SPECTRAL GAP PROPERTY FOR RANDOM DYNAMICS ON THE REAL LINE AND MULTIFRACTAL ANALYSIS OF GENERALISED TAKAGI FUNCTIONS

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ABSTRACT. We consider the random iteration of finitely many expanding $\mathscr{C}^{1+\varepsilon}$ diffeomorphisms on the real line without a common fixed point. We derive the spectral gap property of the associated transition operator acting on spaces of Hölder continuous functions. As an application we introduce generalised Takagi functions on the real line and we perform a complete multifractal analysis of the pointwise Hölder exponents of these functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we investigate the independent and identically-distributed (i.i.d.) random dynamical systems on the real line. The theory of dynamical systems is used to describe various subjects in basically all areas of natural and social sciences. Since nature and any other environment have a lot of random terms, it is very natural and important not only to consider the dynamics of iteration of one map, but also to consider random dynamics. Many researchers in various fields have found and investigated many kinds of new phenomena in random dynamics which cannot hold in deterministic dynamics. These phenomena arise from the effect of randomness or noise and they are called *randomness-induced phenomena* or *noise-induced phenomena* ([JS15, JS17, Sum11, Sum13]). Under certain conditions, because of the effect of randomness or noise, the chaoticity of the system becomes milder, but the system still has some complexity. Hence regarding such random dynamical systems, our aim is to investigate the *gradation between chaos and order*.

To find and to study quantities describing the gradation between chaos and order, we combine ideas of random dynamical systems, ergodic theory (in particular, thermodynamic formalism), iterated function systems, and fractal geometry. More precisely, for any random dynamical system in our setting, there exists an exponent $\alpha_- \in (0,1)$ such that for each α with $0 < \alpha < \alpha_-$, the transition operator of the system behaves well (e.g., it has a spectral gap property) on the space $\mathscr{C}^\alpha$ of α-Hölder continuous functions endowed with α-Hölder norm, but for each α with $\alpha_- < \alpha < 1$, the transition operator of the system does not behave well (Theorem 1.1, Corollary 1.5). This quantity α_- describes the gradation between chaos and order for the system. Furthermore, to provide a refined gradation, we investigate the pointwise Hölder exponents of the limit state functions (i.e., fixed points of the transition operator) and their (higher order) partial derivatives with respect to the probability parameters. It turns out that the pointwise Hölder exponents have a complicated fine structure which can be suitably investigated using the multifractal analysis and the concept of fractal dimension (Theorems 1.3, 1.4). The objects appearing in the multifractal analysis also describe the gradation between chaos and order for the system. Moreover, we present a new general framework to study a large class of fractal functions. In particular, we shed new light on the regularity properties of the classical Takagi function in our framework (see Theorem 1.4 and Proposition 1.6).

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Throughout, let $I := \{1, \dots, s+1\}$, $s \geq 1$, and let $f_i : \mathbb{R} \to \mathbb{R}$, $i \in I$, be a family of $\mathscr{C}^{1+\varepsilon}$ diffeomorphisms with ε-Hölder continuous derivatives for some $\varepsilon > 0$. We say that $(f_i)_{i\in I}$ is *expanding* if there exists $\lambda > 1$ such that $f_i'(x) \geq \lambda > 1$, for all $x \in \mathbb{R}$ and $i \in I$. The family $(f_i)_{i\in I}$ *has no common fixed point* if there exists no $x \in \mathbb{R}$ such that $f_i(x) = x$ for all $i \in I$.

We denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the two-point compactification of $\mathbb{R}$ endowed with a metric d on $\overline{\mathbb{R}}$ which is strongly equivalent to the Euclidean metric on compact subsets of $\mathbb{R}$, that is, for each compact set $K \subset \mathbb{R}$ there exists a constant $C > 0$ such that $C^{-1}|x-y| \leq d(x,y) \leq C|x-y|$, for all $x, y \in K$. For $i \in I$ we extend the definition of f_i from $\mathbb{R}$ to $\overline{\mathbb{R}}$ by setting $f_i(\pm\infty) := \pm\infty$. We say that $(f_i)_{i\in I}$ is *contracting near infinity* if there exist neighborhoods $V^{\pm}$ of $\pm\infty$ such that $\mathrm{Lip}(f_{i|V^{\pm}}) < 1$, $i \in I$. Here, for $D \subset \overline{\mathbb{R}}$ and $g : D \to \mathbb{R}$, we have set $\mathrm{Lip}(g) := \sup_{x,y\in D, x\neq y} d\left(g(x), g(y)\right) / d(x,y)$. Note that if $(f_i)_{i\in I}$ is contracting near infinity then $\mathrm{Lip}(f_i) < \infty$ for each $i \in I$. We refer to Section 7 for details about the property of contraction near infinity.

Throughout, we assume that $(f_i)_{i\in I}$ is expanding, has no common fixed point, and is contracting near infinity. For $\mathbf{p} = (p_1, \dots, p_s) \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$, let $p_{s+1} := 1 - \sum_{i=1}^s p_i$. Let $\mathscr{C}(\overline{\mathbb{R}})$ denote the Banach space of continuous functions endowed with the supremum norm $\|\cdot\|_\infty$. Define the transition operator

$$M_{\mathbf{p}} : \mathscr{C}\left(\overline{\mathbb{R}}\right) \to \mathscr{C}\left(\overline{\mathbb{R}}\right), \quad M_{\mathbf{p}} h = \sum_{i\in I} p_i \cdot h \circ f_i,\ h \in \mathscr{C}\left(\overline{\mathbb{R}}\right).$$

For $\alpha > 0$ let $\mathscr{C}^\alpha(\overline{\mathbb{R}})$ denote the Banach space of α-Hölder continuous functions (see Section 2). Note that $M_{\mathbf{p}}\left(\mathscr{C}^\alpha(\overline{\mathbb{R}})\right) \subset \mathscr{C}^\alpha(\overline{\mathbb{R}})$. To state our first main result we say that $M_{\mathbf{p}}$ has the *spectral gap property* if its spectrum consists of finitely many eigenvalues of modulus one, and the rest of the spectrum is contained in a ball of radius strictly less than one. We say that $(f_i)_{i\in I}$ satisfies the *separating condition* if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$, we have $f_i^{-1}(\overline{O}) \cap f_j^{-1}(\overline{O}) = \varnothing$. For the definition of the bottom of the spectrum $\alpha_- = \alpha_-(\mathbf{p})$ we refer to Section 2.2. For $\mathbf{a} \in \mathbb{R}^s$ and $\delta > 0$ we denote by $B(\mathbf{a}, \delta) \subset \mathbb{R}^s$ the open ball of radius δ with center $\mathbf{a}$ in $\mathbb{R}^s$.

Theorem 1.1 (Theorem 2.4 and Theorem 2.15)**.** *For every $\mathbf{p}_0 \in (0,1)^s$ there exist $\delta > 0$ and $\alpha > 0$ such that $M_{\mathbf{p}} : \mathscr{C}^\alpha(\overline{\mathbb{R}}) \to \mathscr{C}^\alpha(\overline{\mathbb{R}})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$. If $(f_i)_{i\in I}$ satisfies the separating condition, then the previous assertion holds for any $\alpha < \alpha_-(\mathbf{p}_0)$.*

By combining with the perturbation theory of linear operators we can derive that the probability of tending to infinity $T := T_{\mathbf{p}} : \overline{\mathbb{R}} \to [0,1]$ (see (2.1) below for the definition) depends real analytically on $\mathbf{p}$ (see Theorem 2.4 for the detailed statement). This allows us to make the following definition. Let $\mathbb{N}_0 := \{0, 1, \dots\}$.

Definition 1.2. We denote by $\mathscr{T} := \mathscr{T}_{\mathbf{p}}$ the $\mathbb{R}$-vector space of generalised Takagi functions generated by

$$C_{\mathbf{n}}(x) := C_{\mathbf{n},\mathbf{p}}(x) := \frac{\partial^{\sum_{i=1}^s n_i}}{\partial u_1^{n_1} \partial u_2^{n_2} \dots \partial u_s^{n_s}} T_{(u_1,\dots,u_s)}(x)\big|_{(u_1,\dots,u_s)=\mathbf{p}}, \quad \mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_0^s,\ x \in \overline{\mathbb{R}}.$$

We say that an element $C \in \mathscr{T}$ is non-trivial if there exists $(\beta_{\mathbf{n}})_{\mathbf{n}} \neq 0$ such that $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}}$. For the reason why we call the elements of $\mathscr{T}$ generalised Takagi functions, we refer to Remark 2.17. We then proceed to investigate the regularity of the elements of $\mathscr{T}$. The pointwise Hölder exponent of $C \in \mathscr{T}$ at $x \in \mathbb{R}$ is denoted by $\mathrm{H\ddot{o}l}(C, x)$ (see (3.5) below for the definition). We denote by J the Julia set of $(f_i)_{i\in I}$ (see Section 2.1). For $\mathbf{p} \in (0,1)^s$ we define $\alpha_+ = \alpha_+(\mathbf{p})$ in Section 2.2. We say that $(f_i)_{i\in I}$ satisfies the open set condition if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$ we have $f_i^{-1}(O) \cap f_j^{-1}(O) = \varnothing$. By t^* we denote the Legendre transform of the function t defined implicitly by a certain topological pressure functional (see Section 3.2). Denote by $\dim_H(A)$ the Hausdorff dimension of a set $A \subset \mathbb{R}$ with respect to the Euclidean metric.

Theorem 1.3 (Theorem 3.16)**.** *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C \in \mathscr{T}$ be non-trivial. Then we have for all $\alpha \in [\alpha_-, \alpha_+]$,*

$$\dim_H \{x \in J \mid \text{Höl}(C,x) = \alpha\} = -t^*(-\alpha),$$

and for $\alpha \notin [\alpha_-, \alpha_+]$ we have $\{x \in J \mid \text{Höl}(C,x) = \alpha\} = \varnothing$. The function $g(\alpha) := -t^*(-\alpha)$ is continuous and concave on $[\alpha_-, \alpha_+]$. If $\alpha_- < \alpha_+$ then g is real-analytic and positive on (α_-, α_+) and satisfies $g'' < 0$ on (α_-, α_+).

We prove the following result regarding the global Hölder continuity of elements of $\mathscr{T}$.

Theorem 1.4 (Theorem 4.1, Corollary 4.3)**.** *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C \in \mathscr{T}$ be non-trivial. Then we have $\alpha_- = \sup\left\{\alpha \geq 0 \mid C \in \mathscr{C}^{\alpha}(\overline{\mathbb{R}})\right\}$. Further, we have $T \in \mathscr{C}^{\alpha_-}\left(\overline{\mathbb{R}}\right)$.*

The following corollary indicates that the random dynamical system generated by $(f_i)_{i\in I}, (p_i)_{i\in I}$ still has some kind of complexity.

Corollary 1.5 (Corollary 4.4)**.** *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. If $\alpha_- < 1$ then $\lim_{n\to\infty} \|M_{\mathbf{p}}^n\|_\alpha = \infty$, for each $\alpha_- < \alpha < 1$, where $\|M_{\mathbf{p}}^n\|_\alpha$ denotes the operator norm of $M_{\mathbf{p}}^n$ on $\mathscr{C}^{\alpha}(\overline{\mathbb{R}})$.*

Regarding the existence of points of non-differentiability of elements of $\mathscr{T}$ we prove the following. Let $\mathbf{e}_k \in \mathbb{N}_0^s$ denote the k-th unit vector in $\mathbb{N}_0^s$, $1 \leq k \leq s$. We use C_m to denote $C_{(m)}$ for $m \in \mathbb{N}_0$.

Proposition 1.6 (Proposition 5.1)**.** *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition.*

(1) If $\alpha_- < 1$, then there exists a dense subset $E \subset J$ of positive Hausdorff dimension such that, for every non-trivial $C \in \mathscr{T}$ and every $x \in E$, C is not differentiable at x.
(2) If $\alpha_- = 1$, $s = 1$ and f_1' and f_2' are constant functions, then C_m is nowhere differentiable on J, for every $m \geq 1$.

For applications of our results to conjugacies of interval maps we refer to Section 6. In fact, if $(f_i)_{i\in I}$ satisfies the open set condition, then the probability of tending to infinity $T_{\mathbf{p}}$ can also be characterised as the conjugacy map between the expanding dynamical system defined by $(f_i)_{i\in I}$ on J and the dynamical system given by the piecewise linear map on $[0,1]$ with $(s+1)$ full branches and slopes given by $(1/p_i)_{i\in I}$ (see Lemma 6.1). By the rigidity dichotomy in [JKPS09, Theorem 1.2], if J is an interval, then either $\alpha_-(\mathbf{p}) = \alpha_+(\mathbf{p})$ and $T_{\mathbf{p}}$ is a $\mathscr{C}^{1+\varepsilon}$-diffeomorphism, or $\alpha_-(\mathbf{p}) < \alpha_+(\mathbf{p})$ and the set of non-differentiability points of $T_{\mathbf{p}}$ has positive Hausdorff dimension. For general families $(f_i)_{i\in I}$ satisfying the open set condition, we can show that $\alpha_-(\mathbf{p}) = \alpha_+(\mathbf{p})$ if and only if $T_{\mathbf{p}} \in \mathscr{C}^{\dim_H(J)}(\overline{\mathbb{R}})$ (see Section 6).

Higher order derivatives of the classical Takagi function have been considered in [AK06], where it is shown that the classical Takagi function and the higher order derivatives of the Lebesgue singular function for $p = 1/2$ are nowhere differentiable and convex Lipschitz ([MW86]). These results are covered by our general theory. Namely, since for this special case we have $\alpha_- = 1$ and $s = 1$, the non-differentiability follows from Proposition 1.6 (2). That these functions are α-Hölder continuous, for every $\alpha < 1$, follows from Theorem 1.4. In fact, we can also derive that the functions are convex Lipschitz (see Remark 4.2).

Generalised Takagi functions (with respect to the parameter $\mathbf{p}$) have also been introduced in [HY84]. As already observed in [AK06], the higher-order derivatives of $T_{\mathbf{p}}$ of a system $(f_i)_{i\in I}$ with constant derivatives are not covered by the setting in [HY84]. In [SS91] it is shown that the Lebesgue singular function depends real analytically on the parameter, and its higher order derivatives are considered. We point out that our general definition of $\mathscr{T}$ is a far-reaching generalisation of the above concept, where we consider an arbitrary finite number of $\mathscr{C}^{1+\varepsilon}$ diffeomorphisms and arbitrary linear combinations of higher order partial derivatives of the probability of tending to infinity with $s \geq 1$ parameters.

We remark that in the previous works of the authors [JS15, JS17] we dealt with the random complex dynamical systems satisfying the separating condition. However, in this paper, we deal with random dynamical systems on the real line satisfying the open set condition. Note that the separating condition implies the open set condition. When we deal with general systems satisfying the open set condition, we have to overcome new difficulties. In fact, the relation between the pointwise Hölder exponents of elements of $\mathscr{T}$ and the corresponding dynamical quantities is much more involved than in the case of the separating condition. We developed several new ideas (see Propositions 3.11 and 3.15) to overcome these difficulties. In the case of the separating condition we show that α_- is the supremum of the exponents α for which the transition operator has the spectral gap property on $\mathscr{C}^{\alpha}$ by developing some idea from [JS17] and providing a new approach.

In Section 2, we derive the spectral gap property for the transition operator associated with random dynamical systems on the real line. In Section 3, we perform a complete multifractal analysis of the pointwise Hölder exponents of the elements of $\mathscr{T}$ associated with $(f_i)_{i\in I}$ and $\mathbf{p} \in (0,1)^s$. In Section 4, we investigate the global Hölder continuity of the elements of $\mathscr{T}$. In Section 5, we study the (non-)differentiability of the elements of $\mathscr{T}$. In Section 6, we show how our results are related to interval conjugacy maps. In Section 7 (Appendix), we will show that, by modifying the $(f_i)_{i\in I}$ near infinity, we can always assume that an expanding family $(f_i)_{i\in I}$ is contracting near infinity with respect to a metric d which is strongly equivalent to the Euclidean metric on compact subsets of $\mathbb{R}$.

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2. Spectral gap property

In this section we derive the spectral gap property for the transition operator associated with random dynamical systems on the real line induced by a family $(f_i)_{i\in I}$.

2.1. **General results .** Let $I^* := \bigcup_{n\in\mathbb{N}} I^n$. For $\omega \in I^*$ we denote by $|\omega|$ the unique $n \in \mathbb{N}$ such that $\omega \in I^n$. For $\omega = (\omega_1,\dots,\omega_n) \in I^n$ we let $f_{(\omega_1,\dots,\omega_n)} := f_{\omega_n} \circ \cdots \circ f_{\omega_1}$. Let $\Sigma := I^{\mathbb{N}}$. Also, for $\omega \in \Sigma$ and $n \in \mathbb{N}$ we put $\omega_{|n} := (\omega_1,\dots,\omega_n) \in I^n$. Since $(f_i)_{i\in I}$ is contracting near infinity, there exist neighborhoods $V^{\pm}$ of $\pm\infty$ in $\overline{\mathbb{R}}$ such that, for each $x \in V^+$ (resp. $x \in V^-$) we have for all $\omega \in \Sigma$,

$$f_{\omega_{|n}}(x) \to +\infty \quad (\text{resp.} f_{\omega_{|n}}(x) \to -\infty), \quad \text{as } n \to \infty.$$

We put

$$V := V^+ \cup V^-.$$

Denote by $G := \langle f_1, \dots f_{s+1}\rangle := \{f_\omega \mid \omega \in I^*\}$ the semigroup generated by $f_1,\dots,f_{s+1}$ where the semigroup operation is the composition of functions. The *Julia set of G* is defined as

$$J := \left\{x \in \overline{\mathbb{R}} \mid G \text{ is not equicontinuous in any neighborhood of } x \text{ with respect to } d\right\}.$$

Note that the inverse maps $(f_i^{-1})_{|\overline{\mathbb{R}}\setminus V}, i \in I$, form a contracting conformal iterated function system (see e.g. [Fal03, MU03]) and J is the limit set (or attractor) of this system.

For $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$ we define

$$J_\omega := \bigcap_{n\in\mathbb{N}} \left(f_{\omega_{|n}}\right)^{-1} \left(\overline{\mathbb{R}} \setminus V\right).$$

Note that J_ω is a singleton because $(f_i)_{i\in I}$ is expanding. We define the coding map $\pi : \Sigma \to \mathbb{R}$ given by

$$\bigcap_{n\in\mathbb{N}} (f_{\omega_{|n}})^{-1}(\overline{\mathbb{R}} \setminus V) = \{\pi(\omega)\}, \quad \omega \in \Sigma.$$

It is easy to see that

$$J = \bigcup_{\omega\in\Sigma} J_\omega = \pi(\Sigma).$$

The *kernel Julia set of* G ([Sum11]) is given by

$$\mathrm{J}_{\mathrm{ker}} := \bigcap_{g\in G} g^{-1}(J) \subset J.$$

For $\mathbf{p} = (p_1, \dots, p_s) \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$, let $p_{s+1} := 1 - \sum_{i=1}^s p_i$. Let $\mu_{\mathbf{p}}$ denote the $(p_1, \dots, p_s, p_{s+1})$ Bernoulli measure on Σ.

