 3.81 hold.*

(ii) *Let $\tau \in \mathcal{A}$. Let Ω_τ be the set defined in Theorem 3.81. Then $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ and*

$$\Omega_\tau = \{y \in \mathbb{C} \mid \tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \text{ s.t. } \gamma_{n,1}(y) = \infty\}) = 0\}.$$

Moreover, there exists a constant $\rho_\tau \in (0,1)$ such that for each $z \in F^0_{pt}(\tau) = \Omega_\tau$, there exists a Borel subset $C_{\tau,z}$ of X_τ with $\tilde{\tau}(C_{\tau,z}) = 1$ satisfying that for each $\gamma \in C_{\tau,z}$, there exists a constant $\zeta = \zeta(\tau, z, \gamma) > 0$ such that

$$d(\gamma_{n,1}(z), Q_g \cup_{L \in \text{Min}(G_\tau, \hat{\mathbb{C}}), L \text{ is attracting for } \tau, L \cap Q_g = \emptyset} L) \leq \zeta \rho_\tau^n \text{ for all } n \in \mathbb{N}.$$

(iii) *For each* $\tau \in \mathcal{A}$, *we have* $\infty \in J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$ *and* $J_{\ker}(G_\tau) = \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\} \cup \{\infty\}$. *In particular,* $\emptyset \neq J^0_{pt}(\tau)$ *and* $J_{\ker}(G_\tau) \neq \emptyset$ *for each* $\tau \in \mathcal{A}$. *Also, if we set* $\mathcal{A}^f := \{\tau \in \mathcal{A} \mid \sharp\Gamma_\tau < \infty\}$, *then* $\mathcal{A}^f$ *is dense in* $(\mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W}), \mathcal{O})$. *Moreover, if, in addition to the assumptions of our theorem,* g/g' *is not a polynomial of degree one, then for each* $\tau \in \mathcal{A}^f$, *we have* $\overline{J^0_{pt}(\tau)} = J(G_\tau)$ *which is perfect.*

(iv) *Let* $\mathcal{A}_{conv} := \{\tau \in \mathcal{A} \mid \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = \{\{x\} \mid x \in Q_g\} \cup \ \{\infty\}\}\}$. *Then* $\mathcal{A}_{conv}$ *is open in* $\mathcal{A}$.

(v) *Let* $\tau \in \mathcal{A}_{conv}$. *Then we have* $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ *and* $\max_{x \in Q_g} e^{\chi(\tau, \{x\})} < 1$. *Moreover, for each* $\alpha \in (\max_{x \in Q_g} e^{\chi(\tau,\{x\})}, 1)$ *and for each* $z \in F^0_{pt}(\tau) = \Omega_\tau$, *there exists a Borel subset* $C_{\tau,z,\alpha}$ *of* X_τ *with* $\tilde{\tau}(C_{\tau,z,\alpha}) = 1$ *satisfying that for each* $\gamma \in C_{\tau,z,\alpha}$, *there exist an element* $x = x(\tau, z, \alpha, \gamma) \in Q_g$ *and a constant* $\xi = \xi(\tau, z, \alpha, \gamma) > 0$ *such that*

$$d(\gamma_{n,1}(z), x) \leq \xi \alpha^n \quad \textit{for all } n \in \mathbb{N}. \tag{67}$$

Also, for $\tilde{\tau}$*-a.e.* $\gamma \in X_\tau$, *we have* $\mathrm{Leb}_2(J_\gamma) = 0$ *and for each* $z \in F_\gamma$, *there exists an element* $x = x(\tau, \gamma, z) \in Q_g$ *such that*

$$d(\gamma_{n,1}(z), x) \to 0 \ \textit{as } n \to \infty. \tag{68}$$

Moreover, for each $x \in Q_g$ *and for each* $z \in \Omega_\tau$, *we have*

$$\lim_{w \in \hat{\mathbb{C}}, w \to z} T_{x,\tau}(w) = T_{x,\tau}(z). \tag{69}$$

Furthermore, we have

$$\tilde{\tau}(\{\gamma \in X_\tau \mid \exists n \in \mathbb{N} \ s.t. \ \gamma_{n,1}(z) = \infty\}) + \sum_{x \in Q_g} T_{x,\tau}(z) = 1 \ \textit{for all } z \in \hat{\mathbb{C}}, \tag{70}$$

and we have

$$\sum_{x \in Q_g} T_{x,\tau}(z) > 0 \ \ \textit{for all } z \in \hat{\mathbb{C}} \setminus J_{\ker}(G_\tau) = \mathbb{C} \setminus \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\}. \tag{71}$$

In particular, for any subset B *of* $\mathbb{C}$ *with* $\sharp B \geq \deg(g)$, *there exists an element* $z \in B$ *such that* $\sum_{x \in Q_g} T_{x,\tau}(z) > 0$.

(vi) *Let* $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W})$ *and suppose that* $\mathrm{int}(\Gamma_\tau) \supset \{f_\lambda \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\}$ *with respect to the topology in* $\mathcal{Y}$. *Then* $\tau \in \mathcal{A}_{conv}$. *In particular, the statements regarding* (67), (68), (69), (70) *and* (71) *hold for* τ.

(vii) *Let* $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W})$ *and suppose that* $\mathrm{int}(\Gamma_\tau) \supset \{f_\lambda \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\}$ *with respect to the topology in* $\mathcal{Y}$ *and* τ *is absolutely continuous with respect to the* 2*-dimensional Lebesgue measure on* $\mathcal{Y} \cong \Lambda$ *(e.g., let* τ *be the normalized* 2*-dimensional Lebesgue measure on the set* $\{f_\lambda \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq r\}$ *where* $\frac{1}{2} < r < 1$, *under the identification* $\mathcal{Y} \cong \Lambda$*). Then* $\tau \in \mathcal{A}_{conv}$ *and the statements regarding* (67), (68), (69), (70) *and* (71) *hold for* τ. *Moreover, we have*

$$\Omega_\tau = \mathbb{C} \setminus \{z_0 \in \mathbb{C} \mid g'(z_0) = 0, g(z_0) \neq 0\}. \tag{72}$$

In particular, we have $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \deg(g) - 1$, *and for any subset* B *of* $\mathbb{C}$ *with* $\sharp B \geq \deg(g)$, *there exists an element* $z \in B$ *such that*

$$\sum_{x \in Q_g} T_{x,\tau}(z) = 1. \tag{73}$$

Furthermore, for each $\varphi \in C(\hat{\mathbb{C}})$ and for each $z \in \Omega_\tau$, we have

$$M_\tau^n(\varphi)(z) \to \sum_{x \in Q_g} T_{x,\tau}(z)\varphi(x) \text{ as } n \to \infty \tag{74}$$

and this convergence is uniform on any compact subset of Ω_τ.

Proof. When g/g' is a polynomial of degree one, then it is easy to see that statements (i)-(vii) hold. Thus we may assume that g/g' is not a polynomial. By Lemma 4.3, Theorem 3.81, the proof of Lemma 3.30 and Corollary 3.84, statements (i)–(v) hold.

