Quantum behavior of the integrated density of states for the uniform magnetic field and a randomly perturbed lattice

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Abstract. – For the Schrödinger operators on $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$ with the uniform magnetic field and the scalar potentials located at all sites of a randomly perturbed lattice, the asymptotic behavior of the integrated density of states at the infimum of the spectrum is investigated. The randomly perturbed lattice is the model considered by Fukushima and this describes an intermediate situation between the ordered lattice and the Poisson random field. In this paper the scalar potentials are assumed to decay rapidly and the effect of the kinetic part are investigated.

1. Introduction

Let

$$\mathcal{H} = \Big(i\frac{\partial}{\partial x_1} - \frac{Bx_2}{2}\Big)^2 + \Big(i\frac{\partial}{\partial x_2} + \frac{Bx_1}{2}\Big)^2 - B$$

be the Landau Hamiltonian on $L^2(\mathbb{R}^2)$ with the uniform magnetic field $B > 0$ subtracted B so that the lowest eigenvalue is 0, where $i = \sqrt{-1}$. Let $V_\xi(x) = \sum_{q\in\mathbb{Z}^2} u(x - q - \xi_q)$ be a random potential on $\mathbb{R}^2$, where $\xi = (\xi_q)_{q\in\mathbb{Z}^2}$ is a collection of independently and identically distributed $\mathbb{R}^2$-valued random variables with the distribution

$$P_\theta(\xi_q \in dx) = \exp(-|x|^\theta)dx/Z(\theta), \tag{1.1}$$

$\theta \in (0,\infty)$, $Z(\theta)$ is the normalizing constant, and u is a nonnegative function belonging to the Kato class K_2 (cf. [2] p-53). We will consider the random Schrödinger operators

$$H_\xi = \mathcal{H} + V_\xi, \tag{1.2}$$

and the restriction $\mathcal{H}_\xi$ of $\mathcal{H}$ to the complement of $T(\xi) = \overline{\bigcup_{q\in\mathbb{Z}^2} B(q + \xi_q : r_0)}$ by the Dirichlet boundary condition, where $r_0 \in (0,\infty)$ and $B(p : r) := \{x \in \mathbb{R}^2 : |x - p| < r\}$ is the open ball with the center p

and the radius r. V_ξ and $T(\xi)$ are soft and hard obstacles, respectively. We will consider the integrated density of states $(N(\lambda))_{\lambda\geq 0}$ and $(\mathcal{N}(\lambda))_{\lambda\geq 0}$ of H_ξ and $\mathcal{H}_\xi$, respectively: $(N(\lambda))_{\lambda\geq 0}$ is defined by

$$R^{-2}N_{\xi,R}(\lambda) \longrightarrow N(\lambda) \qquad \text{as } R\to\infty \tag{1.3}$$

for any point of continuity of $N(\lambda)$ and almost all ξ, where $N_{\xi,R}(\lambda)$ is the number of eigenvalues not exceeding λ of the self-adjoint operator H_{ξ,Λ_R} on the L^2 space on the cube $\Lambda_R := (-R/2,R/2)^2$ with the Dirichlet boundary condition. $N(\lambda)$ exists as a deterministic increasing function (cf. [2], [13]). $(\mathcal{N}(\lambda))_{\lambda\geq 0}$ is similarly defined for $\mathcal{H}_\xi$.

In this note, we first prove the following:

Theorem 1. (i) *If*

$$\operatorname*{essinf}_{|x|\leq R} u(x) > 0 \tag{1.4}$$

for some $R>0$ and

$$\varlimsup_{|x|\to\infty} |x|^{-2}\log u(x) = -\infty, \tag{1.5}$$

then we have

$$\varliminf_{\lambda\downarrow 0}\Big(\log\frac{1}{\lambda}\Big)^{-(1+\theta/2)}\log N(\lambda) \geq \frac{-2^{2+\theta/2}\pi}{(\theta+1)(\theta+2)B^{1+\theta/2}} \tag{1.6}$$

and

$$\varlimsup_{\lambda\downarrow 0}\Big(\log\frac{1}{\lambda}\Big)^{-1}\Big(\log\log\frac{1}{\lambda}\Big)^{-n}\log N(\lambda) = -\infty \tag{1.7}$$

for any $n\in\mathbb{N}$. Moreover if $\theta>4$, then we have

$$\varlimsup_{\lambda\downarrow 0}\Big(\log\frac{1}{\lambda}\Big)^{-(1+(\theta-4)/6)}\log N(\lambda) \leq \frac{-K}{B^{1+(\theta-4)/6}}, \tag{1.8}$$

where K is a finite constant independent of B. The same estimates hold for $(\mathcal{N}(\lambda))_{\lambda\geq 0}$.

(ii) *If* (1.4) *for any $R\geq 1$ and*

$$u(x) = \exp\Big(\frac{-|x|^\alpha}{C_0}(1+o(1))\Big) \tag{1.9}$$

as $|x|\to\infty$ with $\alpha=2$, then we have

$$\varliminf_{\lambda\downarrow 0}\Big(\log\frac{1}{\lambda}\Big)^{-(1+\theta/2)}\log N(\lambda) \geq \frac{-2\pi}{(\theta+1)(\theta+2)}\Big(\frac{2}{B}+C_0\Big)^{1+\theta/2}. \tag{1.10}$$

We next formulate the 3-dimensional problem by referring to the corresponding result [9] for the Poisson case. We write any element $\boldsymbol{x}$ of $\mathbb{R}^3$ as $(x_\perp, x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and set

$$\|\boldsymbol{x}\|_p^{\boldsymbol{\theta}} := \begin{cases} \||x_\perp|^{\theta_\perp}, |x_3|^{\theta_3}\|_p = (|x_\perp|^{\theta_\perp p} + |x_3|^{\theta_3 p})^{1/p} \text{ if } p \in [1,\infty), \\ |x_\perp|^{\theta_\perp} \vee |x_3|^{\theta_3} \text{ if } p = \infty, \end{cases}$$

for arbitrarily fixed $\boldsymbol{\theta} = (\theta_\perp, \theta_3) \in (0,\infty)^2$ and $p \in [1,\infty]$. Let $\boldsymbol{V}_\xi(\boldsymbol{x}) = \sum_{\boldsymbol{q}\in\mathbb{Z}^3} \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}_{\boldsymbol{q}})$ be a random potential on $\mathbb{R}^3$, where $\boldsymbol{u}$ is a nonnegative function belonging to the Kato class K_3 (cf. [2] p-53), $\boldsymbol{\xi} = (\boldsymbol{\xi}_{\boldsymbol{q}})_{\boldsymbol{q}\in\mathbb{Z}^3}$ is a collection of independently and identically distributed $\mathbb{R}^3$-valued random variables with the distribution

$$P_{\boldsymbol{\theta}}(\boldsymbol{\xi}_{\boldsymbol{q}} \in d\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_p^{\boldsymbol{\theta}}) d\boldsymbol{x}/Z(\boldsymbol{\theta}, p) \tag{1.11}$$

and $Z(\boldsymbol{\theta}, p)$ is the normalizing constant. Let

$$\boldsymbol{\mathcal{H}} = \Big(i\frac{\partial}{\partial x_1} - \frac{Bx_2}{2}\Big)^2 + \Big(i\frac{\partial}{\partial x_2} + \frac{Bx_1}{2}\Big)^2 - B - \frac{\partial^2}{\partial x_3^2}$$

be the direct sum of the Landau Hamiltonian on $L^2(\mathbb{R}^2)$ subtracted the magnetic field B and the Laplacian on $L^2(\mathbb{R})$. As in the 2-dimensional case and as in [20], we will consider the integrated density of states $(\boldsymbol{N}(\lambda))_{\lambda\geq 0}$ of the random Schrödinger operator

$$\boldsymbol{H}_{\boldsymbol{\xi}} = \boldsymbol{\mathcal{H}} + \boldsymbol{V}_{\boldsymbol{\xi}}, \tag{1.12}$$

and that $(\boldsymbol{\mathcal{N}}(\lambda))_{\lambda\geq 0}$ of the restriction $\boldsymbol{\mathcal{H}}_{\boldsymbol{\xi}}$ of $\boldsymbol{\mathcal{H}}$ to the complement of $\boldsymbol{T}(\boldsymbol{\xi}) = \overline{\bigcup_{\boldsymbol{q}\in\mathbb{Z}^3} \boldsymbol{B}(\boldsymbol{q} + \boldsymbol{\xi}_{\boldsymbol{q}} : r_0)}$ by the Dirichlet boundary condition, where $r_0 \in (0,\infty)$ and $\boldsymbol{B}(\boldsymbol{p} : r) := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x} - \boldsymbol{p}| < r\}$ is the open ball with the center $\boldsymbol{p} \in \mathbb{R}^3$ and the radius r.

