Asymptotic behavior of the integrated density of states for random point fields associated with certain Fredholm determinants

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Abstract. – Asymptotic behavior of the integrated density of states of a Schrödinger operator with positive potentials located around all sample points of some random point field at the infimum of the spectrum is investigated. The random point field is taken from a subclass of the class given by Shirai and Takahashi in terms of the Fredholm determinant. In the subclass, the obtained leading orders are same with the well known results for the Poisson point fields, and the character of the random field appears in the leading constants. The random point field associated with the sine kernel and the Ginibre random point field are well studied examples not included in the above subclass, though they are included in the class by Shirai and Takahashi. By applying the results on asymptotics of the hole probability for these random fields, the corresponding asymptotic behaviors of the densities of the states are also investigated in the case where the single site potentials have compact supports. The same method also applies to another well studied example, the zeros of a Gaussian random analytic function.

1. Introduction

In [24], Shirai and Takahashi introduced a class of random point fields by the character that the associated Laplace transform is represented explicitly by Fredholm determinants of certain integral operators. This representation may be regarded as analogies of that for the Poisson point process. For the Poisson point process, the representation is the key to determine the asymptotic behavior of the integrated density of states. Indeed the famous proof of the Lifshits behavior by the Donsker-Varadhan's theory is the application of the theory to the representaion [5], [18]. In this paper we first intend to extend the results

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for the Poisson point process to random point fields in a suitable subclass of Shirai and Takahashi's class. The first attempt does not apply the most interesting examples in Shirai and Takahashi's class. Those are the random point field associated with the sine kernel and the Ginibre random point field. Then we next intend to extend to these examples.

Let K be an integral operator with the kernel $(K(x,y))_{x,y\in\mathbb{R}^d}$ such that

(i) $(x,y)\mapsto K(x,y)$ is continuous;

(ii) $K(x,y)$ is square integrable in x and y for each fixed y and x, respectively;

(iii) $K(x+x_0,y+x_0)=K(x,y)\exp(i\ell_{x_0}(x-y))$ for any $x,y,x_0\in\mathbf{R}^d$, where $i=\sqrt{-1}$ and ℓ_{x_0} is a real linear function depending on x_0;

(iv) K is a nonnegative Hermitian operator.

We take $\alpha\in\{2/m: m\in\mathbb{N}\}\cup\{-1/m: m\in\mathbb{N}\}$ so that $I+\alpha K$ is positive semi-definite if $\alpha<0$. For this number α and the integrable operator K, let $\mu_{\alpha,K}$ be the Borel probability measure on the space $\mathcal{P}(\mathbb{R}^d)$ of nonnegative integer valued Radon measures on $\mathbb{R}^d$ with vague topology such that

$$\begin{aligned}&\int_{\mathcal{P}(\mathbf{R}^d)}\mu_{\alpha,K}(d\xi)\exp\Big(-\int f(x)\xi(dx)\Big)\\ &=\exp\Big(\frac{-1}{\alpha}\operatorname{Tr}\log(I+\alpha\sqrt{1-e^{-f}}K\sqrt{1-e^{-f}})\Big)\end{aligned}\tag{1.1}$$

for any nonnegative continuous function f on $\mathbb{R}^d$. On the existence of $\mu_{2,K}$, we do not have a general theory for the complex setting (cf. [23]). However, if we restrict to a real valued K, then Theorem 1.8 of Shirai and Takahashi [24] states that $\mu_{2,K}$ is the Poisson point process with the intensity $X(x)^2dx$, where X is a centered Gaussian random field such that $E[X(x)X(y)]=K(x,y)$. $\mu_{-1,K}$ is the determinantal point process with the kernel K. The existence of $\mu_{-1,K}$ in a general complex setting is assured by Theorem 3 of Soshinikov [26]. For any $m\in\mathbb{N}$, $\mu_{\alpha/m,K}$ is the m-fold convolution of $\mu_{\alpha,K/m}$ (See the proof of Lemma 3.3 in [24]). As $\alpha\to 0$, the quantity in (1.1) for $\mu_{\alpha,K}$ converges to that for the Poisson point process with the intensity $K(0,0)dx$ (See (1.4) in [24]).

At each sample point of the random measure ξ, we put a single site potential u, for which we assume the boundedness, the nonnegativity, the continuity and the integrability: we define the random scalar potential

$$V_\xi(x)=\int u(x-y)\xi(dy),$$

define the Schrödinger operator as the self-adjoint operator

$$H_\xi = -h\Delta + V_\xi \tag{1.2}$$

on the space $L^2(\mathbb{R}^d)$ of the square integrable functions, and consider its integrated density of states $N(\lambda)$ ($\lambda \in \mathbb{R}$) by the thermodynamic limit

$$\frac{1}{|\Lambda_R|}\#\{[0,\lambda] \cap \mathrm{spec}(H^D_{\xi,\Lambda_R})\} \longrightarrow N(\lambda) \qquad \text{as } R \to \infty. \tag{1.3}$$

In (1.2) h is a positive constant to indicate the quantum effect. In (1.3) Λ_R is a box $(-R/2, R/2)^d$, spec(A) is the spectral set for any self-adjoint operator A, and H^D_{ξ,Λ_R} is the self-adjoint operator defined by restricting H_ξ to Λ_R with the Dirichlet boundary condition. The existence and uniqueness of $N(\lambda)$ is proven by standard methods (cf. [3], [22]). This function increases only on the spectral set of H_ξ and the gradient reflects the density of the spectrum. Thus this function represents the distribution of the spectrum. We here remark that the both of the spectrum $\sigma(H_\xi)$ of H_ξ and the integrated density of states $N(\lambda)$ are proven to be independent of the sample value of ξ by the ergodicity. The spectrum of random operator including H_ξ is an important object in the research on the Anderson localization. However our model H_ξ is one of the random displacement models, for which the research on the Anderson localization is given only for the Poisson case by a highly technical method based on Bourgain's idea (cf. Germinet, Hislop and Klein [10], [11], [12], Bourgain and Kenig [2]). For the proof of the Anderson localization, the results on the behavior of the integrated density of states as in our paper are applied to obtain the initial estimate to prove in the induction steps. As a more fundamental result, $\sigma(H_\xi) = [0, \infty)$ is proven for the Poisson case (Theorem (5.34) in Pastur and Figotin [22], Ando, Iwatsuka, Kaminaga and Nakano [1]). Extending this result to our case is another remained problem.

In this paper we investigate the asymptotic behavior of $N(\lambda)$ at the infimum of the spectrum of H_ξ. For this subject, we have many results in many situations [3], [15], [22], [31]. We first assume the strict positivity

(P) $\quad I + \alpha K$ is strictly positive definite if $\alpha < 0$.

Then we can write the leading terms explicitly. The leading orders are same with those for the Poisson point process and the differences appear in the leading constants. The leading constants also tend to those for the Poisson point process as α tends to 0. The results are Theorems 1, 2, 3 below. If the strict positivity (P) is not satisfied and $\alpha < 0$, then we lost the explicit leading terms, from which we conjecture that the leading orders are also different with those for the Poisson point process. This case includes the following well-studied cases: $\alpha = -1$ and

$$K(x, y) = K_s(x, y) := \frac{\sin \pi(x-y)}{\pi(x-y)} \text{ for } x, y \in \mathbb{R} \tag{1.4}$$

or

$$\begin{aligned} K(x, y) &= K_g(x, y) \\ &:= \frac{1}{\pi} \exp\Big(-\frac{|x|^2}{2} - \frac{|y|^2}{2} + (x_1 - ix_2)(y_1 + iy_2) \Big) \text{ for } x, y \in \mathbf{R}^2. \end{aligned} \tag{1.5}$$

The kernel (1.4) is called the sine kernel, and μ_{-1,K_g} with the kernel (1.5) is called the Ginibre random point field (cf.[13]). μ_{-1,K_s} and μ_{-1,K_g} describe the equilibrium states of the infinite number of Brownian particles interacting via 1 and 2-dimensional Coulomb potentials, respectively. For these aspects and other relating aspects, we refer Soshnikov [26]. In particular, the asymptotics of the hole probabilities $\mu_{-1,K_s}(\xi((-R, R)) = 0)$ and $\mu_{-1,K_g}(\xi(B(R)) = 0)$ as $R \to \infty$ are known, where $B(R) = \{x \in \mathbb{R}^2 : |x| < R\}$. From these we know the leading terms of corresponding $N(\lambda)$ as $\lambda \downarrow 0$ if $\operatorname{supp} u$ is compact. The results are Theorem 4 and (6.1) below. The asymptotics of the hole probability $\mu_{-1,K_g}(\xi(B(R)) = 0)$ to obtain the asymptotics of the integrated density of states is same for another famous example, the case that μ_{-1,K_g} is replaced by the probability distribution μ_{GAF} of $\sum_{a \in \mathcal{X}} \delta_a$ for the sets $\mathcal{X}$ of the zeros of a Gaussian analytic function

$$f_{GFA}(z) = \sum_{n=0}^{\infty} (X_n + iY_n) \sqrt{\frac{L^n}{n!}} z^n \text{ for } z = x_1 + ix_2 \in \mathbf{C}, \tag{1.6}$$

where $L \in (0, \infty)$ and $\{X_n, Y_n\}_n$ are independently and identically distributed random valuables obeying the normal distribution N(0,1/2) with the mean 0 and the valiance 1/2. Moreover the asymptotics of $N(\lambda)$ for the compact support single site potential is determined until the leading constant in the 1-dimensional case and is determined until the leading order in the 2-dimensional case. Thus we state the results in the general form as Theorems 4 and 5 below. In the higher dimensional cases, the author conjecture that more precise information of the random field may be necessary since complex states may contribute to the leading term of the asymptotics of $N(\lambda)$. For this aspect, refer [8] for example. The obtained leading orders for the sine kernel and the Ginibre field are different with those for the Poisson point process, and the results mean that the low lying spectrum becomes thinner.

The difference of the behavior of these particle from those without interactions are studied recently (cf. Osada [20]). If the interaction is absent, the Poisson point process can describe the equilibrium states. For a milder interaction, Sznitman [28] determined the asymptotic behavior of the survival probability of the Brownian motion among the traps around the sample points of the corresponding Gibbs measure. This result is equivalent with that of the asymptotic behavior of $N(\lambda)$ as $\lambda \downarrow 0$ is determined in the case where $\mu_{\alpha,K}$ is replaced by the Gibbs measure and $\operatorname{supp} u$ is compact. In this case, the leading order of the corresponding asymptotics of the density of states are same with the Poisson case. The equivalence holds for general random point fields and our results also determine the survival probability of the Brownian motion among the traps around the sample points of our random point fields.

