A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane

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Abstract. –

For the white noise ξ on $\mathbb{R}^2$, an operator corresponding to a limit of $-\Delta + \xi_\varepsilon + c_\varepsilon$ as $\varepsilon \to 0$ is realized as a self-adjoint operator, where, for each $\varepsilon > 0$, c_ε is a constant, ξ_ε is a smooth approximation of ξ defined by $\exp(\varepsilon^2\Delta)\xi$, and Δ is the Laplacian. This result is a variant of result's obtained by Allez and Chouk, Mouzard, and Ugurcan. The proof in this paper is based on the heat semigroup approach of the paracontrolled calculus, referring the proof by Mouzard. For the obtained operator, the spectral set is shown to be $\mathbb{R}$.

1. Introduction

Our motivation is to study the spectral properties of the Schrödinger operator

$$-\Delta + V(x)$$

on the configuration space $\mathbb{R}^2$, in the case that the potential V is the white noise ξ: $\xi = (\xi(x))_{x\in\mathbb{R}^2}$ is a Gaussian random field on $\mathbb{R}^2$ such that $\mathbb{E}[\xi(x)] = 0$ and $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$ for any $x, y \in \mathbb{R}^2$, where δ is the Dirac delta distribution. However the irregularity of the white noise ξ brings difficulty to define the operator as a self-adjoint operator. If the configuratio space $\mathbb{R}^2$ is replaced by $\mathbb{R}$, then the irregularity is mild so that the Schrödinger operator is realized as a self-adjoint operator and we have many related results. For this aspect, refer the works by Fukushima and Nakao [8] and Minami [17]. For the multidimensional cases, we know that some renormalization techniques are needed by related works which are well developed recently as follows: Hairer developed the theory of regularity structures [11], Gubinelli, Imkeller and Perkowski developed the paracontrolled calculus in [9], and Kupiainen developed the theory of renormalization group [15]. They studied many stochastic partial differential equations as the stochastic quantization equation for ϕ^4_3 Euclidean quantum field theory, the generalized continuous parabolic Anderson models, the Kardar-Parisi-Zhang type equation, the Navier-Stokes equation with very singular forcings and so on. In particular the continuous parabolic Anderson models correspond to consider the heat semigroups generated by the Schrödinger operators. In the equations, the white noise ξ is replaced by $\xi_\varepsilon + c_\varepsilon$ and the existence of the limit as $\epsilon \to 0$ of the solution $u_\varepsilon(t,x)$ of

$$\partial_t u_\varepsilon(t,x) = (\Delta - \xi_\varepsilon(x) - c_\varepsilon)u_\varepsilon(t,x) \text{ for } t > 0, \text{ and } u_\varepsilon(0,x) = u_0(x),$$

is proven, where ξ_ε is a smooth approximation of ξ and c_ε is a constant satisfying $c_\varepsilon \to \infty$ as $\epsilon \to 0$. These works started in the case that the configuration space is replaced by a compact space as $\mathbb{R}^2/\mathbb{Z}^2$. For the extension to noncompact spaces, Hairer and Labbé studied the generalized continuous parabolic Anderson models on $\mathbb{R}^2$ and $\mathbb{R}^3$ and the stochastic heat equation on $\mathbb{R}$ [12], [13], and Dahlqvist, Diehl and Driver extended to 2 dimensional closed manifold [6]. On the other hand, Bailleul, Bernicot and Frey developed the paracontrolled calculus using the heat semigroup so that the calculus can be applied widely, and applied the calculus to the generalized continuous parabolic Anderson models on 2 or 3 dimensional manifolds and the multiplicative Burgers equation on 3 dimensional manifolds [2], [3]. In their theory, the approximation ξ_ε of ξ is defined by the heat semigroup as $\exp(\varepsilon^2\Delta)\xi$. The constant c_ε is replaced by a function. As for the Schrödinger operator, Allez and Chouk proved the self-adjointness of the operator corresponding to the limit of $-\Delta + \xi_\varepsilon + c_\varepsilon$ as $\epsilon \to 0$ in the case that the configuration space is replaced by the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ [1]. They used the paracontrolled calculus by Gubinelli, Imkeller and Perkowski, and they showed also the discreteness of the spectrum and some results on the asymptotic distributions of the eigenvalues. Gubinelli, Ugurcan and Zachhuber extended the results to the 3-dimensional torus $\mathbb{R}^3/\mathbb{Z}^3$ and apply to study some nonlinear Schrödinger and wave equations [10]. Ugurcan extended the results to $\mathbb{R}^2$ where c_ε is replaced by a function [21]. Labbé extended the results to the corresponding operators on $(-1,1)^2$ and $(-1,1)^3$ with the periodic and Dirichlet boundary conditions by applying the theory on regularity structures [16]. Mouzard extended the results to the case that the configuration space is a compact 2-dimensional manifold by the heat semigroup approach to the paracontrolled calculus by Bailleul, Bernicot and Frey [18]. On the other hand, main topics on the Schrödinger operator with random potentials have been the Anderson localization. In that topics, the spectral structure is discussed for the Schrödinger operators with stationary random potentials defined on the Euclidean space $\mathbb{R}^d$ (cf. [4], [19]).

In this paper we prove the self-adjointness of the operator corresponding to the limit of $-\Delta + \xi_\varepsilon + c_\varepsilon$ as $\epsilon \to 0$ in the case that the configuration space is $\mathbb{R}^2$ and c_ε is a constant by referring the methods in

Mouzard [18]. One vantage point of the heat semigroup approach is that the effects to a paraproduct from the two functions decay exponentially as the distance between the supports of the two functions becomes larger. To use this point, we introduce a partition of unity. The convergence to the operator holds in the strong resolvent sense as is discussed in Proposition 4.1 below, which is weak to obtain spectral results. Then to show that the spectral set is $\mathbb{R}$, we construct Weyl sequence on domains where the white noise is close to constants by referring the usual methods to identify the spectral set of the Schrödinger operator with ergodic random potentials (cf. Pastur and Figotin [19] Section 5d).

The organization of this paper is as follows. In Section 2 we give the definition of our operator and state the theorem. In Section 3 we prepare basic estimates to apply the paracontrolled calculus. In Section 4 we prove the theorem on the self-adjointness. In Section 5 we show that the spectral set of our operator is $\mathbb{R}$.

2. The framework and the results

We use a partition of unity to extend the results on compact spaces to a noncompact space: we take a $[0,1]$-valued smooth function χ_0 on $\mathbb{R}^2$ such that

$$\sum_{a\in\mathbb{Z}^2} \chi_a^2 \equiv 1 \text{ on } \mathbb{R}^2$$

and the support of χ_0 is included in Λ_2, where $\chi_a(x) = \chi_0(x-a)$ for any $a \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$, and $\Lambda_r = (-r/2, r/2)^2$ for any $r > 0$. For each $a \in \mathbb{Z}^2$, let $\Lambda_r(a) = a + \Lambda_r$. Referring the paracontrolled calculus in Mouzard [18], we fix a large even natural number b and consider operators

$$Q_t^{(c)} := \frac{(-t\Delta)^c}{(c-1)!} e^{t\Delta} \text{ and } P_t^{(c)} := I - \int_0^t \frac{ds}{s} Q_s^{(c)} = \sum_{j=0}^{c-1} \frac{(-t\Delta)^j}{j!} e^{t\Delta}$$

for $c \in [1,b] \cap \mathbb{N}$, where Δ is the Laplacian on $\mathbb{R}^2$, and I is the identity operator. For $k \in [0, 2b] \cap \mathbb{Z}$, let $StGC^k$ be the set of families of operators of the form

$$((\sqrt{t}\partial_1)^{\alpha_1} (\sqrt{t}\partial_2)^{\alpha_2} P_t^{(c)})_{t\in(0,1]}$$

with $c \in [1,b] \cap \mathbb{N}$ and $\alpha_1, \alpha_2 \in \mathbb{Z}$ satisfying $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = k$. These families of operators are colled as the standard families of Gaussian operators with cancellation of order k. For this family, general operators as $(-t\Delta)^{\alpha/2} e^{t\Delta}$ may be included. However we consider only differential operators times a heat semigroup since it is enough for our purpose and the commutator of a differnetial operator with a multiplication of a smooth function is simple. We also set

$$StGC^{I} = \bigcup_{k \in I \cap \mathbb{Z}} StGC^{k}$$

for any interval I in $[0, \infty)$.

Referring [3] and [18], we decompose the product as follows:

$$fg = P_f g + \Pi(f,g) + P_g f + P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g)),$$

for appropriate distributions f, g on $\mathbb{R}^2$, where

$$P_f g := \sum_{\nu} c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)) \tag{2.1}$$

with a finite subset $\{(c_\nu, Q^{1,\nu}, Q^{2,\nu}, P^\nu)\}_\nu$
of $\mathbb{R} \times StGC^{[b/2,2b]} \times StGC^{[b/2,2b]} \times StGC^{[0,b/2)}$,

and

$$\Pi(f,g) := \sum_{\mu} c_\mu \int_0^1 \frac{dt}{t} P_t^{\mu}((Q_t^{1,\mu} f)(Q_t^{2,\mu} g)) \tag{2.2}$$

with a finite subset $\{(c_\mu, Q^{1,\mu}, Q^{2,\mu}, P^\mu)\}_\mu$
of $\mathbb{R} \times StGC^{[b/2,2b]} \times StGC^{[b/2,2b]} \times StGC^{[0,b/2)}$. $P_f g$ is called as a paraproduct and is well-defined as a distribution for any distributions f and g. $\Pi(f,g)$ is called as a resonating term and we need sufficient regularity properties of f or g to give a meaning for $\Pi(f,g)$. We use also

$${}_hP_f g := \sum_{\nu} c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)h),$$

where h is another appropriate distribution. We use

$$\Delta^{-loc} := -\int_0^1 dt e^{t\Delta}, \tag{2.3}$$

which is an approximation of the inverse of the Laplacian satisfying

$$\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^{\Delta}$$

and the integral kernel has a Gaussian bound:

$$\sup_{|x-y|\geq 1} \frac{\log|\Delta^{-loc}(x,y)|}{|x-y|^2} < 0.$$

We use the commutators:

$$C(f,g,h) := \Pi(\Delta^{-loc} P_f g, h) - f\Pi(\Delta^{-loc} g, h) \tag{2.4}$$

and

$$S(f,g,h) := P_h(\Delta^{-loc}P_f g) - {}_fP_h(\Delta^{-loc}g) \tag{2.5}$$

for any appropriate distributions f, g and h on $\mathbb{R}^2$.

We use the Besov space $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)$ with parameters $p, q \in [1,\infty], \alpha \in (-2b, 2b)$, defined by the completion of $C_0^{\infty}(\mathbb{R}^2)$ with respect to the norm

$$\begin{aligned} &\|f\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)} \\ :=&\|e^{\Delta}f\|_{L^p(\mathbb{R}^2:dx)} \\ &+\sup\{\|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)} : Q \in StGC^{(|\alpha|,2b]}\}. \end{aligned} \tag{2.6}$$

for any $f \in C_0^{\infty}(\mathbb{R}^2)$, where $C_0^{\infty}(\mathbb{R}^2)$ is the smooth functions with compact supports. $\mathcal{C}^{\alpha}(\mathbb{R}^2) := \mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)$ is called as the Besov α-Hölder space, and $\mathcal{H}^{\alpha}(\mathbb{R}^2) = \mathcal{B}^{\alpha}_{2,2}(\mathbb{R}^2)$ is the Sobolev space with the index α.

It is known that $\chi_a\xi$ is $C^{-1-\epsilon}(\mathbb{R}^2)$-valued for any $\epsilon > 0$ and $a \in \mathbb{Z}^2$. As in Theorem 2.1 in [18], there exists a random field Y_ξ such that

$$\lim_{\varepsilon\to 0}\mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_\xi)\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$$

for any $p \in [1,\infty)$, $\epsilon > 0$ and $a \in \mathbb{Z}^2$, where $\xi_\varepsilon := e^{\varepsilon^2\Delta}\xi$ is a smooth approximation of ξ and

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon) - \mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)].$$

Now we define a Schrödinger type operator as follows: we define a domain by

$$\begin{aligned} \mathrm{Dom}_0(\widetilde{H^{\xi}}) :=\Big\{ &u \in \bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0 \\ &\qquad\qquad\qquad\qquad \text{for any } \epsilon > 0, \\ &\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a\Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0\Big\} \end{aligned}$$

and, for $u \in \mathrm{Dom}_0(\widetilde{H^\xi})$, we define

$$
\begin{aligned}
&\widetilde{H^\xi} u \\
&= -\Delta\Phi_\xi(u) + P_\xi\Phi_\xi(u) + \Pi(\Phi_\xi(u), \xi) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi)) \\
&\quad + e^\Delta P_u\xi + e^\Delta{}_u P_\xi(\Delta^{-loc}\xi) + e^\Delta P_u Y_\xi \\
&\quad + C(u,\xi,\xi) + S(u,\xi,\xi) \\
&\quad + P_{Y_\xi}u + \Pi(u, Y_\xi) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_\xi)) \\
&\quad + P_\xi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi)) + \Pi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi), \xi) \\
&\quad + P_\xi(\Delta^{-loc}P_u Y_\xi) + \Pi(\Delta^{-loc}P_u Y_\xi, \xi),
\end{aligned}
\tag{2.7}
$$

where

$$\Phi_\xi(u) := u - \Delta^{-loc}P_u\xi - \Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi) - \Delta^{-loc}P_u Y_\xi.$$

Now our main results in this paper are stated as follows:

Theorem 1. *The operator* $\widetilde{H^\xi}$ *with the domain* $\mathrm{Dom}_0(\widetilde{H^\xi})$ *in* (2.7) *is essentially self-adjoint on* $L^2(\mathbb{R}^2)$.

Theorem 2. *The spectral set of the closure* $\overline{\widetilde{H^\xi}}$ *of the operator in* (2.7) *is* $\mathbb{R}$.

In the rest of this section, we will discuss the motivation of the definition in (2.7)

The first formal object was the operator

$$H^\xi = -\Delta + \xi.$$

To erase the singularity of $\xi \in C^{-1-\epsilon}_{loc}(\mathbb{R}^2)$ in $H^\xi u$, we assume $u \in \mathcal{H}^{1-\epsilon}_{loc}(\mathbb{R}^2)$ so that $\Delta u \in \mathcal{H}^{-1-\epsilon}_{loc}(\mathbb{R}^2)$, where $(\mathcal{B}^\alpha_{p,q})_{loc}(\mathbb{R}^2) := \{f$: a distribution on $\mathbb{R}^2$ s.t. $\chi_a f \in \mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)$ for any $a \in \mathbb{Z}^2\}$ for each Besov space $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)$. In the decomposition

$$\xi u = P_u\xi + P_\xi u + \Pi(u,\xi) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi)),$$

$\Pi(u,\xi)$ is not defined, and $P_u\xi$ and $P_\xi u$ may not belong to $L^2_{loc}(\mathbb{R}^2)$. To erase the singularity of $P_u\xi$, we assume

$$u = \Delta^{-loc}P_u\xi + u^{(\#)}$$

with $u^{(\#)} \in \mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)$. Then we have

$$H^{\xi}u = -\Delta u^{(\#)} + e^{\Delta}P_u\xi + P_{\xi}u + \Pi(u,\xi) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi))$$
$$= -\Delta u^{(\#)} + P_{\xi}(\Delta^{-loc}P_u\xi) + \Pi(\Delta^{-loc}P_u\xi,\xi)$$
$$+ e^{\Delta}P_u\xi + P_{\xi}u^{(\#)} + \Pi(u^{(\#)},\xi) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi))$$

We use the commutators to rewrite the second and third terms as

$$_uP_{\xi}(\Delta^{-loc}\xi) + u\Pi(\Delta^{-loc}\xi,\xi)$$
$$+ S(u,\xi,\xi) + C(u,\xi,\xi)$$

to move the function u to outer parts of the products. Now, as the ill defined term $\Pi(\Delta^{-loc}\xi,\xi)$ is separated from the function u, we replace this by Y_{ξ}. Then the operator H^{ξ} is replaced by $\widetilde{H^{\xi}}$. To erase the singularity of $_uP_{\xi}(\Delta^{-loc}\xi)$ and Y_{ξ}, we assume

$$u^{(\#)} = \Delta^{-loc}{}_uP_{\xi}(\Delta^{-loc}\xi) + \Delta^{-loc}P_uY_{\xi}) + u^{\#}$$

with $u^{\#} \in \mathcal{H}^2(\mathbb{R}^2)$. Then $u^{\#} = \Phi_{\xi}(u)$ and we obtain the definition of (2.7).

3. Estimates for products and commutators

We first prove the following estimates of the inverse operator of the Laplacian. Our basic methods are included in the proof.

Lemma 3.1. *There exists $C \in (0,\infty)$ satisfying the following: for any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$, there exist $C_{\alpha,\epsilon} \in (0,\infty)$ such that*

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)} \le C_{\alpha,\epsilon}\|\chi_{a_2}f\|_{\mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)}\exp(-C|a_1-a_2|^2) \tag{3.1}$$

for any $a_1,a_2 \in \mathbb{Z}^2$ and $f \in \mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)$.

Proof. Since

$$I = \int_0^{\infty}\frac{dt}{t}(-t\Delta)^n e^{t\Delta}/\Gamma(n) = \int_0^2\frac{dt}{t}(-t\Delta)^n e^{t\Delta}/\Gamma(n) + p(\Delta)e^{2\Delta},$$

with a polynomial $p(\cdot)$ of the degree $n-1$, we have only to estimate

$$\|e^{\Delta}\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{L^2(\mathbb{R}^2)} \tag{3.2}$$

and

$$\|t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2}e^{t\Delta}\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{L^2(\mathbb{R}^2\times[0,2]:dxdt/t)} \tag{3.3}$$

with a large n_1+n_2. For (3.3), we have only to estimate

$$\Big\|t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2}e^{t\Delta}\chi_{a_1}\int_0^1 ds\Delta^m e^{(s+1)\Delta}\chi_{a_2}f\Big\|_{L^2(\mathbb{R}^2\times[0,2]:dxdt/t)} \tag{3.4}$$

with $m\in\{0,1,\dots n-1\}$ and

$$\begin{aligned}&\Big\|t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2}e^{t\Delta}\chi_{a_1}\\ &\times\int_0^1 ds\int_0^1\frac{dr}{r}(r\Delta)^n e^{(s+r)\Delta}\chi_{a_2}f\Big\|_{L^2(\mathbb{R}^2\times[0,2]:dxdt/t)}\end{aligned} \tag{3.5}$$

with a large n. We consider (3.5). By the integration by parts, we have

$$\begin{aligned}&\Big\|t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2}e^{t\Delta}\chi_{a_1}\\
&\times\int_0^1 ds\int_0^1\frac{dr}{r}(r\Delta)^n e^{(s+r)\Delta}\chi_{a_2}f\Big\|_{L^\infty(\mathbb{R}^2\times[0,1])}\\
\le&\sup_{0<t\le 2}\iint_{t\le s+r}ds\frac{dr}{r}t^{-\alpha/2}\Big(\frac{t}{s+r}\Big)^{(n_1+n_2)/2}\Big(\frac{r}{s+r}\Big)^n\\
&\times\|e^{t\Delta}(\sqrt{s+r}\partial_{x_1})^{n_1}(\sqrt{s+r}\partial_{x_2})^{n_2}\\
&\times\chi_{a_1}((s+r)\Delta)^n e^{(s+r)\Delta}\chi_{a_2}f\|_{L^\infty(\mathbb{R}^2)}\\
&+\sup_{0<t\le 2}\iint_{t\ge s+r}ds\frac{dr}{r}t^{-\alpha/2}\Big(\frac{r}{t}\Big)^m\Big(\frac{r}{s+r}\Big)^{n-m}\\
&\times\Big\|\int dy\{(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2}(t\Delta_y)^m e^{t\Delta}(x,y)\chi_{a_1}(y)\}\\
&\times\{((s+r)\Delta)^{n-m}e^{(s+r)\Delta}\chi_{a_2}f\}(y)\Big\|_{L^\infty(\mathbb{R}^2)}\\
\le& c_1\Big(\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}\sup_{x\in\mathbb{R}^2,v\in(0,2)}|v^{(2-\alpha)/2}\\
&\times((\sqrt{v}\partial_{x_1})^{m_1}(\sqrt{v}\partial_{x_2})^{m_2}(v\Delta)^n e^{v\Delta}\chi_{a_2}f)(x)|\\
&+\sup_{x\in\mathbb{R}^2,v\in(0,2)}|v^{(2-\alpha)/2}((v\Delta)^{n-m}e^{v\Delta}\chi_{a_2}f)(x)|\Big),\end{aligned}$$

where $m \in \mathbb{N} \cap (0 \vee (-\alpha/2), n \wedge (n-\alpha/2)]$. Moreover by changing the order of the integrations, we have

$$\begin{aligned}
&\Big\| t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2} e^{t\Delta}\chi_{a_1} \\
&\times \int_0^1 ds \int_0^1 \frac{dr}{r}(r\Delta)^n e^{(s+r)\Delta}\chi_{a_2} f\Big\|_{L^1(\mathbb{R}^2\times[0,2]:dxdt/t)} \\
\leq &\int_0^2 \frac{dt}{t} \iint_{t\leq s+r} ds\frac{dr}{r} t^{-\alpha/2}\Big(\frac{t}{s+r}\Big)^{(n_1+n_2)/2}\Big(\frac{r}{s+r}\Big)^n \\
&\times |e^{t\Delta}(\sqrt{s+r}\partial_{x_1})^{n_1}(\sqrt{s+r}\partial_{x_2})^{n_2}\chi_{a_1}((s+r)\Delta)^n e^{(s+r)\Delta}\chi_{a_2} f| \\
&+ \int_0^2 \frac{dt}{t} \iint_{t\geq s+r} ds\frac{dr}{r} t^{-\alpha/2}\Big(\frac{r}{t}\Big)^m\Big(\frac{r}{s+r}\Big)^{n-m} \\
&\times \Big| \int dy \{(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2}(t\Delta_y)^n e^{t\Delta}(x,y)\chi_{a_1}(y)\}\{e^{(s+r)\Delta}\chi_{a_2} f\}(y)\Big| \\
\leq &c_2\Big(\sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2} \|v^{(2-\alpha)/2}((\sqrt{v}\partial_{x_1})^{m_1}(\sqrt{v}\partial_{x_2})^{m_2} \\
&\times (v\Delta)^n e^{v\Delta}\chi_{a_2} f)(x)\|_{L^1(\mathbb{R}^2\times[0,2]:dxdv/v)} \\
&+ \|v^{(2-\alpha)/2}((v\Delta)^{n-m} e^{v\Delta}\chi_{a_2} f)(x)\|_{L^1(\mathbb{R}^2\times[0,2]:dxdv/v)}\Big).
\end{aligned}$$