For this we prove the following:

Theorem 2. *We assume*

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{C_0}{\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}}}(1 + o(1)) \tag{1.13}$$

as $|\boldsymbol{x}| \to \infty$ *for some* $C_0 \in (0,\infty)$, $\tilde{p} \in [1,\infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0,\infty)^2$ *satisfying*

$$\frac{2}{\alpha_\perp} + \frac{3}{\alpha_3} < 1. \tag{1.14}$$

We set

$$\mu_1(\alpha_\perp, \boldsymbol{\theta}) = \frac{3}{\alpha_\perp - 2} + \frac{1}{2} + \frac{3\theta_\perp}{2(\alpha_\perp - 2)} \vee \frac{\theta_3}{2} \tag{1.15}$$

and

$$\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta}) = \frac{2/\alpha_\perp}{1-1/\alpha_3-2/\alpha_\perp} + \frac{1}{2} + \frac{\theta_\perp/\alpha_\perp}{1-1/\alpha_3-2/\alpha_\perp} \wedge \frac{\theta_3}{2}. \tag{1.16}$$

Then we have

$$\underline{\lim}_{\lambda\downarrow 0}\lambda^{\mu_1(\alpha_\perp,\boldsymbol{\theta})}\log \boldsymbol{N}(\lambda) > -\infty, \tag{1.17}$$

$$\overline{\lim}_{\lambda\downarrow 0}\lambda^{\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})}\log \boldsymbol{N}(\lambda) < 0\ , \tag{1.18}$$

$$\underline{\lim}_{\lambda\downarrow 0}\tfrac{\lambda^{(1+\theta_3)/2}}{\log(1/\lambda)}\log \boldsymbol{\mathcal{N}}(\lambda) > -\infty \tag{1.19}$$

and

$$\overline{\lim}_{\lambda\downarrow 0}\sqrt{\lambda}\log \boldsymbol{\mathcal{N}}(\lambda) < 0. \tag{1.20}$$

If $\operatorname{supp}\boldsymbol{u}$ *is compact, then* (1.19) *and* (1.20) *hold by replacing* $\boldsymbol{\mathcal{N}}(\lambda)$ *by* $\boldsymbol{N}(\lambda)$.

As in [20], the above results are extensions of the results in [7] and [8], where the same problem is considered in the case without magnetic fields. As is discussed in [7] and [8], our model describes an intermediate situation between a completely ordered situation and a completely disordered situation since the point process $\{q+\xi_q\}_{q\in\mathbb{Z}^2}$ converges weakly to the Poisson point process with the intensity 1 as $\theta\to 0$ and converges weakly to the lattice $\mathbb{Z}^2$ as $\theta\to\infty$ by slightly modifying the definition as $P_\theta(\xi_q\in dx) = \exp(-(1+|x|)^\theta)dx/Z(\theta)$, which brings no essential changes for our results. The results in [7] and [8] shows that the leading term of the integrated density of states also tends to those for the Poisson case as $\theta\to 0$ and decays as $\theta\to\infty$ which reflects that the infimum of the spectrum is strictly positive if the perturbations $\{\xi_q\}$ of sites are all bounded. In the case with uniform magnetic fields the asymptotics of the integrated density of states has been investigated mainly for the Poisson case. For this topic and the relation with other topics, refer to a recent survey by Kirsch and Metzger [14]. The first result was given by Broderix, Hundertmark, Kirsch and Leschke [1]: they determined the leading term for the case where $d=2$, $u(x) = C_0|x|^{-\alpha}(1+o(1))$ as $|x|\to\infty$ is satisfied for some $\alpha>2$ and $C_0>0$ and the point process $\{q+\xi_q\}_{q\in\mathbb{Z}^2}$ is replaced by the Poisson point process. As is discussed in [10] and [20], this leading term coincides with that of the classical integrated density of states, which depends only on the scalar potential, as in Pastur's case [17] without magnetic fields. Then Erdős [5] treated the same case where the single site potential u is replaced by a function with a compact support and he determined the corresponding leading term of the integrated density of states, which depends only on the magnetic field and the intensity of the point process and is independent of other precise informations on the single site potential as in Nakao's case [15] without magnetic field referring to Donsker and Varadhan's result [3]. On this behavior we may say that the quantum effect appears. The borderline between the classical and quantum behaviors was determined by Hupfer, Leschke and Warzel [10]. The borderline corresponds to the case of (1.9) with $\alpha = 2$. They also determined the leading term for the case of (1.9) with $\alpha \in (0, 2)$. The leading term for the borderline case was determined by Erdős [6]. The leading term for the classical case was determined also in the 3-dimensional case by Hundertmark, Kirsch and Warzel [9]. For the 3-dimensional case, results appearing the quantum effect were obtained by Warzel [21], where general bounds and the leading order for special cases were obtained. In this note we try to extend the theory to our setting. We treat simple classical cases in [20] and remaining cases in this note. Our results in this note give only upper and lower estimates. By these upper estimates and Theorems 6.1 and 6.2 in [20], we see that the quantum effect appears in the following five cases: (i) (1.4) for any $R \geq 1$ and (1.9) with $\alpha \geq \theta + 2$, (ii) (1.4) for any $R \geq 1$ and (1.9) with $\alpha > 6$ and $\theta > 4$, (iii) $\operatorname{supp} u$ is a nonempty compact set, (iv) $\operatorname{essinf}_{|\boldsymbol{x}| \leq R} \boldsymbol{u}(\boldsymbol{x}) > 0$ for any $R \geq 1$ and (1.13) with (1.14), and (v) $\operatorname{supp} \boldsymbol{u}$ is a nonempty compact set. We conjecture that the leading terms are close to our lower bounds in the 2-dimensional cases and are close to our upper bounds in the 3-dimensional case. One reason is that the bounds tend to the corresponding leading terms given in [5], [6] and [14] for the Poisson case as $\theta \to 0$. Thus the borderline between the classical and quantum behaviors is expected to be the case of Theorem 1 (ii) and the case of $2/\alpha_\perp + 3/\alpha_3 = 1$ in Theorem 2 as in the Poisson case.

The organization of this note is as follows. We prove Theorem 1 in Sections 2, 3 and 4: we prove the lower estimates in Section 2, the upper estimate (1.8) in Section 3 and the upper estimate (1.7) in Section 4. We next prove Theorem 2 in Sections 5 and 6: we prove the lower estimate in Section 5 and the upper estimate in Section 6.

2. Lower estimates for the 2-dimensional case

In this section we give lower estimates for Theorem 1. Let

$$\widetilde{N}(t) = \int_0^\infty e^{-t\lambda} dN(\lambda).$$

(1.10) is proven by the following, which we prove by referring to [1] and [10]:

Proposition 2.1. *If* (1.9) *holds with* $\alpha = 2$*, then we have*

$$\varliminf_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{(\log t)^{1+\theta/2}} \geq \frac{-2\pi}{(\theta+1)(\theta+2)}\Big(\frac{2}{B} + C_0\Big)^{1+\theta/2}. \tag{2.1}$$

The basic inequality for the proof is the following extension of (3.5) in [1]:

Proposition 2.2. $\widetilde{N}(t) \geq \widetilde{N}_1(t)/(4\pi t)$*, where*

$$\widetilde{N}_1(t) = \int_{\Lambda_1} dx\ E_\theta\bigg[\exp\Big(-t\int_{\mathbb{R}^2}\phi_B(y)V_\xi(x+y)dy\Big)\bigg]$$

and $\phi_B(x) = \exp(-B|x|^2/2)B/(2\pi)$.

Proof. In (2.4) of [20] we obtain

$$\lim_{\Omega\downarrow 0}\frac{\Omega^2 t}{\pi}E_\theta[\mathrm{Tr}[\exp(-t(H_\xi + \Omega^2|x|^2))]] = \widetilde{N}(t). \tag{2.2}$$

Then the rest of the proof is same with that of (3.5) in [1]. □

Proof of Proposition 2.1. $\widetilde{N}_1(t)$ in Proposition 2.2 is rewritten as

$$\int_{\Lambda_1} dx E_\theta\bigg[\exp\Big(-t\sum_{q\in\mathbb{Z}^2}u_B(x-q-\xi_q)\Big)\bigg],$$

where $u_B = u * \phi_B$. By Lemma 3.5 (ii) in [10], we have

$$\lim_{|x|\to\infty}|x|^{-2}\log u_B(x) = -1/(C_0 + 2/B) =: -1/C_B.$$

By the same lower estimate of Section 3 in [20], we obtain

$$\log \widetilde{N}_1(t) \geq -t\exp(-(1-\varepsilon)^4R^2/C_B) - (R+3\varepsilon+1)^{\theta+2}\frac{2\pi}{(\theta+1)(\theta+2)}$$

for large enough R. By setting $R = \sqrt{C_B(\log t)}(1-\varepsilon)^{-2}$, we can complete the proof. □

Proof of "(1.10) *implies* (1.6)*".* For any $\eta > 0$, we can take a single site potential u_η satisfying $u_\eta \geq u$ and (1.9) where α and C_0 is replaced by 2 and η, respectively. The corresponding integrated density of states $N_\eta(\lambda)$ satisfies $N(\lambda) \geq N_\eta(\lambda)$ and

$$\varliminf_{\lambda\downarrow 0}\Big(\log\frac{1}{\lambda}\Big)^{-(1+\theta/2)}\log N_\eta(\lambda) \geq \frac{-2\pi}{(\theta+1)(\theta+2)}\Big(\frac{2}{B}+\eta\Big)^{1+\theta/2}.$$

Since η is arbitrary, we obtain (1.6). □

3. An upper estimate appearing quantum effects for the 2-dimensional case

In this section we prove the following upper estimate which is enough for (1.8):

Proposition 3.1. *If $\theta > 4$ and $u(x) = C_0 1_{B(r_0)}(x)$, then we have*

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{(\log t)^{(2+\theta)/6}} \leq \frac{-C_1}{B^{(2+\theta)/6}} \tag{3.1}$$

for some positive constant C_1 independent of B, where $B(r_0) := B(0 : r_0)$.

We reduce the proof to an estimate of the lowest eigenvalue of an operator with the Dirichlet boundary condition following Section 6 in Erdős [5]: for any $\beta, \varepsilon_1 \in (0, \infty)$,

$$\varlimsup_{t\uparrow\infty} \frac{\log \widetilde{N}(t)}{(\log t)^{(2+\theta)/6}} \leq \varlimsup_{t\uparrow\infty} \frac{1}{(\log t)^{(2+\theta)/6}} \log \int_{\Lambda_1} dz E_\theta \Big[\exp\Big(\frac{\varepsilon_1 - t}{4} \lambda_1(H^{B+\beta}_{\xi,B(z:s)}) \Big) \Big]. \tag{3.2}$$

In (3.2), $s = \sqrt{(8/\beta)\log t}$ and $H^{B+\beta}_{\xi,B(z:s)}$ is $H_{\xi,B(z:s)}$ where B is replaced by $B+\beta$. The proof of (3.2) can be given by the same method of the proof of Theorem 6.3 in Erdős [5], where we used the estimate

$$\mathrm{Tr}\Big[\exp\Big(-\frac{t}{4} H^{B+\beta}_{\xi,B(z:s)} \Big) \Big] \leq \exp\Big(\frac{\varepsilon_1 - t}{4} \lambda_1(H^{B+\beta}_{\xi,B(z:s)}) \Big) \frac{cs^2}{\varepsilon_1} \exp\Big(\frac{\varepsilon_1(B+\beta)}{4} \Big)$$

instead of (3.4) in [5].

