Spectral structure of the Laplacian on a covering graph

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There are a lot of researches on the spectrum of the discrete Laplacian on an infinite graph in various areas. One of main topics among them is to characterize the spectral structure in terms of a certain geometric property of the graph. We focus on the possibility of the absence of eigenvalues and give a criterion for this in terms of combinatorial property of a graph. A graph G is a connected, locally finite graph and $A(G)$ is the set of its oriented edges. For an edge $e \in A(G)$, the origin vertex and the terminal one of e are denoted by $o(e)$ and $t(e)$, respectively. We define the discrete Laplacian by

$$\Delta_G f(x) = \frac{1}{m(x)} \sum_{e \in A_x(G)} \left(f(t(e)) - f(o(e)) \right),$$

where $A_x(G) = \{e \in A(G) \,|\, o(e) = x\}$, $m(x) = \#A_x(G)$. Here we consider the spectrum of $-\Delta_G$ on the Hilbert space

$$\ell^2(V(G), m) = \ell^2(V(G)) = \{f : V(G) \to \mathbf{C} \,|\, \langle f, f \rangle_V < \infty\}$$

with the inner product

$$\langle f_1, f_2 \rangle = \sum_{x \in V(G)} f_1(x) \overline{f_2(x)} m(x).$$

Now we assume that G is a infinite graph having a finitely generated group Γ of graph-automorphisms satisfying Γ acts freely on G and $M = \Gamma \backslash G$ is finite. Here Δ_G and $m = m_G$ on G is Γ-invariant, so graph G can be considered as a normal covering graph of a finite graph M and Δ_G as the lift of Δ_M.

The following is our main theorem for regular graphs.

Theorem 1. *Assume G is the maximal abelian covering graph of some finite graph M, that is, $\Gamma \cong H_1(M, \mathbf{Z})$, the 1-homology group with coefficients in $\mathbf{Z}$. If $M = \Gamma \backslash G$ is either even-regular graph or odd-regular bipartite graph, then the spectrum of $-\Delta_G$ has no eigenvalues and is absolutely continuous.*

It should be noted that the spectrum, as a set, of $-\Delta_G$ coincides with the whole interval $[0, 2]$ under the same assumption as in the above ([HS]). Hence we may say the spectrum is completely determined for such a family of graphs. The above is a corollary of the following theorem for M having a "2-factor", which is a spanning subgraph with degree 2.

Theorem 2. *Assume G is the maximal abelian covering graph of some finite graph M. If M has a 2-factor, then the spectrum of $-\Delta_G$ has no eigenvalues and is absolutely continuous.*

References

[**HS**] Yu. Higuchi and T. Shirai, *Some spectral and geometric properties for infinite graphs*, AMS Contemp. Math. **347** (2004), 29–56.

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