

Generalized string equations for Hurwitz numbers

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1. Hurwitz numbers of Riemann sphere

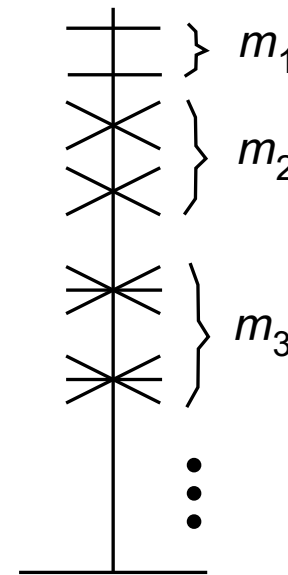
The Hurwitz numbers enumerate topologically nonequivalent finite ramified coverings $\pi : \Gamma \rightarrow \Gamma_0$ of a Riemann surface Γ_0 . In the following, we consider the case where $\Gamma_0 = \mathbf{CP}^1$.

Partition as ramification data

In a neighborhood of the fiber $\pi^{-1}(P)$ of a point P , Γ looks like a union of cyclic coverings of degree μ_1, μ_2, \dots . They give a partition

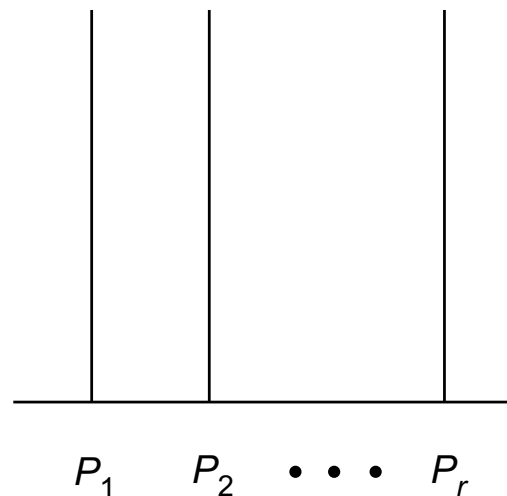
$$\mu = (\mu_1, \mu_2, \dots) = (1^{m_1} 2^{m_2} \dots)$$

of the degree d of the covering.



Hurwitz numbers

Given a positive integer d , a partition $\mu^{(1)}, \dots, \mu^{(r)}$ of d and r points P_1, \dots, P_r of \mathbf{CP}^1 , we consider coverings $\pi : \Gamma \rightarrow \mathbf{CP}^1$ of degree d that are ramified over these points of ramification type $\mu^{(1)}, \dots, \mu^{(r)}$.



There are only a finite number of topologically nonequivalent coverings of this type. The Hurwitz number counts the equivalence classes $[\pi]$ with weight $|\text{Aut}(\pi)|$:

$$H_d(\mu^{(1)}, \dots, \mu^{(r)}) = \sum_{[\pi]} \frac{1}{|\text{Aut}(\pi)|}$$

Formula (Burnside's theorem)

$$H_d(\mu^{(1)}, \dots, \mu^{(r)}) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{k=1}^r f_\lambda(\mu^{(k)}),$$

$$\dim \lambda = \chi_\lambda(C(1^d)), \quad f_\lambda(\mu) = \frac{\chi_\lambda(C(\mu))}{\dim \lambda} |C(\mu)|,$$

where $\chi_\lambda(C)$ denotes the irreducible character (class function) of the symmetric group S_d for the partition λ , $C(\mu)$ the conjugacy class of cycle type $\mu = (1^{m_1} 2^{m_2} \dots)$, and $|C(\mu)|$ the cardinality of $C(\mu)$ as a subset of S_d :

