Old and new tales about Lifshitz tails

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In the mid sixties I. Lifshitz argued that:

1) The integrated density of states $N(E)$ for an ordered system in dimension d behaves like:

$$N(E) \ \sim \ C \ (E - E_0)^{\frac{d}{2}}$$

for $E \searrow E_0$ where E_0 is the infimum of the spectrum.

2) The integrated density of states $N(E)$ for a disordered system in dimension d behaves like:

$$N(E) \ \sim \ C \ \mathrm{e}^{(E-E_0)^{-\gamma}}$$

for $E \searrow E_0$ with γ, the *Lifshitz exponent* equal to $\frac{d}{2}$.

We review some of the mathematical papers on Lifshitz tails starting with the famous paper by Donsker and Varadhan to recent developments.
To be specific let us consider a random Schrödinger operator with potential:

$$V_\omega \ = \ \sum_{i \in \mathbb{Z}^d} q_i(\omega)\, f(x-i)$$

The random variables q_i are independent and identically distributed with a common law P_0.

If the single site potential f decays faster than $(1+|x|)^{-(d+2)}$ then the Lifshitz exponent is, indeed, $\frac{d}{2}$. However, if f behaves near infinity like $C\,(1+|x|)^{-\alpha}$ with $\alpha > d+2$ then we have $\gamma = \frac{d}{\alpha - d}$.

There are also recent results on f decaying in a non homogeneous way, namely like $(1+|x_1|)^{-\alpha_1}$ in some directions x_1 and like $(1+|x_2|)^{-\alpha_2}$ in the other directions x_1.

If we add a homogeneous magnetic field to the random potential V_ω then the Lifshitz exponent is smaller, as a rule. In fact, one can prove that $\gamma = \frac{d}{\alpha}$ for *all* α whenever $f \sim C\,(1+|x|)^{-\alpha}$. In particular $\gamma = 0$ for compactly supported f.

For details, see the review cited below and references therein.

Reference: Kirsch, W.; Metzger, B.: The Integrated Density of States for Random Schrödinger Operators, math-ph/0608066 in arxiv.org