In the case that $\operatorname{supp} u$ is not compact, the sufficient upper estimate for the asymtotics of $N(\lambda)$ as $\lambda \downarrow 0$ has been obtained by applying the Donsker and Varadhan's large deviation theory for the Poisson case [18], [19]. For the application, the key step is the compactification of the configuration space. In a similar setting, the corresponding compactification is introduced in Section 10.C in [22], where the same asymptotic problem is considered for the case that the potential is given as a smoothed square of a Gaussian random field. However the same method seems to need extra conditions in our case if $\alpha < 0$. In this paper, we apply another compactification by Gärtner and König [9]. For this aspect, see Section 4 and Remark 4.1 below.

The organization of this paper is as follows. In the following three sections, we assume the positivity (P) and give the leading term for the asymptotics of $N(\lambda)$ as $\lambda \downarrow 0$: in Section 2, we treat the case that the effect of the potential is strong and the leading term is determined mainly by the classical effect. In Section 3, we treat the case that the effect of the potential is weak and the leading term is determined mainly by the quantum effect. In Section 4, we treat the critical case between the above two cases, where the leading term is determined by both the classical and the quantum effects. This situation is common with the same problem for the Poisson point process. In the last two sections, we treat cases without the condition (P): in Section 5, we treat the random field μ_{-1,K_s} with the sine kernel K_s. In Section 6, we treat the Ginibre random point field μ_{-1,K_g}. In Appendix A, we determine the asymptotics of the survival probability of the Brownian motion among the traps around the sample points of our random point fields.

2. Slowly decreasing potentials with the condition (P)

Our first result is the following:

Theorem 1. *In the above setting, we further assume the condition (P), the positivity of u, and that* $u(x) = C_0|x|^{-\beta}(1 + o(1))$ *as* $|x| \to \infty$ *with some* $\beta \in (d, d+2)$*. Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^\gamma \log N(\lambda) = -C_{d,\beta} C_0^\gamma C_{d,\beta,\alpha,K}^{\gamma+1}, \tag{2.1}$$

where $\gamma = d/(\beta - d)$, $C_{d,\beta} = (\beta - d)d^\gamma/\beta^{\gamma+1}$,

$$C_{d,\beta,\alpha,K} = |S^{d-1}| \int_0^\infty dr r^{d-1} \frac{1}{\alpha} \{\log(I + \alpha(1 - \exp(-r^{-\beta}))K)\}(0, 0), \tag{2.2}$$

$|S^{d-1}|$ *is the* $d-1$*-dimensional volume of the* $d-1$*-dimensional unit surface* $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$*, and* $(\{\log(I + \alpha(1 - \exp(-r^{-\beta}))K)\}(x, y))_{x,y \in \mathbb{R}^d}$ *is the integral kernel of the operator* $\log(I + \alpha(1 - \exp(-r^{-\beta}))K)$

Remark 2.1. When K is the convolution operator defined by $K(x, y) = k(x - y)$, the constant in (2.2) is written in terms of the Fourier transform

$$\widehat{k}(\zeta) = \int_{\mathbb{R}^d} \exp(-2\pi i \zeta \cdot x) k(x) dx$$

as

$$\frac{1}{\alpha}\{\log(I + \alpha t K)\}(0, 0) = \int_{\mathbb{R}^d} d\zeta \frac{1}{\alpha} \log(1 + \alpha t \widehat{k}(\zeta)),$$

since

$$\begin{aligned} K^n(0,0) &= \int k(-x_1) \Big(\prod_{j=1}^{n-2} k(x_j - x_{j+1}) \Big) k(x_{n-1}) \prod_{j=1}^{n} dx_j \\ &= \int \exp\Big(-2\pi i x_1 \cdot \zeta_1 + \sum_{j=1}^{n-2} 2\pi i (x_j - x_{j+1}) \cdot \zeta_j \\ &\qquad + 2\pi i x_{n-1} \cdot \zeta_n \Big) \Big(\prod_{j=1}^{n} \widehat{k}(\zeta_j) dx_j d\zeta_j \Big) \\ &= \int \exp\Big(-\sum_{j=1}^{n-1} 2\pi i x_j \cdot (\zeta_j - \zeta_{j+1}) \Big) \Big(\prod_{j=1}^{n} \widehat{k}(\zeta_j) dx_j d\zeta_j \Big) \\ &= \int \widehat{k}(\zeta)^n d\zeta \end{aligned}$$

for any $n \in \mathbb{N}$, where $(K^n(x, y))_{x,y \in \mathbb{R}^d}$ is the integral kernel of the operator K^n.

The order γ is same the case that $\mu_{\alpha,K}$ is replaced by the Poisson point process. The constant $C_{d,\beta,\alpha,K}$ converges to that for the Poisson point process with the intensity $K(0, 0)dx$. The result in the Poisson case was proven by Pastur [21]. The results are independent of the constant h. In fact these asymptotics coincide with those of the corresponding classical integrated density of states defined by

$$N_c(\lambda) = \int \mu_{\alpha,K}(d\xi)[|\{(x,p) \in \Lambda_R \times \mathbb{R}^d : H_{\xi,c}(x,p) \le \lambda\}|](2\pi\sqrt{h}R)^{-d}$$

for any $R \in \mathbb{N}$, where $|\cdot|$ is the $2d$-dimensional Lebesgue measure and

$$H_{\xi,c}(x,p) = \sum_{j=1}^{d} p_j^2 + V_\xi(x)$$

is the classical Hamiltonian (cf. [16]). Therefore we may say that only the classical effect from the scalar potential determines the leading term for $\beta < d+2$.

Proof. We first investigate the leading term as $t \to \infty$ of the Laplace transform of the integrated density of states represented by the expectation of the diagonal part of the integral kernel of the heat semigroup generated by the Schrödinger operator

$$\widetilde{N}(t) := \int_{\mathbb{R}} e^{-t\lambda} dN(\lambda) = \int \mu_{\alpha,K}(d\xi) \exp(-tH_\xi)(0,0) \tag{2.3}$$

((5.17) in [22]) and the Feynman-Kac formula

$$\begin{aligned}&\exp(-tH_\xi)(x,y)\\ =&E_{0,x}^{2ht,y}\Big[\exp\Big(-\frac{1}{2h}\int_0^{2ht} V_\xi(w(s))ds\Big)\Big]\exp\Big(-\frac{|x-y|^2}{4ht}\Big)\frac{1}{(4\pi ht)^{d/2}},\end{aligned}$$

where $E_{0,x}^{2ht,y}$ is the expectation with respect to the d-dimensional Brownian motion w conditioned that $w(0) = x$ and $w(2ht) = y$.

For the upper estimate, we use the bound

$$\widetilde{N}(t) \le \widetilde{N}_1(t)(4\pi ht)^{-d/2} \tag{2.4}$$

in Theorem (9.6) in [22], where

$$\widetilde{N}_1(t) = \int \mu_{\alpha,K}(d\xi)\exp(-tV_\xi(0)).$$

By (1.1), this is rewritten as

$$\widetilde{N}_1(t) = \exp\Big(\frac{-1}{\alpha}\operatorname{Tr}\log(I + \alpha\sqrt{1-e^{-tu(-\cdot)}}K\sqrt{1-e^{-tu(-\cdot)}})\Big). \tag{2.5}$$

By Mercer's theorem, we have

$$\begin{aligned}&\mathrm{Tr}[(\sqrt{1-e^{-tu(-\cdot)}}K\sqrt{1-e^{-tu(-\cdot)}})^n]\\&=\int_{\mathbb{R}^{dn}}\Big(\prod_{j=1}^n dx_j(1-e^{-tu(-x_j)})\Big)K(x_1,x_2)K(x_2,x_3)\cdots K(x_n,x_1)\end{aligned}$$

for any $n\in\mathbb{N}$. By changing the variable as $x_1\to t^{1/\beta}x_1$, the right hand side is rewritten as

$$\begin{aligned}&t^{d/\beta}\int_{\mathbb{R}^d}dx_1(1-e^{-tu(-t^{1/\beta}x_1)})\int_{\mathbb{R}^{d(n-1)}}\Big(\prod_{j=2}^n dx_j(1-e^{-tu(-x_j)}\Big)\\&\times K(t^{1/\beta}x_1,x_2)K(x_2,x_3)\cdots K(x_n,t^{1/\beta}x_1).\end{aligned}$$

By changing the variable as $x_j-t^{1/\beta}x_1\to x_j$ for $2\le j\le n$, this is rewritten as

$$\begin{aligned}&t^{d/\beta}\int_{\mathbb{R}^d}dx_1(1-e^{-tu(-t^{1/\beta}x_1)})\int_{\mathbb{R}^{d(n-1)}}\Big(\prod_{j=2}^n dx_j(1-e^{-tu(-x_j-t^{1/\beta}x_1)}\Big)\\&\times K(0,x_2)K(x_2,x_3)\cdots K(x_{n-1},x_n)K(x_n,0).\end{aligned}$$

By the square integrability of $K(x,y)$ in each variable, we have

$$\begin{aligned}&\lim_{t\to\infty}t^{-d/\beta}\,\mathrm{Tr}[(\sqrt{1-e^{-tu(-\cdot)}}K\sqrt{1-e^{-tu(-\cdot)}})^n]\\&=\int_{\mathbf{R}^d}dx(1-\exp(-C_0|x|^{-\beta}))^nK^n(0,0).\end{aligned}\tag{2.6}$$

Since the function $\log(1+\alpha t)/(\alpha t)$ of the variable t is uniformly approximated by polynomials on the interval $[0,\|K\|_{op}]$, we have

$$\begin{aligned}&\lim_{t\to\infty}t^{-d/\beta}\frac{1}{\alpha}\,\mathrm{Tr}\log(I+\alpha\sqrt{1-e^{-tu(-\cdot)}}K\sqrt{1-e^{-tu(-\cdot)}})\\&=C_0^{d/\beta}C_{d,\beta,\alpha,K}.\end{aligned}\tag{2.7}$$

Indeed (2.7) is proven as follows: for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

$$\sup_{t\in[0,\|K\|_{op}]}\left|\frac{1}{\alpha t}\log(1+\alpha t)-\sum_{n=0}^N\frac{(-\alpha t)^n}{n+1}\right|<\varepsilon.$$