$$|C(\mu)| = d! / z_\mu, \quad z_\mu = \prod_{i \geq 1} m_i! i^{m_i}.$$

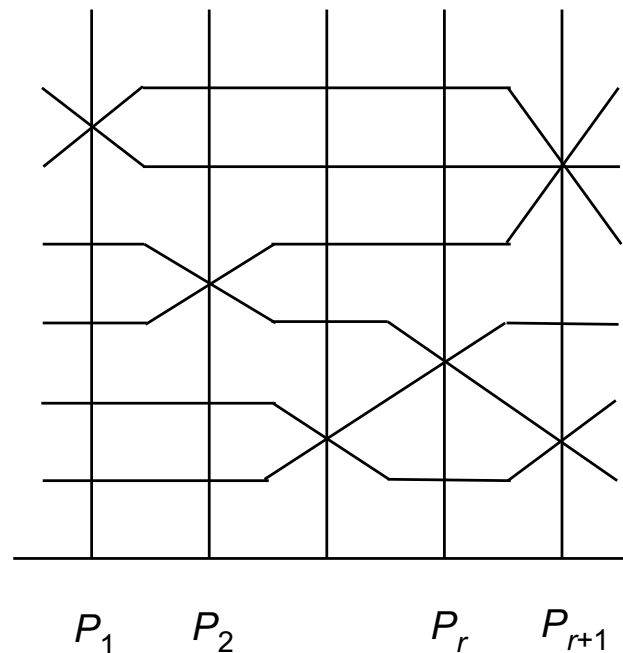
2. Generating functions of Hurwitz numbers²

Generating function of almost simple Hurwitz numbers

$$H_d(\underbrace{1^{d-2}2, \dots, 1^{d-2}2}_r, \mu)$$

Introduce variables β, Q and $\mathbf{x} = (x_1, x_2, \dots)$, the power sums $p_k = \sum_{i \geq 1} x_i^k$ and their products $p_\mu = p_{\mu_1} p_{\mu_2} \dots$. Define the generating function $Z(\mathbf{x})$ as

$$Z(\mathbf{x}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=d} H_d(\underbrace{1^{d-2}2, \dots, 1^{d-2}2}_r, \mu) \frac{\beta^r}{r!} Q^d p_\mu$$

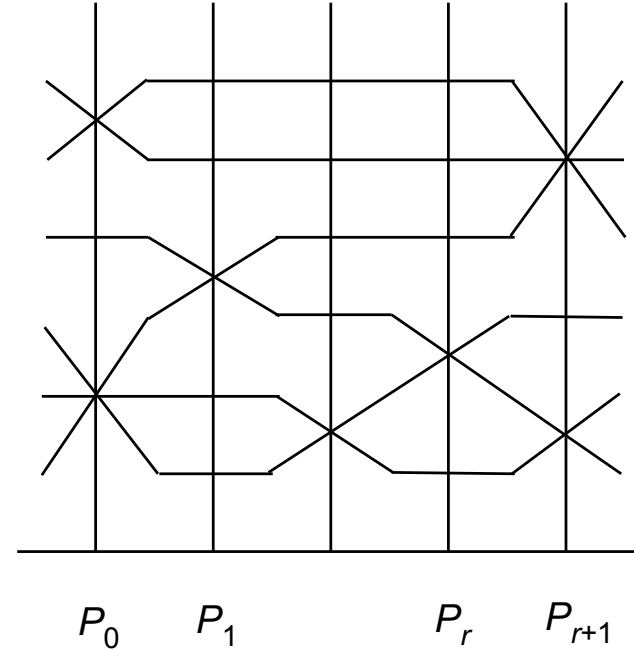


Generating function of double Hurwitz numbers

$$H_d(\mu, \underbrace{1^{d-2}2, \dots, 1^{d-2}2}_r, \bar{\mu})$$

Introduce yet another set of variables $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots)$ and their power sums $\bar{p}_k = \sum_{i \geq 1} \bar{x}_i^k$. Define the generating function $Z(\mathbf{x}, \bar{\mathbf{x}})$ as

$$Z(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{r=0}^{\infty} \sum_{d=0}^{\infty} \sum_{|\mu|=|\bar{\mu}|=d} H_d(\mu, \underbrace{1^{d-2}2, \dots, 1^{d-2}2}_r, \bar{\mu}) \frac{\beta^r}{r!} Q^d p_{\mu} \bar{p}_{\bar{\mu}}$$



Change of variables for KP and Toda hierarchies

- $\mathbf{t} = (t_1, t_2, \dots)$ for KP hierarchy:

$$t_k = \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k$$

- $\mathbf{t} = (t_1, t_2, \dots)$ and $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots)$ for Toda hierarchy:

$$t_k = \frac{p_k}{k} = \frac{1}{k} \sum_{i \geq 1} x_i^k, \quad \bar{t}_k = -\frac{\bar{p}_k}{k} = -\frac{1}{k} \sum_{i \geq 1} \bar{x}_i^k$$