The obstacles are reduced to the hard obstacles by the following:

Lemma 3.1. *If $u(x) = C_0 1_{B(2r_0)}(x)$, then we have*

$$\lambda_1(H_{\xi,B(z:s)}) \geq \frac{C_0}{2} \wedge \frac{\lambda_1(\mathcal{H}_{B(z:s)\setminus T(\xi)})}{4(1 + cr_0^{-2})}. \tag{3.3}$$

Proof. We represent the Landau Hamiltonian by the creation and annihilation operators: $\mathcal{H} = \mathcal{A}^*\mathcal{A}$, where $\mathcal{A} = (i\partial/(\partial x_1) - Bx_2/2) + i(i\partial/(\partial x_2) + Bx_1/2)$. Thus the lowest eigenvalue has the representaion

$$\lambda_1(H_{\xi,B(z:s)}) = \|\mathcal{A}\varphi_0\|^2 + (V_\xi\varphi_0, \varphi_0),$$

where φ_0 is a normalized ground state of $H_{\xi,B(z:s)}$. Since

$$(V_\xi\varphi_0, \varphi_0) \geq C_0 \|\varphi_0\|^2_{L^2(B(z:s)\cap\bigcup_{q\in\mathbb{Z}^2} B(q+\xi_q:2r_0))},$$

we have

$$\|\varphi_0\|^2_{L^2(B(z:s)\cap\bigcup_{q\in\mathbb{Z}^2} B(q+\xi_q:2r_0))} \leq \lambda_1(H_{\xi,B(z:s)})/C_0.$$

On the other hand, we take a smooth function ϑ on $\mathbb{R}^2$ such that $\vartheta = 0$ on $\bigcup_{q\in\mathbb{Z}^2} B(q+\xi_q : r_0)$, $\vartheta = 1$ on $(\bigcup_{q\in\mathbb{Z}^2} B(q+\xi_q : 3r_0/2))^c$, $0 \le \vartheta \le 1$ and $|\nabla\vartheta|^2 \le 9/r_0^2$. Then we have

$$\begin{aligned}\|\mathcal{A}\varphi_0\|^2 &\ge \|\vartheta\mathcal{A}\varphi_0\|^2 \ge \frac{1}{2}\|\mathcal{A}\vartheta\varphi_0\|^2 - \frac{9}{r_0^2}\|\varphi_0\|^2_{L^2(B(z:s)\cap\bigcup_{q\in\mathbb{Z}^2} B(q+\xi_q:3r_0/2))} \\ &\ge \frac{1}{2}\lambda_1(\mathcal{H}_{B(z:s)\setminus T(\xi)})\|\vartheta\varphi_0\|^2 - \frac{9}{r_0^2}\frac{\lambda_1(H_{\xi,B(z:s)})}{C_0}.\end{aligned}$$

Moreover we have

$$\|\vartheta\varphi_0\|^2 \ge 1 - \int_{B(z:s)\cap\bigcup_{q\in\mathbb{Z}^2} B(q+\xi_q:3r_0/2)} |\varphi_0(x)|^2 dx \ge \Big(1 - \frac{\lambda_1(H_{\xi,B(z:s)})}{C_0}\Big)_+.$$

Therefore, if $1 - \lambda_1(H_{\xi,B(z:s)})/C_0 \ge 1/2$, then we have

$$\lambda_1(H_{\xi,B(z:s)}) \ge \frac{1}{4}\lambda_1(\mathcal{H}_{B(z:s)\setminus T(\xi)}) - \frac{9}{r_0^2}\frac{\lambda_1(H_{\xi,B(z:s)})}{C_0},$$

from which we obtain (3.3). □

To estimate $\lambda_1(\mathcal{H}_{B(z:s)\setminus T(\xi)})$, we develop Erdős's isoperimetric inequality stating

$$\lambda_1(\mathcal{H}_D) \ge \lambda_1(\mathcal{H}_{B(L)}) \tag{3.4}$$

for any bounded domain D of $\mathbb{R}^2$ with the area $|D| = |B(L)| = \pi L^2$ and a smooth boundry [4]. The right hand side is dominated from below by $\exp(-BL^2(1+\varepsilon)/2)$ for sufficiently large L, where ε is an arbitrarily small positive constant. We need more precise estimates for more complicated domains. In this note we prove the following:

Proposition 3.2. *Let D be a bounded domain of $\mathbb{R}^2$ with the area $|D| = \pi L^2$ whose boundary is a finite union of smooth curves. Let ρ be the radius of the largest disk contained in D. Then we have*

$$\lambda_1(\mathcal{H}_D) \ge \frac{C_1}{L^2}\exp(-C_2 BL(1\vee\rho)^3), \tag{3.5}$$

where C_1 and C_2 are universal positive constants.

Proof. By the same proof of the inequality (17) in [4], we have

$$\begin{aligned}\lambda_1(\mathcal{H}_D) \ge \inf\Big\{ &\int |e^{-\Lambda(\psi(x))}\nabla e^{\Lambda(\psi(x))}\psi(x)|^2 dx \Big/ \int \psi(x)^2 dx \\ &: 0 \le \psi \in C^\infty(\overline{D}) \cap C^\omega(D), \psi = 0 \text{ on } \partial D, \\ &\Lambda(c) = B\int_c^\infty d\xi |\{\psi > \xi\}| \Big(\int_{\{\psi=\xi\}} |\nabla\psi(x)| L(dx) \Big)^{-1} \Big\},\end{aligned}$$

where $C^\omega(D)$ is the set of all analytic functions on D and $L(\cdot)$ is the measure corresponding to the length of curves. Λ is a nonnegative, strictly monotone decreasing, continuous function on the range of ψ. It is also included in the set C_*^∞ of all real valued functions on $\mathbb{R}$ which are smooth everywhere except for finitely many points. As in the equations (19) and (20) in [4], we rewrite the quatities in the infimum as follows:

$$\int |e^{-\Lambda(\psi(x))}\nabla e^{\Lambda(\psi(x))}\psi(x)|^2 dx = B\int_0^\infty (\Theta'(b))^2 e^{-2b}F(b)db$$

and

$$\int \psi(x)^2 dx = \int_0^\infty \Theta(b)^2 e^{-2b}F'(b)db,$$

where $\Theta(b) = \Lambda^{-1}(b)e^b$ and $F(b) = |\{x \in D : h(x) < b\}|$ with $h(x) = \Lambda(\psi(x))$. We now apply Lemma 3.2 below. Then we have

$$L(\{x : h(x) = b\}) \geq C_1(1\vee\rho)\sqrt{|\{x : h(x) < b\}|\Big(1 - \Big(\frac{\sqrt{|\{x : h(x) < b\}|} - \rho\sqrt{\pi}}{\sqrt{|\{x : h(x) < b\}|} + \rho\sqrt{\pi}}\Big)_+^2\Big)^{-1}}$$

and

$$F'(b) \geq \frac{C_2(1\vee\rho)^2}{B\Big(1 - \Big(\frac{\sqrt{F(b)}-\rho\sqrt{\pi}}{\sqrt{F(b)}+\rho\sqrt{\pi}}\Big)_+^2\Big)}.$$

Therefore (23) in [4] is rewritten as follows:

$$\begin{aligned}
&\lambda_1(\mathcal{H}_D)\\
&\geq \inf\Big\{B\int_0^{b_0} (\Theta'(b))^2 e^{-2b}F(b)db\Big(\int_0^{b_0} \Theta(b)^2 e^{-2b}F'(b)db\Big)^{-1}\\
&\quad : F, \Theta \in C_*^\infty, \Theta \geq 0, \Theta \text{ is bounded, } F \text{ is strictly monotone increasing,}\\
&\quad F'(b) \geq \frac{C_2(1\vee\rho)^2}{B\Big(1 - \Big(\frac{\sqrt{F(b)}-\rho\sqrt{\pi}}{\sqrt{F(b)}+\rho\sqrt{\pi}}\Big)_+^2\Big)} \text{ for a.a. } 0 < b < b_0,\\
&\quad F(0) = 0, F(b_0) = |D|, \Theta(b_0) = 0\Big\}.
\end{aligned}$$

As in [4], if we set $h^*(r) = F^{-1}(\pi r^2)$, $a(r) = (h^*)'(r)$ and $q(r) = \Theta(h^*(r))\exp(-h^*(r))$, then q, $q' \in L^2((0,L), rdr)$ and $q(L) = 0$. (25) and (26) in [4] also hold:

$$2\pi\int_0^L (q'(r) + a(r)q(r))^2 rdr \leq B\int (\Theta'(b))^2 e^{-2b}F(b)db$$

and

$$2\pi\int_0^L q(r)^2 rdr = \int (\Theta(b))^2 e^{-2b}F'(b)db.$$

(24) in [4] is rewritten as

$$0 \leq a(r) = (h^*)'(r) = \frac{2\pi r}{F'(F^{-1}(\pi r^2))} \leq C_3(1\vee\rho)^2 Br\Big(1 - \Big(\frac{r-\rho}{r+\rho}\Big)_+^2\Big) =: a_\rho(r).$$