Proposition 2.1. *We have* $\mathrm{J}_{\mathrm{ker}} = \varnothing$*. Moreover, we have* $\mu_{\mathbf{p}}(\{\omega \in \Sigma \mid x \in J_\omega\}) = 0$*, for each* $x \in \mathbb{R}$*.*

Proof. Since $(f_i)_{i\in I}$ is expanding without a common fixed point, for all $x \in \mathbb{R}$ there exists $g_x \in G$ such that $g_x(x) \in V$. Hence, $\mathrm{J}_{\mathrm{ker}} = \varnothing$. By [Sum11, Lemma 4.6] we have $\mu_{\mathbf{p}}(\{\omega \in \Sigma \mid x \in J_\omega\}) = 0$, for each $x \in \mathbb{R}$. $\square$

Recall that $M_{\mathbf{p}}$ is *almost periodic* if $(M_{\mathbf{p}}^n h)_{n\geq 1}$ is relatively compact in $\mathscr{C}(\overline{\mathbb{R}})$, for each $h \in \mathscr{C}(\overline{\mathbb{R}})$. The dual operator of $M_{\mathbf{p}}$ is given by $M_{\mathbf{p}}^* : \mathscr{M}_1(\overline{\mathbb{R}}) \to \mathscr{M}_1(\overline{\mathbb{R}})$, where $\mathscr{M}_1(\overline{\mathbb{R}})$ denotes the space of Borel probability measures on $\overline{\mathbb{R}}$ endowed with the topology of weak convergence. We define the compact subset

$$J_{\mathrm{meas}} := \left\{m \in \mathscr{M}_1(\overline{\mathbb{R}}) \mid (M_{\mathbf{p}}^*)^n_{n\geq 1} \text{ is not equicontinuous in any neighbourhood of } m\right\} \subset \mathscr{M}_1(\overline{\mathbb{R}}).$$

The following fact is a special case of [Sum11, Proposition 4.7, Lemma 4.2(6)].

Proposition 2.2. *We have that* $J_{meas} = \varnothing$ *and that* $M_{\mathbf{p}} : \mathscr{C}(\overline{\mathbb{R}}) \to \mathscr{C}(\overline{\mathbb{R}})$ *is almost periodic.*

By a well-known result of Ljubich ([Lju83]) on almost periodic operators, we have

$$\mathscr{C}(\overline{\mathbb{R}}) = \left\{h \in \mathscr{C}(\overline{\mathbb{R}}) \mid \|M_{\mathbf{p}}^n h\|_\infty \to 0, \text{ as } n \to \infty\right\} \oplus \overline{\operatorname{span}\left\{h \in \mathscr{C}(\overline{\mathbb{R}}) \mid \exists \rho \in \mathbb{S}^1 \mid M_{\mathbf{p}} h = \rho h\right\}}.$$

As in [Sum11] we define the probability of tending to infinity

$$T_{\mathbf{p}} : \overline{\mathbb{R}} \to [0,1], \quad T_{\mathbf{p}}(x) := \mu_{\mathbf{p}}\left\{\omega \in \Sigma \mid \lim_{n\to\infty} f_{\omega_{|n}}(x) = \infty\right\}. \tag{2.1}$$

It follows from Proposition 2.1 and the dominated convergence theorem that for every $h \in \mathscr{C}(\overline{\mathbb{R}})$ that

$$\lim_{n\to\infty} M_{\mathbf{p}}^n h(x) = \lim_{n\to\infty} \int h \circ f_{\omega_n} \circ \cdots \circ f_{\omega_1}(x)\, d\mu_{\mathbf{p}}(\omega) = T_{\mathbf{p}}(x) h(\infty) + (1 - T_{\mathbf{p}}(x)) h(-\infty). \tag{2.2}$$

Hence,

$$\operatorname{span}\left\{h \in \mathscr{C}(\overline{\mathbb{R}}) \mid \exists \rho \in \mathbb{S}^1 \mid M_{\mathbf{p}} h = \rho h\right\} = \mathbb{R} T_{\mathbf{p}} \oplus \mathbb{R} 1$$

and we have by Ljubich's result,

$$\|M_{\mathbf{p}}^n h - (h(\infty) - h(-\infty)) T_{\mathbf{p}} - h(-\infty) 1\|_\infty \to 0, \quad \text{as } n \to \infty.$$

For $\alpha > 0$ we say that $h : \overline{\mathbb{R}} \to \mathbb{R}$ is α-Hölder continuous if

$$V_\alpha := \sup_{x \neq y} \left\{ \frac{|h(x) - h(y)|}{d(x,y)^\alpha} \right\} < \infty.$$

We say that a function is Hölder continuous if it is α-Hölder continuous for some $\alpha > 0$. We denote by $\mathscr{C}^\alpha(\overline{\mathbb{R}})$ the Banach space of α-Hölder continuous maps on $\overline{\mathbb{R}}$ endowed with the α-Hölder norm

$$\|h\|_\alpha := V_\alpha + \|h\|_\infty, \quad h \in \mathscr{C}^\alpha(\overline{\mathbb{R}}).$$

The following lemma is the key to derive the spectral gap property of $M_{\mathbf{p}}$ on $\mathscr{C}^\alpha(\overline{\mathbb{R}})$. The proof is inspired by [Sum13]. For $n \in \mathbb{N}$ and $\omega \in I^n$ we will use the notation

$$p_\omega := p_{\omega_1} \cdot \dots \cdot p_{\omega_n}.$$

Hence, we have for $n \in \mathbb{N}$ and $h \in \mathscr{C}\left(\overline{\mathbb{R}}\right)$

$$M_{\mathbf{p}}^n h = \sum_{\omega \in I^n} p_\omega \cdot (h \circ f_\omega) = \int h \circ f_{\omega_{|n}} \, d\mu_{\mathbf{p}}(\omega).$$

Lemma 2.3. *For every $\mathbf{p}_0 \in (0,1)^s$ there exist $\delta > 0$, $\alpha > 0$, $n \in \mathbb{N}$ and constants $0 < c < 1$ and $C > 0$ such that, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ and for every $h \in \mathscr{C}^\alpha(\overline{\mathbb{R}})$,*

$$\left| M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y) \right| \leq (c\|h\|_\alpha + C\|h\|_\infty)\, d(x,y)^\alpha, \quad x,y \in \overline{\mathbb{R}}.$$

Proof. Recall that $V = V^+ \cup V^-$. Since $(f_i)_{i \in I}$ is expanding without a common fixed point, we have that, for all $x \in \mathbb{R}$ there exists $g_x \in G$ and a compact neighborhood U_x of x in $\overline{\mathbb{R}}$ such that $g_x(U_x) \subset V$. Since $\overline{\mathbb{R}}$ is compact, there exist $t \in \mathbb{N}$ and $x_1, \dots, x_t \in \overline{\mathbb{R}}$ such that $\overline{\mathbb{R}} = \bigcup_{j=1}^t \operatorname{Int}(U_{x_j})$, where $\operatorname{Int}(A)$ denotes the interior of a set $A \subset \overline{\mathbb{R}}$. Since $\bigcup_{g \in G} g(V) \subset V$ we may assume that there exists $r \in \mathbb{N}$ such that, for each $j = 1, \dots, t$ there exists $\beta^j \in I^r$ with $g_{x_j} = f_{\beta^j}$. For $\omega \in I^*$ with $|\omega| = \ell$ we denote by $[\omega] := \{\tau \in \Sigma \mid \tau_1 = \omega_1, \dots, \tau_\ell = \omega_\ell\}$ the cylinder set of ω. Let

$$a := a(\mathbf{p}) := \max\left\{1 - \mu_{\mathbf{p}}([\beta^j]) \mid 1 \leq j \leq t\right\} < 1.$$

Recall that $\operatorname{Lip}(g) < \infty$, for every $g \in G$. Let

$$\Lambda := 2 \cdot \max\left\{\max\left\{\operatorname{Lip}(f_\omega) \mid \omega \in I^r\right\}, 1\right\} \geq 2.$$

Let $R > 0$ be a Lebesgue number of the covering $\left(\operatorname{Int}(U_{x_j})\right)_{1 \leq j \leq t}$ of $\overline{\mathbb{R}}$. Let $\alpha > 0$ such that

$$\eta := a\Lambda^\alpha < 1.$$

Let $n \in \mathbb{N}$ to be determined later. Let $x, y \in \overline{\mathbb{R}}$. Since $M_{\mathbf{p}}^n$ has norm one, we may assume that $d(x,y) < R$. Let

$$n(x,y) := \max\left\{k \geq 0 \mid \Lambda^{-k} R > d(x,y)\right\}.$$

If $n(x,y) < n$, then $d(x,y) \geq \Lambda^{-n} R$ and the desired estimate follows with $C := 2\Lambda^{n\alpha} R^{-\alpha}$. Now, we consider the case $n(x,y) \geq n$. Then we have $d(x,y) < \Lambda^{-n} R$. Consequently, for $j \leq n$ and $\omega \in I^{jr}$ we have $d(f_\omega(x), f_\omega(y)) \leq R$. By the definition of R there exists $i_0 \in \{1, \dots, t\}$ such that $B(x,R) \subset U_{x_{i_0}}$. Let $A(0) := [\beta^{i_0}] \subset \Sigma$ and $B(0) := \Sigma \setminus [\beta^{i_0}]$. We define inductively, for $j \geq 1$,

$$A(j) := \left\{\omega \in B(j-1) \mid \exists i \in \{1, \dots, t\} \text{ such that } B(f_{\omega_{|rj}}(x), R) \subset U_{x_i} \text{ and } \left(\omega_{rj+1}, \dots \omega_{r(j+1)}\right) = \beta^i\right\}$$

and $B(j) := B(j-1) \setminus A(j)$. We have

$$\left|M_{\mathbf{p}}^{rn} h(x) - M_{\mathbf{p}}^{rn} h(y)\right| \leq \left|\sum_{j=0}^{n-1} \int_{A(j)} h(f_{\omega_{|rn}}(x)) - h(f_{\omega_{|rn}}(y)) d\mu_{\mathbf{p}}(\omega)\right| + \left|\int_{B(n-1)} h(f_{\omega_{|rn}}(x)) - h(f_{\omega_{|rn}}(y)) d\mu_{\mathbf{p}}(\omega)\right|.$$

Since $\mu_{\mathbf{p}}(A(j)) \leq a^j$ for every $j \leq n-1$, and $B(f_{\omega_{|r(j+1)}}(x), R) \subset V$, for every $\omega \in A(j)$, we can estimate with

$$S := \max\left\{\mathrm{Lip}(f_{i|V^+}), \mathrm{Lip}(f_{i|V^-}) \mid i \in I\right\} < 1 \quad \text{and } \tilde{c} := \max\left\{\eta, S^{r\alpha}\right\} < 1,$$

$$\begin{aligned}
\int_{A(j)} \left| h(f_{\omega_{|rn}}(x)) - h(f_{\omega_{|rn}}(y)) \right| d\mu_{\mathbf{p}}(\omega) &\leq a^j \sup_{\omega \in A(j)} \left| h(f_{\omega_{|rn}}(x)) - h(f_{\omega_{|rn}}(y)) \right| \\
&\leq a^j \|h_{|V}\|_\alpha \sup_{\omega \in A(j)} \left\{ \left(S^{(n-j-1)r} d\left(f_{\omega_{|r(j+1)}}(x), f_{\omega_{|r(j+1)}}(y) \right) \right)^\alpha \right\} \\
&\leq a^j S^{(n-j-1)r\alpha} \Lambda^{(j+1)\alpha} \|h_{|V}\|_\alpha d(x,y)^\alpha \\
&= \eta^j (S^{r\alpha})^{n-j-1} \Lambda^\alpha \|h_{|V}\|_\alpha d(x,y)^\alpha \leq \tilde{c}^{n-1} \Lambda^\alpha \|h_{|V}\|_\alpha d(x,y)^\alpha.
\end{aligned}$$

Finally, we verify that

$$\begin{aligned}
\int_{B(n-1)} \left| h(f_{\omega_{|rn}}(x)) - h(f_{\omega_{|rn}}(y)) \right| d\mu_{\mathbf{p}}(\omega) &\leq \mu\left(B(n-1)\right) \sup_{\omega \in B(n-1)} \left| h \circ f_{\omega_{|rn}}(x) - h \circ f_{\omega_{|rn}}(y) \right| \\
&\leq a^n \|h\|_\alpha \sup_{\omega \in B(n-1)} d\left(f_{\omega_{|rn}}(x), f_{\omega_{|rn}}(y) \right)^\alpha \\
&\leq a^n \|h\|_\alpha \Lambda^{n\alpha} d(x,y)^\alpha \leq \eta^n \|h\|_\alpha d(x,y)^\alpha.
\end{aligned}$$

We have thus shown that

$$\left| M_{\mathbf{p}}^{rn} h(x) - M_{\mathbf{p}}^{rn} h(y) \right| \leq \left(n\tilde{c}^{n-1} \Lambda^\alpha \|h_{|V}\|_\alpha + \eta^n \|h\|_\alpha \right) d(x,y)^\alpha.$$

For n sufficiently large, the assertion of the lemma follows. It is clear that $a = a(\mathbf{p})$ and thus, α and the other constants involved, depend continuously on $\mathbf{p}$. Therefore, the assertion of the lemma holds with locally uniform constants. The proof is complete. □

Remark. It follows from the proof that result of the previous lemma holds for every $\alpha < -\log(a)/\log(\Lambda)$.

Theorem 2.4. *For every $\mathbf{p}_0 \in (0,1)^s$ there exists $\delta > 0$ and $\alpha > 0$ such that $M_{\mathbf{p}} : \mathscr{C}^\alpha(\overline{\mathbb{R}}) \to \mathscr{C}^\alpha(\overline{\mathbb{R}})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$. In particular, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ we have $T_{\mathbf{p}} \in \mathscr{C}^\alpha(\overline{\mathbb{R}})$ and the convergence*

$$\|M_{\mathbf{p}}^n h - (h(\infty) - h(-\infty)) T_{\mathbf{p}} - h(-\infty) 1\|_\alpha \to 0, \quad \text{as } n \to \infty,$$

is exponentially fast. Moreover, the map $\mathbf{p} \mapsto T_{\mathbf{p}} \in \mathscr{C}^\alpha(\overline{\mathbb{R}})$ is real-analytic on $B(\mathbf{p}_0, \delta)$.

Proof. By Lemma 2.3 there exist $\delta > 0$, $\alpha > 0$, $n \in \mathbb{N}$, $0 < c < 1$ and $C > 0$ such that, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ and for every $h \in \mathscr{C}^\alpha(\overline{\mathbb{R}})$, we have the Ionescu-Tulcea and Marinescu inequality

$$\|M_{\mathbf{p}}^n h\|_\alpha \leq c\|h\|_\alpha + (C+1)\|h\|_\infty.$$

Therefore, the theorem follows from the well-known result of [ITM50] in tandem with the perturbation theory for linear operators ([Kat76]). □

Remark. We remark that a result similar to Theorem 2.4 has been obtained in [Sum13, Theorem 3.30] in the framework of random complex dynamical systems. In this paper, we deal with random real one-dimensional dynamical systems. Regarding the proof of Theorem 2.4, we obtain a simple and straightforward proof by using the Ionescu-Tulcea and Marinescu inequality.

Corollary 2.5. *For every $\mathbf{p}_0 \in (0,1)^s$ there exists $\delta > 0$ and $\alpha > 0$ such that $\mathscr{T}_{\mathbf{p}} \subset \mathscr{C}^\alpha(\overline{\mathbb{R}})$ for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$.*

The next lemma can be proved exactly as in [JS17, Lemma 4.1].

Lemma 2.6. *For every $n \in \mathbb{N}_0^s$ we have*

$$C_{\mathbf{n}} = M_{\mathbf{p}} C_{\mathbf{n}} + \sum_{i=1}^{s} n_i \left(C_{\mathbf{n}-\mathbf{e}_i} \circ f_i - C_{\mathbf{n}-\mathbf{e}_i} \circ f_{s+1} \right).$$

The Bernoulli measure $\mu_{\mathbf{p}}$ on Σ defines the probability measure $\tilde{\mu}_{\mathbf{p}} = \mu_{\mathbf{p}} \circ \pi^{-1}$ on J with distribution function

$$F_{\mathbf{p}} : \overline{\mathbb{R}} \to [0,1], \quad F_{\mathbf{p}}(x) = \tilde{\mu}_{\mathbf{p}} \{(-\infty, x]\}.$$

Lemma 2.7. *We have $M_{\mathbf{p}} F_{\mathbf{p}} = F_{\mathbf{p}}$.*

Proof. Clearly, $\mu_{\mathbf{p}}$ is a self-similar measure on Σ, i.e.,

$$\mu_{\mathbf{p}} = \sum_{i=1}^{s+1} p_i \mu_{\mathbf{p}} \circ \sigma_i^{-1},$$

where $\sigma_i : \Sigma \to \Sigma$ is given by $\sigma_i(\omega) := i\omega$. Using that $f_i^{-1} \circ \pi = \pi \circ \sigma_i$, we obtain for every Borel set $B \subset \mathbb{R}$,

$$\begin{aligned}
\tilde{\mu}_{\mathbf{p}}(B) = \mu_{\mathbf{p}}\left(\pi^{-1}(B)\right) &= \sum_{i=1}^{s+1} p_i \mu_{\mathbf{p}} \circ \sigma_i^{-1}\left(\pi^{-1}(B)\right) \\
&= \sum_{i=1}^{s+1} p_i \mu_{\mathbf{p}} \circ \pi^{-1} \left(f_i^{-1}\right)^{-1}(B) = \sum_{i=1}^{s+1} p_i \tilde{\mu}_{\mathbf{p}} \left(f_i(B) \cap J\right).
\end{aligned}$$

Setting $B := (-\infty, x]$ the lemma follows. □

Note that $M_{\mathbf{p}}$ can be defined for every Borel measurable function.