Schur functions are redefined as functions of these variables:

$$s_\lambda(\mathbf{x}) = s_\lambda[\mathbf{t}], \quad s_\lambda(\bar{\mathbf{x}}) = s_\lambda[-\bar{\mathbf{t}}] \quad (\text{Zinn-Justin's notation})$$

Generating functions in terms of Schur functions

Use Frobenius' formula $\sum_{|\mu|=d} \frac{\chi_\lambda(C(\mu))}{z_\mu} p_\mu = s_\lambda(\mathbf{x})$ to rewrite the generating functions to sums over partitions λ of arbitrary length:

$$Z(\mathbf{x}) = \sum_{\lambda} \frac{\dim \lambda}{|\lambda|!} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda(\mathbf{x}) = \sum_{\lambda} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda[\mathbf{t}] s_\lambda[1, 0, \dots],$$

$$Z(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{\lambda} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda(\mathbf{x}) s_\lambda(\bar{\mathbf{x}}) = \sum_{\lambda} e^{\beta \kappa_\lambda / 2} Q^{|\lambda|} s_\lambda[\mathbf{t}] s_\lambda[-\bar{\mathbf{t}}].$$

Remark: Relevant formulae

$$\frac{\dim \lambda}{|\lambda|!} = s_\lambda[1, 0, \dots], \quad f_\lambda(1^{d-2}2) = \kappa_\lambda = \sum_{i \geq 1} \lambda_i (\lambda_i - 2i + 1).$$

Cut-and-join operator

$$M_0 = \frac{1}{2} \sum_{k,l=1}^{\infty} \left(k l t_k t_l \frac{\partial}{\partial t_{k+l}} + (k+l) t_{k+l} \frac{\partial^2}{\partial t_k \partial t_l} \right)$$

- Schur functions are eigenfunctions of M_0 :

$$M_0 s_\lambda[\mathbf{t}] = \frac{\kappa_\lambda}{2} s_\lambda[\mathbf{t}].$$

- $Z[\mathbf{t}] = Z(\mathbf{x})$ and $Z[\mathbf{t}, \bar{\mathbf{t}}] = Z(\mathbf{x}, \bar{\mathbf{x}})$ can be expressed as

$$Z[\mathbf{t}] = e^{\beta M_0} e^{Q t_1},$$

$$Z[\mathbf{t}, \bar{\mathbf{t}}] = e^{\beta M_0} \exp \left(- \sum_{k=1}^{\infty} Q^k k t_k \bar{t}_k \right).$$

Generating functions as tau functions

- $Z[\mathbf{t}]$ is a tau function of **the KP hierarchy** (Kazarian & Lando, Goulden & Jackson, \dots).
- $Z[\mathbf{t}, \bar{\mathbf{t}}]$ is a tau function of **the Toda hierarchy** at a point, say $s = 0$, of the lattice (Okounkov). In other words, $Z[\mathbf{t}, \bar{\mathbf{t}}]$ is a tau function of **the 2-component KP hierarchy**.

Remark: Any tau function $\tau(s, \mathbf{t}, \bar{\mathbf{t}})$ of the Toda hierarchy is a sequence of tau functions (indexed by $s \in \mathbf{Z}$) of the 2-component KP hierarchy.

3. Fermionic representation of tau functions

Fermionic operators and Fock space

- creation-annihilation operators ψ_i, ψ_i^* ($i \in \mathbf{Z}$)

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i+j,0}, \quad \psi_i \psi_j + \psi_j \psi_i = 0, \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

- ground states in charge s sector of the Fock space

$$\langle s | = \langle -\infty | \cdots \psi_{s-1}^* \psi_s^*, \quad |s\rangle = \psi_{-s} \psi_{-s+1} \cdots | -\infty \rangle$$

- Fermion bilinears

$$J_k = \sum_{n \in \mathbf{Z}} : \psi_{-n+k} \psi_n^* :, \quad L_0 = \sum_{n \in \mathbf{Z}} n : \psi_{-n} \psi_n^* :, \quad W_0 = \sum_{n \in \mathbf{Z}} n^2 : \psi_{-n} \psi_n^* :$$

Remark: $M_0 \leftrightarrow \frac{1}{2} \sum_{n \in \mathbf{Z}} (n - 1/2)^2 : \psi_{-n} \psi_n^* : = \frac{1}{2} W_0 - \frac{1}{2} L_0 + \frac{1}{8} J_0$

