2 Dimensional $O(N)$ Spin Model and Localization Type Problems

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1: The $O(N)$ spin model

It is shown [1, 2] that the two-point correlation functions $< s_0 s_x >$ of 2D $O(N)$ (classical) spin model ($s_x \in S^{N-1}$) is represented as an average of the Green's function $G^{(\psi)}(0,x)$ which depends on real random potentials [2] $\{\psi(x); x \in Z^2\}$ with a pure imaginary coefficient $i\kappa$:

$$\begin{aligned} < s_0 s_x > &= \int G^{(\psi)}(0,x) d\mu(\psi) \\ G^{(\psi)}(0,x) &= \frac{1}{-\Delta + m^2 + i\kappa\psi}(0,x), \quad (\kappa = 2/\sqrt{N}) \end{aligned}$$

where Δ is the Laplacian defined on the lattice space Z^2 $((\Delta)_{xy} = -4\delta_{x,y} + \delta_{|x-y|,1})$ and $\{\psi(x); x \in Z^2\}$ are random variables which obey the a distribution $d\mu(\psi)$ which is close to a Gaussian probability distribution but not exactly:

$$\begin{aligned} d\mu(\psi) &= F(\psi) \prod d\psi_x \\ F(\psi) &= \det{}_3{}^{-N/2}(1 + i\kappa G\psi) \exp[- < \psi, G^{\circ 2}\psi >] \end{aligned}$$

where $G(x,y) = G^{(\psi=0)}(x,y)$, $G^{\circ 2}(x,y) \equiv (G(x,y))^2$ and m^2 is chosen so that $G(0) = [-\Delta + m^2](0) = \beta$. Here $\beta > 0$ is the inverse temperature of the system and then $m \sim e^{-2\pi\beta}$ is very small. It is a subtle and longstanding problem [5, 6] to prove (or disprove) that $< s_0 s_x >$ decreases exponentially fast as $|x| \to \infty$ if $N \geq 3$ in two dimensions.

2: Result

Our idea is to apply the argument of the Anderson localization to show that $< s_0 s_x >$ decreaeses exponentially fast. Assume that the interaction is restricted to a finite rectangular region $\Lambda \subset Z^2$ and study the limit $\Lambda \to Z^2$. We decompose Λ into many small squares $\Lambda = \cup_{i=1}^n \Delta_i$ and apply the Feshbach formula to obtain $\det{}^{-N/2}(1 + i\kappa G_\Lambda \psi_\Lambda)$ in the following form:

$$\left[\prod_{i=1}^{n-1} \det{}^{-N/2}\left(1 + W(\Delta_i, \Lambda_i)\right)\right] \prod_{i=1}^{n} \det{}^{-N/2}\left(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}\right) \tag{1}$$

where $\kappa = 2/\sqrt{N}$, $\Lambda_k = \cup_{i=k+1}^n \Delta_i$, $G_\Delta = \chi_\Delta G \chi_\Delta$, ${}_\Lambda G_\Delta = \chi_\Lambda G \chi_\Delta$ and

$$W(\Delta_i, \Lambda_i) = -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}} G_{\Delta_i,\Lambda_i}\psi_{\Lambda_i} \frac{1}{1 + i\kappa G_{\Lambda_i}\psi_{\Lambda_i}} G_{\Lambda_i,\Delta_i}\psi_{\Delta_i}$$

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$[G_\Lambda]^{-1}$ is a Laplacian with free boundary conditions at $\partial\Lambda_i$. To apply cluster expansion to prove the long-standing conjecture argued in [3, 5, 6], we will have to show $W(\Delta, \Lambda)$ are small. This work is the first step in this direction.

We prove that $([G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i})^{-1}$ behaves as massive Green's functions which decrease fast (localization). The measure restricted to each block is

$$d\mu_\Delta = \det{}_3^{-N/2}(1 + i\kappa G_\Delta\psi_\Delta)\exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta)] \prod_{x\in\Delta} d\psi(x) \tag{2}$$

which is regarded as a perturbed Gaussian measure which is not necessarily real positive. Put $d\nu = \prod d\mu_\Delta$ and we use $d\nu$ for $d\mu$ as the first approximation. Then we have the following theorem which exhibits the generations of large mass m_{eff}:

Theorem *Let*

$$G^{(ave)}(x,y) \equiv \int G^{(\psi)}(x,y)d\nu(\psi) \tag{3}$$

Then

$$G^{(ave)}(x,y) \sim \frac{1}{-\Delta + m_{eff}^2}(x,y) \tag{4}$$

where $m_{eff}^2 = O(1/N\beta)$.

See [4] for the proof.

References

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