Therefore we have

$$\lambda_1(\mathcal{H}_D) \geq \inf \Big\{ 2\pi \int_0^L (q'(r) + a(r)q(r))^2 r dr \Big(2\pi \int_0^L q(r)^2 r dr \Big)^{-1}$$
$$: q, q' \in L^2((0,L), rdr), q(L) = 0, q(r) \geq 0, 0 \leq a(r) \leq a_\rho(r) \Big\}.$$

By Lemma 3.1 in [4], we have

$$\lambda_1(\mathcal{H}_D) \geq \inf \Big\{ 2\pi \int_0^L (q'(r) + a_\rho(r)q(r))^2 r dr \Big(2\pi \int_0^L q(r)^2 r dr \Big)^{-1}$$
$$: q, q' \in L^2((0,L), rdr), q(L) = 0, q(r) \geq 0 \Big\}. \tag{3.6}$$

We rewrite q as $q(r) = \exp(-A_\rho(r))Q(r)$, where

$$A_\rho(r) = \int_0^r a_\rho(s) ds.$$

Since $A_\rho(r)$ is monotone increasing, by the uniform estimate, we have

$$\lambda_1(\mathcal{H}_D) \geq \inf \Big\{ 2\pi \int_0^L Q'(r)^2 \exp(-2A_\rho(r)) r dr \Big/ 2\pi \int_0^L Q(r)^2 \exp(-2A_\rho(r)) r dr$$
$$: Q, Q' \in L^2((0,L), rdr), Q(L) = 0, Q(r) \geq 0 \Big\}$$
$$\geq \lambda_1(-\Delta_{B(L)}) \exp(-2A_\rho(L)),$$

where $\lambda_1(-\Delta_{B(L)})$ is the lowest eigenvalue of the Dirichlet Laplacian of $B(L)$. By the scaling, we have $\lambda_1(-\Delta_{B(L)}) = \lambda_1(-\Delta_{B(1)})/L^2$. By a simple calculation, we have

$$A_\rho(r) \leq C_4 B(1 \vee \rho)^3 L. \tag{3.7}$$

Thus we obtain (3.5). □

The following is the estimate used in the proof of the last proposition:

Lemma 3.2. *Suppose that the domain D is defined by $B(R) \setminus \overline{\bigcup_{i\in\mathbb{N}} B(a_i : r_0)}$ for some $0 < r_0 < R < \infty$ and $\{a_i : i \in \mathbb{N}\} \subset \mathbb{R}^2$ and that ρ is the radius of the largest open disk contained in D. Then for any domain $\mathcal{D}$ contained in D such that its boundary $\partial\mathcal{D}$ is a finite union of rectifiable Jordan curves, we have*

$$(1 + c\rho)^2 \Big(1 - \Big(\frac{\sqrt{L(\partial\mathcal{D})} - \rho\sqrt{\pi}}{\sqrt{L(\partial\mathcal{D})} + \rho\sqrt{\pi}} \Big)^2 \Big) \geq \frac{4\pi|\mathcal{D}|}{L(\partial\mathcal{D})}^2, \tag{3.8}$$

where c is a finite constant depending only on r_0.

Proof. This estimate has its origin in the classical isoperimetric inequality stating $L(\partial\mathcal{D})^2 \geq 4\pi|\mathcal{D}|$ for any domain $\mathcal{D}$ in $\mathbb{R}^2$ bounded by a finite union of rectifiable Jordan curves. This inequality has been improved for more complicated domains. Many such inequalities are known as Bonnesen type isoperimetric inequalities (see Ossermann [16] and references therein). Among them we apply the following inequality by Ossermann: if $\mathcal{D}$ is a domain of $\mathbb{R}^2$ bounded by a rectifiable Jordan curve, then we have

$$L(\partial\mathcal{D})^2 - 4\pi|\mathcal{D}| \geq L(\partial\mathcal{D})^2\Big(\frac{\mathcal{R}-\rho}{\mathcal{R}+\rho}\Big)^2, \tag{3.9}$$

where $\mathcal{R}$ is the radius of the smallest disc including $\mathcal{D}$ and ρ is the radius of the largest disc included in $\mathcal{D}$ ((23) in Ossermann [16]). To apply (3.9), $\mathcal{D}$ should be simply connected. Now for any domain $\mathcal{D}$ in $D = B(R) \setminus \overline{\bigcup_{i\in\mathbb{N}} B(a_i : r_0)}$, we classify its holes to two groups: let $\{H_k\}_{k=1}^K$ and $\{\widehat{H_i}\}_{i\in\mathcal{I}}$ be simply connected closed domains such that

$$H_k \cap \overline{\bigcup_{i\in\mathbb{N}} B(a_i : r_0)} \neq \emptyset, \quad \widehat{H_i} \cap \overline{\bigcup_{i\in\mathbb{N}} B(a_i : r_0)} = \emptyset$$

and $\mathcal{D} + \bigcup_{k=1}^K H_k + \bigcup_{i\in\mathcal{I}} \widehat{H_i}$ is a simply connected domain. Therefore $\{H_k\}_{k=1}^K$ are holes intersecting the holes of D and $\{\widehat{H_i}\}_{i\in\mathcal{I}}$ are holes apart from the holes of D. We may erase the holes $\{\widehat{H_i}\}_{i\in\mathcal{I}}$ to replace $\mathcal{D}$ by the domain $\widehat{\mathcal{D}} := \mathcal{D} + \bigcup_{i\in\mathcal{I}} \widehat{H_i}$, since the radius of the largest disc included in $\widehat{\mathcal{D}}$ is still ρ. For the holes $\{H_k\}_{k=1}^K$, we have $K \leq L(\partial\mathcal{D})/(2\pi r_0)$, since $L(\partial\mathcal{D}) \geq 2\pi r_0$. By erasing K numbers of line segments $\{C_k\}_{k=1}^K$ of the length $L(C_k)$ less than or equal to 2ρ, the domain $\widehat{\mathcal{D}}$ becomes a disjoint union of finite number of simply connected domains $\{\mathcal{D}_j\}_{j=1}^J$: $\widehat{\mathcal{D}} \setminus \sum_{k=1}^K C_k = \sum_{j=1}^J \mathcal{D}_j$. Then Ossermann's inequality (3.9) gives

$$L(\partial\mathcal{D}_j) \geq \sqrt{4\pi|\mathcal{D}_j|\Big/\Big(1 - \Big(\frac{\sqrt{|\mathcal{D}_j|} - \rho\sqrt{\pi}}{\sqrt{|\mathcal{D}_j|} - \rho\sqrt{\pi}}\Big)\Big)}$$

by dominating also the radius of the smallest disc including $\mathcal{D}_j$ by $\sqrt{|\mathcal{D}_j|/\pi}$. By using Lemma 3.3 below, we have

$$L(\partial\widehat{\mathcal{D}}) + 2\sum_{k=1}^K L(c_k) \geq \sqrt{4\pi|\widehat{\mathcal{D}}|\Big/\Big(1 - \Big(\frac{\sqrt{|\widehat{\mathcal{D}}|} - \rho\sqrt{\pi}}{\sqrt{|\widehat{\mathcal{D}}|} - \rho\sqrt{\pi}}\Big)\Big)},$$

from which we easily obtain (3.8). $\square$

Lemma 3.3. *Let $H(F_1, \cdots, F_n)$ be a function of $F_1, \ldots, F_n \geq 0$ defined by*

$$H(F_1, \cdots, F_n) = \sum_{i=1}^n \left\{ \frac{F_i}{1 - \Big(\frac{\sqrt{F_i}-\beta}{\sqrt{F_i}+\beta}\Big)_+^2} \right\}^{1/2}$$

with $\beta \in (0,\infty)$*. Then we have*

$$H(F_1,\cdots,F_n) \geq H\Big(\sum_{i=1}^n F_i, 0,\cdots,0\Big).$$

Proof. Let $H_1(f) := H(F_1+f, F_2-f, F_3,\cdots,F_n)$. This function is increasing in small $f \geq 0$. By using also the symmetry in $(F_1,\cdots,F_n)$, we can complete the proof. $\square$

Finally we complete the proof of Proposition 3.1.

Proof of Proposition 3.1. We have only to show

$$\begin{aligned}(3.10)\qquad &\overline{\lim_{t\uparrow\infty}} \frac{1}{(\log t)^{(2+\theta)/6}} \log \int_{\Lambda_1} dz E_\theta\Big[\exp\Big(c_1 \frac{\varepsilon_1 - t}{s^2}\exp(-c_2(B+\beta)\rho_1(z,s,\xi)^3 s)\Big)\Big] \\ &\leq -K_1(\theta)/B^{(2+\theta)/6}\end{aligned}$$

for some β by (3.2), Lemma 3.1 and Proposition 3.2, where $\rho_1(z,s,\xi) := 1 \vee \rho(z,s,\xi)$, $\rho(z,s,\xi)$ is the radius of the largest disk contained in $B(z:s)\setminus T(\xi)$, c_1, $c_2 \in (0,\infty)$ and $K_1(\theta) \in (0,\infty)$ depends on θ. For this, we have only to show

$$(3.11)\qquad \overline{\lim_{t\uparrow\infty}} \frac{1}{(\log t)^{(2+\theta)/6}} \log \int_{\Lambda_1} dz \int_0^{s^{-2}} dR e^{c_3(\varepsilon_1 - t)R} P_\theta(\widehat{R} \leq \rho_1(z,s,\xi))$$

is less than or equal to the right hand side of (3.10), where $\widehat{R} = \{(\log(R^{-1}s^{-2}))/(c_2(B+\beta)s)\}^{1/3}$. The qunatity in (3.11) is dominated from above by

$$(3.12)\qquad -\underline{\lim_{t\uparrow\infty}} \frac{1}{(\log t)^{(2+\theta)/6}} \inf_{0<R\leq s^{-2}, z\in\Lambda_1} \{c_3(t-\varepsilon_1)R - \log P_\theta(\widehat{R} \leq \rho_1(z,s,\xi))\}$$