Then we have

$$\begin{aligned}&\left|\frac{1}{t^{d/\beta}}\,\mathrm{Tr}\log(I+\alpha\sqrt{1-e^{-tu(-\cdot)}}K\sqrt{1-e^{-tu(-\cdot)}})-\frac{1}{\alpha}\int_{\mathbf{R}^d}dx\{\log(I+\alpha(1-\exp(-C_0|x|^{-\beta}))K)\}(0,0)\right|\\&\le I_1+I_2+I_3,\end{aligned}$$

where

$$I_1 = \Big| \frac{1}{t^{d/\beta}} \operatorname{Tr} \log(I + \alpha\sqrt{1 - e^{-tu(-\cdot)}} K \sqrt{1 - e^{-tu(-\cdot)}})$$

$$- \sum_{n=0}^{N} \frac{(-\alpha)^n}{(n+1)t^{d/\beta}} \operatorname{Tr}[(\sqrt{1 - e^{-tu(-\cdot)}} K \sqrt{1 - e^{-tu(-\cdot)}})^{n+1}] \Big|$$

$$I_2 = \Big| \sum_{n=0}^{N} \frac{(-\alpha)^n}{(n+1)t^{d/\beta}} \operatorname{Tr}[(\sqrt{1 - e^{-tu(-\cdot)}} K \sqrt{1 - e^{-tu(-\cdot)}})^{n+1}]$$

$$- \int_{\mathbf{R}^d} dx \sum_{n=0}^{N} \frac{(-\alpha)^n}{n+1} (1 - \exp(-C_0|x|^{-\beta}))^{n+1} K^{n+1}(0,0) \Big|$$

and

$$I_3 = \Big| \int_{\mathbf{R}^d} dx \sum_{n=0}^{N} \frac{(-\alpha)^n}{n+1} (1 - \exp(-C_0|x|^{-\beta}))^{n+1} K^{n+1}(0,0)$$

$$- \frac{1}{\alpha} \int_{\mathbf{R}^d} dx \{\log(I + \alpha(1 - \exp(-C_0|x|^{-\beta}))K)\}(0,0) \Big|$$

By (2.6) with $n = 1$, we have

$$I_1 \le \frac{\varepsilon}{t^{d/\beta}} \operatorname{Tr}[\sqrt{1 - e^{-tu(-\cdot)}} K \sqrt{1 - e^{-tu(-\cdot)}}] \le c_1 \varepsilon$$

with some $c_1 \in (0, \infty)$ independent of t. By (2.6) with general n, we have $I_2 \to 0$ as $t \to \infty$. We also have

$$I_3 \le \varepsilon \int_{\mathbf{R}^d} dx \{\log(I + \alpha(1 - \exp(-C_0|x|^{-\beta}))K)\}(0,0).$$

By these we obtain (2.7). From (2.7), we have

$$\varlimsup_{t\to\infty} t^{-d/\beta} \log \widetilde{N}(t) \le \varlimsup_{t\to\infty} t^{-d/\beta} \log \widetilde{N}_1(t) \le -C_0^{d/\beta} C_{d,\beta,\alpha,K},$$

where $\|\cdot\|_{op}$ is the operator norm.

For the lower estimate, we use the bound

$$\widetilde{N}(t) \ge R^{-d} \exp(-th\|\nabla\psi_R\|_2^2) \widetilde{N}_2(t) \tag{2.8}$$

which holds for any $R \ge 1$ and $\psi_1 \in C_0^\infty(\Lambda_1)$ such that $\|\psi_1\|_2 = 1$, where $\psi_R = \psi_1(\cdot/R)/R^{d/2}$, $\|\cdot\|_2$ is the L^2-norm, and

$$\widetilde{N}_2(t) = \int \mu_{\alpha,K}(d\xi) \exp\Big(- t \int dx \psi_R(x)^2 V_\xi(x) \Big)$$

(Theorem (9.6) in [22]). As in (2.5), we have

$$\widetilde{N}_2(t) = \exp\Big(\frac{-1}{\alpha} \operatorname{Tr} \log \Big(I + \alpha \Big\{ 1 - \exp\Big(- t \int dx \psi_R(x)^2 u(x - \cdot) \Big) \Big\}^{1/2}$$

$$\times K \Big\{ 1 - \exp\Big(- t \int dx \psi_R(x)^2 u(x - \cdot) \Big) \Big\}^{1/2} \Big) \Big).$$

We take R as $t^{1/\beta}$. Then, as in the upper estimate, we have

$$\begin{aligned}
&\mathrm{Tr}\Big[\Big(\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\\
&\times K\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\Big)^n\Big]\\
=&t^{d/\beta}\int_{\mathbb{R}^d}dy_1\Big\{1-\exp\Big(-\int dx_1\psi_1(x_1)^2tu(t^{1/\beta}(x_1-y_1))\Big)\Big\}\\
&\times\int_{\mathbb{R}^{d(n-1)}}\Big(\prod_{j=2}^n dy_j\Big\{1-\exp\Big(-\int dx_j\psi_1(x_j)^2tu(t^{1/\beta}(x_j-y_1)-y_j))\Big)\Big\}\\
&\times K(0,y_2)K(y_2,y_3)\cdots K(y_{n-1},y_n)K(y_n,0)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to\infty}t^{-d/\beta}\,\mathrm{Tr}\Big[\Big(\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\\
&\times K\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\Big)^n\Big]\\
=&\int_{\mathbb{R}^d}dy\Big(1-\exp\Big(-\int\frac{dxC_0\psi_1(x)^2}{|x-y|^\beta}\Big)\Big)^nK^n(0,0)
\end{aligned}$$

for any $n\in\mathbb{N}$. Thus we have

$$\begin{aligned}
&\lim_{t\to\infty}t^{-d/\beta}\frac{1}{\alpha}\,\mathrm{Tr}\log\Big(I+\alpha\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\\
&\times K\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\Big)\\
=&\frac{1}{\alpha}\int_{\mathbb{R}^d}dy\Big\{\log\Big(I+\alpha\Big(1-\exp\Big(-\int\frac{dxC_0\psi_1(x)^2}{|x-y|^\beta}\Big)\Big)K\Big)\Big\}(0,0),
\end{aligned}\tag{2.9}$$

where

$$\Big(\Big\{\log\Big(I+\alpha\Big(1-\exp\Big(-\int\frac{dxC_0\psi_1(x)^2}{|x-y|^\beta}\Big)\Big)K\Big\}(x',y')\Big)_{x',y'\in\mathbb{R}^d}$$

is the integral kernel of the operator

$$\log\Big(I+\alpha\Big(1-\exp\Big(-\int\frac{dxC_0\psi_1(x)^2}{|x-y|^\beta}\Big)\Big)K\Big).$$

Since $\beta<d+2$, we have

$$\begin{aligned}
&\varliminf_{t\to\infty}t^{-d/\beta}\log\widetilde{N}(t)\geq\varliminf_{t\to\infty}t^{-d/\beta}\log\widetilde{N}_2(t)\\
\geq&-\frac{1}{\alpha}\int_{\mathbb{R}^d}dy\Big\{\log\Big(I+\alpha\Big(1-\exp\Big(-\int\frac{dxC_0\psi_1(x)^2}{|x-y|^\beta}\Big)\Big)K\Big)\Big\}(0,0).
\end{aligned}$$

Since ψ_1 is arbitrary, we have

$$\lim_{t\to\infty}t^{-d/\beta}\log\widetilde{N}(t)=-C_0^{d/\beta}C_{d,\beta,\alpha,K}.$$

Now we can complete the proof by the Tauberian theorem. □

3. Rapidly decreasing potentials with the condition (P)

Our second result is the following:

Theorem 2. *In the above setting, we further assume the condition (P), the nonnegativity of u, the existence of $\varepsilon_0, r_0 > 0$ such that $u \geq \varepsilon_0$ on $B(r_0)$, and that $u(x) = o(|x|^{-d-2})$ as $|x| \to \infty$. Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{d/2} \log N(\lambda) = -h^{d/2} \lambda_1(-\Delta_{B(1)})^{d/2} C_{\alpha,K} |B(1)|, \tag{3.1}$$

where

$$C_{\alpha,K} = \frac{1}{\alpha}\{\log(I + \alpha K)\}(0,0),$$

$(\{\log(I + \alpha K)\}(x,y))_{x,y \in \mathbb{R}^d}$ *is the integral kernel of the operator* $\log(1 + \alpha K)$, $|B(1)|$ *is the volume of the d-dimensional unit ball* $B(1) = \{x \in \mathbb{R}^d : |x| \leq 1\}$, $-\Delta_{B(1)}$ *is the Dirichlet Laplacian on* $B(1)$ *and* $\lambda_1(-\Delta_{B(1)})$ *is the least eigenvalue of* $-\Delta_{B(1)}$.

The result is also same with that for the case that $\mu_{\alpha,K}$ is replaced by the Poisson point process with the intensity $C_{\alpha,K}$. That theorem is well known as one of the successful application of the Donsker and Varadhan's large deviation theorem: Nakao [18] showed that the behavior was essentially proved in Donsker and Varadhan [6].

Contrarily to the results in the last subsection, the result in these theorems depend on h and the right hand side of (3.1) are strictly less than that of (2.1). Therefore we may say that the quantum effect appears in Theorem 2. The existence of such behavior was pointed out by the physicist, Lifschits [17] and is called the Lifschitz behavior.

Proof.

As in the last subsection, we investigate the leading term as $t \to \infty$ of the Laplace transform $\widetilde{N}(t)$ and use the Tauberian theorem.