Thus by the interpolation, we have

$$\begin{aligned}
&\Big\| t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2} e^{t\Delta}\chi_{a_1} \\
&\times \int_0^1 ds \int_0^1 \frac{dr}{r}(r\Delta)^n e^{(s+r)\Delta}\chi_{a_2} f\Big\|_{L^2(\mathbb{R}^2\times[0,1]:dxdt/t)} \\
\leq &c_3\|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}.
\end{aligned}$$

By a similar and simpler method, we have

$$\begin{aligned}
&\Big\| t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2} e^{t\Delta}\chi_{a_1} \int_0^1 ds \Delta^m e^{(s+1)\Delta}\chi_{a_2} f\Big\|_{L^2(\mathbb{R}^2\times[0,2]:dxdt/t)} \\
\leq &c_4\|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}.
\end{aligned}$$

Thus the quantity in (3.3) is dominated by $\|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}$. The quantity in (3.2) is also dominated by the same quantity. Thus we obtain (3.1) without the exponential term.

When $|a_1 - a_2|_\infty \geq 3$, we have

$$\begin{aligned}
&\|t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2} e^{t\Delta}\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{L^2(\mathbb{R}^2\times[0,2]:dxdt/t)} \\
\leq &c_5\|t^{(n_1+n_2-\alpha)/2}\|_{L^2(\times[0,2]:dt/t)} \sum_{m_1=0}^{n_1}\sum_{m_2=0}^{n_2}\int_0^1 ds\|\partial_{x_1}^{m_1}\partial_{x_2}^{m_2} e^{s\Delta}\chi_{a_2} f\|_{L^2(\Lambda_2(a_1):dx)}.
\end{aligned}$$

For any $k \in \mathbb{N}$, we have

$$\|\partial_{x_1}^{m_1}\partial_{x_2}^{m_2} e^{s\Delta}\chi_{a_2} f\|_{L^2(\Lambda_2(a_1):dx)}$$

$$\le c_{k,1}\Big(\int_{\Lambda_2(a_1)} dx \sum_{\ell=0}^{k}\int_{\Lambda_2(a_2)} dy \Big|\nabla_y^{\otimes \ell}\partial_{x_1}^{m_1}\partial_{x_2}^{m_2} e^{s\Delta}(x,y)\Big|^2 \|\chi_{a_2} f\|_{\mathcal{H}^{-k}(\mathbb{R}^2)}^2\Big)^{1/2}$$

$$\le c_{k,2}\exp(-c_6 d(\Lambda_2(a_1),\Lambda_2(a_2))^2/s)\|\chi_{a_2} f\|_{\mathcal{H}^{-k}(\mathbb{R}^2)}$$

and

$$\|t^{-\alpha/2}(\sqrt{t}\partial_{x_1})^{n_1}(\sqrt{t}\partial_{x_2})^{n_2} e^{t\Delta}\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{L^2(\mathbb{R}^2\times[0,2]:dxdt/t)}$$

$$\le c_{k,3}\exp(-c_6|a_1-a_2|^2)\|\chi_{a_2} f\|_{\mathcal{H}^{-k}(\mathbb{R}^2)}.$$

We also have

$$\|e^{\Delta}\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{L^2(\mathbb{R}^2:dx)}$$

$$\le c_{k,4}\exp(-c_6|a_1-a_2|^2)\|\chi_{a_2} f\|_{\mathcal{H}^{-k}(\mathbb{R}^2)}.$$

By combining these estimates, we can complete the proof. □

For the paraproduct and resonating terms, we have the following estimate as in Propositions 1.4 and 1.7 in Mouzard [18]:

Lemma 3.2. *There exists $C \in (0,\infty)$ satisfying the following:*

(i) *For any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$, there exist $C_\alpha, C_{\alpha,\epsilon} \in (0,\infty)$ such that*

$$\|\chi_{a_1} P_{\chi_{a_2} g}(\chi_{a_3} f)\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}$$

$$\le C_\alpha \|\chi_{a_3} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^\infty(\mathbb{R}^2)}\exp(-C(|a_1-a_2|^2+|a_1-a_3|^2))$$

for any $a_1, a_2, a_3 \in \mathbb{Z}^2$, $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$ and $g \in L^\infty(\mathbb{R}^2)$, and

$$\|\chi_{a_1} P_{\chi_{a_2} g}(\chi_{a_3} f)\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)}$$

$$\le C_{\alpha,\epsilon} \|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^2(\mathbb{R}^2)}\exp(-C(|a_1-a_2|^2+|a_1-a_3|^2))$$

for any $f \in C^\alpha(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R}^2)$.

(ii) *For any $\alpha \in (-\infty,0)$ and $\beta \in \mathbb{R}$, there exists $C_{\alpha,\beta} \in (0,\infty)$ such that*

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)}$$

$$\le C_{\alpha,\beta} \|\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{H}^{\beta}(\mathbb{R}^2)}\exp(-C(|a_1-a_2|^2+|a_1-a_3|^2))$$

for any $a_1, a_2, a_3 \in \mathbb{Z}^2$, $f \in C^\alpha(\mathbb{R}^2)$ *and* $g \in \mathcal{H}^\beta(\mathbb{R}^2)$, *and*

$$\begin{aligned}&\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)}\\ \leq & C_{\alpha,\beta} \|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))\end{aligned}$$

for any $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$ *and* $g \in C^\beta(\mathbb{R}^2)$.

(iii) *For any* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha + \beta > 0$, *there exists* $C_{\alpha,\beta} \in (0, \infty)$ *such that*

$$\begin{aligned}&\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)}\\ \leq & C_{\alpha,\beta} \|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))\end{aligned}$$

for any $a_1, a_2, a_3 \in \mathbb{Z}^2$, $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$ *and* $g \in \mathcal{C}^\beta(\mathbb{R}^2)$.

(iv) *For any* $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ *and* $\epsilon \in (0, 1)$, *there exists* $C_{\alpha,\beta,\epsilon} \in (0, \infty)$ *such that*

$$\begin{aligned}&\|\chi_{a_1 \chi_{a_2} h} P_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)}\\ \leq & C_{\alpha,\beta,\epsilon} \|\chi_{a_3} f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|\chi_{a_4} g\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)}\\ & \times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))\end{aligned}$$

for any $a_1, a_2, a_3, a_4 \in \mathbb{Z}^2$, $f \in \mathcal{C}^\alpha(\mathbb{R}^2)$, $g \in \mathcal{C}^\beta(\mathbb{R}^2)$ *and* $h \in L^2(\mathbb{R}^2)$.

To treat white noise, we prepare the following (cf. Theorem 2.1 and Proposition 2.2 in Mouzard [18]):

Lemma 3.3. (i) *For any* $\varepsilon \in (0, 1)$, *we take an approximation of the white noise by smooth random fields as* $\xi_\varepsilon := e^{\varepsilon^2 \Delta} \xi$, *which satisfies*

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|\chi_a(\xi_\varepsilon - \xi)\|^p_{C^{-1-\epsilon}(\mathbb{R}^2)}] = 0$$

for any $a \in \mathbb{Z}^2$ *and* $p \in [1, \infty)$. *Then, there exists a random field* Y_ξ *such that*

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_\xi)\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$$

for any $p \in [1, \infty)$ *and* $a \in \mathbb{Z}^2$, *where*

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc} \xi_\varepsilon, \xi_\varepsilon) - \mathbb{E}[\Pi(\Delta^{-loc} \xi_\varepsilon, \xi_\varepsilon)].$$

(ii) *For any* $\epsilon \in (0, 1)$ *and almost all* ξ, *there exist* $C_{\epsilon,\xi} \in (0, \infty)$ *such that*

$$\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi} (\log(2 + |a|))^2$$

and

$$\|\chi_a Y_\xi\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi} \log(2 + |a|)$$

for any $a \in \mathbb{Z}^2$.

Proof. (i) The proof is same with that of Theorem 2.1 in [18].

(ii) For any $\epsilon \in (0,1)$, there exists $h, k, p_\epsilon, M \in (0,\infty)$ such that

$$\mathbb{E}\Big[\exp\Big\{h\Big(\|\chi_a Y_\xi\|^{p_\epsilon}_{C^{-\epsilon}(\mathbb{R}^2)}\Big)^{1/p_\epsilon}\Big\}\Big] \le M$$

for any $a \in \mathbb{Z}^2$, as in Proposition 2.2 in Mouzard [18] (cf. Fernique [7]). Now, for the event

$$\Xi_a := \Big\{\xi : \exp\Big\{h\Big(\|\chi_a Y_\xi\|^{p_\epsilon}_{C^{-\epsilon}(\mathbb{R}^2)}\Big)^{1/p_\epsilon}\Big\} \ge (2+|a|)^3\Big\},$$

we have

$$\sum_{a\in\mathbb{Z}^2} \mathbb{P}(\Xi_a) \le \sum_{a\in\mathbb{Z}^2} \frac{M}{(2+|a|)^3} < \infty.$$

Thus by the Borel-Cantelli lemma, we can complete the proof for Y_ξ. By the same method, we can prove the inequality for ξ. □

Remark 3.1. The normalizing constant $c_\varepsilon := -\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]$ to define Y_ξ and $\widetilde{H^\xi}$ is writen as

$$\begin{aligned} c_\varepsilon &= -\mathbb{E}[(\Delta^{-loc}\xi_\varepsilon)\xi_\varepsilon] + \mathbb{E}[P_1^{(b)}((P_1^{(b)}\Delta^{-loc}\xi_\varepsilon)(P_1^{(b)}\xi_\varepsilon))] \\ &= \frac{1}{4\pi}\log\Big(1+\frac{1}{2\varepsilon^2}\Big) \\ &\quad - \sum_{j,k=0}^{b-1}\Big(\frac{(-\Delta)^{j+k}}{j!k!}\Delta^{-loc}e^{2(1+\varepsilon^2)\Delta}\Big)(0,0), \end{aligned} \tag{3.6}$$

since

$$\mathbb{E}[P_{\Delta^{-loc}\xi_\varepsilon}\xi_\varepsilon] = \mathbb{E}[P_{\xi_\varepsilon}(\Delta^{-loc}\xi_\varepsilon)] = 0.$$

The second term in the right hand side of (3.6) converges as $\varepsilon \to 0$.

As in Propositions 1.9 and 1.11 in Mouzard [18], we have the following:

Lemma 3.4. *There exists* $C \in (0,\infty)$ *satisfying the following:*

(i) *For any* $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}, \gamma \in (-\infty, 0)$ *such that* $\beta+\gamma < 0$ *and* $\alpha+\beta+\gamma > 0$*, there exist* $C_{\epsilon,\alpha,\beta,\gamma} \in (0,\infty)$ *such that*

$$\begin{aligned} &\|\chi_{a_1} C(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\ \le& C_{\epsilon,\alpha,\beta,\gamma}\|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)}\|\chi_{a_4} h\|_{\mathcal{C}^\gamma(\mathbb{R}^2)} \\ &\times \exp(-C(|a_1-a_2|^2 + |a_1-a_3|^2 + |a_1-a_4|^2)), \end{aligned}$$

for any $a_1, a_2, a_3, a_4 \in \mathbb{Z}^2$, $f \in \mathcal{H}^{\alpha}(\mathbb{R}^2)$, $g \in \mathcal{C}^{\beta-2}(\mathbb{R}^2)$ *and* $h \in \mathcal{C}^{\gamma}(\mathbb{R}^2)$, *where* $C^a(\cdot,\cdot,\cdot)$ *is the commutator defined in* (2.4).

(ii) *For any* $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}$, $\gamma \in (-\infty, 0)$ *such that* $\beta + \gamma < 0$ *and* $\alpha + \beta + \gamma > 0$, *there exist* $C_{\epsilon,\alpha,\beta,\gamma} \in (0,\infty)$ *such that*

$$\begin{aligned}
&\|\chi_{a_1} S(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\
\leq & C_{\epsilon,\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|\chi_{a_4} h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \\
& \times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2 + |a_1 - a_4|^2)),
\end{aligned}$$

for any $a_1, a_2, a_3, a_4 \in \mathbb{Z}^2$, $f \in \mathcal{H}^{\alpha}(\mathbb{R}^2)$, $g \in \mathcal{C}^{\beta-2}(\mathbb{R}^2)$ *and* $h \in \mathcal{C}^{\gamma}(\mathbb{R}^2)$, *where* $S^a(\cdot,\cdot,\cdot)$ *is the commutator defined in* (2.5).

The commutators used in [2] and [18] are

$$C_M(f, g, h) := \Pi(\Delta^{-loc} P_f \Delta g, h) - f\Pi(g, h).$$

and

$$S_M(f, g, h) := P_h(\Delta^{-loc} P_f \Delta g) - P_f(P_h g).$$

Then our commutators $C(f, g, h)$ and $S(f, g, h)$ are modifications of $C_M(f, \Delta^{-1} g, h)$ and $S_M(f, \Delta^{-1} g, h)$, respectively. The main difference is the operator Δ^{-1} acting on g. Thus in the right hand side of the inequalities in the above lemma, the norm $\|\cdot\|_{\mathcal{C}^{\beta-2}}$ appears instead of $\|\cdot\|_{\mathcal{C}^{\beta}}$. As for the commutator S, the second term in the right hand side is modificated so that the complicated structure of the paraproduct appears once. Our second factor ${}_f P_h g$ is estimated similarly for $P_h g$ as is shown in Lemma 3.2 (iv). The proof of Lemma 3.4 (ii) is essentially given in [2] since the estimate of $S_M(f, g, h)$ in [2] was obtained by estimating $P_h(\Delta^{-loc} P_f \Delta g) -_f P_h g$ and ${}_f P_h g - P_f(P_h g)$. Similarly

$$C_M(f, g, h) := \Pi(\widetilde{P_f} g, h) - h\Pi(f, g)$$

is also estimated by dividing to $\Pi(\Delta^{-loc} P_f \Delta g, h) - \Pi_h(f, g)$ and $\Pi_h(f, g) - h\Pi(f, g)$, where

$$\Pi_h(f, g) = \sum_{\mu} c_{\mu} \int_0^1 \frac{dt}{t} P_t^{\mu}((Q_t^{1,\mu} f)(Q_t^{2,\mu} g)h).$$

This is also defined by using the structure of the paraproduct only once. However we do not use $\Pi_h(f, g)$ and choose the commutator C referring [18], since the key point of this paper is modifying the operator by introducing Y_{ξ}. For this modification, the commutator C is used.

Proof of Lemma 3.4. (i) The inequality without the exponential term is obtained by modifying the proof of Propositions B.4 treating the setting of the Hölder norms in Mouzard [18] to our setting of the Sobolev and Hölder norms. When $|a_1 - a_2| \vee |a_1 - a_3| \vee |a_1 - a_4|$ is large, we have

$$\begin{aligned}
&\|\chi_{a_1}\Pi(\Delta^{-loc}P_{\chi_{a_2}f}\chi_{a_3}g, \chi_{a_4}h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\
\leq & c_1\|\chi_{a_2}f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3}g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)}\|\chi_{a_4}h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \\
&\times \exp(-c_2(|a_1-a_2|^2 + |a_1-a_3|^2 + |a_1-a_4|^2)).
\end{aligned}$$

When $|a_1 - a_2| \vee |a_1 - a_4|$ is large, we have

$$\begin{aligned}
&\|\chi_{a_1}\chi_{a_2}f\Pi(\Delta^{-loc}\chi_{a_3}g, \chi_{a_4}h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\
\leq & c_3\|\chi_{a_2}f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3}g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)}\|\chi_{a_4}h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \\
&\times \exp(-c_4(|a_1-a_2|^2 + |a_1-a_3|^2))\chi_{|a_1-a_4|_\infty \leq 2}.
\end{aligned}$$

By combining these estimates, we can complete the proof.

(ii) The inequality without the exponential term is obtained by modifying the proof of Proposition 38 treating the setting of the Hölder norms in Bailleul and Bernicot [2] to our setting of the Sobolev and Hölder norms. When $|a_1 - a_2| \vee |a_1 - a_5| \vee |a_2 - a_3| \vee |a_2 - a_4|$ is large, we have

$$\begin{aligned}
&\|\chi_{a_1}P_{\chi_{a_4}h}(\Delta^{-loc}P_{\chi_{a_2}f}\chi_{a_3}g)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\
\leq & c_1\|\chi_{a_2}f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3}g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)}\|\chi_{a_4}h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \\
&\times \exp(-c_2(|a_1-a_2|^2 + |a_1-a_3|^2 + |a_1-a_4|^2)).
\end{aligned}$$

When $|a_1 - a_2| \vee |a_1 - a_3| \vee |a_1 - a_4|$ is large, we have

$$\begin{aligned}
&\|\chi_{a_1\chi_{a_2}f}P_{\chi_{a_4}h}(\Delta^{-loc}\chi_{a_3}g)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\
\leq & c_3\|\chi_{a_2}f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3}g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)}\|\chi_{a_4}h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \\
&\times \exp(-c_4(|a_1-a_2|^2 + |a_1-a_3|^2 + |a_1-a_4|^2)).
\end{aligned}$$

By combining these estimates, we can complete the proof. □

We use also the following products:

$$P^s_f g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q^{1,\nu}_t((P^\nu_t f)(Q^{2,\nu}_t g))$$

and

$${}_hP^s_f g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q^{1,\nu}_t((P^\nu_t f)(Q^{2,\nu}_t g)h).$$

for appropriate distributions f, g and h on $\mathbb{R}^2$. As in Proposition 2.3 in [18], we have the following:

Lemma 3.5. *There exists $C \in (0,\infty)$ satisfying the following:*

(i) *For any $\beta < \gamma$, there exists $C_{\beta,\gamma} \in (0,\infty)$ such that*

$$\begin{aligned}
&\|\chi_{a_1} P^s_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^\beta(\mathbb{R}^2)}\\
\le& C_{\beta,\gamma} s^{(\gamma-\beta)/2} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)}\\
&\times \exp(-C(d(\Lambda_2(a_1),\Lambda_2(a_2))^2 + d(\Lambda_2(a_1),\Lambda_2(a_3))^2)/s))
\end{aligned}$$

for any $s \in [0,1]$, $a_1, a_2, a_3 \in \mathbb{Z}^2$, $f \in L^2(\mathbb{R}^2)$ and $g \in C^\gamma(\mathbb{R}^2)$.