The probability can be estimated as follows:

$$\begin{aligned} &P_\theta(\widehat{R} \leq \rho_1(z,s,\xi)) \leq P_\theta(B(a:\widehat{R}) \subset B(s)\setminus T(\xi) \text{ for some } a \in B(s)) \\ \leq &\sum_{a\in B(s)\cap(\varepsilon_2\mathbb{Z}^2)} P_\theta(|\xi_q| \geq \widehat{R} + r_0 - |q-a| - \varepsilon_2 \text{ for } \forall q \in B(a:\widehat{R}-\varepsilon_2)\cap\mathbb{Z}^2) \\ \leq &\#\{B(s)\cap(\varepsilon_2\mathbb{Z}^2)\}\exp\Big(-(1-\varepsilon_3)\sum_{q\in B(a:\widehat{R}-\varepsilon_2)\cap\mathbb{Z}^2}(\widehat{R}+r_0-|q-a|-\varepsilon_2)_+^\theta + c_4\widehat{R}^2\Big) \\ \leq &c_5 s^2 \exp\Big(-(1-\varepsilon_3)\frac{2\pi(\widehat{R}-1-\varepsilon_2)_+^{\theta+2}}{(\theta+1)(\theta+2)} + c_4\widehat{R}^2\Big), \end{aligned}$$

where ε_2 and ε_3 are arbitrarily small positive constants, and c_4 and c_5 are finite constants depending ε_2 and ε_3. Thus the quantity (3.12) is less than or equal to

$$(3.13)\qquad -\underline{\lim_{t\uparrow\infty}} \frac{1}{(\log t)^{(2+\theta)/6}} \inf_{0<R\leq s^{-2}} \Big\{c_3(t-\varepsilon_1)R + c_6\Big(\Big(\frac{1}{s}\log\frac{1}{Rs^2}\Big)^{1/3} - c_7\Big)_+^{\theta+2}\Big\},$$

where $c_6 = (1-2\varepsilon_3)2\pi\{(\theta+1)(\theta+2)(c_2(B+\beta))^{(\theta+2)/3}\}^{-1}$ and $c_7 = (1+\varepsilon_2)(c_2(B+\beta))^{1/3}$. For an arbitrary positive number v, a sufficient condition for $((\log(R^{-1}s^{-2}))/s)^{1/3} - c_7 \geq vc_7$ is $R \leq \exp(-s(v+1)^3 c_7^3)/s^2$. Then we have

$$\Big(\frac{1}{s}\log\frac{1}{Rs^2}\Big)^{1/3} - c_7 \geq \frac{v}{v+1}\Big(\frac{1}{s}\log\frac{1}{Rs^2}\Big)^{1/3}$$

and

$$\begin{aligned}&\inf\Big\{c_3(t-\varepsilon_1)R + c_6\Big(\Big(\frac{1}{s}\log\frac{1}{Rs^2}\Big)^{1/3} - c_7\Big)_+^{\theta+2} : 0 < R \leq \exp(-s(v+1)^3c_7^3)/s^2\Big\}\\ &\geq \inf\{F(R) : 0 < R \leq \exp(-s(v+1)^3c_7^3)/s^2\},\end{aligned}$$

where

$$F(R) = c_3(t-\varepsilon_1)R + c_8\Big(\frac{1}{s}\log\frac{1}{Rs^2}\Big)^{(\theta+2)/3}$$

with $c_8 = c_6(v/(v+1))^{\theta+2}$. The infimum of $F(R)$ is attained at $R = R(t)$ satisfying

$$R(t) = \frac{c_8(\theta+2)}{3c_3(t-\varepsilon_1)s^{(\theta+2)/3}}\Big(\log\frac{1}{R(t)s^2}\Big)^{(\theta-1)/3}.$$

Since $R(t) = (1+o(1))(\log t)^{(\theta-4)/6}/t$ as $t\to\infty$, $R(t) \leq \exp(-s(v+1)^3c_7^3)/s^2$ and

$$\inf\{F(R) : 0 < R \leq \exp(-s(v+1)^3c_7^3)/s^2\} = F(R(t))$$

for sufficiently large t. It is easy to see that

$$\lim_{t\to\infty}\frac{F(R(t))}{(\log t)^{(2+\theta)/6}} = c_8\Big(\frac{\beta}{8}\Big)^{(2+\theta)/6}.$$

On the other hand, by only the effect of the first term, we have

$$\begin{aligned}&\inf\{c_3(t-\varepsilon_1)R + c_6\Big(\Big(\frac{1}{s}\log\frac{1}{Rs^2}\Big)^{1/3} - c_7\Big)_+^{\theta+2} : \exp(-s(v+1)^3c_7^3)/s^2 \leq R \leq s^{-2}\Big\}\\ &\geq c_3\frac{t-\varepsilon_1}{s^2}\exp(-s(v+1)^3c_7^3) \geq c_9\sqrt{t}.\end{aligned}$$

Thus the quantity (3.13) is less than or equal to $-c_8(\beta/8)^{(2+\theta)/6}$. By taking the limit $\varepsilon_3 \downarrow 0$ and $v\to\infty$, we see that the quantity (3.13) is less than or equal to

$$\frac{-2\pi}{(\theta+1)(\theta+2)}\Big(\frac{\beta}{8c_2^2(B+\beta)^2}\Big)^{(2+\theta)/6}.$$

This becomes the optimal value, the right hand side of (3.10), when $\beta = B$. $\square$

4. A general upper estimate for the 2-dimensional case

In this section we prove the following upper estimate which is enough for (1.7):

Proposition 4.1. *If $u(x) = C_0 1_{B(r_0)}(x)$ with $C_0, r_0 \in (0,\infty)$, then we have*

$$\overline{\lim_{t\uparrow\infty}} \frac{\log \widetilde{N}(t)}{(\log t)(\log\log t)^n} = -\infty \text{ for any } n \in \mathbb{N}. \tag{4.1}$$

To prove this proposition, we modify Erdős's upper estimate for the Poisson case [5]. For the estimate, he applied his method of enlargement of obstacles referring to Sznitman's theory (cf. [19]). As in [5], we assume $0 < r_0 < 1$. We fix β and $\varepsilon > 0$ arbitrarily, and take $s = \sqrt{(8/\beta)\log t}$, $\ell = 10\sqrt{(\log t)/B}$ and $b > 10r_0$ specified later. In his theory, the points $\{q+\xi_q\}$ corresponding to the centers of obstacles are classified to two groups: $\ell\Lambda_1(m) \ni q+\xi_q$ is defined to be "good" if

$$\begin{aligned} &|(\ell\Lambda_1(m)) \cap \overline{B(q+\xi_q : 10^{k+1}b)} \cap \bigcup_{q'+\xi_{q'}\in\ell\Lambda_1(m)} \overline{B(q'+\xi_{q'} : b)}| \\ &\geq \frac{\varepsilon}{9}|(\ell\Lambda_1(m)) \cap \overline{B(q+\xi_q : 10^{k+1}b)}| \end{aligned}$$

for any $k \in \mathbb{Z}_+ \cap [0, (\log(\ell/(2b)))/(\log 10)]$, and the other $\ell\Lambda_1(m) \ni q+\xi_q$ is defined to be "bad". Then bad obstacles $B(q+\xi_q : r_0)$ are erased and good obstacles $B(q+\xi_q : r_0)$ are enlarged to $B(q+\xi_q : b)$. In his theory, b is fixed to be a constant. We now take b as an increasing function of t. A sufficient condition for his theory to be generalized to this setting is

$$\lim_{t\to\infty} b^2 \left(\frac{\ell}{b}\right)^{-k(\varepsilon,b)} = 0, \tag{4.2}$$

where $k(\varepsilon, b) = (4\log 10)^{-1}\log\{(1-c(b)p(\varepsilon))^{-1}\}$, $c(b) = P(\inf\{t : r(t) = r_1/b\} < \inf\{t : r(t) = 6\}$, $r(t)$ is a 2-dimensional Bessel process starting at 2, $r_1 \in (0,1)$ and $p(\varepsilon)$ is a $[0,1]$-valued function decaying as $\varepsilon \downarrow 0$ (cf. Lemma 7.5 in [5]). Indeed the condition (4.2) is sufficient to obtain (7.23) in [5] from the last estimate in the proof of Lemma 7.5 (i) in [5]. By a result on 1-dimensional diffusion process, we can rewrite

$$c(b) = (\log 3)\Big/\log\frac{6b}{r_1}$$

(cf. Theorem VI-3.1 in [11], [12]). Then we see that $b = (\log\log t)^\gamma$ satisfies (4.2) for any $\gamma > 0$. The rest of the proof is same.