We now prove (vi). Let $\Theta := \{L \in \mathrm{Min}(\langle f_1 \rangle, \hat{\mathbb{C}}) \mid L \subset \mathbb{C} \setminus Q_g, L \text{ is attracting for } \delta_{f_1}\}$. Then each L is an attracting periodic cycle of f_1. Let $L \in \Theta$. If the period p_L of (f, L) is equal to 1, then there exists an element $x \in Q_g$ with $L = \{x\}$. However, this is a contradiction. Hence, we have $p_L \geq 2$. In particular, two different points of L never belong to the same connected component of $F(f_1)$.

We now let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W})$ and suppose $\mathrm{int}(\Gamma_\tau) \supset \Gamma_0$ with respect to the topology in $\mathcal{Y}$, where $\Gamma_0 := \{f_\lambda \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\}$. Let $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$ be an element such that $\mathrm{int}(\Gamma) \supset \Gamma_0$, $\mathrm{int}(\Gamma_\tau) \supset \Gamma$, and Γ is close enough to Γ_0 with respect to the Hausdorff metric. We now use the arguments in the proof of Theorem 3.81 and we modify them a little. By Lemma 4.3, we have the following claim.

<u>Claim 1.</u> Let $h \in \Gamma$ and $x \in Q_g$. Then we have $h(x) = x$ and $\|Dh_x\|_s < 1$. Also, $h(\infty) = \infty$ and $\|Dh_\infty\|_s > 1$.

We now prove the following claim.

<u>Claim 2.</u> Let $L \in \mathrm{Min}(\langle \Gamma_0 \rangle, \hat{\mathbb{C}})$ and suppose $L \subset \mathbb{C} \setminus Q_g$. Then L is not attracting for Γ_0.

To prove this claim, suppose that there exists an element $L \in \mathrm{Min}(\langle \Gamma_0 \rangle, \hat{\mathbb{C}})$ with $L \subset \mathbb{C} \setminus Q_g$ which is attracting for Γ_0. Then there exists an element $L_0 \in \Theta$ with $L_0 \subset L$. We have that the period of (f_1, L_0) is not 1. Let $B := \max\{|f_1(x) - x| \mid x \in L_0\} > 0$. Let $x_0 \in L_0$ be an element such that $|f_1(x_0) - x_0| = B$. Then we have

$$\{f_\lambda(x_0) \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\} = \{x_0 - \lambda \frac{g(x_0)}{g'(x_0)} \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\} = \{z \in \mathbb{C} \mid |z - f_1(x_0)| \leq \frac{1}{2}B\}. \tag{75}$$

Let $f_0 = Id$. By (75) and the fact $|f_1(x_0) - f_1^2(x_0)| \leq B$, we obtain that

$$\begin{aligned}
& \{f_\lambda(f_1(x_0)) \mid \lambda \in [0,1]\} \\
= \ & \{f_\lambda(f_1(x_0)) \mid \lambda \in [0, \frac{1}{2}]\} \cup \{f_\lambda(f_1(x_0)) \mid \lambda \in [\frac{1}{2}, 1]\} \\
\subset \ & \{z \in \mathbb{C} \mid |z - f_1(x_0)| \leq \frac{1}{2}|f_1(x_0) - f_1^2(x_0)|\} \cup \{f_\lambda(f_1(x_0)) \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\} \\
\subset \ & \{f_\lambda(x_0) \mid |\lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\} \cup \{f_\lambda(f_1(x_0)) \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq \frac{1}{2}\} \subset L.
\end{aligned}$$

Moreover, since two different points $f_1(x_0)$ and $f_1^2(x_0)$ in L_0 cannot belong to the same connected component of $F(f_1)$, we have that $\{f_\lambda(f_1(x_0)) \mid \lambda \in [0,1]\} \cap J(f_1) \neq \emptyset$. From these arguments, it follows that $L \cap J(\langle \Gamma_0 \rangle) \neq \emptyset$. However, this contradicts the assumption that L is attracting for Γ_0. Thus we have proved Claim 2.

We now prove the following claim.

<u>Claim 3.</u> We have $\mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}}) = \{\{x\} \mid x \in Q_g\} \cup \{\{\infty\}\}$.

This claim is proved by combining Claims 1,2 and Lemma 3.71.

By using Claim 3, Lemma 4.3 and the arguments in the part from Claims 6, 7 and the last in the proof of Theorem 3.76, we obtain that τ is weakly mean stable, τ satisfies the assumptions of Theorem 3.80, and there is no $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \subset \mathbb{C} \setminus Q_g$. By Theorem 3.80, it follows that $\tau \in \mathcal{A}_{conv}$. Thus we have proved statement (vi) in our theorem.

Statement (vii) follows from statements (i), (ii), (iv), (v), (vi) and Theorem 3.65.

Thus we have proved our theorem. $\square$

Remark 4.5. Let g be a non-constant polynomial. We say that g is **normalized** if the set $\{z_0 \in \mathbb{C} \mid g(z_0) = 0\}$ is contained in $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Note that if $g \in \mathcal{P}$ is normalized, then g' is also a normalized polynomial (see [1, page 29]). Thus, for a normalized polynomial $g \in \mathcal{P}$, for a random relaxed Newton's method scheme $(\mathcal{Y}, \mathcal{W})$ for g, if $\tau \in \mathcal{M}_{1,c}(\mathcal{Y}, \mathcal{W})$ is an element such that $\mathrm{int}(\Gamma_\tau) \supset \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq \frac{1}{2}\}$ and τ is absolutely continuous with respect to the 2-dimensional Lebesgue measure on $\mathcal{Y} \cong \{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$, then for any $z_0 \in \mathbb{C} \setminus \mathbb{D}$, for $\tilde{\tau}$-a.e. γ, $\gamma_{n,1}(z_0)$ converges to a root of g as $n \to \infty$.

5 Examples

In this section, we give some examples to which we can apply our main theorems.

Example 5.1. Let $\mathcal{Y}$ be a weakly nice subset of $\mathcal{P}$ with respect to some holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of polynomial maps. Suppose that $\cap_{j=1}^m S(\mathcal{W}_j) = \{\infty\}$. Then $\mathcal{Y}$ is nice with respect to $\{\mathcal{W}_j\}_{j=1}^m$ and $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ satisfies the assumptions of Lemma 3.53. Thus by Lemma 3.53, the set $\mathcal{A} := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) \mid \tau \text{ is mean stable}\}$ is open and dense in $\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with respect to the topology in $\mathcal{O}$. In particular, all statements (i)–(xi) of Theorem 3.81 hold for any $\tau \in \mathcal{A}$ and the set Ω_τ in Theorem 3.81 is equal to $\hat{\mathbb{C}}$.

We give some examples of $\mathcal{Y}$ which are mild, non-exceptional and strongly nice and satisfies the assumptions in Theorem 3.81.