(ii) *For any $\beta, \gamma_1, \gamma_2 \in \mathbb{R}$ satisfying $\gamma_1 \le 0$ and $\beta < \gamma_1 + \gamma_2$, there exists $C_{\beta,\gamma_1,\gamma_2} \in (0,\infty)$ such that*

$$\begin{aligned}
&\|\chi_{a_1} \chi_{a_2} h P^s_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^\beta(\mathbb{R}^2)}\\
\le& C_{\beta,\gamma_1,\gamma_2} s^{(\gamma_1+\gamma_2-\beta)/2} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)}\\
&\times \exp(-C(d(\Lambda_2(a_1),\Lambda_2(a_2))^2 + d(\Lambda_2(a_2),\Lambda_2(a_3))^2\\
&+ d(\Lambda_2(a_2),\Lambda_2(a_4))^2)/s)
\end{aligned}$$

for any $s \in [0,1]$, $a_1, a_2, a_3, a_4 \in \mathbb{Z}^2$, $f \in C^{\gamma_1}(\mathbb{R}^2)$, $g \in C^{\gamma_2}(\mathbb{R}^2)$ and $h \in L^2(\mathbb{R}^2)$.

Lemma 3.6. *There exists $C \in (0,\infty)$ satisfying the following:*

(i) *For any $\beta, \gamma \in \mathbb{R}$, there exists $C_{\beta,\gamma} \in (0,\infty)$ such that*

$$\begin{aligned}
&\|\chi_{a_1}(P_{\chi_{a_2} f}(\chi_{a_3} g) - P^s_{\chi_{a_2} f}(\chi_{a_3} g))\|_{\mathcal{H}^\beta(\mathbb{R}^2)}\\
\le& \begin{cases}
\dfrac{C_{\beta,\gamma}}{s^{(\beta-\gamma)/2}} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \\
\quad \times \exp(-C(|a_1-a_2|^2 + |a_1-a_3|^2)) \text{ if } \beta > \gamma,\\
C_{\beta,\gamma} \Big(\log \dfrac{1}{s}\Big) \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \\
\quad \times \exp(-C(|a_1-a_2|^2 + |a_1-a_3|^2)) \text{ if } \beta = \gamma,\\
C_{\beta,\gamma} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \\
\quad \times \exp(-C(|a_1-a_2|^2 + |a_1-a_3|^2)) \text{ if } \beta < \gamma,
\end{cases}
\end{aligned}$$

for any $s \in [0,1]$, $a_1, a_2, a_3 \in \mathbb{Z}^2$, $f \in L^2(\mathbb{R}^2)$ and $g \in C^\gamma(\mathbb{R}^2)$.

(ii) *For any $\gamma_1 \le 0, \gamma_2, \beta \in \mathbb{R}$, there exists $C_{\beta,\gamma_1,\gamma_2} \in (0,\infty)$ such that*

$$\|\chi_{a_1}({}_{\chi_{a_2}h}P_{\chi_{a_3}f}(\chi_{a_4}g) - {}_{\chi_{a_2}h}\, P^s_{\chi_{a_3}f}(\chi_{a_4}g))\|_{\mathcal{H}^\beta(\mathbb{R}^2)}$$

$$\le \begin{cases} \dfrac{C_{\beta,\gamma_1,\gamma_2}}{s^{(\beta-\gamma_1-\gamma_2)/2}}\|\chi_{a_3}f\|_{C^{\gamma_1}(\mathbb{R}^2)}\|\chi_{a_4}g\|_{C^{\gamma_2}(\mathbb{R}^2)}\|\chi_{a_2}h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times\exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2)) \\ \qquad \textit{if } \beta > \gamma_1+\gamma_2, \\ C_{\beta,\gamma_1,\gamma_2}\Big(\log\dfrac{1}{s}\Big)\|\chi_{a_3}f\|_{C^{\gamma_1}(\mathbb{R}^2)}\|\chi_{a_4}g\|_{C^{\gamma_2}(\mathbb{R}^2)}\|\chi_{a_2}h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times\exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2)) \\ \qquad \textit{if } \beta = \gamma_1+\gamma_2, \\ C_{\beta,\gamma_1,\gamma_2}\|\chi_{a_3}f\|_{C^{\gamma_1}(\mathbb{R}^2)}\|\chi_{a_4}g\|_{C^{\gamma_2}(\mathbb{R}^2)}\|\chi_{a_2}h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times\exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2)) \\ \qquad \textit{if } \beta < \gamma_1+\gamma_2, \end{cases}$$

for any $s \in [0,1]$, $a_1, a_2, a_3, a_4 \in \mathbb{Z}^2$, $f \in C^{\gamma_1}(\mathbb{R}^2)$, $g \in C^{\gamma_2}(\mathbb{R}^2)$ and $h \in L^2(\mathbb{R}^2)$.

We prepare also the following:

Lemma 3.7. *For any $\alpha \ge 0$, there exists $c_\alpha \in (0,\infty)$ such that*

$$\Big\|\sum_{a\in\mathbb{Z}^2} f_a\Big\|^2_{\mathcal{H}^\alpha(\mathbb{R}^2)} \le c_\alpha \sum_{a\in\mathbb{Z}^2}\|f_a\|^2_{\mathcal{H}^\alpha(\mathbb{R}^2)}$$

for any $f_a \in \mathcal{H}^\alpha(\mathbb{R}^2)$ such that $\operatorname{supp} f_a \subset \Lambda_2(a)$, $a \in \mathbb{Z}^2$.

Proof. For any $Q \in StGC^k$ with $k \in (|\alpha|, 2b] \cap \mathbb{Z}$, we should estimate

$$I_0 := \Big\|t^{-\alpha/2}Q_t\sum_{a\in\mathbb{Z}^2} f_a\Big\|^2_{L^2(\mathbb{R}^2\times[0,1],dxdt/t)} \le 2(I_1+I_2),$$

where

$$I_1 := \Big\|t^{-\alpha/2}\sum_{a\in\mathbb{Z}^2} 1_{\Lambda_3(a)}Q_t f_a\Big\|^2_{L^2(\mathbb{R}^2\times[0,1],dxdt/t)}$$

and

$$I_2 := \Big\|t^{-\alpha/2}\sum_{a\in\mathbb{Z}^2} 1_{\Lambda_3(a)^c}Q_t f_a\Big\|^2_{L^2(\mathbb{R}^2\times[0,1],dxdt/t)}.$$

Since $\overline{\Lambda_3(a)} \cap \overline{\Lambda_3(a')} \neq \emptyset$ implies $|a-a'|_\infty \leq 3$, the first term is estimated as

$$\begin{aligned} I_1 \leq & \sum_{a,a' \in \mathbb{Z}^2 : |a-a'|_\infty \leq 3} \left\| t^{-\alpha/2} 1_{\Lambda_3(a)} Q_t f_a \right\|_{L^2(\mathbb{R}^2 \times [0,1], dxdt/t)} \\ & \times \left\| t^{-\alpha/2} 1_{\Lambda_3(a')} Q_t f_{a'} \right\|_{L^2(\mathbb{R}^2 \times [0,1], dxdt/t)} \\ \leq & 49 \sum_{a \in \mathbb{Z}^2} \|f_a\|^2_{\mathcal{H}^\alpha(\mathbb{R}^2)}. \end{aligned}$$

By

$$|Q_t(x,y)| \leq \frac{c_1}{t} \exp\Big(-\frac{|x-y|^2}{c_2 t} \Big),$$

we have

$$\begin{aligned} I_2 \leq & c_3 \int_0^1 \frac{dt}{t^{3+\alpha}} \sum_{a,a' \in \mathbb{Z}^2} \int dy |f_a(y)| \int dy' |f_{a'}(y')| \\ & \times \int_{\Lambda_3(a)^c \cap \Lambda_3(a')^c} dx \exp\Big(-\frac{|x-y|^2}{c_2 t} - \frac{|x-y'|^2}{c_2 t} \Big). \end{aligned}$$

Since

$$|x-y|^2 + |x-y'|^2 \geq |y-y'|^2/2$$

and

$$|y-y'|_\infty \geq |a-a'|_\infty - 2$$

for any $x \in \Lambda_3(a)^c \cap \Lambda_3(a')^c$, $y \in \Lambda_2(a)$ and $y' \in \Lambda_2(a')$, we have

$$\begin{aligned} I_2 \leq & c_4 \int_0^1 \frac{dt}{t^{2+\alpha}} \exp\Big(-\frac{c_5}{t} \Big) \sum_{a,a' \in \mathbb{Z}^2} \int dy |f_a(y)| \\ & \times \int dy' |f_{a'}(y')| \exp\Big(-c_6(|a-a'|_\infty - 2)_+^2 \Big) \\ \leq & c_7 \sum_{a \in \mathbb{Z}^2} \|f_a\|^2_{L^2(\mathbb{R}^2, dx)} \end{aligned}$$

and

$$I_0 \leq c_8 \sum_{a \in \mathbb{Z}^2} \|f_a\|^2_{\mathcal{H}^\alpha(\mathbb{R}^2)}.$$

By a similar and simpler method, we obtain

$$\Big\| e^\Delta \sum_{a \in \mathbb{Z}^2} f_a \Big\|^2_{L^2(\mathbb{R}^2, dx)} \leq c_9 \sum_{a \in \mathbb{Z}^2} \|f_a\|^2_{\mathcal{H}^\alpha(\mathbb{R}^2)}$$

and we can complete the proof.

□

4. Proof of Theorem 1

For $\boldsymbol{s} = (s(a), s_1(a,a'), s_2(a))_{a,a'\in\mathbb{Z}^2} \in [0,1]^{\mathbb{Z}^2} \times [0,1]^{\mathbb{Z}^2\times\mathbb{Z}^2} \times [0,1]^{\mathbb{Z}^2}$ specified later, we set

$$\Phi_\xi^{\boldsymbol{s}}(u) := u - \underline{\Phi_\xi^{\boldsymbol{s}}}(u),$$

where

$$\underline{\Phi_\xi^{\boldsymbol{s}}}(u) := \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s(a)}(\chi_a^2\xi) + \sum_{a,a'\in\mathbb{Z}^2} \Delta^{-loc}{}_u P_{\chi_a^2\xi}^{s_1(a,a')}(\Delta^{-loc}\chi_{a'}^2\xi)$$
$$+ \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s_2(a)}(\chi_a^2 Y_\xi)$$

for appropriate distributions u on $\mathbb{R}^2$. By Lemma 3.2, Lemma 3.3 (ii) and Lemma 3.5, we have the following:

Lemma 4.1. *For any $\epsilon \in (0,1)$ and almost all ξ, there exist $s(\epsilon,\xi), s_1(\epsilon,\xi), s_2(\epsilon,\xi) \in (0,1)$ and $M, M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ such that*

$$\|\chi_a \underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}}(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le \delta \sum_{a'\in\mathbb{Z}^2} \exp(-M|a-a'|^2)\|\chi_{a'}^2 u\|_{L^2(\mathbb{R}^2)}, \tag{4.1}$$

for any $\delta \ge 0$, where $\boldsymbol{s}(\epsilon,\xi,\delta) = (s(a;\epsilon,\xi,\delta), s_1(a,a';\epsilon,\xi,\delta), s_2(a;\epsilon,\xi,\delta))_{a,a'\in\mathbb{Z}^2}$ is

$$s(a;\epsilon,\xi,\delta) = s(\epsilon,\xi)\Big(\frac{\delta}{(\log(2+|a|))^2}\Big)^{M(\epsilon)},$$
$$s_1(a,a';\epsilon,\xi,\delta) = s_1(\epsilon,\xi)\Big(\frac{\delta}{(\log(2+|a|))^2(\log(2+|a'|))^2}\Big)^{M_1(\epsilon)}$$

and

$$s_2(a;\epsilon,\xi,\delta) = s_2(\epsilon,\xi)\Big(\frac{\delta}{\log(2+|a|)}\Big)^{M_2(\epsilon)}.$$

By this lemma and Lemma 3.7, we obtain the finite constant $C_{\xi,\epsilon}$, which may depend ξ and ϵ, such that

$$\|\underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}}(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le C_{\xi,\epsilon}\delta\|u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}. \tag{4.2}$$

Thus for $\delta \in (0, 1/C_{\xi,\epsilon})$, there exists the inverse

$$(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1} = \sum_{n=0}^\infty (\underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}})^n$$

such that

$$\|(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1}(v)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le \|v\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}/(1-C_{\xi,\epsilon}\delta) \tag{4.3}$$

for any $v \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$.

We use also Lemma 3.6. Then we have the following:

Lemma 4.2. (i) *For any* $\varepsilon \in (0,1)$, *we set*

$$\mathrm{Dom}_\epsilon(\widetilde{H^\xi}) := \Big\{ u \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\}.$$

Then we have $\mathrm{Dom}_\epsilon(\widetilde{H^\xi}) = \mathrm{Dom}_0(\widetilde{H^\xi})$.

(ii) $\mathrm{Dom}_0(\widetilde{H^\xi})$ *is dense in* $L^2(\mathbb{R}^2)$.

Proof. (i) For any $u \in \mathrm{Dom}_\epsilon(\widetilde{H^\xi})$ and $\epsilon' \in (0,\epsilon)$, we will show that $u \in \mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)$ and that

$$\limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)} < 0.$$

By Lemma 4.1, we have

$$\begin{aligned}
&\|\chi_a u\|_{\mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)} \\
\le &\|\chi_a \Phi_\xi^{\boldsymbol{s}(\epsilon',\xi,\delta)}(u)\|_{\mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)} \\
&+ \sum_{n=1}^\infty \delta^n \sum_{a_1,a_2,\dots,a_n \in \mathbb{Z}^2} \exp\Big(-c_1 \sum_{j=1}^n |a_{j-1}-a_j|^2\Big) \|\chi_{a_n} \Phi_\xi^{\boldsymbol{s}(\epsilon',\xi,\delta)}(u)\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

where $a_0 = a$. By Lemma 3.6 and Lemma 3.3, we have

$$\begin{aligned}
&\|\chi_a(\Phi_\xi(u) - \Phi_\xi^{\boldsymbol{s}(\epsilon',\xi,\delta)}(u))\|_{\mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)} \\
\le &c_2(\log(2+|a|))^4 \sum_{a' \in Z^2} \|\chi_{a'} u\|_{L^2(\mathbb{R}^2)} \exp(-c_3|a-a'|^2).
\end{aligned}$$

For small enough $m > 0$, there exists $c_4 \in (0,\infty)$ such that

$$\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}, \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} \le c_4 e^{-m|a|}$$

for any $a \in \mathbb{Z}^2$ since $u \in \mathrm{Dom}_\epsilon(\widetilde{H^\xi})$. Thus we have

$$\|\chi_a \Phi_\xi^{\boldsymbol{s}(\epsilon',\xi,\delta)}(u)\|_{\mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)} \le c_5 e^{-c_6|a|},$$

and

$$
\begin{aligned}
&\|\chi_a u\|_{\mathcal{H}^{1-\epsilon'}(\mathbb{R}^2)}\\
\leq & c_7 \sum_{n=0}^{\infty}(c_8\delta)^n \sum_{a_n\in\mathbb{Z}^2} \exp\Big(-\frac{c_9}{n}|a-a_n|^2 - c_{10}|a_n|\Big)\\
\leq & c_{11}\sum_{n=0}^{\infty}(c_{12}\delta)^n \exp(-c_{13}|a|)\\
\leq & c_{14}\exp(-c_{13}|a|)
\end{aligned}
$$

by taking δ as sufficiently small numbers. Thus we can complete the proof.

(ii) We will show that $\mathrm{Dom}_\epsilon(\widetilde{H^\xi})$ is dense in $L^2(\mathbb{R}^2)$ for arbitrarily taken $\epsilon\in(0,1)$. For any $R\in(1,\infty)$, $u\in C_0^\infty(\Lambda_R)$ and $\varepsilon\in(0,1)$, we set $u_\varepsilon := (\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1}(\Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u))$. For this, we have

$$
\begin{aligned}
&\|\chi_a u_\varepsilon\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}\\
\leq & \Big\|\chi_a \Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u))\Big\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}\\
&+\sum_{n=1}^{\infty}(c_1\delta)^n\sum_{a_n\in\mathbb{Z}^2}\exp\Big(-\frac{c_2}{n}|a-a_n|^2\Big)\Big\|\chi_{a_n}\Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u))\Big\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

By the estimate (4.2) in the case of $\xi=\xi_\varepsilon$, we have

$$
\|\chi_a\Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} + \delta\sum_{a'\in\mathbb{Z}^2}\exp(-M|a-a'|^2)\|\chi_{a'}u\|_{L^2(\mathbb{R}^2)}.
$$

Thus we have $u_\varepsilon\in\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$ and

$$
\limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a u_\varepsilon\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0.
$$

In the decomposition

$$
\Phi_\xi(u_\varepsilon) = \Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u) + (\Phi_\xi - \Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})(u_\varepsilon),
$$

the each term is estimated by Lemma 3.1, Lemma 3.5 and Lemma 3.6 as follows:

$$
\begin{aligned}
&\|\chi_a \Delta^{-loc} P^{s(a';\epsilon,\xi,\delta)}_{\chi_{a'} u}(\chi_{a'}\xi_\varepsilon)\|_{\mathcal{H}^2(\mathbb{R}^2)} \\
&\le c_3 \|\chi_{a'} u\|_{L^2(\mathbb{R}^2)} \|\chi_{a'}\xi_\varepsilon\|_{\mathcal{C}^1(\mathbb{R}^2)} \exp(-c_4|a-a'|^2), \\
&\|\chi_a \Delta^{-loc}{}_{\chi_{a''}u} P^{s_1(a',a'';\epsilon,\xi,\delta)}_{\chi_{a'}\xi_\varepsilon}(\chi_{a'} X^{a''}_{\xi_\varepsilon})\|_{\mathcal{H}^2(\mathbb{R}^2)} \\
\le & c_5 \|\chi_{a''} u\|_{L^2((\mathbb{R}^2)} \|\chi_{a'}\xi_\varepsilon\|_{\mathcal{C}^{-1}(\mathbb{R}^2)} \|\chi_{a''}\xi_\varepsilon\|_{\mathcal{C}^0(\mathbb{R}^2)} \exp(-c_6(|a-a'|^2+|a-a''|^2)), \\
&\|\chi_a \Delta^{-loc} P^{s_2(a';\epsilon,\xi,\delta)}_{\chi_{a''} u}(\chi_{a'} Y_{\xi_\varepsilon})\|_{\mathcal{H}^2(\mathbb{R}^2)} \\
\le & c_7 \|\chi_{a''} u\|_{L^2(\mathbb{R}^2)} \|\chi_{a'} Y_{\xi_\varepsilon}\|_{\mathcal{C}^1(\mathbb{R}^2)} \exp(-c_8(|a-a'|^2+|a-a''|^2)) \\
&\|\chi_a \Delta^{-loc}(P_{\chi_{a'} u_\varepsilon}(\chi_{a'}\xi) - P^{s(a';\epsilon,\xi,\delta)}_{\chi_{a'} u_\varepsilon}(\chi_{a'}\xi))\|_{\mathcal{H}^2(\mathbb{R}^2)} \\
&\le \frac{c_9}{s(a';\epsilon,\xi,\delta)^{(1+\epsilon)/2}} \|\chi_{a'} u_\varepsilon\|_{L^2(\mathbb{R}^2)} \|\chi_{a'}\xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \exp(-c_{10}|a-a'|^2), \\
&\|\chi_a \Delta^{-loc}({}_{\chi_{a''}u} P_{\chi_{a'}\xi}(\Delta^{-loc}\chi_{a''}\xi) - {}_{\chi_{a''}u} P^{s_1(a',a'';\epsilon,\xi,\delta)}_{\chi_{a'}\xi}(\Delta^{-loc}\chi_{a''}\xi))\|_{\mathcal{H}^2(\mathbb{R}^2)} \\
\le & \frac{c_{11}}{s_1(a',a'';\epsilon,\xi,\delta)^{1+\epsilon}} \|\chi_{a''} u_\varepsilon\|_{L^2((\mathbb{R}^2)} \|\chi_{a'}\xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \|\Delta^{-loc}\chi_{a''}\xi\|_{\mathcal{C}^{3-\epsilon}(\mathbb{R}^2)} \\
&\times \exp(-c_{12}(|a-a'|^2+|a-a''|^2)),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\chi_a \Delta^{-loc}(P_{\chi_{a''} u}(\chi_{a'} Y_\xi) - P^{s_2(a';\epsilon,\xi,\delta)}_{\chi_{a''} u}(\chi_{a'} Y_\xi))\|_{\mathcal{H}^2(\mathbb{R}^2)} \\
\le & \frac{c_{13}}{s_2(a';\epsilon,\xi,\delta)^{1+\epsilon}} \|\chi_{a''} u_\varepsilon\|_{L^2(\mathbb{R}^2)} \|\chi_{a'} Y_\xi\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \exp(-c_{14}(|a-a'|^2+|a-a''|^2)).
\end{aligned}
$$

Thus we have $\Phi_\xi(u_\varepsilon) \in \mathcal{H}^2(\mathbb{R}^2)$,

$$
\limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u_\varepsilon)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0,
$$

and $u_\varepsilon \in \mathrm{Dom}_\epsilon(\widetilde{H^\xi})$.

