Uniformly perfect and hereditarily non uniformly perfect analytic and conformal non-autonomous attractor sets

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ABSTRACT
Conditions are given which imply that certain non-autonomous analytic iterated function systems (NIFS's) in the complex plane $\mathbb{C}$ have uniformly perfect attractor sets, while other conditions imply the attractor is pointwise thin, and thus hereditarily non uniformly perfect. Examples are given to illustrate the main theorems, as well as to indicate how they generalize other results. Examples are also given to illustrate how possible generalizations of corresponding results for autonomous IFS's do not hold in general in this more flexible setting. Further, applications to non-autonomous Julia sets are given.

Lastly, since our definition of NIFS is in some ways more general than others found in the literature, a careful analysis is given to show when certain familiar relationships still hold, along with detailed examples showing when other relationships do not hold.

1. Introduction and statements of the main theorems

The aim of this paper is two-fold, the first is a thickness result while the second relates to a corresponding notion of thinness. In particular, we present conditions that imply the attractors in $\mathbb{C}$ of certain non-autonomous iterated function systems are uniformly perfect, and then, looking to the other extreme, give conditions for attractors to be pointwise thin (and thus hereditarily non uniformly perfect).

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Uniformly perfect sets, which are defined in Section 4, were introduced by A. F. Beardon and Ch. Pommerenke in 1978 in [1]. Such sets cannot be separated by annuli that are too large in modulus (equivalently, large ratio of outer to inner radius). Thus, uniform perfectness, in a sense, measures how thick a set is near each of its points and is related in spirit to many other notions of thickness such as Hausdorff content and dimension, logarithmic capacity and density, Hölder regularity, and positive injectivity radius for Riemann surfaces. For an excellent survey of uniform perfectness and how it relates to these and other such notions see Pommerenke [2] and Sugawa [3].

The concept of hereditarily non uniformly perfect was introduced in [4] and can be thought of as a thinness criterion for sets which is a strong version of failing to be uniformly perfect. In particular, a compact set $E \subset \mathbb{C}$ is called *hereditarily non uniformly perfect* (HNUP) if no subset of E is uniformly perfect. Often a compact set is shown to be HNUP by showing it satisfies the stronger property of *pointwise thinness* (see Definition 4.5). This is done in several examples in [4,5], and will be done in this paper each time a set is shown to be HNUP.

When the maps are all complex analytic and the IFS is *autonomous* (see Section 3), uniform perfectness results of the type we seek are found in [6]. We also note that [7] includes related results for similar systems (which require an open set condition). Certain constructions in [4] are *non-autonomous* iterated function systems shown to have uniformly perfect attractors (though those examples were not presented as attractors, but rather as Cantor-like constructions - see Example 5.1 in this paper), while other examples there are not uniformly perfect. We look to generalize those results here, and we begin by following [8] to introduce the main framework and definitions (with some key differences) of *non-autonomous iterated function systems* (NIFS's). We also note that attractors of NIFS's are often Moran-set constructions (see [9] for a good exposition of such).

Definition 1.1 (NIFS). Let (U, X) be a pair where U is a non-empty open connected subset of $\mathbb{C}$ and $X \subset U$ is compact. A *non-autonomous iterated function system* (NIFS) Φ on the pair (U, X) is given by a sequence $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$, where each $\Phi^{(j)}$ is a collection of non-constant functions $(\varphi_i^{(j)} : U \to X)_{i \in I^{(j)}}$ such that there exists $0 < s < 1$ and a metric d on U where $d(\varphi(z), \varphi(w)) \leq s d(z, w)$ for all $z, w \in X$ and all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$. We also stipulate that d induces the Euclidean topology on X. Thus this system is uniformly contracting on the forward invariant (see definition below) metric space (X, d).

Definition 1.2 (Forward Invariant). A set $\widetilde{X} \subseteq U$ is called *forward invariant under* Φ when $\varphi(\widetilde{X}) \subseteq \widetilde{X}$ for all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$.

We define a NIFS and its corresponding attractor set (see Definition 1.4) to be *analytic* (respectively, *conformal*) if all the maps are complex analytic (respectively, conformal) on U. Note that here and throughout conformal means analytic and one-to-one (globally on U, not just locally).

Important differences from [8] in the above setup are: 1) We do not impose that X have other geometric properties such as convexity or a smooth boundary. 2) The maps do not need to be conformal. In fact, they do not even need to be locally conformal. 3) In [8], the focus is on certain measures and dimension of the attractor sets, and so it is required that each $I^{(j)}$ be a finite or countably infinite index set. We, however, do not make any such assumption. 4) We do not in general impose an *open set condition*,

2

and, in fact, there can be substantial overlap in sets of the form $\varphi_a^{(j)}(X)$ and $\varphi_b^{(j)}(X)$. However, for several of our HNUP results we shall require the *Strong Separation Condition*: $\varphi_a^{(j)}(X) \cap \varphi_b^{(j)}(X) = \emptyset$, for each $j \in \mathbb{N}$ and distinct $a, b \in I^{(j)}$. 5) The main object of interest to this paper is the *analytic* NIFS, and so the condition imposed that each φ map U into X allows us, under this condition of analyticity, to take the metric d to be the hyperbolic metric on U (see Section 4).

Given an NIFS, we wish to study the limit set (or attractor) which we can define after the next definition.

Definition 1.3 (Words). For each $k \in \mathbb{N}$, we define the symbolic spaces

$$I^k := \prod_{j=1}^{k} I^{(j)} \qquad \text{and} \qquad I^\infty := \prod_{j=1}^{\infty} I^{(j)}.$$

Note that a k-tuple $(\omega_1, \ldots, \omega_k) \in I^k$ may be identified with the corresponding word $\omega_1 \ldots \omega_k$. When $\omega^* \in I^\infty$ has $\omega_j^* = \omega_j$ for $j = 1, \ldots, k$, we call ω^* an *extension* of $\omega = \omega_1 \ldots \omega_k \in I^k$.

Definition 1.4. For all $k \in \mathbb{N}$ and $\omega = \omega_1 \cdots \omega_k \in I^k$, we define $\varphi_\omega := \varphi_{\omega_1}^{(1)} \circ \cdots \circ \varphi_{\omega_k}^{(k)}$ with

$$X_\omega := \varphi_\omega(X) \qquad \text{and} \qquad X_k := \bigcup_{\omega \in I^k} X_\omega.$$

The *limit set* (or *attractor*) of Φ is defined as

$$J = J(\Phi) := \bigcap_{k=1}^{\infty} X_k.$$

Remark 1. The attractor J does not have to be compact. For example, J is not compact for the autonomous system (see Section 3) given in Example 4.3 of [6]. However, if each index set $I^{(j)}$ is finite, then each X_k is compact and hence so is J.

In order to state the main results, Theorems 1.5 and 1.6 (regarding uniform perfectness) and Theorems 1.7 and 1.9 and Corollary 1.8 (regarding pointwise thinness), we first present the following notation.

Given an NIFS $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$ on some (U, X), we note that by excluding $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(j-1)}$, the sequence $\Phi^{(j)}, \Phi^{(j+1)}, \Phi^{(j+2)}, \ldots$ also forms an NIFS (which formally would be $\widetilde{\Phi}^{(1)}, \widetilde{\Phi}^{(2)}, \widetilde{\Phi}^{(3)}, \ldots$ where each $\widetilde{\Phi}^{(k)} = \Phi^{(k+j-1)}$). The new NIFS would then induce sets as in Definition 1.4, which we denote as $X_\omega^{(j)}, X_k^{(j)}$, and $J^{(j)}$ with the superscript used to indicate the relationship to the original NIFS. In particular, for the original NIFS the sets X_k may also be denoted $X_k^{(1)}$. See Example 2.7, illustrated in Figure 1, noting that the superscript indicates the column and the subscript indicates the row where a given set resides (noting that row 0 refers to the top row).

Theorem 1.5. *Let Φ be a conformal NIFS on (U, X). Suppose*

(i) (Möbius Condition) each map in $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$ is Möbius, and

3

(ii) *(Two Point Separation Condition) there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains (not necessarily distinct) maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that for some (not necessarily distinct) $z_a, z_b \in J^{(j+1)}$ we have $|\varphi_a^{(j)}(z_a) - \varphi_b^{(j)}(z_b)| \geq \delta$, and*

(iii) *(Derivative Condition) there exists $\eta > 0$ such that for all $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$ we have $|\varphi'| \geq \eta$ on X.*

Then each $\overline{J^{(j)}}$ is uniformly perfect. Furthermore, for a given (U, X), the modulus of any annulus separating any $\overline{J^{(j)}}$ is bounded above by a constant depending only on δ and η.

Remark 2. Instead of verifying the Two Point Separation Condition as stated, it is often easier to check any of the increasingly stronger conditions:

(1) there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains at least two maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that for some $z \in J^{(j+1)}$ we have $|\varphi_a^{(j)}(z) - \varphi_b^{(j)}(z)| \geq \delta$,

(2) there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains at least two maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that for <u>all $z \in X$</u> we have $|\varphi_a^{(j)}(z) - \varphi_b^{(j)}(z)| \geq \delta$,

(3) there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains at least two maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that the images $\varphi_a^{(j)}(X)$ and $\varphi_b^{(j)}(X)$ are at least a distance δ apart.

Note that (3) is much weaker than what in the literature is often called the *Strong Separation Condition* for *finite autonomous systems*, which can be equivalently stated as such: there exists $\delta > 0$ such that for *all distinct maps* $\varphi_a^{(j)}, \varphi_b^{(j)} \in \Phi^{(j)}$, for $j \in \mathbb{N}$, the images $\varphi_a^{(j)}(X)$ and $\varphi_b^{(j)}(X)$ are at least a distance δ apart. This condition involving δ is clearly stronger than the Strong Separation Condition stated earlier just before Definition 1.3.

We also note that this Two Point Separation Condition shows that, for each $j \in \mathbb{N}$, $\text{diam}(J^{(j)}) \geq \delta$ since for any $z_a, z_b \in J^{(j+1)}$ and $\varphi_a^{(j)}, \varphi_b^{(j)} \in \Phi^{(j)}$, we have, by the inclusion proved in Remark 10, $\varphi_a^{(j)}(z_a), \varphi_b^{(j)}(z_b) \in J^{(j)}$. In the proof of Theorem 1.5, the Two Point Separation Condition is only used to obtain a uniform lower bound on $\text{diam}(J^{(j)})$.

The next result employs the notation $\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_n = \{f_1 \circ f_2 \circ \cdots \circ f_n : f_i \in \Gamma_i\}$ for families of maps $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$.

Theorem 1.6. *Suppose Φ is an analytic NIFS such that $\overline{J^{(n)}}$, for some integer $n > 1$, is uniformly perfect (e.g., when the NIFS given by $\Phi^{(n)}, \Phi^{(n+1)}, \Phi^{(n+2)}, \ldots$, satisfies the hypotheses of Theorem 1.5). Suppose also that $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(n-1)}$ is finite. Then $\overline{J(\Phi)}$ is uniformly perfect.*

We now present the main results regarding pointwise thinness, which is defined in Section 4 along with other relevant terms. Theorem 1.7 and Corollary 1.8 concern conformal NIFS's having the Strong Separation Condition, and Theorem 1.9 concerns analytic NIFS's which do not require the Strong Separation Condition but do require a certain type of separation condition. In order to present these results precisely, we must first introduce the projection map.

Remark 3 (Projection Map). Consider $\omega^* \in I^\infty$ and note that the compact sets $\varphi_{\omega_1^* \cdots \omega_n^*}(X)$ decrease with $\text{diam}_d(\varphi_{\omega_1^* \cdots \omega_n^*}(X)) \leq s^n \text{diam}_d(X) \to 0$ as $n \to \infty$. Hence

4

$\cap_{n=1}^\infty \varphi_{\omega_1^* \cdots \omega_n^*}(X)$ contains just a single point that we call $\pi(\omega^*)$. Note that $\pi(\omega^*) \in J$ since it clearly belongs to each $\varphi_{\omega_1^* \cdots \omega_n^*}(X) \subseteq X_n$. We call $\pi_\Phi : I^\infty \to J$ the *projection map*.