Example 5.2. For each $q \in \mathbb{N}$ with $q \geq 2$, let $\mathcal{P}_q := \{f \in \mathcal{P} \mid \deg(f) = q\}$. Let $(q_1, \ldots, q_m) \in \mathbb{N}^m$ with $q_1 < q_2 < \cdots < q_m$ and let $\mathcal{W}_j = \{f\}_{f \in \mathcal{P}_{q_j}}, j = 1, \ldots, m$ and let $\mathcal{Y} = \cup_{j=1}^m \mathcal{P}_{q_j}$. In this case, $S(\mathcal{W}_j) = \{\infty\}$. Thus by Example 5.1, the set $\mathcal{A} := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m) \mid \tau \text{ is mean stable}\}$ is open and dense in $\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ with respect to the topology in $\mathcal{O}$ and the set Ω_τ in Theorem 3.81 is equal to $\hat{\mathbb{C}}$.

Example 5.3. Let $q \in \mathbb{N}$ with $q \geq 2$ and let $\mathcal{W} = \{z^q + c\}_{c \in \mathbb{C}}$. Let $\mathcal{Y} = \{z^q + c \mid c \in \mathbb{C}\}$. In this case, $S(\mathcal{W}) = \{\infty\}$. Thus by Example 5.1, the set $\mathcal{A} := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W}) \mid \tau \text{ is mean stable}\}$ is open and dense in $\mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W})$ with respect to the topology in $\mathcal{O}$ and the set Ω_τ in Theorem 3.81 is equal to $\hat{\mathbb{C}}$.

We now give an important example of $\mathcal{Y}$ to which we can apply Theorems 3.76 and 3.81 but in which $\Omega_\tau \neq \hat{\mathbb{C}}$ for any τ in an open subset of $\mathcal{A}$, where $\mathcal{A}$ is the set in Theorems 3.76 and 3.81.

Example 5.4. Let $\mathcal{W} = \{\lambda z(1-z)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ and let $\mathcal{Y} = \{\lambda z(1-z) \in \mathcal{P}_2 \mid \lambda \in \mathbb{C} \setminus \{0\}\}$. In this case, $S(\mathcal{W}) = \{0, 1, \infty\}$ and $S(\mathcal{W}) \setminus \{\infty\} = \{0, 1\} \neq \emptyset$. It is easy to see that $\mathcal{Y}$ is a mild subset of $\mathcal{P}$ and $\mathcal{Y}$ is non-exceptional and strongly nice with respect to holomorphic family $\mathcal{W}$. Thus the statements of Theorems 3.76, 3.81 hold. Let $\mathcal{A}$ be the largest open and dense subset of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W}), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all statements (i)-(v) of Theorem 3.76 and all statements (i)–(ix) of Theorem 3.81 hold. Since each element of $\mathcal{Y}$ is a quadratic polynomial, for each $\tau \in \mathcal{A}$, exactly one of the followings holds.

- Type (I). $\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = \{\{0\}, \{\infty\}\}$.
- Type (II). $\mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = \{\{0\}, \{\infty\}, L_\tau\}$, where L_τ is an attracting minimal set with $L_\tau \neq \{0\}, \{\infty\}$.

If $\tau \in \mathcal{A}$ is of type (I), then Theorem 3.76 (v) and Theorem 3.65 imply that

$$M_\tau^n(\varphi)(y) \to T_{0,\tau}(y)\varphi(0) + T_{\infty,\tau}(y)\varphi(\infty) \text{ as } n \to \infty, \text{ for each } y \in \hat{\mathbb{C}}, \varphi \in C(\hat{\mathbb{C}}) \quad (76)$$

i.e., $(M_\tau^*)^n(\delta_y) \to T_{0,\tau}(y)\delta_0 + T_{\infty,\tau}(y)\delta_\infty$ as $n \to \infty$. If $\tau \in \mathcal{A}$ is of type (II), then Theorem 3.76 (v) and Theorem 3.65 imply that

$$M_\tau^{nr_\tau}(\varphi)(y) \to T_{0,\tau}(y)\varphi(0) + T_{\infty,\tau}(y)\varphi(\infty) + \sum_{j=1}^{r_\tau} \alpha((L_\tau)_j, y) \int \varphi \, d\omega_{L_\tau,j} \text{ as } n \to \infty \tag{77}$$

for each $y \in \hat{\mathbb{C}}$ and for each $\varphi \in C(\hat{\mathbb{C}})$, where $r_\tau = \dim_{\mathbb{C}}(U_{\tau,L_\tau})$ (the period of (τ, L_τ), see Lemma 3.60 and Definition 3.61), and $\{(L_\tau)_j\}_{j=1}^{r_\tau}$, $\{\omega_{L_\tau,j}\}_{j=1}^{r_\tau}$ are elements coming from Theorem 3.65.

Note that there exists an element $\tau \in \mathcal{A}$ of type (I). For example, let $g_0(z) = \lambda_0 z(1-z) \in \mathcal{Y}$ where $0 < |\lambda_0| < 1$ and let $\tau_0 = \delta_{g_0}$. Then any element $\tau \in \mathcal{A}$ which is close enough to τ_0 is of type (I). Also, there exists an element $\tau \in \mathcal{A}$ of type (II). For example, let $g_1 \in \mathcal{Y}$ be an element which has an attracting periodic cycle with period $p \geq 2$. Let $\tau_1 = \delta_{g_1}$. Then any element $\tau \in \mathcal{A}$ which is close enough to τ_1 is of type (II) with $r_\tau = p$.

We now classify elements $\tau \in \mathcal{A}$ of type (I) into the following three types.

- Type (Ia). $0 \in F(G_\tau)$ and $\{0\} \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is attracting for τ.
- Type (Ib). $0 \in J_{\ker}(G_\tau)$ and $\chi(\tau, \{0\}) < 0$.
- Type (Ic). $0 \in J_{\ker}(G_\tau)$ and $\chi(\tau, \{0\}) > 0$.

We first remark that for each type $(*)$ above, there exists an element $\tau \in \mathcal{A}$ of type $(*)$. In fact, for the above τ_0, any element $\tau \in \mathcal{A}$ which is close enough to τ_0 is of type (Ia). Also, let $g_3(z) = \frac{1}{2}z(1-z) \in \mathcal{Y}, g_4(z) = 6z(1-z) \in \mathcal{Y}$ and let $\tau_2 := p_1\delta_{g_3} + p_2\delta_{g_4}$, where $(p_1, p_2) \in (0,1)^2$ with $p_1 + p_2 = 1$, $p_1 \log \frac{1}{2} + p_2 \log 6 < 0$. Then any element $\tau \in \mathcal{A}$ which is close enough to τ_3 is of type (Ib). Moreover, let $\tau_3 := q_1\delta_{g_3} + q_2\delta_{g_4}$, where $(q_1, q_2) \in (0,1)^2$ with $q_1 + q_2 = 1$, $q_1 \log \frac{1}{2} + q_2 \log 6 > 0$. Then any element $\tau \in \mathcal{A}$ which is close enough to τ_3 is of type (Ic). Hence for each type $(*)$, there exists an element $\tau \in \mathcal{A}$ of type $(*)$.

For each type $(*)$=(Ia), (Ib), (Ic), (II), we set $\mathcal{A}_*$ the set of element $\tau \in \mathcal{A}$ of type $(*)$. We show the folloing claim.

