Interacting Brownian motions related to random matrices

Graduate School of Mathematics, Kyushu University, Hirofumi OSADA

Let S be a connected open set in $\mathbb{R}^d$. Interacting Brownian motions (IBMs) are infinitely dimensional diffusion processes with free potential $\Phi: S \to \mathbb{R}$ and interaction potential $\Psi: \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$.

These diffusions are given by the following infinitely dimensional stchastic differential equations (SDEs):

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\in\mathbb{Z},\, j\neq i} \nabla\Psi(X_t^i - X_t^j)dt \qquad (i \in \mathbb{Z}) \tag{1}$$

In SDE (1) we label the particles and the state space of diffusion is $(\mathbb{R}^d)^{\mathbb{Z}}$.

The purpose of this talk is to present a method to solve SDEs describing IBMs. Our method is an application of Dirichlet form theory. Indeed, we first construct the associated unlabeled dynamics by using the Dirichlet forms. Then we solve the SDEs with a conceptual method.

We are in particular motivated by the following three SDEs. All of them are related to random matrix theory. Because of the long range interaction of these models, the behavior of the diffusions are different from that of Gibbsian case. As an example, we will investigate the tagged particle problem of the last SDE (Gin).

Dyson's model: Let $S = \mathbb{R}$.

$$dX_t^i = dB_t^i + [\sum_{j\in\mathbb{Z},\ j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j}]\, dt \qquad (i \in \mathbb{Z}), \tag{Dy}$$

Bessel IBM: Let $S = (0, \infty)$. Let $\alpha > 1$ be a constant.

$$dX_t^i = dB_t^i + \{-\frac{1}{2} + \frac{\alpha}{2X_t^i} + \sum_{j\in\mathbb{N},\ j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j}\}\, dt \qquad (i \in \mathbb{N}), \tag{Be}$$

Ginibre IBM: Let $S = \mathbb{C}$. Let $Z_t^i = X_t^i + \sqrt{-1}Y_t^i \in \mathbb{C}$. Let B_t^i be a 2-D Brownian motion.

$$dZ_t^i = dB_t^i + \{-Z_t^i + \sum_{j\in\mathbb{Z},\ j\neq i}^{\infty} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2}\}\, dt \qquad (i \in \mathbb{Z}), \tag{Gin}$$

We remark interaction potentials of these SDEs are logarithmic fuctions:

$$\Psi(x) = -2\log|x|$$

So the stationary measures are not Gibbs measures. The difficulty of the problems is the presence of singular drifts coming from the *log potentials*.

We remark the stationary measures of the unlabeled dynamics of the SDEs (Dy), (Be), and (Gin) are given by the determinantal random point fields μ_{Dy}, μ_{Be} and μ_{Gin} with *sine*, *Bessel* and *exponential* kernels, respectively. It is well known that these random point fields are the scaling limits of the spectrum of random matricies.

We remark that the measures μ_{Dy} and μ_{Gin} are translation invariant.

The stationary distribution of the last SDE are the scaling limit of the spectrum of Ginibre Ensemble. This model describe the one component plasma in the plane with the 2D Couloumb interaction.

When we use the Dirichlet form approach to IBMs, it is convenient to unlabel the particles. So we recall the definition of *configurations* on S. Let

$$\Theta = \{\theta = \sum_i \delta_{x_i}\, ;\, \{x_i\} \text{ are sequences in } S \text{ such that } \theta(\{|x| \le r\}) < \infty \text{ for all } r\}.$$

The set Θ is called the configuration space on $\mathbb{R}^d$ and equipped with the vague topology. From the labeled solution (X_t^i) we construct Θ-valued diffusion $\mathbb{X}$ as follows;

$$\mathbb{X}_t = \sum_{i\in\mathbb{Z}} \delta_{X_t^i} \tag{2}$$

Then the diffusion have invariant probability measures.

Let μ be a probability measures on Θ. Let

$$\mathcal{E}(f,g) = \int_\Theta \mathbb{D}[f,g]d\mu, \qquad \mathbb{D}[f,g](\theta) = \frac{1}{2}\sum_i \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_i}, \tag{3}$$

$$\mathcal{D}_\infty = \bigcup_{r=1}^\infty \mathcal{D}_r, \quad \mathcal{D}_r = \{f \in L^2(\Theta,\mu); f \text{ is } \pi_r\text{-measurable, smooth } \mathcal{E}(f,f) < \infty\}.$$

Theorem 1. (1) *Assume μ is one of the determinantal random point fields with kernels sine or Bessel or exponential kernels. Then $(\mathcal{E},\mathcal{D}_\infty,L^2(\Theta,\mu))$ is closable, and its closure is quasi-regular Dirichlet forms.* (2) *The SDEs* (Dy), (Be) *and* (Gin) *have a solution for q.e. initial configuration.*

We next investigate the trajectry-wise property of (Gin).

It is known that the tagged particle Z_t^i of IBM with translation invariant stationary measures converges to constant multiple of the Brownian motion under the diffusive scaling.

$$\lim_{\epsilon\to 0} \epsilon Z^i_{t/\epsilon^2} = \sqrt{2a}B_t$$

Here a is a non-negative definite matrix called self-diffusion matrix.

The self-diffusion matrix a is zero if $d = 1$ and the particles have a hard core. It is known that a is positive definite if $d \geq 2$, the stationary measures are Gibbsian and the particle have a convex hard core. It is believed that a is always positive definite whenever $d \geq 2$ and the stationary measures are Gibbsian.

Since the unlabeled dynamics of SDE (Gin) has a translation invariant probability measure, its diffusive limit $\epsilon Z^i_{t/\epsilon^2}$ converge to constant multiple of the Brownian motion.

The interaction of the above SDE are quite strong compare with the Gibbsian case. Hence the long time behavior of the dynamics is very different that of interacting Brownian motions related to Gibbs measures. We indeed have the following results:

Theorem 2 (logarithmic growth of the tagged particles)**.** *Let Z_t^i be a tagged particle of* (Gin)*. Then for q.e. θ*

$$0 < \limsup_{t\to\infty} \frac{|Z_t^i|}{\log t} < \infty \qquad P_\theta\text{-a.s.}. \tag{4}$$

In particular, $a = 0$.

I believe that the tagged particle of Dyson's model has similar logarithmic behavior. I have however not proved it yet.