Further note that for any non-empty compact $\widetilde{X} \subseteq X$ that is forward invariant under Φ, we have that $\cap_{n=1}^\infty \varphi_{\omega_1^* \cdots \omega_n^*}(\widetilde{X}) = \cap_{n=1}^\infty \varphi_{\omega_1^* \cdots \omega_n^*}(X)$ since each is a singleton set with the left set being a subset of the right set. We summarize this by saying that the projection map π_Φ is independent of the choice of non-empty compact forward invariant set X.

Theorem 1.7. *Let Φ be a conformal NIFS on (U, X), with X connected, satisfying the Strong Separation Condition and the following*

Separating Annuli Condition *there exists a sequence of conformal annuli $\{A_{j_n}\}_{n \in \mathbb{N}}$, where each A_{j_n} and the bounded component of $\mathbb{C} \setminus A_{j_n}$ are in X, such that for all $n \in \mathbb{N}$ the annulus A_{j_n} separates $X_1^{(j_n)}$ where $\mathrm{mod}\, A_{j_n} \to \infty$ as $n \to \infty$.*

For each $n \in \mathbb{N}$, choose $m_n \in I^{(j_n)}$ such that the set $\varphi_{m_n}^{(j_n)}(X)$ is surrounded by A_{j_n} (which can be done since X is connected and A_{j_n} separates $X_1^{(j_n)}$), and fix $\omega = (\omega_1, \omega_2, \dots) \in I^\infty$ such that $\omega_{j_n} = m_n$ for all $n \in \mathbb{N}$. Then, J is pointwise thin at $\pi_\Phi(\omega)$.

Remark 4. The Separating Annuli Condition can be visualized in Figure 1 in Example 2.7 by considering annuli A_j of maximum modulus separating the two components in each $X_1^{(j)}$ (in row 1), noting that $\mathrm{mod}\, A_j \to \infty$ exactly when $a_j \to 0$.

Corollary 1.8. *Let Φ be a conformal NIFS on (U, X), with X connected, satisfying the Strong Separation Condition. Suppose along some subsequence j_n, we have $2 \le \#\Phi^{(j_n)} < \infty$ for all $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$,*

$$b_{j_n} = \min\{\mathrm{dist}(\varphi_i^{(j_n)}(X), \partial X) : i \in I^{(j_n)}\},$$

$$\delta_{j_n} = \min\{\mathrm{dist}(\varphi_a^{(j_n)}(X), \varphi_b^{(j_n)}(X)) : a, b \in I^{(j_n)} \ with \ a \ne b\}$$

and

$$\eta_{j_n} = \max\{\mathrm{diam}(\varphi_i^{(j_n)}(X)) : i \in I^{(j_n)}\}.$$

Suppose for some $c > 1$, we have $\delta_{j_n} \le c b_{j_n}$ for all $n \in \mathbb{N}$. Further suppose $\frac{\delta_{j_n}}{\eta_{j_n}} \to \infty$ as $n \to \infty$. Then, $J = \pi_\Phi(I^\infty)$ is pointwise thin (and thus HNUP when J is compact).

Remark 5. Since each $\delta_{j_n} \le \mathrm{diam}(X)$, we see that we may choose $c = \frac{\mathrm{diam}(X)}{\inf_n\{b_{j_n}\}} > 1$ when $\inf_n\{b_{j_n}\} > 0$.

Remark 6. Since each $\delta_{j_n} \le \mathrm{diam}(X)$, we see that for $\frac{\delta_{j_n}}{\eta_{j_n}} \to \infty$ we must have $\eta_{j_n} \to 0$. In such a situation then, Φ cannot satisfy the Derivative Condition, a critical assumption in the proof of the uniform perfectness of J in Theorem 1.5 (see Remark 17).

Remark 7. Corollary 1.8 applies much more generally when we recall that one can

combine stages in the manner described in Remark 11. Specifically, we may show $J(\Phi)$ is pointwise thin by applying Corollary 1.8 to any $\widetilde{\Phi}$ created by combining stages in Φ. This technique of combining stages is used later to analyze Example 5.2.

Theorem 1.9. *Suppose*

(i) Φ *is an analytic NIFS such that* $\overline{J^{(n)}}$, *for some integer* $n > 1$, *is pointwise thin (e.g., when the NIFS given by* $\Phi^{(n)}, \Phi^{(n+1)}, \Phi^{(n+2)}, \ldots$, *satisfies the hypotheses of Corollary 1.8 with each* $\Phi^{(j)}$ *finite), and*

(ii) $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(n-1)}$ *is finite with* $\varphi_a(\overline{J^{(n)}}) \cap \varphi_b(\overline{J^{(n)}}) = \emptyset$ *for all distinct* $\varphi_a, \varphi_b \in \widetilde{\Phi}^{(1)}$ *(e.g., when* Φ *satisfies the Strong Separation Condition), and*

(iii) *for every* $\varphi_a \in \widetilde{\Phi}^{(1)}$ *and* $z \in \overline{J^{(n)}}$, *we have that* z *is the only point of* $\overline{J^{(n)}}$ *which maps to* $\varphi_a(z)$ *(e.g., when each map in* $\widetilde{\Phi}^{(1)}$ *is conformal).*

Then $\overline{J(\Phi)}$ *is pointwise thin.*

The remainder of the paper is organized as follows. Section 2 establishes the preliminary results needed later as well as provides several examples, including Example 2.7 which graphically highlights key relationships. This section also identifies some important aspects that show how the systems of study in this paper can be more delicate than related systems found in the literature. Section 3 reviews known results for autonomous attractors (where $I^{(j)}$ and $\Phi^{(j)}$ are independent of j) and relates them to the main results for non-autonomous attractors stated in Theorems 1.5 and 1.6. Section 4 contains basic results and definitions. Section 5 presents some examples to demonstrate why the possible generalizations of results for autonomous systems (presented as Theorems A-C in Section 3) do not hold for general NIFS's. In Section 5 we show that our main results generalize Theorem 4.1 of [4]. Section 6 contains applications of Corollary 1.8 to non-autonomous Julia sets along polynomial sequences. Section 7 is then used to prove Theorems 1.5 and 1.6 on uniform perfectness, and Theorem 1.7, Corollary 1.8, and Theorem 1.9 on pointwise thinness.

2. Key preliminaries regarding the projection map, dependence on X, invariance conditions, and stage combination

We begin this section by establishing some notation.

Notation to be used throughout: Let q be a metric on $\mathbb{C}$. For a set $F \subseteq \mathbb{C}$, we define its *diameter* to be $\operatorname{diam}_q F = \sup\{q(z, w) : z, w \in F\}$ and *ϵ-ball about F* to be $B_q(F, \epsilon) = \{z : \operatorname{dist}_q(z, F) < \epsilon\}$ where $\operatorname{dist}_q(z, F) = \inf\{q(z, w) : w \in F\}$. Also, for $w \in \mathbb{C}$ and $r > 0$ we define the *disk* and *circle*, respectively, by $\Delta_q(w, r) = \{z : q(z, w) < r\}$ and $C_q(w, r) = \{z : q(z, w) = r\}$. If no metric is noted, then it is assumed that the metric is the Euclidean metric. Lastly, the open unit disk in $\mathbb{C}$ is denoted $\mathbb{D}$.

Remark 8 (Pieces of X_k)**.** The limit set $J = \cap_{k=1}^{\infty} X_k$ is a decreasing intersection of the X_k, but an important facet of each X_k is that it is the union of what we call the *pieces* of X_k, each of which must contain both a limit point and a fixed point. More precisely, note that for any $k \in \mathbb{N}$ and $\omega = \omega_1 \cdots \omega_k \in I^k$, we have that the *piece* $\varphi_\omega(X)$ of X_k, for which $\operatorname{diam}_d(\varphi_\omega(X)) \leq s^k \operatorname{diam}_d(X)$, contains both the fixed point of the contraction φ_ω and the point $\pi_\Phi(\omega^*) \in J$ for any extension $\omega^* \in I^\infty$ of ω. Note also that the pieces of X_k are not necessarily components of X_k since the pieces may overlap in general.

6

In the NIFS systems studied in [8] (see Definition and Lemma 2.4 of [8], which makes key use of the open set condition - something we do not impose here), it must be the case that $\pi_\Phi(I^\infty) = J$. We do not necessarily have this in all cases (see Example 2.3), but we do note that the additional assumption of the Strong Separation Condition would allow the proof in [8] to apply. In all cases, however, we do have the following result.

Lemma 2.1. *Let $J'(\Phi) = \{z : \phi_\omega(z) = z$ for some ω in some $I^k\}$ where Φ is a NIFS on (U, X). Then $J(\Phi) \subseteq \overline{J'(\Phi)}$, and hence $\overline{J(\Phi)} \subseteq \overline{J'(\Phi)}$. Also,*

$$\overline{J(\Phi)} = \overline{\pi_\Phi(I^\infty)},$$

and so, if $\pi_\Phi(I^\infty)$ is compact, then $J(\Phi) = \pi_\Phi(I^\infty)$.

We note that in the non-autonomous case, unlike in the autonomous case (see Claim 1), J' does not necessarily have to be a subset of J, or even of $\overline{J}$. See Example 5.2.

Proof. Let $z \in J$ and $\delta > 0$. Choose k such that $s^k \text{diam}_d(X) < \delta$. Since $z \in J \subseteq X_k$, there exists $\omega \in I^k$ such that $z \in \varphi_\omega(X)$. Extend ω to any $\omega^* \in I^\infty$ and note that, as stated in Remark 8, $\varphi_\omega(X)$ contains both the fixed point of the contraction φ_ω and the point $\pi_\Phi(\omega^*) \in J$. Since $\varphi_\omega(X) \subseteq \Delta_d(z, s^k \text{diam}_d(X)) \subseteq \Delta_d(z, \delta)$, we conclude $J \subseteq \overline{J'(\Phi)} \cap \overline{\pi_\Phi(I^\infty)}$. This and the definition of π_Φ yield that $\overline{J} \subseteq \overline{\pi_\Phi(I^\infty)} \subseteq \overline{J}$.

The final statement follows since if $\pi_\Phi(I^\infty)$ is compact, we have $J(\Phi) \subseteq \overline{J(\Phi)} = \overline{\pi_\Phi(I^\infty)} = \pi_\Phi(I^\infty) \subseteq J(\Phi)$. $\qquad\square$

In certain examples, it is convenient to change the set X to a more convenient forward invariant compact set. The following result shows that such a change to X, though it may affect J (see Example 2.4), will not affect $\overline{J}$, the central object of study for this paper.

Lemma 2.2. *Let $\widetilde{X} \neq \emptyset$ be a compact subset of X that is forward invariant under NIFS Φ on (U, X). Then, calling $\widetilde{X}_k := \bigcup_{\omega \in I^k} \varphi_\omega(\widetilde{X})$, we have*

$$\overline{J(\Phi)} = \overline{\bigcap_{k=1}^\infty X_k} = \overline{\bigcap_{k=1}^\infty \widetilde{X}_k}.$$

Hence, if each $\widetilde{X}_k$ is compact, then $J(\Phi) = \bigcap_{k=1}^\infty X_k = \bigcap_{k=1}^\infty \widetilde{X}_k$.

Proof. Since, as was noted in Remark 3, the projection map π_Φ is independent of the choice of non-empty compact forward invariant set X, the first result follows immediately from Lemma 2.1.