For the lower estimate, we still use the bound (2.8). In this case, we take $R = t^{1/(d+2)}$. Then we have

$$\begin{aligned}
&\lim_{t\to\infty} t^{-d/(d+2)} \operatorname{Tr}\Big[\Big(\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2} \\
&\times K\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\Big)^n\Big] \\
=&|\operatorname{supp}\psi_1|K^n(0,0), \\
&\lim_{t\to\infty} t^{-d/(d+2)}\frac{1}{\alpha}\operatorname{Tr}\log\Big(I+\alpha\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2} \\
&\times K\Big\{1-\exp\Big(-t\int dx\psi_R(x)^2u(x-\cdot)\Big)\Big\}^{1/2}\Big) \\
=&C_{\alpha,K}|\operatorname{supp}\psi_1|,
\end{aligned}$$

and

$$\begin{aligned}
&\varliminf_{t\to\infty} t^{-d/(d+2)}\log\widetilde{N}(t) \ge -h\|\nabla\psi_1\|_2^2 + \varliminf_{t\to\infty} t^{-d/(d+2)}\log\widetilde{N}_2(t) \\
&\ge -h\|\nabla\psi_1\|_2^2 - C_{\alpha,K}|\operatorname{supp}\psi_1|.
\end{aligned}$$

We here used also the condition $u \ge \varepsilon_0$ on $B(r_0)$. Since ψ_1 is arbitrary, we have

$$\begin{aligned}
&\varliminf_{t\to\infty} t^{-d/(d+2)}\log\widetilde{N}(t) \ge -\inf_{R>0}\Big\{\frac{h\lambda_1(-\Delta^D_{B(1)})}{R^2} + C_{\alpha,K}R^d|B(1)|\Big\} \\
=&-(C_{\alpha,K}|B(1)|)^{2/(d+2)}\frac{d+2}{2}\Big(\frac{2}{d}h\lambda_1(-\Delta^D_{B(1)})\Big)^{d/(d+2)} =: -C_{\alpha,K,d,h}.
\end{aligned}$$

The upper estimate was firstly obtained by the Donsker and Varadhan's large deviation theory for the Poisson case. By the same theory, we can treat the present case as in our proof of the next theorem, Theorem 3. We here discuss a second proof following Sznitman's coarse graining method for the Poisson case [29]: we follow the second version of the upper bound in Section 4.5 in [29]. This method is useful when the single site potential u has a compact support. Moreover the vantage point of this method is applicable to the case that the single site potentials u around the sample points of ξ are replaced by the Dirichlet boundary condition on the nonpolar sets around the sample points. For these aspects, refer Appendix A below. As in [18] and [29], the problem is reduced to the upper estimate of

$$\widetilde{N}_3(t) = \int \mu_{\alpha,K}(d\xi)E\Big[\exp\Big(-\frac{1}{2h}\int_0^{2ht}V_\xi(w(s))ds\Big)\Big], \tag{3.2}$$

where E is the expectation with respect to the d-dimensional Brownian motion w starting at 0. It is enough to show

$$\varlimsup_{t\to\infty} t^{-d/(d+2)}\log\widetilde{N}_3(t) \le -C_{\alpha,K,d,h=1/2}$$

in the case that $h = 1/2$ and $u = C_0 1_{B(r_0)}$ for some $C_0, r_0 \in (0, \infty)$, where $B(r_0) = \{x \in \mathbb{R}^2 : |x| < r_0\}$. As in [29], we take $\varepsilon = t^{-1/(d+2)}$ and use the scaling property of the Brownian motion to rewrite as

$$\widetilde{N}_3(t) = \int \mu_{\alpha,K_\varepsilon}(d\xi) E\Big[\exp\Big(- \int_0^\tau ds \int \xi(dx) u^\varepsilon(w(s) - x)\Big)\Big], \tag{3.3}$$

where $\tau = t^{d/(d+2)}$, K_ε is the integral operator with the kernel $(K(x/\varepsilon, y/\varepsilon)/\varepsilon^d)_{x,y\in\mathbb{R}^d}$, and $u^\varepsilon = u(\cdot/\varepsilon)/\varepsilon^2$. Now the only difference with the Poisson case is the law of the point process ξ. Thus we have only to modify Proposition 4.2 and the probabilistic estimate in (5.81) in Chapter 4 of [29] if $d \geq 2$. A sufficient modification can be obtained since the following property means our point process behaves similarly as the original Poisson point process with the intensity $C_{\alpha,K}/\varepsilon^d$: for any Borel set A in $\mathbb{R}^d$ with $|A| < \infty$, we have

$$\begin{aligned}
&\mu_{\alpha,K_\varepsilon}(\xi(A) = 0) = \lim_{s\to\infty} \int \mu_{\alpha,K_\varepsilon}(d\xi) \exp(-s\xi(A)) \\
&= \lim_{s\to\infty} \exp\Big(\frac{-1}{\alpha} \operatorname{Tr} \log(I + \alpha\sqrt{1 - \exp(-s1_A)} K_\varepsilon \sqrt{1 - \exp(-s1_A)})\Big) \\
&= \exp\Big(\frac{-1}{\alpha} \operatorname{Tr} \log(I + \alpha 1_A K_\varepsilon 1_A)\Big)
\end{aligned}$$

and

$$\lim_{t\to\infty} \varepsilon^d \log \mu_{\alpha,K_\varepsilon}(\xi(A) = 0) = -C_{\alpha,K}|A| \tag{3.4}$$

as in (2.7). The proof for the case of $d = 1$ is given by Theorem 3.1 in Chapter 3 of [29]. □

4. Critically decreasing potentials with the condition (P)

For the critical case between the last two subsections, we prove the following:

Theorem 3. *In the above setting, we further assume the condition (P), the positivity of u, and that $u(x) = C_0|x|^{-d-2}(1 + o(1))$ as $|x| \to \infty$. Then we have*

$$\lim_{\lambda\downarrow 0} \lambda^{d/2} \log N(\lambda) = \frac{-2d^{d/2}}{(d+2)^{1+d/2}} C(h, C_0, \alpha, K)^{1+d/2},$$

where $C(h, C_0, \alpha, K)$ is

$$\inf\Big\{h \int_{\mathbb{R}^d} |\nabla\psi(x)|^2 dx + \Phi(\psi^2) : \psi \in C^\infty(\mathbb{R}^d), \int \psi(x)^2 dx = 1\Big\},$$

$$\begin{aligned}
&\Phi(\phi) \\
&:= \int_{\mathbf{R}^d} dy \frac{1}{\alpha}\Big\{ \log\Big(I + \alpha\Big(1 - \exp\Big(- C_0 \int_{\mathbf{R}^d} \frac{\phi(x)dx}{|x - y|^{d+2}}\Big)\Big)K\Big)\Big\}(0, 0)
\end{aligned} \tag{4.1}$$

for any nonnegative integrable function ϕ*, and*

$$\Big(\Big\{\log\Big(I+\alpha\Big(1-\exp\Big(-C_0\int_{\mathbb{R}^d}\frac{\phi(x)dx}{|x-y|^{d+2}}\Big)\Big)K\Big)\Big\}(x',y')\Big)_{x',y'\in\mathbb{R}^d}$$

is the integral kernel of the operator

$$\log\Big(I+\alpha\Big(1-\exp\Big(-C_0\int_{\mathbb{R}^d}\frac{\phi(x)dx}{|x-y|^{d+2}}\Big)\Big)K\Big).$$

The order $d/2$ is also same with that for the case that $\mu_{\alpha,K}$ is replaced by the Poisson point process with the intensity $C_{\alpha,K}$. The quantity $C(h,C_0,\alpha,K)$ also converges to that for the Poisson point process with the intensity $K(0,0)dx$. The result for the Poisson case was obtained in Ôkura [19]. The results show that both of the quantum effect and the potential appear in the leading term.

Proof.

As in the last two subsections, we investigate the leading term as $t\to\infty$ of the Laplace transform $\widetilde{N}(t)$ and use the Tauberian theorem.

For the lower estimate we modify the proof for the lower estimate of Theorem 1 as $\psi_1\in C_0^\infty(\Lambda_s)$ for any $s\geq 1$ and $R=t^{1/(d+2)}$. Then we obtain

$$\varliminf_{t\to\infty} t^{-d/(d+2)}\log\widetilde{N}(t)\geq -C'(h,C_0,\alpha,K),$$

where $C'(h,C_0,\alpha,K)$ is the quantity obtained by replacing $C^\infty(\mathbb{R}^d)$ by the space $C_0^\infty(\mathbb{R}^d)$ of all smooth functions with compact supports in the definition of $C(h,C_0,\alpha,K)$. Since $\Phi(\psi^2)<\infty$ for any $\psi\in C_0^\infty(\mathbb{R}^d)$, we have $C(h,C_0,\alpha,K)\leq C'(h,C_0,\alpha,K)<\infty$. For any $\varepsilon>0$, we have $\psi_\varepsilon\in C^\infty(\mathbb{R}^d)$ such that $\|\psi_\varepsilon\|_2=1$ and $h\|\nabla\psi_\varepsilon\|_2^2+\Phi(\psi_\varepsilon^2)\leq C(h,C_0,\alpha,K)+\varepsilon$. By the integrability, we have $R_\varepsilon\in(0,\infty)$ such that $\|1_{B(R_\varepsilon)^c}\psi_\varepsilon\|_2\leq\varepsilon$. We now take $\zeta_\varepsilon\in C_0^\infty(\mathbb{R}^d)$ such that $\zeta_\varepsilon=1$ on $B(R_\varepsilon)$ and $|\nabla\zeta_\varepsilon|\leq\varepsilon$. Then we have $\widetilde{\psi_\varepsilon}:=\zeta_\varepsilon\psi_\varepsilon/\|\zeta_\varepsilon\psi_\varepsilon\|_2\in C_0^\infty(\mathbb{R}^d)$ such that $\|\widetilde{\psi_\varepsilon}\|_2=1$. By a simple calculation, we have $\|\nabla\widetilde{\psi_\varepsilon}\|_2\leq(\|\nabla\psi_\varepsilon\|_2+\varepsilon)/(1-\varepsilon)$. We also have $\Phi(\widetilde{\psi_\varepsilon}^2)\leq\Phi(\psi_\varepsilon^2/\|\zeta_\varepsilon\psi_\varepsilon\|_2^2)=\Psi(\psi_\varepsilon^2)(1+o(1))$ as $\varepsilon\downarrow 0$. Thus we obtain $h\|\nabla\widetilde{\psi_\varepsilon}\|_2^2+\Phi(\widetilde{\psi_\varepsilon}^2)\leq C(h,C_0,\alpha,K)(1+o(1))$ as $\varepsilon\downarrow 0$. Therefore $C'(h,C_0,\alpha,K)=C(h,C_0,\alpha,K)$ and we obtained the necessary lower estimate.