Lemma 2.8. *$T_{\mathbf{p}}$ is the unique bounded Borel measurable function such that $M_{\mathbf{p}} T_{\mathbf{p}} = T_{\mathbf{p}}$ and $T_{\mathbf{p}|V^-} = 0$ and $T_{\mathbf{p}|V^+} = 1$. In particular, $T_{\mathbf{p}} = F_{\mathbf{p}}$.*

Proof. By (2.2) we have that $M_{\mathbf{p}} T_{\mathbf{p}} = T_{\mathbf{p}}$. Clearly, $T_{\mathbf{p}}$ is bounded and measurable, and satisfies $T_{\mathbf{p}|V^-} = 0$ and $T_{\mathbf{p}|V^+} = 1$. Let h be another bounded Borel measurable function such that $M_{\mathbf{p}} h = h$ and $h_{|V^-} = 0$ and $h_{|V^+} = 1$. Note that (2.2) in fact holds for every bounded Borel measurable function which is continuous at $\{\pm\infty\}$. Therefore, we have

$$h(x) = \lim_{n\to\infty} M_{\mathbf{p}}^n h(x) = h(\infty) T_{\mathbf{p}}(x) + h(-\infty)(1 - T_{\mathbf{p}}(x)) = T_{\mathbf{p}}(x),$$

which proves the asserted uniqueness. To prove that $T_{\mathbf{p}} = F_{\mathbf{p}}$ we note that $F_{\mathbf{p}}$ is bounded and Borel measurable, $M_{\mathbf{p}} F_{\mathbf{p}} = F_{\mathbf{p}}$, $F_{\mathbf{p}|V^-} = 0$ and $F_{\mathbf{p}|V^+} = 1$. The assertion of the lemma follows. □

The following fact follows immediately from the definition of $F_{\mathbf{p}}$.

Fact 2.9. *$F_{\mathbf{p}}$ (and hence, $T_{\mathbf{p}}$) is locally constant precisely on $\mathbb{R} \setminus J$.*

2.2. **Improved spectral gap property for systems with separating condition.** In this section we derive an improved spectral gap property for systems satisfying the separating condition. We define the potentials

$$\varphi : \Sigma \to \mathbb{R}, \quad \varphi(\omega) := -\log\left|f'_{\omega_1}(\pi(\omega))\right|, \quad \text{and} \quad \psi := \psi_{\mathbf{p}} : \Sigma \to \mathbb{R}, \quad \psi(\omega) := \log p_{\omega_1}.$$

We define the shift map $\sigma : \Sigma \to \Sigma$, $\sigma((\omega_1, \omega_2, \dots)) := (\omega_2, \omega_3, \dots)$. For $u : \Sigma \to \mathbb{R}$ and $n \in \mathbb{N}$ we denote by $S_n u := \sum_{k=0}^{n-1} u \circ \sigma^k$ the nth ergodic sum. Further we let

$$\alpha_- := \alpha_-(\mathbf{p}) := \inf_{\omega\in\Sigma} \liminf_{n\to\infty} \frac{S_n \psi_{\mathbf{p}}(\omega)}{S_n \varphi(\omega)}, \quad \alpha_+ := \alpha_+(\mathbf{p}) := \sup_{\omega\in\Sigma} \limsup_{n\to\infty} \frac{S_n \psi_{\mathbf{p}}(\omega)}{S_n \varphi(\omega)}, \tag{2.3}$$

and we refer to α_- as the bottom of the spectrum.

Lemma 2.10. *The map $\mathbf{p} \mapsto \alpha_-(\mathbf{p})$ is lower semi-continuous on $(0,1)^s$.*

Proof. Let $\mathbf{p}_0 \in (0,1)^s$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\psi_{\mathbf{p}} - \psi_{\mathbf{p}_0}\| < \varepsilon / \min|\varphi|$ for every $\mathbf{p}$ with $|\mathbf{p} - \mathbf{p}_0| < \delta$. Let $\mathbf{p}$ with $|\mathbf{p} - \mathbf{p}_0| < \delta$, $\omega \in \Sigma$ and $n_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k\to\infty} \frac{S_{n_k}\psi_{\mathbf{p}}(\omega)}{S_{n_k}\varphi(\omega)} = \alpha_-(\mathbf{p}).$$

The existence of such $\omega \in \Sigma$ and (n_k) follows from [Sch99]. We have

$$\frac{S_{n_k}\psi_{\mathbf{p}}(\omega)}{S_{n_k}\varphi(\omega)} = \frac{S_{n_k}\psi_{\mathbf{p}_0}(\omega)}{S_{n_k}\varphi(\omega)} + \frac{S_{n_k}\psi_{\mathbf{p}}(\omega) - S_{n_k}\psi_{\mathbf{p}_0}(\omega)}{S_{n_k}\varphi(\omega)}.$$

Since

$$\left| \frac{S_{n_k}\psi_{\mathbf{p}}(\omega) - S_{n_k}\psi_{\mathbf{p}_0}(\omega)}{S_{n_k}\varphi(\omega)} \right| \leq \frac{n_k \|\psi_{\mathbf{p}} - \psi_{\mathbf{p}_0}\|}{n_k \min|\varphi|} < \varepsilon \quad \text{and} \quad \liminf_{k\to\infty} \frac{S_{n_k}\psi_{\mathbf{p}_0}(\omega)}{S_{n_k}\varphi(\omega)} \geq \alpha_-(\mathbf{p}_0),$$

we conclude that

$$\alpha_-(\mathbf{p}) = \lim_{k\to\infty} \frac{S_{n_k}\psi_{\mathbf{p}}(\omega)}{S_{n_k}\varphi(\omega)} \geq \liminf_{k\to\infty} \frac{S_{n_k}\psi_{\mathbf{p}_0}(\omega)}{S_{n_k}\varphi(\omega)} - \varepsilon \geq \alpha_-(\mathbf{p}_0) - \varepsilon.$$

□

Remark. Similarly, one can show that $\mathbf{p} \mapsto \alpha_+(\mathbf{p})$ is upper semi-continuous on $(0,1)^s$.

Lemma 2.11. *There exists $\eta < 1$ such that for every compact set $K \subset \overline{\mathbb{R}} \setminus J$ there exists a constant $C_K < \infty$ such that for all $x, y \in K$ belonging to the same connected component of $\overline{\mathbb{R}} \setminus J$, we have for every $\omega \in I^*$,*

$$d(f_\omega(x), f_\omega(y)) \leq C_K \eta^{|\omega|} d(x,y).$$

Proof. There exists $N = N(K) \in \mathbb{N}$ such that for all $n \geq N$, $\omega \in I^n$ and all $x, y \in K$ belonging to the same component of $\overline{\mathbb{R}} \setminus J$, we have either $f_\omega(x), f_\omega(y) \in V^+$ or $f_\omega(x), f_\omega(y) \in V^-$. For otherwise, there exists $\tau \in \Sigma$ such that $f_{\tau_{|n}}(x) \in \overline{\mathbb{R}} \setminus V$, for all $n \in \mathbb{N}$, contradicting that $x \notin J$. Let $S := \max\left\{\mathrm{Lip}(f_{i|V^+}), \mathrm{Lip}(f_{i|V^-}) \mid i \in I\right\}$. Since $(f_i)_{i\in I}$ is contracting near infinity, we have $S < 1$. For $n \geq N$ and $\omega \in I^n$ we have

$$d\left(f_\omega(x), f_\omega(y)\right) \leq S^{n-N} d\left(f_{\omega_{|N}}(x), f_{\omega_{|N}}(y)\right) \leq S^{n-N} \left(\max_{i\in I} \mathrm{Lip}(f_i)\right)^N d\left(x,y\right) = S^n C_K d(x,y),$$

where we have set $C_K := S^{-N} \left(\max_{i\in I} \mathrm{Lip}(f_i)\right)^N$. The estimate for $n < N$ can be shown similarly. □

Definition 2.12. We say that $(f_i)_{i\in I}$ satisfies the *separating condition* if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i, j \in I$ with $i \neq j$, we have $f_i^{-1}(\overline{O}) \cap f_j^{-1}(\overline{O}) = \varnothing$. If the separating condition holds, then we may always assume that O is bounded and that $J \subset O$.

The proof of the following lemma is standard and therefore omitted (see e.g., [MU03]).

Lemma 2.13 (Bounded distortion)**.** *Let Ω be a bounded open set such that $f_i^{-1}(\Omega) \subset \Omega$ for all $i \in I$. Then we have*

$$D := D(\Omega) := \sup\left\{ \frac{\left|f_\gamma'(v_1)\right|}{\left|f_\gamma'(v_2)\right|} \middle| \gamma \in I^*, v_1, v_2 \in f_\gamma^{-1}(\overline{\Omega}) \right\} < \infty. \tag{2.4}$$

Lemma 2.14. *Suppose that $(f_i)_{i\in I}$ satisfies the separating condition. Let $\mathbf{p}_0 \in (0,1)^s$ and $0 < \alpha < \alpha_-(\mathbf{p}_0)$. Then there exist $\delta > 0$, $n \in \mathbb{N}$ and constants $0 < c < 1$ and $C > 0$ such that, for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$, $h \in \mathscr{C}^\alpha(\overline{\mathbb{R}})$ and for all $x, y \in \overline{\mathbb{R}}$,*

$$\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| \leq \left(c\|h\|_\alpha + C\|h\|_\infty\right) d(x,y)^\alpha.$$

Proof. Suppose that the separating condition holds with bounded open interval O. We may assume that $J \subset O$ and that there exists $r_0 > 0$ such that for all $i, j \in I$ with $i \neq j$,

$$\overline{f_i^{-1}(B(O,r_0))} \subset B(O,r_0) \quad \text{and } f_j\left(\overline{f_i^{-1}(B(O,r_0))}\right) \subset \overline{\mathbb{R}} \setminus J,$$

where $B(O,r_0) := \bigcup_{u \in O} B(u,r_0)$. For $i \neq j$ we define the compact sets

$$K_{i,j} := f_j\left(\overline{f_i^{-1}(B(O,r_0))}\right) \quad \text{and} \quad K' := \bigcup_{i,j \in I,\, i \neq j} K_{i,j} \subset \overline{\mathbb{R}} \setminus J.$$

Let $\delta := d(\overline{\mathbb{R}} \setminus O, J)$. Since $J \subset O$ is compact, we have $\delta > 0$. Define the compact set

$$K := K' \cup \left\{u \in \overline{\mathbb{R}} \setminus J, d(u,J) \geq \delta/2\right\} \subset \overline{\mathbb{R}} \setminus J.$$

Since $\overline{O}$ is compact, by modifying r_0 if necessary, we may assume that

$$\forall x' \in \overline{O}\, \forall y' \in B(x',r_0)\, \forall i \in I:\ d(f_i(x'), f_i(y')) < \delta/2. \tag{2.5}$$

Let $x, y \in \overline{\mathbb{R}}$ and $n \in \mathbb{N}$ sufficiently large (to be determined later). We now distinguish two cases. First suppose that $x \notin O$. We may assume that $d(x,y) \leq \delta/2$. Hence, x and y are contained in the compact set $\left\{u \in \overline{\mathbb{R}} \mid d(u,J) \geq \delta/2\right\} \subset K \subset \overline{\mathbb{R}} \setminus J$ and belong to the same connected component of $\overline{\mathbb{R}} \setminus J$. Therefore, by Lemma 2.11, we have

$$\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| \leq \sum_{|\tau|=n} p_\tau \left|h(f_\tau(x)) - h(f_\tau(y))\right| \leq \sum_{|\tau|=n} p_\tau \|h\|_\alpha \left(C_K \eta^n\right)^\alpha d(x,y)^\alpha \leq \|h\|_\alpha \left(C_K \eta^n\right)^\alpha d(x,y)^\alpha.$$

For n sufficiently large, we have $\left(C_K \eta^n\right)^\alpha < 1$. This finishes the proof in the case when $x \notin O$.

Next we consider the remaining case $x \in O$. Let

$$\ell_x := \sup\left\{\ell \geq 0 \mid \exists \omega \in I^* \cup \{\varnothing\} \text{ such that } f_{\omega_{|j}}(x) \in O \ \text{ for any } 0 \leq j \leq \ell\right\} \in \mathbb{N} \cup \{0, \infty\}.$$

Then there exists a unique $(\omega_1, \dots, \omega_{\ell_x}) \in I^{\ell_x}$ such that $f_{\omega_{\ell_x}} \cdots f_{\omega_1}(x) \in O$. Here, if $\ell_x = 0$, then we set $f_{\omega_{\ell_x}} \cdots f_{\omega_1}(x) = x$. We distinguish two subcases (a) $\ell_x \geq n$ and (b) $\ell_x < n$. We begin with subcase (a). For all $j = 0, 1, 2, \dots$ let

$$B_j(x) := f_{\omega_{|j}}^{-1}(B(f_{\omega_{|j}}(x), r_0)),$$

where $B_0(x) := B(x, r_0)$. We may assume that $y \in B_n(x)$. We have the decomposition

$$I^n = \bigcup_{j \leq n} \bigcup_{|\tau| = n-j, \tau_1 \neq \omega_{j+1}} \left\{\omega_{|j}\tau\right\}.$$

For all $j < n$ and $\tau_1 \in I \setminus \left\{\omega_{j+1}\right\}$ we have

$$f_{\tau_1}\left(f_{\omega_{|j}}(B_n(x))\right) \subset f_{\tau_1}\left(f_{\omega_{j+1}}^{-1}(O, r_0)\right) \subset K' \subset \overline{\mathbb{R}} \setminus J.$$

By Lemma 2.11, for all $j < n$ and $\tau \in I^{n-j}$ with $\tau_1 \neq \omega_{j+1}$, we have

$$\begin{aligned} d\left(f_\tau\left(f_{\omega_{|j}}(x)\right), f_\tau\left(f_{\omega_{|j}}(y)\right)\right) &\leq C_{K'} \eta^{n-j-1} d\left(f_{\tau_1}(f_{\omega_{|j}}(x)), f_{\tau_1}(f_{\omega_{|j}}(y))\right) \\ &\leq C_{K'} \operatorname{Lip}(f_{\tau_1}) \eta^{n-j-1} d\left(f_{\omega_{|j}}(x), f_{\omega_{|j}}(y)\right). \end{aligned}$$

Put $C_1 := C_{K'} \max_{i\in I} \mathrm{Lip}(f_i)$. Since h is Hölder continuous, we can assert that

$$\begin{aligned}
\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| &= \left|\sum_{j\le n} p_{\omega|_j} \sum_{|\tau|=n-j,\tau_1\neq\omega_{j+1}} p_\tau \left(h\left(f_\tau\left(f_{\omega|_j}(x)\right)\right) - h\left(f_\tau\left(f_{\omega|_j}(y)\right)\right)\right)\right| \\
&\le \sum_{j\le n} p_{\omega|_j} \sum_{|\tau|=n-j,\tau_1\neq\omega_{j+1}} p_\tau \|h\|_\alpha C_1^\alpha (\eta^{n-j-1})^\alpha d\left(f_{\omega|_j}(x), f_{\omega|_j}(y)\right)^\alpha \\
&\le C_1^\alpha \|h\|_\alpha \sum_{j\le n} p_{\omega|_j} \left(\eta^{n-j-1}\right)^\alpha d\left(f_{\omega|_j}(x), f_{\omega|_j}(y)\right)^\alpha .
\end{aligned}$$

Using that d is strongly equivalent to the Euclidean metric on the compact set $\overline{B(O,r_0)} \subset \mathbb{R}$ and combining with the bounded distortion estimate in (2.4) of Lemma 2.13 with $\Omega := B(O,r_0)$, we deduce the existence of $D_0 < \infty$ such that

$$\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| \le C_1^\alpha D_0^\alpha \|h\|_\alpha \sum_{j\le n} p_{\omega|_j} \left(\eta^{n-j-1}\right)^\alpha |f'_{\omega|_j}(x)|^\alpha d(x,y)^\alpha.$$

The following was proved in [JS17, (6.2) in the proof of Theorem 1.3]. There exists a $C(\varphi,\psi_{\mathbf{p}})$, which depends continuously on $\mathbf{p}\in(0,1)^s$, such that for all $j\in\mathbb{N}$ and for all $\omega\in\Sigma$,

$$\mathrm{e}^{S_j\psi_{\mathbf{p}}(\omega)} \le C(\varphi,\psi_{\mathbf{p}})\mathrm{e}^{\alpha_-(\mathbf{p})S_j\varphi(\omega)}. \tag{2.6}$$

Let $\omega = (\omega_1,\dots,\omega_{\ell_x},\omega_1,\dots,\omega_{\ell_x},\dots)\in\Sigma$. By combining (2.6) with the bounded distortion estimate in (2.4) we have for some D_1 with $D_1 > D_0$

$$p_{\omega|_j} \le D_1^{\alpha_-(\mathbf{p})} C(\varphi,\psi_{\mathbf{p}}) |f'_{\omega|_j}(x)|^{-\alpha_-(\mathbf{p})}.$$

Hence, we obtain

$$\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| \le C_1^\alpha D_1^{\alpha+\alpha_-(\mathbf{p})} C(\varphi,\psi_{\mathbf{p}}) \|h\|_\alpha \sum_{j\le n} \eta^{\alpha(n-j-1)} |f'_{\omega|_j}(x)|^{\alpha-\alpha_-(\mathbf{p})} d(x,y)^\alpha.$$

For $\delta>0$ sufficiently small and $\mathbf{p}\in B(\mathbf{p}_0,\delta)$ we have $\sup_{\mathbf{p}\in B(\mathbf{p}_0,\delta)}(\alpha-\alpha_-(\mathbf{p}))<0$ by Lemma 2.10. Also we can define $C(\varphi,\psi) := \sup_{\mathbf{p}\in B(\mathbf{p}_0,\delta)} C(\varphi,\psi_{\mathbf{p}}) < \infty$. Since $|f'_{\omega|_j}(x)| \ge \lambda^j$ there exist $\tilde\eta < 1$ and $C_2<\infty$ such that for all $j\le n$ and $\mathbf{p}\in B(\mathbf{p}_0,\delta)$,

$$\eta^{\alpha(n-j-1)} |f'_{\omega|_j}(x)|^{\alpha-\alpha_-(\mathbf{p})} \le C_2 \tilde\eta^n.$$

Therefore,

$$|M^n h(x) - M^n h(y)| \le C_1^\alpha D_1^{\alpha+\alpha_-(\mathbf{p})} C(\varphi,\psi)\|h\|_\alpha C_2 n\tilde\eta^n d(x,y)^\alpha.$$

Put $c := C_1^\alpha D_1^{\alpha+\alpha_-(\mathbf{p})} C(\varphi,\psi)\|h\|_\alpha C_2 n\tilde\eta^n$. For n sufficiently large we have $c<1$. Thus, assuming subcase (a), we have derived the desired estimate.

Finally, to complete the proof, let us consider the subcase (b) when $\ell_x < n$. We may assume that $y\in B_{\ell_x}(x)$. We estimate

$$\begin{aligned}
\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| \le &\left|\sum_{j<\ell_x} p_{\omega|_j} \sum_{|\tau|=n-j,\tau_1\neq\omega_{j+1}} p_\tau \left(h\left(f_\tau\left(f_{\omega|_j}(x)\right)\right) - h\left(f_\tau\left(f_{\omega|_j}(y)\right)\right)\right)\right| \\
&+ p_{\omega|_{\ell_x}} \left|\sum_{|\tau|=n-\ell_x} p_\tau \left(h\left(f_\tau\left(f_{\omega|_{\ell_x}}(x)\right)\right) - h\left(f_\tau\left(f_{\omega|_{\ell_x}}(y)\right)\right)\right)\right|.
\end{aligned}$$

The first summand on the right-hand side satisfies an α-Hölder condition with $c<1$ for n large by the same arguments as in (a) above. Finally, to deal with the second summand, let $x' := f_{\omega|_{\ell_x}}(x)$ and $y' := f_{\omega|_{\ell_x}}(y)$. Since $y\in B_{\ell_x}(x)$ we have that $d(x',y')<r_0$. By the definition of ℓ_x we have that $f_{\tau_1}(x')\notin O$ for all $\tau_1\in I$.