5. Lower estimates for the 3-dimensional case

In this section we prove (1.17). We first assume (1.13) with $\boldsymbol{\alpha} \in (0,\infty)^2$. For $R_\perp$ and $R_3 \in \mathbb{N}$, we consider the event

$$
\begin{aligned}
&\Big\{|q_\perp| \le 3R_\perp \text{ and } |q_3| \le 3R_3 \Rightarrow |q_\perp + \xi_{q,\perp}| \ge 2R_\perp \text{ or } |q_3 + \xi_{q,3}| \ge 2R_3, \\
&\text{"}|q_\perp| > 3R_\perp \text{ or } |q_3| > 3R_3\text{" and } \Big(\frac{|q_\perp|}{R_\perp}\Big)^{\alpha_\perp} \ge \Big(\frac{|q_3|}{R_3}\Big)^{\alpha_3} \Rightarrow |\xi_{\boldsymbol{q},\perp}| \le K_\perp |q_\perp|, \\
&\text{"}|q_\perp| > 3R_\perp \text{ or } |q_3| > 3R_3\text{" and } \Big(\frac{|q_\perp|}{R_\perp}\Big)^{\alpha_\perp} < \Big(\frac{|q_3|}{R_3}\Big)^{\alpha_3} \Rightarrow |\xi_{\boldsymbol{q},3}| \le K_3 |q_3|\Big\},
\end{aligned}
\tag{5.1}
$$

where $K_\perp = (1 - 3^{-(1\wedge(\alpha_3/\alpha_\perp))})/2$ and $K_3 = (1 - 3^{-(1\wedge(\alpha_\perp/\alpha_3))})/2$. We have

$$
\boldsymbol{V}_{\boldsymbol{\xi}}(\boldsymbol{x}) \le c_1 R_\perp^2 R_3 (R_\perp^{\alpha_\perp} \wedge R_3^{\alpha_3})^{-1} \tag{5.2}
$$

for $\boldsymbol{x} \in \Lambda_{2R_\perp} \times I(R_3)$ on this event. Here and in the following, $I(R) = (-R, R)$ for any $R > 0$. To prove (5.2) we divide the summation as

$$
\begin{aligned}
\boldsymbol{V}_{\boldsymbol{\xi}}(\boldsymbol{x}) = &\sum_{\boldsymbol{q} \in \mathbb{Z}^3 \cap (B(3R_\perp) \times I(3R_3))} \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}_{\boldsymbol{q}}) \\
&+ \sum_{\boldsymbol{q} \in \mathbb{Z}^3 \setminus (B(3R_\perp) \times I(3R_3)) : (|q_\perp|/R_\perp)^{\alpha_\perp} \ge (|q_3|/R_3)^{\alpha_3}} \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}_{\boldsymbol{q}}) \\
&+ \sum_{\boldsymbol{q} \in \mathbb{Z}^3 \setminus (B(3R_\perp) \times I(3R_3)) : (|q_\perp|/R_\perp)^{\alpha_\perp} < (|q_3|/R_3)^{\alpha_3}} \boldsymbol{u}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}_{\boldsymbol{q}}).
\end{aligned}
\tag{5.3}
$$

The first term in the right hand side is easily dominated from above by $R_\perp^2 R_3/(R_\perp^{\alpha_\perp} \wedge R_3^{\alpha_3})$. Since $\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}} \ge |x_\perp|^{\alpha_\perp}$, the second term is dominated by

$$
\sum_{\boldsymbol{q} \in \mathbb{Z}^3 \setminus (B(3R_\perp) \times I(3R_3)) : (|q_\perp|/R_\perp)^{\alpha_\perp} \ge (|q_3|/R_3)^{\alpha_3}} \Big((1 - K_\perp)|q_\perp| - R_\perp\Big)^{-\alpha_\perp}.
$$

By taking the summaton with respect to q_3 and replacing the summation by the integration, this is dominated by

$$
\int_{q_\perp \in \mathbb{R}^2 : |q_\perp| > 3^{1\wedge(\alpha_3/\alpha_\perp)} R_\perp} \frac{R_3(|q_\perp|/R_\perp)^{\alpha_\perp/\alpha_3}}{\Big((1 - K_\perp)|q_\perp| - R_\perp\Big)^{\alpha_\perp}} dq_\perp.
$$

Since $1 > 2/\alpha_\perp + 1/\alpha_3$ and $(1 - K_\perp)3^{1\wedge(\alpha_3/\alpha_\perp)} - 1 > 0$, this is dominated by $R_3 R_\perp^{2-\alpha_\perp}$. Simirarly the third term in the right hand side of (5.3) is dominated by $R_\perp^2 R_3^{1-\alpha_3}$. Therefore we obtain (5.2).

The probability is estimated as

$$
\log P_{\boldsymbol{\theta}}(\text{ the event (5.1) occurs }) \ge -c_2 R_\perp^2 R_3 (R_\perp^{\theta_\perp} \vee R_3^{\theta_3}). \tag{5.4}
$$

Indeed we have

$$
\begin{aligned}
&\log P_{\boldsymbol{\theta}}(\text{ the event (5.1) occurs })\\
=&\sum_{\boldsymbol{q}\in\mathbb{Z}^3\cap(B(3R_\perp)\times I(3R_3))}\log P_{\boldsymbol{\theta}}(|q_\perp+\xi_{\boldsymbol{q},\perp}|\geq 2R_\perp \text{ or } |q_3+\xi_{\boldsymbol{q},3}|\geq 2R_3)\\
&+\sum_{\boldsymbol{q}\in\mathbb{Z}^3\setminus(B(3R_\perp)\times I(3R_3)):(|q_\perp|/R_\perp)^{\alpha_\perp}\geq(|q_3|/R_3)^{\alpha_3}}\log P_{\boldsymbol{\theta}}(|\xi_{\boldsymbol{q},\perp}|\leq K_\perp|q_\perp|)\\
&+\sum_{\boldsymbol{q}\in\mathbb{Z}^3\setminus(B(3R_\perp)\times I(3R_3)):(|q_\perp|/R_\perp)^{\alpha_\perp}<(|q_3|/R_3)^{\alpha_3}}\log P_{\boldsymbol{\theta}}(|\xi_{\boldsymbol{q},3}|\leq K_3|q_3|).
\end{aligned}\tag{5.5}
$$

The first term in the right hand side is dominated from below by

$$R_\perp^2R_3\log P_{\boldsymbol{\theta}}(|\xi_{\mathbf{0},\perp}|\geq 5R_\perp \text{ or } |\xi_{\mathbf{0},3}|\geq 5R_3).$$

Since the probability is rewritten as

$$
\begin{aligned}
&P_{\boldsymbol{\theta}}(|\xi_{\mathbf{0},\perp}|\geq 5R_\perp \text{ or } |\xi_{\mathbf{0},3}|\geq 5R_3)\\
=&R_\perp^2R_3\int_{|y_\perp|\geq 5 \text{ or } |y_3|\geq 5}\exp(-\|\,|R_\perp y_\perp|^{\theta_\perp},|R_3y_3|^{\theta_3}\|_p)\frac{d\boldsymbol{y}}{Z(\boldsymbol{\theta},p)}\\
=&R_\perp^2R_3P_{\boldsymbol{\theta}}(|\xi_{\mathbf{0},\perp}|\geq 5 \text{ or } |\xi_{\mathbf{0},3}|\geq 5)\\
&\times E\Big[\exp(-(\|\,|R_\perp\xi_{\mathbf{0},\perp}|^{\theta_\perp},|R_3\xi_{\mathbf{0},3}|^{\theta_3}\|_p-\|\boldsymbol{\xi}\|_p^{\boldsymbol{\theta}})\Big||\xi_{\mathbf{0},\perp}|\geq 5 \text{ or } |\xi_{\mathbf{0},3}|\geq 5\Big],
\end{aligned}
$$

this probability is dominated from below by $\exp(-c_3(R_\perp^{\theta_\perp}\vee R_3^{\theta_3}))$. Therefore the first term in the right hand side of (5.5) is dominated from below by $-R_\perp^2R_3(R_\perp^{\theta_\perp}\vee R_3^{\theta_3})$. The second and the third terms in the right hand side (5.5) is dominated from below by $-R_3\exp(-c_4R_\perp^{\theta_\perp})$ and $-R_\perp^2\exp(-c_5R_3^{\theta_3})$, respectively, by using $\log(1-X)\geq -2X$ for $0\leq X\leq 1/2$. Therefore we obtain (5.4).

We now recall

$$
\begin{aligned}
\boldsymbol{N}(\lambda)\geq\frac{1}{8R_\perp^2R_3}P_{\boldsymbol{\theta}}\Big(&\lambda_1(\mathcal{H}_{B(R_\perp)})+\lambda_1\Big(-\Big(\frac{d^2}{dx^2}\Big)_{I(R_3)}\Big)\\
&+\int d\boldsymbol{x}|\psi_{R_\perp}(x_\perp)|^2\phi_{R_3}(x_3)^2\boldsymbol{V}_{\boldsymbol{\xi}}(\boldsymbol{x})\leq\lambda\\
&\text{and the event (5.1) occurs}\Big),
\end{aligned}\tag{5.6}
$$

where $\mathcal{H}_{B(R_\perp)}$ and $(d^2/(dx^2))_{I(R_3)}$ are the restrictions to the disk $B(R_\perp)$ and the interval $I(R_3)$, respectively, of $\mathcal{H}$ and the $d^2/(dx^2)$, respectively, by the Dirichlet boundary condition, and $\psi_{R_\perp}$ and ϕ_{R_3} are the normalized eigenfunctions corresponding to $\mathcal{H}_{B(R_\perp)}$ and $(d^2/(dx^2))_{I(R_3)}$, respectively (cf. Theorem (5.25) in [18]). By Erdős's bound

$$\lambda_1(\mathcal{H}_{B(R_\perp)})\leq\exp\Big(-\frac{B}{2}R_\perp^2(1-\varepsilon_1)\Big)\quad\text{and}\quad\lambda_1\Big(-\Big(\frac{d^2}{dx^2}\Big)_{(-R_3,R_3)}\Big)=\Big(\frac{\pi}{2R_3}\Big)^2,$$

we have

$$\log \boldsymbol{N}\Big(\exp\Big(-\frac{B}{2}R_\perp^2(1-\varepsilon_1)\Big)+\Big(\frac{\pi}{2R_3}\Big)^2+\frac{c_1R_\perp^2R_3}{R_\perp^{\alpha_\perp}\wedge R_3^{\alpha_3}}\Big)\ge -c_6R_\perp^2R_3(R_\perp^{\theta_\perp}\vee R_3^{\theta_3}),$$

where ε_1 is an arbitrarily fixed positive constant.