To obtain the upper estimate, we apply the Donsker and Varadhan's large deviation theory as in Gärtner and König [9] for the seccond term for the negative Poisson potential. The idea of Gärtner and König is to apply the estimate (4.3) below. Before this, we use the reduction in Ôkura [19]: for arbitrarily fixed $C_1\in(0,C_0)$, we take a nonnegative continuous function ρ and a nonnegative Borel measurable function v such that $\int\rho(x)dx=1$, $\operatorname{supp}\rho\subset B(1)$, $v(x)=C_1|x|^{-d-2}(1+o(1))$ as $|x|\to\infty$, and $u \geq \rho * v$, where $(\rho * v)(x) = \int \rho(x-y)v(y)dy$ is the convolution. Then the proof is reduced to show

$$\overline{\lim_{t\to\infty}}\, t^{-d/(d+2)} \log \widetilde{N_1}(t) \leq -C(h, C_1, \alpha, K),$$

where $\widetilde{N_1}(t)$ is the Laplace transform of the integrated density of states for the operator H_ξ where the single site potential u is replaced by $\rho * v$. As in (3.3), we take $\varepsilon = t^{-1/(d+2)}$ and $\tau = t^{d/(d+2)}$ and use the scaling property of the Brownian motion to rewrite as

$$\widetilde{N_1}(t) = \int \mu_{\alpha,K_\varepsilon}(d\xi) E_{0,0}^{2h\tau,0}\Big[\exp\Big(-\frac{1}{2h}\int_0^{2h\tau} ds V_\xi^\varepsilon(w(s))\Big)\Big]\frac{1}{(4\pi ht)^{d/2}},$$

where

$$V_\xi^\varepsilon(x) = \int \xi(dy)\int dz \rho_\varepsilon(x-y-z)v^\varepsilon(z),$$

$\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^d$ and $v^\varepsilon(x) = v(x/\varepsilon)/\varepsilon^2$. For $R(t) = t\mathcal{R}$ with a positive constant $\mathcal{R}$, we have

$$P_{0,0}^{2h\tau,0}\Big(\sup_{0\leq s\leq 2h\tau}|w(s)| \geq R(h\tau)\Big)\frac{1}{(4\pi ht)^{d/2}} \leq \exp(-c_1\tau\mathcal{R}^2),$$

which decays faster than $\widetilde{N}(t)$ if $\mathcal{R}$ is sufficiently large. Thus the problem is reduced to the asymptotics of

$$\begin{aligned}\widetilde{N_2}(t) = \int \mu_{\alpha,K_\varepsilon}(d\xi) E_{0,0}^{2h\tau,0}\Big[&\exp\Big(-\frac{1}{2h}\int_0^{2h\tau} ds V_\xi^\varepsilon(w(s))\Big)\\ &: \sup_{0\leq s\leq 2h\tau}|w(s)| < R(h\tau)\Big]\frac{1}{(4\pi ht)^{d/2}}.\end{aligned}$$

By

$$\int_0^{2h\tau} ds V_\xi^\varepsilon(w(s)) \geq \int_\delta^{2h\tau-\delta} ds V_\xi^\varepsilon(w(s))$$

and

$$e^{\delta\Delta/2}(x,y) \leq 1/(2\pi\delta)^{d/2},$$

with small $\delta > 0$, the problem is reduced to the asymptotics of

$$\begin{aligned}\widetilde{N_3}(t) = \int \mu_{\alpha,K_\varepsilon}(d\xi)&\int_{B(R(h\tau))} dx_1 \int_{B(R(h\tau))} dx_2\\ &\times \exp\Big(-\frac{|x_1-x_2|^2}{4h\tau}\Big)\frac{1}{(4\pi h\tau)^{d/2}}\\ &\times E_{0,x_1}^{2h\tau,x_2}\Big[\exp\Big(-\frac{1}{2h}\int_0^{2h\tau} ds V_\xi^\varepsilon(w(s))\Big)\\ &: \sup_{0\leq s\leq 2h\tau}|w(s)| < R(h\tau)\Big].\end{aligned} \tag{4.2}$$

We now apply the estimate in Proposition 1 in Gärtner and König [9]: by (3.2) in [9],

$$\int_{\Lambda_r}\frac{d\theta}{|\Lambda_r|}\int_0^{2h\tau} ds \Phi_r(w(s)-\theta) \leq \frac{c_2 h^2\tau}{r},$$

where

$$\Phi_r(x) = \sum_{k\in\mathbb{Z}^d} h|\nabla\eta(rk+x)|^2$$

and η is a real smooth function such that $\eta = 1$ on Λ_{r-1}, $\eta = 0$ on Λ_{r+1}^c and

$$\sum_{k\in\mathbb{Z}^d} \eta(rk+x)^2 = 1$$

on $\mathbb{R}^d$. By this estimate and and the Jensen inequality, we have

$$\begin{aligned}
\widetilde{N_3}(t) \leq& \exp\Big(\frac{c_2 h\tau}{2r}\Big)\int_{\Lambda_r}\frac{d\theta}{|\Lambda_r|}\int \mu_{\alpha,K_\varepsilon}(d\xi)\int_{B(R(h\tau))}dx_1\int_{B(R(h\tau))}dx_2 \\
&\times \exp\Big(-\frac{|x_1-x_2|^2}{4h\tau}\Big)\frac{1}{(4\pi h\tau)^{d/2}} \\
&\times E_{0,x_1}^{2h\tau,x_2}\Big[\exp\Big(-\frac{1}{2h}\int_0^{2h\tau}ds(\Phi_r(w(s)-\theta)+V_\xi^\varepsilon(w(s)))\Big) \\
&\qquad : \sup_{0\leq s\leq 2h\tau}|w(s)|<R(h\tau)\Big] \\
\leq& \exp\Big(\frac{c_2 h\tau}{2r}\Big)|B(R(h\tau))|\int_{\Lambda_r}\frac{d\theta}{|\Lambda_r|}\int \mu_{\alpha,K_\varepsilon}(d\xi) \\
&\times \exp\Big(-\tau\lambda_1(-h\Delta+\Phi_r(\cdot-\theta)+V_\xi^\varepsilon)_{B(R(h\tau))}\Big).
\end{aligned}$$

By (3.1) in [9], we have

$$\begin{aligned}
&\lambda_1(-h\Delta+\Phi_r(\cdot-\theta)+V_\xi^\varepsilon)_{B(R(h\tau))} \\
&\geq \inf\{\lambda_1(-h\Delta+V_\xi^\varepsilon)_{\Lambda_{r+1}(rk+\theta)} : k\in\mathbb{Z}^d \text{ s.t. } \Lambda_{r+1}(rk+\theta)\cap B(R(h\tau))\neq\emptyset\}.
\end{aligned} \tag{4.3}$$

By using also the stationarity of our random field, we have

$$\widetilde{N_3}(t) \leq \frac{c_3 R(h\tau)^{2d}}{r^d}\exp\Big(\frac{c_2 h\tau}{2r}\Big)\int \mu_{\alpha,K_\varepsilon}(d\xi)\exp(-\tau\lambda_1(-h\Delta+V_\xi^\varepsilon)_{\Lambda_{r+1}}).$$

Now, for the uniform ergodicity condition $(\overline{U})$ in P-113 of [4] for the large deviation theory, we will replace the Dirichlet condition by the periodic boundary condition:

$$\lambda_1(-h\Delta+V_\xi^\varepsilon)_{\Lambda_{r+1}} \geq \lambda_1(-h\Delta+V_\xi^\varepsilon)_{\Lambda_{r+1}}^p,$$

where $(-h\Delta+V_\xi^\varepsilon)_{\Lambda_{r+1}}^P$ is the self-adjoint operator associated to the closed extension of the quadratic form

$$C_P^\infty(\overline{\Lambda_{r+1}})\times C_P^\infty(\overline{\Lambda_{r+1}})\ni(\phi,\psi)\mapsto\int_{\Lambda_{r+1}}dx\{h(\nabla\phi(x))\cdot(\nabla\psi(x))+V_\xi^\varepsilon(x)\phi(x)\psi(x)\}$$

on $C_P^\infty(\overline{\Lambda_{r+1}}) = \{\phi : \overline{\Lambda_{r+1}} \to \mathbb{R} : C^\infty, \phi = \widetilde{\phi}$ on Λ_{r+1} for some $\widetilde{\phi} \in C^\infty(\mathbb{T}^d_{r+1})\}$ and $\mathbb{T}^d_{r+1} = (\mathbb{R}/((r+1)\mathbb{Z}))^d$ is the torus. Thus we obtain

$$\widetilde{N_3}(t) \le \frac{c_3 R(h\tau)^{2d}}{r^d} \exp\Big(\frac{c_2 h\tau}{2r}\Big) \int \mu_{\alpha,K_\varepsilon}(d\xi) \operatorname{Tr}[\exp(-\tau(-h\Delta + V^\varepsilon_\xi)^P_{\Lambda_{r+1}})].$$

As in (4.2), the problem is reduced to the asymptotics of

$$\begin{aligned}\widetilde{N_4}(t) =& \frac{c_3 R(h\tau)^{2d}}{r^d} \exp\Big(\frac{c_2 h\tau}{2r}\Big) \int \mu_{\alpha,K_\varepsilon}(d\xi) \int_{\Lambda_{r+1}} dx \int_{\Lambda_{r+1}} dx' \\ & \times \exp(-\tau(-h\Delta + V^\varepsilon_\xi)^P_{\Lambda_{r+1}})(x,x') \\ =& \frac{c_3 R(h\tau)^{2d}}{r^d} \exp\Big(\frac{c_2 h\tau}{2r}\Big) \int_{\Lambda_{r+1}} dx E_{0,x}[\exp(-\tau \mathcal{V}_t(\mathcal{R}_\varepsilon \ell_{2h\tau}))],\end{aligned}$$

where

$$\begin{aligned}\mathcal{V}_t(\phi) =& \frac{-1}{\tau} \log \int \mu_{\alpha,K_\varepsilon}(d\xi) \exp\Big(-\tau \iint \phi(y+z)\xi(dy) v^\varepsilon(z)dz\Big) \\ =& \frac{1}{\alpha\tau} \operatorname{Tr}\log\Big\{I + \alpha\Big(1 - \exp\Big(-\tau\int \phi(\cdot+z)v^\varepsilon(z)dz\Big)\Big)^{1/2} \\ & \times K_\varepsilon \Big(1 - \exp\Big(-\tau\int \phi(\cdot+z)v^\varepsilon(z)dz\Big)\Big)^{1/2}\Big\}\end{aligned}$$

is the functional on the space $L^1(\mathbb{T}^d_{r+1} \to [0,\infty))$ of all nonnegative integrable functions ϕ on $\mathbb{T}^d_{r+1}$, $\mathcal{R}_\varepsilon$ is the integral operator from the space $\mathcal{P}(\mathbb{T}^d_{r+1})$ of all probability measures μ on $\mathbb{T}^d_{r+1}$ to $L^1(\mathbb{T}^d_{r+1} \to [0,\infty))$ defined by

$$(\mathcal{R}_\varepsilon \mu)(x) = \int \rho_\varepsilon(y-x)\mu(dy),$$

and $E_{0,x}$ is the expectation of the local time

$$\ell_{2h\tau}(dx) = \frac{1}{2h\tau}\int_0^{2h\tau} \delta_{w(s)}(dx)$$

of the Brownian motion $\{w(s) : 0 \le s \le 2h\tau\}$ on $\mathbb{T}^d_{r+1}$ starting from x. Now we will apply a Varadhan's lemma on large deviations as Proposition (10.7) in Pastur and Figotin [22]:

$$\varlimsup_{t\to\infty} \frac{1}{\tau} \log \widetilde{N}_4(t) \le \frac{c_2 h}{2r} - \inf\Big\{I(\phi) + \mathcal{V}(\phi) : \phi \in L^1(\mathbb{T}^d_{r+1} \to [0,\infty)), \int \phi(x)dx = 1\Big\}, \tag{4.4}$$

where

$$I(\phi) = \int h|\nabla\sqrt{\phi(x)}|^2 dx$$

and

$$\mathcal{V}(\phi) = \int_{\mathbb{R}^d} dx \frac{1}{\alpha}\Big\{\log\Big(I + \alpha\Big(1 - \exp\Big(-\int_{\mathbb{T}^d_{r+1}} \frac{C_0\phi(y)dy}{|x-y|^{d+2}}\Big)\Big)K\Big)\Big\}(0,0).$$

