Large deviation principle for certain spatially lifted Gaussian rough path

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要約　近年 M. Hairer が急速に研究を進めている Hairer 流のラフ確率偏微分方程式 (rough SPDE) 理論の枠組において, Schilder 型の大偏差原理を証明する。ラフパス理論の確率論的部分においては, この種の大偏差原理は中心的な話題であるが, rough SPDE 理論では (いかなる流派においても) 前例がないと思う。Hairer 理論においては, 2 変数ガウス過程を空間変数に関してラフパス持ち上げしたものを空間的ラフパスと呼び, これが主役を演じるのだが, 論文によって, 考える 2 変数ガウス過程が違っている。本講演では, 最も典型的な例である確率熱方程式の解の持ち上げに対して, Schilder 型の大偏差原理を証明する。証明は通常のラフパス理論においてもっとも一般性の高い手法である Friz-Victoir の議論を「2 変数化」することにより与える。

In rough path theory of T. Lyons, the notion of paths is generalized to a great extent and so is that of ordinary differential equations. They are called rough paths and rough differential equations (RDEs), respectively. The solution map of an RDE is called an Itô map, which is defined for every rough path and, moreover, is continuous with respect to the topology of rough path space (Lyons' continuity theorem). As a result, stochastic differential equations (SDEs) in the usual sense are made deterministic or "dis-randomized".

Even though Itô maps are deterministic, the probabilistic aspect of the theory is still very important undoubtedly. In a biased view of the author, a large deviation principle of Schilder type is a central issue in stochastic analysis on rough path spaces. This kind of large deviations was first shown by Ledoux, Qian, and Zhang (2002) for the law of Brownian rough path. Combined with Lyons' continuity theorem, this result immediately recovers well-known Freidlin-Wentzell type large deviations for solutions of SDEs. Since then many papers have been published on this subject.

Naturally, one would like to apply rough path theory to stochastic PDEs. There have been some successful attempts. In this paper, we focus on M. Hairer's theory [1, 3, 4], which is based on M. Gubinelli's "algebraic" rough integration theory. In Hairer's theory, rough path theory is used for the space variable $x \in S^1 = \mathbf{R}/\mathbf{Z}$ for each fixed time variable $t > 0$. This is surprising because almost everyone regarded solutions of stochastic PDEs as processes indexed by the time-variable t that take values in function spaces of the space-variable x and then modify and apply infinite dimensional rough path theory. Not only his point of view is novel, but his theory also turned out to be very powerful when he rigorously solved KPZ equation in the periodic case for the first time [2].

Under these circumstances, it seems natural and necessary to develop stochastic analysis in this framework. In this paper we will prove a large deviation principle of Schilder type for the spatial lift of the (scaled) solution ψ of the stochastic heat equation on S^1. This process ψ plays a crucial role in [3, 4]. To our knowledge, a large deviation principle is new in rough stochastic PDE theories of any kind.

Now we introduce our setting. Let us recall the stochastic heat equation on S^1. As usual $S^1 = \mathbf{R}/\mathbf{Z}$ is regarded as $[0, 1]$ with the two end points identified and $\triangle = \triangle_{S^1}$ stands for the periodic Laplacian. Let $\xi^i = \xi(t, x)^i$ $(1 \leq i \leq d)$ are independent copies of the space-time white noise associated with $L^2([0, T] \times S^1)$ with the (formal) covariance $\mathbb{E}[\xi(t, x)^i \xi(s, y)^j] = \delta_{ij} \cdot \delta_{t-s} \cdot \delta_{x-y}$. Let $\psi = \psi(t, x)$ be a unique solution of the following $\mathbf{R}^d$-valued stochastic PDE.

$$\partial_t \psi = \triangle_x \psi + \xi \qquad \text{with} \qquad \psi(0, x) \equiv 0.$$

Then, $\psi = (\psi(t, x))_{0 \leq t \leq T, 0 \leq x \leq 1}$ is a two-parameter continuous Gaussian process. It was shown in [3] that, (i) for each t, $x \mapsto \psi(t, x)$ admits a natural lift to a geometric rough path $(x, y) \mapsto \Psi(t; x, y)$ a.s. and (ii) there exists a modification of Ψ such that $t \mapsto \Psi(t; \bullet, \star)$ is continuous in the geometric rough path space a.s. In Hairer's theory, a solution of a rough stochastic PDE is obtained as a continuous image of Ψ. Therefore, it is important to analyze (the law of) Ψ.

Let $1/3 < \alpha < 1/2$. We denote by $G\Omega^H_\alpha(\mathbf{R}^d)$ the α-Hölder geometric rough path space over $\mathbf{R}^d$. The first level path of $X \in G\Omega^H_\alpha(\mathbf{R}^d)$ is a usual path in $\mathbf{R}^d$ which starts at 0. Let $G\hat{\Omega}^H_\alpha(\mathbf{R}^d) \cong \mathbf{R}^d \times G\Omega^H_\alpha(\mathbf{R}^d)$ be the α-Hölder geometric rough path space in an extended sense so that information of the initial values of the first level paths are added. For each t, the random variable $\Psi(t;\bullet,\star)$ takes values in this Polish space $G\hat{\Omega}^H_\alpha(\mathbf{R}^d)$. Let $\mathcal{P}_\infty G\hat{\Omega}^H_\alpha(\mathbf{R}^d) = C([0,T], G\hat{\Omega}^H_\alpha(\mathbf{R}^d))$ be the continuous path space over $G\hat{\Omega}^H_\alpha(\mathbf{R}^d)$. Its topology is given by the uniform convergence in t as usual. The random variable Ψ takes values in this Polish space and hence its law is a probability measure on this space.

Introduce a small parameter $0 < \varepsilon \le 1$. Let $\varepsilon\Psi$ is the dilatation of Ψ by ε, which is equal to the natural lift of $\varepsilon\psi$, anyway. Denote by ν_ε the law of $\varepsilon\Psi$ on $\mathcal{P}_\infty G\hat{\Omega}^H_\alpha(\mathbf{R}^d)$. Our main result is the following:

Main result: *For any $\alpha \in (1/3, 1/2)$, the family $(\nu_\varepsilon)_{0<\varepsilon\le 1}$ of probability measures on $\mathcal{P}_\infty G\hat{\Omega}^H_\alpha(\mathbf{R}^d)$ satisfies a large deviation principle as $\varepsilon \searrow 0$ with a good rate function I.*

Here, we give a few quick remarks. The rate function I takes the usual form. So, we omit its explicit form. Just like in the usual rough path theory, the continuity of the Itô map and the contraction principle for LDP immediately imply Freidlin-Wentzell type LDP for solutions of rough SPDEs as in [3].

We show the main result by developping an extended version of Friz-Victoir's method for Schilder-type LDP for Gaussian rough path (2007).

References

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