When each $\widetilde{X}_k$ is compact, the second result follows since $J(\Phi) \subseteq \overline{J(\Phi)} = \overline{\bigcap_{k=1}^\infty X_k} = \overline{\bigcap_{k=1}^\infty \widetilde{X}_k} = \bigcap_{k=1}^\infty \widetilde{X}_k \subseteq \bigcap_{k=1}^\infty X_k = J(\Phi)$. $\qquad\square$

Example 2.3 (Projection map $\pi_\Phi : I^\infty \to J$ not onto)**.** Let $X = [0, 1]$ be the unit interval. Let $\Phi^{(1)} = \{f_1, f_2, f_3, \dots\}$ where $f_n(z) = \frac{z}{3} + e_n$ with $e_n = \frac{1}{3} - \frac{1}{3^n}$. Note that $e_1 = 0$ and $0 < e_n < \frac{1}{3}$ for all $n \geq 2$. Let $\Phi^{(k)} = \{f_1\}$ for all $k \geq 2$.

Technically speaking, one should first establish an open set $U \subseteq \mathbb{C}$ (e.g., $\Delta(0, 10)$) and corresponding compact subset X (e.g., $\overline{\Delta(0, 9)}$) to satisfy the NIFS condition that each function map U into X. And then afterwards use Lemma 2.2 to replace X by

the forward invariant interval $[0, 1]$ without altering the limit set J. However, in later examples we forgo such details leaving it for the reader to quickly check that such a procedure can be validly executed.

We now show $\frac{1}{3} \in J \setminus \pi_\Phi(I^\infty)$. Since, for each $n \in \mathbb{N}$, we have $\frac{1}{3} \in [e_n, \frac{1}{3}] = [e_n, \frac{1}{3^n} + e_n] = f_n \circ f_1^{n-1}(X) \subseteq X_n$, we see $\frac{1}{3} \in J$. However, for each $\omega \in I^\infty$ there must be some $f_n \in \Phi^{(1)}$ such that $\{\pi_\Phi(\omega)\} = \cap_{k=1}^\infty f_n \circ f_1^{k-1}(X) = \cap_{k=1}^\infty [e_n, \frac{1}{3^k} + e_n] = \{e_n\} \neq \{\frac{1}{3}\}$. Hence $\pi_\Phi(I^\infty) = \{e_n : n \in \mathbb{N}\}$, and so $\pi_\Phi(I^\infty) \neq J = \{e_n : n \in \mathbb{N}\} \cup \{1/3\}$, where the equality follows from Lemma 2.1.

Example 2.4 (J depends on X). Let $X = [-1, 1]$ and $\widetilde{X} = [0, 1]$. For each $n \in \mathbb{N}$, set $z_n = \frac{1}{2^n - 1} > 0$ and $f_n(z) = \frac{1}{2}(z - z_n) + z_n$. Clearly, each of X and $\widetilde{X}$ is forward invariant under each contraction f_n. We consider the (autonomous) system generated where each $\Phi^{(k)} = \{f_n : n \in \mathbb{N}\}$. Considering $\widetilde{X}_k$ given as in Lemma 2.2, it is clear that $0 \notin \widetilde{X}_1$ since, for all n, we see $0 \notin [\frac{z_n}{2}, \frac{1 + z_n}{2}] = f_n(\widetilde{X})$. However, for all $n \in \mathbb{N}$, since the n-th iterate $f_n^n(z) = \frac{1}{2^n}(z - z_n) + z_n$, we see $0 \in [0, f_n^n(1)] = [f_n^n(-1), f_n^n(1)] = f_n^n(X) \subseteq X_n$. Hence $0 \in \cap_{n=1}^\infty X_n \setminus \cap_{n=1}^\infty \widetilde{X}_n$, showing that J does depend on the choice of forward invariant non-empty compact set X (something which cannot happen in the NIFS systems studied in [8] where, as noted, $J = \pi_\Phi(I^\infty)$ must hold).

Remark 9 (Invariance Condition). Note that for any $j \geq 1$ and $k \geq 0$, we unpack the relevant definitions (defining each $X_0^{(j)} = X$) to see the following invariance condition

$$\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(X_k^{(j+1)}) = X_{k+1}^{(j)}, \tag{1}$$

which is illustrated in Figure 1 as a way of relating the diagonally adjacent sets $X_{k+1}^{(j)}$ and $X_k^{(j+1)}$.

Remark 10. Letting $k \to \infty$ in the invariance condition (1) leads one to wonder if we must always have $\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) = J^{(j)}$. While this is not true in general, we do always get the inclusion

$$\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) = \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(\bigcap_{k=1}^\infty X_k^{(j+1)}) \subseteq \bigcup_{i \in I^{(j)}} \bigcap_{k=1}^\infty \varphi_i^{(j)}(X_k^{(j+1)})$$

$$\subseteq \bigcap_{k=1}^\infty \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(X_k^{(j+1)}) = \bigcap_{k=1}^\infty X_{k+1}^{(j)} = J^{(j)}.$$

Now consider Example 2.3 to see that equality above does not follow. Since $J^{(2)} = \{0\}$, $\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(J^{(2)}) = \{e_n : n \in \mathbb{N}\} \neq \{e_n : n \in \mathbb{N}\} \cup \{\frac{1}{3}\} = J^{(1)}$. Additionally, $\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}\left(\overline{J^{(2)}}\right) \neq \overline{J^{(1)}}$.

Additional hypotheses, however, lead to the following result.

Lemma 2.5. *Let Φ be a NIFS on (U, X) and let $j \in \mathbb{N}$. When $\Phi^{(j)}$ is finite, we have*

$$\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}\left(\overline{J^{(j+1)}}\right) = \overline{J^{(j)}}.$$

8

Hence, when $\Phi^{(j)}$ is finite and $J^{(j+1)}$ is compact (e.g., when all $\Phi^{(k)}$, for $k \geq j$, are finite), we see that $\bigcup_{i \in I^{(j)}} \varphi_i^{(j)} \left(J^{(j+1)} \right) = J^{(j)}$.

The following result, which can easily be shown, will be used in the proof.

Lemma 2.6. *Suppose $f : X \to Y$ is continuous and compact sets $A_n \subseteq X$ form a decreasing sequence. Then $f(\cap_{n=1}^{\infty} A_n) = \cap_{n=1}^{\infty} f(A_n)$.*

Proof. To prove the first statement it suffices to consider $j = 1$. Letting $I_1^{\infty} = \prod_{k=1}^{\infty} I^{(k)}$ and $I_2^{\infty} = \prod_{k=2}^{\infty} I^{(k)}$, we define the respective projection maps $\pi_1 : I_1^{\infty} \to J^{(1)}$ and $\pi_2 : I_2^{\infty} \to J^{(2)}$. We first note that

$$\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{\infty})) = \pi_1(I_1^{\infty}),$$

since

$$
\begin{aligned}
\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{\infty})) &= \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}\Big(\bigcup_{\omega \in I_2^{\infty}} \{\pi_2(\omega)\}\Big) = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}\Big(\bigcup_{\omega \in I_2^{\infty}} \bigcap_{n=2}^{\infty} \varphi_{\omega_2 \cdots \omega_n}(X)\Big) \\
&= \bigcup_{i \in I^{(1)}} \bigcup_{\omega \in I_2^{\infty}} \varphi_i^{(1)}\Big(\bigcap_{n=2}^{\infty} \varphi_{\omega_2 \cdots \omega_n}(X)\Big) \\
&= \bigcup_{i \in I^{(1)}} \bigcup_{\omega \in I_2^{\infty}} \bigcap_{n=2}^{\infty} \varphi_i^{(1)}(\varphi_{\omega_2 \cdots \omega_n}(X)) \\
&= \bigcup_{i \in I^{(1)}} \bigcup_{\omega \in I_2^{\infty}} \bigcap_{n=1}^{\infty} \varphi_{i \cdot \omega_2 \cdots \omega_n}(X) = \bigcup_{\omega^* \in I_1^{\infty}} \{\pi_1(\omega^*)\} = \pi_1(I_1^{\infty}),
\end{aligned}
$$

where Lemma 2.6 was used with regard to $\varphi_i^{(1)}$ and the decreasing compact sets $\varphi_{\omega_2 \cdots \omega_n}(X)$.

Then, using Lemma 2.1, we see

$$
\begin{aligned}
\overline{J^{(1)}} &= \overline{\pi_1(I_1^{\infty})} = \overline{\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{\infty}))} = \bigcup_{i \in I^{(1)}} \overline{\varphi_i^{(1)}(\pi_2(I_2^{\infty}))} \\
&= \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}\left(\overline{\pi_2(I_2^{\infty})} \right) = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}\left(\overline{J^{(2)}} \right),
\end{aligned}
$$

where we used the facts that the union is finite, each $\varphi_i^{(1)}$ is continuous, and the set $\overline{\pi_2(I_2^{\infty})}$ is compact.

The final statement of the lemma follows since, if $\Phi^{(j)}$ is finite and $J^{(j+1)}$ is compact, then $J^{(j)} \subseteq \overline{J^{(j)}} = \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}\left(\overline{J^{(j+1)}} \right) = \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}\left(J^{(j+1)} \right) \subseteq J^{(j)}$, where the last inclusion is justified by Remark 10. $\qquad \square$

Example 2.7. Let $X = [0, 1]$ denote the closed unit interval. Consider a sequence (a_j) such that each $0 < a_j \leq 1/3$, and define maps $\varphi_1^{(j)}(z) = a_j z$ and $\varphi_2^{(j)}(z) = a_j(z-1)+1$. Then the families of maps $\Phi^{(j)} = \{\varphi_1^{(j)}, \varphi_2^{(j)}\}$ define an NIFS. See Figure 1.
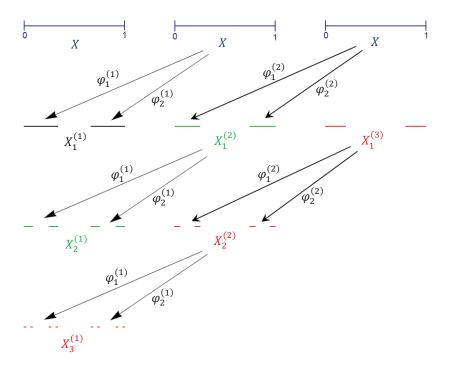
Figure 1. Table illustrating Example 2.7 with $a_1 = \frac{1}{3}, a_2 = \frac{1}{4}$, and $a_3 = \frac{1}{5}$. Note that sets in each column decrease down to the corresponding limit set, i.e., for each $j \in \mathbb{N}$ we have $\cap_{k=1}^{\infty} X_k^{(j)} = J^{(j)}$. Also, note that diagonally adjacent sets $X_{k+1}^{(j)}$ and $X_k^{(j+1)}$ are related by the invariance condition (1) in Remark 9.

Remark 11 (Combining Stages). It will be useful later to analyze a limit set of some NIFS Φ by first combining stages. Here we present what this means, in particular, showing that this does not alter the limit set. Recall, for families of maps $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, we have $\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_n = \{f_1 \circ f_2 \circ \cdots \circ f_n : f_i \in \Gamma_i\}$.

Given an NIFS $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$ on some (U, X), we can create a new NIFS by combining finite strings of stages as follows. Consider any strictly increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers and define a new NIFS $\widetilde{\Phi}$ by $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(k_1)}$, $\widetilde{\Phi}^{(2)} = \Phi^{(k_1+1)} \circ \cdots \circ \Phi^{(k_2)}$, and, in general for $n > 1$, $\widetilde{\Phi}^{(n)} = \Phi^{(k_{n-1}+1)} \circ \cdots \circ \Phi^{(k_n)}$.

Notice that $\widetilde{\Phi}$ inherits all the defining properties of an NIFS from Φ. Furthermore, $J(\widetilde{\Phi}) = \bigcap_{n=1}^{\infty} X_{k_n} = \bigcap_{k=1}^{\infty} X_k = J(\Phi)$, since the sets X_k are decreasing.

3. Review of Autonomous Attractors

In this section we review known results for autonomous attractors and relate them to the main results for non-autonomous attractors stated in Theorems 1.5 and 1.6.

