HEREDITARILY NON UNIFORMLY PERFECT ANALYTIC AND CONFORMAL NON-AUTONOMOUS ATTRACTOR SETS

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ABSTRACT. Conditions are given which imply that certain non-autonomous analytic iterated function systems in the complex plane $\mathbb{C}$ have pointwise thin, and thus hereditarily non uniformly perfect, attractor sets. Examples are given to illustrate the main theorem, as well as to indicate how it generalizes other results. Applications to non-autonomous Julia sets are also given.

1. INTRODUCTION

This paper can be regarded as a complementary paper to [3]. Whereas the focus of [3] is to give conditions for an analytic non-autonomous iterated function system (NIFS) to have a uniformly perfect attractor, this paper looks to the other extreme and gives conditions for an analytic NIFS to have a hereditarily non uniformly perfect (HNUP) attractor.

We exclude from our focus analytic *autonomous* systems since results found in [7] show that such an attractor is often uniformly perfect (see also [4] for uniformly perfectness results regarding similar autonomous systems). Certain constructions in [8] are *non-autonomous* iterated function systems shown to have HNUP attractors. (Those examples were not presented as attractors, but rather as Cantor-like constructions. However, Example 2.1 (see also [3]) shows how they can be viewed as non-autonomous attractors.) We look to generalize such results here, and we begin, as done in [3], by following [6] to introduce the main framework and definitions (with some key differences) of NIFS's. We also note that attractors of NIFS's are often Moran-set constructions (see [10] for good exposition of such).

1.1. **NIFS's.** A *non-autonomous iterated function system* (NIFS) Φ on the pair (U, X) is given by a sequence $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \dots$, such that each $\Phi^{(j)}$ is a collection of non-constant functions $(\varphi_i^{(j)} : U \to X)_{i \in I^{(j)}}$, where each function maps the non-empty open connected set $U \subset \mathbb{C}$ into a compact set $X \subset U$ such that there exists $0 < s < 1$ and a metric d on U with $d(\varphi(z), \varphi(w)) \leq sd(z, w)$

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for all $z, w \in X$ and all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$. We also require d to induce the Euclidean topology on X, and note that the system is uniformly contracting on the metric space (X, d). The system is called *autonomous* if $I^{(j)}$ and $\Phi^{(j)}$ are independent of j.

We define a NIFS and its corresponding attractor set (see Definition 1.3) to be *analytic* (respectively, *conformal*) if all the maps are analytic (respectively, conformal) on U. Note that here and throughout conformal means analytic and one-to-one (globally on U, not just locally). The main object of interest to this paper is the *analytic* NIFS, and so the condition imposed that each φ map U into X allows us, under this condition of analyticity, to take the metric d to be the hyperbolic metric on U (see Section 2).

Important differences from [6] in the above setup include that we do not impose an *open set condition* in our definition. However, for several of our results we shall require the following even stronger condition.

Definition 1.1 (Strong Separation Condition)**.** We say that NIFS Φ on (U, X) satisfies the *Strong Separation Condition* when

$$\varphi_a^{(j)}(X) \cap \varphi_b^{(j)}(X) = \emptyset,$$

for each $j \in \mathbb{N}$ and distinct $a, b \in I^{(j)}$.

Given an NIFS, we wish to study the limit set (or attractor) which we define with the help of the next definition.

Definition 1.2 (Words)**.** For each $k \in \mathbb{N}$, we define the symbolic spaces

$$I^k := \prod_{j=1}^{k} I^{(j)} \qquad \text{and} \qquad I^\infty := \prod_{j=1}^{\infty} I^{(j)}.$$

Note that a k-tuple $(\omega_1, \dots, \omega_k) \in I^k$ may be identified with the corresponding word $\omega_1 \cdots \omega_k$. When $\omega^* \in I^\infty$ has $\omega_j^* = \omega_j$ for $j = 1, \dots, k$, we call ω^* an *extension* of $\omega = \omega_1 \dots \omega_k \in I^k$.