By (2.5) we have $d(f_{\tau_1}(y'),J) \geq \delta/2 > 0$. Since moreover $f_{\tau_1}(x')$ and $f_{\tau_1}(y')$ are in the same component of $\overline{\mathbb{R}} \setminus J$, Lemma 2.11 implies that

$$d\left(f_\tau\left(f_{\omega|_{\ell_x}}(x)\right), f_\tau\left(f_{\omega|_{\ell_x}}(y)\right)\right) \leq C_K \eta^{n-\ell_x} d(f_{\omega|_{\ell_x}}(x), f_{\omega|_{\ell_x}}(y)).$$

The rest of the proof runs as in subcase (a) above. The proof is complete. □

We thus obtain the following strengthening of Theorem 2.4 when the separating condition holds.

Theorem 2.15. *Suppose that $(f_i)_{i\in I}$ satisfies the separating condition. Let $\mathbf{p}_0 \in (0,1)^s$ and $\alpha < \alpha_-(\mathbf{p}_0)$. There exists $\delta > 0$ such that $M_{\mathbf{p}} : \mathscr{C}^\alpha(\overline{\mathbb{R}}) \to \mathscr{C}^\alpha(\overline{\mathbb{R}})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$. Moreover, the map $\mathbf{p} \mapsto T_{\mathbf{p}} \in \mathscr{C}^\alpha(\overline{\mathbb{R}})$ is real-analytic on $B(\mathbf{p}_0, \delta)$.*

Corollary 2.16. *Suppose that $(f_i)_{i\in I}$ satisfies the separating condition. Let $\mathbf{p}_0 \in (0,1)^s$ and $\alpha < \alpha_-(\mathbf{p}_0)$. There exists $\delta > 0$ such that $\mathscr{T}_{\mathbf{p}} \subset \mathscr{C}^\alpha(\overline{\mathbb{R}})$ for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$.*

Remark 2.17. By Proposition 4.5 below it will turn out that in many cases, $\mathscr{T}_p \nsubseteq \mathscr{C}^{\alpha_-(p)}(\overline{\mathbb{R}})$. Hence, by the previous corollary, the spectral gap property stated in Theorem 2.15 is sharp. In particular, this is the case for the classical Takagi function. Namely, let $f_1(x) = 2x$, $f_2(x) = 2x - 1$. Then $J = [0,1]$ and $T_{1/2}(x) = x$ for $x \in J$ and C_1 is a multiple of the classical Takagi function on J. Hence, $\alpha_- = 1$, $T_{1/2}$ is Lipschitz continuous, and C_1 is α-Hölder continuous for every $\alpha < 1$, but not Lipschitz continuous. For further examples, see Proposition 4.5.

3. Multifractal Analysis of the Pointwise Hölder exponent

In this section we perform a complete multifractal analysis of the pointwise Hölder exponents of the elements of $\mathscr{T}$ associated with $(f_i)_{i\in I}$ and $\mathbf{p} \in (0,1)^s$. We begin by providing the necessary terminology which has been introduced in [JS17]. We use $\mathbf{n} = (n_1, \ldots, n_s)$ to denote an element of $\mathbb{N}_0^s$. Let $\mathbf{e}_i \in \mathbb{N}_0^s$ denote the element whose ith component is 1 and all other components are 0. An element $A \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ is represented as $A = (A_{\mathbf{x},\mathbf{y}})_{(\mathbf{x},\mathbf{y})\in\mathbb{N}_0^s\times\mathbb{N}_0^s}$, where $A_{\mathbf{x},\mathbf{y}} \in \mathbb{R}$, and such an element A is called an ($\mathbb{N}_0^s$-)matrix. $A_{\mathbf{x},\mathbf{y}}$ is called the $(\mathbf{x},\mathbf{y})$-component of A. We denote by $1_{\mathbf{n},\mathbf{m}} \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ the matrix such that for every $(\mathbf{x},\mathbf{y}) \in \mathbb{N}_0^s \times \mathbb{N}_0^s$ the $(\mathbf{x},\mathbf{y})$-component of $1_{\mathbf{n},\mathbf{m}}$ is given by

$$(1_{\mathbf{n},\mathbf{m}})_{\mathbf{x},\mathbf{y}} = \begin{cases} 1, & \mathbf{x} = \mathbf{n},\ \mathbf{y} = \mathbf{m} \\ 0, & \text{else.} \end{cases}$$

In order to investigate $\mathscr{T}$ we define the matrix cocycle $A_0 : \Sigma \times \mathbb{N} \to \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ given by

$$A_0(\omega,1) := \begin{cases} \sum_{\mathbf{n}\in\mathbb{N}_0^s} (p_{\omega_1} 1_{\mathbf{n},\mathbf{n}} + n_{\omega_1} 1_{\mathbf{n},\mathbf{n}-\mathbf{e}_{\omega_1}}), & \omega_1 \in \{1,\ldots,s\} \\ \sum_{\mathbf{n}\in\mathbb{N}_0^s} (p_{\omega_1} 1_{\mathbf{n},\mathbf{n}} - \sum_{i=1}^s n_i 1_{\mathbf{n},\mathbf{n}-\mathbf{e}_i}), & \omega_1 = s+1 \end{cases}$$

and for $k \in \mathbb{N}$,

$$A_0(\omega,k) := A_0(\omega,1) A_0(\sigma\omega,1) \ldots A_0(\sigma^{k-1}\omega,1) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}.$$

Here, the matrix product $A_0(\tau,1)A_0(\upsilon,1) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ is for $\tau, \upsilon \in \Sigma$ and $\mathbf{l}, \mathbf{m} \in \mathbb{N}_0^s$ given by

$$(A_0(\tau,1) \cdot A_0(\upsilon,1))_{\mathbf{l},\mathbf{m}} := \sum_{\mathbf{k}\in\mathbb{N}_0^s} (A_0(\tau,1))_{\mathbf{l},\mathbf{k}} \cdot (A_0(\upsilon,1))_{\mathbf{k},\mathbf{m}}. \tag{3.1}$$

Note that the sum in (3.1) is actually a finite sum. Further matrix products in the definition of $A_0(\omega,k)$ are defined in the same way. We also define

$$A(\omega,k) := (p_{\omega_{|k}})^{-1} A_0(\omega,k) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}.$$

Moreover, for all $a,b \in \mathbb{R}$ we define the matrix

$$U(a,b) := (u_{\mathbf{n}}(a,b))_{\mathbf{n}\in\mathbb{N}_0^s} \in \mathbb{R}^{\mathbb{N}_0^s} \quad \text{given by } u_{\mathbf{n}}(a,b) := C_{\mathbf{n}}(a) - C_{\mathbf{n}}(b).$$

Remark. In (3.1) and in the following we make use of the product of matrices with an infinite index set. These matrix products will always be well defined, since either the first factor of the product possesses at most finitely many non-zero entries in each row, or the second factor contains at most finitely many non-zero entries in each column.

Since the above definitions of A_0 and A coincide with the ones given in [JS17] we immediately obtain the following two lemmas. For $\mathbf{n},\mathbf{m} \in \mathbb{N}_0^s$ we write $\mathbf{n} \le \mathbf{m}$ if $n_i \le m_i$ for each $1 \le i \le s$.

Lemma 3.1 ([JS17], Lemma 4.5)**.** *Let $\omega \in \Sigma$ and $k \in \mathbb{N}$. Then $A(\omega,k)_{\mathbf{n},\mathbf{n}} = 1$ for every $\mathbf{n} \in \mathbb{N}_0^s$. Also, $A(\omega,k)_{\mathbf{n},\mathbf{m}} = 0$ unless $\mathbf{m} \le \mathbf{n}$.*

Lemma 3.2 ([JS17], Lemma 4.8)**.** *Let $\omega \in I^{\mathbb{N}}$ and $k \in \mathbb{N}$. Put $m_i := m_i(k) := \operatorname{card}\left\{1 \le j \le k : \omega_j = i\right\}$ for $1 \le i \le s+1$. Let $\mathbf{m} = (m_i)_{i=1}^s \in \mathbb{N}_0^s$. Let $\mathbf{q}$, $\mathbf{r} \in \mathbb{N}_0^s$ with $\mathbf{0} \le \mathbf{r} \le \mathbf{q}$. Then there exists a constant $K \ge 1$ which depends on $\mathbf{q}$ and the probability vector $\mathbf{p}$ but not on k such that*

$$|A(\omega,k)_{\mathbf{q},\mathbf{r}}| \le K \left(\prod_{i=1}^{s} \tilde{m}_i^{q_i - r_i}\right) \tilde{m}_{s+1}^{|\mathbf{q}|-|\mathbf{r}|} \text{ and } |A(\omega,k)_{\mathbf{q},\mathbf{r}}| \le K k^{|\mathbf{q}|},$$

where $\tilde{m}_j := \max\{1, m_j\}$ for $1 \le j \le s+1$. If $\omega_j \ne s+1$ for all $1 \le j \le k$ and $m_i > q_i - r_i$ for all $1 \le i \le s$, then there exists $K' > 0$ depending only on $\mathbf{q}$ such that

$$A(\omega,k)_{\mathbf{q},\mathbf{r}} \ge K' \prod_{i=1}^{s} m_i^{q_i - r_i}.$$

Definition 3.3. We say that $(f_i)_{i\in I}$ satisfies the open set condition if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$, for all $i \in I$, and for all $i,j \in I$ with $i \ne j$ we have $f_i^{-1}(O) \cap f_j^{-1}(O) = \varnothing$.

We write $A \le B$ for subsets $A,B \subset \mathbb{R}$ if $a \le b$ for every $a \in A$ and $b \in B$.

Remark. If $(f_i)_{i\in I}$ satisfies the open set condition, then we will always assume that $f_i^{-1}(\overline{O}) \le f_j^{-1}(\overline{O})$ for all $i,j \in I$ with $i < j$.

The purpose of the above definitions is the following.

Lemma 3.4. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $k \in \mathbb{N}$, $\omega \in I^k$ and $x,y \in f_\omega^{-1}(\overline{O})$. Then we have $U(x,y) = A_0(\overline{\omega},k)U(f_\omega(x), f_\omega(y))$. Here, we set $\overline{\omega} := (\omega_1 \dots \omega_k, \omega_1 \dots \omega_k \dots) \in \Sigma$.*

Proof. The assertion can be shown as in [JS17, Lemma 4.7] if we observe that $u_{\mathbf{n}}(f_\tau(x), f_\tau(y)) = 0$, for all $\tau \in I^k$ with $\tau \ne \omega$ and for all $\mathbf{n} \in \mathbb{N}_0^s$. To prove this, note that by the open set condition we have that $[f_\tau(x), f_\tau(y)] \cap \overline{O}$ has at most one point. Consequently, $T_{\mathbf{p}}$ is constant on $[f_\tau(x), f_\tau(y)]$ and thus every $C_{\mathbf{n}}$ is constant on $[f_\tau(x), f_\tau(y)]$, $\mathbf{n} \in \mathbb{N}_0^s$. □

Recall that $G = \langle f_1, \dots, f_{s+1}\rangle$ and write $G(x) := \{g(x) \mid g \in G\}$.

Lemma 3.5. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $x_0 \in J$. Then there exist $a,b \in (J \cap O) \setminus G(x_0)$ such that b is arbitrarily close to a and $T_{\mathbf{p}}(a) \ne T_{\mathbf{p}}(b)$.*

Proof. First recall the well-known fact that J is a non-empty perfect subset of $\mathbb{R}$, since $\{f_1,\dots,f_{s+1}\}$ does not have a common fixed point in $\mathbb{R}$. Thus, every neighborhood of any point of J contains uncountably many points in J. Let $x_0\in J$ and $\varepsilon>0$. Since $J\subset\overline{O}$ and $G(x_0)$ is countable, there exists $a\in(J\cap O)\setminus G(x_0)$. Since $T_{\mathbf{p}}$ is not locally constant at a and T_p is locally constant on $\overline{\mathbb{R}}\setminus J$ by Fact 2.9, there exists $c\in B(a,\varepsilon/2)\cap J\cap O$ such that $T_{\mathbf{p}}(c)\neq T_{\mathbf{p}}(a)$. Finally, since $T_{\mathbf{p}}$ is continuous at c by Theorem 2.4, $G(x_0)$ is countable and every neighborhood of c in J contains uncountably many points, there exists $b\in(B(c,\varepsilon/2)\cap J\cap O)\setminus G(x_0)$ such that $T_{\mathbf{p}}(b)\neq T_{\mathbf{p}}(a)$. Clearly, we also have $b\in B(a,\varepsilon)\cap O$. □

By using Lemma 3.5 we can extend the methods used in [JS17]. The following lemma is the analogue of [JS17, Lemma 4.9].

Lemma 3.6. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $x_0\in J$ and let $\varepsilon>0$. Let $\mathbf{n}\in\mathbb{N}_0^s$ and set $n:=|\mathbf{n}|:=\sum_{i=1}^s n_i$. Then there exists a constant $K>0$ such that for every $k\in\mathbb{N}$ there exist points $a_k\in B(x_0,\varepsilon)\cap J\setminus\{x_0\}$ and $b_k\in B(x_0,\varepsilon)\setminus\{x_0\}$ with $u_{\mathbf{0}}(a_k,b_k)\neq 0$ such that for $\mathbf{0}\le\mathbf{q}\le\mathbf{n}$,*

$$K^{-1}k^{\sum_{i=1}^s q_i(n+1)^{i-1}}\le\frac{u_{\mathbf{q}}(a_k,b_k)}{u_{\mathbf{0}}(a_k,b_k)}\le Kk^{\sum_{i=1}^s q_i(n+1)^{i-1}}.$$

Proof. Let $\varepsilon>0$. There exists $\omega\in\Sigma$ such that $\pi(\omega)=x_0$. Since $(f_i)_{i\in I}$ is expanding and the open set O is bounded, there exists $r\in\mathbb{N}$ such that, with $\tau:=\omega_1\dots\omega_r\in I^r$, we have $\operatorname{diam}(f_\tau^{-1}(\overline{O}))<\varepsilon$. By Lemma 3.5 there exist $a,b\in(J\cap O)\setminus G(x_0)$, $a\neq b$, such that $u_{\mathbf{0}}(a,b)=T_{\mathbf{p}}(a)-T_{\mathbf{p}}(b)\neq 0$. For each $k\in\mathbb{N}$ and $1\le i\le s$ we set $m_i(k):=k^{(n+1)^{i-1}}$ and $\xi_k:=(1^{m_1(k)},2^{m_2(k)},\dots,s^{m_s(k)})\in I^{\sum_{i=1}^s m_i(k)}$ where $u^m:=(u,u,\dots,u)\in I^m$ for $u\in\{1,\dots,s+1\}$. Then we define $\tilde{a_k}:=f_{\xi_k}^{-1}(a)$, $\tilde{b_k}:=f_{\xi_k}^{-1}(b)$ as well as $a_k:=f_\tau^{-1}(\tilde{a_k})$, $b_k:=f_\tau^{-1}(\tilde{b_k})$. By Lemma 3.4 and the fact that

$$u_{\mathbf{0}}(a_k,b_k)=p_\tau\cdot u_{\mathbf{0}}(\tilde{a}_k,\tilde{b}_k)$$

it follows that

$$(u_{\mathbf{0}}(a_k,b_k))^{-1}U(a_k,b_k)=\left(u_{\mathbf{0}}(\tilde{a}_k,\tilde{b}_k)\right)^{-1}A(\overline{\tau},r)U(\tilde{a}_k,\tilde{b}_k).\tag{3.2}$$

Similarly, we obtain that

$$\left(u_{\mathbf{0}}(\tilde{a}_k,\tilde{b}_k)\right)^{-1}U(\tilde{a}_k,\tilde{b}_k)=(u_{\mathbf{0}}(a,b))^{-1}A\left(\overline{\xi_k},\sum_{i=1}^s m_i(k)\right)U(a,b).\tag{3.3}$$

By combining the previous two equalities (3.2) and (3.3) we have

$$(u_{\mathbf{0}}(a_k,b_k))^{-1}U(a_k,b_k)=(u_{\mathbf{0}}(a,b))^{-1}A(\overline{\tau},r)A\left(\overline{\xi_k},\sum_{i=1}^s m_i(k)\right)U(a,b).$$

Since $\xi_k\in\{1,\dots,s\}^*$ and $u_{\mathbf{0}}(a,b)\neq 0$ it follows from Lemmas 3.1 and 3.2 that for $\mathbf{q}\le\mathbf{n}$,

$$\left(A\left(\overline{\xi_k},\sum_{i=1}^s m_i(k)\right)U(a,b)\right)_{\mathbf{q}}\asymp\prod_{i=1}^s(m_i(k))^{q_i}\asymp k^{\sum_{i=1}^s q_i(n+1)^{i-1}}\text{ as }k\to\infty,$$

where for any two non-negative functions $\phi_1(k)$ and $\phi_2(k)$ of $k\in\mathbb{N}$, we write $\phi_1(k)\asymp\phi_2(k)$ as $k\to\infty$ if there exists a constant $D>1$ such that $D^{-1}\phi_2(k)\le\phi_1(k)\le D\phi_2(k)$ for every $k\in\mathbb{N}$. From this and Lemma 3.1 we conclude that as $k\to\infty$,

$$\left(A(\overline{\tau},r)A\left(\overline{\xi_k},\sum_{i=1}^s m_i(k)\right)U(a,b)\right)_{\mathbf{q}}=\sum_{\mathbf{r}\le\mathbf{q}}A(\overline{\tau},r)_{\mathbf{q},\mathbf{r}}\left(A\left(\overline{\xi_k},\sum_{i=1}^s m_i(k)\right)U(a,b)\right)_{\mathbf{r}}\asymp k^{\sum_{i=1}^s q_i(n+1)^{i-1}}.$$

The proof is complete. □

We are now in the position to derive the following key lemma. Since the proof follows closely the arguments given in [JS17, Lemma 5.2] we comment only on the necessary modifications. Note that in [JS17, Lemma 5.2] the Julia set J_ω should be replaced by $J(G)$.