Claim 1. For each $(*)$=(Ia), (Ib), (Ic), (II), the set $\mathcal{A}_*$ is a non-empty open subset of $\mathcal{A}$. Also, $\mathcal{A} = \amalg_* \mathcal{A}_*$, where $\amalg$ denotes the disjoint union.

To show this claim, we first remark that we have already shown that each $\mathcal{A}_*$ is non-empty and $\mathcal{A} = \cup_* \mathcal{A}_*$. By [37, Lemma 5.2], the sets $\mathcal{A}_{Ia}, \mathcal{A}_{II}$ are open in $\mathcal{A}$. Also, since $\chi(\tau, \{0\})$ is continuous with respect to $\tau \in \mathcal{A}$, we see that $\mathcal{A}_{Ic}$ is open in $\mathcal{A}$. Finally, since each $\tau \in \mathcal{A}$ is weakly mean stable, for each $\tau \in \mathcal{A}_{Ib}$, there exists an element $g \in \Gamma_\tau$ with $|g'(0)| > 1$. From this, we obtain that $\mathcal{A}_{Ib}$ is open in $\mathcal{A}$. Thus we have proved Claim 1.

We now show the following claim.

Claim 2. For each $\tau \in \mathcal{A}_{II}$ and for each $g \in \Gamma_\tau$, we have $|g'(0)| > 1$. In particular, $0 \in J_{\ker}(G_\tau)$ and $\chi(\tau, \{0\}) > 0$.

To show this claim, let $\tau \in \mathcal{A}_{II}$ and $g \in \Gamma_\tau$. Then g has an attracting periodic cycle in L_τ, which does not meet 0. Thus $|g'(0)| > 1$. Hence we have proved Claim 2.

For each $\tau \in \mathcal{A}$, we have that τ is weakly mean stable. We now show the following claim.

Claim 3. Each element $\tau \in \mathcal{A}_{Ia}$ is mean stable. However, each element $\tau \in \mathcal{A}_{Ib} \cup \mathcal{A}_{Ic} \cup \mathcal{A}_{II}$ is weakly mean stable but not mean stable.

To show this claim, let $\tau \in \mathcal{A}_{Ia}$. Since each minimal set is attracting, τ is mean stable. We now let $\tau \in \mathcal{A}_{Ib} \cup \mathcal{A}_{Ic} \cup \mathcal{A}_{II}$. Then $0 \in J_{\ker}(G_\tau)$. Thus τ is not mean stable. Hence we have proved Claim 3.

For each $\tau \in \mathcal{A}_{Ia}$, the convergence in (76) is uniform on $y \in \hat{\mathbb{C}}$ since τ is mean stable.

Let $\mathcal{U} := \{\tau \in \mathcal{A} \mid \chi(\tau, \{0\}) > 0\} = \mathcal{A}_{Ic} \cup \mathcal{A}_{II}$. Then $\mathcal{U}$ is a non-empty open subset of $\mathcal{A}$. We now prove the following claim.

Claim 4. For each $\tau \in \mathcal{U}$, the set Ω_τ (with $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$) in Theorem 3.81 is not equal to $\hat{\mathbb{C}}$. In particular, $\emptyset \neq J_{pt}^0(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$.

To prove this claim, let $\tau \in \mathcal{U}$. Then $\chi(\tau, \{0\}) > 0$. By the definition of Ω_τ, we obtain that $0 \in \hat{\mathbb{C}} \setminus \Omega_\tau$. Also, by Theorem 3.81 (vii), we have $J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$. Hence we have proved Claim 4.

We now prove the following claim. Note that the set $\overline{\{\tau \in \mathcal{U} \mid \sharp\Gamma_\tau < \infty\}}$ is dense in $\mathcal{U}$.

Claim 5. Let $\tau \in \mathcal{U}$ with $\sharp\Gamma_\tau < \infty$. Then we have $\overline{J^0_{pt}(\tau)} = J(G_\tau)$ and this is a perfect set. Also, there exists an element $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ such that letting $\varphi_L \in C(\hat{\mathbb{C}})$ be any element such that $\varphi_L|_L = 1$ and $\varphi_L|_{L'} = 0$ for any $L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L' \neq L$, the convergence in (42) in Theorem 3.65 for $\varphi = \varphi_L$ is not uniform in any open subset V of $\hat{\mathbb{C}}$ with $V \cap J(G_\tau) \neq \emptyset$.

This claim follows from Claim 4 and Corollary 3.84. We have proved Claim 5.

We now prove the following claim.

Claim 6. For each $\tau \in \mathcal{A}_{Ia}$, the functions $T_{0,\tau}, T_{\infty,\tau}$ are continuous on $\hat{\mathbb{C}}$ and there exists a neighborhood V of 0 such that $T_{0,\tau}|_V \equiv 1$ and $T_{\infty,\tau}|_V \equiv 0$. Also, for each $\tau \in \mathcal{A}_{Ib}$, the functions $T_{0,\tau}$ and $T_{\infty,\tau}$ are continuous on $\hat{\mathbb{C}}$ and $T_{0.\tau}(0) = 1$, $T_{\infty,\tau}(0) = 0$, but for any neighborhood V of 0, we have $T_{0,\tau}|_V \not\equiv 1$ and $T_{\infty,\tau}|_V \not\equiv 0$.

To prove this claim, let $\tau \in \mathcal{A}_{Ia} \cup \mathcal{A}_{Ib}$. Then by Theorem 3.66 (or Theorem 3.81), the functions $T_{0,\tau}, T_\infty$ are continuous. If $\tau \in \mathcal{A}_{Ia}$, then $0 \in F(G_\tau)$ and since the functions $T_{0,\tau}$ and $T_{\infty,\tau}$ are locally constant (see [36, Theorem 3.15] or Theorems 3.76 and 3.65 (vi)), there exists a neighborhood V of 0 such that $T_{0,\tau}|_V \equiv 1$ and $T_{\infty,\tau}|_V \equiv 0$. We now suppose $\tau \in \mathcal{A}_{Ib}$. Let $F_\infty(G_\tau)$ be the connected component of $F(G_\tau)$ with $\infty \in F_\infty(G_\tau)$. Then $T_{\infty,\tau}|_{F_\infty(G_\tau)} \equiv 1$. Let V be any neighborhood of 0. Since $0 \in J(G_\tau)$, there exist an element $z \in V$ and an element $g \in G_\tau$ such that $g(z) \in F_\infty(G_\tau)$. Let $(\gamma_1 \ldots, \gamma_n) \in \Gamma_\tau^n$ be an element such that $g = \gamma_n \circ \cdots \circ \gamma_1$. Then there exists a neighborhood Λ of $(\gamma_1, \ldots, \gamma_n)$ in Γ_τ^n such that for each $(\alpha_1, \ldots, \alpha_n) \in \Lambda$, $\alpha_n \circ \cdots \circ \alpha_1(z) \in F_\infty(G_\tau)$. It implies that $T_{\infty,\tau}(z) \geq (\otimes_{j=1}^n \tau)(\Lambda) > 0$. Therefore $T_{\infty,\tau}|_V \not\equiv 0$. Since $T_{0,\tau} + T_{\infty,\tau} = 1$, it follows that $T_{0,\tau}|_V \not\equiv 1$. Thus we have proved Claim 6.