By specifying as $R_\perp = R_3^\beta$ with $\beta\in(0,\infty)$, we have

$$\log \boldsymbol{N}(c_7/R_3^{2\wedge\{(\alpha_\perp-2)\beta-1\}\wedge(\alpha_3-1-2\beta)})\ge -c_8R_3^{2\beta+1+(\beta\theta_\perp)\vee\theta_3}$$

and $\log \boldsymbol{N}(\lambda)\ge -c_9/\lambda^{f(\beta;\boldsymbol{\alpha},\boldsymbol{\theta})}$, where

$$f(\beta;\boldsymbol{\alpha},\boldsymbol{\theta})=\frac{2\beta+1+(\beta\theta_\perp)\vee\theta_3}{2\wedge\{(\alpha_\perp-2)\beta-1\}\wedge(\alpha_3-1-2\beta)}.$$

The function $f(\beta;\boldsymbol{\alpha},\boldsymbol{\theta})$ attains $\mu_1(\alpha_\perp,\boldsymbol{\theta})$ at $\beta=3/(\alpha_\perp-2)$. Therefore we obatin (1.17).

For the operator $\boldsymbol{\mathcal{H}}_{\boldsymbol{\xi}}$, we consider a simpler event, (5.1) with $\alpha_\perp=\alpha_3$. On this event we have $\boldsymbol{T}(\boldsymbol{\xi})\cap(\Lambda_{2R_\perp}\times I(R_3))=\emptyset$ for large $R_\perp$ and R_3. Thus we have

$$\log \boldsymbol{\mathcal{N}}\Big(\exp\Big(-\frac{B}{2}R_\perp^2(1-\varepsilon_1)\Big)+\Big(\frac{\pi}{2R_3}\Big)^2\Big)\ge -c_6R_\perp^2R_3(R_\perp^{\theta_\perp}\vee R_3^{\theta_3}).$$

By taking $R_\perp=2\sqrt{(\log R_3)/(B(1-\varepsilon_1))}$ so that $\exp(-BR_\perp^2(1-\varepsilon_1)/2)\le R_3^{-2}$, we have

$$\log \boldsymbol{\mathcal{N}}(c_7/R_3^2)\ge -c_8R_3^{1+\theta_3}\log R_3$$

and $\log \boldsymbol{\mathcal{N}}(\lambda)\ge -c_9\lambda^{-(1+\theta_3)/2}\log(1/\lambda)$.

6. Upper estimates for the 3-dimensional case

In this section we prove (1.18). We may assume that $\boldsymbol{u}$ is continuous since the essential condition is only (1.13). Following Proposition 5.11 in Warzel [21], we have

$$\widetilde{\boldsymbol{N}}(t)\le\frac{B}{2\pi(1-e^{-2Bt})}\int_{\boldsymbol{\Lambda}_1}d\boldsymbol{x}E_{\boldsymbol{\theta}}[\exp(-t(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot)))(x_3,x_3)],\tag{6.1}$$

where $\exp(-t(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot)))(x_3,x_3')$ is the integral kernel of the heat semigroup generated by the Schrödinger operator $-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot)$ on $L^2(\mathbb{R})$ and $\boldsymbol{\Lambda}_R:=(-R/2,R/2)^3$ for any $R>0$. By the Neumann-bracketing, the right hand side is dominated by

$$\begin{aligned}&\frac{B}{2\pi(1-e^{-2Bt})R_3}\int_{\Lambda_1}dx_\perp E_{\boldsymbol{\theta}}[\mathrm{Tr}[\exp(-t(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot))^N_{I(R_3/2)})]]\\ \le&\frac{c_1}{R_3}\int_{\Lambda_1}dx_\perp E_{\boldsymbol{\theta}}[\exp(-(t-\varepsilon_1)\lambda_1(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot))^N_{I(R_3/2)})],\end{aligned}\tag{6.2}$$

where $(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot))^N_{I(R_3/2)}$ is the restriction to the interval $I(R_3/2)$ by the Neumann boundary condition and ε_1 is an arbitrary small positive constant.

The eigenvalue is estimated as

$$
\begin{aligned}
&\lambda_1(-\partial_3^2 + \boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot))^N_{I(R_3/2)} \\
&\ge \inf\Big\{\lambda_1\Big(-\partial_3^2 + \sum_{j=1}^M \boldsymbol{u}(\boldsymbol{x}-\boldsymbol{b}_j)\Big)^N_{I(R_3/2)} : \boldsymbol{b}_1,\ldots,\boldsymbol{b}_M \in \Lambda_{R_\perp}\times I(R_3/2)\Big\} \\
&\ge \inf\Big\{\lambda_1\Big(-\frac{d^2}{dt^2} + \frac{c_2 M}{\|R_\perp^{\alpha_\perp}, |t-b|^{\alpha_3}\|_{\tilde{p}}}\Big)^N_{I(R_3/2)} : b\in I(R_3/2)\Big\},
\end{aligned}
\tag{6.3}
$$

where $M = \#\{\boldsymbol{b}\in\mathbb{Z}^3\cap(\Lambda_{R_\perp}\times I(R_3/2)) : b+\xi_{\boldsymbol{b}}\in\Lambda_{R_\perp}\times I(R_3/2)\}$. We take small positive numbers ε_2, ε_3 and $\varepsilon_\perp$. Then

$$
\varepsilon_2 R_\perp^2 R_3 \ge \#\{\boldsymbol{b}\in\mathbb{Z}^3\cap(\Lambda_{R_\perp}\times I(R_3/2)) : |\xi_{\boldsymbol{b},\perp}|_\infty > \varepsilon_\perp R_\perp/2 \text{ or } |\xi_{\boldsymbol{b},3}| > \varepsilon_3 R_3/2\}
\tag{6.4}
$$

implies

$$
M > \{(1-\varepsilon_\perp)^2(1-\varepsilon_3)-\varepsilon_2\}R_\perp^2 R_3.
\tag{6.5}
$$

Indeed since $M \ge \#\{\boldsymbol{b}\in\mathbb{Z}^3\cap(\Lambda_{(1-\varepsilon_\perp)R_\perp}\times I((1-\varepsilon_3)R_3/2)) : |\xi_{\boldsymbol{b},\perp}|_\infty \le \varepsilon_\perp R_\perp/2 \text{ and } |\xi_{\boldsymbol{b},3}| \le \varepsilon_3 R_3/2\}$, the right hand side of (6.4) is less than or equal to

$$
\begin{aligned}
&\#\{\boldsymbol{b}\in\mathbb{Z}^3\cap(\Lambda_{(1-\varepsilon_\perp)R_\perp}\times I((1-\varepsilon_3)R_3/2)) : |\xi_{\boldsymbol{b},\perp}|_\infty > \varepsilon_\perp R_\perp/2 \text{ or } |\xi_{\boldsymbol{b},3}| > \varepsilon_3 R_3/2\} \\
&> (1-\varepsilon_\perp)^2(1-\varepsilon_3)R_\perp^2 R_3 - M.
\end{aligned}
$$

(6.5) implies the right hand side of (6.3) is greater than or equal to

$$
\inf\Big\{\lambda_1\Big(-\frac{d^2}{dt^2} + \frac{c_3 R_\perp^2 R_3}{\|R_\perp^{\alpha_\perp}, |t-b|^{\alpha_3}\|_{\tilde{p}}}\Big)^N_{I(R_3/2)} : b\in I(R_3/2)\Big\}.
$$

This equals

$$
\frac{1}{h^2}\inf\Big\{\lambda_1\Big(-\frac{d^2}{dt^2} + \frac{c_3 R_\perp^2 R_3}{h^{\alpha_3-2}\|R_\perp^{\alpha_\perp}/h^{\alpha_3}, |t-b|^{\alpha_3}\|_{\tilde{p}}}\Big)^N_{I(R_3/(2h))} : b\in I(R_3/(2h))\Big\}
\tag{6.6}
$$

for any $h\in(0,\infty)$ by changing the variables. Referring $\lambda_1(-d^2/(dt^2)+1_{[0,1]}/R)^N_{[0,R]} \ge c_4/R^2$ for any $R\ge 2$ and the condition (1.14), we let $h=(R_\perp R_3)^{2/(\alpha_3-1)}$ and

$$
c_5 \le R_\perp^{\alpha_\perp(\alpha_3-1)/(2\alpha_3)-1} \le R_3
\tag{6.7}
$$

so that $h^{\alpha_3-2}/(R_\perp^2 R_3) = R_3/h \ge 2$ and $R_\perp^{\alpha_\perp}/h^{\alpha_3}\le 1$. Thus the quantity in (6.6) is dominated from below by R_3^{-2} and we obtain $\lambda_1(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot))^N_{I(R_3/2)} \ge c_6/R_3^2$. Therefore we have

$$
\begin{aligned}
&P_{\boldsymbol{\theta}}(\lambda_1(-\partial_3^2+\boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp,\cdot))^N_{I(R_3/2)} < c_6/R_3^2) \\
\le &P_{\boldsymbol{\theta}}(\#\{\boldsymbol{b}\in\mathbb{Z}^3\cap(\Lambda_{R_\perp}\times I(R_3/2)) : |\xi_{\boldsymbol{b},\perp}|_\infty > \varepsilon_\perp R_\perp/2 \text{ or } |\xi_{\boldsymbol{b},3}| > \varepsilon_3 R_3/2\} > \varepsilon_2 R_\perp^2 R_3).
\end{aligned}
$$

Since the event in the probability on the right hand side implies

$$\sum_{\boldsymbol{b}\in\mathbb{Z}^3\cap(\Lambda_{R_\perp}\times I(R_3/2))} \|\boldsymbol{\xi_b}\|_p^{\boldsymbol{\theta}} \geq \varepsilon_2 R_\perp^2 R_3 \Big\{ \Big(\frac{\varepsilon_\perp R_\perp}{2}\Big)^{\theta_\perp} \wedge \Big(\frac{\varepsilon_3 R_3}{2}\Big)^{\theta_3} \Big\},$$

we have

$$P_{\boldsymbol{\theta}}(\lambda_1(-\partial_3^2 + \boldsymbol{V_\xi}(x_\perp,\cdot))_{I(R_3/2)}^N < c_6/R_3^2) \leq \exp(-c_7 R_\perp^2 R_3 (R_\perp^{\theta_\perp} \wedge R_3^{\theta_3})). \tag{6.8}$$