Lemma 3.7. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C=\sum_{\mathbf{n}}\beta_{\mathbf{n}}C_{\mathbf{n}}\in\mathscr{T}$ be non-trivial. Let $j(k)\to\infty$ be a sequence of positive integers. Let $\omega\in\Sigma$. For every $x\in J$ and for any non-empty neighbourhood V of x in $\mathbb{R}$ there exist $a,b\in V\cap O$ with $a\neq b$ such that*

$$\eta:=\limsup_{k\to\infty}\left|\sum_{\mathbf{m}}\sum_{\mathbf{n}}\beta_{\mathbf{n}}A(\omega,j(k))_{\mathbf{n},\mathbf{m}}u_{\mathbf{m}}(a,b)\right|\in(0,\infty].$$

Proof. There exists $\mathbf{n}_{\max}\in\mathbb{N}_0^s$ such that $\beta_{\mathbf{n}}=0$ for all $\mathbf{n}\geq\mathbf{n}_{\max}$. By Lemma 3.1, it is easy to see that the matrix $(A(\omega,j(k))_{\mathbf{n},\mathbf{m}})_{\mathbf{n}\leq\mathbf{n}_{\max},\mathbf{m}\leq\mathbf{n}_{\max}}$ is invertible. Since $(\beta_{\mathbf{n}})_{\mathbf{n}\leq\mathbf{n}_{\max}}\neq 0$ we conclude that, for all $k\in\mathbb{N}$,

$$\lambda(k):=(\lambda_{\mathbf{m}}(k))_{\mathbf{m}\leq\mathbf{n}_{\max}}:=\left(\sum_{\mathbf{n}\leq\mathbf{n}_{\max}}\beta_{\mathbf{n}}A(\omega,j(k))_{\mathbf{n},\mathbf{m}}\right)_{\mathbf{m}\leq\mathbf{n}_{\max}}\neq 0.$$

Let $\varepsilon>0$ and suppose by way of contradiction that $\eta=0$ for all $a,b\in B(x_0,\varepsilon)\setminus\{x_0\}$ with $a\neq b$. Proceeding exactly as in the proof of [JS17, Lemma 5.2] one defines $\lambda:=(\lambda_{\mathbf{m}})_{\mathbf{m}\leq\mathbf{n}_{\max}}$ as a limit point of the sequence $(\lambda(k)/\|\lambda(k)\|)_{k\geq 1}$ and observes that $\|\lambda\|=1$ and

$$\sum_{\mathbf{m}\leq\mathbf{n}_{\max}}\lambda_{\mathbf{m}}u_{\mathbf{m}}(a,b)=0,\ \text{for all } a,b\in B(x_0,\varepsilon)\setminus\{x_0\}. \tag{3.4}$$

To derive the desired contradiction one verifies that there exist $(a_{\mathbf{r}})_{\mathbf{r}\leq\mathbf{n}_{\max}},(b_{\mathbf{r}})_{\mathbf{r}\leq\mathbf{n}_{\max}}$ with $a_{\mathbf{r}},b_{\mathbf{r}}\in B(x_0,\varepsilon)\setminus\{x_0\}$, for every $\mathbf{r}\leq\mathbf{n}_{\max}$ such that the matrix

$$(u_{\mathbf{q}}(a_{\mathbf{r}},b_{\mathbf{r}}))_{\substack{\mathbf{r}\leq\mathbf{n}_{\max}\\ \mathbf{q}\leq\mathbf{n}_{\max}}}$$

is invertible. Hence, it follows from (3.4) that $\lambda=0$ contradicting $\|\lambda\|=1$. The existence of the vectors $(a_{\mathbf{r}})_{\mathbf{r}\leq\mathbf{n}_{\max}}$ and $(b_{\mathbf{r}})_{\mathbf{r}\leq\mathbf{n}_{\max}}$ as stated above is demonstrated in [JS17, Proposition 4.11]. The key is to combine Lemma 3.6 with the idea of the Vandermonde determinant (see also [JS17, Lemma 4.10]). □

3.1. **Dynamical characterisation of pointwise Hölder exponents.** In this section we provide a dynamical characterisation of the pointwise Hölder exponent of non-trivial elements of $\mathscr{T}$. The *pointwise Hölder exponent* of C at $x\in\mathbb{R}$ is defined as

$$\mathrm{H\ddot{o}l}(C,x):=\sup\left\{\alpha>0\mid\limsup_{y\to x}\frac{|C(y)-C(x)|}{|y-x|^{\alpha}}<\infty\right\}\in[0,\infty].$$

We remark that it is possible that $\mathrm{H\ddot{o}l}(C,x)>1$. In this case, C is differentiable at x and $C'(x)=0$. In fact, in many examples there exists a set $A\subset J$ of positive Hausdorff dimension such that for every $x\in A$ we have $\mathrm{H\ddot{o}l}(C,x)>1$. We also note that if $\mathrm{H\ddot{o}l}(C,x)<1$ then C is not differentiable at x. There exist examples for which there exist sets $A,B\subset J$ of positive Hausdorff dimension such that for each $x\in A$ we have $\mathrm{H\ddot{o}l}(C,x)>1$ whereas for each $x\in B$ we have $\mathrm{H\ddot{o}l}(C,x)<1$. For example, let $s=1$ and suppose that (f_1,f_2) satisfies the open set condition. For $p_1>0$ sufficiently close to zero we have that $\alpha_-<1$ and $\alpha_+>1$ by (2.3). Then the existence of the sets A,B as stated above follows from Theorem 3.16.

By [JS15, Lemma 5.1]) we have for every $x\in\mathbb{R}$,

$$\mathrm{H\ddot{o}l}(C,x)=\liminf_{r\to 0}\frac{\log\sup_{y\in B(x,r)}|C(y)-C(x)|}{\log r}. \tag{3.5}$$

Remark 3.8. In the proof of [JS15, Lemma 5.1]) we have demonstrated that

$$\text{Höl}(C,x) = \liminf_{y\to x} \frac{\log|C(x)-C(y)|}{\log|x-y|}.$$

From this, it is straightforward to derive (3.5). For the convenience of the reader, we give the details. For a bounded function $h : V \to \mathbb{R}$ defined on some domain $V \subset \mathbb{R}$ and $x \in V$ we will show that

$$\liminf_{y\to x} \frac{\log|h(x)-h(y)|}{\log|x-y|} = \liminf_{r\to 0} \frac{\log \sup_{y\in B(x,r)} |h(y)-h(x)|}{\log r}. \tag{3.6}$$

Denote the left-hand side of (3.6) by H, the right-hand side of (3.6) by H'. We will show that $H = H'$. First observe that if h is not continuous at x then $H = H' = 0$. Now, assume that h is continuous at x. Let (y_n) be a sequence with $y_n \neq x$, and $y_n \to x$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\log|h(y_n)-h(x)|}{\log|y_n-x|} = H.$$

We may assume that $r_n := |y_n - x| < 1$ for all $n \geq 1$. It follows that

$$H' \leq \liminf_{n\to\infty} \frac{\log \sup_{y\in B(x,r_n)} |h(y)-h(x)|}{\log r_n} \leq \liminf_{n\to\infty} \frac{\log|h(y_n)-h(x)|}{\log|y_n-x|} = H.$$

To prove the reverse inequality, let (r_n) be a sequence with $r_n > 0$ and $r_n \to 0$, as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\log \sup_{y\in B(x,r_n)} |h(y)-h(x)|}{\log r_n} = H'.$$

Then there exists a sequence (y_n) with $y_n \in B(x,r_n)$ such that

$$\lim_{n\to\infty} \frac{\log|h(y_n)-h(x)|}{\log r_n} = H'.$$

Since h is continuous at x we may assume that $\log|h(y_n)-h(x)| < 0$ for all $n \geq 1$. Hence, we have

$$H \leq \liminf_{n\to\infty} \frac{\log|h(y_n)-h(x)|}{\log|y_n-x|} \leq \liminf_{n\to\infty} \frac{\log|h(y_n)-h(x)|}{\log r_n} = H'.$$

This completes the proof of $H = H'$.

We proceed with an upper bound for the pointwise Hölder exponent.

Proposition 3.9. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}} \in \mathscr{T}$ be non-trivial. For every $x \in J$ we have*

$$\text{Höl}(C,x) \leq \inf_{\omega\in\pi^{-1}(x)} \liminf_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)}.$$

Proof. Let $x \in J$ and $\omega \in \pi^{-1}(x)$. Since J is compact, there exists a sequence (j_k) tending to infinity and $x_0 \in J$ such that

$$\alpha := \liminf_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)} = \lim_{k\to\infty} \frac{S_{j_k}\psi(\omega)}{S_{j_k}\varphi(\omega)} \quad \text{and} \quad \lim_{k\to\infty} f_{\omega_{|j_k}}(x) = x_0 \in J.$$

By Lemma 3.7 we may assume that there exist $\varepsilon > 0$, $\eta_0 > 0$ and points $a,b \in B(x_0,\varepsilon) \cap O$ with $a \neq b$ such that for all k sufficiently large,

$$\left| \sum_{\mathbf{m}} \sum_{\mathbf{n}} \beta_{\mathbf{n}} A(\omega, j(k))_{\mathbf{n},\mathbf{m}} u_{\mathbf{m}}(a,b) \right| \geq \eta_0 > 0.$$

Define $y_k := (f_{\omega_{|j_k}})^{-1}(a)$ and $z_k := (f_{\omega_{|j_k}})^{-1}(b)$. By Lemma 3.4 we have

$$C(y_k) - C(z_k) = \sum_{\mathbf{n}} \beta_{\mathbf{n}} \left(U(y_k,z_k)\right)_{\mathbf{n}} = p_{\omega_{|j(k)}} \sum_{\mathbf{n}} \beta_{\mathbf{n}} \left(A(\omega, j(k))U(a,b)\right)_{\mathbf{n}}.$$

Hence, we can estimate

$$\liminf_{k\to\infty}\frac{\log|C(y_k)-C(z_k)|}{\log|y_k-z_k|}\leq\liminf_{k\to\infty}\frac{S_{j(k)}\psi(\omega)+\log\eta_0}{S_{j(k)}\varphi(\omega)}=\alpha.$$

Finally, proceeding exactly as in [JS17, Proof of Lemma 5.3 (5.4), (5.5) and (5.6)], the statement of the lemma follows. □

Since $\text{Höl}(C,x)<\infty$ by Proposition 3.9, we can conclude the following.

Corollary 3.10. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C\in\mathscr{T}$ be non-trivial. Then C is not locally constant at any point of J and thus $\mathscr{T}$ is the direct sum of vector spaces $\oplus_{\mathbf{n}\in\mathbb{N}_0^s}\mathbb{R}C_{\mathbf{n}}$.*

Next we provide lower bounds for the pointwise Hölder exponent. In the following proposition we define for $\omega\in\Sigma$ and $x=\pi(\omega)$ the sequences

$$\delta_n:=\delta_n(\omega):=d\left(f_{\omega_{|n}}(x),\partial O\right),\quad\text{ and } s_n:=s_n(\omega):=\delta_n\cdot\left|f'_{\omega_{|n}}(x)\right|^{-1}\cdot D^{-1},$$

where ∂O refers to the boundary of the open interval O and $D=D(O)$ refers to the bounded distortion constant in Lemma 2.13.

Proposition 3.11. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C\in\mathscr{T}$ be non-trivial. Let $\omega\in\Sigma$ be a sequence which is not eventually constant and let $x=\pi(\omega)\in J$. Then we have*

$$\text{Höl}(C,x)\cdot\left(1+\frac{\liminf_{n\to\infty}n^{-1}\log\delta_n}{\max\varphi}\right)\geq\liminf_{n\to\infty}\frac{S_n\psi(\omega)}{S_n\varphi(\omega)}.$$

Proof. By (3.5) there exists a sequence (r_k) tending to zero such that

$$\text{Höl}(C,x)=\lim_{k\to\infty}\frac{\log\sup_{y\in B(x,r_k)}|C(x)-C(y)|}{\log r_k}.$$

Let $n_k:=\max\{n\geq1\mid s_n>r_k\}$. Since ω is not eventually constant, we have $s_n>0$ for each $n\in\mathbb{N}$. Hence, $(n_k)\to\infty$, as $k\to\infty$. By the definition of n_k we have

$$B(x,r_k)\subset B_k:=\left(f_{\omega_{|n_k}}\right)^{-1}\left(B\left(f_{\omega_{|n_k}}(x),\delta_{n_k}\right)\right),\quad k\in\mathbb{N}.$$

Hence, if $r_k<1$ then

$$\frac{\log\sup_{y\in B(x,r_k)}|C(x)-C(y)|}{\log r_k}\geq\frac{\log\sup_{y\in B_k}|C(x)-C(y)|}{\log r_k}.$$

Also, by the definition of n_k, we have $s_{n_k+1}\leq r_k$. Therefore, we have

$$\frac{\log\sup_{y\in B_k}|C(x)-C(y)|}{\log r_k}\geq\frac{\log\sup_{y\in B_k}|C(x)-C(y)|}{\log s_{n_k+1}}=\frac{\log\sup_{y\in B_k}|C(x)-C(y)|}{-\log D+S_{n_k+1}\varphi(\omega)+\log\delta_{n_k+1}}.$$

By Lemmas 3.4 and 3.2 there exist constants K' and q, which are independent of k, such that

$$\sup_{y\in B_k}|C(x)-C(y)|\leq K'p_{\omega_{|n_k}}n_k^q.$$

We conclude that

$$\begin{aligned}\frac{\log\sup_{y\in B_k}|C(x)-C(y)|}{-\log D+S_{n_k+1}\varphi(\omega)+\log\delta_{n_k+1}}&\geq\frac{S_{n_k}\psi(\omega)+q\log(n_k)+\log K'}{-\log D+S_{n_k+1}\varphi(\omega)+\log\delta_{n_k+1}}.\\&=\frac{S_{n_k}\psi(\omega)\cdot\left(1+(q\log(n_k)+\log K')/S_{n_k}\psi(\omega)\right)}{S_{n_k+1}\varphi(\omega)\cdot\left(1+\left(-\log D+\log\delta_{n_k+1}\right)/S_{n_k+1}\varphi(\omega)\right)}\end{aligned}$$

and the claim follows by letting k tend to infinity. □

The following result is the analogue of [JS17, Lemma 5.1 and Lemma 5.3].

Corollary 3.12. *Suppose that $(f_i)_{i\in I}$ satisfies the separating condition. Let $C \in \mathscr{T}$ be non-trivial. Then for all $\omega \in \Sigma$ and $x = \pi(\omega)$, we have*

$$\liminf_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)} = \mathrm{H\ddot{o}l}(C,x).$$

Proof. Since $\inf_{n\in\mathbb{N}} \delta_n > 0$, we have $\liminf_{n\to\infty} n^{-1}\log\delta_n = 0$. Hence, $\liminf_{n\to\infty} S_n\psi(\omega)/S_n\varphi(\omega) \le \mathrm{H\ddot{o}l}(C,x)$ by Proposition 3.11. The converse inequality follows from Proposition 3.9. □

Let us end this section with the following remark concerning Proposition 3.11 and Corollary 3.12.

Remark 3.13. It is not difficult to find examples of systems $(f_i)_{i\in I}$ satisfying the open set condition with limit points $x = \pi(\omega) \in J$, $\omega \in \Sigma$, such that

$$\mathrm{H\ddot{o}l}(C_0,x) < \lim_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)}.$$

We refer to [JS20, Example 3.1] for the details. Hence, in contrast to systems satisfying the separating condition (see Corollary 3.12), the dynamical characterization of the pointwise Hölder exponent in terms of quotients of ergodic sums does not always hold for systems satisfying the open set condition. However, Proposition 3.11 can be used to establish a dynamical characterization for almost every limit point with respect to suitable reference measures. Moreover, it turns out in Theorem 3.16 below that the dimension spectrum of the pointwise Hölder exponents coincides with the spectrum of quotients of ergodic sums of the potentials φ and ψ.

3.2. **Dimension spectrum of pointwise Hölder exponents.** We define

$$\mathscr{F}(\alpha) := \mathscr{F}_{\mathbf{p}}(\alpha) := \pi\left\{\omega \in \Sigma \mid \lim_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)} = \alpha\right\}.$$

Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. It is well known that the multifractal spectrum is complete ([Sch99]), that is, there exist $\alpha_-, \alpha_+ \in \mathbb{R}$ such that $\mathscr{F}(\alpha) \neq \varnothing$ if and only if $\alpha \in [\alpha_-, \alpha_+]$. For every $\beta \in \mathbb{R}$ there exists a unique $t(\beta) \in \mathbb{R}$ such that $\mathscr{P}(t(\beta)\varphi + \beta\psi) = 0$, where $\mathscr{P}(u)$ refers to the topological pressure of a continuous function u with respect to the dynamical system (Σ, σ) (see [Wal82]). Note that φ is Hölder continuous since the dynamical system is expanding with $\mathscr{C}^{1+\varepsilon}$ branches. Also, ψ is Hölder continuous as it only depends on the first coordinate. By well-known results from thermodynamic formalism for Hölder continuous potentials, it follows that the function t is real-analytic and convex function with $t'(\beta) = -\int \psi\, d\mu_\beta / \int \varphi\, d\mu_\beta$ where μ_β denotes the unique Gibbs probability measure on Σ associated with $t(\beta)\varphi + \beta\psi$. Moreover, with α_- and α_+ given by (2.3), we have that the function t satisfies $t'' > 0$ if and only if $\alpha_- < \alpha_+$, and have that $\alpha_- = \alpha_+$ if and only if $\delta\varphi$ and ψ are cohomologous, where

$$\delta := t(0) = \dim_H(J).$$

Here, we say that $\delta\varphi$ and ψ are cohomologous if there exists a continuous function $\kappa : \Sigma \to \mathbb{R}$ such that $\delta\varphi = \psi + \kappa - \kappa\circ\sigma$. Note that we have $-t'(\mathbb{R}) = (\alpha_-, \alpha_+)$ if $\alpha_- < \alpha_+$, and $-t'(\mathbb{R}) = \{\alpha_-\}$, otherwise. We define the level sets

$$\mathscr{F}^{\#}(\alpha) := \begin{cases} \pi\left\{\omega \in \Sigma \mid \limsup_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)} \ge \alpha\right\}, & \alpha \ge \alpha_0 \\ \pi\left\{\omega \in \Sigma \mid \liminf_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)} \le \alpha\right\}, & \alpha < \alpha_0, \end{cases}$$

where we have set $\alpha_0 := \int \psi\, d\mu_0 / \int \varphi\, d\mu_0$. We denote the convex conjugate of t ([Roc70]) by

$$t^*(u) := \sup\{\beta u - t(\beta) \mid \beta \in \mathbb{R}\} \in \mathbb{R} \cup \{+\infty\}.$$

It is well-known (see e.g. [Pes97, Sch99]) that for $\alpha \in [\alpha_-, \alpha_+]$,

$$\dim_H(\mathscr{F}(\alpha)) = \dim_H\left(\mathscr{F}^{\#}(\alpha)\right) = -t^*(-\alpha) \geq 0, \tag{3.7}$$

$-t^*(-\alpha) > 0$ for $\alpha \in (\alpha_-, \alpha_+)$ if $\alpha_- < \alpha_+$, and that $\mathscr{F}(\alpha) = \mathscr{F}^{\#}(\alpha) = \varnothing$ for $\alpha \notin [\alpha_-, \alpha_+]$. To prove this, it is shown that if $\beta \in \mathbb{R}$ and $\alpha = -t'(\beta)$, then for the corresponding Gibbs measure μ_β we have $\mu_\beta \circ \pi^{-1}(\mathscr{F}(\alpha)) = 1$ and

$$\dim_H(\mathscr{F}(\alpha)) = \dim_H(\mu_\beta \circ \pi^{-1}) > 0. \tag{3.8}$$

We refer to [JS15, JS17] for a closely related framework for random complex dynamical systems. See in particular [JS15, Remark 3.14, Proposition 4.4, Theorem 5.3]. If $\alpha_- = \alpha_+$ then $\mathscr{F}(\alpha_-) = J$ and for every $\beta \in \mathbb{R}$,

$$\dim_H(\mathscr{F}(\alpha_-)) = \dim_H(\mu_\beta \circ \pi^{-1}) = \dim_H(J) = t(0) = \delta.$$

Corollary 3.14. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C \in \mathscr{T}$ be non-trivial. Then for all $\alpha \in [\alpha_-, \alpha_+]$ we have*

$$\dim_H\{x \in J \mid \text{Höl}(C,x) = \alpha\} \geq -t^*(-\alpha).$$

Proof. It is well known that, for each $\alpha \in [\alpha_-, \alpha_+]$, there exists an ergodic Borel probability measure μ with $\int \psi\, d\mu / \int \varphi\, d\mu = \alpha$ and $\dim_H(\mu \circ \pi^{-1}) = -t^*(-\alpha) \geq 0$ (see e.g. [Pes97, Sch99]). Suppose that $-t^*(-\alpha) > 0$; otherwise there is nothing to prove. Following [Pat97] we have for μ-a.e. $\omega \in \Sigma$ and $x = \pi(\omega)$, $\lim_{n\to\infty} n^{-1} \log \delta_n = 0$, where δ_n is defined prior to Proposition 3.11. Hence, for μ-a.e. $\omega \in \Sigma$ and $x = \pi(\omega)$, $\text{Höl}(C,x) \geq \alpha$ by Proposition 3.11. Moreover, by Proposition 3.9 we have $\text{Höl}(C,x) \leq \alpha$ μ-a.e. We conclude that $\mu \circ \pi^{-1}(\{x \in J \mid \text{Höl}(C,x) = \alpha\}) = 1$. Thus, $\dim_H\{x \in J \mid \text{Höl}(C,x) = \alpha\} \geq \dim_H(\mu \circ \pi^{-1}) = -t^*(-\alpha)$. $\square$

The following proposition is an extension of results of Allaart ([All17]) for self-similar measures.