We now prove the following claim.

Claim 7. Let $\tau \in \mathcal{A}_{Ic}$. Then for each $z \in \Omega_\tau$, where Ω_τ is the subset of $\hat{\mathbb{C}}$ defined in Theorem 3.81, we have $T_{\infty,\tau}(z) = 1$. Also, $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$.

To prove this claim, by Theorem 3.81, we have $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$. Also, by the definition of Ω_τ, the result $T_{0,\tau} + T_\infty = 1$ on $\hat{\mathbb{C}}$ and Lemma 3.36, we see that $T_{\infty,\tau}(y) = 1$ for each $y \in \Omega_\tau$. Thus we have proved Claim 7.

We now prove the following claim.

Claim 8. Let $\tau \in \mathcal{A}$. Then $\tau \in \mathcal{A}_{Ic}$ if and only if for $\tilde{\tau}$-a.e.$\gamma \in X_\tau$, we have $\mathrm{Leb}_2(K_\gamma) = 0$, where K_γ denotes the filled-in Julia set of γ, i.e., $K_\gamma := \{z \in \mathbb{C} \mid \{\gamma_{n,1}(z)\}_{n=1}^\infty$ is bounded in $\mathbb{C}\}$.

To prove this claim, let $\tau \in \mathcal{A}_{Ic}$. Then by Claim 7 and the Fubini theorem, for $\tilde{\tau}$-a.e.$\gamma \in X_\tau$, we have $\mathrm{Leb}_2(K_\gamma) = 0$. We now suppose that $\tau \in \mathcal{A}$ and for $\tilde{\tau}$-a.e.$\gamma \in X_\tau$, we have $\mathrm{Leb}_2(K_\gamma) = 0$. Then by the Fubini theorem, we obtain that for Leb_2-a.e. $z \in \hat{\mathbb{C}}$, we have $T_{\infty,\tau}(z) = 1$. Therefore by Claim 7, $\tau \notin \mathcal{A}_{Ia} \cup \mathcal{A}_{Ib} \cup \mathcal{A}_{II}$. Hence by Claim 1, we obtain $\tau \in \mathcal{A}_{Ic}$. Thus we have proved Claim 8.

We also give some further examples to which we can apply Theorems 3.76 and 3.81.

Example 5.5. Let $Q = \{x_1, \ldots, x_n\}$ be any non-empty finite subset of $\mathbb{C}$, where $x_1, \ldots, x_n$ are mutually distinct points. Let $f(z) = a\prod_{j=1}^n (z - x_j) \in \mathcal{P}$, where $a \in \mathbb{C} \setminus \{0\}$. Then we have $\{z_0 \in \mathbb{C} \mid f(z_0) = 0\} = Q$ and if $z_0 \in \mathbb{C}, f(z_0) = 0$, then $f'(z_0) \neq 0$. Let $\mathcal{W} = \{z + \lambda f(z)\}_{\lambda \in \mathbb{C} \setminus \{0\}}$ and let $\mathcal{Y} = \{z + \lambda f(z) \in \mathcal{P} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$. In this case, $S(\mathcal{W}) = Q \cup \{\infty\}$ and $S(\mathcal{W}) \cap \mathbb{C} = \{z_0 \in \mathbb{C} \mid f(z_0) = 0\} = Q \neq \emptyset$. By Lemma 4.2, we obtain that $\mathcal{Y}$ is a mild subset of $\mathcal{P}$, the set $\mathcal{Y}$ is strongly nice and non-exceptional with respect to holomorphic family $\mathcal{W}$ and $(\mathcal{Y}, \mathcal{W})$ satisfies the assumptions of Theorems 3.76, 3.81. Thus there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W}), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all statements (i)–(x) in Theorem 3.81 hold for τ. In particular, each $\tau \in \mathcal{A}$ is weakly mean stable. Let $f_\lambda(z) = z + \lambda f(z)$. Then we have

$$f'_\lambda(z) = 1 + \lambda f'(z). \tag{78}$$

Let $\mathcal{A}_+ := \{\tau \in \mathcal{A} \mid \exists L \in \mathrm{Min}(G_\tau, Q) \text{ s.t. } \chi(\tau, L) > 0\}$. Also, let $\mathcal{A}_{+,all} := \{\tau \in \mathcal{A} \mid$ for all $L \in \mathrm{Min}(G_\tau, Q)$ we have $\chi(\tau, L) > 0\}$. Moreover, let $\mathcal{A}^f := \{\tau \in \mathcal{A} \mid \sharp\Gamma_\tau < \infty\}$, $\mathcal{A}_+^f := \mathcal{A}_+ \cap \mathcal{A}^f$, and $\mathcal{A}_{+.all}^f := \mathcal{A}_{+,all} \cap \mathcal{A}^f$.

We now show the following claim.

Claim 1. The sets $\mathcal{A}_+$ and $\mathcal{A}_{+,all}$ are non-empty open subsets of $\mathcal{A}$ (and thus they are non-empty open subsets of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \mathcal{W}), \mathcal{O})$). Also, $\mathcal{A}_+^f$ is dense in $\mathcal{A}_+$ and $\mathcal{A}_{+,all}^f$ is dense in $\mathcal{A}_{+,all}$. Moreover, for each $\tau \in \mathcal{A}_+$, we have $\emptyset \neq \cup_{L \in H_{+,\tau}} L \subset J_{pt}^0(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$, where Ω_τ and $H_{+,\tau}$ are the sets defined in Theorem 3.81, and for each $\tau \in \mathcal{A}_{+,all}$, we have $Q \subset J_{pt}^0(\tau)$. Furthermore, for each $\tau \in \mathcal{A}_+^f$, we have $\overline{J_{pt}^0(\tau)} = J(G_\tau)$ which is a perfect set.

To prove this claim, it is easy to see that $\mathcal{A}_+$ and $\mathcal{A}_{+.all}$ are open in $\mathcal{A}$. By (78) and the fact $f'(x) \neq 0$ for each $x \in Q$, if $|\lambda_0|$ is large enough, then letting $\tau_0 := \delta_{f_{\lambda_0}}$, we have $\chi(\tau_0, \{x\}) > 0$ for each $x \in Q$. Therefore for each $\tau \in \mathcal{A}$ which is close enough to τ_0 and for each $x \in Q$, we have $\chi(\tau, \{x\}) > 0$. Thus $\mathcal{A}_+ \supset \mathcal{A}_{+,all} \neq \emptyset$. The rest statements follow from Theorem 3.81 and Corollary 3.84. Thus we have proved Claim 1.

Let $\mathcal{A}_{-,all} := \{\tau \in \mathcal{A} \mid$ for all $L \in \mathrm{Min}(G_\tau, Q)$ we have $\chi(\tau, L) < 0\}$. We now prove the following claim.

Claim 2. The set $\mathcal{A}_{-,all}$ is a non-empty open subset of $\mathcal{A}$ and $\mathcal{A}_{-,all} \cap \mathcal{A}_{+,all} = \emptyset$.