Since

$$
u_\varepsilon - u = (\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_\xi)^{-1}(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi_\varepsilon}(u) - \Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_\xi(u)),
$$

we have

$$
\begin{aligned}
&\|u_\varepsilon - u\|_{L^2(\mathbb{R}^2)}^2 \le c_{15} \|\Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u) - \Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u)\|_{L^2(\mathbb{R}^2)}^2 \\
\le & c_{16} \sum_{a\in\mathbb{Z}^2} \|\chi_a(\Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u) - \Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u))\|_{L^2(\mathbb{R}^2)}^2 \\
\le & c_{17} \sum_{a\in\mathbb{Z}^2} \Big\{ \sum_{a'\in\mathbb{Z}^2} (s(a';\epsilon,\xi,\delta)^{(1-\epsilon)/2} \|\chi_{a'} u\|_{L^2(\mathbb{R}^2)} \\
&\quad \times \|\chi_{a'}(\xi_\varepsilon - \xi)\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \exp(-c_{18}|a-a'|^2) \\
&\quad + \sum_{a',a''\in\mathbb{Z}^2} s_1(a',a'';\epsilon,\xi,\delta)^{1-\epsilon} \|\chi_{a''} u\|_{L^2(\mathbb{R}^2)} \\
&\qquad\qquad \times (\|\chi_{a'}\xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \|\chi_{a''}(\xi_\varepsilon - \xi)\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \\
&\qquad\qquad\quad + \|\chi_{a'}(\xi - \xi_\varepsilon)\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \|\chi_{a''}\xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}) \\
&\quad \times \exp(-c_{19}(|a-a''|^2 + |a'-a''|^2)\Big) \\
&\quad + \sum_{a'\in\mathbb{Z}^2} s_2(a';\epsilon,\xi,\delta)^{1-\epsilon/2} \|\chi_{a'} u\|_{L^2(\mathbb{R}^2)} \|\chi_{a'}(Y_{\xi_\varepsilon} - Y_\xi)\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \\
&\quad\quad \times \exp(-c_{20}|a-a'|^2) \Big\}^2.
\end{aligned}
$$

We take a sequence $\{\varepsilon(m)\}_{m\in\mathbb{N}}$ such that $\varepsilon(m) \to 0$ and $\|\chi_a(\xi_{\varepsilon(m)} - \xi)\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \to 0$ and $\|\chi_a(Y_{\xi_{\varepsilon(m)}} - Y_\xi)\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \to 0$ as $m \to \infty$ for any $a \in \mathbb{Z}^2$ and for almost all ξ. Then we have $\|u_{\varepsilon(m)} - u\|_{L^2(\mathbb{R}^2)} \to 0$ as $m \to \infty$ for almost all ξ. Since $C_0^\infty(\Lambda_R)$ is dense in $L^2(\Lambda_R)$, we can complete the proof. □

For any $R \in \mathbb{N}$, we set

$$
\mathrm{Dom}(\widetilde{H_R^\xi}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,R}(u) \in \mathcal{H}^2(\mathbb{R}^2) \Big\}
$$

and, for $\phi \in \mathrm{Dom}(\widetilde{H_R^\xi})$, we set

$$\begin{aligned}
&\widetilde{H_R^\xi} u \\
=& -\Delta\Phi_{\xi,R}(u) + P_{\xi_R}(\Phi_{\xi,R}(u)) + \Pi(\Phi_{\xi,R}(u), \xi_R) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi_R)) \\
&+ e^{\Delta} P_u \xi_R + e^{\Delta}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R) + e^{\Delta} P_u Y_{\xi,R} \\
&+ C(u, \xi_R, \xi_R) + S(u, \xi_R, \xi_R) \\
&+ P_{Y_{\xi,R}} u + \Pi(u, Y_{\xi,R}) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_{\xi,R})) \\
&+ P_{\xi_R}(\Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R)) + \Pi(\Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R), \xi_R) \\
&+ P_{\xi_R}(\Delta^{-loc} P_u Y_{\xi,R}) + \Pi(\Delta^{-loc} P_u Y_{\xi,R}, \xi_R),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{\xi,R}(u) &:= u - \Delta^{-loc} P_u \xi_R - \Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R) - \Delta^{-loc} P_u Y_{\xi,R}, \\
\xi_R &:= \sum_{a \in \mathbb{Z}^2 \cap \Lambda_R} \chi_a^2 \xi, \\
\xi_{\varepsilon,R} &:= \sum_{a \in \mathbb{Z}^2 \cap \Lambda_R} \chi_a^2 e^{\varepsilon^2 \Delta} \xi, \\
Y_{\xi_\varepsilon,R} &:= \Pi(\Delta^{-loc}\xi_{\varepsilon,R}, \xi_{\varepsilon,R}) - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\varepsilon,R}, \xi_{\varepsilon,R})],
\end{aligned}$$

and $Y_{\xi,R}$ is a random field such that

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon,R} - Y_{\xi,R})\|_{C^{-\epsilon}(\mathbb{R}^2)}^p] = 0$$

for any $p \in [1, \infty)$, $\epsilon > 0$ and $a \in \mathbb{Z}^2$. For these, Lemma 3.3 (ii) and Lemma 4.1 are modified as follows:

Lemma 4.3. (i) *For any $\epsilon \in (0,1)$ and almost all ξ, there exist $C_{\epsilon,\xi}, C'_{\epsilon,\xi}, C''_{\epsilon,\xi}, C_\epsilon, C'_\epsilon, C''_\epsilon \in (0,\infty)$ such that*

$$\begin{aligned}
\|\chi_a Y_{\xi,R}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le& C_{\epsilon,\xi} \log(2+|a|) \exp(-C_\epsilon d(a, \Lambda_R)^2) \\
\le& C'_{\epsilon,\xi} \log(2+R) \exp(-C'_\epsilon d(a, \Lambda_R)^2)
\end{aligned}$$

and

$$\|\chi_a(Y_\xi - Y_{\xi,R})\|_{C^{-\epsilon}(\mathbb{R}^2)} \le C''_{\epsilon,\xi} \log(2+|a|) \exp(-C''_\epsilon d(a, \Lambda_R^c)^2)$$

for any $a \in \mathbb{Z}^2$ and $R \in \mathbb{N}$.

(ii) *For any $R \in \mathbb{N}, \epsilon, \delta \in (0,1]$, $u \in L^2(\mathbb{R}^2)$ and almost all ξ, we set*

$$s(R;\epsilon,\xi,\delta) = s(\epsilon,\xi)\Big(\frac{\delta}{(\log(2+R))^2}\Big)^{M(\epsilon)},$$
$$s_1(R;\epsilon,\xi,\delta) = s_1(\epsilon,\xi)\Big(\frac{\delta}{(\log(2+R))^4}\Big)^{M_1(\epsilon)},$$
$$s_2(R;\epsilon,\xi,\delta) = s_2(\epsilon,\xi)\Big(\frac{\delta}{\log(2+R)}\Big)^{M_2(\epsilon)},$$

and

$$\underline{\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R}}(u) := \Delta^{-loc} P_u^{s(R,\epsilon,\xi,\delta)} \xi_R + \Delta^{-loc}{}_u P_{\xi_R}^{s_1(R,\epsilon,\xi,\delta)}(\Delta^{-loc}\xi_R) + \Delta^{-loc} P_u^{s_2(R,\epsilon,\xi,\delta)} Y_{\xi,R},$$

where $s(\epsilon,\xi), s_1(\epsilon,\xi), s_2(\epsilon,\xi) \in (0,1)$ and $M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ are given in Lemma 4.1. Then we have

$$\|\chi_a \underline{\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R}}(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le \delta \sum_{a'\in\mathbb{Z}^2} \exp(-M(|a-a'|^2 + d(a',\Lambda_R)^2))\|\chi_{a'} u\|_{L^2(\mathbb{R}^2)}. \tag{4.4}$$

Thus, as in (4.2), we have

$$\|(\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R})^{-1}(v)\|_{\mathcal{H}^{1-\epsilon}} \le \|v\|_{\mathcal{H}^{1-\epsilon}}/(1 - C_{\xi,\epsilon}\delta) \tag{4.5}$$

for any $\delta \in (0, 1/C_{\xi,\epsilon})$ and $v \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$, where $\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R}(u) = u - \underline{\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R}}(u)$ for any $u \in L^2(\mathbb{R}^2)$. Then, as in Lemma 4.2, we have the following:

Lemma 4.4. (i) *For any $\varepsilon \in (0,1)$, we set*

$$\mathrm{Dom}_\epsilon(\widetilde{H^\xi_R}) := \Big\{u \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,R}(u) \in \mathcal{H}^2(\mathbb{R}^2)\Big\}.$$

Then we have $\mathrm{Dom}_\epsilon(\widetilde{H^\xi_R}) = \mathrm{Dom}(\widetilde{H^\xi_R})$.

(ii) $\mathrm{Dom}(\widetilde{H^\xi_R})$ *is dense in $L^2(\mathbb{R}^2)$.*

Moreover, as in Proposition 2.8 in [18], we can show the following:

Lemma 4.5. (i) *For any $\epsilon \in (0,1), R \in \mathbb{N}$, almost all ξ and any $u \in \mathrm{Dom}(\widetilde{H^\xi_R})$, we have*

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} = 0$$

and

$$\lim_{\varepsilon\to 0} \|\widetilde{H^\xi_R} u - \widetilde{H^{\xi_\varepsilon}_R} u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 0,$$

where $u_\varepsilon := (\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi_\varepsilon,R})^{-1}(\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R}(u)) \in \mathrm{Dom}(\widetilde{H^{\xi_\varepsilon}_R})$ with arbitrarily fixed $\delta \in (0,1)$.

(ii) *For almost all* ξ, *any* $u \in \mathrm{Dom}_0(\widetilde{H^\xi})$ *and* $\epsilon \in (0,1)$, *we have*

$$\limsup_{R\to\infty} \frac{1}{R} \log \|u_R - u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0$$

and

$$\limsup_{R\to\infty} \frac{1}{R} \log \|\widetilde{H^\xi_R} u_R - \widetilde{H^\xi} u\|_{L^2(\mathbb{R}^2)} < 0,$$

where, for each $R \in \mathbb{N}$, $u_R := (\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi,R})^{-1}(\Phi^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}_{\xi}(u)) \in \mathrm{Dom}(\widetilde{H^\xi_R})$ *with arbitrarily fixed* $\delta \in (0,1)$.

By this lemma, we have the following:

Lemma 4.6. (i) $(\widetilde{H^\xi_R} u, v)_{L^2(\mathbb{R}^2)} = (u, \widetilde{H^\xi_R} v)_{L^2(\mathbb{R}^2)}$ *for any* $u, v \in \mathrm{Dom}(\widetilde{H^\xi_R})$.

(ii) $(\widetilde{H^\xi} u, v)_{L^2(\mathbb{R}^2)} = (u, \widetilde{H^\xi} v)_{L^2(\mathbb{R}^2)}$ *for any* $u, v \in \mathrm{Dom}_0(\widetilde{H^\xi})$.

On the other hand, as in Proposition 2.6 in [18], we have the following:

Lemma 4.7. *For any* $R \in \mathbb{N}$, $\delta > 0$ *and almost all* ξ, *there exists* $c(\xi, \delta, R) \in (0,\infty)$ *such that*

$$\|\Phi_{\xi,R}(u)\|_{L^2(\mathbb{R}^2)} \le c(\xi,\delta,R)\|u\|_{L^2(\mathbb{R}^2)},$$

$$(1-\delta)\|\Delta\Phi_{\xi,R}(u)\|_{L^2(\mathbb{R}^2)} \le \|\widetilde{H^\xi_R} u\|_{L^2(\mathbb{R}^2)} + c(\xi,\delta,R)\|u\|_{L^2(\mathbb{R}^2)},$$

and

$$\|\widetilde{H^\xi_R} u\|_{L^2(\mathbb{R}^2)} \le (1+\delta)\|\Delta\Phi_{\xi,R}(u)\|_{L^2(\mathbb{R}^2)} + c(\xi,\delta,R)\|u\|_{L^2(\mathbb{R}^2)}$$

for any $u \in \mathrm{Dom}(\widetilde{H^\xi_R})$.

Then, as in Proposition 2.7 in [18], we have the following:

Lemma 4.8. *The operator* $\widetilde{H^\xi_R}$ *with the domain* $\mathrm{Dom}(\widetilde{H^\xi_R})$ *is a closed operator on* $L^2(\mathbb{R}^2)$.

Moreover, as in Proposition 2.9 in [18], we have the following:

Lemma 4.9. *For any* $R \in \mathbb{N}$, $s \in (0,1]$, *and almost all* ξ, *there exists* $k(\xi, s, R) \in (0,\infty)$ *such that*

$$s\|\nabla\Phi_{\xi,R}(u)\|^2_{L^2(\mathbb{R}^2)} \le (u, (\widetilde{H^\xi_R} + k(\xi,s,R))u)_{L^2(\mathbb{R}^2)} \tag{4.6}$$

for any $u \in \mathrm{Dom}(\widetilde{H^\xi_R})$.

Now we can show the following:

Lemma 4.10. *The operator* $\widetilde{H^\xi_R}$ *with the domain* $\mathrm{Dom}(\widetilde{H^\xi_R})$ *is self-adjoint on* $L^2(\mathbb{R}^2)$.

Proof. By Lemma 4.9, $(\varphi,\varphi')_{(1/2)} := (\varphi, (\widetilde{H_R^\xi} + k(\xi,s,R)+1)\varphi')_{L^2(\mathbb{R}^2)}$ for any $\varphi,\varphi' \in \mathrm{Dom}(\widetilde{H_R^\xi})$ is an inner product of $\mathrm{Dom}(\widetilde{H_R^\xi})$. We take $\{\varphi_n\}_n \subset \mathrm{Dom}(\widetilde{H_R^\xi})$ so that this is a complete orthonormal basis of the completion $\overline{\mathrm{Dom}(\widetilde{H_R^\xi})}^{\|\cdot\|_{(1/2)}}$ of $\mathrm{Dom}(\widetilde{H_R^\xi})$ with respect to this inner product. For any $0 \neq \psi \in \mathrm{Dom}(\widetilde{H_R^\xi})$, since

$$\psi = \sum_n (\psi,\varphi_n)_{(1/2)}\varphi_n$$

converges in $\overline{\mathrm{Dom}(\widetilde{H_R^\xi})}^{\|\cdot\|_{(1/2)}}$, this converges also in $L^2(\mathbb{R}^2)$ and it holds that

$$\begin{aligned}
0 \neq \|\psi\|_{L^2(\mathbb{R}^2)}^2 &= \lim_{N\to\infty} \Big(\psi, \sum_{n=1}^N (\psi,\varphi_n)_{(1/2)}\varphi_n\Big)_{L^2(\mathbb{R}^2)} \\
&= \lim_{N\to\infty} \Big(\psi, \sum_{n=1}^N (\psi,\varphi_n)_{L^2}(\widetilde{H_R^\xi} + k(\xi,s,R)+1)\varphi_n\Big)_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Thus we have $\psi \notin (\mathrm{Ran}(\widetilde{H_R^\xi} + k(\xi,s,R)+1))^\perp$. By considering the contraposition, we have $(\mathrm{Ran}(\widetilde{H_R^\xi} + k(\xi,s,R)+1))^\perp \cap \overline{\mathrm{Dom}(\widetilde{H_R^\xi})}^{\|\cdot\|_{(1/2)}} = \{0\}$ and $\overline{\mathrm{Dom}(\widetilde{H_R^\xi})}^{\|\cdot\|_{(1/2)}} \subset \mathrm{Ran}(\widetilde{H_R^\xi} + k(\xi,s,R)+1)$. Since $\widetilde{H_R^\xi}$ is densely defined and closed by Lemma 4.2 and Lemma 4.8, we have $\mathrm{Ran}(\widetilde{H_R^\xi}+k(\xi,s,R)+1) = L^2(\mathbb{R}^2)$. □

We prepare the following Combes-Thomas type estimate (cf. [5]):

Lemma 4.11. *For almost all ξ, there exist $C_\xi, C'_\xi \in (0,\infty)$ and $m \in \mathbb{N}$ such that*

$$\|\chi_{\Lambda_1(a)}(\widetilde{H_R^\xi}+i)^{-1}\chi_{\Lambda_1(b)}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq C_\xi \exp\Big(-\frac{C'_\xi|a-b|}{(\log(2+R))^m}\Big), \tag{4.7}$$

for any $a,b \in \mathbb{R}^2$ and $R \in \mathbb{N}$, where $\|\cdot\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}$ is the operator norm on $L^2(\mathbb{R}^2)$, and $\chi_{\Lambda_1(a)}$ is the operators of multiplying the characteristic function of the square $\Lambda_1(a)$.

Proof. For any $v \in \mathbb{R}^2$, since

$$\begin{aligned}
&e^{-v\cdot x}(\widetilde{H_R^\xi}+i)^{-1}e^{v\cdot x} \\
=&(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}(1-(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}2v\cdot\nabla(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2})^{-1}(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2},
\end{aligned}$$

we have

$$\|e^{-v\cdot x}(\widetilde{H_R^\xi}+i)^{-1}e^{v\cdot x}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq \sum_{n=0}^\infty \|(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}2v\cdot\nabla(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}\|^n_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}.$$

By Lemma 4.12 below, we have

$$\|e^{-v\cdot x}(\widetilde{H_R^\xi}+i)^{-1}e^{v\cdot x}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq 2$$

if $|v| \leq 1$ and

$$|v| \leq 1/(8C_\xi(\log(2+R))^m).$$

With this v, we have

$$\|\chi_{\Lambda_1(a)}(\widetilde{H_R^\xi}+i)^{-1}\chi_{\Lambda_1(b)}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq 2 \sup_{a'\in\Lambda_1(a), b\in\Lambda_1(b)} \exp(v\cdot(a'-b')).$$

By taking v appropriately, we obtain (4.7). □

Lemma 4.12. *For almost all ξ, there exist $C_\xi \in (0,\infty)$ and $m \in \mathbb{N}$ such that*

$$\|(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}v\cdot\nabla(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq C_\xi(\log(2+R))^m|v|(1+|v|)^2$$

for any $v \in \mathbb{R}^2$ and $R \in \mathbb{N}$.