The system Φ in Definition 1.1 is called *autonomous* (and thus just called an IFS) if $I^{(j)}$ and $\Phi^{(j)}$ are independent of j, i.e., each $\Phi^{(j)} = \{g_i : i \in I\}$ for some index set I. In such an instance we use the notation A for the attractor instead of J in order to give a notational reminder that we are in a very special (and previously well-studied) case. For such an autonomous system, we let $G = \langle g_i : i \in I \rangle$ denote the set of all finite compositions of generating maps $\{g_i : i \in I\}$, and, following [6], simply say $G = \langle g_i : i \in I \rangle$ is an IFS on (U, X).

Claim 1. When Φ is autonomous, the attractor set $A = J$ given in Definition 1.4 satisfies $A \supseteq A'$ and $\overline{A} = \overline{A'}$, the closure of A' in the Euclidean topology (equivalently given by the metric d), where $A' = A'(G) := \{z : \text{there exists } g \in G \text{ such that } g(z) = z\}$ is the set of (attracting) fixed points of G.

Note that in [6] the attractor set was *defined* to be $\overline{A'}$ and not defined in terms of X_k as in Definition 1.4. This claim, however, shows that the closures of the sets given by the two definitions yield the same set.

Proof. Let $z \in A'$. Since the system is autonomous, there exist some $k \in \mathbb{N}$ and $\omega \in I^k$ such that $\phi_\omega(z) = z$. Clearly then for each n we see that $z \in \phi_\omega^n(X) \in X_{kn}$, where ϕ_ω^n denotes the nth iterate of ϕ_ω (note that the autonomous condition is used here). Hence $z \in \cap_{n=1}^\infty X_{kn} = \cap_{k=1}^\infty X_k = J = A$. Thus $A' \subseteq A$, and so $\overline{A'} \subseteq \overline{A}$.

The reverse inclusion follows from Lemma 2.1. $\qquad\square$

If each $\Phi^{(j)} = \{g_1, \ldots, g_N\}$, a situation we call the *finite autonomous* case, then the attractor A is the unique non-empty compact subset of U that has the *self-similarity* property given by

$$A = \bigcup_{i=1}^{N} g_i(A) \tag{2}$$

(see [10], p. 724). We note that in this finite autonomous case, the sets $X_k^{(j)}$, and $J^{(j)}$ are all independent of j (in Example 2.7 illustrated in Figure 1 this would amount to sets across rows being identical because $a_1 = a_2 = a_3 = \ldots$). Furthermore, the invariance shown in Remark 9 then becomes $\cup_{i=1}^N g_i(X_k) = X_{k+1}$, which by taking the limit as $k \to \infty$ in a suitable space produces (2) (see [10] or apply Lemma 2.5).

Remark 12. We also point out that in [11,12] the limit set J of a *conformal* IFS is defined a bit differently, but with a clear connection to our definition. See [11,12] for a discussion on the Hausdorff dimension, packing dimension, and other properties of limit sets of their conformal IFS's.

In [13] certain *autonomous conformal* attractor sets are shown to be uniformly perfect, when the generating maps are Möbius. Then in [6] a collection of results regarding uniform perfectness are given for *autonomous analytic* attractor sets. The motivation for the current paper is to explore to what degree, if any, these results generalize to the non-autonomous case. Hence we first state the major results from [6].

Theorem A (Corollary 1.1 in [6])**.** Let $G = \langle g_i : i \in I \rangle$ be an analytic IFS on (U, X) such that there exists $\eta > 0$ where $|g_i'| \geq \eta$ on A for all $i \in I$. If A has infinitely many points, then A is uniformly perfect.

Theorem B (Corollary 1.2 in [6])**.** Let $G = \langle g_i : i \in I \rangle$ be a conformal IFS on (U, X) such that there exists $\eta > 0$ where $|g_i'| \geq \eta$ on A for all $i \in I$. If A contains more than one point, then A is uniformly perfect.

Theorem C (Corollary 1.3 in [6])**.** Let $G = \langle g_1, \ldots, g_N \rangle$ be a conformal IFS on (U, X). If A contains more than one point, then A is uniformly perfect.

The proofs of Theorems A-C in [6], which consider only autonomous systems, heavily rely on the facts (i) $A' \subseteq A$, and (ii) A is forward invariant under G, i.e., for every

$a \in A$ and $g \in G$ we have $g(a) \in A$ (Lemma 2.2 in [6]). The main complicating features of the non-autonomous systems we wish to consider in this paper are that these properties do not hold or generalize in a way that allows for the techniques in [6] to be easily adapted to such more general systems (see Example 5.2 and Remark 18). Here, however, we do prove Theorem 1.5 regarding conformal NIFS's and Theorem 1.6 regarding analytic NIFS's.

Section 5 presents some examples to demonstrate why the possible generalizations of Theorems A-C do not hold for general NIFS's, in particular, showing that both (i) and (ii) can fail.

4. Definitions and basic facts

The main object of interest to this paper is the *analytic* NIFS. This allows us, via the next result used similarly in [6], to employ the hyperbolic metric in the definition of NIFS. In particular, any sequence $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$, such that each $\Phi^{(j)}$ is a collection of non-constant *complex analytic* functions $(\varphi_i^{(j)} : U \to X)_{i \in I^{(j)}}$, where each function maps the non-empty open connected set $U \subset \mathbb{C}$ into a compact set $X \subset U$, will automatically be uniformly contracting with respect to the hyperbolic metric on U. Note that $U \subseteq \mathbb{C}$ must support a hyperbolic metric since U cannot be the plane or punctured plane else the image of U under a non-constant analytic map would have to be dense in $\mathbb{C}$.

Lemma 4.1 (Lemma 2.1 of [6]). *If the analytic function φ maps an open connected set $U \subset \mathbb{C}$ into a compact set $X \subset U$, then there exists $0 < s < 1$, which depends on U and X only, such that $d(\varphi(z), \varphi(w)) \leq s d(z, w)$ for all $z, w \in X$ where d is the hyperbolic metric defined on U.*

Remark 13. Let Φ be an analytic NIFS on (U, X). Note that, for each $x \in X$, the hyperbolic disk $\Delta_d(x, 2 \cdot \text{diam}(X)) \subset U$ contains X and is connected (being the continuous image of a connected hyperbolic disk in $\Delta(0, 1)$). Hence, $\widetilde{X} = \overline{\bigcup_{x \in X} \Delta_d(x, 2 \cdot \text{diam}(X))}$ is connected (and compact). Further, since X is forward invariant under Φ, then so is $\widetilde{X}$ since analytic maps cannot increase hyperbolic distances. We note then that Lemma 2.2 (with the roles of X and $\widetilde{X}$ reversed) allows us to replace X by the connected $\widetilde{X}$ without altering $\overline{J}$.

We call a doubly connected domain A in $\mathbb{C}$ that can be conformally mapped onto a true (round) annulus $\text{Ann}(w; r, R) = \{z : r < |z - w| < R\}$, for some $0 < r < R$, a *conformal annulus* with the *modulus* of A given by $\text{mod } A = \log(R/r)$, noting that R/r is uniquely determined by A (see, e.g., the version of the Riemann mapping theorem for multiply connected domains in [14]).

Definition 4.2. A conformal annulus A is said to *separate* a set $F \subset \mathbb{C}$ if $F \cap A = \emptyset$ and F intersects both components of $\mathbb{C} \setminus A$.

Definition 4.3. A compact subset $F \subset \mathbb{C}$ with two or more points is *uniformly perfect* if there exists a uniform upper bound on the modulus of each conformal annulus which separates F.

Remark 14. Because of the following well-known lemma (see, e.g., Theorem 2.1 of [15]), we can equivalently characterize uniformly perfect sets in terms of only true

annuli: A compact subset $F \subset \mathbb{C}$ with two or more points is *uniformly perfect* if there exists a uniform upper bound on the modulus of each *true* annulus (centered at a point in F, if we choose) which separates F.

Lemma 4.4. *Any conformal annulus $A \subset \mathbb{C}$ of sufficiently large modulus contains an essential true annulus B (i.e., B separates the boundary of A) with $\mathrm{mod}\, A = \mathrm{mod}\, B + O(1)$. Since, for any $R > 3r$ and any $w' \in \overline{\Delta(w,r)}$, the true annulus $\mathrm{Ann}(w'; 2r, R-r)$ is an essential annulus of $\mathrm{Ann}(w; r, R)$, we may choose B to be centered at any given point in the bounded component of $\mathbb{C} \setminus A$.*

Remark 15. For the case when the conformal annulus $A \subset \overline{\mathbb{C}}$ contains infinity the above lemma can be modified to read as: Any conformal annulus $A \subset \overline{\mathbb{C}}$ of sufficiently large modulus contains an essential true annulus B (i.e., B separates the boundary of A) with $\mathrm{mod}\, A = 2\mathrm{mod}\, B + O(1)$. To see this note that A contains two disjoint essential conformal annuli A' and A'' each with half the modulus of A, at most one of which, say, A'' can contain infinity. This can be observed by considering mapping A conformally onto $\mathrm{Ann}(0; 1, R)$, and then taking the preimages of $\mathrm{Ann}(0; 1, \sqrt{R})$ and $\mathrm{Ann}(0; \sqrt{R}, R)$ inside of A. By applying Lemma 4.4 to A' we can obtain our desired result.

Recall that a compact set $E \subset \mathbb{C}$ is called *hereditarily non uniformly perfect* (HNUP) if no subset of E is uniformly perfect. Often a set is shown to be HNUP by showing it satisfies the following stronger property of *pointwise thinness*. This is done in several examples in [4], and will be done in Example 5.2. Also, certain non-autonomous Julia sets in [5] and in Theorem 6.1 are shown to be HNUP this way (where it is worth noting that the Julia sets constructed are limit sets of conformal NIFS's).

Definition 4.5. A set $E \subset \mathbb{C}$ is *pointwise thin at $z \in E$* if there exists a sequence of conformal annuli A_n each of which separates E, has z in the bounded component of its complement, and such that $\mathrm{mod}\, A_n \to +\infty$ while the Euclidean diameter of A_n tends to zero. A set $E \subset \mathbb{C}$ is called *pointwise thin* when it is pointwise thin at each of its points.

Note that any pointwise thin compact set is HNUP since none of its points can lie in a uniformly perfect subset. Also note that if E is pointwise thin, then $\overline{E}$ is pointwise thin at each point of E, but not necessarily pointwise thin at each point of $\overline{E}$. Furthermore, the property of pointwise thinness is strictly stronger than the property of being HNUP. The following example illustrates both points.

Example 4.6 (Closure of pointwise thin is not pointwise thin, pointwise thin is strictly stronger than HNUP)**.** The set $E = \{2^{-n} : n \in \mathbb{N}\}$ is trivially pointwise thin, but its closure $\overline{E}$ is not pointwise thin at 0 since the reader can check that the modulus of any round annulus separating $\overline{E}$ and containing 0 must be bounded by $\log 2$. Clearly, $\overline{E}$ is HNUP since it is countable.