To prove this claim, it is easy to see $\mathcal{A}_{-,all} \cap \mathcal{A}_{+,all} = \emptyset$ and $\mathcal{A}_{-,all}$ is open in $\mathcal{A}$. For each $x \in Q$, combining (78), the fact $f'(x) \neq 0$ and the method above, we see that there exists an element $\lambda_x \in \mathbb{C} \setminus \{0\}$ such that $f'_{\lambda_x}(x) = 0$. Let $\tau_1 = \sum_{x \in Q} \frac{1}{n} \delta_{f_{\lambda_x}}$. Then $\chi(\tau_1, \{x\}) = -\infty$ for each $x \in Q$. Hence for each $\tau \in \mathcal{A}$ which is close enough to τ_1, we have $\chi(\tau, \{x\}) < 0$ for all $x \in Q$. Thus $\mathcal{A}_{-,all} \neq \emptyset$. Hence we have proved Claim 2.

We now prove the following claim.

Claim 3. Let $\tau \in \mathcal{A}_{-,all}$. Then for each $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$, we have $\chi(\tau, L) < 0$, and each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$ is attracting τ. Thus τ satisfies all assumptions of Theorem 3.66 and all conclusions in Theorem 3.66 hold. In particular, $J_{pt}^0(\tau) = \emptyset$ and $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.

This claim follows from Theorem 3.81, the fact τ is weakly mean stable and Theorem 3.66.

Example 5.6. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $w = e^{2\pi i/n} \in \mathbb{C}$. For each $i = 1, \ldots, n$, let $\mathcal{W}_i = \{w^i(z + \lambda(z^n - 1))\}_{\lambda \in \mathbb{C} \setminus \{0\}}$. Let $i_1, \ldots, i_m \in \{1, \ldots, n\}$ with $i_1 < i_2 \cdots < i_m$. Let $\mathcal{Y} = \cup_{j=1}^m \{w^{i_j}(z + \lambda_j(z^n - 1)) \in \mathcal{P} \mid \lambda_j \in \mathbb{C} \setminus \{0\}\}$. For each $j = 1, \ldots, m$, let $\Lambda_j := \mathbb{C} \setminus \{0\}$ and let $f_{j,\lambda_j}(z) = w^{i_j}(z + \lambda_j(z^n - 1))$ for each $z \in \hat{\mathbb{C}}, \lambda_j \in \Lambda_j$. Let $\mathcal{W}_j = \{f_{j,\lambda_j}\}_{\lambda_j \in \Lambda_j}$ for each $j = 1, \ldots, m$. We show the following claim.

Claim 1. $\mathcal{Y}$ is a mild subset of $\mathcal{P}$ and $\mathcal{Y}$ is strongly nice and non-exceptional with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of polynomial maps and $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$ satisfies the assumptions of Theorem 3.81.

To prove this claim, we first note that $S(\mathcal{W}_j) = \{w^k \mid k = 1, \ldots, n\} \cup \{\infty\}$ for each $j = 1, \ldots, m$. Hence we have $\cap_{j=1}^m S(\mathcal{W}_j) \cap \mathbb{C} = \{w^k \mid k = 1, \ldots, n\} \neq \emptyset$. For each $w^k \in \cap_{j=1}^m S(\mathcal{W}_j)$ and for each $j = 1, \ldots, m$ and for each $\lambda_j \in \mathbb{C} \setminus \{0\}$, we have $f_{j,\lambda_j}(w^k) = w^{k+i_j} \in \cap_{j=1}^m S(\mathcal{W}_j)$. Thus for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$, we have $G_\tau(\cap_{j=1}^m S(\mathcal{W}_j) \cap \mathbb{C}) \subset \cap_{j=1}^m S(\mathcal{W}_j) \cap \mathbb{C}$. Let $Q := \cap_{j=1}^m S(\mathcal{W}_j) \cap \mathbb{C} = \{w^k \mid k = 1, \ldots, m\}$. For each $j = 1, \ldots, m$, let $\alpha_j : Q \to Q$ be the map defined by $\alpha_j(z) = w^{i_j} \cdot z, z \in Q$. Then for each $j = 1, \ldots, m$ and for each $\lambda_j \in \Lambda_j$, we have $f_{j,\lambda_j}|_Q = \alpha_j$. Since the semigroup $\{\alpha_j^n \mid n \in \mathbb{N}\}$ is a cyclic group generated by α_j, there exists an element $n_j \in \mathbb{N}$ such that $\alpha_j^{-1} = \alpha_j^{n_j}$. Therefore we obtain that Q is equal to the union of minimal sets of the semigroup generated by $\{\alpha_1, \ldots, \alpha_m\}$. Thus $Q = \cup_{L \in \mathrm{Min}(G_\tau, Q)} L$ for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Hence there is no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\})$. Moreover, for each $z \in Q$, for each $j = 1, \ldots, m$ and for each $\lambda_j \in \Lambda_j$, we have

$$f'_{j,\lambda_j}(z) = w^{i_j}(1 + \lambda_j n z^{n-1}). \tag{79}$$

Hence $\mathcal{Y}$ is strongly nice with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of polynomial maps. Moreover, by the formula $f'_{j,\lambda_j}(z) = w^{i_j}(1 + \lambda_j n z^{n-1})$. above, it is easy to see that $\mathcal{Y}$ is non-exceptional with respect to $\{\mathcal{W}_j\}_{j=1}^m$. Thus we have proved Claim 1.

Let $\mathcal{A}$ be the open and dense subset of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ given in Theorem 3.81. Then for each $\tau \in \mathcal{A}$, all statements (i)–(xi) in Theorem 3.81 hold. In particular, any $\tau \in \mathcal{A}$ is weakly mean stable. Let $\mathcal{A}_+ := \{\tau \in \mathcal{A} \mid \exists L \in \text{Min}(G_\tau, Q) \text{ s.t. } \chi(\tau, L) > 0\}$. Also, let

$$\mathcal{A}_{+,all} := \{\tau \in \mathcal{A} \mid \text{ for all } L \in \text{Min}(G_\tau, Q) \text{ we have } \chi(\tau, L) > 0\}.$$

Moreover, let $\mathcal{A}^f := \{\tau \in \mathcal{A} \mid \sharp\Gamma_\tau < \infty\}$, $\mathcal{A}_+^f := \mathcal{A}_+ \cap \mathcal{A}^f$, and $\mathcal{A}_{+.all}^f := \mathcal{A}_{+,all} \cap \mathcal{A}^f$.

We now show the following claim.

Claim 2. The sets $\mathcal{A}_+$ and $\mathcal{A}_{+,all}$ are non-empty open subsets of $\mathcal{A}$ (and thus they are non-empty open subsets of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$). Also, $\mathcal{A}_+^f$ is dense in $\mathcal{A}_+$ and $\mathcal{A}_{+,all}^f$ is dense in $\mathcal{A}_{+,all}$. Moreover, for each $\tau \in \mathcal{A}_+$, we have $\emptyset \neq \cup_{L \in H_{+,\tau}} L \subset J_{pt}^0(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$, where Ω_τ and $H_{+,\tau}$ are the sets defined in Theorem 3.81, and for each $\tau \in \mathcal{A}_{+,all}$, we have $Q \subset J_{pt}^0(\tau)$. Furthermore, for each $\tau \in \mathcal{A}_+^f$, we have $\overline{J_{pt}^0(\tau)} = J(G_\tau)$ which is a perfect set.