Proof. We write as

$$\begin{aligned}
&\|(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}v\cdot\nabla(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}\\
&= \sup_{\|\varphi\|_{L^2(\mathbb{R}^2)}=\|\psi\|_{L^2(\mathbb{R}^2)}=1} |(\widetilde{\varphi}, v\cdot\nabla\widetilde{\psi})_{L^2(\mathbb{R}^2)}|,
\end{aligned}$$

where $\widetilde{\varphi} := (\widetilde{H_R^\xi}-|v|^2-i)^{-1/2}\varphi$ and $\widetilde{\psi} := (\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}\psi$. By

$$\sum_{\ell=0}^{n-1}\frac{(-\Delta)^\ell}{\Gamma(\ell+1)}e^{\Delta} + \int_0^1 \frac{dt}{t}(-t\Delta)^n e^{t\Delta}/\Gamma(n) = 1,$$

we have

$$|(\widetilde{\varphi}, v\cdot\nabla\widetilde{\psi})_{L^2(\mathbb{R}^2)}| \leq c_1|v|\|\widetilde{\varphi}\|_{L^2(\mathbb{R}^2)}\|\widetilde{\psi}\|_{L^2(\mathbb{R}^2)} + \Big|\Big(\int_0^1 \frac{dt}{t}\frac{(-t\Delta)^n}{\Gamma(n)}e^{t\Delta}\widetilde{\varphi}, v\cdot\nabla\int_0^1\frac{ds}{s}\frac{(-s\Delta)^n}{\Gamma(n)}e^{s\Delta}\widetilde{\psi}\Big)_{L^2(\mathbb{R}^2)}\Big|.$$

For the second term, we rewrite as

$$\begin{aligned}
&\Big(\int_0^1\frac{dt}{t}(-t\Delta)^n e^{t\Delta}\widetilde{\varphi}, v\cdot\nabla\int_0^1\frac{ds}{s}(-s\Delta)^n e^{s\Delta}\widetilde{\psi}\Big)_{L^2(\mathbb{R}^2)}\\
&= \int_0^1\frac{dt}{t^{3/2}}\int_0^t\frac{ds}{s}\Big(\sqrt{t}v\cdot\nabla(-t\Delta)^n e^{t\Delta}\widetilde{\varphi}, (-s\Delta)^n e^{s\Delta}\widetilde{\psi}\Big)_{L^2(\mathbb{R}^2)}\\
&\quad + \int_0^1\frac{ds}{s^{3/2}}\int_0^s\frac{dt}{t}\Big((-t\Delta)^n e^{t\Delta}\widetilde{\varphi}, \sqrt{s}v\cdot\nabla(-s\Delta)^n e^{s\Delta}\widetilde{\psi}\Big)_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Then we obtain

$$|(\widetilde{\varphi}, v\cdot\nabla\widetilde{\psi})_{L^2(\mathbb{R}^2)}| \leq c_2|v|\|\widetilde{\varphi}\|_{\mathcal{H}^{(1+\epsilon)/2}(\mathbb{R}^2)}\|\widetilde{\psi}\|_{\mathcal{H}^{(1+\epsilon)/2}(\mathbb{R}^2)}$$

and

$$\|(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}v\cdot\nabla(\widetilde{H_R^\xi}-|v|^2+i)^{-1/2}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq c_3|v|\sup_{\|\varphi\|_{L^2(\mathbb{R}^2)}=1}\|\widetilde{\varphi}\|^2_{\mathcal{H}^{(1+\epsilon)/2}(\mathbb{R}^2)}.$$

By (4.5), we have

$$\|\widetilde{\varphi}\|_{\mathcal{H}^{(1+\epsilon)/2}(\mathbb{R}^2)} \leq c_4 \|\Phi_{\xi,R}^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}(\widetilde{\varphi})\|_{\mathcal{H}^{(1+\epsilon)/2}(\mathbb{R}^2)}$$

$$\leq c_5(\|\Phi_{\xi,R}^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}(\widetilde{\varphi})\|_{L^2(\mathbb{R}^2)} + \|\nabla\Phi_{\xi,R}^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}(\widetilde{\varphi})\|_{L^2(\mathbb{R}^2)}).$$

By (4.4), we have

$$\|\Phi_{\xi,R}^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}(\widetilde{\varphi})\|_{L^2(\mathbb{R}^2)} \leq c_6\|\widetilde{\varphi}\|_{L^2(\mathbb{R}^2)}.$$

By Lemma 3.3, Lemma 3.6 and Lemma 4.3, there exists $m_0 \in \mathbb{N}$ and $c_7 \in (0,\infty)$ such that

$$\|\nabla(\Phi_{\xi,R}(\widetilde{\varphi}) - \Phi_{\xi,R}^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}(\widetilde{\varphi}))\|_{L^2(\mathbb{R}^2)} \leq \|\Phi_{\xi,R}(\widetilde{\varphi}) - \Phi_{\xi,R}^{\boldsymbol{s}(R,\epsilon,\xi,\delta)}(\widetilde{\varphi})\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq c_7(\log(2+R))^{m_0}\|\widetilde{\varphi}\|_{L^2(\mathbb{R}^2)}.$$

By Lemma 4.9, we have

$$\|\nabla\Phi_{\xi,R}(\widetilde{\varphi})\|_{L^2(\mathbb{R}^2)} \leq c_8(1+|v|)\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Then we can complete the proof. □

Then we can prove the theorem:

Proof of Theorem 1. For any $f \in \mathrm{Ran}(\widetilde{H^\xi}+i)^\perp$, we consider

$$\|f\|_{L^2(\mathbb{R}^2)}^2 = \lim_{R\to\infty}(f, \widetilde{\chi_R} f)_{L^2(\mathbb{R}^2)}, \tag{4.8}$$

where $\widetilde{\chi_R}$ is a $[0,1]$-valued smooth function on $\mathbb{R}^2$ such that $\widetilde{\chi_R}=0$ on $\mathbb{R}^2\setminus\Lambda_R$ and $\widetilde{\chi_R}=1$ on Λ_{R-1}. For any $L\in\mathbb{N}$, we set $\varphi_{R,L} := (\widetilde{H_{R+L}^\xi}+i)^{-1}\widetilde{\chi_R}f \in \mathrm{Dom}(\widetilde{H_{R+L}^\xi})$ and $\widetilde{\varphi_{R,L}} := (\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1}(\Phi_{\xi,R+L}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(\varphi_{R,L}))$ with arbitrarily fixed $\epsilon\in(0,1)$ and $\delta\in(0,1/C_\epsilon)$, where C_ϵ is the constant given in (4.2). We will show that $\widetilde{\varphi_{R,L}} \in \mathrm{Dom}_0(\widetilde{H^\xi})$. By Lemma 4.11, we have

$$\|\chi_a\varphi_{R,L}\|_{L^2(\mathbb{R}^2)} \leq c_1(\log(2+R+L))^{c_2}\exp\Big(\frac{-c_3 d(a,\Lambda_R)}{(\log(2+R+L))^{c_4}}\Big)\|\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2)}. \tag{4.9}$$

From this inequality, the methods as in the proof of Lemma 4.2 are enough to obtain only the exponential decay in a of $\|\chi_a\widetilde{\varphi_{R,L}}\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$ and $\|\chi_a\Phi_\xi(\widetilde{\varphi_{R,L}})\|_{\mathcal{H}^2(\mathbb{R}^2)}$ for each fixed L. However our proof of the self-adjointness uses the decay in L of $\{\|\chi_a\widetilde{\varphi_{R,L}}\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} : a\in\mathbb{Z}^2\setminus\Lambda_{R+L}\}$ and $\{\|\chi_a\Phi_\xi(\widetilde{\varphi_{R,L}})\|_{\mathcal{H}^2(\mathbb{R}^2)} : a \in$

$\mathbb{Z}^2 \setminus \Lambda_{R+L}\}$. For this purpose, we here give sharper estimates. We start with

$$\begin{aligned}
&\|\chi_a \Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \\
\le & c_5\Big(\frac{1}{t^{(1+\epsilon)/2}} \sum_{a_1\in\mathbb{Z}^2\cap\overline{\Lambda_4(a)}} \|\chi_{a_1}\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{L^2(\mathbb{R}^2)} \\
&+ t^{(1-\epsilon)/2} \sum_{a_1\in\mathbb{Z}^2\cap\overline{\Lambda_2(a)}} \|\chi_{a_1}\Delta\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{L^2(\mathbb{R}^2)}\Big)
\end{aligned}$$

for any $t \in (0,\infty)$. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&\|\chi_{a_1}(\Delta\Phi_{\xi,R+L}(\varphi_{R,L}) + \widetilde{H^{\xi}_{R+L}}\varphi_{R,L})\|_{L^2(\mathbb{R}^2)} \\
\le & c_6 \sum_{a_2\in\mathbb{Z}^2} \{(\log(2+|a_2|))^2 \|\chi_{a_2}\Phi_{\xi,R+L}(\varphi_{R,L})\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \\
&+ (\log(2+|a_2|))^4 \|\chi_{a_2}\varphi_{R,L}\|_{\mathcal{H}^{\epsilon}(\mathbb{R}^2)} \\
&+ (\log(2+|a_2|))^6 \|\chi_{a_2}\varphi_{R,L}\|_{L^2(\mathbb{R}^2)}\} \exp(-c_7(|a_1-a_2|^2 + d(a_2,\Lambda_{R+L})^2))\Big)
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\|\chi_{a_2}\underline{\Phi_{\xi,R+L}}(\varphi_{R,L})\|_{\mathcal{H}^{\epsilon}(\mathbb{R}^2)} \\
\le & c_8 \sum_{a_3\in\mathbb{Z}^2} (\log(2+|a_3|))^4 \|\chi_{a_3}\varphi_{R,L}\|_{L^2(\mathbb{R}^2)} \exp(-c_9(|a_2-a_3|^2 + d(a_3,\Lambda_{R+L})^2)).
\end{aligned}$$

By

$$\varphi_{R,L} = \Phi_{\xi,R+L}(\varphi_{R,L}) - \underline{\Phi_{\xi,R+L}}(\varphi_{R,L})$$

and

$$\widetilde{H^{\xi}_{R+L}}\varphi_{R,L} = \widetilde{\chi_R} f - i\varphi_{R,L},$$

we have

$$\begin{aligned}
&\|\chi_{a_2}\varphi_{R,L}\|_{\mathcal{H}^{\epsilon}(\mathbb{R}^2)} \\
\le & \|\chi_{a_2}\Phi_{\xi,R+L}(\varphi_{R,L})\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \\
&+ c_{10} \sum_{a_3\in\mathbb{Z}^2} (\log(2+|a_3|))^4 \|\chi_{a_3}\varphi_{R,L}\|_{L^2(\mathbb{R}^2)} \exp(-c_{11}(|a_2-a_3|^2 + d(a_3,\Lambda_{R+L})^2)).
\end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
&\|\chi_{a_1}\Delta\Phi_{\xi,R+L}(\varphi_{R,L})\|_{L^2(\mathbb{R}^2)}\\
\le&\|\chi_{a_1}\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2)}\\
&+c_{12}\sum_{a_2\in\mathbb{Z}^2}((\log(2+|a_2|))^4\|\chi_{a_2}\Phi_{\xi,R+L}(\varphi_{R,L})\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\\
&+(\log(2+|a_2|))^8\|\chi_{a_2}\varphi_{R,L}\|_{L^2(\mathbb{R}^2)})\exp(-c_{13}(|a_1-a_2|^2+d(a_2,\Lambda_{R+L})^2)).
\end{aligned}
\tag{4.12}
$$

Thus, from (4.9), we obtain

$$
\begin{aligned}
&\|\chi_a\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\\
\le&c_{14}\frac{(\log(2+R+L))^{c_{15}}}{t^{(1+\epsilon)/2}}\exp\Big(\frac{-c_{16}d(a,\Lambda_R)}{(\log(2+R+L))^{c_{17}}}\Big)\\
&+c_{18}t^{(1-\epsilon)/2}(\log(2+R+L))^{c_{19}}\\
&\times\sum_{a_1\in\mathbb{Z}^2}\|\chi_{a_1}\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\exp(-c_{20}(|a-a_1|^2+d(a_1,\Lambda_{R+L})^2)).
\end{aligned}
\tag{4.13}
$$

By iterating the estimates, we have

$$
\begin{aligned}
&\|\chi_a\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\\
\le&c_{14}\frac{(\log(2+R+L))^{c_{15}}}{t^{(1+\epsilon)/2}}\exp\Big(-\frac{c_{16}d(a_j,\Lambda_R)}{(\log(2+R+L))^{c_{17}}}\Big)\\
&+\sum_{j=1}^{n-1}c_{14}\frac{(\log(2+R+L))^{c_{15}}}{t^{(1+\epsilon)/2}}(c_{18}t^{(1-\epsilon)/2}(\log(2+R+L))^{c_{19}})^j\\
&\times\sum_{a_1,\dots,a_j\in\mathbb{Z}^2}\exp\Big(-c_{20}\sum_{k=1}^{j}(|a_{k-1}-a_k|^2+d(a_k,\Lambda_{R+L})^2)-\frac{c_{16}d(a_j,\Lambda_R)}{(\log(2+R+L))^{c_{17}}}\Big)\\
&+(c_{18}t^{(1-\epsilon)/2}(\log(2+R+L))^{c_{19}})^n\\
&\times\sum_{a_1,\dots,a_n\in\mathbb{Z}^2}\|\chi_{a_n}\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\exp\Big(-c_{20}\sum_{k=0}^{n-1}(|a_k-a_{k+1}|^2+d(a_{k+1},\Lambda_{R+L})^2)\Big)\\
\le&\frac{(\log(2+R+L))^{c_{15}}}{t^{(1+\epsilon)/2}}\exp\Big(-\frac{c_{16}d(a,\Lambda_R)}{(\log(2+R+L))^{c_{17}}}\Big)\sum_{j=0}^{n-1}c_{14}(c_{21}t^{(1-\epsilon)/2}(\log(2+R+L))^{c_{19}})^j\\
&+(c_{21}t^{(1-\epsilon)/2}(\log(2+R+L))^{c_{19}})^nc_{22}\|\Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}
\end{aligned}
$$

for any $n\in\mathbb{N}$, where $a_0=a$. For any $\widehat{\delta}\in(0,1)$, we take t as

$$
t=\widehat{\delta}^{2/(1-\epsilon)}(c_{21}(\log(2+R+L))^{c_{19}})^{-2/(1-\epsilon)}.
$$

Then, by taking the limit $n \to \infty$, we obtain

$$\|\chi_a \Phi_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \le c_{23}(\log(2+R+L))^{c_{24}} \exp\Big(\frac{-c_{25} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{26}}}\Big), \tag{4.14}$$

where the constants c_j depend on $\widehat{\delta}$. Thus, from (4.10), (4.11) and (4.12), we have

$$\|\chi_a \varphi_{R,L}\|_{\mathcal{H}^{\epsilon}(\mathbb{R}^2)} \le c_{27}(\log(2+R+L))^{c_{28}} \exp\Big(\frac{-c_{29} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{26}}}\Big),$$

$$\|\chi_a \Delta\Phi_{\xi,R+L}(\varphi_{R,L})\|_{L^2(\mathbb{R}^2)} \le c_{30}(\log(2+R+L))^{c_{31}} \exp\Big(\frac{-c_{32} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{26}}}\Big),$$

and

$$\begin{aligned}&\|\chi_a(\Delta\Phi_{\xi,R+L}(\varphi_{R,L}) + \widetilde{H^{\xi}_{R+L}\varphi_{R,L}})\|_{L^2(\mathbb{R}^2)} \\ \le& c_{31}(\log(2+R+L))^{c_{32}} \exp\Big(\frac{-c_{33} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{34}}}\Big).\end{aligned}$$

By using also the estimates

$$\|\chi_a \Phi_{\xi,R+L}(\varphi_{R,L})\|_{L^2(\mathbb{R}^2)} \le c_{35}(\log(2+R+L))^{c_{36}} \exp\Big(\frac{-c_{37} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{38}}}\Big)\|\widetilde{\chi_R f}\|_{L^2(\mathbb{R}^2)}.$$

and

$$\|\chi_a \widetilde{H^{\xi}_{R+L}\varphi_{R,L}}\|_{L^2(\mathbb{R}^2)} \le c_{39}(\log(2+R+L))^{c_{40}} \exp\Big(\frac{-c_{41} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{42}}}\Big)\|\widetilde{\chi_R f}\|_{L^2(\mathbb{R}^2)}.$$

obtained by Lemma 3.2 and (4.9), we obtain

$$\|\chi_a \Phi_{\xi,R+L}(\varphi_{R,L})\|_{\mathcal{H}^2(\mathbb{R}^2)} \le c_{43}(\log(2+R+L))^{c_{44}} \exp\Big(\frac{-c_{45} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{46}}}\Big). \tag{4.15}$$

By Lemma 3.3 and Lemma 3.6, we have

$$\|\chi_a(\Phi_{\xi,R+L}(\varphi_{R,L}) - \Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi,R+L}(\varphi_{R,L}))\|_{\mathcal{H}^2(\mathbb{R}^2)} \le c_{47}(\log(2+R+L))^{c_{48}} \exp\Big(\frac{-c_{49} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{50}}}\Big)$$

and

$$\|\chi_a \Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi,R+L}(\varphi_{R,L})\|_{\mathcal{H}^2(\mathbb{R}^2)} \le c_{51}(\log(2+R+L))^{c_{52}} \exp\Big(\frac{-c_{53} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{54}}}\Big).$$

As in the proof of Lemma 4.2 (i), we have

$$\begin{aligned}&\|\chi_a \widetilde{\varphi_{R,L}}\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \\ \le&\|\chi_a \Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi,R+L}(\varphi_{R,L})\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \\ &+\sum_{n=1}^{\infty}\delta^n \sum_{a_1,a_2,\dots,a_n\in\mathbb{Z}^2} \exp\Big(-c_{55}\sum_{j=1}^{n}|a_{j-1}-a_j|^2\Big)\|\chi_{a_n}\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi,R+L}(\varphi_{R,L})\|_{L^2(\mathbb{R}^2)} \\ \le& c_{56}(\log(2+R+L))^{c_{57}} \exp\Big(\frac{-c_{58} d(a,\Lambda_R)}{(\log(2+R+L))^{c_{59}}}\Big)\end{aligned} \tag{4.16}$$

for small enough $\delta > 0$, where $a_0 = a$. By Lemma 3.3 and Lemma 3.6, we have

$$\|\chi_a(\Phi_\xi(\widetilde{\varphi_{R,L}}) - \Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}(\widetilde{\varphi_{R,L}}))\|_{\mathcal{H}^2(\mathbb{R}^2)} \leq c_{60}(\log(2+R+L))^{c_{61}} \exp\Big(\frac{-c_{62}d(a,\Lambda_R)}{(\log(2+R+L))^{c_{63}}}\Big)$$

and

$$\|\chi_a\Phi_\xi(\widetilde{\varphi_{R,L}})\|_{\mathcal{H}^2(\mathbb{R}^2)} \leq c_{64}(\log(2+R+L))^{c_{65}} \exp\Big(\frac{-c_{66}d(a,\Lambda_R)}{(\log(2+R+L))^{c_{67}}}\Big). \tag{4.17}$$

Thus we obtain $\widetilde{\varphi_{R,L}} \in \mathrm{Dom}_0(\widetilde{H^\xi})$ and sufficiently sharp estimates of $\|\chi_a\widetilde{\varphi_{R,L}}\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$ and $\|\chi_a\Phi_\xi(\widetilde{\varphi_{R,L}})\|_{\mathcal{H}^2(\mathbb{R}^2)}$ for our proof of the self-adjointness.

