LOCALIZATION FOR SCHRÖDINGER OPERATORS WITH POISSON RANDOM POTENTIAL

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Consider an electron moving in an amorphous medium with randomly placed identical impurities, each impurity creating a local potential. For a fixed configuration of the impurities, described by the countable set $X \subset \mathbb{R}^d$ giving their locations, this motion is described by the Schrödinger equation $-i\partial_t \psi_t = H_X \psi_t$ with the Hamiltonian

$$H_X := -\Delta + V_X \quad \text{on } L^2(\mathbb{R}^d),$$

where the potential is given by

$$V_X(x) := \sum_{\zeta \in X} u(x - \zeta),$$

with $u(x - \zeta)$ being the single-site potential created by the impurity placed at $\zeta$. Since the impurities are randomly distributed, the configuration $X$ is a random countable subset of $\mathbb{R}^d$, and hence it is modeled by a point process on $\mathbb{R}^d$.

The Poisson Hamiltonian is the random Schrödinger operator $H_X$ in (1) with $X$ a Poisson process on $\mathbb{R}^d$ with density $\rho > 0$. The potential $V_X$ is then a Poisson random potential. Poisson Hamiltonians may be the most natural random Schrödinger operators in the continuum as the distribution of impurities in various samples of material is naturally modeled by a Poisson process. A mathematical proof of the existence of localization in two or more dimensions has been a long-standing open problem.

In this lecture I discuss the following theorem proved by F. Germinet, P. Hislop and myself:

**Theorem 1** (Germinet, Hislop and Klein). Let $H_X$ be a Poisson Hamiltonian on $L^2(\mathbb{R}^d)$ with density $\rho > 0$. Then there exist $E_0 = E_0(\rho) > 0$ and $m = m(\rho) > 0$ for which the following holds $\mathbb{P}$-a.e.: The operator $H_X$ has pure point spectrum in $[0, E_0]$ with exponentially localized eigenfunctions with rate of decay $m$, i.e., if $\phi$ is an eigenfunction of $H_X$ with eigenvalue $E \in [0, E_0]$ we have

$$\| \chi_x \phi \| \leq C_{X, \phi} e^{-m|x|}, \quad \text{for all } x \in \mathbb{R}^d.$$  

Moreover, there exist $\tau > 1$ and $s \in ]0, 1[$ such that for all eigenfunctions $\psi, \phi$ (possibly equal) with the same eigenvalue $E \in [0, E_0]$ we have

$$\| \chi_x \psi \| \| \chi_y \phi \| \leq C_X \| T^{-1} \psi \| \| T^{-1} \phi \| e^{(y)\tau} e^{-|x-y|^s}, \quad \text{for all } x, y \in \mathbb{Z}^d.$$  

In particular, the eigenvalues of $H_X$ in $[0, E_0]$ have finite multiplicity, and $H_X$ exhibits dynamical localization in $[0, E_0]$, that is, for any $p > 0$ we have

$$\sup_t \| (x)^p e^{-itH_X} \chi_{[0, E_0]} \| \chi_0 \|_2^2 < \infty.$$