ON THE MONOTONICITY OF $\mathcal{L}_0$-COST ALONG BACKWARD HEAT FLOW

TAKAFUMI AMABA AND KAZUMASA KUWADA

Let M be a d-dimensional connected manifold. Assume we are given a complete Ricci flow

$$\frac{dg(t)}{dt} = -2\mathrm{Ric}_{g(t)}$$

on M, that is, we are given a family $(g(t))_{0\le t\le T}$ of Riemannian metrics on M, satisfying the above equation, each of which makes M a complete Riemannian manifold. When M is closed, this (modified by a diffeomorphism) can be interpreted as a gradient flow of the Perelman's $\mathcal{F}$-functional (see [3]) defined by

$$\mathcal{F}(g,f) := \int_M \big\{R_g + |\nabla^{g(t)} f|_g^2\big\} e^{-f} \mathrm{dvol}_g$$

under the constraint that $e^{-f}\mathrm{dvol}_g$ is fixed. Here g and f are any Riemannian metrics and smooth functions on M respectively. The quantity R_g denotes the scalar curvature with respect to the Riemannian structure g.

On the other hand, from an optimal-transport point of view, Lott [2] defined the $\mathcal{L}_0$ functional by

$$\mathcal{L}_0(\gamma) := \frac{1}{2}\int_{t'}^{t''} \big\{|\dot\gamma(t)|_{g(t)}^2 + R_{g(t)}(\gamma(t))\big\}\mathrm{d}t$$

where $0 \le t' < t'' \le T$ and $\gamma : [t', t''] \to M$ is a smooth curve in M. From this lagrangian, a Riemannian distance-like function $L_0^{t',t''}$ and the $\mathcal{L}_0$-transportation cost can be defined by

$$L_0^{t',t''}(m', m'') := \inf_\gamma \mathcal{L}_0(\gamma), \quad m' \neq m'' \text{ in } M$$

where the infimum is taken among smooth curves $\gamma : [t', t''] \to M$ such that $\gamma(t') = m'$ and $\gamma(t'') = m''$, and

$$C_0^{t',t''}(\mu', \mu'') := \inf_{\pi \in \prod(\mu',\mu'')} \int_{M\times M} L_0^{t',t''}(m', m'')\pi(\mathrm{d}m', \mathrm{d}m'')$$

respectively, where μ' and μ'' are two Borel probability measures on M and $\prod(\mu', \mu'')$ is the set of all couplings of them.

The Perelman's $\mathcal{F}$-functional and the $\mathcal{L}_0$-transportational cost are related (see Lott [2]) by the equation

$$\lim_{t''\to t'} \frac{1}{t''-t'} C_0^{t',t''}(\alpha(t'), \alpha(t'')) = \mathcal{F}\big(g(t'), \frac{\mathrm{d}\alpha(t')}{\mathrm{dvol}_{g(t')}}\big)$$

where α is a curve in the space of probability measures on M satisfying the backward heat equation

$$\frac{\partial \alpha}{\partial t} = - \triangle \alpha.$$

By using this relation, Lott [2] gave an optimal-transport theoretical proof to the monotonicity of the $\mathcal{F}$-functional along the Ricci flow (although this is immediate from the Perelman's gradient flow interpretation of Ricci flow). In fact, heuristically, he proved the monotonicity of the $\mathcal{L}_0$-transportational cost along the backward heat flow, which is valid at least under the condition where Otto's calculus works well.

In this paper, we investigated a probabilistic proof of the Lott's result for a deeper understanding. Let

$$0 \leq t_0' < t_1' \leq T, \quad 0 \leq t_0'' < t_1'' \leq T$$

with $t_0' < t_0''$ and $t_1' - t_0' = t_1'' - t_0''$. It will be helpful to think that we have two worlds governed by Ricci flows $(M, g(t'))_{t_0' \leq t \leq t_1'}$ and $(M, g(t'))_{t_0'' \leq t \leq t_1''}$.

Theorem 1. *Assume that our Ricci flow satisfies the condition*

$$\sup_{(t,m) \in [t_0', t_1''] \times M} |\mathrm{Rm}_{g(t)}(m)|_{g(t)} < \infty$$

where Rm_g *is the Riemannian curvature tensor with respect to the Riemannian structure* g*. Then for each* $(m', m'') \in M \times M$*, there exists a coupling of* $g(t_1' - s)$*-Brownian motion* $X = (X_s)_{0 \leq s \leq t_1' - t_0'}$ *starting from* m' *and* $g(t_1'' - s)$*-Brownian motion* $Y = (Y_s)_{0 \leq s \leq t_1'' - t_0''}$ *starting from* m'' *such that* $s \mapsto L_0^{t_1'-s, t_1''-s}(X_s, Y_s)$ *is a supermartingale.*

Remark 1. (1) This result is an analogy of Kuwada-Philipowski [1] in which they discussed an existence of such a coupling of Brownian motions fitting to the relationship of Perelman's $\mathcal{W}$-functional and the transportation cost described by the Perelman's $\mathcal{L}$-geometry. (2) From Theorem 1, the monotonicity of $\mathcal{L}_0$-cost along the backward heat equation follows. (3) This also gives an extension of Lott's monotonicity result because Lott's framework assumed the closedness of M but we do not. Instead we assumed the space-time boundedness of the Riemannian curvature.

References

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[2] Lott, John *Optimal transport and Perelman's reduced volume.* (English summary) Calc. Var. Partial Differential Equations 36 (2009), no. 1, 49-84.

[3] Perelman, Grisha, *The entropy formula for the Ricci flow and its geometric application.*, arXiv:math/0211159 [math.DG]

(T. Amaba) Ritsumeikan University
E-mail address: capca0310@gmail.com

(K. Kuwada)Ochanomizu University