Lemma 4.7. *Suppose $A = \mathrm{Ann}(z; r, R)$, for some $z \in \mathbb{C}$ and $0 < r < R$, is a true annulus separating J, where $J = \cap_{k=1}^{\infty} X_k$ is the attractor of some NIFS Φ. Fix $0 < \delta < \frac{R-r}{2}$. Then the annulus $B = \mathrm{Ann}(z; r + \delta, R - \delta) \subset A$ separates some X_k. Hence, given any $0 < \epsilon < \mathrm{mod}\, A$, we can choose $\delta > 0$ such that $\mathrm{mod}\, B = \log(\frac{R-\delta}{r+\delta}) = \log(\frac{R}{r}) - \epsilon = \mathrm{mod}\, A - \epsilon$, where B separates some X_k.*

Proof. Since A separates J and B is an essential subannulus of A, both components

of $\mathbb{C} \setminus B$ must meet J, and therefore must meet each $X_k \supseteq J$. We complete the proof by showing that $B \cap X_k = \emptyset$ for some k. Suppose not. Now fix k and choose $z_k \in X_k \cap B$. Hence there exists $\omega \in I^k$ such that $z_k \in \varphi_\omega(X)$. Since $\operatorname{diam}_d(\varphi_\omega(X)) \leq s^k \operatorname{diam}_d(X)$ (see Remark 8), we have that $\varphi_\omega(X) \subseteq \Delta_d(z_k, s^k \operatorname{diam}_d(X)) \subset A$ for k sufficiently large (since $z_k \in \overline{B} \subset A$ and d generates the Euclidean topology on X). Since $\varphi_\omega(X)$ must contain a point of J (see Remark 8), we see that $A \cap J \neq \emptyset$ and thus A does not separate J, which is a contradiction. $\qquad \square$

Lemma 4.8. *Suppose $A = \operatorname{Ann}(z; r, R)$, for some $z \in \mathbb{C}$ and $0 < r < R$, separates $E \subseteq X \subset \mathbb{C}$ where $\operatorname{diam}(X) < \infty$ and $R \geq 2 \cdot \operatorname{diam}(X)$. Then $\frac{R}{r} \leq 2$.*

Proof. Since A separates E, there exist $x_1, x_2 \in E$ with $|x_1 - z| \geq R$ and $|x_2 - z| \leq r$. Hence $2 \cdot \operatorname{diam}(X) - r \leq R - r \leq |x_1 - x_2| \leq \operatorname{diam}(E) \leq \operatorname{diam}(X)$, which gives that $\operatorname{diam}(X) \leq r$. Again using that $R - r \leq \operatorname{diam}(X)$, we see that $\frac{R-r}{r} \leq \frac{\operatorname{diam}(X)}{r} \leq 1$, which gives $\frac{R}{r} \leq 2$ as desired. $\qquad \square$

The following is a result that seems to be well understood by many but, since a reference could not be found, we provide a proof here.

Proposition 4.9. *Let $f : U \to \mathbb{C}$ be non-constant and analytic on open connected $U \subset \mathbb{C}$. Suppose that $E \subset U$ is uniformly perfect. Then $f(E)$ is uniformly perfect.*

This result follows from the fact that *locally* non-constant analytic maps are either conformal or behave like $z \mapsto z^k$ for some $k \in \mathbb{N}$, which can distort the modulus of an annulus by at most a factor of k.

Proof. The local behavior of non-constant analytic maps clearly implies that, since E is perfect, so is $f(E)$. We now suppose towards a contradiction that $f(E)$ is not uniformly perfect. Hence there exists true annuli $A_n = \operatorname{Ann}(w_n; r_n, R_n)$ which separate $f(E)$ with $R_n/r_n \to \infty$.

By Lemma 4.4, we may assume each $w_n \in f(E)$. Since $f(E)$ is perfect, it follows that $R_n \to 0$ (see, e.g., Lemma 2.7 of [6]).

By compactness of both $f(E)$ and E, and passing to a subsequence if necessary, we may assume there exists $w_0 \in f(E)$ such that $w_n \to w_0$ and $z_0, z_n \in E$ such that $z_n \to z_0$ with each $f(z_n) = w_n$.

Suppose $f'(z_0) \neq 0$. Thus there exists a local branch h of f^{-1} defined on some neighborhood of w_0. Hence, the conformal annuli $h(A_n)$, for large n, must then separate E, which is a contradiction since E is uniformly perfect and $\operatorname{mod} h(A_n) = \operatorname{mod} A_n \to \infty$.

Now suppose $f'(z_0) = 0$, and choose k such that f maps z_0 to w_0 with multiplicity $k > 1$. By pre- and post- composing with translations, we may assume $z_0 = w_0 = 0$, and so there exists a conformal map g defined on a neighborhood of 0 such that $gfg^{-1}(z) = z^k$ (see, e.g., Theorem 6.10.1 of [16]). It suffices to consider two cases: Case(i) Each A_n surrounds $w_0 = 0$, and Case (ii) No A_n surrounds $w_0 = 0$.

Case (i): From each conformal annulus $g(A_n)$ of large modulus (and so for all large n), we apply Lemma 4.4 to extract an essential true annulus $B_n = \operatorname{Ann}(0; s_n, S_n) \subseteq g(A_n)$ of modulus $\operatorname{mod} B_n = \operatorname{mod} A_n - K$, for some fixed $K > 0$. Since $A'_n = \operatorname{Ann}(0; s_n^{1/k}, S_n^{1/k})$ maps by $z \mapsto z^k$ onto $B_n \subseteq g(A_n)$, we must have that each conformal annulus $g^{-1}(A'_n)$ surrounds $z_0 = 0$ and $\operatorname{mod} g^{-1}(A'_n) = \operatorname{mod}(A'_n) = \frac{1}{k} \operatorname{mod} B_n \to \infty$, which is a contradiction since each $g^{-1}(A'_n)$ separates the uniformly perfect set E.

Case (ii): Again for each conformal annulus $g(A_n)$ of large modulus (and so for all large n), we apply Lemma 4.4 to extract an essential true annulus $B_n = \text{Ann}(g(w_n); s_n, S_n) \subseteq g(A_n)$ of modulus $\text{mod } B_n = \text{mod } A_n - K$, for some fixed $K > 0$. Note that no $\Delta(g(w_n), S_n)$ contains 0. Hence, the map $z \mapsto z^k$ has k well-defined inverse branches on B_n, one of which must map B_n to a conformal annulus B'_n surrounding $g(z_n)$. And so, $g^{-1}(B'_n)$ is a conformal annulus surrounding z_n and separating E, with modulus $\text{mod } g^{-1}(B'_n) = \text{mod } B'_n = \text{mod } B_n = \text{mod } A_n - K$. This is a contradiction since E is uniformly perfect and $\text{mod } A_n \to \infty$. □

The following result can be proven using the style of argument used to prove Proposition 4.9, and so we omit the details.

Proposition 4.10. *Let $f : U \to \mathbb{C}$ be non-constant and analytic on open connected $U \subset \mathbb{C}$. Suppose that compact $E \subset U$ is pointwise thin at $z \in E$ and z is the only point of E which maps to $f(z)$. Then $f(E)$ is pointwise thin at $f(z) \in f(E)$.*

The following example shows that in the above hypothesis it is critical that z is the only point of E which maps to $f(z)$.

Example 4.11 (Analytic image of pointwise thin is not pointwise thin). Letting each $x_n = 2^{-n^2}$, we set $E_1 = \cup_{n \geq 0}[x_{2n+1}, x_{2n}] \cup \{0\}$ and $E_2 = \cup_{n \geq 0}[x_{2n+2}, x_{2n+1}] \cup \{0\}$. By considering the annuli $\text{Ann}(0; x_{2n+2}, x_{2n+1})$, one can show that E_1 is pointwise thin at 0. Similarly, E_2 can be shown to be pointwise thin at 0. Note also that $E_1 \cup E_2 = [0, 1]$. Hence both $F_1 = E_1 \cap (\{2^{-n} : n \in \mathbb{N}\} \cup \{0\})$ and $F_2 = E_2 \cap (\{2^{-n} : n \in \mathbb{N}\} \cup \{0\})$ are pointwise thin (at each point). Using the principal branch to define $g(z) = \sqrt{z+1}$, we see by Proposition 4.10 that $g(F_1)$ is pointwise thin. Similarly, we consider $-g(z) = -\sqrt{z+1}$ to show that $-g(F_2)$ is pointwise thin. Note that g and $-g$ are branches of the inverse of $f(z) = z^2 - 1$. Letting $E = g(F_1) \cup -g(F_2)$, we then see that E is compact and pointwise thin (and thus pointwise thin at both 1 and -1), but $f(E) = F_1 \cup F_2 = \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$ is not pointwise thin at $f(1) = f(-1) = 0$ (as noted in Example 4.6).

We close this section with a remark that relates to Proposition 4.10.

Remark 16. In [17], Shiga discusses the quasiconformal equivalence of Cantor sets which appear as limits sets of non-autonomous IFSs and Julia sets of rational maps. Many complex analysts are interested in the complements of various kind of Cantor sets, since the complements of Cantor sets are good examples of Riemann surfaces of infinite type.

We note the following:

(a) If a compact set K in the plane is uniformly perfect and a compact set L in the plane is not uniformly perfect, then there is no quasiconformal map $g : \mathbb{C} \to \mathbb{C}$ such that $g(K) = L$.

(b) It is an open problem whether there are multiple quasiconformal equivalence classes of pointwise thin limit sets of non-autonomous IFSs.

5. Examples

In this section we provide examples to show that possible generalizations of Theorems A-C of Section 3 to the non-autonomous case do not hold. Specifically, we show that none of the following Statements 1-3 hold. Examples to illustrate Theorem 1.5

15

and Corollary 1.8 are also given, along with an analysis of how these results generalize Theorem 4.1 of [4].

Statement 1: (Generalization of Theorem A) Let $\Phi^{(1)}, \Phi^{(2)}, \ldots$ be an analytic NIFS on (U, X) such that there exists $\eta > 0$ with $|\varphi'| \geq \eta$ on X for all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$. If J has infinitely many points, then J is uniformly perfect.

Statement 2: (Generalization of Theorem B) Let $\Phi^{(1)}, \Phi^{(2)}, \ldots$ be a conformal NIFS on (U, X) such that there exists $\eta > 0$ with $|\varphi'| \geq \eta$ on X for all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$. If J contains more than one point, then J is uniformly perfect.

Statement 3: (Generalization of Theorem C) Let $\Phi^{(1)}, \Phi^{(2)}, \ldots$ be a conformal NIFS on (U, X) such that there is a uniform bound on the cardinality of $\Phi^{(j)}$. If J contains more than one point, then J is uniformly perfect.

Example 5.1. Each set $I_{\bar{a}}$ in Theorem 4.1 of [4] is a limit set of a NIFS suitably chosen as follows. Set $X = [0, 1]$, fix $m \in \{2, 3, \ldots\}$, and choose $0 < a \leq \frac{1}{m+1}$. Fix a sequence $\bar{a} = (a_1, a_2, \ldots)$ such that $0 < a_k \leq a$ for $k = 1, 2, \ldots$. For each $k \in \mathbb{N}$, set $\Phi^{(k)}$ to be the collection $\{\varphi_1^{(k)}, \ldots, \varphi_m^{(k)}\}$ of linear maps, each with derivative a_k, such that the images $\varphi_1^{(k)}(X), \ldots, \varphi_m^{(k)}(X)$ are m equally spaced subintervals of X with $\varphi_1^{(k)}(X) = [0, a_k]$ and $\varphi_m^{(k)}(X) = [1 - a_k, 1]$. Example 2.7, illustrated in Figure 1, is such an NIFS (with $m = 2$). Each set X_k then coincides with what [4] calls I_k, and consists of m^k so-called basic intervals. And the limit set J then coincides with what [4] calls $I_{\bar{a}}$.

Theorem 4.1(1) of [4] shows that J is perfect, but pointwise thin (and thus HNUP) when $\liminf a_k = 0$. We now show that this also follows from Corollary 1.8. In order to use this corollary we set $U = \Delta(\frac{1}{2}, 0.7)$ and $X = \overline{\Delta}(\frac{1}{2}, 0.6)$, recalling that Lemma 2.2 shows that J is unchanged by this change of X from $[0, 1]$. Selecting a subsequence $a_{k_n} \to 0$, the reader can quickly check that $\inf_n\{b_{k_n}\} > 0, \inf_n\{\delta_{k_n}\} > 0$, and $\eta_{k_n} = a_{k_n} \cdot \mathrm{diam}(X) \to 0$, and thus Corollary 1.8 applies (since Φ clearly satisfies the Strong Separation Condition). We also note that when $\liminf a_k = 0$, Corollary 1.8 shows J is pointwise thin even when the strict setup above is considerably relaxed (e.g., the sets $\varphi_1^{(k)}([0, 1]), \ldots, \varphi_m^{(k)}([0, 1])$ do not need to be *equally spaced* subintervals of $[0, 1]$).