We take $\beta \in (0,\infty)$ to specify as $R_\perp = R_3^\beta$. Then the condition (6.7) becomes $\beta \leq (2/\alpha_\perp)/(1 - 2/\alpha_\perp - 1/\alpha_3) =: \overline{\beta}$ and the right hand side of (6.8) becomes $\exp(-c_7 R_3^{2f(\beta;\boldsymbol{\theta})})$, where $f(\beta;\boldsymbol{\theta}) = \beta + 1/2 + ((\beta\theta_\perp)\wedge \theta_3)/2$. The function $f(\beta;\boldsymbol{\theta})$ attains its maximum $\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})$ defined in (1.16) at $\beta = \overline{\beta}$. Thus our optimal estimate is

$$P_{\boldsymbol{\theta}}(\lambda_1(-\partial_3^2 + \boldsymbol{V_\xi}(x_\perp,\cdot))_{I(R_3/2)}^N < c_6/R_3^2) \leq \exp(-c_7 R_3^{2\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})}).$$

By (6.2), we have

$$\widetilde{\boldsymbol{N}}(t) \leq \frac{c_1}{R_3}\exp(-c_7 R_3^{2\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})}) + \frac{c_1}{R_3}\exp\Big(-\frac{(t-\varepsilon_1)c_6}{R_3^2}\Big).$$

Since

$$\widetilde{\boldsymbol{N}}(t) = t\int_0^\infty d\lambda e^{-t\lambda}\boldsymbol{N}(\lambda) \geq t\int_{\eta/R_3^2}^\infty d\lambda e^{-t\lambda}\boldsymbol{N}\Big(\frac{\eta}{R_3^2}\Big) = \exp\Big(-t\frac{\eta}{R_3^2}\Big)\boldsymbol{N}\Big(\frac{\eta}{R_3^2}\Big),$$

we have $\boldsymbol{N}(\eta/R_3^2) \leq \exp(-c_8 R_3^{2\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})})$ for large R_3 by taking $\eta = c_6/2$ and $t = c_7 R_3^{2+2\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})}/(2\eta)$. By taking $\lambda = \eta/R_3^2$, we obtain (1.18).

For $\boldsymbol{\mathcal{N}}$ and the case that $\operatorname{supp}\boldsymbol{u}$ is compact, we may assume $\boldsymbol{u} = C_0 1_{\overline{B(r_\perp)}\times\overline{I(r_3)}}$ with $0 < r_\perp, r_3 < 1/2$. Then we apply a standard Brownian estimate to reduce to an estimate of the eigenvalue of the operator with the Dirichlet boundary condition:

$$\begin{aligned}\widetilde{\boldsymbol{N}}(t) \leq &\frac{B}{2\pi(1-e^{-2Bt})}\int_{\Lambda_1} dx_\perp E_{\boldsymbol{\theta}}[\mathrm{Tr}[\exp(-t(-\partial_3^2 + \boldsymbol{V_\xi}(x_\perp,\cdot))_{I(t)}]] + e^{-c_0 t}\\ \leq &c_1\int_{\Lambda_1} dx_\perp E_{\boldsymbol{\theta}}[\exp(-(t-\varepsilon_1)\lambda_1(-\partial_3^2 + \boldsymbol{V_\xi}(x_\perp,\cdot))_{I(t)})] + e^{-c_0 t}\end{aligned}$$

(cf. [12] Section 1.7). Then we use Theorem 3.1 in the page 123 in [19] to have

$$\lambda_1(-\partial_3^2 + \boldsymbol{V_\xi}(x_\perp,\cdot))_{I(t)} \geq \pi^2/(\sup_k |I_k| + c_2)^2,$$

where $\{I_k\}_k$ are the random open disjoint intervals such that

$$\sum_k I_k = I(t) \setminus \bigcup_{\boldsymbol{q}\in\mathbb{Z}^3: q_\perp + \xi_{\boldsymbol{q},\perp}\in\overline{B(x_\perp : r_\perp)}} [q_3 + \xi_{\boldsymbol{q},3} - r_3, q_3 + \xi_{\boldsymbol{q},3} + r_3]$$

and $|I_k|$ is the length of I_k. If $\sup_k |I_k| \geq s$, then there exists $p \in \mathbb{Z} \cap I(t)$ such that

$$[p, p+s-2] \cap \bigcup_{\boldsymbol{q} \in \mathbb{Z}^3 : q_\perp + \xi_{\boldsymbol{q},\perp} \in \overline{B(x_\perp : r_\perp)}} [q_3 + \xi_{\boldsymbol{q},3} - r_3, q_3 + \xi_{\boldsymbol{q},3} + r_3] = \emptyset.$$

Then we have $(0,0,q) + \boldsymbol{\xi}_{(0,0,q)} \notin \overline{B(x_\perp : r_\perp)} \times [p - r_3, p + s - 2 + r_3]$ for any $q \in [p - r_3, p + s - 2 + r_3] \cap \mathbb{Z}$. Thus the probability of this event is estimated as

$$P_{\boldsymbol{\theta}}(\sup_k |I_k| \geq s) \leq \sum_{p \in \mathbb{Z} \cap I(t)} \prod_{q \in [p - r_3 + 1, p + s - 3 + r_3] \cap \mathbb{Z}} P_{\boldsymbol{\theta}}(\xi_{(0,0,q_3),\perp} \notin \overline{B(x_\perp : r_\perp)} \text{ or } |\xi_{(0,0,q_3),3}| > 1)$$

$$\leq 2t \exp(-c_3(s + 2r_3 - 4))$$

if $s \geq 4$. Therefore we have

$$P_{\boldsymbol{\theta}}(\lambda_1(-\partial_3^2 + \boldsymbol{V}_{\boldsymbol{\xi}}(x_\perp, \cdot))_{I(t)} \geq \pi^2/(s + c_2)^2) \leq 2t \exp(-c_3(s + 2r_3 - 4))$$

for $s \geq 4$. The rest of the proof is same.

References

1. Broderix, K., Hundertmark, D., Kirsch, W., and Leschke, H., The fate of Lifshits tails in magnetic fields *J. Statist. Phys.*, **80** (1995), 1–22.
2. Carmona, R. and Lacroix, J., *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990.
3. Donsker, M. D. and Varadhan, S. R. S., Asymptotics for the Wiener sausage *Comm. Pure Appl. Math.*, **28** (1975), 525–565.
4. Erdős, L., Rayleigh-type isoperimetric inequality with a homogeneous magnetic field, *Calc. Var. Partial Differential Equations*, **4** (1996), 283–292.
5. Erdős, L., Lifschitz tail in a magnetic field: the nonclassical regime, *Probab. Theory Related Fields*, **112** (1998), 321–371.
6. Erdős, L., Lifschitz tail in a magnetic field: coexistence of classical and quantum behavior in the borderline case *Probab. Theory Related Fields*, **121** (2001), 219–236.
7. Fukushima, R., Brownian survival and Lifshitz tail in perturbed lattice disorder *J. Funct. Anal.*, **256** (2009), 2867–2893.
8. Fukushima, R. and Ueki, N., Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice, *in preparation,* Available at `http://www.math.h.kyoto-u.ac.jp/~ueki/Former-revised.pdf`
9. Hundertmark, D., Kirsch, W. and Warzel, S., Classical magnetic Lifshits tails in three space dimensions: impurity potentials with slow anisotropic decay, *Markov Process. Related Fields*, **9** (2003), 651–660.
10. Hupfer, T., Leschke, H. and Warzel, S., Poissonian obstacles with Gaussian walls discriminate between classical and quantum Lifshits tailing in magnetic fields, *J. Statist. Phys.*, **97** (1999), 725–750.
11. Ikeda, N. and Watanabe, S., *Stochastic differential equations and diffusion processes*, North-Holland, Amsterdam, 1989.
12. Itô, K. and McKean, Jr. H. P., *Diffusion processes and their sample paths*, Springer-Verlag, Berlin, 1974.
13. Kirsch, W. and Martinelli, F., On the density of states of Schrödinger operators with a random potential, *J. Phys. A,* **15** (1982), 2139–2156.

14. Kirsch, W., Metzger, B., The integrated density of states for random Schrödinger operators, *In: Spectral theory and mathematical physics (a Festschrift in honor of Barry Simon's 60th birthday), Proc. Sympos. Pure Math.*, **76** (Amer. Math. Soc., Providence, RI 2007), 649–696.

15. Nakao, S., On the spectral distribution of the Schrödinger operator with random potential, *Japan. J. Math. (N.S.)*, **3** (1977), 111–139.

16. Osserman, R., Bonnesen-style isoperimetric inequalities, *Amer. Math. Monthly,* **86** (1979), 1–29.

17. Pastur, L. A., The behavior of certain Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential, *Teoret. Mat. Fiz.*, **32** (1977), 88–95.

18. Pastur, L. and Figotin, A., *Spectra of random and almost-periodic operators*, Springer-Verlag, Berlin, 1992.

19. Sznitman, A.-S., *Brownian motion, obstacles and random media*, Springer-Verlag, Berlin, 1998.

20. Ueki, N., Classical behavior of the integrated density of states for the uniform magnetic field and a randomly perturbed lattice, *in preparation.* Available at `http://www.math.h.kyoto-u.ac.jp/~ueki/Frenkel-magnet-classical1.pdf`

21. Warzel, S., Lifshits tails in magnetic fields, *Ph-D Thesis, Universität Erlangen-Nürnberg*, 2001.

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