To prove this claim, by (79), we obtain that $\mathcal{A}_+$ and $\mathcal{A}_{+,all}$ are non-empty. It is easy to see that $\mathcal{A}_+$ and $\mathcal{A}_{+,all}$ are open in $\mathcal{A}$. The rest statements follow from Theorem 3.81 and Corollary 3.84.

Let $\mathcal{A}_{-,all} := \{\tau \in \mathcal{A} \mid \text{ for all } L \in \text{Min}(G_\tau, Q) \text{ we have. } \chi(\tau, L) < 0\}$. We now prove the following claims.

Claim 3. The set $\mathcal{A}_{-,all}$ is a non-empty open subset of $\mathcal{A}$ and $\mathcal{A}_{-,all} \cap \mathcal{A}_{+,all} = \emptyset$.

Claim 4. Let $\tau \in \mathcal{A}_{-,all}$. Then for each $L \in \text{Min}(G_\tau, J_{\ker}(G_\tau))$, we have $\chi(\tau, L) < 0$, and each $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$ is attracting τ. Thus τ satisfies all assumptions of Theorem 3.66 and all conclusions in Theorem 3.66 hold. In particular, $J_{pt}^0(\tau) = \emptyset$ and $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.

These claims 3,4 can be shown by (79) and the method in Example 5.5.

Example 5.7. Let $x_1, \ldots, x_u \in \mathbb{C}$ be mutually distinct points with $u \geq 2$. Let $a \in \mathbb{C} \setminus \{0\}$ and let $g(z) = a \prod_{j=1}^u (z - x_j)$. Let $Q = \{x_1, \ldots, x_u\}$. Let $P_1, \ldots, P_m$ be mutually distinct non-constant polynomials and suppose that $P_j(Q) \subset Q$ for each $j = 1, \ldots .m$. Also, suppose that $Q = \cup_{L \in \text{Min}(\langle P_1, \ldots, P_m \rangle, Q)} L$. Note that we have the following claim.

Claim 1. For any finite subset Q of $\mathbb{C}$, we can take such elements $P_1, \ldots, P_m$.

To prove this claim, we remark that for any map $\varphi : Q \to Q$, there exists a polynomial P such that $P|_Q = \varphi$ on Q. This fact can be shown by using van der Monde determinant argument. Thus the statement of Claim 1 holds.

For each $j = 1, \ldots, m$, let $\Lambda_j := \mathbb{C} \setminus \{0\}$ and for each $\lambda_j \in \Lambda_j$, let $f_{j,\lambda_j}(z) = P_j(z + \lambda_j g(z))$. Let $\mathcal{W}_j = \{f_{j,\lambda_j}\}_{\lambda_j \in \Lambda_j}$ and let $\mathcal{Y} = \cup_{j=1}^m \{f_{j,\lambda_j} \mid \lambda_j \in \Lambda_j\}$. Then $\mathcal{Y}$ is a weakly nice subset of $\mathcal{P}$ with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of polynomials. We now prove the following claim.

Claim 2. We have $\cap_{j=1}^m S(\mathcal{W}_j) = Q \cup \{\infty\}$. Moreover, $\mathcal{Y}$ is a mild subset of $\mathcal{P}$ and $\mathcal{Y}$ is non-exceptional and strongly nice with respect to holomorphic families $\{\mathcal{W}_j\}_{j=1}^m$ of polynomial maps. Hence, there exists the largest open and dense subset $\mathcal{A}$ of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$ such that for each $\tau \in \mathcal{A}$, all statements (i)–(xi) in Theorem 3.81 hold. In particular, any $\tau \in \mathcal{A}$ is weakly mean stable.

We give the proof of this claim. Since $\mathcal{Y} \subset \mathcal{P}$, the set $\mathcal{Y}$ is mild. For each $x \in Q$, for each $j = 1, \ldots, m$ and for each $\lambda_j \in \Lambda_j$, we have $f_{j,\lambda_j}(x) = P_j(x)$. Thus for each $j = 1, \ldots, m$, we have $S(\mathcal{W}_j) = Q \cup \{\infty\}$. Hence $\cap_{j=1}^m S(\mathcal{W}_j) = Q \cup \{\infty\}$. Also, by the property of $\{P_j\}_{j=1}^m$, we have that for each $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$, we have $Q = \cup_{L \in \text{Min}(G_\tau, Q)} L$. Hence there is no peripheral cycle for $(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m)$. Also, we have

$$f'_{j,\lambda_j}(x) = P'_j(x)(1 + \lambda_j g'(x)) \text{ for all } x \in Q, j = 1, \ldots, m, \lambda_j \in \Lambda_j. \tag{80}$$

Therefore $\mathcal{Y}$ is strongly nice with respect to $\{\mathcal{W}_j\}_{j=1}^m$. By (80), it is easy to see that $\mathcal{Y}$ is non-exceptional with respect to $\{\mathcal{W}_j\}_{j=1}^m$. By Theorem 3.81, the statement of Claim 2 holds. Thus we have proved Claim 2.

We define subsets $\mathcal{A}_+, \mathcal{A}_{+,all}, \mathcal{A}^f$, $\mathcal{A}_+^f, \mathcal{A}_{+,all}^f, \mathcal{A}_{-,all}$ of $\mathcal{A}$ in the same way as that of Example 5.6. Then by (80) and the arguments in Examples 5.6 and 5.5, we obtain the following claims.

Claim 3. The set $\mathcal{A}_{-,all}$ is a non-empty open subset of $\mathcal{A}$ and $\mathcal{A}_{-,all} \cap \mathcal{A}_{+,all} = \emptyset$.
Claim 4. Let $\tau \in \mathcal{A}_{-,all}$. Then for each $L \in \mathrm{Min}(G_\tau, J_{\ker}(G_\tau))$, we have $\chi(\tau, L) < 0$, and each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \not\subset J_{\ker}(G_\tau)$ is attracting τ. Thus τ satisfies all assumptions of Theorem 3.66 and all conclusions in Theorem 3.66 hold. In particular, $J^0_{pt}(\tau) = \emptyset$ and $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.
Claim 5. Suppose that $P'_j(x) \neq 0$ for any $j = 1, \ldots, m$ and for any $x \in Q$. Then the sets $\mathcal{A}_+$ and $\mathcal{A}_{+,all}$ are non-empty open subsets of $\mathcal{A}$ (and thus they are non-empty open subsets of $(\mathfrak{M}_{1,c}(\mathcal{Y}, \{\mathcal{W}_j\}_{j=1}^m), \mathcal{O})$). Also, $\mathcal{A}^f_+$ is dense in $\mathcal{A}_+$ and $\mathcal{A}^f_{+,all}$ is dense in $\mathcal{A}_{+,all}$. Moreover, for each $\tau \in \mathcal{A}_+$, we have $\emptyset \neq \cup_{L \in H_{+,\tau}} L \subset J^0_{pt}(\tau) = \hat{\mathbb{C}} \setminus \Omega_\tau$, where Ω_τ and $H_{+,\tau}$ are the sets defined in Theorem 3.81, and for each $\tau \in \mathcal{A}_{+,all}$, we have $Q \subset J^0_{pt}(\tau)$. Furthermore, for each $\tau \in \mathcal{A}^f_+$, we have $\overline{J^0_{pt}(\tau)} = J(G_\tau)$ which is a perfect set.