For (4.4), we should show the sufficient condition:

$$\varliminf_{t\to\infty} \mathcal{V}_t(\phi_\tau) \geq \mathcal{V}(\phi_\infty) \tag{4.5}$$

for any $\{\phi_\tau\}_{\tau\in[1,\infty]} \subset L^1(\mathbb{T}^d_{r+1} \to [0,\infty))$ such that $\|\phi_\tau\|_{L^1} = 1$, $\phi_\tau \to \phi_\infty$ in L^1 as $\tau \to \infty$, and $I(\phi_\infty) < \infty$. The condition (4.5) is reduced to

$$\lim_{t\to\infty} \mathcal{V}^\eta_t(\phi_\tau) = \mathcal{V}(\phi_\infty) \tag{4.6}$$

since $\mathcal{V}_t \geq \mathcal{V}^\eta_t$, where $\eta > 0$,

$$\begin{aligned}\mathcal{V}^\eta_t(\phi) =& \frac{-1}{\tau} \log \int \mu_{\alpha,K_\varepsilon}(d\xi) \exp\Big(-\tau \int \phi(y+z)\xi(dy)(\zeta_\eta v)^\varepsilon(z)dz\Big) \\ =& \frac{1}{\alpha\tau} \operatorname{Tr}\log\Big\{I + \alpha\Big(1-\exp\Big(-\tau\int \phi(\cdot+z)(\zeta_\eta v)^\varepsilon(z)dz\Big)\Big)^{1/2} \\ & \times K_\varepsilon\Big(1-\exp\Big(-\tau\int \phi(\cdot+z)(\zeta_\eta v)^\varepsilon(z)dz\Big)\Big)^{1/2}\Big\},\end{aligned}$$

and ζ_η is a $[0,1]$-valued continuous function on $\mathbb{R}^d$ such that $\zeta_\eta(x) = 1$ on $\{x : |x| \geq 2\eta\}$ and $\zeta_\eta(x) = 0$ on $\{x : |x| \leq \eta\}$. For any $\delta > 0$, we take a polynomial P_δ such that

$$\sup_{s\in[0,\|K\|_{op}]} \Big|\frac{\log(1+\alpha s)}{\alpha s} - P_\delta(s)\Big| \leq \delta$$

and define

$$\begin{aligned}\mathcal{V}^{\eta,\delta}_t(\phi) =& \frac{1}{\tau} \operatorname{Tr}\Big[\widetilde{P_\delta}\Big(\Big(1-\exp\Big(-\tau\int \phi(\cdot+z)(\zeta_\eta v)^\varepsilon(z)dz\Big)\Big)^{1/2} K_\varepsilon \\ & \times \Big(1-\exp\Big(-\tau\int \phi(\cdot+z)(\zeta_\eta v)^\varepsilon(z)dz\Big)\Big)^{1/2}\Big)\Big],\end{aligned}$$

and

$$\mathcal{V}^\delta(\phi) = \int_{\mathbb{R}^d} dx \Big\{\widetilde{P_\delta}\Big(\Big(1-\exp\Big(-\int_{\mathbb{T}^d_{r+1}} \frac{C_0\phi(y)dy}{|x-y|^{d+2}}\Big)\Big)K\Big)\Big\}(0,0),$$

where $\widetilde{P_\delta}(s) = sP_\delta(s)$. Then we have

$$\sup_{\phi\in L^1(\mathbb{T}^d_{r+1}\to[0,\infty)):\|\phi\|_{L^1}=1,t\geq 1} |\mathcal{V}^{\eta,\delta}_t(\phi) - \mathcal{V}^\eta_t(\phi)| \leq c_4\delta.$$

and

$$\sup_{\phi\in L^1(\mathbb{T}^d_{r+1}\to[0,\infty)):\|\phi\|_{L^1}=1,t\geq 1} |\mathcal{V}(\phi) - \mathcal{V}^\delta(\phi)| \leq c_4\delta.$$

Thus the condition (4.6) is reduced to

$$\lim_{t\to\infty} \mathcal{V}^{\eta,n}_t(\phi_\tau) = \mathcal{V}^n(\phi_\infty) \tag{4.7}$$

for any $n \in \mathbb{N}$. where

$$\mathcal{V}_t^{\eta,n}(\phi) = \frac{1}{\tau} \operatorname{Tr} \Big[\Big(\Big(1 - \exp\Big(-\tau \int \phi(\cdot + z)(\zeta_\eta v)^\varepsilon(z) dz \Big)\Big)^{1/2} K_\varepsilon \\ \times \Big(1 - \exp\Big(-\tau \int \phi(\cdot + z)(\zeta_\eta v)^\varepsilon(z) dz \Big)\Big)^{1/2} \Big)^n \Big],$$

and

$$\mathcal{V}^n(\phi) = \int_{\mathbb{R}^d} dx \Big\{ \Big(\Big(1 - \exp\Big(- \int_{\mathbb{T}^d_{r+1}} \frac{C_0 \phi(y) dy}{|x-y|^{d+2}} \Big)\Big) K \Big)^n \Big\}(0,0).$$

(4.7) is straightforward and we obtain (4.4). Since r is arbitrary, we can complete the proof. □

Remark 4.1. In (4.4), the error term by the compactification appears in a simple form $c_2 h/(2r)$. This is negligible without extra conditions. However if we follow the method in Section 10.c in [22], then the error term may depend on t, and extra conditions on K may be necessary to show that the term is negligible.

5. The sine kernel

We next treat examples not satisfying the condition (P). We first treat 1-dimensional cases. We assume $\alpha = -1$, the compactness of $\operatorname{supp} u$,

$$\varliminf_{x \downarrow 0} \int_0^x u(y) \frac{dy}{x} > 0, \qquad \varliminf_{x \downarrow 0} \int_{-x}^0 u(y) \frac{dy}{x} > 0, \tag{5.1}$$

and that K is the sine kernel (1.4). Then we will prove

$$\lim_{\lambda \downarrow 0} \lambda \log N(\lambda) = \frac{-\pi^4 h}{8}.$$

For the proof, the key fact is

$$\mu_{-1,K_s}(\xi([0,R]) = 0) = \exp\Big(- \frac{\pi^2 R^2}{8}(1 + o(1)) \Big) \text{ as } R \to \infty \tag{5.2}$$

(Dyson [7], Widom [34]). The same asymptotics where the constant $\pi^2/8$ is replaced by other positive constants are obtained for a generalization called as the point process Sine_β. As for the definition and properties of the point process Sine_β, see Valkó and Virág [32]. $\beta = 2$ corresponds to our process μ_{-1,K_s}. Thus we summarize the result in a general form:

Theorem 4. *Let μ be any Borel probability measure on the space $\mathcal{P}(\mathbb{R})$ of nonnegative integer valued Radon measures on $\mathbb{R}$ with vague topology such that μ is stationary and ergodic under the shift of the space valuable: $\mu(\xi \circ \tau_a \in B)$ is independent of $a \in \mathbb{R}$ and any Borel subset B of $\mathcal{P}(\mathbb{R})$ such that $\mu(\{\xi : \xi \circ \tau_a \in B\} \triangle B) = 0$ for any $a \in \mathbb{R}$ are trivial as $\mu(B) = 0$ or 1, where τ_a is the transform*

$\tau_a x = x + a$ for any $x \in \mathbb{R}$, and $B \triangle B' = (B \setminus B') \cup (B' \setminus B)$ for any Borel subsets B, B' of $\mathcal{P}(\mathbb{R})$. Under this probability μ, we consider the Schrödinger operator H_ξ in (1.2) and its integrated density of states $N(\lambda)$ ($\lambda \in \mathbb{R}$) defined by (1.3). We further assume $d = 1$, the compactness of $\operatorname{supp} u$, (5.1), and that μ satisfy

$$\mu(\xi([0,R]) = 0) = \exp(-R^H(H' + o(1))) \text{ as } R \to \infty \tag{5.3}$$

with some H and $H' \in (0, \infty)$. Then we have

$$\lim_{\lambda \downarrow 0} \lambda^{H/2} \log N(\lambda) = -H'(\pi\sqrt{h})^H.$$

This result coincides with that of Theorem 2 when $\mu = \mu_{\alpha,K}$.

Proof.

By the Dirichlet and Neumann bracketing, we have

$$\begin{aligned} &\frac{1}{|\Lambda_L|} \int \mu(d\xi) \#\{[0,\lambda] \cap \operatorname{spec}(H^{\xi,D}_{\Lambda_L})\} \\ &\leq N(\lambda) \\ &\leq \frac{1}{|\Lambda_L|} \int \mu(d\xi) \#\{[0,\lambda] \cap \operatorname{spec}(H^{\xi,N}_{\Lambda_L})\} \end{aligned} \tag{5.4}$$

((5.41) in [22]). From the upper bound, we have

$$N(\lambda) \leq \mu(\lambda_1(H^{\xi,N}_{\Lambda_L}) \leq \lambda) \#\{[0,\lambda] \cap \operatorname{spec}(-h\Delta^N_{\Lambda_L})\}$$

((10.10) in [22]). We easily obtain

$$\#\{[0,\lambda] \cap \operatorname{spec}(-h\Delta^N_{\Lambda_L})\} \leq \operatorname{Tr}[\exp(1 + h\Delta^N_{\Lambda_L}/\lambda)] = e \int_{\Lambda_L} \exp(h\Delta^N_{\Lambda_L}/\lambda)(x,x)dx \leq c_0 L \sqrt{\frac{\lambda}{h}}.$$

We now apply Theorem 3.1 in the page 123 in [29]:

$$\lambda_1(H^{\xi,N}_{\Lambda_L}) \geq \frac{h\pi^2}{(\sup_k |I_k| + c_1)^2}$$

for any $L \geq c_0$, where $\{I_k\}_k$ is the random open intervals such that $\sum_k I_k = \Lambda_L \setminus \operatorname{supp} \xi$. Then we have

$$\begin{aligned} &\mu(\lambda_1(H^{\xi,N}_{\Lambda_L}) \leq \lambda) \\ \leq &\mu\Big(\sup_k |I_k| \geq \pi\sqrt{\frac{h}{\lambda}} - c_1\Big) \\ \leq &\sum_{p \in \mathbb{Z} \cap \Lambda_L} \mu\Big(\xi\Big(\Big[p, p + \pi\sqrt{\frac{h}{\lambda}} - c_1 - 2\Big]\Big) = 0\Big) \\ \leq &c_2 |\Lambda_L| \mu\Big(\xi\Big(\Big[0, \pi\sqrt{\frac{h}{\lambda}} - c_1 - 2\Big]\Big) = 0\Big). \end{aligned} \tag{5.5}$$