Tau function for double Hurwitz numbers

The special $GL(\infty)$ element

$$g = e^{\beta W_0/2} Q^{L_0}$$

determines a tau function

$$\tau(s, \mathbf{t}, \bar{\mathbf{t}}) = \langle s | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) e^{\beta W_0/2} Q^{L_0} \exp \left(- \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) | s \rangle$$

of the Toda hierarchy. This tau function has the Schur function expansion

$$\begin{aligned} \tau(s, \mathbf{t}, \bar{\mathbf{t}}) &= e^{\beta s(s+1)(2s+1)/12} Q^{s(s+1)/2} \\ &\quad \times \sum_{\lambda} e^{\beta \kappa_{\lambda}/2} (e^{\beta(2s+1)/2} Q)^{|\lambda|} s_{\lambda}[\mathbf{t}] s_{\lambda}[-\bar{\mathbf{t}}] \end{aligned}$$

and reduces to $Z[\mathbf{t}, \bar{\mathbf{t}}]$ upon renormalizing Q and setting $s = 0$.

4. Generalized string equations

Intertwining relations of fermion bilinears

$g = e^{-\beta W_0/2} Q^{L_0}$ intertwines special fermion bilinears as

$$J_k g = g Q^k e^{-\beta k^2/2} \sum_{n \in \mathbf{Z}} e^{\beta k n} : \psi_{-n+k} \psi_n^* :,$$

$$g J_{-k} = Q^k e^{\beta k^2/2} \sum_{n \in \mathbf{Z}} e^{\beta k n} : \psi_{-n-k} \psi_n^* : g$$

Remark: These relations play a role in an integrable structure of **the melting crystal model** as well.

Lax and Orlov-Schulman operators

$$L = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(-n+1)\partial_s}, \quad \bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \sum_{n=1}^{\infty} \bar{u}_n e^{(n-1)\partial_s},$$

$$M = \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \quad \bar{M} = - \sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n$$

Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \quad k = 1, 2, \dots$$

same equations replacing $L \rightarrow M, \bar{L}, \bar{M}$

Twisted canonical commutation relations

$$[L, M] = L, \quad [\bar{L}, \bar{M}] = \bar{L}$$

Theorem

- The generalized string equations

$$L^k = Q^k e^{-\beta k^2 / 2} \bar{L}^k e^{\beta k \bar{M}}, \quad \bar{L}^{-k} = Q^k e^{\beta k^2 / 2} L^{-k} e^{\beta k M}$$

hold for $k = 1, 2, \dots$.

- These equations reduce to the lowest ones

$$L = Q e^{-\beta/2} \bar{L} e^{\beta \bar{M}}, \quad \bar{L}^{-1} = Q e^{\beta/2} L^{-1} e^{\beta M}$$

Remark: Generalized string equations in $c = 1$ string theory

$$L = \bar{L} \bar{M} + \text{const.} \bar{L}, \quad \bar{L}^{-1} = L^{-1} M + \text{const.} L^{-1}$$

5. Classical limit of generalized string equations

Classical (= dispersionless) limit of Toda hierarchy

- Introduce a new parameter \hbar , allow the tau function itself to depend on \hbar , and **assume** that the rescaled tau function $\tau_{\hbar}(s, \mathbf{t}, \bar{\mathbf{t}}) = \tau(\hbar, \hbar^{-1}s, \hbar^{-1}\mathbf{t}, \hbar^{-1}\bar{\mathbf{t}})$ behaves as

$$\log \tau_{\hbar}(s, \mathbf{t}, \bar{\mathbf{t}}) \sim \hbar^{-2} \mathcal{F}(s, \mathbf{t}, \bar{\mathbf{t}}) + O(\hbar^{-1}) \quad (\hbar \rightarrow 0).$$

$\mathcal{F}(s, \mathbf{t}, \bar{\mathbf{t}})$ is called “free energy”, etc.

