KdV-flow and Floquet exponents

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ABSTRACT: The KdV equation is

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x},$$

and n-soliton solutions for the KdV equation are given for $m_i, \eta_i > 0$ by

$$u(t,x) = -2D_x^2 \log \det(I + A),\ A = \left(\frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-(\eta_i+\eta_j)x + 4\left(\eta_i^3 + \eta_j^3\right)t}\right)_{1 \le i,j \le n}.$$

For each fixed $t \in \mathbf{R}$, $u(t, \cdot)$ is a reflectionless potential which appears in one–dimensional scattering theory. Marchenko considered the compact uniform closure $\Omega\left([-\lambda_0, 0]\right)$ of reflectionless potentials. We assume here that the Schrödinger operator $L(u)$ with potential $u \in \Omega\left([-\lambda_0, 0]\right)$ has its spectrum in $[-\lambda_0, \infty)$. M.Sato and Y.Sato established a unified approach for a large class of completely integrable systems. Let

$$\Gamma = \left\{ \begin{array}{c} g;\ g(z) \text{ is holomorphic on } \mathbf{D},\ g(0) = 1,\ g(z) \neq 0 \text{ for } \forall z \in \mathbf{D}, \\ \text{takes real values on } \mathbf{R} \text{ and } g(-z) = g(z)^{-1} \text{ for } \forall z \in \mathbf{D} \end{array} \right\},$$

where $\mathbf{D}$ is the closed unit disc. We construct a homomorphism K between the group Γ and the group of all homeomorphism on $\Omega\left([-\lambda_0, 0]\right)$ by applying Sato's theory. This K induces the shift operation if we choose $g_x(z) = e^{-xz} \in \Gamma$ and solutions for the KdV equation if $g_{x,t}(z) = e^{-xz+4tz^3} \in \Gamma$. Any other higher order KdV equation can be solved in this way on $\Omega\left([-\lambda_0, 0]\right)$. It is also known that for $u \in \Omega\left([-\lambda_0, 0]\right)$, $L(u)$ and $L(K(g)u)$ are unitarily equivalent.

Since $\Omega\left([-\lambda_0, 0]\right)$ is compact and the KdV flow $\{K(g)\}_{g\in\Gamma}$ is commutative, the space of all probability measures on $\Omega\left([-\lambda_0, 0]\right)$ invariant with respect to $\{K(g)\}_{g\in\Gamma}$ is a non-empty compact convex set. We can define the Floquet exponent

$$w_\mu(\lambda) = -\frac{1}{2}\int_{\Omega([-\lambda_0,0])} g_\lambda(0,0,u)^{-1}\mu(du),$$

where $g_\lambda(x,y,u)$ is the Green function of the Schrödinger operator with potential u. It is known that Floquet exponents arising from finite band spectrum determine uniquely a $\{K(g)\}$ −invariant probability measures.

Open problem: To what extent does the Floquet exponent w_μ characterize the measure μ ?