Theorem 4.1(2) of [4] shows that J is uniformly perfect when $\liminf a_k > 0$. This also follows from Theorem 1.5, noting that we may choose $\eta = \inf a_k > 0$ to satisfy the Derivative Condition and choose $\delta = 1 - 2a$ to satisfy the Two Point Separation Condition (even when $\liminf a_k = 0$) since the images $\varphi_1^{(k)}(X)$ and $\varphi_m^{(k)}(X)$ are always a distance $1 - 2a_k$ apart. We also note that when $\liminf a_k > 0$, Theorem 1.5 shows J is uniformly perfect even when the strict setup above is considerably relaxed. For example, the sets $\varphi_1^{(k)}(X), \ldots, \varphi_m^{(k)}(X)$ do not need to be *equally spaced* subintervals of X. In fact, these sets could even overlap, as long as the Two Point Separation Condition is met (and $\liminf a_k > 0$), and J would still be uniformly perfect.

Remark 17. Note that Example 2.7, with each $a_j = \frac{1}{j+2}$, shows that Statement 3 does not hold since J would then be perfect but also be HNUP. It also illustrates that the Derivative Condition in Theorem 1.5 is critical, even when all the other conditions are met.

Example 5.2. Again, let $X = [0, 1]$. Set $f_1(z) = \frac{z}{3}, f_2(z) = \frac{z+2}{3}$ and $f_3(z) = \frac{1}{3}(z - \frac{1}{2}) + \frac{1}{2}$. We fix a sequence of postive integers (l_j), and then create Φ by choosing $\Phi^{(1)} = \{f_1, f_2\}, \Phi^{(2)} = \Phi^{(3)} = \cdots = \Phi^{(1+l_1)} = \{f_3\}, \Phi^{(1+l_1+1)} = \{f_1, f_2\}, \Phi^{(1+l_1+2)} =$
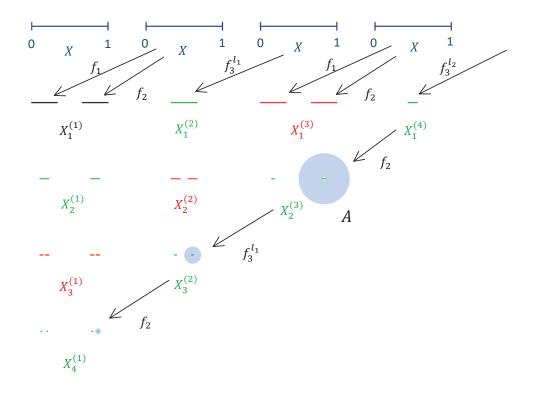
16

Figure 2. Table illustrating $\widetilde{\Phi}$ in Example 5.2, where $l_1 = 1$ and $l_2 = 2$.

$\Phi^{(1+l_1+3)} = \cdots = \Phi^{(1+l_1+1+l_2)} = \{f_3\}$, etc. Hence, defining $L_0 = 0$ and $L_n = \sum_{j=1}^{n}(1 + l_j)$, we have, for each $n = 0, 1, 2, \ldots,$ $\Phi^{(L_n+1)} = \{f_1, f_2\}$ and $\Phi^{(L_n+i)} = \{f_3\}$ for $2 \leq i \leq 1 + l_{n+1}$.

We prove the following dichotomy.

Claim: We have that $\sup l_j = +\infty$ implies J is perfect but pointwise thin (and thus HNUP), whereas $\sup l_j < +\infty$ implies J is uniformly perfect.

We now consider a related NIFS $\widetilde{\Phi}$ such that $J(\widetilde{\Phi}) = J(\Phi)$ by combining stages of consecutive $\Phi^{(j)}$ which equal $\{f_3\}$ (see Remark 11). Specifically, we have $\widetilde{\Phi}^{(1)} = \Phi^{(1)} = \{f_1, f_2\}, \widetilde{\Phi}^{(2)} = \Phi^{(2)} \circ \Phi^{(3)} \circ \cdots \circ \Phi^{(l_1+1)} = \{f_3^{l_1}\}, \widetilde{\Phi}^{(3)} = \Phi^{(1+l_1+1)} = \{f_1, f_2\}, \widetilde{\Phi}^{(4)} = \{f_3^{l_2}\}, \ldots,$ noting each iterate $f_3^{l_n}(z) = \frac{1}{3^{l_n}}(z - \frac{1}{2}) + \frac{1}{2}$. More succinctly we have for each $n \in \mathbb{N}$, $\widetilde{\Phi}^{(2n-1)} = \{f_1, f_2\}$ and $\widetilde{\Phi}^{(2n)} = \{f_3^{l_n}\}$. We now replace Φ by $\widetilde{\Phi}$, hence the $X_n^{(j)}$ and I^j below formally are constructed in reference to $\widetilde{\Phi}$ (see Figure 2).

We now suppose $\sup l_j < +\infty$ and prove J is uniformly perfect. Again we combine stages, this time doing so in order to utilize Theorem 1.5. Create NIFS Ψ with $J(\Psi) = J(\widetilde{\Phi}) = J(\Phi)$ by stipulating that, for each $k \in \mathbb{N}$, $\Psi^{(k)} = \widetilde{\Phi}^{(2k-1)} \circ \widetilde{\Phi}^{(2k)} = \{f_1 \circ f_3^{l_k}, f_2 \circ f_3^{l_k}\}$. Since the images $f_1 \circ f_3^{l_k}(X) \subseteq f_1(X) = [0, 1/3]$ and $f_2 \circ f_3^{l_k}(X) \subseteq f_2(X) = [2/3, 1]$ are always separated by $\delta = 1/3$, we see that the Two Point Separation Condition (with respect to Ψ) is met. Further the Derivative Condition (with respect to Ψ) is also met (when $\sup l_j < +\infty$, but not when $\sup l_j = +\infty$) since each map in $\Psi^{(k)}$ is linear with derivative $\frac{1}{3^{l_k+1}}$. From Theorem 1.5 it then follows that $J(\Psi)$ is uniformly perfect.

We now suppose that $\sup l_j = +\infty$ in order to show $J(\Phi) = J(\Psi)$ is perfect but pointwise thin. Perfectness follows from the fact that the diameter of each component of $X_{2n}^{(1)}$ shrinks to zero as $n \to \infty$ and each component of $X_{2n}^{(1)}$ contains two components

17

of $X_{2n+2}^{(1)}$. Now note that we may take Ψ to be an NIFS on $(U, \widetilde{X})$ with $U = \Delta(\frac{1}{2}, 0.7)$ and $\widetilde{X} = \overline{\Delta}(\frac{1}{2}, 0.6)$ Select a subsequence $l_{k_n} \to \infty$. Since the images $f_1 \circ f_3^{l_k}(\widetilde{X}) \subseteq f_1(\widetilde{X}) \subset \text{Int}(\widetilde{X})$ and $f_2 \circ f_3^{l_k}(\widetilde{X}) \subseteq f_2(\widetilde{X}) \subset \text{Int}(\widetilde{X})$, the reader can quickly check that Ψ clearly satisfies the Strong Separation Condition and $\inf_n\{b_{k_n}\} > 0, \inf_n\{\delta_{k_n}\} > 0$, and $\eta_{k_n} = \frac{\text{diam}(\widetilde{X})}{3^{l_{k_n}+1}} \to 0$ (since each map in $\Psi^{(k_n)}$ is linear with derivative $\frac{1}{3^{l_{k_n}+1}}$). Hence, Corollary 1.8 applies to show $J(\Phi) = J(\Psi)$ is pointwise thin.

Remark 18. Example 5.2 shows that (when $\sup l_j = +\infty$) $J(\Phi)$ can be perfect yet fail to be uniformly perfect even when Φ (but not the modified NIFS $\widetilde{\Phi}$) satisfies both the Derivative Condition and Möbius Condition of Theorem 1.5. This example shows that the Two Point Separation Condition in Theorem 1.5 is critical, and also shows that none of the above Statements 1-3 hold. We also note that $J' = \{z : \phi_\omega(z) = z$ for some ω in some $I^k\}$ is not a subset of J (e.g., 0 is a fixed point of f_1 but is not in J). Hence, also J is not forward invariant under the maps ϕ_ω for $\omega \in I^k$. Compared with statements (i) and (ii) as given for autonomous IFSs near the end of Section 3, we note that the non-autonomous situation is far more delicate.

6. Applications to Non-Autonomous Julia Sets

Given a sequence of complex polynomials (P_j), define its *Fatou set* $\mathcal{F} = \mathcal{F}((P_j))$ by

$$\mathcal{F} = \{z \in \overline{\mathbb{C}} : \{P_n \circ \cdots \circ P_2 \circ P_1\}_{n=1}^\infty \text{ is a normal family on some neighborhood of } z\}$$

where we take our neighborhoods with respect to the spherical topology on $\overline{\mathbb{C}}$. We then define the *Julia set* $\mathcal{J} = \mathcal{J}((P_j))$ to be the complement $\overline{\mathbb{C}} \setminus \mathcal{F}$.

Theorem 6.1. *Let f be a polynomial on $\mathbb{C}$ of degree at least 2. Suppose f has no critical values in the closed unit disk $\overline{\mathbb{D}}$ and that $f^{-1}(\overline{\mathbb{D}}) \subset \mathbb{D}$. Fixing a sequence $a_j \in \mathbb{C}$ with each $|a_j| > 1$, we define polynomials $P_j(z) = a_j f(z)$. Then*

(1) $\mathcal{J}((P_j))$ is uniformly perfect if and only if $\limsup |a_j| < \infty$, and
(2) $\mathcal{J}((P_j))$ is pointwise thin (and HNUP) if and only if $\limsup |a_j| = \infty$.

Remark 19. For $a, c \in \mathbb{C}$ with $|c| > 1$ and $|a| - |c| > 1$, one may choose $f(z) = az^2 + c$ in the above theorem. Note then that $|z| \geq 1$ implies $|f(z)| = |az^2 + c| \geq |a| - |c| > 1$, i.e., $f(\mathbb{C} \setminus \mathbb{D}) \subseteq \mathbb{C} \setminus \overline{\mathbb{D}}$, which gives that $f^{-1}(\overline{\mathbb{D}}) \subset \mathbb{D}$. Also, clearly the sole critical value of f is $c \notin \overline{\mathbb{D}}$. Hence applying the above theorem with such an f and a suitable sequence (a_j) with $\limsup |a_j| = \infty$, we can create a simple sequence of polynomials with pointwise thin (and thus HNUP) Julia set without the complicated arguments presented in [5].

Proof. (1) The Julia set of a bounded sequence of polynomials is known to be uniformly perfect (see Theorem 1.21 of [18]).

(2) Suppose $\limsup |a_j| = \infty$, and choose a subsequence a_{j_n} such that $|a_{j_n}| \to \infty$. We complete the proof by showing $\mathcal{J}((P_j))$ is pointwise thin and compact. Calling d the degree of f, we note that f has d well defined inverse branches $f_1, \ldots, f_d$, on some open connected set $U = \Delta(0, 1 + \epsilon) \supset \overline{\mathbb{D}}$ since all critical values of f lie outside of $\overline{\mathbb{D}}$. Furthermore, we note that we may choose U such that $f^{-1}(U) \subset \overline{\mathbb{D}}$. Hence, each P_j has d well defined inverse branches on U given by $\varphi_i^{(j)}(z) = f_i(\frac{z}{a_j})$ for $i = 1, \ldots, d$.