Since $(\widetilde{H^\xi}+i)\widetilde{\varphi_{R,L}} \in \mathrm{Ran}(\widetilde{H^\xi}+i)$, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty}(f, (\widetilde{H^\xi_{R+L}}+i)\varphi_{R,L} - (\widetilde{H^\xi}+i)\widetilde{\varphi_{R,L}})_{L^2(\mathbb{R}^2)}. \tag{4.18}$$

As in the proof of Lemma 4.2 we have

$$\begin{aligned}&\|\chi_a(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}(\varphi_{R,L}) - \Phi_{\xi,R+L}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(\varphi_{R,L})\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}\\ \leq& c_{68}(\log(2+R+L))^{c_{69}} \exp\Big(-c_{70}\Big(d(a,\Lambda_{R+L}^c)^2 + \frac{d(a,\Lambda_R)}{(\log(2+R+L))^{c_{71}}}\Big)\Big),\end{aligned}$$

and

$$\|\chi_a(\varphi_{R,L} - \widetilde{\varphi_{R,L}})\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq c_{72}(\log(2+R+L))^{c_{73}} \exp\Big(-c_{74}\frac{L+d(a,\Lambda_R)}{(\log(2+R+L))^{c_{75}}}\Big). \tag{4.19}$$

We next consider

$$\|\widetilde{H^\xi_{R+L}}\varphi_{R,L} - \widetilde{H^\xi}\widetilde{\varphi_{R,L}}\|_{L^2(\mathbb{R}^2)} \leq \sum_{j=1}^{16} I_j,$$

where

$$
\begin{aligned}
I_1 =& \|\Delta\Phi_{\xi,R+L}(\varphi_{R,L}) - \Delta\Phi_{\xi}(\widetilde{\varphi_{R,L}})\|_{L^2(\mathbb{R}^2)},\\
I_2 =& \|P_{\xi_{R+L}}\Phi_{\xi,R+L}(\varphi_{R,L}) - P_{\xi}\Phi_{\xi}(\widetilde{\varphi_{R,L}})\|_{L^2(\mathbb{R}^2)},\\
I_3 =& \|\Pi(\Phi_{\xi,R+L}(\varphi_{R,L}),\xi_{R,L}) - \Pi(\Phi_{\xi}(\widetilde{\varphi_{R,L}}),\xi)\|_{L^2(\mathbb{R}^2)},\\
I_4 =& \|P_1^{(b)}((P_1^{(b)}\varphi_{R,L})(P_1^{(b)}\xi_{R+L})) - P_1^{(b)}((P_1^{(b)}\widetilde{\varphi_{R,L}})(P_1^{(b)}\xi))\|_{L^2(\mathbb{R}^2)},\\
I_5 =& \|e^{\Delta}P_{\varphi_{R,L}}\xi_{R+L} - e^{\Delta}P_{\widetilde{\varphi_{R,L}}}\xi\|_{L^2(\mathbb{R}^2)},\\
I_6 =& \|e^{\Delta}{}_{\varphi_{R,L}}P_{\xi_{R+L}}(\Delta^{-loc}\xi_{R+L}) - e^{\Delta}{}_{\widetilde{\varphi_{R,L}}}P_{\xi}(\Delta^{-loc}\xi)\|_{L^2(\mathbb{R}^2)},\\
I_7 =& \|e^{\Delta}P_{\varphi_{R,L}}Y_{\xi,R+L} - e^{\Delta}P_{\widetilde{\varphi_{R,L}}}Y_{\xi}\|_{L^2(\mathbb{R}^2)},\\
I_8 =& \|C(\varphi_{R,L},\xi_{R+L},\xi_{R+L}) - C(\widetilde{\varphi_{R,L}},\xi,\xi)\|_{L^2(\mathbb{R}^2)},\\
I_9 =& \|S(\varphi_{R,L},\xi_{R+L},\xi_{R+L}) - S(\widetilde{\varphi_{R,L}},\xi,\xi)\|_{L^2(\mathbb{R}^2)},\\
I_{10} =& \|P_{\chi^e_{a'}Y_{\xi,R+L}}\varphi_{R,L} - P_{Y_\xi}\widetilde{\varphi_{R,L}}\|_{L^2(\mathbb{R}^2)},\\
I_{11} =& \|\Pi(\varphi_{R,L},Y_{\xi,R+L}) - \Pi(\widetilde{\varphi_{R,L}},Y_{\xi})\|_{L^2(\mathbb{R}^2)},\\
I_{12} =& \|P_1^{(b)}((P_1^{(b)}\varphi_{R,L})(P_1^{(b)}Y_{\xi,R+L})) - P_1^{(b)}((P_1^{(b)}\widetilde{\varphi_{R,L}})(P_1^{(b)}Y_{\xi}))\|_{L^2(\mathbb{R}^2)},\\
I_{13} =& \|P_{\xi_{R+L}}(\Delta^{-loc}{}_{\varphi_{R,L}}P_{\xi_{R+L}}(\Delta^{-loc}\xi_{R+L})) - P_{\xi}(\Delta^{-loc}{}_{\widetilde{\varphi_{R,L}}}P_{\xi}(\Delta^{-loc}\xi))\|_{L^2(\mathbb{R}^2)},\\
I_{14} =& \|\Pi(\Delta^{-loc}{}_{\varphi_{R,L}}P_{\xi_{R+L}}(\Delta^{-loc}\xi_{R+L}),\xi_{R+L}) - \Pi(\Delta^{-loc}{}_{\widetilde{\varphi_{R,L}}}P_{\xi}(\Delta^{-loc}\xi),\xi)\|_{L^2(\mathbb{R}^2)},\\
I_{15} =& \|P_{\xi_{R+L}}(\Delta^{-loc}P_{\varphi_{R,L}}Y_{\xi,R+L}) - P_{\xi}(\Delta^{-loc}P_{\widetilde{\varphi_{R,L}}}Y_{\xi})\|_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

and

$$
I_{16} = \|\Pi(\Delta^{-loc}P_{\chi_{a''}\varphi_{R,L}}Y_{\xi,R+L},\xi_{R+L}) - \Pi(\Delta^{-loc}P_{\chi_{a''}\widetilde{\varphi_{R,L}}}Y_{\xi},\xi)\|_{L^2(\mathbb{R}^2)}.
$$

Since

$$
\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi,R+L}(\varphi_{R,L}) = \Phi^{\boldsymbol{s}(\epsilon,\xi,\delta)}_{\xi}(\widetilde{\varphi_{R,L}}),
$$

we have

$$
I_1 \le c_{76}\sum_{j=1}^{6} I_{1,j},
$$

where

$$I_{1,1} = \Big\| \sum_{a\in\mathbb{Z}^2\cap\Lambda_{R+L}} (P_{\varphi_{R,L}-\widetilde{\varphi_{R,L}}}(\chi_a^2\xi) - P^{s(a;\epsilon,\xi,\delta)}_{\varphi_{R,L}-\widetilde{\varphi_{R,L}}}(\chi_a^2\xi))\Big\|_{L^2(\mathbb{R}^2)},$$

$$I_{1,2} = \Big\| \sum_{a\in\mathbb{Z}^2\setminus\Lambda_{R+L}} (P_{\widetilde{\varphi_{R,L}}}(\chi_a^2\xi) - P^{s(a;\epsilon,\xi,\delta)}_{\widetilde{\varphi_{R,L}}}(\chi_a^2\xi))\Big\|_{L^2(\mathbb{R}^2)},$$

$$I_{1,3} = \Big\| \sum_{a,a'\in\mathbb{Z}^2\cap\Lambda_{R+L}} ({}_{\varphi_{R,L}-\widetilde{\varphi_{R,L}}}P_{\chi_a\xi}(\Delta^{-loc}\chi_{a'}^2\xi) - {}_{\varphi_{R,L}-\widetilde{\varphi_{R,L}}}P^{s_1(a,a';\epsilon,\xi,\delta)}_{\chi_a\xi}(\Delta^{-loc}\chi_{a'}^2\xi))\Big\|_{L^2(\mathbb{R}^2)},$$

$$I_{1,4} = \Big\| \sum_{(a,a')\in\mathbb{Z}^2\times\mathbb{Z}^2\setminus\Lambda_{R+L}\times\Lambda_{R+L}} ({}_{\widetilde{\varphi_{R,L}}}P_{\chi_a^2\xi}(\Delta^{-loc}\chi_{a'}^2\xi) - {}_{\widetilde{\varphi_{R,L}}}P^{s(a,a';\epsilon,\xi,\delta)}_{\chi_a^2\xi}(\Delta^{-loc}\chi_{a'}^2\xi))\Big\|_{L^2(\mathbb{R}^2)},$$

$$I_{1,5} = \Big\| \sum_{a\in\mathbb{Z}^2} (P_{\varphi_{R,L}-\widetilde{\varphi_{R,L}}}(\chi_a^2 Y_{\xi,R+L}) - P^{s_2(a;\epsilon,\xi,\delta)}_{\varphi_{R,L}-\widetilde{\varphi_{R,L}}}(\chi_a^2 Y_{\xi,R+L}))\Big\|_{L^2(\mathbb{R}^2)},$$

and

$$I_{1,6} = \Big\| \sum_{a\in\mathbb{Z}^2} (P_{\widetilde{\varphi_{R,L}}}(\chi_a^2(Y_{\xi,R+L}-Y_\xi) - P^{s_2(a;\epsilon,\xi,\delta)}_{\widetilde{\varphi_{R,L}}}(\chi_a^2(Y_{\xi,R+L}-Y_\xi))\Big\|_{L^2(\mathbb{R}^2)}.$$

To estimate the each term, we apply Lemma 3.6, Lemma 3.3 and Lemma 4.1. For $I_{1,1}, I_{1,3}$ and $I_{1,5}$, we apply (4.19), and, for $I_{1,2}, I_{1,4}$ and $I_{1,6}$, we apply (4.16). Then we have

$$I_1 \le c_{76}(R+\log(2+R+L))^{c_{77}} \exp\Big(\frac{-c_{78}L}{(\log(2+R+L))^{c_{79}}}\Big),$$

which converges to 0 as $L\to\infty$. Similar methods show that $\{I_j\}_{2\le j\le 16}$ also converges to 0 as $L\to\infty$ by Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.6, Lemma 4.1, (4.19), (4.16), and (4.17). Thus the right hand side of (4.8) is zero, and we obtain $\mathrm{Ran}(\widetilde{H^\xi}+i)^\perp = \{0\}$ (cf.[20]). □

In [18], the obtained operator is shown to be the limit of the smooth approximation in the norm resolvent sense, and many results on the spectrum are obtained from this fact (cf. Proposition 2.14 in [18]). In our case we obtain only the following results on the convergence in the strong resolvent sense:

Proposition 4.1. (i) *The closure $\overline{\widetilde{H^{\xi_\varepsilon}}}$ of the operator $\widetilde{H^{\xi_\varepsilon}}$ with the domain $C_0^\infty(\mathbb{R}^2)$ converges to the closure $\overline{\widetilde{H^\xi}}$ of the operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_0(\widetilde{H^\xi})$ in the strong resolvent sense as $\varepsilon\to 0$.*

(ii) *The self-adjoint oeprator $\widetilde{H^\xi_R}$ in Lemma 4.10 converges to the closure $\overline{\widetilde{H^\xi}}$ of the operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_0(\widetilde{H^\xi})$ in the strong resolvent sense as $R\to\infty$.*

Proof. (i) For any $v\in L^2(\mathbb{R}^2)$, we set $u := (\overline{\widetilde{H^\xi}}+i)^{-1}v$. Then, for any $\eta,\epsilon>0$, there exists $u_0\in \mathrm{Dom}_\epsilon(\widetilde{H^\xi})$ such that

$$\|u-u_0\|_{L^2(\mathbb{R}^2)}, \|\overline{\widetilde{H^\xi}}u - \widetilde{H^\xi}u_0\|_{L^2(\mathbb{R}^2)} < \eta.$$

As in Lemma 4.5, we set $u_\varepsilon := (\Phi_{\xi_\varepsilon}^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1}(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u_0))$. Then we have

$$\lim_{\varepsilon\to 0}\|u_0-u_\varepsilon\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}=0$$

and

$$\lim_{\varepsilon\to 0}\|\widetilde{H^\xi}u_0-\widetilde{H^{\xi_\varepsilon}}u_\varepsilon\|_{L^2(\mathbb{R}^2)}=0.$$

Thus we have

$$\begin{aligned}
&\|(\widetilde{H^{\xi_\varepsilon}}+i)^{-1}v-(\overline{\widetilde{H^\xi}}+i)^{-1}v\|_{L^2(\mathbb{R}^2)}\\
\le&\|(\widetilde{H^{\xi_\varepsilon}}+i)^{-1}(\overline{\widetilde{H^\xi}}+i)(u-u_0)\|_{L^2(\mathbb{R}^2)}+\|(\widetilde{H^{\xi_\varepsilon}}+i)^{-1}(\overline{\widetilde{H^\xi}}+i)u_0-u_\varepsilon\|_{L^2(\mathbb{R}^2)}\\
&+\|u_\varepsilon-u_0\|_{L^2(\mathbb{R}^2)}+\|u_0-u\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

This is less than 3η for sufficiently small ε.

(ii) In the proof of (i), we replace u_ε by $u_R := (\Phi_{\xi,R}^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1}(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)}(u_0))$. Then the rest of the proof is same with that of (i).

□

5. Proof of Theorem 2

In this section we prove Theorem 2: we show that the spectral set of $\overline{\widetilde{H^\xi}}$ is $\mathbb{R}$.

On the 2-dimensional flat torus $\mathbb{T}_L^2 := \mathbb{R}^2/(L\mathbb{Z})^2$ with any $L\in\mathbb{N}$, we take an orthonormal basis $\{\varphi_{\boldsymbol{n}}^L\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ of $L^2(\mathbb{T}_L^2)$ defined by

$$\varphi_{(n_1,n_2)}^L(x_1,x_2)=\phi_{n_1}^L(x_1)\phi_{n_2}^L(x_2)$$

and

$$\phi_{n_1}^L(x_1)=\begin{cases}\sqrt{2/L}\cos(2\pi n_1x_1/L) & \text{for } 0<n_1\in\mathbb{Z},\\ \sqrt{1/L} & \text{for } n_1=0,\\ \sqrt{2/L}\sin(2\pi n_1x_1/L) & \text{for } 0>n_1\in\mathbb{Z}.\end{cases}$$

Then any white noise ξ^L on $\mathbb{T}_L^2$ is represented as

$$\xi^L(x)=\sum_{\boldsymbol{n}\in\mathbb{Z}^2}X_{\boldsymbol{n}}(\xi^L)\varphi_{\boldsymbol{n}}^L(x)$$

in the Besov Hölder space $\mathcal{C}^{-1-\epsilon}(\mathbb{T}_L^2)$ on $\mathbb{T}_L^2$ for any $\epsilon>0$, where $\{X_{\boldsymbol{n}}(\xi^L)\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ is a system of independently identically distributed random variables having the standard normal distribution. Let $\widetilde{\chi_L}$ and $\widetilde{\chi_L}^c$

be $[0,1]$-valued smooth function on $\mathbb{R}^2$ such that $\widetilde{\chi_L}=0$ on $\mathbb{R}^2\setminus\Lambda_L$, $\widetilde{\chi_L}=1$ on Λ_{L-1} and $\widetilde{\chi_L}^2+(\widetilde{\chi_L}^c)^2=1$ on $\mathbb{R}^2$.

We represent the white noise as

$$\xi=\widetilde{\chi_L}\xi^L+\widetilde{\chi_L}^c\xi^{L,c}, \tag{5.1}$$

where ξ^L and $\xi^{L,c}$ are white noises on $\mathbb{T}_L^2$ and $\mathbb{R}^2$, respectively, such that ξ^L and $\xi^{L,c}$ are independent as random fields. For any $N\in\mathbb{N}$, we decompose ξ^L as

$$\xi^L=\xi^L_{N\geq}+\xi^L_{N<}, \tag{5.2}$$

where

$$\xi^L_{N\geq}(x)=\sum_{\boldsymbol{n}\in\mathbb{Z}^2\cap\Lambda_N}X_{\boldsymbol{n}}(\xi^L)\varphi^L_{\boldsymbol{n}}(x)$$

and

$$\xi^L_{N<}(x)=\sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N}X_{\boldsymbol{n}}(\xi^L)\varphi^L_{\boldsymbol{n}}(x).$$

For the random field $\widetilde{\xi^L_{N<}}:=\widetilde{\chi_L}\xi^L_{N<}+\widetilde{\chi_L}^c\xi^{L,c}$, we set $(\widetilde{\xi^L_{N<}})_\varepsilon=e^{\varepsilon^2\Delta}\widetilde{\xi^L_{N<}}$,

$$Y_{\xi,\varepsilon,L,N<}:=\Pi(\Delta^{-loc}(\widetilde{\xi^L_{N<}})_\varepsilon,(\widetilde{\xi^L_{N<}})_\varepsilon)-\mathbb{E}[\Pi(\Delta^{-loc}(\widetilde{\xi^L_{N<}})_\varepsilon,\chi_a(\widetilde{\xi^L_{N<}})_\varepsilon)]),$$

and $Y_{\xi,L,N<}$ is a random variable such that

$$\lim_{\varepsilon\to0}\mathbb{E}[\|\chi_a(Y_{\xi,\varepsilon,L,N<}-Y_{\xi,L,N<}\|^p_{C^{-\epsilon}(\mathbb{R}^2)}]=0$$

for any $p\in[1,\infty)$, $\epsilon>0$ and $a\in\mathbb{Z}^2$. We note the relation

$$\begin{aligned}Y_{\xi,L,N<}=&\,Y_{\widetilde{\xi^L}}-\Pi(\Delta^{-loc}\widetilde{\xi^L_{N<}},\widetilde{\chi_L}\xi^L_{N\geq})-\Pi(\Delta^{-loc}\widetilde{\chi_L}\xi^L_{N\geq},\widetilde{\xi^L_{N<}})\\&-\Pi(\Delta^{-loc}\widetilde{\chi_L}\xi^L_{N\geq},\widetilde{\chi_L}\xi^L_{N\geq})+\sum_{\boldsymbol{n}\in\mathbb{Z}^2\cap\Lambda_N}\Pi(\Delta^{-loc}\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}},\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}).\end{aligned}$$

We modify the operator as follows:

$$\begin{aligned}\mathrm{Dom}_\epsilon(\widetilde{H^{\xi,L,N<}}):=&\Big\{u\in\mathcal{H}^{1-\epsilon}(\mathbb{R}^2):\limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}<0,\\&\Phi_{\xi,L,N<}(u)\in\mathcal{H}^2(\mathbb{R}^2),\limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a\Phi_{\xi,L,N<}(u)\|_{\mathcal{H}^2(\mathbb{R}^2)}<0\Big\},\\\Phi_{\xi,L,N<}(u):=&u-\Delta^{-loc}P_u\widetilde{\xi^L_{N<}}-\Delta^{-loc}{}_uP_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}})-\Delta^{-loc}P_uY_{\xi,L,N<}\end{aligned}$$

and

$$
\begin{aligned}
&\widetilde{H^{\xi,L,N<}}u \\
&= -\Delta\Phi_{\xi,L,N<}(u) + P_{\widetilde{\xi^L_{N<}}}\Phi_{\xi,L,N<}(u) + \Pi(\Phi_{\xi,L,N<}(u), \widetilde{\xi^L_{N<}}) \\
&\quad + P^{(b)}_1((P^{(b)}_1 u)(P^{(b)}_1 \widetilde{\xi^L_{N<}})) \\
&\quad + e^{\Delta}P_u\widetilde{\xi^L_{N<}} + e^{\Delta}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}) + e^{\Delta}P_u Y_{\xi,L,N<} \\
&\quad + C(u, \widetilde{\xi^L_{N<}}, \widetilde{\xi^L_{N<}}) + S(u, \widetilde{\xi^L_{N<}}, \widetilde{\xi^L_{N<}}) \\
&\quad + P_{Y_{\xi,L,N<}}u + \Pi(u, Y_{\xi,L,N<}) + P^{(b)}_1((P^{(b)}_1 u)(P^{(b)}_1 Y_{\xi,L,N<})) \\
&\quad + P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}})) \\
&\quad + \Pi(\Delta^{-loc}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}), \widetilde{\xi^L_{N<}}) \\
&\quad + P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}P_u Y_{\xi,L,N<}) + \Pi(\Delta^{-loc}P_u Y_{\xi,L,N<}, \widetilde{\xi^L_{N<}}),
\end{aligned}
$$

Then we have the following:

Lemma 5.1. *For any L and $N \in \mathbb{N}$, we have*

$$\mathrm{Dom}_\epsilon(\widetilde{H^\xi}) = \mathrm{Dom}_\epsilon(\widetilde{H^{\xi,L,N<}})$$

and

$$\widetilde{H^\xi}u = (\widetilde{H^{\xi,L,N<}} + \widetilde{\chi_L}\xi^L_{N\geq} - Y^{L,N\geq})u$$

for any $u \in \mathrm{Dom}_\epsilon(\widetilde{H^\xi})$, where

$$Y^{L,N\geq} := \sum_{\boldsymbol{n}\in\mathbb{Z}^2\cap\Lambda_N} \Pi(\Delta^{-loc}\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}, \widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}).$$

The term $Y^{L,N\geq}$ may diverge as $N \to \infty$. However we have the following bound:

Lemma 5.2. *For any $\eta \in (0,1)$, there exists $c_\eta \in [0,\infty)$ such that*

$$\sup_{x\in\mathbb{R}^2} |Y^{L,N\geq}(x)| \leq c_\eta (N/L)^\eta$$

for any L and $N \in \mathbb{N}$.