Here the choice of L is not restrictive. We choose $L = \pi^2 h/\lambda$ so that $|\Lambda_L|$ is not so big and the probability event in the above inequality is not empty. Then we obtain

$$\varlimsup_{\lambda\downarrow 0} \lambda^{H/2} \log N(\lambda) \le -H'(\pi\sqrt{h})^H.$$

From the lower bound, we have

$$N(\lambda) \ge \frac{1}{|\Lambda_L|}\mu(\lambda_1(H^{\xi,D}_{\Lambda_L}) \le \lambda).$$

If $\xi(\Lambda_L + \operatorname{supp} u) = 0$, then

$$\lambda_1(H^{\xi,D}_{\Lambda_L}) = \lambda_1((-h\Delta^D_{\Lambda_L}) = h\Big(\frac{\pi}{|\Lambda_L|}\Big)^2.$$

Thus by taking $L = \pi\sqrt{h/\lambda}$, we have

$$N(\lambda) \ge \frac{1}{\pi}\sqrt{\frac{\lambda}{h}}\mu(\xi(\Lambda_{\pi\sqrt{h/\lambda}} + \operatorname{supp} u) = 0).$$

Thus we obtain

$$\varliminf_{\lambda\downarrow 0} \lambda^{H/2} \log N(\lambda) \ge -H'(\pi\sqrt{h})^H.$$

□

6. 2 dimensional examples: the Ginibre random point field and the zeros of a Gaussian analytic function

We next treat a 2-dimensional example: in the above setting, we further assume $d = 2$, $\alpha = -1$, the compactness of $\operatorname{supp} u$, $u \ge \varepsilon_0 1_{B(r_0)}$ with some $\varepsilon_0, r_0 \in (0,1)$, and that K is given by (1.5): μ_{-1,K_g} is the Ginibre random point field. Then we will prove

$$\lim_{\lambda\downarrow 0} \frac{\log|\log N(\lambda)|}{\log\lambda} = -2. \tag{6.1}$$

For the proof, the key fact is

$$\lim_{r\to\infty} \frac{1}{r^4}\log\mu(\xi(B(r)) = 0) = \frac{-1}{4}. \tag{6.2}$$

when $\mu = \mu_{-1,K_g}$ (cf. Proposition 7.2.1 in [14]). The same asymptotics is known for another famous example, the case that μ is replaced by the probability distribution μ_{GAF} of $\sum_{a\in\mathcal{X}}\delta_a$ for the sets $\mathcal{X}$ of the zeros of a Gaussian analytic function

$$f_{GFA}(z) = \sum_{n=0}^{\infty}(X_n + iY_n)\sqrt{\frac{L^n}{n!}}z^n, \tag{6.3}$$

where $L \in (0,\infty)$ and $\{X_n, Y_n\}_n$ are independently and identically distributed random valuables obeying the normal distribution N(0,1/2) with the mean 0 and the valiance 1/2. This is a result obtained by Sodin and Tsirelson [25] (cf. Theorem 7.2.3 in [14]). Thus we summarize the result in the general form.

Theorem 5. *Let μ be any Borel probability measure on the space $\mathcal{P}(\mathbb{R}^2)$ of nonnegative integer valued Radon measures on $\mathbb{R}^2$ with vague topology such that μ is stationary and ergodic under the shift of the space valuable: $\mu(\xi \circ \tau_a \in B)$ is independent of $a \in \mathbb{R}^2$ and any Borel subset B of $\mathcal{P}(\mathbb{R}^2)$ such that $\mu(\{\xi : \xi \circ \tau_a \in B\} \triangle B) = 0$ for any $a \in \mathbb{R}^2$ are trivial as $\mu(B) = 0$ or 1, where τ_a is the transform $\tau_a x = x + a$ for any $x \in \mathbb{R}^d$, and $B \triangle B' = (B \setminus B') \cup (B' \setminus B)$ for any Borel subsets B, B' of $\mathcal{P}(\mathbb{R}^2)$. Under this probability μ, we consider the Schrödinger operator H_ξ in (1.2) and its integrated density of states $N(\lambda)$ ($\lambda \in \mathbb{R}$) defined by (1.3). We further assume the compactness of* $\operatorname{supp} u$, *$u \geq \varepsilon_0 1_{B(r_0)}$ with some $\varepsilon_0, r_0 \in (0,1)$, and that μ satisfy the asymptotics*

$$\lim_{r\to\infty} \frac{1}{r^H} \log \mu(\xi(B(r)) = 0) = -H'. \tag{6.4}$$

with some H and $H' \in (0,\infty)$. Then we have

$$\lim_{\lambda\downarrow 0} \frac{\log|\log N(\lambda)|}{\log \lambda} = -\frac{H}{2}. \tag{6.5}$$

Remark 6.1. (i) This result coincides with that from Theorem 2 when $\mu = \mu_{\alpha,K}$.

(ii) More detailed upper and lower bounds are obtained as in (6.6) and (6.7).

(iii) For the probability μ_{GAF} associated with the zeros of a Gaussian analytic function (6.3), the stationarity and the ergodicity under the shift τ_a, $a \in \mathbb{R}^2$ are proven, for example, in Propositions 2.3.4 and 2.3.7 in [14]. In the statement of Proposition 2.3.7, the invariance under the rotations are also assumed, However the rotations invariance are not used in the proof.

For the upper bound in Theorem 5, we apply the following:

Lemma 6.1. *For any $c_0, r_0 \in (0,\infty)$, there exists $R(c_0, r_0) \in (0,\infty)$ depending only on c_0 and r_0 such that $\inf_{b\in\Lambda_R} \lambda_1(-\Delta + c_0 1_{B(b,r_0)})^N_{\Lambda_R} \geq 1/(4R^2 \log R)$ for any $R \geq R(c_0, r_0)$, where $B(b, r_0) = \{x \in \mathbb{R}^2 : |x - b| < r_0\}$.*

This is a version of Lemma 3.15 in [8].

Proof of Theorem 5.

We use also the Dirichlet and Neumann bracketing (5.4). For the upper bound, we apply

$$\begin{aligned}&\#\Big\{\Big[0, \frac{h}{4R^2\log R}\Big] \cap \mathrm{spec}(-h\Delta^N_{\Lambda_R})\Big\} \le \mathrm{Tr}[\exp(1+(4R^2\log R)\Delta^N_{\Lambda_R})]\\ &= e\int_{\Lambda_L} \exp((4R^2\log R)\Delta^N_{\Lambda_R})(x,x)dx \le \frac{c_0}{\log R} \le c_0\end{aligned}$$

and Lemma 6.1 to obtain

$$\begin{aligned}N\Big(\frac{h}{4R^2\log R}\Big) \le& c_0\mu\Big(\lambda_1(\lambda : H^{\xi,N}_{\Lambda_R}) \le \frac{h}{4R^2\log R}\Big)\\ \le& c_0\mu(\xi(\Lambda_R)=0)\\ \le& c_0\mu(\xi(B(R/2))=0)\end{aligned}$$

if $R \ge R(\varepsilon_0/h, r_0)$. Thus by (6.4), for any $\varepsilon > 0$, there exists $R(\varepsilon, h) \in (0,\infty)$ such that

$$\frac{1}{R^H}\log N\Big(\frac{h}{4R^2\log R}\Big) \le \frac{-H'+\varepsilon}{2^H}$$

for any $R \ge R(\varepsilon, h) \vee R(\varepsilon_0/h, r_0)$. This is interpreted as

$$\frac{1}{\mathcal{R}^{-1}(h/(4\lambda))^H}\log N(\lambda) \le \frac{-H'+\varepsilon}{2^H}$$

if $h/(4\lambda) \ge \mathcal{R}(R(\varepsilon, h) \vee R(\varepsilon_0/h, r_0))$, where $\mathcal{R}(R) = R^2\log R$ and $\mathcal{R}^{-1}$ is its inverse function. Since

$$\mathcal{R}^{-1}\Big(\frac{h}{4\lambda}\Big) = \sqrt{\frac{h}{2\lambda(\log h - \log(4\lambda) - \log\log\mathcal{R}^{-1}(h/(4\lambda)))}},$$

we have

$$\varlimsup_{\lambda\downarrow 0}\Big(\lambda\log\frac{1}{\lambda}\Big)^{H/2}\log N(\lambda) \le -H'\frac{h^{H/2}}{2^{3H/2}}. \tag{6.6}$$

For the lower bound, we proceed as in the proof of Theorem 4: if $\xi(\Lambda_L + \mathrm{supp}\, u) = 0$, then

$$\lambda_1(\lambda : H^{\xi,D}_{\Lambda_L}) = \lambda_1(-h\Delta^D_{\Lambda_L}) = \frac{2h\pi^2}{|\Lambda_L|}.$$

Thus by taking $L = \pi\sqrt{2h/\lambda}$, we have

$$\begin{aligned}N(\lambda) &\ge \frac{1}{|\Lambda_L|}\mu(\lambda_1(\lambda : H^{\xi,D}_{\Lambda_L}) \le \lambda)\\ &\ge \frac{\lambda}{2h\pi^2}\mu(\xi(\Lambda_{\pi\sqrt{2h/\lambda}} + \mathrm{supp}\, u) = 0).\end{aligned}$$

Thus we obtain

$$\varliminf_{\lambda\downarrow 0}\lambda^{H/2}\log N(\lambda) \ge -H'\frac{\pi^H h^{H/2}}{2^{H/2}}. \tag{6.7}$$

□

Appendix A. The survival probability of the Brownian motion in our random point fields

In this section, we interpret the results in this paper to those on the survival probability of the Brownian motion among the traps around the sample points of our random point fields.

Let P_0 be the probability measure of the standard d-dimensional Brownian motion $\{w(t)\}_{t\geq 0}$ independent with the random point field $\mu_{\alpha,K}$. Let τ_O be its hitting time to

$$O^{\xi} := \bigcup_{a \in \mathrm{supp}\ \xi} (a + O),$$

where O is a nonpolar compact set. Our object in this subsection is the asymptotic behavior of

$$S_{\alpha,K}(t) := (\mu_{\alpha,K} \otimes P_0)(\tau_O \geq t)$$

as $t \to \infty$.