Proposition 3.15. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. For every $\alpha \in [\alpha_-, \alpha_0]$ and every $C \in \mathscr{T}$ we have*

$$\dim_H\{x \in J \mid \text{Höl}(C,x) = \alpha\} \leq -t^*(-\alpha).$$

Further, if $\alpha < \alpha_-$ then $\{x \in J \mid \text{Höl}(C,x) = \alpha\} = \varnothing$.

Proof. We use $\text{diam}(A) := \sup\{d(x,y) \mid x,y \in A\}$ to denote the diameter of a set $A \subset \mathbb{R}$. We first observe that by the Hölder continuity of φ, there exists a constant $D \geq 1$ such that

$$\text{diam}(\pi([\gamma i])) \geq D^{-1}\text{diam}(\pi([\gamma]))$$

for all $i \in I$ and for all $\gamma \in I^*$. Let $c \in J$ and $x = \pi(\omega) \in J$ with $\text{Höl}(C,x) = \alpha$. By (3.5) there exists $r_k \to 0$ such that

$$\alpha = \lim_{k\to\infty} \log \sup_{y\in B(x,r_k)} |C(y) - C(x)| / \log r_k.$$

Define $n_k := \min\left\{n \geq 1 \mid \text{diam}(\pi([\omega_{|n}])) < D^2 r_k\right\}$, $k \geq 1$. If

$$(\omega_1, \ldots, \omega_{n_k}) = (\tau j (s+1)^{\ell_k}),$$

for some $\tau \in I^*$, $j \leq s$ and $\ell_k \geq 1$ then we define $\nu_k, \nu_k' \in I^*$ by

$$\nu_k := (\tau j (s+1)^{\ell_k - 1}) = (\omega_1, \ldots, \omega_{n_k - 1}), \quad \nu_k' := (\tau (j+1) 1^{\ell_k'}),$$

where $\ell_k' \geq 1$ is given by

$$\ell_k' := \max\{\ell \geq 1 \mid \operatorname{diam}(\pi([\tau(j+1)1^{\ell}])) \geq r_k\}.$$

Similarly, if $(\omega_1, \ldots, \omega_{n_k}) = (\tau(j+1)1^{\ell_k})$, then we define $\nu_k := (\omega_1, \ldots, \omega_{n_k-1})$ and $\nu_k' := (\tau j(s+1)^{\ell_k'})$, where $\ell_k' := \max\{\ell \geq 1 \mid \operatorname{diam}(\pi([\tau j(s+1)^{\ell}])) \geq r_k\}$. Note that the sequence ℓ_k' is well-defined. If $\omega_{n_k} \notin \{1, s+1\}$ then we define $\nu_k := \nu_k' := (\omega_1, \ldots, \omega_{n_k-1})$. Let $n_k' := |\nu_k'|$.

It is important to note that, by the definition of ν_k and ν_k', we have as $k \to \infty$,

$$\operatorname{diam}(\pi[\nu_k]) \asymp \operatorname{diam}(\pi[\nu_k']) \asymp r_k. \tag{3.9}$$

Moreover, there exists a constant $E \geq 0$ such that

$$\ell_k' \cdot \varphi(\overline{1}) - E \leq \ell_k \cdot \varphi(\overline{s+1}) \leq \ell_k' \cdot \varphi(\overline{1}) + E. \tag{3.10}$$

To prove (3.10) suppose that $(\omega_1, \ldots, \omega_{n_k}) = (\tau j(s+1)^{\ell_k})$. The other case $(\omega_1, \ldots, \omega_{n_k}) = (\tau(j+1)1^{\ell_k})$, can be handled analogously. By the Hölder continuity of φ we have, as $k \to \infty$,

$$\operatorname{diam}(\pi[\nu_k]) = \operatorname{diam}\left(\pi\left[(\tau j(s+1)^{\ell_k-1})\right]\right) \asymp \operatorname{diam}(\pi[\tau])\, e^{S_{\ell_k}\varphi(\overline{(s+1)})}$$

$$\operatorname{diam}(\pi[\nu_k']) = \operatorname{diam}\left(\pi\left[\tau(j+1)1^{\ell_k'}\right]\right) \asymp \operatorname{diam}(\pi[\tau])\, e^{S_{\ell_k'}\varphi(\overline{1})},$$

which proves (3.10).

We will show that for $k \geq 1$

$$B(x, r_k) \cap J \subset \pi([\nu_k]) \cup \pi([\nu_k']).$$

First suppose that $(\omega_1, \ldots, \omega_{n_k}) = (\tau j(s+1)^{\ell_k})$. Then $\pi[\nu_k] = \pi\left[(\tau j(s+1)^{\ell_k-1})\right] \supset \pi\left[(\tau j(s+1)^{\ell_k-1}1)\right]$. Since $x \in \pi\left[(\tau j(s+1)^{\ell_k})\right]$ we have $x \geq \max \pi\left[(\tau j(s+1)^{\ell_k-1}1)\right]$. By the definition of n_k we have

$$\operatorname{diam}\left(\pi\left[(\tau j(s+1)^{\ell_k-1}1)\right]\right) \geq D^{-1}\operatorname{diam}\left(\pi\left[(\tau j(s+1)^{\ell_k-1})\right]\right) = D^{-1}\operatorname{diam}\left(\pi\left[(\omega_1, \ldots, \omega_{n_k-1})\right]\right) \geq Dr_k.$$

Hence,

$$\pi[\nu_k] \supset [x, x-r_k] \cap J.$$

Further, by the definition of ℓ_k' we have

$$\operatorname{diam}(\pi[\nu_k']) = \operatorname{diam}\left(\pi\left[\tau(j+1)1^{\ell_k'}\right]\right) \geq r_k,$$

so $[x, x+r_k] \cap J \subset \pi[\nu_k] \cup \pi[\nu_k']$. This proves that $B(x, r_k) \cap J \subset \pi([\nu_k]) \cup \pi([\nu_k'])$.

Let $\varepsilon > 0$. We will derive from our assumption $\text{Höl}(C, x) = \alpha$ that there exists $N \geq 1$ such that for all $k \geq N$,

$$\frac{S_{|\nu_k|}\psi(\overline{\nu_k})}{S_{|\nu_k|}\varphi(\overline{\nu_k})} \leq \alpha + \varepsilon \quad \text{or} \quad \frac{S_{|\nu_k'|}\psi(\overline{\nu_k'})}{S_{|\nu_k'|}\varphi(\overline{\nu_k'})} \leq \alpha + \varepsilon. \tag{3.11}$$

To prove (3.11), we first note that by Lemmas 3.4 and 3.2, there exist constants $K' \geq 1$ and $q \in \mathbb{N}_0$ such that for every $\nu \in I^*$,

$$\sup_{y_1, y_2 \in \pi([\nu])} |C(y_1) - C(y_2)| \leq K' \exp(S_{|\nu|}\psi(\overline{\nu}))|\nu|^q.$$

Suppose for a contradiction that (3.11) does not hold. Then, by passing to a subsequence of (n_k) we may assume that for all k and for all $\nu \in \{\nu_k, \nu_k'\}$ we have $S_{|\nu|}\psi(\overline{\nu}) / S_{|\nu|}\varphi(\overline{\nu}) \geq \alpha + \varepsilon$, and hence, by enlarging the constant K' if necessary, we have

$$\sup_{y_1, y_2 \in \pi([\nu])} |C(y_1) - C(y_2)| \leq K' \exp(S_{|\nu|}\psi(\overline{\nu}))|\nu|^q \leq K' r_k^{\alpha+\varepsilon}|\nu|^q.$$

Since $B(x,r_k)\cap J \subset \pi([\nu_k])\cup\pi([\nu_k'])$, we conclude that

$$\lim_{k\to\infty}\log\sup_{y\in B(x,r_k)}|C(y)-C(x)|/\log r_k \geq \alpha+\varepsilon.$$

This contradiction proves (3.11). Let $\beta>0$, $\eta>0$ and let $u=t(\beta)+\beta(\alpha+\varepsilon)+\eta$. Note that by (3.7) we have $u>t(\beta)+\beta(\alpha)\geq 0$. We define

$$\mathscr{C}_{\alpha+\varepsilon}:=\left\{\tau\in I^*\mid \frac{S_{|\tau|}\psi(\overline{\tau})}{S_{|\tau|}\varphi(\overline{\tau})}\leq\alpha+\varepsilon\right\}.$$

We obtain a covering $\mathscr{C}$ of $\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}$ by images of cylinders of sufficiently small diameters as follows. For each $x\in J$ with $\mathrm{H\ddot{o}l}(C,x)=\alpha$ we define the sequence (n_k) and the sequences of cylinders (ν_k) and (ν_k') as above. This means in particular that (3.11) holds and $x\in\pi([\nu_k])$ for every $k\geq 1$. We then pick an image of a cylinder $\nu(x):=\pi([\nu_k])$ of sufficiently small diameter. This defines a covering of $\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}$ given by

$$\mathscr{C}:=\{\nu(x)\mid x\in J:\mathrm{H\ddot{o}l}(C,x)=\alpha\}.$$

To verify that the corresponding sum of diameters $\sum_{\nu\in\mathscr{C}}\mathrm{diam}(\pi([\nu]))^u$ converges, we proceed as follows. If $\nu\notin\mathscr{C}_{\alpha+\varepsilon}$ then, by (3.9) and (3.11), we can replace ν by $\nu'\in\mathscr{C}_{\alpha+\varepsilon}$ satisfying $\mathrm{diam}(\pi([\nu]))\asymp\mathrm{diam}(\pi([\nu']))$. This defines a map $\nu\mapsto\nu'$ from $\mathscr{C}\setminus\mathscr{C}_{\alpha+\varepsilon}$ to $\mathscr{C}_{\alpha+\varepsilon}$. Since the involved numbers ℓ_k and ℓ_k' in the definition of ν_k and ν_k' satisfy (3.10), we have that every element of $\mathscr{C}_{\alpha+\varepsilon}$ is taken at most a uniformly bounded number of times under the map $\nu\mapsto\nu'$. Since

$$\sum_{\omega\in\mathscr{C}_{\alpha+\varepsilon}}\mathrm{diam}(\pi([\omega]))^u\asymp\sum_{\omega\in\mathscr{C}_{\alpha+\varepsilon}}\mathrm{e}^{(t(\beta)+\beta(\alpha+\varepsilon)+\eta)S_{|\omega|}\varphi(\overline{\omega})}\leq\sum_{\omega\in\mathscr{C}_{\alpha+\varepsilon}}\mathrm{e}^{(t(\beta)+\eta)S_{|\omega|}\varphi(\overline{\omega})+\beta S_{|\omega|}\psi(\overline{\omega})}<\infty, \tag{3.12}$$

we therefore conclude that the u-dimensional Hausdorff measure of $\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}$ is finite. To prove that the last sum in (3.12) is finite, first recall that by the definition of $t(\beta)$ we have that $\mathscr{P}(t(\beta)\varphi+\beta\psi)=0$. Further, since $\varphi<0$ we conclude that $c\mapsto\mathscr{P}(c\varphi+\beta\psi)$ is strictly decreasing. Whence, $a:=\mathscr{P}((t(\beta)+\eta)\varphi+\beta\psi)<0$. By the definition of topological pressure this implies that there exists a constant b such that $\sum_{n=1}^{\infty}\sum_{\omega\in I^n}e^{S_n((t(\beta)+\eta)\varphi+\beta\psi)(\overline{\omega})}\leq b\sum_{n=1}^{\infty}e^{na/2}<\infty$. To complete the proof of the proposition, first assume that $\alpha\in[\alpha_-,\alpha_0]$. Since ε and η are arbitrary, it follows that

$$\dim_H\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}\leq\inf_{\beta>0}\{t(\beta)+\beta\alpha\}=-t^*(-\alpha),$$

where we have used $\alpha\leq\alpha_0$ for the last equality. Finally, if $\alpha<\alpha_-$ then $\mathscr{C}_{\alpha+\varepsilon}=\varnothing$ if $\alpha+\varepsilon<\alpha_-$. By the above construction of the covering of $\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}$, it thus follows that $\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}=\varnothing$. The proof is complete. □

Theorem 3.16. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C\in\mathscr{T}$ be non-trivial. Then we have for all $\alpha\in[\alpha_-,\alpha_+]$,*

$$\dim_H\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}=-t^*(-\alpha),$$

and for $\alpha\notin[\alpha_-,\alpha_+]$ we have $\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}=\varnothing$. The function $g(\alpha)=-t^*(-\alpha)$ is continuous and concave on $[\alpha_-,\alpha_+]$. If $\alpha_-<\alpha_+$ then g is real-analytic and positive on (α_-,α_+) and satisfies $g''<0$ on (α_-,α_+).

Proof. By Corollary 3.14, we have $\dim_H\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}\geq -t^*(-\alpha)$ for $\alpha\in[\alpha_-,\alpha_+]$. By Proposition 3.9 we have for every $\alpha\in\mathbb{R}$,

$$\{x\in J\mid \mathrm{H\ddot{o}l}(C,x)=\alpha\}\subset\pi\left(\left\{\omega\in\Sigma\mid\liminf_{n\to\infty}\frac{S_n\psi(\omega)}{S_n\varphi(\omega)}\geq\alpha\right\}\right). \tag{3.13}$$

We distinguish two cases. If $\alpha \geq \alpha_0$, then by the definition of $\mathscr{F}^{\#}(\alpha)$ we have $\{x \in J \mid \text{Höl}(C,x) = \alpha\} \subset \mathscr{F}^{\#}(\alpha)$. Hence, by (3.7), we have $\dim_H(\{x \in J \mid \text{Höl}(C,x) = \alpha\}) \leq \dim_H(\mathscr{F}^{\#}(\alpha)) = -t^*(-\alpha)$ for $\alpha \geq \alpha_0$. Also, by (3.13), we have $\{x \in J \mid \text{Höl}(C,x) = \alpha\} = \varnothing$ if $\alpha > \alpha_+$. For $\alpha \leq \alpha_0$ the remaining assertions follow from Proposition 3.15. For the proof of the well-known properties of g we refer to ([Pes97], [Sch99], see also [JS15]). The proof is complete. □

Remark 3.17. In [BKK16] the pointwise Hölder exponent of affine zipper curves generated by contracting affine mappings is investigated. The multifractal dimension spectrum is obtained only for $\alpha \geq \alpha_0$. In a recent paper of Allaart ([All17]) the complete multifractal spectrum of pointwise Hölder exponents for curves associated with selfsimilar measures is obtained.

4. GLOBAL HÖLDER CONTINUITY

In this section we investigate the global Hölder continuity of the elements of $\mathscr{T}$. The first statement of the next theorem has been obtained for the Minkowski's question mark function in [KS08], for distributions of conformal iterated function systems satisfying the separating condition in [KS09], and for expanding circle diffeomorphisms in [JKPS09].

Theorem 4.1. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Then we have the following.*

(1) $T_{\mathbf{p}} \in \mathscr{C}^{\alpha_-}(\overline{\mathbb{R}})$.
(2) $\mathscr{T} \subset \bigcap_{\alpha<\alpha_-} \mathscr{C}^{\alpha}(\overline{\mathbb{R}})$.

Proof. We only verify the desired Hölder continuity at points $x, y \in J$. That this is sufficient can be seen as follows. If $y \in \overline{\mathbb{R}} \setminus J$, then either there exists $u \in J$ between x and y and the desired Hölder continuity follows from the triangle inequality, or we have $T_{\mathbf{p}}(x) = T_{\mathbf{p}}(y)$ (resp. $C(x) = C(y)$). We may assume that $O = (\text{Fix}(f_1), \text{Fix}(f_{s+1}))$, where $\text{Fix}(f_k)$ denotes the unique fixed point of f_k in $\mathbb{R}$.

Let $x, y \in J$ with $x < y$. Let $\omega \in I^* \cup \{\varnothing\}$ and $i, j \in I$ with $i < j$ such that $f_\omega(x) \in \overline{O}$, $f_\omega(y) \in \overline{O}$ and $f_i(f_\omega(x)) \in \overline{O}$ and $f_j(f_\omega(y)) \in \overline{O}$. Note that such (ω, i, j) always exists because $x, y \in J$ and $x < y$.