For each $j \in \mathbb{N}$, let $\Phi^{(j)} = \{\varphi_1^{(j)}, \ldots, \varphi_d^{(j)}\}$ and note that these families form an NIFS Φ on (U, X) where $X = \overline{\mathbb{D}}$. For each j, note that $\varphi_i^{(j)}(X) = f_i(\overline{\Delta}(0, \frac{1}{|a_j|})) \subset f_i(X) \subset \text{Int}(X)$ for $i = 1, \ldots, d$. Hence, Φ satisfies the Strong Separation Condition and, using the notation of Corollary 1.8, we also see that for each $n \in \mathbb{N}$,

$$b_{j_n} \geq b_0 := \min\{\text{dist}(f_i(X), \partial X) : i \in \{1, \ldots, d\}\} > 0,$$

$$\delta_{j_n} \geq \delta_0 := \min\{\text{dist}(f_a(X), f_b(X)) : a, b \in \{1, \ldots, d\} \text{ with } a \neq b\} > 0$$

and

$$\eta_{j_n} = \max\{\text{diam}(\varphi_i^{(j_n)}(X)) : i \in \{1, \ldots, d\}\}$$

$$= \max\{\text{diam}(f_i(\overline{\Delta}(0, \frac{1}{|a_{j_n}|}))) : i \in \{1, \ldots, d\}\} \to 0.$$

Since $\inf\{b_{j_n}\} > 0$, Corollary 1.8 yields that $J(\Phi)$ is pointwise thin since $\frac{\delta_{j_n}}{\eta_{j_n}} \geq \frac{\delta_0}{\eta_{j_n}} \to \infty$. Further, we note that $J(\Phi)$ is compact since each $I^{(j)}$ is finite.

The result then follows by showing that $\mathcal{J}((P_j)) = J(\Phi)$. Note that $J(\Phi) = \{z \in \mathbb{C} : P_j \circ \cdots \circ P_1(z) \in \overline{\mathbb{D}} \text{ for each } j\}$. Also note that $\mathbb{C} \setminus \overline{\mathbb{D}}$ is forward invariant under each P_j, and so it follows from Montel's Theorem that $\mathbb{C} \setminus J(\Phi) \subseteq \mathcal{F}((P_j))$, i.e., $\mathcal{J}((P_j)) \subseteq J(\Phi)$. Since $J(\Phi)$ is pointwise thin, it is clear that $J(\Phi)$ has no interior. This implies that any $z \in J(\Phi)$, which necessarily has as its orbit contained in the compact subset $f_1(X) \cup \cdots \cup f_d(X)$ of $\mathbb{D}$, must be arbitrarily close to points whose orbits escape $\overline{\mathbb{D}}$. Hence, $J(\Phi) \subseteq \mathcal{J}((P_j))$. $\qquad \square$

Corollary 6.2. *Let f be a polynomial on $\mathbb{C}$ of degree at least 2. Suppose f has no critical values in $\overline{\mathbb{D}}$ and that $f^{-1}(\overline{\mathbb{D}}) \subset \mathbb{D}$. Let τ be a probability measure on $\mathbb{C} \setminus \overline{\mathbb{D}}$ with unbounded support. Then for almost all sequences $(a_j) \in \prod_{j=1}^{\infty}(\mathbb{C} \setminus \overline{\mathbb{D}})$ with respect to $\tilde{\tau} = \bigotimes_{j=1}^{\infty} \tau$, the maps $P_j = a_j \cdot f$ define a sequence of polynomials whose Julia set $\mathcal{J}((P_j))$ is pointwise thin.*

Proof. For $N \in \mathbb{N}$, set $B_N = \{(a_j) : |a_j| \leq N \text{ for all } j\}$ and note that since τ has unbounded support, $\tilde{\tau}(B_N) = 0$ by the law of large numbers. Hence, $\tilde{\tau}(\cup_{N \in \mathbb{N}} B_N) = 0$, i.e., the set of bounded sequences has $\tilde{\tau}$-measure zero. The result then follows from Theorem 6.1. $\qquad \square$

7. Proof of the Main Theorems

In this section we first prove Theorems 1.5 and 1.6 regarding uniform perfectness, and then prove Theorem 1.7, Corollary 1.8, and Theorem 1.9 regarding pointwise thinness.

We begin by proving a crucial lemma that will be key in providing a uniform Lipschitz constant for certain locally defined inverse maps.

Lemma 7.1. *Let $\mathcal{F}$ be a collection of analytic functions mapping a non-empty open set $U \subset \mathbb{C}$ into a compact set $X \subset U$ such that there exists $\eta > 0$ where for all $f \in \mathcal{F}$ we have $|f'| \geq \eta$ on X. Then there exists $r_0 > 0$ such that for every $f \in \mathcal{F}$ and $x \in X$,*

we have $|g'| \leq \frac{2}{\eta}$ on $\Delta(f(x), r_0)$ where g is the local branch of the inverse of f such that $g(f(x)) = x$.

Note that this lemma does not require the maps $f \in \mathcal{F}$ to be Möbius, or even globally conformal on U.

Proof. First note that by compactness, there exists $r > 0$ such that for all $x \in X$ we have $\Delta(x, r) \subseteq U$. Applying Lemma 2.3 of [6], where $M > 0$ is taken large enough so that $X \subset \Delta(0, M)$, we see that for some $\rho > 0$ each $f \in \mathcal{F}$ is one-to-one on $\Delta(x, \rho)$ for every $x \in X$. (Note that ρ is independent of $f \in \mathcal{F}$ and $x \in X$.) By the Koebe distortion theorem (see, e.g., Theorem 1.6 of [19]), there exists $0 < r_1 < \rho$ such that for every $f \in \mathcal{F}$ and $x \in X$, we have $|f'| \geq \frac{\eta}{2}$ on $\Delta(x, r_1)$. By the Koebe 1/4 Theorem, for each $x \in X$ we then see that $f(\Delta(x, r_1)) \supseteq \Delta(f(x), \frac{r_1 \eta}{4})$. Hence, calling $r_0 = \frac{r_1 \eta}{4}$ we have that a branch g of f^{-1} is defined on $\Delta(f(x), r_0)$ such that $g(f(x)) = x$ and has $|g'| \leq \frac{2}{\eta}$ there. $\qquad\square$

Remark 20. Under the hypotheses of Theorem 1.5, the Derivative Condition along with the distortion theorems used in the proof of the above lemma yield that $\inf\{\mathrm{diam}(\varphi(X)) : \varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}\} > 0$. To see this, choose $x_0 \in X$ and $r > 0$ such that $\Delta(x_0, r) \subset X$ (note that X must have interior since it contains the open sets $\varphi(U)$ for all $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$). Fixing $r < \rho$ from the above proof, we see that by the Koebe 1/4 Theorem, $\varphi(X) \supseteq \varphi(\Delta(x_0, r)) \supseteq \Delta(\varphi(x_0), \frac{r\eta}{4})$ for all $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$, which justifies the claim.

Proof of Theorem 1.5. We begin by replacing X, if it is not connected, by the connected $\widetilde{X} \subset U$ as in Remark 13, noting that the hypotheses are still met. Indeed, the Möbius and Two Point Separation Conditions are clearly still satisfied with respect to $\widetilde{X} \supset X$. The Derivative Condition also still holds with respect to $\widetilde{X} \supset X$ though not as trivially. We show this by contradiction. Assume $\varphi'_n(z_n) \to 0$ as $n \to \infty$ where each $z_n \in \widetilde{X}$ and each $\varphi_n \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$. By compactness we may suppose $z_n \to z_0 \in \widetilde{X}$. Since by Montel's Theorem, the family $\cup_{j \in \mathbb{N}} \Phi^{(j)}$ is normal on U, we may suppose φ_n converges normally on U to some map φ. Hence, we must have $\varphi'(z_0) = 0$. Since each map in $\cup_{j \in \mathbb{N}} \Phi^{(j)}$ is Möbius, and thus one-to-one on U, we see by Hurwitz's Theorem that φ must be constant. This implies that for any $x \in X$, we must have $\varphi'_n(x) \to \varphi'(x) = 0$, but this contradicts the Derivative Condition on X which gives that each $|\varphi'_n(x)| \geq \eta$.

It suffices to prove $\overline{J^{(1)}}$ is uniformly perfect since clearly each sub-NIFS of Φ which generates $J^{(j)}$ must also satisfy conditions (i)-(iii). First note that by Remark 2 we see that $\mathrm{diam}(J^{(1)}) \geq \delta$ and so $J = J^{(1)}$ has more than one point. Recalling Remark 14 and Remark 15, we consider a *true* annulus A_1 which separates $\overline{J}$ and which has modulus large enough so that any conformal annulus $B \subset \overline{\mathbb{C}}$ with $\mathrm{mod}\, B \geq \mathrm{mod}\, A_1 - 1$ contains an essential true annulus $B' \subset B$ such that $\mathrm{mod}\, B' = \frac{1}{3}\mathrm{mod}\, B$. Since the true annulus A_1 must also separate $J = \cap_{k=1}^{\infty} X_k$, we apply Lemma 4.7 to obtain a true annulus $A \subset A_1$ which separates some X_{k_0} and has $\mathrm{mod}\, A = \mathrm{mod}\, A_1 - 1$. We complete the proof by showing that there exists an upper bound on $\mathrm{mod}\, A$.

Recall the superscript notation of Section 1, in particular, that $X_{k_0}^{(1)} = X_{k_0}$. By the invariance condition (1) in Remark 9, we have $\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(X_{k_0-1}^{(2)}) = X_{k_0}^{(1)}$, and so there must be some $\varphi_{i_1}^{(1)} \in \Phi^{(1)}$ such that A surrounds some point of $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$ (i.e., the bounded component of $\mathbb{C} \setminus A$ contains a point of $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$). Since A separates $X_{k_0}^{(1)}$,
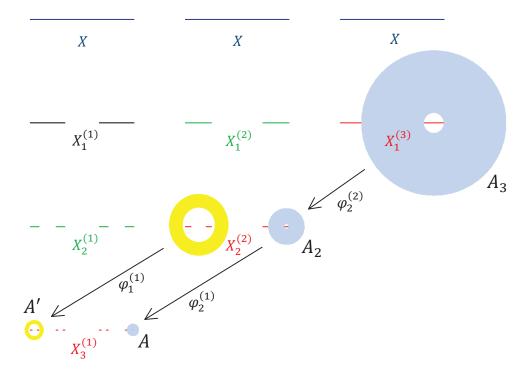
Figure 3. Illustration of the proof of Theorem 1.5 using the system of Example 2.7. Note that A and $A_2 = (\varphi_2^{(1)})^{-1}(A)$ are both of Case (II) type, whereas A' is of Case (II) type, but $(\varphi_1^{(1)})^{-1}(A')$ is of Case (I) type.

we must have one of two cases: Case (I) A surrounds all of $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$, or Case (II) A separates $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$. See Figure 3.

Case (I): Write $A = \text{Ann}(z; r, R)$ and suppose it surrounds all of $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$. Hence $(\varphi_{i_1}^{(1)})^{-1}(\overline{\Delta(z,r)}) \supseteq X_{k_0-1}^{(2)} \supseteq J^{(2)}$, which by the Two Point Separation Condition (see Remark 2) gives that $\text{diam}\left((\varphi_{i_1}^{(1)})^{-1}(\overline{\Delta(z,r)})\right) \geq \delta$.

From Lemma 4.8 it follows that we only need to consider cases where $R < 2 \cdot \text{diam}(X)$. We now establish an upper bound for $\text{mod } A = \log(R/r)$ by finding a positive lower bound for r.

Notice that due to the Derivative Condition and Lemma 7.1, there exists $r_0 > 0$ such that for any $x \in X$ and $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$, we have $|(\varphi^{-1})'| \leq \frac{2}{\eta}$ on $\Delta(\varphi(x), r_0)$.

We now suppose $r < \min\{\frac{\delta\eta}{4}, \frac{r_0}{2}\}$, from which we derive a contradiction, thus producing a lower bound for r and completing the proof for Case (I). Since $\overline{\Delta(z,r)}$ meets $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$, we may choose $x_0 \in X_{k_0-1}^{(2)} \subseteq X$ such that $\varphi_{i_1}^{(1)}(x_0) \in \overline{\Delta(z,r)} \subset \overline{\Delta(\varphi_{i_1}^{(1)}(x_0), 2r)} \subset \Delta(\varphi_{i_1}^{(1)}(x_0), r_0)$. Since $\left|\left((\varphi_{i_1}^{(1)})^{-1}\right)'\right| \leq \frac{2}{\eta}$ on $\Delta(\varphi_{i_1}^{(1)}(x_0), r_0)$ which contains the convex set $\overline{\Delta(z,r)}$, we see that $\text{diam}\left((\varphi_{i_1}^{(1)})^{-1}(\overline{\Delta(z,r)})\right) \leq \frac{4r}{\eta} < \delta$, which is a contradiction.