Proof. By the L^∞-version of Lemma 3.2 (iii), we have

$$\begin{aligned}&\sup_{\Lambda_2(a')}|\Pi(\chi_a\Delta^{-loc}\chi_{a'}\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}},\chi_a\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}})|\\ \le& c_1\|\chi_a\Delta^{-loc}\chi_{a'}\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}\|_{C^{1+\eta}(\mathbb{R}^2)}\|\chi_a\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}\|_{C^{-1+\eta}(\mathbb{R}^2)}\exp(-c_2|a-a'|^2)\\ \le& c_3\|\chi_{a'}\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}\|_{C^{-1+\eta}(\mathbb{R}^2)}\|\chi_a\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}\|_{C^{-1+\eta}(\mathbb{R}^2)}\exp(-c_2|a-a'|^2)\end{aligned}$$

and

$$\|\chi_a\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}\|_{C^{-1+\eta}(\mathbb{R}^2)}\le c_4L^{-2\eta}|\boldsymbol{n}|^{-1+2\eta},$$

from which we obtain the bound. □

For any $N,R,L\in\mathbb{N}$ satisfying $R+2\le L$, the lowest eigenvalue $\lambda(L,N,R)$ of the operator $-\Delta-Y^{L,N\ge}$ on the domain Λ_R with the Dirichlet boundary condition is estimated by this lemma as

$$|\lambda(L,N,R)|\le c'_\eta\Big(1\vee\frac{N}{L}\Big)^\eta. \tag{5.3}$$

For any $\varepsilon\in(0,1)$, we take a function $\varphi_{\varepsilon,R}\in C^\infty(\mathbb{R}^2)$ such that $\operatorname{supp}\varphi_{\varepsilon,R}\subset\Lambda_R$, $\|\varphi_{\varepsilon,R}\|_{L^2(\mathbb{R})}=1$ and

$$\|((-\Delta-Y^{L,N\ge})-\lambda(L,N,R))\varphi_{\varepsilon,R}\|_{L^2(\mathbb{R}^2)}<\varepsilon. \tag{5.4}$$

Then this function also has the estimate

$$\|\varphi_{\varepsilon,R}\|_{\mathcal{H}^2(\mathbb{R}^2)}\le c''_\eta\Big(1\vee\frac{N}{L}\Big)^\eta. \tag{5.5}$$

We use the euality

$$(-\Delta-Y^{L,N\ge})-\lambda(L,N,R)=(-\Delta+r(\lambda,L,N,R)\varphi^L_{\boldsymbol{0}}-Y^{L,N\ge})-\lambda \tag{5.6}$$

for any $\lambda\in\mathbb{R}$, where

$$r(\lambda,L,N,R):=L(\lambda-\lambda(L,N,R)).$$

We fix R arbitrarily.

For any $\varepsilon,\epsilon\in(0,1)$, $\lambda\in\mathbb{R}$ and $N,R,L\in\mathbb{N}$ satisfying $R+2\le L$, we define the event $E(\varepsilon,\epsilon,\lambda,L)$ by

$$\Big\{\xi:\text{In the representation of (5.1) and (5.2) with } N=L^{10},\text{ it holds that}$$
$$|X_{\boldsymbol{0}}(\xi^L)-r(\lambda,L,N,R)|,|X_{\boldsymbol{n}}(\xi^L)|\le\varepsilon/N^2\text{ for any }\boldsymbol{n}\in\mathbb{Z}^2\cap\Lambda_N\setminus\{\boldsymbol{0}\},\text{ and}$$
$$\|\chi_a\widetilde{\xi^L_{N<}}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}\vee\|\chi_aY_{\xi,L,N<}\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)}\le 1_{\Lambda_{L/2}}(a)L^{-\epsilon}+1_{\Lambda^c_{L/2}}(a)|a|^\epsilon$$
$$\text{for any }a\in\mathbb{Z}^2.\Big\}.$$

The positivity of this event is proven as in the proof of Lemma 3.3:

Lemma 5.3. *For any $\varepsilon, \epsilon \in (0,1)$, $\lambda \in \mathbb{R}$, and $R \in \mathbb{N}$, there exists $L_0 \in \mathbb{N}$ such that $\mathbb{P}(E(\varepsilon, \epsilon, \lambda, L)) > 0$ for any $L_0 \le L \in \mathbb{N}$.*

Proof. We devide the probability as follows:

$$\mathbb{P}(E(\varepsilon, \epsilon, \lambda, L)) \ge I_0\Big(1 - \sum_{a \in \mathbb{Z}^2} I_a - \sum_{a \in \mathbb{Z}^2} J_a\Big),$$

where

$$I_0 = \mathbb{P}(|X_{\mathbf{0}}(\xi^L) - r(\lambda, L, N, R)| \le \varepsilon/N^2) \prod_{\boldsymbol{n} \in \mathbb{Z}^2 \cap \Lambda_N \setminus \{\mathbf{0}\}} \mathbb{P}(|X_{\boldsymbol{n}}(\xi^L)| \le \varepsilon/N^2),$$

$$I_a = \mathbb{P}(\|\chi_a \widetilde{\xi^L_{N<}}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \ge 1_{\Lambda_{L/2}}(a) L^{-\epsilon} + 1_{\Lambda^c_{L/2}}(a)|a|^{\epsilon})$$

and

$$J_a = \mathbb{P}(\|\chi_a Y_{\xi, L, N<}\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \ge (1_{\Lambda_{L/2}}(a) L^{-\epsilon} + 1_{\Lambda^c_{L/2}}(a)|a|^{\epsilon})).$$

I_0 is positive for any $\varepsilon > 0$ and $N \in \mathbb{N}$. By taking $p = 4/\epsilon$ and $\epsilon_0 = \epsilon/2$, we have

$$\sum_{a \in \mathbb{Z}^2} I_a \le c_1\Big(\sum_{a \in \mathbb{Z}^2 \cap \Lambda_{L/2}} \mathbb{E}[\|\chi_a \xi^L_{N<}\|^p_{\mathcal{B}^{-1-\epsilon_0}_{p,p}(\mathbb{R}^2)}] L^4 + \sum_{a \in \mathbb{Z}^2 \setminus \Lambda_{L/2}} \mathbb{E}[\|\chi_a \widetilde{\xi^L_{N<}}\|^p_{\mathcal{B}^{-1-\epsilon_0}_{p,p}(\mathbb{R}^2)}] |a|^{-4}\Big).$$

As in Lemma 3.3, we have

$$\sup_{a \in \mathbb{Z}^2, L \in \mathbb{N}} \mathbb{E}[\|\chi_a \widetilde{\xi^L_{N<}}\|^p_{\mathcal{B}^{-1-\epsilon_0}_{p,p}(\mathbb{R}^2)}] < \infty.$$

Moreover, for any $Q \in StGC^r(\mathbb{R}^2)$ with $r \in \mathbb{N}$, we have

$$
\begin{aligned}
&\mathbb{E}[\|t^{(1+\epsilon_0)/2} Q_t \widetilde{\chi_L} \chi_a \xi^L_{N<}\|^p_{L^p(\mathbb{R}^2\times[0,1]:dxdt/t)}] \\
\le & c_2 \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} dx \Big\{ t^{1+\epsilon_0} \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N, \iota\in\{1,2\}} ((Q_t \widetilde{\chi_L} \chi_a \\
& \times \partial_\iota (-\Delta_{\mathbb{T}^2_L})^{-(1-\epsilon_0)/2} \varphi^L_{\boldsymbol{n}})(x))^2 \Big(\frac{L}{|\boldsymbol{n}|}\Big)^{\epsilon_0} \Big\}^{p/2} \\
\le & c_3 \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} dx \Big\{ t^{\epsilon_0} \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N, \iota\in\{1,2\}} \Big(\int_0^1 \frac{dr}{r^{(1+\epsilon_0)/2}} \\
& \times (\sqrt{t}\partial_\iota Q_t \widetilde{\chi_L} \chi_a \exp(r\Delta_{\mathbb{T}^2_L}) \varphi^L_{\boldsymbol{n}})(x) \Big)^2 \Big(\frac{L}{|\boldsymbol{n}|}\Big)^{\epsilon_0} \Big\}^{p/2} \qquad (5.7) \\
& + c_3 \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} dx \Big\{ t^{1+\epsilon_0} \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N, \iota\in\{1,2\}} \Big(\int_0^1 \frac{dr}{r^{(1+\epsilon_0)/2}} \\
& \times (Q_t \widetilde{\chi_{L,a,\iota}} \exp(r\Delta_{\mathbb{T}^2_L}) \varphi^L_{\boldsymbol{n}})(x) \Big)^2 \Big(\frac{L}{|\boldsymbol{n}|}\Big)^{\epsilon_0} \Big\}^{p/2} \\
& + c_3 \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} dx \Big\{ t^{\epsilon_0} \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N, \iota\in\{1,2\}} ((Q_t \widetilde{\chi_L} \chi_a \partial_\iota \varphi^L_{\boldsymbol{n}})(x))^2 \\
& \times \Big(\int_1^\infty \frac{dr}{r^{(1+\epsilon_0)/2}} \exp\Big(\frac{-c_4 r N^2}{L^2}\Big)\Big)^2 \Big(\frac{L}{|\boldsymbol{n}|}\Big)^{\epsilon_0} \Big\}^{p/2},
\end{aligned}
$$

where, for each L, a, ι, $\widetilde{\chi_{L,a,\iota}}$ is a smooth function with a support in $\Lambda_2(a) \cap \Lambda_L$. The first term in the right hand side is dominated by

$$
\begin{aligned}
&\Big(\frac{L}{N}\Big)^2 \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} dx \Big\{ t^{\epsilon_0} \sum_{\iota\in\{1,2\}} \int_0^1 \frac{dr_1}{r_1^{(1+\epsilon_0)/2}} \int_0^1 \frac{dr_2}{r_2^{(1+\epsilon_0)/2}} \\
& \times ((\sqrt{t}\partial_\iota Q_t)\widetilde{\chi_L}\chi_a \exp(-(r_1+r_2)\Delta_{\mathbb{T}^2_L}) \chi_a \widetilde{\chi_L} (\sqrt{t}\partial_\iota Q_t)^*)(x,x) \Big\}^{p/2} \\
\le & c_5 \Big(\frac{L}{N}\Big)^2 \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} dx \Big\{ t^{\epsilon_0} \int_{\Lambda_2(a)\cap\Lambda_L} \frac{dx_1}{t} \exp\Big(-\frac{|x-x_1|^2}{c_6 t}\Big) \int_0^1 \frac{dr_1}{r_1^{(1+\epsilon_0)/2}} \\
& \times \int_0^1 \frac{dr_2}{r_2^{(1+\epsilon_0)/2}} \int_{\Lambda_2(a)\cap\Lambda_L} \frac{dx_2}{r_1+r_2} \sum_{y\in\mathbb{Z}^2} \exp\Big(-\frac{|x_1-x_2+Ly|^2}{4(r_1+r_2)}\Big) \frac{1}{t} \exp\Big(-\frac{|x_2-x|^2}{c_6 t}\Big) \Big\}^{p/2} \\
\le & c_7 L^2/N^2.
\end{aligned}
$$

The other terms are also similarly estimated. Therefore we obtain

$$
\mathbb{E}[\|\widetilde{\chi_L}\chi_a \xi^L_{N<}\|^p_{\mathcal{B}^{-1-\epsilon_0}_{p,p}(\mathbb{R}^2)}] \le c_8 L^2/N^2.
$$

Thus there exist L_0 such that

$$
\sum_{a\in\mathbb{Z}^2} I_a < \frac{1}{2}
$$

for any $L \geq L_0$. Simirarly we have

$$\sum_{a\in\mathbb{Z}^2} J_a \leq c_9 \sum_{a\in\mathbb{Z}^2} \mathbb{E}[\|\chi_a Y_{\xi,L,N<}\|^p_{\mathcal{B}^{-\epsilon_0}_{p,p}(\mathbb{R}^2)}] \times \Big(1_{\Lambda_{L/2}}(a) L^4 + 1_{\Lambda^c_{L/2}}(a)|a|^{-4} \Big).$$

As in Lemma 3.3, we have

$$\sup_{a\in\mathbb{Z}^2} \mathbb{E}[\|\chi_a Y_{\xi,L,N<}\|^p_{\mathcal{B}^{-\epsilon_0}_{p,p}(\mathbb{R}^2)}] < \infty.$$

To obtain sharper estimates, we consider the approximation $Y_{\xi,\varepsilon,L,N<}$. For any $Q \in StGC^r(\mathbb{R}^2)$ with $r \in \mathbb{N} \cap (0, 2b]$, by the hypercontractivity, we have

$$\begin{aligned}
&\mathbb{E}[\|t^{\epsilon_0/2} Q_t \chi_a Y_{\xi,\varepsilon,L,N<}\|^p_{L^p(\mathbb{R}^2\times[0,1]:dxdt/t)}] \\
\leq & c_{10} \int_0^1 \frac{dt}{t} t^{\epsilon_0 p/2} \int_{\mathbb{R}^2} dx \mathbb{E}[|(Q_t\chi_a Y_{\xi,\varepsilon,L,N<})(x)|^2]^{p/2}.
\end{aligned}$$

Moreover by the Gaussian property, we have

$$\begin{aligned}
&\mathbb{E}[|(Q_t\chi_a Y_{\xi,\varepsilon,L,N<})(x)|^2] \\
=&\sum_{\mu,\underline{\mu}} \int_0^1 \frac{ds}{s} \int_{\mathbb{R}^2} dx_1 (Q_t\chi_a P^\mu_s)(x,x_1) \int_0^1 \frac{d\underline{s}}{\underline{s}} \int_{\mathbb{R}^2} d\underline{x_1} (Q_t\chi_a P^{\underline{\mu}}_{\underline{s}})(x,\underline{x_1}) \\
&\times (\mathbb{E}[(Q^{1,\mu}_s \Delta^{-loc}(\widetilde{\xi^L_{N<}})_\varepsilon)(x_1)(Q^{1,\underline{\mu}}_{\underline{s}} \Delta^{-loc}(\widetilde{\xi^L_{N<}})_\varepsilon)(\underline{x_1})] \\
&\times \mathbb{E}[(Q^{2,\mu}_s(\widetilde{\xi^L_{N<}})_\varepsilon)(x_1)(Q^{2,\underline{\mu}}_{\underline{s}}(\widetilde{\xi^L_{N<}})_\varepsilon)(\underline{x_1})] \\
&+ \mathbb{E}[(Q^{1,\mu}_s \Delta^{-loc}(\widetilde{\xi^L_{N<}})_\varepsilon)(x_1)(Q^{2,\underline{\mu}}_{\underline{s}}(\widetilde{\xi^L_{N<}})_\varepsilon)(\underline{x_1})] \\
&\times \mathbb{E}[(Q^{1,\underline{\mu}}_{\underline{s}} \Delta^{-loc}(\widetilde{\xi^L_{N<}})_\varepsilon(\underline{x_1})(Q^{2,\mu}_s(\widetilde{\xi^L_{N<}})_\varepsilon)(x_1)]) \\
\leq & 2\sum_{n,n'} \Big(\sum_\mu \int_0^1 \frac{ds}{s} \int_{\mathbb{R}^2} dy (Q_t\chi_a P^\mu_s)(x,x_1) \\
&\times (Q^{1,\mu}_s \Delta^{-loc} e^{\varepsilon^2\Delta}\Phi^L_n)(x_1)(Q^{2,\mu}_s e^{\varepsilon^2\Delta}\Phi^L_{n'})(x_1) \Big)^2,
\end{aligned}$$

where $\{\Phi^L_n\}_n = \{\widetilde{\chi_L}\varphi^L_{\boldsymbol{n}}, \widetilde{\chi_L}^c\varphi_m : \boldsymbol{n}, \boldsymbol{n'} \in \mathbb{Z}^2 \setminus \Lambda_N, m \in \mathbb{N}\}$ and $\{\varphi_m : m \in \mathbb{N}\}$ is a complete orthonormal basis of $L^2(\mathbb{R}^2)$. As in (5.7), we have

$$\mathbb{E}[|(Q_t\chi_a Y_{\xi,\varepsilon,L,N<})(x)|^2] \leq c_{11} \sum_{i=1}^4 \mathcal{I}_i(t,x),$$

where

$$
\begin{aligned}
\mathcal{I}_1(t,x) =& \Big(\frac{L}{N}\Big)^{2\epsilon_1} \sum_{n,\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N,\iota\in\{1,2\}} \Big(\sum_{\mu} \int_0^1 \frac{ds}{s^{3/2}} \int_{\mathbb{R}^2} dx_1 (Q_t\chi_a P_s^{\mu})(x,x_1) \\
& \times (Q_s^{1,\mu}\Delta^{-loc} e^{\varepsilon^2\Delta}\Phi_n^L)(x_1) \\
& \times (\sqrt{s}\partial_\iota Q_s^{2,\mu} e^{\varepsilon^2\Delta}\widetilde{\chi_L} \int_0^1 \frac{dr}{r^{(1+\epsilon_1)/2}} \exp(r\Delta_{\mathbb{T}_L^2})\varphi_{\boldsymbol{n}}^L)(x_1)\Big)^2 \\
\mathcal{I}_2(t,x) =& \Big(\frac{L}{N}\Big)^{2\epsilon_1} \sum_{n,\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N,\iota\in\{1,2\}} \Big(\sum_{\mu} \int_0^1 \frac{ds}{s} \int_{\mathbb{R}^2} dx_1 (Q_t\chi_a P_s^{\mu})(x,x_1) \\
& \times (Q_s^{1,\mu}\Delta^{-loc} e^{\varepsilon^2\Delta}\Phi_n^L)(x_1) \\
& \times (Q_s^{2,\mu} e^{\varepsilon^2\Delta}\widetilde{\chi_{L,\iota}} \int_0^1 \frac{dr}{r^{(1+\epsilon_1)/2}} \exp(r\Delta_{\mathbb{T}_L^2})\varphi_{\boldsymbol{n}}^L)(x_1)\Big)^2 \\
\mathcal{I}_3(t,x) =& \Big(\frac{L}{N}\Big)^{2\epsilon_1} \sum_{n,\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N,\iota\in\{1,2\}} \Big(\sum_{\mu} \int_0^1 \frac{ds}{s} \int_{\mathbb{R}^2} dx_1 (Q_t\chi_a P_s^{\mu})(x,x_1) \\
& \times (Q_s^{1,\mu}\Delta^{-loc} e^{\varepsilon^2\Delta}\Phi_n^L)(x_1) \\
& \times (Q_s^{2,\mu} e^{\varepsilon^2\Delta}\widetilde{\chi_L}\partial_\iota\varphi_{\boldsymbol{n}}^L)(x_1)\Big)^2 \Big(\int_1^\infty \frac{dr}{r^{(1+\epsilon_1)/2}} \exp\Big(\frac{-c_{12} r N^2}{L^2}\Big)\Big)^2 \\
\mathcal{I}_4(t,x) =& \sum_{n,m} \Big(\sum_{\mu} \int_0^1 \frac{ds}{s} \int_{\mathbb{R}^2} dx_1 (Q_t\chi_a P_s^{\mu})(x,x_1) \\
& \times (Q_s^{1,\mu}\Delta^{-loc} e^{\varepsilon^2\Delta}\Phi_n^L)(x_1)(Q_s^{2,\mu} e^{\varepsilon^2\Delta}\widetilde{\chi_L}^c\varphi_m)(x_1)\Big)^2,
\end{aligned}
$$