We first consider the case when $j = i+1$. Let

$$\ell := \sup\left\{n \geq 0 \mid f_{s+1}^n(f_i(f_\omega(x))) \in \overline{O}\right\} \quad \text{and} \quad \ell' := \sup\left\{n \geq 0 \mid f_1^n(f_j(f_\omega(y))) \in \overline{O}\right\}.$$

We define

$$\xi := \max(f_{\omega i})^{-1}(\overline{O}) \quad \text{and} \quad \xi' := \min(f_{\omega j})^{-1}(\overline{O}).$$

We will verify that there exists a uniform constant $D' \geq 1$ such that

$$|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(\xi)| \leq C(\varphi, \psi_{\mathbf{p}}) D' d(x, \xi)^{\alpha_-(\mathbf{p})} \tag{4.1}$$

and

$$\left|T_{\mathbf{p}}(\xi') - T_{\mathbf{p}}(y)\right| \leq C(\varphi, \psi_{\mathbf{p}}) D' d(\xi', y)^{\alpha_-(\mathbf{p})}. \tag{4.2}$$

By the triangle inequality we have that

$$|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(y)| \leq |T_{\mathbf{p}}(x) - T_{\mathbf{p}}(\xi)| + \left|T_{\mathbf{p}}(\xi) - T_{\mathbf{p}}(\xi')\right| + \left|T_{\mathbf{p}}(\xi') - T_{\mathbf{p}}(y)\right|,$$

which proves the first assertion in the case when $j = i+1$ because $T_{\mathbf{p}}$ is constant on $[\xi, \xi']$.

We only verify (4.1), the proof of (4.2) is completely analogous. To prove (4.1) we may assume that $\ell < \infty$ because if $\ell = \infty$ then $x = \xi$ and (4.1) holds trivially. By the definition of ℓ there exists $1 \leq k < s+1$ such that

$$x \in f_\omega^{-1} f_i^{-1} f_{s+1}^{-\ell} f_k^{-1}(\overline{O}) \leq f_\omega^{-1} f_i^{-1} f_{s+1}^{-\ell} f_{s+1}^{-1}(\overline{O}) \subset f_\omega^{-1} f_i^{-1}(\overline{O}).$$

We conclude that

$$d(x,\xi) \geq \operatorname{diam}\left(f_\omega^{-1} f_i^{-1} f_{s+1}^{-\ell} f_{s+1}^{-1}(\overline{O})\right). \tag{4.3}$$

Define $\tau := \omega i (s+1)^\ell$. Clearly, we have $[x,\xi] \subset f_\tau^{-1}(\overline{O})$. Since the open set condition holds,

$$|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(\xi)| \leq p_\tau. \tag{4.4}$$

By (2.6) there exists a constant $C(\varphi, \psi_{\mathbf{p}})$ such that

$$p_\tau = \mathrm{e}^{S_{|\tau|}\psi_{\mathbf{p}}(\overline{\tau})} \leq C(\varphi, \psi_{\mathbf{p}}) \mathrm{e}^{\alpha_-(\mathbf{p}) S_{|\tau|}\varphi(\overline{\tau})}. \tag{4.5}$$

By the bounded distortion property (Lemma 2.13) and (4.3), there exists a uniform constant $D' \geq 1$ such that $\mathrm{e}^{\alpha_-(\mathbf{p}) S_{|\tau|}\varphi(\overline{\tau})} \leq D' d(x,\xi)^{\alpha_-(\mathbf{p})}$. We have thus shown (4.1). It remains to consider the case when $j > i+1$. Using that $[x,y] \subset f_\omega^{-1}(\overline{O})$ we obtain as above

$$|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(y)| \leq p_\omega \leq C(\varphi, \psi_{\mathbf{p}}) \mathrm{e}^{\alpha_-(\mathbf{p}) S_{|\omega|}\varphi(\overline{\omega})}.$$

Here, we set $p_\varnothing = 1$ and $S_{|\varnothing|}\varphi(\overline{\varnothing}) := 0$. By our assumption that $j > i+1$ we have $d(x,y) \geq \operatorname{diam}(f_\omega^{-1} f_{i+1}^{-1}(O))$, which implies that there exists a constant $D'' \geq 1$ such that

$$\mathrm{e}^{\alpha_-(\mathbf{p}) S_{|\tau|}\varphi(\overline{\omega})} \leq D'' d(x,y)^{\alpha_-(\mathbf{p})}.$$

The proof of the first assertion is complete. To prove the second assertion, it is sufficient to verify it for $C_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{N}_0^s$. To this end, we replace the estimate in (4.4) by the following estimate. By Lemma 3.4 and Lemma 3.2 there exists a constant $K' \geq 1$ (which only depends on $\mathbf{p}$ and $|\mathbf{n}|$) such that

$$|C_{\mathbf{n}}(x) - C_{\mathbf{n}}(\xi)| \leq K' |\tau|^{|\mathbf{n}|} p_\tau. \tag{4.6}$$

Then proceeding as above, for every $\alpha < \alpha_-(\mathbf{p})$ there exists $D''' \geq 1$ such that

$$|C_{\mathbf{n}}(x) - C_{\mathbf{n}}(\xi)| \leq K' |\tau|^{|\mathbf{n}|} C(\varphi, \psi_{\mathbf{p}}) \mathrm{e}^{\alpha_-(\mathbf{p}) S_{|\tau|}\varphi(\overline{\tau})} \leq K' |\tau|^{|\mathbf{n}|} C(\varphi, \psi_{\mathbf{p}}) D''' d(x,\xi)^\alpha \mathrm{e}^{(\alpha_-(\mathbf{p})-\alpha) S_{|\tau|}\varphi(\overline{\tau})}. \tag{4.7}$$

Since $\alpha - \alpha_-(\mathbf{p}) < 0$, we have $K' |\tau|^{|\mathbf{n}|} C(\varphi, \psi_{\mathbf{p}}) D''' \mathrm{e}^{(\alpha_-(\mathbf{p})-\alpha) S_{|\tau|}\varphi(\overline{\tau})} \to 0$ uniformly as $|\tau| \to \infty$. Hence, $C_{\mathbf{n}}$ is α-Hölder continuous. □

Remark 4.2. By the bounded distortion property of (f_i) (Lemma 2.13) it is not difficult to see that $|\tau| \asymp -\log d(x,\xi)$. By combining this estimate with (4.5) and (4.6), we can derive that $|C_{\mathbf{n}}(x) - C_{\mathbf{n}}(y)| \ll (-\log d(x,y))^{|\mathbf{n}|} d(x,y)^{\alpha_-(\mathbf{p})}$. If $\alpha_- = 1$ this implies that $C_{\mathbf{n}}$ is convex Lipschitz ([MW86]). In fact, this property was derived in [MW86] for the classical Takagi function, and for the higher order derivatives of the Lebesgue singular function in [AK06]. The property of convex Lipschitz can be used to prove that the graph of $C_{\mathbf{n}}$ has Hausdorff dimension one if $\alpha_- = 1$.

Recall that we have $\alpha_- \leq \delta := \dim_H(J)$ with equality if and only if $\alpha_- = \alpha_+$ ([Pes97, Chapter 21, see in particular, Figure 17b]).

Corollary 4.3. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Then for every non-trivial $C \in \mathscr{T}$ we have*

$$\alpha_- = \sup\left\{\alpha \geq 0 \mid C \in \mathscr{C}^\alpha(\overline{\mathbb{R}})\right\} \leq \delta.$$

The equality $\alpha_- = \alpha_+$ occurs if and only if $T_{\mathbf{p}} \in \mathscr{C}^\delta(\overline{\mathbb{R}})$.

Proof. The first assertion follows from Theorem 3.16 and Theorem 4.1. Now suppose that $\alpha_- = \alpha_+$. Hence, $\alpha_- = \delta$ and $T_{\mathbf{p}} \in \mathscr{C}^{\delta}(\overline{\mathbb{R}})$ by Theorem 4.1. Conversely, suppose that $T_{\mathbf{p}} \in \mathscr{C}^{\delta}(\overline{\mathbb{R}})$. Then, by the first assertion of Corollary 4.3, we have $\alpha_- \geq \delta$. Hence, $\alpha_- = \alpha_+$. □

Corollary 4.4. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. If $\alpha_- < 1$ then, for each $1 > \alpha > \alpha_-$ there exists $\phi \in \mathscr{C}^{\alpha}\left(\overline{\mathbb{R}}\right)$ such that $\|M_{\mathbf{p}}^n \phi\|_{\alpha} \to \infty$, as $n \to \infty$.*

Proof. Let $\phi \in \mathscr{C}^{\alpha}\left(\overline{\mathbb{R}}\right)$ such that $\phi(\infty) = 1$ and $\phi(-\infty) = 0$. Then $\|M_{\mathbf{p}}^n \phi - T_{\mathbf{p}}\|_{\infty} \to 0$ as $n \to \infty$. Now suppose for a contradiction that $\liminf_{n\to\infty} \|M_{\mathbf{p}}^n \phi\|_{\alpha} < \infty$. Then there exists a sequence (n_j) in $\mathbb{N}$ tending to infinity and a constant $M < \infty$ such that $\|M_{\mathbf{p}}^{n_j} \phi\|_{\alpha} < M$ for each j. Hence, $|M_{\mathbf{p}}^{n_j} \phi(x) - M_{\mathbf{p}}^{n_j} \phi(y)| \leq M d(x,y)^{\alpha}$ for all j and $x, y \in \mathbb{R}$. Letting $j \to \infty$ gives $|T_{\mathbf{p}}(x) - T_{\mathbf{p}}(y)| \leq M d(x,y)^{\alpha}$ for all $x, y \in \mathbb{R}$. We have thus shown that $T_{\mathbf{p}} \in \mathscr{C}^{\alpha}\left(\overline{\mathbb{R}}\right)$ giving the desired contradiction to Theorem 1.4. □

In the following, we will denote $C_{(n)}$ by C_n if $s = 1$ and $n \in \mathbb{N}_0$.

Proposition 4.5. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition.*

(1) If $\alpha_- = \alpha_+$ then $\mathscr{T} \cap \mathscr{C}^{\alpha_-}(\overline{\mathbb{R}}) = \mathbb{R} T_{\mathbf{p}}$.
(2) If $s = 1$ and each f_i has constant derivative, then $C_n \notin \mathscr{C}^{\alpha_-}(\overline{\mathbb{R}})$ for every $n \geq 1$.

Proof. We first prove (1). For $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^s$ we use $\preceq$ to denote the lexicographical order, that is, we write $\mathbf{n} \preceq \mathbf{m}$ if $n_k < m_k$ where $k := \min\{1 \leq i \leq s \mid n_i \neq m_i\}$ or if $\mathbf{n} = \mathbf{m}$. Since the lexicographical order is a total order, each non-trivial $C \in \mathscr{T}$ has a representation as $C = \sum_{\mathbf{n} \preceq \mathbf{n}_{\max}} \beta_{\mathbf{n}} C_{\mathbf{n}}$ with $\beta_{\mathbf{n}_{\max}} \neq 0$. Suppose that $n := \sum_{i=1}^s (\mathbf{n}_{\max})_i \geq 1$. Our aim is to prove that $C \notin \mathscr{C}^{\alpha_-}(\overline{\mathbb{R}})$. Let $a := \min J$ and $b := \max J$. Define for $k \geq 1$

$$m_i(k) := k^{(n+1)^{i-1}}, \quad 1 \leq i \leq s,$$

and for $\omega(k) := 1^{m_1(k)} 2^{m_2(k)} \ldots s^{m_s(k)} \in I^{\sum_{i=1}^s m_i(k)}$ we let

$$x_k := \left(f_{\omega(k)}\right)^{-1}(a), \quad y_k := \left(f_{\omega(k)}\right)^{-1}(b).$$

Recall that, since $\alpha_- = \alpha_+$, we have $\alpha_- = \alpha_+ = \delta = \dim_H J$, and the potentials $\delta\varphi$ and ψ are cohomologous. It follows that

$$\frac{|C(x_k) - C(y_k)|}{d(x_k, y_k)^{\alpha_-}} \asymp \frac{|C(x_k) - C(y_k)|}{p_{\omega(k)}}, \quad \text{as } k \to \infty.$$

Since $C_{\mathbf{n}}(a) - C_{\mathbf{n}}(b) = \mathbf{0}$, for $\mathbf{n} \neq \mathbf{0}$ and $C_{\mathbf{0}}(a) - C_{\mathbf{0}}(b) = 1$, we conclude by Lemma 3.4 that

$$\frac{C(x_k) - C(y_k)}{p_{\omega(k)}} = \sum_{\mathbf{n} \preceq \mathbf{n}_{\max}} \beta_{\mathbf{n}} \left(A\left(\overline{\omega(k)}, \sum_{i=1}^s m_i(k)\right) U(a,b) \right)_{\mathbf{n}} = \sum_{\mathbf{n} \preceq \mathbf{n}_{\max}} \beta_{\mathbf{n}} \left(A\left(\overline{\omega(k)}, \sum_{i=1}^s m_i(k)\right) \right)_{\mathbf{n},\mathbf{0}}.$$

Hence, by Lemma 3.2 we have as $k \to \infty$

$$\frac{|C(x_k) - C(y_k)|}{p_{\omega(k)}} \asymp \left| \sum_{\mathbf{n} \preceq \mathbf{n}_{\max}} \beta_{\mathbf{n}} \prod_{i=1}^s m_i(k)^{n_i} \right| \asymp \left| \sum_{\mathbf{n} \preceq \mathbf{n}_{\max}} \beta_{\mathbf{n}} k^{\sum_{i=1}^s n_i (n+1)^{i-1}} \right| \asymp |\beta_{\mathbf{n}_{\max}}| k^{\sum_{i=1}^s (\mathbf{n}_{\max})_i (n+1)^{i-1}}.$$

It follows that $|C(x_k) - C(y_k)| / d(x_k, y_k)^{\alpha_-}$ is not bounded, as $k \to \infty$, which implies that $C \notin \mathscr{C}^{\alpha_-}(\overline{\mathbb{R}})$. The proof of (1) is complete.

To prove (2) we can proceed along the same lines. First note that, by our assumptions on f_1 and f_2, φ depends only on the first coordinate. We show that there exists $i \in I$ such that $\psi(\bar{i})/\varphi(\bar{i}) = \alpha_-$. First observe that there exists $i \in I$ such that $\psi(\bar{i})/\varphi(\bar{i}) = \min(\psi/\varphi)$. Thus, for all $n \geq 1$, $S_n\psi(\bar{i})/S_n\varphi(\bar{i}) = \min(\psi/\varphi)$. From the mediant inequality we derive inductively that $S_n\psi/S_n\varphi \geq \min(\psi/\varphi)$ which completes the proof that $\alpha_- = \psi(\bar{i})/\varphi(\bar{i})$. Hence, for the sequences given by $x_k := (f_i)^{-k}(a)$ and $y_k := (f_i)^{-k}(b)$ we have as $k \to \infty$,

$$\frac{|C_n(x_k) - C_n(y_k)|}{d(x_k, y_k)^{\alpha_-}} \asymp \frac{|C_n(x_k) - C_n(y_k)|}{p_i^k} \asymp \left|\left(A(\bar{i}, k)U(a,b)\right)_n\right| \asymp k^n,$$

which tends to infinity, as $k \to \infty$. Again this implies that $C_n \notin \mathscr{C}^{\alpha_-}(\overline{\mathbb{R}})$. The proof is complete. □

5. Non-differentiability

In this section we investigate the (non-)differentiability of the elements of $\mathscr{T}$. Note that $\alpha_- \le \dim_H(J) \le 1$. We denote by Leb the Lebesgue measure on $[0,1]$.

Proposition 5.1. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition.*

(1) If $\alpha_- < 1$ then there exists a dense subset $E \subset J$ with $\dim_H(E) > 0$ such that, for every non-trivial $C \in \mathscr{T}$ and $x \in E$, C is not differentiable at x. If moreover $\alpha_+ < 1$ then $C \in \mathscr{T} \setminus \{0\}$ is nowhere differentiable on J.

(2) If $\alpha_- < 1$ and $\dim_H(J) = 1$ then we have, for every $C \in \mathscr{T}$, $C'(x) = 0$ for Leb-a.e. $x \in J$.

(3) If $\alpha_- = 1$ then $\alpha_+ = 1$ and $\dim_H(J) = 1$. If moreover $s = 1$ and f_1' and f_2' are constant functions, then C_m is nowhere differentiable on J, for every $m \ge 1$.

Proof. First assume that $\alpha_- < 1$. If $\alpha_- < \alpha_+$ then there exists $\alpha \in (\alpha_-, \alpha_+)$ with $\alpha < 1$. Then by Proposition 3.9 we have $\text{Höl}(C,x) \le \alpha < 1$ for all $x \in E := \pi(\mathscr{F}(\alpha))$, and E has the desired properties. If moreover $\alpha_+ < 1$ then we have $\text{Höl}(C,x) \le \alpha_+ < 1$ for every $x \in J$ by Proposition 3.9, which completes the proof of (1).

To prove (2), let $\alpha_- < 1$ and observe that $\dim_H(J) = 1$ implies $\alpha_- < \alpha_+$. Hence, t is strictly convex. Since $t(0) = 1$ and $t(1) = 0$, it follows that $-t'(0) > 1$. Thus, we have $\text{Höl}(C,x) = \int \psi \, d\mu_0 / \int \varphi \, d\mu_0 = -t'(0) > 1$ μ_0-almost everywhere. Since μ_0 is equivalent to Leb, the assertion in (2) follows.