Case (II): Suppose A separates $\varphi_{i_1}^{(1)}(X_{k_0-1}^{(2)})$. Hence, the conformal annulus $A_2 = (\varphi_{i_1}^{(1)})^{-1}(A)$ must separate $X_{k_0-1}^{(2)}$ and must have $\text{mod } A_2 = \text{mod } A$. In terms of Figure 3, we have constructed an annulus A_2 which separates $X_{k_0-1}^{(2)}$ in the picture diago-

nally up and right of the picture of $X_{k_0}^{(1)}$. Note, however, that A_2 will contain ∞ when $\varphi_{i_1}^{(1)}(\infty) \in A$, and so we must allow for this possibility.

Hence we may repeat our process as follows. Since A_2 separates the set $X_{k_0-1}^{(2)} = \bigcup_{i \in I^{(2)}} \varphi_i^{(2)}(X_{k_0-2}^{(3)})$, we must have at least one of two cases: Case (I') one component of $\overline{\mathbb{C}} \setminus A_2$ contains some $\varphi_{i_2}^{(2)}(X_{k_0-2}^{(3)})$ while the other component contains some other $\varphi_{i_2'}^{(2)}(X_{k_0-2}^{(3)})$, or Case (II') A_2 separates some $\varphi_{i_2}^{(2)}(X_{k_0-2}^{(3)})$. If Case (I') holds, extract an essential true annulus $A_2' \subset A_2$ with mod $A_2' = \frac{1}{3}$mod $A_2 = \frac{1}{3}$mod A, which must surround all of either $\varphi_{i_2}^{(2)}(X_{k_0-2}^{(3)})$ or $\varphi_{i_2'}^{(2)}(X_{k_0-2}^{(3)})$, and then bound mod A_2' as in Case (I) above. If Case (II') holds, we repeat the process of Case (II) above, noting that we do not need to first extract a true annulus from A_2.

This process must then end by eventually applying the method of Case (I), or by eventually producing (after k_0 steps) an annulus A_{k_0}, with the same modulus as of A, which separates $X_1^{(k_0)}$. The proof is thus concluded by showing that such a modulus is uniformly bounded independent of the choice of k_0. First, extract an essential true annulus $A_{k_0}' = \text{Ann}(z'; r', R') \subset A_{k_0}$ with mod $A_{k_0}' = \frac{1}{3}$mod $A_{k_0} = \frac{1}{3}$mod A, which necessarily separates $X_1^{(k_0)}$. Again by Lemma 4.8, it is then clear that we only need to produce a lower bound for r'. This follows easily from Remark 20 by noting that $\overline{\Delta(z', r')}$ would need to contain the connected set $\varphi(X)$ for some $\varphi \in \Phi^{(k_0)}$.

Examination of the above proof shows that mod A_1 is bounded above by a constant which depends only on δ and η. $\qquad \square$

Note that the step of extracting a true annulus of one-third the modulus is done only at most once in the above proof.

Proof of Theorem 1.6. By Proposition 4.9, for each $\varphi \in \widetilde{\Phi}^{(1)}$ the set $\phi\left(\overline{J^{(n)}}\right)$ is uniformly perfect. Lemma 2.5 gives that $\overline{J^{(1)}} = \bigcup_{\varphi \in \widetilde{\Phi}^{(1)}} \varphi\left(\overline{J^{(n)}}\right)$, and the result follows since the finite union of uniformly perfect sets is uniformly perfect. $\qquad \square$

We now prove the Theorem 1.7, Corollary 1.8, and Theorem 1.9 regarding pointwise thinness.

Proof of Theorem 1.7. Note that since the NIFS Φ is conformal and both the annulus A_{j_n} and its bounded complementary component lie inside $X \subset U$, we see that $\pi_\Phi(\omega) \in \varphi_{\omega_1 \cdots \omega_{j_n-1}}(\varphi_{m_n}^{(j_n)}(X))$ (see Remark 8) is surrounded by the conformal annulus $A_{j_n}' = \varphi_{\omega_1 \cdots \omega_{j_n-1}}(A_{j_n})$, which separates $\varphi_{\omega_1 \cdots \omega_{j_n-1}}(X_1^{(j_n)}) \subseteq X_{j_n}^{(1)}$. See Figure 4. We claim that $A_{j_n}' \cap X_{j_n}^{(1)} = \emptyset$, from which it follows that A_{j_n}' separates $X_{j_n}^{(1)}$, and thus separates J. Since mod $A_{j_n'} = $ mod $A_{j_n} \to \infty$ with diam$(A_{j_n}') \to 0$, we see that J is pointwise thin at $\pi_\Phi(\omega)$.

To prove the claim, suppose towards a contradiction that A_{j_n}' meets

$$X_{j_n}^{(1)} = \bigcup_{\omega^* \in I^{j_n}} \varphi_{\omega^*}(X) = \bigcup_{\omega_1^* \cdots \omega_{j_n-1}^* \in I^{j_n-1}} \bigcup_{\omega_{j_n}^* \in I^{(j_n)}} \varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(\varphi_{\omega_{j_n}^*}(X))$$

$$= \bigcup_{\omega_1^* \cdots \omega_{j_n-1}^* \in I^{j_n-1}} \varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X_1^{(j_n)}).$$
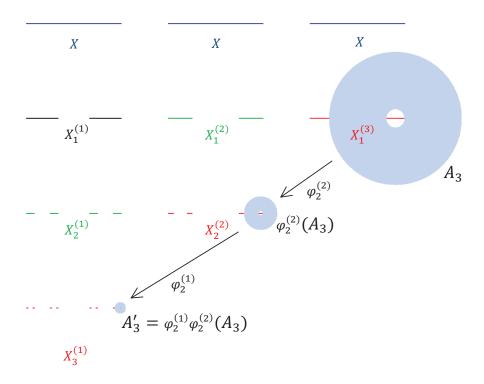
22

Figure 4. Table illustrating the proof of Theorem 1.7 using the system of Example 2.7.

Hence, A'_{j_n} meets $\varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X_1^{(j_n)})$ for some $\omega_1^* \cdots \omega_{j_n-1}^* \in I^{j_n-1}$. Note that $\omega_1^* \cdots \omega_{j_n-1}^* \neq \omega_1 \cdots \omega_{j_n-1}$ since A'_{j_n} separates $\varphi_{\omega_1 \cdots \omega_{j_n-1}}(X_1^{(j_n)})$. However, since $X_1^{(j_n)} \subseteq X$, $A_{j_n} \subseteq X$, and $\varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X) \cap \varphi_{\omega_1 \cdots \omega_{j_n-1}}(X) = \emptyset$ by the strong separation condition (see the discussion preceding Definition 1.3), we see that $\varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X_1^{(j_n)})$ cannot meet $A'_{j_n} = \varphi_{\omega_1 \cdots \omega_{j_n-1}}(A_{j_n})$, which is a contradiction. $\square$

Proof of Corollary 1.8. Pick an arbitrary $\omega \in I^\infty$. For each n, choose some $z_n \in \varphi_{\omega_{j_n}}^{(j_n)}(X)$, and define $A_{j_n} = \text{Ann}(z_n; \eta_{j_n}, \frac{\delta_{j_n}}{c})$, which by definition of η_{j_n} must surround $\varphi_{\omega_{j_n}}^{(j_n)}(X)$. Hence by definition of δ_{j_n}, the annulus A_{j_n} must separate $X_1^{(j_n)}$. Lastly, since $\frac{\delta_{j_n}}{c} \leq b_{j_n} \leq \text{dist}(\varphi_{\omega_{j_n}}^{(j_n)}(X), \partial X)$, we see that $\Delta(z_n, \frac{\delta_{j_n}}{c}) \subseteq X$. Thus by Theorem 1.5, noting that mod $A_{j_n} = \log \frac{\delta_{j_n}}{c\eta_{j_n}} \to \infty$, we see that J is pointwise thin at $\pi_\Phi(\omega)$. The proof is then complete by noting $J = \pi_\Phi(I^\infty)$ since Φ satisfies the Strong Separation Condition (as mentioned just before the statement of Lemma 2.1). $\square$

Proof of Theorem 1.9. Consider the analytic NIFS $\widetilde{\Phi}$ given by $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(n-1)}$ and $\widetilde{\Phi}^{(j)} = \Phi^{(j+n-2)}$ for each $j > 1$. Hence, by Remark 11, we see that $J(\Phi) = J(\widetilde{\Phi})$. By Proposition 4.10, for each $\varphi \in \widetilde{\Phi}^{(1)}$ the set $\phi\left(\overline{J^{(n)}}\right)$ is pointwise thin. Lemma 2.5 gives that $\overline{J^{(1)}} = \bigcup_{\varphi \in \widetilde{\Phi}^{(1)}} \varphi\left(\overline{J^{(n)}}\right)$, and the result follows since the finite disjoint union of compact pointwise thin sets is pointwise thin. $\square$

References

[1] Beardon AF, Pommerenke C. The Poincaré metric of plane domains. J London Math Soc (2). 1978;18(3):475–483. Available from: http://dx.doi.org/10.1112/jlms/s2-18.3.475.

[2] Pommerenke C. Uniformly perfect sets and the Poincaré metric. Arch Math. 1979;32:192–199.

[3] Sugawa T. Uniformly perfect sets – analytic and geometric aspects. Sugaku Expositions. 2003;16(2):225–242.

[4] Stankewitz R, Sumi H, Sugawa T. Hereditarily non uniformly perfect sets. Discrete and Continuous Dynamical Systems S. 2019;12(8).

[5] Comerford M, Stankewitz R, Sumi H. Hereditarily non uniformly perfect non-autonomous Julia sets. Discrete and Continuous Dynamical System - A. 2020;40(1):33–46.

[6] Stankewitz R. Uniformly perfect analytic and conformal attractor sets. Bull London Math Soc. 2001;33(3):320–330.

[7] R. Daniel Mauldin and Mariusz Urbański. *Graph directed Markov systems*, volume 148 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003. Geometry and dynamics of limit sets.

[8] Rempe-Gillen L, Urbański M. Non-autonomous conformal iterated function systems and Moran-set constructions. Trans Amer Math Soc. 2016;368(3):1979–2017. Available from: https://doi.org/10.1090/tran/6490.

[9] Wen Z. Moran sets and Moran classes. Chinese Sci Bull. 2001;46(22):1849–1856. Available from: https://doi.org/10.1007/BF02901155.

[10] Hutchinson J. Fractals and self similarity. Indiana University Math Journal. 1981;30:713–747.

[11] Mauldin R, Urbanski M. Dimensions and measures in the infinite iterated function systems. Proc London Math Soc. 1996;73:105–154.

[12] Mauldin R, Urbanski M. Conformal iterated function systems with applications to the geometry of continued fractions. Trans Amer Math Soc. 1999;351(12):4995–5025.

[13] Stankewitz R. Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's. Proc Amer Math Soc. 2000;128(9):2569–2575.

[14] Ahlfors LV. Conformal invariants: topics in geometric function theory. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg; 1973. McGraw-Hill Series in Higher Mathematics.

[15] McMullen C. Complex dynamics and renormalization. Princeton Univeristy Press; 1994.

[16] Beardon AF. Iterations of rational functions. Springer-Verlag, New York; 1991.

[17] Shiga H. On the quasiconformal equivalence of dynamical Cantor sets. Preprint. https://arxiv.org/abs/1812.07785.

[18] Sumi H. Semi-hyperbolic fibered rational maps and rational semigroups. ErgodTh& Dynam Sys. 2006;26:893–922.

[19] Carleson L, Gamelin TW. Complex dynamics. New York: Springer-Verlag; 1993.