$\widetilde{\chi_{L,\iota}}$ is a smooth function with a support in Λ_L, and $\epsilon_1 \in (0,1)$ are taken arbitrarily. $\mathcal{I}_1(t,x)$ is dominated by

$$
\begin{aligned}
&\Big(\frac{L}{N}\Big)^{2\epsilon_1} \exp(-c_{13}(|x-a|^2 + d(a,\Lambda_L)^2)) \int_0^1 \frac{ds}{s^{3/2}} \int_{\Lambda_2(a)} \frac{dx_0}{t} \exp\Big(\frac{-|x-x_0|^2}{c_{14}t}\Big) \\
&\times \int_{\mathbb{R}^2} \frac{dx_1}{s} \exp\Big(\frac{-|x_0-x_1|^2}{c_{15}s}\Big) \int_{\mathbb{R}^2} \frac{dx_2}{s} \exp\Big(\frac{-|x_1-x_2|^2}{c_{16}s}\Big) \\
&\times \Big(\int_0^s d\sigma + \int_s^1 d\sigma\Big(\frac{s}{\sigma}\Big)^{b/4}\Big) \int_{\mathbb{R}^2} \frac{dx_3}{\sigma} \exp\Big(\frac{-|x_2-x_3|^2}{c_{17}\sigma}\Big) \\
&\times \int_{\mathbb{R}^2} \frac{d\underline{x_3}}{\varepsilon^2} \exp\Big(\frac{-|x_3-\underline{x_3}|^2}{4\varepsilon^2}\Big) \int_0^1 \frac{d\underline{s}}{\underline{s}^{3/2}} \int_{\Lambda_2(a)} \frac{d\underline{x_0}}{t} \exp\Big(\frac{-|x-\underline{x_0}|^2}{c_{14}t}\Big) \\
&\times \int_{\mathbb{R}^2} \frac{d\underline{x_1}}{\underline{s}} \exp\Big(\frac{-|\underline{x_0}-\underline{x_1}|^2}{c_{15}\underline{s}}\Big) \int_{\mathbb{R}^2} \frac{d\underline{x_2}}{\underline{s}} \exp\Big(\frac{-|\underline{x_1}-\underline{x_2}|^2}{c_{16}s}\Big) \\
&\times \Big(\int_0^{\underline{s}} d\underline{\sigma} + \int_{\underline{s}}^1 d\underline{\sigma}\Big(\frac{\underline{s}}{\underline{\sigma}}\Big)^{b/4}\Big) \frac{1}{\underline{\sigma}} \exp\Big(\frac{-|\underline{x_2}-\underline{x_3}|^2}{c_{17}\underline{\sigma}}\Big) \\
&\times \int_{\mathbb{R}^2} \frac{dx_2'}{s} \exp\Big(\frac{-|x_1-x_2'|^2}{c_{18}s}\Big) \int_{\Lambda_L} \frac{dx_3'}{\varepsilon^2} \exp\Big(\frac{-|x_2'-x_3'|^2}{4\varepsilon^2}\Big) \\
&\times \int_{\mathbb{R}^2} \frac{d\underline{x_2'}}{\underline{s}} \exp\Big(\frac{-|\underline{x_1}-\underline{x_2'}|^2}{c_{18}\underline{s}}\Big) \int_{\Lambda_L} \frac{d\underline{x_3'}}{\varepsilon^2} \exp\Big(\frac{-|\underline{x_2'}-\underline{x_3'}|^2}{4\varepsilon^2}\Big) \\
&\times \int_0^1 \frac{dr}{r^{\epsilon_1}} \sum_{y\in\mathbb{Z}^2} \frac{1}{r} \exp\Big(\frac{-|x_3'-\underline{x_3'}-Ly|^2}{c_{19}r}\Big) \\
\le & c_{20}\Big(\frac{L}{N}\Big)^{2\epsilon_1} \exp(-c_{13}(|x-a|^2 + d(a,\Lambda_L)^2)) \\
&\times \int_0^1 \frac{ds}{s^{3/2}} \Big(\int_0^s d\sigma + \int_s^1 d\sigma\Big(\frac{s}{\sigma}\Big)^{b/4}\Big) \int_0^1 \frac{d\underline{s}}{\underline{s}^{3/2}} \\
&\times \Big(\int_0^{\underline{s}} d\underline{\sigma} + \int_{\underline{s}}^1 d\underline{\sigma}\Big(\frac{\underline{s}}{\underline{\sigma}}\Big)^{b/4}\Big) \int_0^1 \frac{dr}{r^{\epsilon_1}} \frac{1}{t+s+\sigma+\underline{\sigma}+\underline{s}} \frac{1}{s+r+\underline{s}} \\
\le & c_{21}\Big(\frac{L}{N}\Big)^{2\epsilon_1} \exp(-c_{13}(|x-a|^2 + d(a,\Lambda_L)^2)) \frac{1}{t^{\epsilon_1}}.
\end{aligned}
$$

The part of $\mathbb{E}[\|t^{\epsilon_0/2}Q_t\chi_a Y_{\xi,\varepsilon,L,N<}\|^p_{L^p(\mathbb{R}^2\times[0,1]:dxdt/t)}]$ dominated by using $\mathcal{I}_1(t,x)$ is dominated by

$$
\int_0^1 \frac{dt}{t} t^{\epsilon_0 p/2} \int_{\mathbb{R}^2} dx \mathcal{I}_1(t,x)^{p/2} \le c_{22}\Big(\frac{L}{N}\Big)^{\epsilon_1 p} \exp(-c_{23} d(a,\Lambda_L)^2).
$$

The other parts are also similarly estimated, and we obtain

$$
\mathbb{E}[\|\chi_a Y_{\xi,L,N<}\|^p_{\mathcal{B}^{-\epsilon_0}_{p,p}(\mathbb{R}^2)}] \le c_{24}(L/N)^{2-\epsilon_2} \exp(-c_{25} d(a,\Lambda_L)^2) + c_{26}\exp(-c d(a,\Lambda^c_{L-1})^2),
$$

where $\epsilon_2 \in (0,1)$ is taken arbitrarily small. Thus, we can take L_0 so that

$$
\sum_{a\in\mathbb{Z}^2} J_a < \frac{1}{2}
$$

for any $L \ge L_0$. □

For

$$\underline{\Phi^{\boldsymbol{s}}_{\xi,L,N<}}(u) := \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P^{s(a)}_u(\chi_a^2 \widetilde{\xi^L_{N<}}) + \sum_{a,a'\in\mathbb{Z}^2} \Delta^{-loc}{}_u P^{s_1(a,a')}_{\chi_a^2 \widetilde{\xi^L_{N<}}}(\Delta^{-loc}\chi_{a'}^2 \widetilde{\xi^L_{N<}})$$
$$+ \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P^{s_2(a)}_u(\chi_a^2 Y_{\xi,L,N<}),$$

we modify Lemma 4.1:

Lemma 5.4. *For any $\epsilon \in (0,1/2)$ and $\xi \in E(\varepsilon,\epsilon,\lambda,L)$, there exist $s(\epsilon,\xi,L)$, $s_1(\epsilon,\xi,L)$, $s_2(\epsilon,\xi,L) \in (0,1)$ and $M, M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ such that*

(5.8)
$$\begin{aligned}&\|\chi_a \underline{\Phi^{\boldsymbol{s}(\epsilon,\xi,L,\delta)}_{\xi,L,N<}}(u)\|_{\mathcal{H}^{1-2\epsilon}(\mathbb{R}^2)}\\ \le&\frac{\delta}{2}\sum_{a'\in\mathbb{Z}^2}(\exp(-Md(a',\Lambda_{L/2})^2)L^{-\epsilon} + \exp(-Md(a',\Lambda^c_{L/2})^2))\exp(-M|a-a'|^2)\|\chi_{a'}u\|_{L^2(\mathbb{R}^2)},\end{aligned}$$

for any $\delta \ge 0$, where $\boldsymbol{s}(\epsilon,\xi,L,\delta) = (s(a;\epsilon,\xi,L,\delta), s_1(a,a';\epsilon,\xi,L,\delta), s_2(a;\epsilon,\xi,L,\delta))_{a,a'\in\mathbb{Z}^2}$ is

$$s(a;\epsilon,\xi,L,\delta) = s(\epsilon,\xi,L)\delta^{M(\epsilon)}(1_{\Lambda_{L/2}}(a) + 1_{\Lambda^c_{L/2}}(a)|a|^\epsilon)^{-2M(\epsilon)},$$

$$\begin{aligned}s_1(a;a';\epsilon,\xi,L,\delta) = s_1(\epsilon,\xi,L)&\delta^{M_1(\epsilon)}(1_{\Lambda_{L/2}}(a) + 1_{\Lambda^c_{L/2}}(a)|a|^\epsilon)^{-2M_1(\epsilon)}\\ &\times (1_{\Lambda_{L/2}}(a') + 1_{\Lambda^c_{L/2}}(a')|a'|^\epsilon)^{-2M_1(\epsilon)}\end{aligned}$$

and

$$s_2(a;\epsilon,\xi,L,\delta) = s_2(\epsilon,\xi,L)\delta^{M_2(\epsilon)}(1_{\Lambda_{L/2}}(a) + 1_{\Lambda^c_{L/2}}(a)|a|^\epsilon)^{-M_2(\epsilon)}.$$

Under the event $E(\varepsilon,\epsilon,\lambda,L)$, we set $\widetilde{\varphi_{\varepsilon,R}} := (\Phi^{\boldsymbol{s}(\epsilon,\xi,L,\delta)}_{\xi,L,N<})^{-1}(\varphi_{\varepsilon,R})$. As in the proof of Theorem 1, we have the following:

Lemma 5.5. *For any $\epsilon \in (0,1/2)$, $\delta \in (0,1)$, $\lambda \in \mathbb{R}$ and $R \in \mathbb{N}$, there exists a positive finite constant $c(\epsilon,\delta,\lambda,R)$, and for these and any $\eta \in (0,1)$, there exists a positive finite constant $c(\eta,\epsilon,\delta,\lambda,R)$ satisfying the following: under the event $E(\varepsilon,\epsilon,\lambda,L)$, $\widetilde{\varphi_{\varepsilon,R}} \in \mathrm{Dom}_{2\epsilon}(\widetilde{H^\xi})$,*

(5.9)
$$\|\widetilde{\varphi_{\varepsilon,R}} - \varphi_{\varepsilon,R}\|_{L^2(\mathbb{R}^2)} \le c(\epsilon,\delta,\lambda,R)L^{-\epsilon}$$

and

(5.10)
$$\|(\widetilde{H^\xi} - \lambda)\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)} \le c(\eta,\epsilon,\delta,\lambda,R)\Big(\varepsilon + \Big(1 \vee \frac{N}{L}\Big)^\eta \frac{1}{L^\epsilon}\Big).$$

Proof. By Lemma 5.4, we have

$$\|\chi_a(\widetilde{\varphi_{\varepsilon,R}} - \varphi_{\varepsilon,R})\|_{\mathcal{H}^{1-2\epsilon}(\mathbb{R}^2)} \le \frac{c_1}{L^\epsilon}\exp(-c_2 d(a,\Lambda_R)). \tag{5.11}$$

(5.9) is a simple consequence of this. By (5.5), we have

$$\|\chi_a\varphi_{\varepsilon,R}\|_{\mathcal{H}^{1-2\epsilon}(\mathbb{R}^2)} \le c_3 1_{\Lambda_{R+2}}(a)\Big(1 \vee \frac{N}{L}\Big)^{\eta(1/2-\epsilon)} \tag{5.12}$$

and

$$\|\chi_a\widetilde{\varphi_{\varepsilon,R}}\|_{\mathcal{H}^{1-2\epsilon}(\mathbb{R}^2)} \le \frac{c_1}{L^\epsilon}\exp(-c_2 d(a,\Lambda_R)) + c_3 1_{\Lambda_{R+2}}(a)\Big(1 \vee \frac{N}{L}\Big)^{\eta(1/2-\epsilon)}. \tag{5.13}$$

We here note that

$$\|\chi_a\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)} \le \frac{c_1}{L^\epsilon}\exp(-c_2 d(a,\Lambda_R)) + 1_{\Lambda_{R+2}}(a), \tag{5.14}$$

since $\varphi_{\varepsilon,R}$ is normalized in $L^2(\mathbb{R}^2)$. By this, $\xi \in E(\varepsilon,\epsilon,\lambda,L)$, Lemma 3.6 and Lemma 5.4, we have

$$\begin{aligned}&\|\chi_a(\Phi_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}}) - \Phi^{\boldsymbol{s}(\epsilon,\xi,L,N,\delta)}_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}}))\|_{\mathcal{H}^2(\mathbb{R}^2)}\\ \le& c_4 L^{-\epsilon}\exp(-c_5 d(a,\Lambda_R)^2).\end{aligned} \tag{5.15}$$

By using also (5.5), we have

$$\begin{aligned}&\|\chi_a\Phi_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}})\|_{\mathcal{H}^2(\mathbb{R}^2)}\\ \le& c_4 L^{-\epsilon}\exp(-c_5 d(a,\Lambda_R)^2) + c_6 1_{\Lambda_{R+2}}(a)\Big(1 \vee \frac{N}{L}\Big)^{\eta}.\end{aligned} \tag{5.16}$$

Thus we have $\widetilde{\varphi_{\varepsilon,R}} \in \mathrm{Dom}_{2\epsilon}(\widetilde{H^\xi})$.

For (5.10), we estimate as

$$\begin{aligned}&\|(\widetilde{H^\xi} - \lambda)\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)}\\ \le&|\lambda|\|\widetilde{\varphi_{\varepsilon,R}} - \varphi_{\varepsilon,R}\|_{L^2(\mathbb{R}^2)} + \|(\lambda - (-\Delta + r(\lambda,L,N,R)\varphi^L_{\mathbf{0}} - Y^{L,N\ge}))\varphi_{\varepsilon,L}\|_{L^2(\mathbb{R}^2)}\\ &+ \|(-\Delta + r(\lambda,L,N,R)\varphi^L_{\mathbf{0}} - Y^{L,N\ge})\varphi_{\varepsilon,R} - \widetilde{H^\xi}\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

The first term of the right hand side is estimated by (5.9). The second term is estimated by (5.4) and (5.6). For the third term, we use Lemma 5.1 to estimate as

$$\begin{aligned}&\|(-\Delta + r(\lambda,L,N,R)\varphi^L_{\mathbf{0}} - Y^{L,N\ge})\varphi_{\varepsilon,R} - \widetilde{H^\xi}\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)}\\ \le&\|\Delta\varphi_{\varepsilon,R} + \widetilde{H^{\xi,L,N<}}\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)}\\ &+ \|r(\lambda,L,N,R)\varphi^L_{\mathbf{0}}\varphi_{\varepsilon,R} - \widetilde{\chi_L}\xi^L_{N\ge}\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)}\\ &+ \|Y^{L,N\ge}(\varphi_{\varepsilon,R} - \widetilde{\varphi_{\varepsilon,R}})\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

By Lemma 5.2 and (5.11), we have

$$\|Y^{L,N\geq}(\varphi_{\varepsilon,R} - \widetilde{\varphi_{\varepsilon,R}})\|_{L^2(\mathbb{R}^2)} \leq c_7 (N/L)^{\eta} L^{-\epsilon}.$$

By $\xi \in E(\varepsilon, \epsilon, \lambda, L)$ and (5.9), we have

$$\|r(\lambda, L, N, R)\varphi_{\mathbf{0}}^L \varphi_{\varepsilon,R} - \widetilde{\chi_L \xi^L_{N\geq}} \widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)} \leq c_8 \Big(\frac{\varepsilon}{L} + \Big(1 \vee \frac{N}{L}\Big)^{\eta} \frac{1}{L^{\epsilon}} \Big).$$

Moreover we estimate each term of the right hand side of

$$\|\Delta \varphi_{\varepsilon,R} + \widetilde{H^{\xi,L,N<}} \widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)} \leq \sum_{j=1}^{16} I_j,$$

where

$$\begin{aligned}
I_1 =& \|\Delta \varphi_{\varepsilon,R} - \Delta \Phi_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}})\|_{L^2(\mathbb{R}^2)},\\
I_2 =& \|P_{\widetilde{\xi^L_{N<}}} \Phi_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}})\|_{L^2(\mathbb{R}^2)},\\
I_3 =& \|\Pi(\widetilde{\xi^L_{N<}}, \Phi_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}}))\|_{L^2(\mathbb{R}^2)},\\
I_4 =& \|P_1^{(b)}((P_1^{(b)}\widetilde{\xi^L_{N<}})(P_1^{(b)}\Phi_{\xi,L,N<}(\widetilde{\varphi_{\varepsilon,R}}))\|_{L^2(\mathbb{R}^2)},\\
I_5 =& \|e^{\Delta} P_{\widetilde{\varphi_{\varepsilon,R}}} \widetilde{\xi^L_{N<}}\|_{L^2(\mathbb{R}^2)},\\
I_6 =& \|e^{\Delta}{}_{\widetilde{\varphi_{\varepsilon,R}}} P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}})\|_{L^2(\mathbb{R}^2)},\\
I_7 =& \|e^{\Delta} P_{\widetilde{\varphi_{\varepsilon,R}}} Y_{\xi,L,N<}\|_{L^2(\mathbb{R}^2)},\\
I_8 =& \|C(\widetilde{\varphi_{\varepsilon,R}}, \widetilde{\xi^L_{N<}}, \widetilde{\xi^L_{N<}})\|_{L^2(\mathbb{R}^2)},\\
I_9 =& \|S(\widetilde{\varphi_{\varepsilon,R}}, \widetilde{\xi^L_{N<}}, \widetilde{\xi^L_{N<}})\|_{L^2(\mathbb{R}^2)},\\
I_{10} =& \|P_{Y_{\xi,L,N<}} \widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)},\\
I_{11} =& \|\Pi(Y_{\xi,L,N<}, \widetilde{\varphi_{\varepsilon,R}})\|_{L^2(\mathbb{R}^2)},\\
I_{12} =& \|P_1^{(b)}((P_1^{(b)} Y_{\xi,L,N<})(P_1^{(b)}\widetilde{\varphi_{\varepsilon,R}}))\|_{L^2(\mathbb{R}^2)},\\
I_{13} =& \|P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}{}_{\widetilde{\varphi_{\varepsilon,R}}} P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}))\|_{L^2(\mathbb{R}^2)},\\
I_{14} =& \|\Pi(\widetilde{\xi^L_{N<}}, \Delta^{-loc}{}_{\widetilde{\varphi_{\varepsilon,R}}} P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}))\|_{L^2(\mathbb{R}^2)},\\
I_{15} =& \|P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc} P_{\widetilde{\varphi_{\varepsilon,R}}} Y_{\xi,L,N<})\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

and

$$I_{16} = \|\Pi(\widetilde{\xi^L_{N<}}, \Delta^{-loc} P_{\widetilde{\varphi_{\varepsilon,R}}} Y_{\xi,L,N<})\|_{L^2(\mathbb{R}^2)}.$$

By (5.15), we have

$$I_1 \leq c_9 L^{-\epsilon}.$$

By $\xi \in E(\varepsilon, \epsilon, \lambda, L)$, (5.16), (5.13), Lemma 3.2 and Lemma 3.4, we have

$$I_2, I_3, I_8, I_9, I_{10}, I_{11}, I_{12} \leq c_{10}\Big(1 \vee \frac{N}{L}\Big)^{\eta} \frac{1}{L^{\epsilon}}.$$

By $\xi \in E(\varepsilon, \epsilon, \lambda, L)$, (5.14), Lemma 3.1 and Lemma 3.2, we have

$$I_4, I_5, I_6, I_7, I_{13}, I_{14}, I_{15}, I_{16} \leq c_{11} L^{-\epsilon}.$$

□

Proof of Theorem 2. For any $x_0 \in \mathbb{Z}^2$, $\varepsilon \in (0, 1)$, $\epsilon \in (0, 1/2)$, $\lambda \in \mathbb{R}$ and $L \in \mathbb{N}$, we set

$$E(x_0, \varepsilon, \epsilon, \lambda, L) := \{\xi : \xi(\cdot - x_0) \in E(\varepsilon, \epsilon, \lambda, L)\}.$$

Then $\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, \epsilon, \lambda, L)$ is $\mathbb{Z}^2$-invariant. Thus by Lemma 5.3 and the ergodicity of the white noise, we have

$$\mathbb{P}\Big(\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, \epsilon, \lambda, L)\Big) = 1.$$

For any $x_0 \in \mathbb{Z}^2$ and $\xi \in E(x_0, \varepsilon, \epsilon, \lambda, L)$, we define

$$\widetilde{\varphi_{\varepsilon,R,x_0}}(x) := \widetilde{\varphi_{\varepsilon,R}}(x; \xi(\cdot - x_0))$$

for any $x \in \mathbb{R}^2$, where $\widetilde{\varphi_{\varepsilon,R}}(\cdot; \xi)$ is the function $\widetilde{\varphi_{\varepsilon,R}}(\cdot)$ used in Lemma 5.5 whose dependence on ξ is denoted. Then we have $\widetilde{\varphi_{\varepsilon,R,x_0}}(\cdot + x_0) \in \mathrm{Dom}_{2\epsilon}(\widetilde{H^{\xi}})$ and

$$\|(\widetilde{H^{\xi}} - \lambda)\widetilde{\varphi_{\varepsilon,R,x_0}}(\cdot + x_0)\|_{L^2(\mathbb{R}^2)} \leq c(\eta, \epsilon, \delta, \lambda, R)\Big(\varepsilon + \Big(1 \vee \frac{N}{L}\Big)^{\eta} \frac{1}{L^{\epsilon}}\Big).$$

In this estimate, ε and η are taken arbitrarily small, and L is taken arbitrarily large. Thus by Weyl's criterion (cf. Hislop and Sigal [14], Theorem 5.10), λ belongs to the spectral set of $\widetilde{\widetilde{H^{\xi}}}$. □

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