The results are the following:

Proposition A.1.

(i) *Under the condition (P), we have*

$$\begin{aligned}\lim_{t\to\infty} & t^{-d/(d+2)} \log S_{\alpha,K}(t) \\ &= -\frac{d+2}{2d^{d/(d+2)}} \lambda_1(-\Delta_{B(1)})^{d/(d+2)} C_{\alpha,K}^{2/(d+2)} |B(1)|^{2/(d+2)},\end{aligned} \tag{A.1}$$

where $C_{\alpha,K}$ is the constant introduced in Theorem 2.

(ii) *For any Borel probability measure μ on the space $\mathcal{P}(\mathbb{R})$ such that μ is stationary and ergodic under the shift of the space valuable $a \in \mathbb{R}$ and that the asymptotics* (5.3) *holds, we have*

$$\lim_{t\to\infty} t^{-H/(2+H)} \log S(t) = -\frac{2+H}{2} \pi^{2H/(2+H)} (H')^{2/(2+H)} H^{-H/(2+H)}, \tag{A.2}$$

where

$$S(t) := (\mu \otimes P_0)(\tau_O \geq t). \tag{A.3}$$

(iii) *For any Borel probability measure μ on the space $\mathcal{P}(\mathbb{R}^2)$ such that μ is stationary and ergodic under the shift of the space valuable $a \in \mathbb{R}^2$ and that the asymptotics* (6.4) *holds, we have*

$$\lim_{t\to\infty} \frac{\log|\log S(t)|}{\log t} = \frac{H}{2+H}, \tag{A.4}$$

where $S(t)$ is defined as in (A.3).

In (iii), the more precise bounds are obtained in (A.9) and (A.10) below. The results in (ii) and (iii) coincides with those from (i) when $\mu = \mu_{\alpha,K}$.

Proof.

(i) Our $S_{\alpha,K}(t)$ corresponds to $\widetilde{N}_3(t)$ in (3.2) with $h = 1/2$ and $u = \infty 1_O$. The upper estimate of $S_{\alpha,K}(t)$ is also proven as in Section 3 and the second version of the upper bound in Section 4.5 in [29].

The lower estimate is easily obtained by

$$S_{\alpha,K}(t) \geq \mu_{\alpha,K}(\xi(B(R + r_0)) = 0)(\exp(t\Delta^D_{B(R)}/2)1)(0) \tag{A.5}$$

for any $R \geq 1$, where r_0 is a number such that $O \subset B(r_0)$. Indeed $\lambda_1(-\Delta^D_{B(R)}) = \lambda_1(-\Delta^D_{B(1)})/R^2$ and

$$\log \mu_{\alpha,K}(\xi(B(R)) = 0) = \frac{-1}{\alpha} \operatorname{Tr} \log(I + \alpha 1_{B(R)} K 1_{B(R)}) = -C_{\alpha,K}|B(R)|(1 + o(1))$$

as $R \to \infty$. Therefore, by taking R appropriately, we can give the lower bound.

(ii) We apply an estimate of probability of exit time of the Brownian motion to obtain

$$S(t) \leq \widetilde{S}(t) + c_1 e^{-c_2 t},$$

where

$$\widetilde{S}(t) = \int \mu(d\xi)(\exp(t\Delta^D_{B(t)\setminus O^\xi}/2)1)(0).$$

By (1.9) in [29], for any $\varepsilon_1 \in (0,1)$, there exists $c_{\varepsilon_1} \in (0,\infty)$ such that

$$\widetilde{S}(t) \leq c_{\varepsilon_1} \int \mu(d\xi) \exp(-t(1-\varepsilon_1)\lambda_1(-\Delta^D_{B(t)\setminus O^\xi})/2). \tag{A.6}$$

We easily see that

$$\lambda_1(-\Delta^D_{B(t)\setminus O^\xi}) \geq \frac{\pi^2}{\sup_k |I_k|^2},$$

where $\{I_k\}_k$ is the random open intervals such that $\sum_k I_k = B(t) \setminus \operatorname{supp} \xi$. Then we have

$$\widetilde{S}(t) \leq c_{\varepsilon_1} \sum_{m=0}^{[t]} \mu(m < \sup_k |I_k| \leq m+1) \exp\Big(\frac{-t(1-\varepsilon_1)\pi^2}{2(m+1)^2}\Big).$$

As in (5.5), we have

$$\mu\Big(m < \sup_k |I_k|\Big) \leq t\mu(\xi([0,(m-2)_+]) = 0).$$

Thus we have

$$\widetilde{S}(t) \leq c_{\varepsilon_1} t^2 \sup_{m \geq 0} \Big\{\mu(\xi([0,(m-2)_+]) = 0) \exp\Big(\frac{-t(1-\varepsilon_1)\pi^2}{2(m+1)^2}\Big)\Big\} \tag{A.7}$$

By (5.3), for any $\varepsilon_2 > 0$, there exists $m_{\varepsilon_2} \in (0, \infty)$ such that

$$\mu(\xi([0, (m-2)_+]) = 0) \le \exp(-H' m^H (1 - \varepsilon_2))$$

and $m + 1 \le (1 + \varepsilon_2)m$ for any $m \ge m_{\varepsilon_2}$. There exists $t_{\varepsilon_2} \in (0, \infty)$ such that the supremum in (A.7) is attained in $(m_{\varepsilon_2}, \infty)$. Then we have

$$\begin{aligned}\widetilde{S}(t) \le & c_{\varepsilon_1} t^2 \exp\Big(- \inf_{m \ge 0} \Big(H' m^H (1 - \varepsilon_2) + \frac{t(1-\varepsilon_1)\pi^2}{2(1+\varepsilon_2)^2 m^2} \Big)\Big) \\ = & c_{\varepsilon_1} t^2 \exp\Big(- (H'(1-\varepsilon_2))^{2/(2+H)} \Big(\frac{t(1+\varepsilon_1)\pi^2}{H(1+\varepsilon_2)^2} \Big)^{H/(2+H)} \frac{2+H}{2}\end{aligned}$$

for $t \ge t_{\varepsilon_2}$. Thus we obtain

$$\overline{\lim_{t\to\infty}}\, t^{-H/(2+H)} \log S(t) \le -\frac{2+H}{2} \pi^{2H/(2+H)} (H')^{2/(2+H)} H^{-H/(2+H)}$$

For the lower estimate, we apply (A.5) and (5.3) to obtain

$$\overline{\lim_{t\to\infty}}\, t^{-H/(2+H)} \log S(t) \ge -\frac{2+H}{2} \pi^{2H/(2+H)} (H')^{2/(2+H)} H^{-H/(2+H)}.$$

(iii) We use the capacity Cap relative to the operator $-\Delta$ on $\mathbb{R}^2$. Then by an extension of Propositions 2.3 and 2.4 in [30], we have

$$\inf_{b \in B(1/8)} \lambda_1(-\Delta^{N,D}_{B(1/4), b+K}) \ge c_0 Cap(K)$$

where, for any $R > 0$ and any compact set K in $B(1/8)$, $-\Delta^{N,D}_{B(R),b+K}$ is the minus Laplacian with the Neumann and the Dirichlet boundary conditions on $\partial B(R)$ and $b + \partial K$, respectively: $-\Delta^{N,D}_{B(R),b+K}$ is the self-adjoint operator associated with the closure of the quadratic form $(\nabla\phi, \nabla\psi)$ with the domain $\phi, \psi \in \{\phi : \overline{B(R)} \to \mathbb{R} :$ smooth, $\phi = 0$ on $b + K\}$. By the scaling, we can modify Lemma 6.1 as

$$\begin{aligned}&\inf_{b \in B(R/2)} \lambda_1(-\Delta^{N,D}_{B(R),b+O}) \\ &= \frac{1}{(4R)^2} \inf_{b \in B(1/8)} \lambda_1(-\Delta^{N,D}_{B(R),b+O/(4R)}) \\ &\ge \frac{c_0}{(4R)^2} Cap\Big(\frac{O}{4R}\Big).\end{aligned}$$

for any $R > 0$ such that $O \subset B(R/2)$. The variational principle in Theorem 4.9 in Sznitman [29] gives

$$\begin{aligned}&Cap(K) \\ =&\Big\{ \inf \Big\{ \int \frac{1}{2\pi} \Big(\log \frac{1}{|x-y|} \Big) \mu(dx)\mu(dy) \;\Big|\; \mu : \text{ a probability measure on } K \Big\} \Big\}^{-1}.\end{aligned}$$

Thus we obtain

$$\inf_{b \in B(R/2)} \lambda_1(-\Delta^{N,D}_{B(R),b+O}) \ge \frac{c_1}{R^2 \log R}. \tag{A.8}$$

for any $R \geq c_2$.

Since

$$\lambda_1(-\Delta^D_{B(t)\setminus O^\xi}) \geq c_3 \min_{a \in B(t)\cap(R/2)\mathbb{Z}^2} \lambda_1(-\Delta^{N,D}_{B(a,R),O^\xi}),$$

we have

$$\widetilde{S}(t) \leq c_{\varepsilon_1} \sum_{a \in B(t)\cap(R/2)\mathbb{Z}^2} \int \mu(d\xi) \exp(-t(1-\varepsilon_1)c_3\lambda_1(-\Delta^{N,D}_{B(a,R),O^\xi})/2)$$

and

$$\log \widetilde{S}(t) \leq (\log \mu(\xi(B(R/2)) = 0)) \vee \frac{-t(1-\varepsilon_1)c_1c_3}{2R^2 \log R} + \log c_{\varepsilon_1} \Big(\frac{t}{R}\Big)^2.$$

By (6.4), we have

$$\log \mu(\xi(B(R/2)) = 0) \leq -H'(R/2)^H(1-\varepsilon_2)$$

for sufficiently large R. Thus by taking

$$R = \Big(\frac{t(1-\varepsilon_1)c_1c_3 2^{H-1}}{(1-\varepsilon_2)\log R}\Big)^{1/(2+H)},$$

we have

$$\overline{\lim_{t\to\infty}} \Big(\frac{\log t}{t}\Big)^{H/(2+H)} \log S(t) \leq \frac{-H'(c_1c_3(2+H))^{H/(2+H)}}{2^{3H/(2+H)}}. \tag{A.9}$$

For the lower estimate, we apply (A.5) and (6.4) to obtain

$$\underline{\lim_{t\to\infty}} t^{-H/(2+H)} \log S(t) \geq -\frac{2+H}{2}\lambda_1(-\Delta^D_{B(1)})^{2/(2+H)}(H')^{2/(H+2)}H^{-H/(2+H)}. \tag{A.10}$$

□

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