Definition 1.3. For all $k \in \mathbb{N}$ and $\omega = \omega_1 \cdots \omega_k \in I^k$, we define $\varphi_\omega := \varphi_{\omega_1}^{(1)} \circ \cdots \circ \varphi_{\omega_k}^{(k)}$ with

$$X_\omega := \varphi_\omega(X) \qquad \text{and} \qquad X_k := \bigcup_{\omega \in I^k} X_\omega.$$

The *limit set* (or *attractor*) of Φ is defined as

$$J = J(\Phi) := \bigcap_{k=1}^{\infty} X_k.$$

Remark 1.1. The attractor J is not necessarily compact (e.g., J is not compact for the autonomous system given in Example 4.3 of [7]). However, if each index set $I^{(j)}$ is finite, then each X_k is compact and hence so is J.

Notation to be used throughout: Let q be a metric. For a set $F \subseteq \mathbb{C}$, we define its *diameter* to be $\text{diam}_q F = \sup\{q(z,w) : z, w \in F\}$. We define the *distance* between sets $E, F \subset \mathbb{C}$ to be $\text{dist}_q(E,F) = \inf\{q(z,w) : z \in E, w \in F\}$. Also, for $w \in \mathbb{C}$ and $r > 0$ we define the *open disk*, *closed disk*, and *circle*, respectively, by $\Delta_q(w,r) = \{z : q(z,w) < r\}, \overline{\Delta}_q(w,r) = \{z : q(z,w) \leq r\}$ and $C_q(w,r) = \{z : q(z,w) = r\}$. If no metric is noted, then it is assumed that the metric is the Euclidean metric. Lastly, the open unit disk in $\mathbb{C}$ is denoted $\mathbb{D}$.

Remark 1.2 (Projection Map). Let $\omega^* \in I^\infty$ be arbitrary. Then the compact sets $\varphi_{\omega_1^* \cdots \omega_n^*}(X)$ decrease with $\text{diam}_d(\varphi_{\omega_1^* \cdots \omega_n^*}(X)) \leq s^n \text{diam}_d(X) \to 0$ as $n \to \infty$ and thus $\cap_{n=1}^\infty \varphi_{\omega_1^* \cdots \omega_n^*}(X)$ contains just a single point that we call $\pi_\Phi(\omega^*)$. Note that $\pi_\Phi(\omega^*) \in J$ since it clearly belongs to each $\varphi_{\omega_1^* \cdots \omega_n^*}(X) \subseteq X_n$. We call $\pi_\Phi : I^\infty \to J$ the *projection map.*

Also note that for any non-empty compact $\widetilde{X} \subseteq X$ which is forward invariant under Φ, i.e., $\varphi(\widetilde{X}) \subseteq \widetilde{X}$ for all $\varphi \in \cup_{j=1}^\infty \Phi^{(j)}$, we have that $\cap_{n=1}^\infty \varphi_{\omega_1^* \cdots \omega_n^*}(\widetilde{X}) = \cap_{n=1}^\infty \varphi_{\omega_1^* \cdots \omega_n^*}(X)$ since each is a singleton set with the set on the left being a subset of the set on the right. We can thus say that the projection map π_Φ is independent of the choice of non-empty compact forward invariant set X.

Remark 1.3 (Pieces of X_k). The limit set $J = \cap_{k=1}^\infty X_k$ is a decreasing intersection sets X_k, an important property of the X_k being that they are unions of what we call *pieces* of the X_k, each of which must contain a limit point. More precisely, note that for any $k \in \mathbb{N}$ and $\omega = \omega_1 \cdots \omega_k \in I^k$, we have that the *piece* $\varphi_\omega(X)$ of X_k, for which $\text{diam}_d(\varphi_\omega(X)) \leq s^k \text{diam}_d(X)$, contains the point $\pi_\Phi(\omega^*) \in J$ for any extension $\omega^* \in I^\infty$ of ω. Note also that the pieces of X_k are not necessarily components of X_k since the pieces may overlap.