- In the Lax formalism, this amounts to replacing the difference operators L, M, \bar{L}, \bar{M} (quantum observables) by **functions** $\mathcal{L}, \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}$ (classical observables):

$$\sum_n a_n(\hbar, s) e^{n\partial_s} \rightarrow \sum_n a_n^{(0)}(s) p^n, \quad a_n(\hbar, s) = a_n^{(0)}(s) + O(\hbar).$$

Lax and Orlov-Schulman functions

$$\mathcal{L} = p + \sum_{n=1}^{\infty} u_n^{(0)} p^{1-n}, \quad \bar{\mathcal{L}}^{-1} = \bar{u}_0^{(0)} p^{-1} + \sum_{n=1}^{\infty} \bar{u}_n^{(0)} p^{n-1},$$

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}, \quad \bar{\mathcal{M}} = - \sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n^{(0)} \bar{\mathcal{L}}^n$$

Lax equations and twisted canonical Poisson relations

$$\frac{\partial \mathcal{L}}{\partial t_k} = \{B_k, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_k} = \{\bar{B}_k, \mathcal{L}\}, \quad k = 1, 2, \dots$$

same equations replacing $\mathcal{L} \rightarrow \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}}$,

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}$$

with respect to **Poisson bracket** $\{F, G\} = p \frac{\partial F}{\partial p} \frac{\partial G}{\partial s} - \frac{\partial F}{\partial s} p \frac{\partial G}{\partial p}$.

Classical limit of generalized string equation

- Rescaling $s, \mathbf{t}, \bar{\mathbf{t}}$ as $s, \mathbf{t}, \bar{\mathbf{t}} \rightarrow \hbar^{-1}s, \hbar^{-1}\mathbf{t}, \hbar^{-1}\bar{\mathbf{t}}$ changes the generalized string equation as

$$L = Qe^{-\beta/2}\bar{L}e^{\beta\hbar^{-1}\bar{M}}, \quad \bar{L}^{-1} = Qe^{\beta/2}L^{-1}e^{\beta\hbar^{-1}M}$$

These equations themselves **do not have** a limit as $\hbar \rightarrow 0$.

- By rescaling β as $\beta \rightarrow \hbar^{-1}\beta$, one can take the classical limit as $\hbar \rightarrow 0$.
- Thus the generalized string equations

$$\mathcal{L} = Q\bar{\mathcal{L}}e^{\beta\bar{\mathcal{M}}}, \quad \bar{\mathcal{L}}^{-1} = Q\mathcal{L}^{-1}e^{\beta\mathcal{M}}$$

in the classical limit is obtained.

Theorem

There is a unique solution of the classical limit of the generalized string equations that has **power series expansion** with respect to $\mathbf{t}, \bar{\mathbf{t}}$:

$$\log \bar{u}_0 = \log Q + \beta s + (\text{terms of positive orders in } \mathbf{t}, \bar{\mathbf{t}}),$$

$$u_n = -\beta n \bar{t}_n \bar{u}_0^n + (\text{terms of higher orders in } \mathbf{t}, \bar{\mathbf{t}}),$$

$$\bar{u}_n = \beta n t_n \bar{u}_0 + (\text{terms of higher orders in } \mathbf{t}, \bar{\mathbf{t}}).$$

(“(0)” is omitted.) This solution is quasi-homogeneous with respect to rescaling

$$\begin{aligned} t_n &\rightarrow c^{-n} t_n, & \bar{t}_n &\rightarrow c^n \bar{t}_n, & s &\rightarrow s, & p &\rightarrow c^{-1} p, \\ u_n &\rightarrow c^n u_n, & \bar{u}_n &\rightarrow c^{-n} \bar{u}_n, & v_n &\rightarrow c^n v_n, & \bar{v}_n &\rightarrow c^{-n} \bar{v}_n. \end{aligned}$$

Solution at special values of t, \bar{t}

When \bar{t} is specialized to $\bar{t}_k = \bar{t}_1 \delta_{k1}$ (which amounts to specializing to almost-simple Hurwitz numbers), \mathcal{L} simplifies as

$$\mathcal{L} = pe^{-\beta \bar{t}_1 \bar{u}_0 p^{-1}},$$

and $\bar{u}_0 (= \partial^2 \mathcal{F} / \partial s^2)$ is determined by the single equation

$$\log \bar{u}_0 = \log Q + \beta s + \beta \sum_{k=1}^{\infty} kt_k \frac{(-k\beta \bar{t}_1 \bar{u}_0)^k}{k!}.$$

Remarkably, the functional structure of \mathcal{L} essentially coincides with the defining equation $z = we^w$ of Lambert's W-function $w = W(z)$ ($\mathcal{L}^{-1} \leftrightarrow z, p^{-1} \leftrightarrow w$). Presumably, this will be related to the **spectral curve** in the sense of Eynard and Orantin.