Asymptotic Expansion of A Gaussian Integral of the Chern-Simons Lagragian

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Let M be a compact oriented smooth 3-manifold, G a simply connected, connected compact simple Lie group, $\mathfrak{g}$ the associated Lie algebra and $\Omega^r(M,\mathfrak{g})$ the space of $\mathfrak{g}$-valued smooth r-forms on M with the inner product $(\ ,\ )$. We may identify $\mathcal{A}$ of connections on a principal G-bundle over M with $\Omega^1(M,\mathfrak{g})$.

The *Chern-Simons integral* of the *Wilson line* $F(A)$ is given by

$$\int_{\mathcal{A}/\mathcal{G}} F(A)\, e^{L(A)}\, \mathcal{D}(A), \tag{0.1}$$

where the Chern-Simons Lagrangian L is defined by

$$L(A) = -\frac{\sqrt{-1}k}{4\pi}\int_M \mathrm{Tr}\left\{A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right\}.$$

Here $\mathcal{D}(A)$ is the *Feynman measure* integrating over all gauge orbits, Tr denotes the trace in the adjoint representation of the Lie algebra $\mathfrak{g}$, and the parameter k is a positive integer called the *level of charges*.

Then we replace $\mathcal{D}(A)$ by a standard Gaussian measure, which is called a Gaussian integral of the Chern-Simons Lagrangian.

Let Q_{A_0} be a twisted Dirac operator coming from the Lorentz gauge fixing of (0.1) and λ_i and $e_i = (e_i^A, e_i^\phi)$, $i = 1, 2, \ldots$ the eigenvalues and eigenvectors of Q_{A_0}.

For a sufficiently large integer p, we define the Hilbert subspace $H_p(\Omega_+)$ of $L^2(\Omega_+) = L^2\left(\Omega^1(M,\mathfrak{g})\oplus\Omega^3(M,\mathfrak{g})\right)$ with new inner product $(\ ,\ )_p$ defined by

$$\left((A,\phi),\,(B,\varphi)\right)_p = \left(A,\,\left(I+Q_{A_0}^2\right)^p B\right) + \left(\phi,\,\left(I+Q_{A_0}^2\right)^p\varphi\right),$$

where I is the identity operator on $L^2(\Omega_+)$.

Now, let $H = H_p(\Omega_+)$ and (B, H, μ) an abstract Wiener space.

Let denote ϵ-regularized Wilson line by $F^\epsilon_{A_0}(x)$, (Mitoma-Nishikawa) and the regularized determinant coming from the Lorentz gauge fixing, by $\det_{Reg}(x)$, (Albeverio-Mitoma).

From now on, we use the brief notations such that

$$\beta_j = \left(1+\lambda_j^2\right)^{-p/2}, h_j = \beta_j e_j \ \text{ and } \ a_j = \beta_j^2\lambda_j.$$

Then a Gaussian integral of the Chern-Simons Lagrangian in an abstract Wiener space setting is defined by

$$
\begin{aligned}
&\frac{1}{Z_G}\int_B \widetilde{F}^{\epsilon}_{A_0}(\frac{1}{\sqrt[3]{k}}x)\exp\left[i\sqrt[3]{k}CS(x)\right] \\
&\times\exp\left[i\sum_{abc=1}^{\infty}\langle x,h_a\rangle\langle x,h_b\rangle\langle x,h_c\rangle\beta_a\beta_b\beta_c T_{abc}\right]\mu(dx),
\end{aligned}
\tag{0.2}
$$

where

$$
Z_G = \int_B \exp\left[i\sqrt[3]{k}CS(x)\right]\mu(dx),
$$

$$
\widetilde{F}^{\epsilon}_{A_0}(x) = F^{\epsilon}_{A_0}(x)\,\det{}_{Reg}(x),
$$

$$
CS(x) = \sum_{j=1}^{\infty} a_j\langle x,h_j\rangle^2 < +\infty,
$$

$$
T_{abc} = -\int_M \mathrm{Tr}\,\frac{1}{6\pi}e^A_a\bigwedge e^A_b\bigwedge e^A_c,
$$

and $\langle\cdot,\cdot\rangle$ denotes the bilinear form of B and its dual space B^*.

By the Fujiwara-Kumano-go method [1, 2], we obtain

Theorem. *If we take for sufficiently large p of H_p in the abstract Wiener space, we have the following asymptotic expansion up to order $2N$:*

$$
\begin{aligned}
&(0.2) = \widetilde{F}^{\epsilon}_{A_0}(0) \\
&+\sum_{s=1}^{N}\left(\sum_{r=1}^{s}\left(\sum_{1\le j_1<j_2<j_3\cdots<j_r<\infty}\left(\sum_{m_1,m_2,\cdots,m_r\ge 1, m_1+m_2+\cdots+m_r=s}\right.\right.\right. \\
&\left(\left(\prod_{q=1}^{r}\frac{1}{2^{m_q}m_q!(1-2i\sqrt[3]{k}a_{j_q})^{m_q}}\nabla^{2m_q}_{h_{j_q}}\right)\right. \\
&\left.\left.\left.\left.\widetilde{F}^{\epsilon}_{A_0}(\frac{1}{\sqrt[3]{k}}x)\exp\left[i\sum_{abc=1}^{\infty}\langle x,h_a\rangle\langle x,h_b\rangle\langle x,h_c\rangle\beta_a\beta_b\beta_c T_{abc}\right]\right)(0)\right)\right)\right) \\
&+O\left(\left(\sum_{j=1}^{\infty}j^{16N+16}\frac{\beta_j^2}{|1-2i\sqrt[3]{k}a_j|}\right)^{N+1}\right)\Bigg\},
\end{aligned}
$$

for sufficiently large k.

References

[1] D. Fujiwara, The stationary phase method with an estimate of the remainder term on a space of large dimension, Nagoya Math. J. **124** (1991), 61-97.

[2] N. Kumano-go, Feynman path integrals as analysis on path space by time slicing approximation, Bull. Sci. Math. 128 (2004), 197-251.