Remark 5.8. As in Example 5.7, we can embed many finite Markov chains into $\mathbb{C}$ as weak attractors (i.e. minimal sets with negative Lyapunov exponents) of one random complex polynomial dynamical system generated by $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ which is weakly mean stable and satisfies all statements in Theorem 3.66 (e.g. $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$).

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References

[1] L. V. Ahlfors, *Complex Analysis* 3rd ed., McGraw-ill International Editions, 1979.

[2] A. Beardon, *Iteration of Rational Functions*, Graduate Texts in Mathematics 132, Springer-Verlag, 1991.

[3] R. Brück, *Connectedness and stability of Julia sets of the composition of polynomials of the form* $z^2 + c_n$, J. London Math. Soc. **61** (2000), 462-470.

[4] R. Brück, *Geometric properties of Julia sets of the composition of polynomials of the form* $z^2 + c_n$, Pacific J. Math., **198** (2001), no. 2, 347–372.

[5] R. Brück, M. Büger and S. Reitz, *Random iterations of polynomials of the form* $z^2 + c_n$*: Connectedness of Julia sets*, Ergodic Theory Dynam. Systems, **19**, (1999), No.5, 1221–1231.

[6] M. Büger, *Self-similarity of Julia sets of the composition of polynomials*, Ergodic Theory Dynam. Systems, **17** (1997), 1289–1297.

[7] M. Büger, *On the composition of polynomials of the form* $z^2 + c_n$, Math. Ann. 310 (1998), no. 4, 661–683.

[8] R. Devaney, *An Introduction to Chaotic Dynamical Systems, Second edition*, Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, 1989.

[9] R. Durrett, Probability, Theory and Examples, 4th ed., Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge, 2010.

[10] J. E. Fornaess and N. Sibony, *Random iterations of rational functions*, Ergodic Theory Dynam. Systems, **11**(1991), 687–708.

[11] Z. Gong, W. Qiu and Y. Li, *Connectedness of Julia sets for a quadratic random dynamical system*, Ergodic Theory Dynam. Systems, (2003), **23**, 1807-1815.

[12] Z. Gong and F. Ren, *A random dynamical system formed by infinitely many functions,* Journal of Fudan University, **35**, 1996, 387–392.

[13] A. Hinkkanen and G. J. Martin, *The Dynamics of Semigroups of Rational Functions I*, Proc. London Math. Soc. (3)**73**(1996), 358-384.

[14] A. Hinkkanen and G. J. Martin, *Julia sets of rational semigroups*, Math. Z., 222 (1996), No. 2, 161-169.

[15] M. Hurley, *Multiple attractors in Newton's method*, Ergodic Theory Dynam. Systems, **6**, (1986), 561–569.

[16] J. Hubbard, D. Schleicher, S. Sutherland, *How to find all roots of complex polynomials by Newton's method*, Invent. Math. 146 (2001), no. 1, 133.

[17] J. Jaerisch and H. Sumi, *Multifractal formalism for expanding rational semigroups and random complex dynamical systems*, Nonlinearity 28 (2015) 2913-2938.

[18] J. Jaerisch and H. Sumi, *Pointwise Hölder exponents of the complex analogues of the Takagi function in random complex dynamics*, Adv. Math. 313 (2017) 839–874.

[19] J. Jaerisch and H. Sumi, *Dynamics of infinitely generated nicely expanding rational semigroups and the inducing method*, to appear in Trans. Amer. Math. Soc., doi: http://dx.doi.org/10.1090/tran/6862. http://arxiv.org/abs/1501.06772.

[20] K. Matsumoto and I. Tsuda, *Noise-induced order*, J. Statist. Phys. 31 (1983) 87-106.

[21] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Dynamical Systems (Valparaiso, 1986) (Lecture Notes in Mathematics vol 1331) (Berlin: Springer) pp 86-117, 1988.

[22] C. T. McMullen, *Families of rational maps and iterative root-finding algorithm*, Ann. of Math., **125**, (1987), 467-493.

[23] C. T. McMullen, *Complex Dynamics and Renormalization*, Annals of Mathematical Studies 135, Princeton University Press, 1994.

[24] J. Milnor, *Dynamics in One Complex Variable, Third Edition*, Annals of Mathematics Studies No. 160, Princeton University Press, 2006.

[25] T. Morita, *Deterministic version lemmas in ergodic theory of random dynamical systems*, Hiroshima Math. J. **18** (1988), 15-29.

[26] M. Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. Éc. Norm. Sup. **20** (1987) 1-29.

[27] R. Stankewitz, *Density of repelling fixed points in the Julia set of a rational or entire semigroup, II*, Discrete Contin. Dyn. Syst. 32 (2012), no. 7, 25832589.

[28] R. Stankewitz and H. Sumi, *Dynamical properties and structure of Julia sets of postcritically bounded polynomial semigroups*, Trans. Amer. Math. Soc., 363 (2011), no. 10, 5293–5319.

[29] D. Sullivan, *Quasiconformal homeomorphism and dynamics I: Solution of Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), 401-418.

[30] H. Sumi, *Skew product maps related to finitely generated rational semigroups,* Nonlinearity, **13**, (2000), 995–1019.

[31] H. Sumi, *Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*, Ergodic Theory Dynam. Systems, (2001), **21**, 563–603.

[32] H. Sumi, *Semi-hyperbolic fibered rational maps and rational semigroups*, Ergodic Theory Dynam. Systems, (2006), **26**, 893–922.

[33] H. Sumi, *Interaction cohomology of forward or backward self-similar systems*, Adv. Math., 222 (2009), no. 3, 729–781.

[34] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups III: classification of semi-hyperbolic semigroups and random Julia sets which are Jordan curves but not quasicircles*, Ergodic Theory Dynam. Systems, (2010), **30**, No. 6, 1869–1902.

[35] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups I: connected components of the Julia sets*, Discrete Contin. Dyn. Sys. Ser. A, Vol. 29, No. 3, 2011, 1205–1244.

[36] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc., (2011), 102 (1), 50–112.

[37] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math. 245 (2013) 137–181.

[38] H. Sumi, *Random complex dynamics and devil's coliseums*, Nonlinearity 28 (2015) 1135-1161.

[39] H. Sumi, *The space of* 2*-generator postcritically bounded polynomial semigroups and random complex dynamics*, Adv. Math., 290 (2016) 809–859.

[40] H. Sumi and M. Urbański, *Real analyticity of Hausdorff dimension for expanding rational semigroups*, Ergodic Theory Dynam. Systems (2010), Vol. 30, No. 2, 601-633.

[41] H. Sumi and M. Urbański, Transversality family of expanding rational semigroups, Adv. Math. 234 (2013) 697–734.