Singularity of Solutions to Schrödinger Equations

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We consider solutions to Schrödinger evolution equation with variable coefficients. We characterize the microlocal singularity of the solution at time $t > 0$ in terms of the initial condition. This is a refinement of the so-called *microlocal smoothing estimates* due to Craig-Kappeler-Strauss, Wunsch, etc.

We consider the Schrödinger evolution equation:

$$i\frac{\partial}{\partial t}u(t,x) = Hu(t,x), \quad t \in \mathbb{R}, x \in \mathbb{R}^n,$$

with $u(0,x) = u_0(x) \in L^2(\mathbb{R}^n)$. The Hamiltonian is

$$H = -\frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j} a_{jk}(x)\partial_{x_k} + V(x) \quad \text{on } L^2(\mathbb{R}^n),$$

where $a_{jk}(x)$, $V(x) \in C^\infty(\mathbb{R}^n;\mathbb{R})$.
Assumption: There is $\mu > 0$ such that $\forall \alpha \in \mathbb{Z}_+^n$, $\exists C_\alpha > 0$,

$$\left|\partial_x^\alpha(a_{jk}(x) - \delta_{jk})\right| \le C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad \left|\partial_x^\alpha V(x)\right| \le C_\alpha \langle x\rangle^{2-\mu-|\alpha|}$$

Here we denote $\langle x\rangle = \sqrt{1+|x|^2}$.
Classical Hamilton Flow: We denote the classical Hamiltonian by $p(x,\xi) = \frac{1}{2}\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k + V(x)$. and the Hamilton flow is denoted by: $\exp tH_p(x_0,\xi_0)$. We also use the geodesic flow: $\exp tH_k$ with $k(x,\xi) = \frac{1}{2}\sum a_{jk}(x)\xi_j\xi_k$ and denote $\exp tH_k(x_0,\xi_0) = (\tilde{y}(t;x_0,\xi_0), \tilde{\eta}(t;x_0,\xi_0))$.
Def. (Nontrapping condition): $(x_0,\xi_0) \in \mathbb{R}^{2n}$ is called *backward nontrapping* if $|\tilde{y}(t,x_0,\xi_0)| \to +\infty$ as $t \to -\infty$.

Then our main result is the following:
Theorem 1: If (x_0,ξ_0) is backward nontrapping, then for $t > 0$,

$$(x_0,\xi_0) \notin WF(u(t)) \iff \begin{cases} \exists a \in C_0^\infty(\mathbb{R}^{2n}) \text{ such that } a(x_0,\xi_0) \ne 0, \text{ and} \\ \|(a_h \circ \exp tH_p)(x,D_x)u_0\| = O(h^\infty) \text{ as } h \downarrow 0. \end{cases}$$

Reference:

Shu Nakamura: Semiclassical Singularity Propagation Property for Schrödinger Equations. (Preprint 2006: `http://jp.arxiv.org/abs/math.AP/0605742`)