Finally, we turn to the proof of (3). Let $s = 1$. Recall that $\alpha_- = 1$ implies $\alpha_- = \alpha_+ = \dim_H(J) = 1$ and that φ is cohomologous to ψ. Hence, there exists a continuous function $h : \Sigma \to \mathbb{R}$ such that $\varphi = \psi + h - h \circ \sigma$. Since f_1' and f_2' are constant functions, $\varphi = \varphi(\omega)$ depends only on the first symbol ω_1 of $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$. We conclude that for every $\omega \in \Sigma$,

$$\varphi(\omega) = \varphi(\overline{\omega_1}) = \psi(\overline{\omega_1}) + h(\overline{\omega_1}) - h \circ \sigma(\overline{\omega_1}) = \psi(\omega).$$

Now, suppose for a contradiction that there exists $m \ge 1$ and $x \in J$ such that C_m is differentiable at x and let $\omega \in \Sigma$ such that $x = \pi(\omega)$. Let $a := \min J$ and $b := \max J$. Define $x_n := (f_{\omega|n})^{-1}(a)$ and $y_n := (f_{\omega|n})^{-1}(b)$, $n \ge 1$. We will verify that the sequence $(\gamma_n)_{n\ge1}$ given by

$$\gamma_n := \frac{C_m(x_n) - C_m(y_n)}{x_n - y_n}, \quad n \ge 1,$$

is not convergent. Since $x_n \le x \le y_n$ and $x_n, y_n \to x$ we obtain the desired contradiction. To prove that (γ_n) is not convergent, note that for all $n \ge 1$,

$$\frac{p_{\omega_{|n}}}{|x_n - y_n|} = \frac{\mathrm{e}^{S_n\psi(\omega) - S_n\varphi(\omega)}}{b-a} = \frac{1}{b-a}.$$

Combining with $C_m(a) = C_m(b) = 0$, $T(a) - T(b) = 1$ and Lemma 3.4 we obtain

$$\gamma_n \cdot (b-a) = -\frac{(U(x_n, y_n))_m}{p_{\omega_{|n}}} = -\left(A(\omega,n)U(a,b)\right)_m = -A(\omega,n)_{m,0}(T(a) - T(b)) = -A(\omega,n)_{m,0}.$$

Let $\delta_{i,j}$ denote the Dirac delta function, for $i,j \in I$. By the definition of the matrix cocycle A we have

$$A(\omega,n)_{1,0} = \sum_{i=1}^{n} A(\sigma^{i-1}(\omega),1)_{1,0} = \sum_{i=1}^{n} \left(\frac{\delta_{1,\omega_i}}{p_1} - \frac{\delta_{2,\omega_i}}{p_2} \right). \tag{5.1}$$

This shows that $(A(\omega,n)_{1,0})$ does not converge as $n \to \infty$ because $\frac{\delta_{1,\omega_i}}{p_1} - \frac{\delta_{2,\omega_i}}{p_2} \in \{1/p_1, -1/p_2\}$. We now proceed inductively to verify that $(A(\omega,n)_{m,0})$ does not converge as $n \to \infty$. It is easy to see that

$$\begin{aligned} A(\omega,n+1)_{m,0} &= A(\omega,n)_{m,0} A(\sigma^n(\omega),1)_{0,0} + A(\omega,n)_{m,1} A(\sigma^n(\omega),1)_{1,0} \\ &= A(\omega,n)_{m,0} + A(\omega,n)_{m,1} A(\sigma^n(\omega),1)_{1,0}. \end{aligned}$$

Hence,

$$A(\omega,n+1)_{m,0} - A(\omega,n)_{m,0} = A(\omega,n)_{m,1} \cdot A(\sigma^n(\omega),1)_{1,0} = A(\omega,n)_{m,1} \cdot \left(\frac{\delta_{1,\omega_{n+1}}}{p_1} - \frac{\delta_{2,\omega_{n+1}}}{p_2} \right).$$

Further, we have

$$\begin{aligned} A(\omega,n)_{m,1} &= \sum_{1 \le i_1 < \cdots < i_{m-1} \le n} \prod_{k=1}^{m-1} A(\sigma^{i_k-1}(\omega),1)_{m-k+1,m-k} \\ &= \sum_{1 \le i_1 < \cdots < i_{m-1} \le n} \prod_{k=1}^{m-1} \left(\frac{\delta_{1,\omega_{i_k}}}{p_1} - \frac{\delta_{2,\omega_{i_k}}}{p_2} \right)(m-k+1) \\ &= m \cdot \sum_{1 \le i_1 < \cdots < i_{m-1} \le n} \prod_{k=1}^{m-1} \left(\frac{\delta_{1,\omega_{i_k}}}{p_1} - \frac{\delta_{2,\omega_{i_k}}}{p_2} \right)(m-k) \\ &= m \cdot A(\omega,n)_{m-1,0}. \end{aligned}$$

Therefore,

$$A(\omega,n+1)_{m,0} - A(\omega,n)_{m,0} = m \cdot A(\omega,n)_{m-1,0} \cdot \left(\frac{\delta_{1,\omega_{n+1}}}{p_1} - \frac{\delta_{2,\omega_{n+1}}}{p_2} \right). \tag{5.2}$$

By induction hypothesis, we may assume that $(A(\omega,n)_{m-1,0})$ is not convergent as $n \to \infty$. Since $\frac{\delta_{1,\omega_i}}{p_1} - \frac{\delta_{2,\omega_i}}{p_2} \in \{1/p_1, -1/p_2\}$ we conclude by (5.2) that $(A(\omega,n)_{m,0})$ is not convergent as $n \to \infty$. The proof of (3) is complete.

□

Remark 5.2. If $\alpha_- = \alpha_+ = 1$, then the nowhere-differentiability of elements of $\mathscr{T}$ stated in Theorem 5.1 (3) will be compared with the fact that $T_{\mathbf{p}}$ is a $\mathscr{C}^{1+\varepsilon}$-diffeomorphism by Corollary 6.3.

6. Conjugacies between Interval maps

In this section we show how our results are related to interval conjugacies. This section is motivated by the results in [JKPS09] for conjugacies between expanding $C^{1+\varepsilon}$ maps on the unit interval with finitely many full branches. Note that in [JKPS09] it is always assumed that the Julia set J is equal to the unit interval.

For $\mathbf{p} \in (0,1)^s$ we define the expanding linear maps $g_1, \dots, g_{s+1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ which are for $i \in I$ given by

$$g_i(x) := \frac{1}{p_i} \left(x - \sum_{1 \le j < i} p_j \right).$$

Clearly, $(g_1, \dots, g_{s+1})$ satisfies our standing assumptions. Moreover, $(g_1, \dots, g_{s+1})$ satisfies the open set condition with the open set $O := (0,1)$ because $g_i^{-1}(O) = (\sum_{j<i} p_j, \sum_{j \le i} p_j)$. The Julia set of $(g_1, \dots, g_{s+1})$

is equal to $[0,1]$. We denote by $\pi_{\mathbf{p}} : \Sigma \to [0,1]$ the coding map of the Julia set of $(g_1,\ldots,g_{s+1})$. Similarly, we define

$$g_{\mathbf{p}} : [0,1] \to [0,1], \quad g_{\mathbf{p}}(x) := g_i(x), \quad \text{where } i = \min\left\{ j \in I \mid g_j(x) \in [0,1] \right\}.$$

Note that $g_{\mathbf{p}}$ is the piecewise linear map on $[0,1]$ with $(s+1)$ full branches and slopes given by $(1/p_i)_{i\in I}$.

Suppose that $(f_1,\ldots,f_{s+1})$ satisfies the open set condition and suppose that $f_i^{-1}(O) \leq f_{i+1}^{-1}(O)$ for all $i \in I$. Denote the Julia set of $(f_1,\ldots,f_{s+1})$ by J and its coding map by $\pi : \Sigma \to J$. We also define

$$f : J \to J, \quad f(x) := f_i(x), \quad \text{where } i = \min\left\{ j \in I \mid f_j(x) \in J \right\}.$$

Further, we define

$$\Phi_{\mathbf{p}} : J \to [0,1], \quad \Phi_{\mathbf{p}}(x) := \pi_{\mathbf{p}}(\omega) \text{ for some / any } \omega \in \pi^{-1}(x).$$

Note that $\Phi_{\mathbf{p}}$ is a well-defined, Borel measurable function satisfying $\Phi_{\mathbf{p}}(\max J) = 1$ and $\Phi_{\mathbf{p}}(\min J) = 0$. Let $\mathscr{E} := \{\omega \in \Sigma \mid \omega \text{ is eventually constant}\}$. For $x \in \pi(\Sigma \setminus \mathscr{E})$ we denote by $\pi^{-1}(x)$ the unique $\omega \in \Sigma \setminus \mathscr{E}$ such that $\pi(\omega) = x$. For $\omega \in \Sigma \setminus \mathscr{E}$ and $x = \pi(\omega)$ we have $f(x) = f_{\omega_1}(x)$ and thus,

$$\pi^{-1}(f(x)) = \sigma(\omega).$$

Further, $\mathbf{g}_{\mathbf{p}}(\pi_{\mathbf{p}}(\omega)) = g_{\omega_1}(\pi_{\mathbf{p}}(\omega))$ implies

$$\pi_{\mathbf{p}} \circ \sigma(\omega) = \mathbf{g}_{\mathbf{p}} \circ \pi_{\mathbf{p}}(\omega).$$

Hence, for $x \in \pi(\Sigma \setminus \mathscr{E})$,

$$\Phi_{\mathbf{p}}(f(x)) = \pi_{\mathbf{p}} \circ \pi^{-1}(f(x)) = \pi_{\mathbf{p}} \circ \sigma(\omega) = \mathbf{g}_{\mathbf{p}} \circ \pi_{\mathbf{p}}(\omega) = \mathbf{g}_{\mathbf{p}}(\Phi_{\mathbf{p}}(x)). \tag{6.1}$$

If J is an interval, the following lemma appears implicitly in [JKPS09, Proof of Proposition 1.4].

Lemma 6.1. *For every* $\mathbf{p} \in (0,1)^s$ *we have that* $\Phi_{\mathbf{p}} = T_{\mathbf{p}|J}$.

Proof. We proceed in two steps. First, we will show that $\Phi_{\mathbf{p}}(x) = T_{\mathbf{p}}(x)$ for $x \in \pi(\Sigma \setminus \mathscr{E})$. Let $\tilde{\Phi}_{\mathbf{p}} : \overline{\mathbb{R}} \to [0,1]$ denote a bounded Borel measurable extension of $\Phi_{\mathbf{p}}$ such that $\tilde{\Phi}_{\mathbf{p}}(y) = 0$ for $y \in [-\infty, \min J]$, and $\tilde{\Phi}_{\mathbf{p}}(y) = 1$ for $y \in [\max J, +\infty]$. Let $\omega \in \Sigma \setminus \mathscr{E}$. For $i < \omega_1$ we have $f_i(\pi(\omega)) \geq \max J$, and for $i > \omega_1$ we have $f_i(\pi(\omega)) \leq \min J$. Since $\Phi_{\mathbf{p}}(\max J) = 1$ and $\Phi_{\mathbf{p}}(\min J) = 0$, the equality in (6.1) yields for $x = \pi(\omega)$

$$M_{\mathbf{p}}(\tilde{\Phi}_{\mathbf{p}})(x) = \sum_{i\in I} p_i \tilde{\Phi}_{\mathbf{p}}(f_i(x)) = \sum_{i<\omega_1} p_i + p_{\omega_1} \cdot \Phi_{\mathbf{p}}(f(x)) + \sum_{i>\omega_1} 0 = \sum_{i<\omega_1} p_i + p_{\omega_1} \cdot \mathbf{g}_{\mathbf{p}}(\Phi_{\mathbf{p}}(x)).$$

Since $x = \pi(\omega)$ and $\omega \in \Sigma \setminus \mathscr{E}$, we have $\mathbf{g}_{\mathbf{p}}(\Phi_{\mathbf{p}}(x)) = g_{\omega_1}(\Phi_{\mathbf{p}}(x))$. Hence, we have for every $x \in \pi(\Sigma \setminus \mathscr{E})$,

$$M_{\mathbf{p}}(\tilde{\Phi}_{\mathbf{p}})(x) = \sum_{i<\omega_1} p_i + p_{\omega_1} \cdot g_{\omega_1}(\Phi_{\mathbf{p}}(x)) = \sum_{i<\omega_1} p_i + p_{\omega_1} \cdot \frac{1}{p_{\omega_1}} \left(\Phi_{\mathbf{p}}(x) - \sum_{i<\omega_1} p_i \right) = \tilde{\Phi}_{\mathbf{p}}(x).$$

Further, for every $x \in [-\infty, \min J] \cup [\max J, +\infty]$ we have $M_{\mathbf{p}}(\tilde{\Phi}_{\mathbf{p}})(x) = \tilde{\Phi}_{\mathbf{p}}(x)$. Let $E := \pi(\Sigma \setminus \mathscr{E}) \cup [-\infty, \min J] \cup [\max J, +\infty]$. Since $f_i(E) \subset E$ for every $i \in I$, we can show inductively that for every $x \in E$ and $n \in \mathbb{N}$,

$$M_{\mathbf{p}}^n(\tilde{\Phi}_{\mathbf{p}})(x) = M_{\mathbf{p}}(M_{\mathbf{p}}^{n-1}\tilde{\Phi}_{\mathbf{p}})(x) = \sum_{i\in I} p_i (M_{\mathbf{p}}^{n-1}\tilde{\Phi}_{\mathbf{p}})(f_i(x)) = \sum_{i\in I} p_i \tilde{\Phi}_{\mathbf{p}}(f_i(x)) = \tilde{\Phi}_{\mathbf{p}}(x).$$

By (2.2) (Remark: (2.2) is valid for any bounded measurable function h on $\overline{R}$ such that $h = 1$ around $+\infty$ and $h = 0$ around $-\infty$) we conclude that for $x \in \pi(\Sigma \setminus \mathscr{E})$,

$$\tilde{\Phi}_{\mathbf{p}}(x) = \lim_{n\to\infty} M_{\mathbf{p}}^n \tilde{\Phi}_{\mathbf{p}}(x) = T_{\mathbf{p}}(x)\tilde{\Phi}_{\mathbf{p}}(\infty) + (1 - T_{\mathbf{p}}(x))\tilde{\Phi}_{\mathbf{p}}(-\infty) = T_{\mathbf{p}}(x).$$

This completes the proof of $\Phi_{\mathbf{p}}(x)=T_{\mathbf{p}}(x)$ for $x\in\pi(\Sigma\setminus\mathscr{E})$. Now, let $\omega\in\mathscr{E}$ and $x=\pi(\omega)$. Let $(\omega^{(n)})\subset\Sigma\setminus\mathscr{E}$ such that $(\omega_1^{(n)},\dots,\omega_n^{(n)})=(\omega_1,\dots,\omega_n)$ and $x_n:=\pi(\omega^{(n)})$, $n\geq 1$. By the continuity of π with respect to the word metric, we have $x_n\to x$ as $n\to\infty$. By the definition of $\Phi_{\mathbf{p}}$ we have $\Phi_{\mathbf{p}}(x_n)=\pi_{\mathbf{p}}(\omega^{(n)})$ and $\Phi_{\mathbf{p}}(x)=\pi_{\mathbf{p}}(\omega)$. So, by the continuity of $\pi_{\mathbf{p}}$ with respect to the word metric, we have $\Phi_{\mathbf{p}}(x_n)\to\Phi_{\mathbf{p}}(x)$ as $n\to\infty$. Since $\Phi_{\mathbf{p}}(x_n)=T_{\mathbf{p}}(x_n)$ by the first part of the proof, and since $T_{\mathbf{p}}$ is continuous by Theorem 2.4, we can conclude that $\Phi_{\mathbf{p}}(x)=T_{\mathbf{p}}(x)$. The proof is complete. □

Theorem 6.2 ([JKPS09, Theorem 1.2]). *Suppose that $(f_i)_{i\in I}$ are $\mathscr{C}^{1+\varepsilon}$-diffeomorphisms satisfying the open set condition. Suppose that J is an interval. Then for every $\mathbf{p}\in(0,1)^s$ the following rigidity dichotomy holds.*

(1) If $\alpha_-=\alpha_+(=1)$ then $\Phi_{\mathbf{p}}$ is a $\mathscr{C}^{1+\varepsilon}$-diffeomorphism.
(2) If $\alpha_-<\alpha_+$ then $\Phi'_{\mathbf{p}}\equiv 0$ Leb-a.e., $\Phi_{\mathbf{p}}\in\mathscr{C}^{\alpha_-}(\overline{\mathbb{R}})$ and the set of non-differentiable points of $\Phi_{\mathbf{p}}$ has positive Hausdorff dimension.

Combining the previous theorem with the fact that $T_{\mathbf{p}}\in\mathscr{C}^{\alpha_-(\mathbf{p})}(J)$ (see Theorem 4.1) we obtain the following corollary from Lemma 6.1, Theorem 6.2 and Corollary 4.3. Recall that we have $\alpha_-(\mathbf{p})\leq\dim_H(J)$ with equality if and only if $\alpha_-=\alpha_+$.

Corollary 6.3. *Suppose that $(f_i)_{i\in I}$ are $\mathscr{C}^{1+\varepsilon}$-diffeomorphisms satisfying the open set condition. Let $\delta:=\dim_H(J)$. For $\mathbf{p}\in(0,1)^s$ we have $\alpha_-(\mathbf{p})=\alpha_+(\mathbf{p})$ if and only if $T_{\mathbf{p}}$ is $\mathscr{C}^{\delta}(\overline{\mathbb{R}})$. If $\alpha_-(\mathbf{p})=1$ then $\alpha_-=\alpha_+=1$, $J=\overline{O}$ and $T_{\mathbf{p}}$ is a $\mathscr{C}^{1+\varepsilon}$-diffeomorphism.*

7. Appendix: Contractions near infinity

The property that $(f_i)_{i\in I}$ is contracting near infinity depends on the choice of the metric d. We will show that, by modifying the $(f_i)_{i\in I}$ near infinity, we can always assume that an expanding family $(f_i)_{i\in I}$ is contracting near infinity with respect to a metric d which is strongly equivalent to the Euclidean metric on compact subsets of $\mathbb{R}$.

We consider the metric d on $\overline{\mathbb{R}}$ induced by the bijection

$$h:\overline{\mathbb{R}}\to[-1,1],\quad h(x):=\frac{x}{1+|x|}.$$

Namely, we set

$$d(x,y):=|h(x)-h(y)|.$$

Note that the metric d generates the topology of the two-point compactification of $\mathbb{R}$. Moreover, d is strongly equivalent to the Euclidean metric on compact subsets of $\mathbb{R}$.

Now suppose that $(f_i)_{i\in I}$ is expanding with expansion rate $\lambda>1$. We can take g_i, $i\in I$, such that $g_i=f_i$ in a neighbourhood of $\overline{\mathbb{R}}\setminus(V_+\cup V_-)$ and

$$g_i'(x)\to\lambda,\quad \text{as } x\to\pm\infty.$$

Then we can prove the following lemma.

Lemma 7.1. *There exist neighbourhoods $V^{\pm}$ of $\pm\infty$ such that* $\mathrm{Lip}(g_{i|V^{\pm}})<1$ *for each $i\in I$.*

Proof. Let $i\in I$. For $x,y\in\mathbb{R}$ if x,y are close to ∞ or x,y are close to $-\infty$, then we have

$$\frac{d(g_i(x),g_i(y))}{d(x,y)}=\frac{|(1+|x|)(1+|y|)|}{(1+|g_i(x)|)|(1+|g_i(y)|)|}\left|\frac{g_i(x)-g_i(y)}{x-y}\right|. \tag{7.1}$$

By our assumptions we have $\limsup_{u\to\infty} u/g_i(u) \le \lambda^{-1}$ and $\lim_{u\to\infty} g_i'(u) \to \lambda$. Hence, it follows from (7.1) that for x, y sufficiently large,

$$\frac{d(g_i(x), g_i(y))}{d(x,y)} < 1.$$

It remains to consider the case when $x = \infty$. The case when $x = -\infty$ is similar and therefore omitted. For y sufficiently large such that $g_i(y) \ge (\lambda - \eta)y$, for some η with $\lambda - \eta > 1$, we have

$$\frac{d(g_i(\infty), g_i(y))}{d(\infty, y)} = \frac{1+y}{1+g_i(y)} \le \frac{1+y}{1+(\lambda-\eta)y} < 1.$$

Hence, there exists a neighbourhood V^+ of $+\infty$ such that $\mathrm{Lip}(g_{i|V^+}) < 1$. □

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