Wong-Zakai approximation of solutions to reflecting stochastic differential equations on domains in Euclidean spaces

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This talk is based on a joint work with Kosuke Sasaki (SPA, 123, 2013, 3800-3827). Let D be a connected domain in $\mathbb{R}^d$. We define the set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x \in \partial D$ by

$$\mathcal{N}_x = \cup_{r>0}\mathcal{N}_{x,r} \tag{1}$$

$$\mathcal{N}_{x,r} = \left\{\boldsymbol{n} \in \mathbb{R}^d \mid |\boldsymbol{n}| = 1, B(x - r\boldsymbol{n}, r) \cap D = \emptyset\right\}, \tag{2}$$

where $B(z,r) = \{y \in \mathbb{R}^d \mid |y-z| < r\}$, $z \in \mathbb{R}^d$, $r > 0$.

Let us recall what Skorohod problem is.

Definition 1 (Skorohod Problem on $\bar{D}$). *Let $w(t)$ $(0 \le t \le T)$ be a continuous path on $\mathbb{R}^d$ with $w(0) \in \bar{D}$. The pair of paths (ξ, ϕ) on $\mathbb{R}^d$ is a solution of a Skorohod problem on $\bar{D}$ associated with w if the following properties hold.*

(i) *$\xi(t)$ $(0 \le t \le T)$ is a continuous path in $\bar{D}$ with $\xi(0) = w(0)$.*

(ii) *It holds that $\xi(t) = w(t) + \phi(t)$ for all $0 \le t \le T$.*

(iii) *$\phi(t)$ $(0 \le t \le T)$ is a continuous bounded variation path on $\mathbb{R}^d$ such that $\phi(0) = 0$ and*

$$\phi(t) = \int_0^t \boldsymbol{n}(s) d\|\phi\|_{[0,s]} \tag{3}$$

$$\|\phi\|_{[0,t]} = \int_0^t 1_{\partial D}(\xi(s)) d\|\phi\|_{[0,s]}. \tag{4}$$

where $\boldsymbol{n}(t) \in \mathcal{N}_{\xi(t)}$ if $\xi(t) \in \partial D$.

In the above, $\|\phi\|_{[0,t]}$ stands for the total variation of ϕ on $[0,t]$. When the solution ξ is unique, we denote $\xi = \Gamma(w)$ and we call the mapping Γ a Skorohod map.

In this talk, we consider domains whose boundary may not be smooth. More precisely, we consider the following conditions (A), (B), (C) on domains following Saisho (PTRF 74, 1987) and Lions-Sznitman (Comm.Pure Appl.Math. 37, 1984).

Definition 2. (A) (*uniform exterior sphere condition*). *There exists a constant $r_0 > 0$ such that*

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset \quad \textit{for any } x \in \partial D. \tag{5}$$

(B) *There exist constants* $\delta > 0$ *and* $\beta \geq 1$ *satisfying:*
for any $x \in \partial D$ *there exists a unit vector* l_x *such that*

$$(l_x, \boldsymbol{n}) \geq \frac{1}{\beta} \qquad \text{for any } \boldsymbol{n} \in \cup_{y \in B(x,\delta)\cap \partial D} \mathcal{N}_y. \tag{6}$$

(C) *There exists a* C_b^2 *function* f *on* $\mathbb{R}^d$ *and a positive constant* γ *such that for any* $x \in \partial D$, $y \in \bar{D}$, $\boldsymbol{n} \in \mathcal{N}_x$ *it holds that*

$$(y - x, \boldsymbol{n}) + \frac{1}{\gamma}\left((Df)(x), \boldsymbol{n}\right)|y - x|^2 \geq 0. \tag{7}$$

Note that the condition (C) holds locally when (A) and (B) hold. Under the assumptions (A) and (B), Saisho proved the existence and uniqueness of the solution of the Skorohod problem which improved the result in Lions-Sznitman* and Tanaka (Hiroshima Math.J.9,1979). Moreover, the Skorohod mapping $\Gamma : w \mapsto \xi$ is 1/2-Hölder continuous map in the uniform norm.

Let us explain the meaning of reflecting SDE. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F}_t$ be the right-continuous filtration with the property that $\mathcal{F}_t$ contains all null sets of $(\Omega, \mathcal{F}, P)$. Let $B(t)$ be an $\mathcal{F}_t$-Brownian motion on $\mathbb{R}^n$. Let $\sigma \in C(\mathbb{R}^d \to \mathbb{R}^n \otimes \mathbb{R}^d)$, $b \in C(\mathbb{R}^d \to \mathbb{R}^d)$ be continuous mappings. We consider an SDE with reflecting boundary condition on $\bar{D}$:

$$X(t) = x + \int_0^t \sigma(X(s))dB(s) + \int_0^t b(X(s))ds + \Phi(t), \tag{8}$$

where $x \in \bar{D}$. We denote this SDE by SDE(σ, b) simply. A pair of $\mathcal{F}_t$-adapted continuous processes $(X(t), \Phi(t))$ is called a solution to (8) if the following holds. Let

$$Y(t) = x + \int_0^t \sigma(X(s))dB(s) + \int_0^t b(X(s))ds \tag{9}$$

Then $(X(\cdot,\omega), \Phi(\cdot,\omega))$ is a solution of the Skorohod problem associated with $Y(\cdot,\omega)$ for almost all $\omega \in \Omega$. Saisho proved that the existence and uniqueness of the solution to (8) under the conditions (A) and (B) and the Lipschitz continuities on σ and b.

Now, we assume $\sigma \in C_b^2$ and $b \in C_b^1$. Let $N \in \mathbb{N}$. We define the Wong-Zakai approximation X^N to the solution X of the SDE$(\sigma, \tilde{b})$, where $\tilde{b}(x) = b(x) + \frac{1}{2}\mathrm{tr}(D\sigma)[\sigma(x)\cdot](\cdot)$ as the solution to the reflecting ODE:

$$X^N(t) = x + \int_0^t \sigma(X^N(s))dB^N(s) + \int_0^t b(X^N(s))ds + \Phi^N(t), \tag{10}$$

where $B^N(t)$ is the piecewise linear approximation of $B(t)$ at times $\{kT/N \mid 0 \leq k \leq N\}$. The following our main theorem improves the weak convergence which were proved in Evans-Stroock (SPA 121, 2011). When ∂D is bounded and smooth, the convergece in probability was proved by Doss and Priouret (Z. Wahrsc. Verw. Gebiete 61, 1982).

Theorem 3 (A-Sasaki)**.** *Assume* $\sigma \in C_b^2$, $b \in C_b^1$ *and conditions* (A), (B) *and* (C). *Let* X *be the solution to* SDE$(\sigma, \tilde{b})$, *where* $\tilde{b} = b + \frac{1}{2}\mathrm{tr}(D\sigma)(\sigma)$. *Let* $0 < \theta < 1$. *There exists a positive constant* $C_{T,\theta}$ *such that for all* $N \in \mathbb{N}$,

$$E\left[\max_{0 \leq t \leq T} |X^N(t) - X(t)|^2\right] \leq C_{T,\theta}\left(\frac{T}{N}\right)^{\theta/6}. \tag{11}$$

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*They assume additional assumptions " admissibility condition on D".