In the NIFS systems studied in [6] (see Definition and Lemma 2.4 there, which makes essential use of the open set condition and a geometric condition on X, neither of which we impose here), it must be the case that $\pi_\Phi(I^\infty) = J$. We do not necessarily have this in all cases (see Example 1.1 of [3]), but we do note that the Strong Separation Condition is strong enough to allow the proof in [6] to apply. Combining this with Lemma 1.1 in [3], gives the following result.

Lemma 1.1. *Let Φ be a NIFS on (U, X). Then,*

$$\overline{J(\Phi)} = \overline{\pi_\Phi(I^\infty)},$$

and so, if $\pi_\Phi(I^\infty)$ is compact, then $J(\Phi) = \pi_\Phi(I^\infty)$. Furthermore, if Φ satisfies the Strong Separation Condition, then $J(\Phi) = \pi_\Phi(I^\infty)$.

In certain examples, it is convenient to change the set X to another forward invariant compact set. The following result shows that such a change to X, though it may affect J (see Example 1.2 of [3]), will not affect $\overline{J}$, the main object we study in this paper.

Lemma 1.2 (Lemma 1.2 in [3])**.** *Let $\widetilde{X} \neq \emptyset$ be a compact subset of X that is forward invariant under NIFS Φ on (U, X), i.e., $\varphi(\widetilde{X}) \subseteq \widetilde{X}$ for all $\varphi \in \cup_{j=1}^\infty \Phi^{(j)}$. Then, calling $\widetilde{X}_k := \bigcup_{\omega \in I^k} \varphi_\omega(\widetilde{X})$, we have*

$$\overline{J(\Phi)} = \overline{\bigcap_{k=1}^\infty X_k} = \overline{\bigcap_{k=1}^\infty \widetilde{X}_k}.$$

Hence, if each $\widetilde{X}_k$ is compact, then $J(\Phi) = \bigcap_{k=1}^\infty X_k = \bigcap_{k=1}^\infty \widetilde{X}_k$.

Given an NIFS $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$ on some (U, X), we note that by excluding $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(j-1)}$, the sequence $\Phi^{(j)}, \Phi^{(j+1)}, \Phi^{(j+2)}, \ldots$ also forms an NIFS (which formally would be $\widetilde{\Phi}^{(1)}, \widetilde{\Phi}^{(2)}, \widetilde{\Phi}^{(3)}, \ldots$ where each $\widetilde{\Phi}^{(k)} = \Phi^{(k+j-1)}$). The new NIFS would then induce sets as in Definition 1.3, which we denote as $X_\omega^{(j)}, X_k^{(j)}$, and $J^{(j)}$ with the superscript used to indicate the relationship to the original NIFS. In particular, for the original NIFS the sets X_k may also be denoted $X_k^{(1)}$. See Example 1.1, illustrated in Figure 1, where the superscript indicates the column and the subscript indicates the row where a given set resides (noting that row 0 refers to the top row).

Remark 1.4 (Invariance Condition)*.* Note that for any $j \geq 1$ and $k \geq 0$, we can unpack the relevant definitions (defining each $X_0^{(j)} = X$) to see the following invariance condition

$$\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(X_k^{(j+1)}) = X_{k+1}^{(j)}, \tag{1.1}$$

which is illustrated in Figure 1 as a way of relating the diagonally adjacent sets $X_{k+1}^{(j)}$ and $X_k^{(j+1)}$.

Additional hypotheses lead to the following result.

Lemma 1.3 (Lemma 1.3 in [3])**.** *Let Φ be a NIFS on (U, X). When $\Phi^{(j)}$ is finite, we have*

$$\bigcup_{i \in I^{(j)}} \varphi_i^{(j)} \left(\overline{J^{(j+1)}}\right) = \overline{J^{(j)}}.$$

Hence, when $\Phi^{(j)}$ is finite and $J^{(j+1)}$ is compact (e.g., when all $\Phi^{(k)}$, for $k \geq j$, are finite), we see that $\bigcup_{i \in I^{(j)}} \varphi_i^{(j)} \left(J^{(j+1)}\right) = J^{(j)}$.

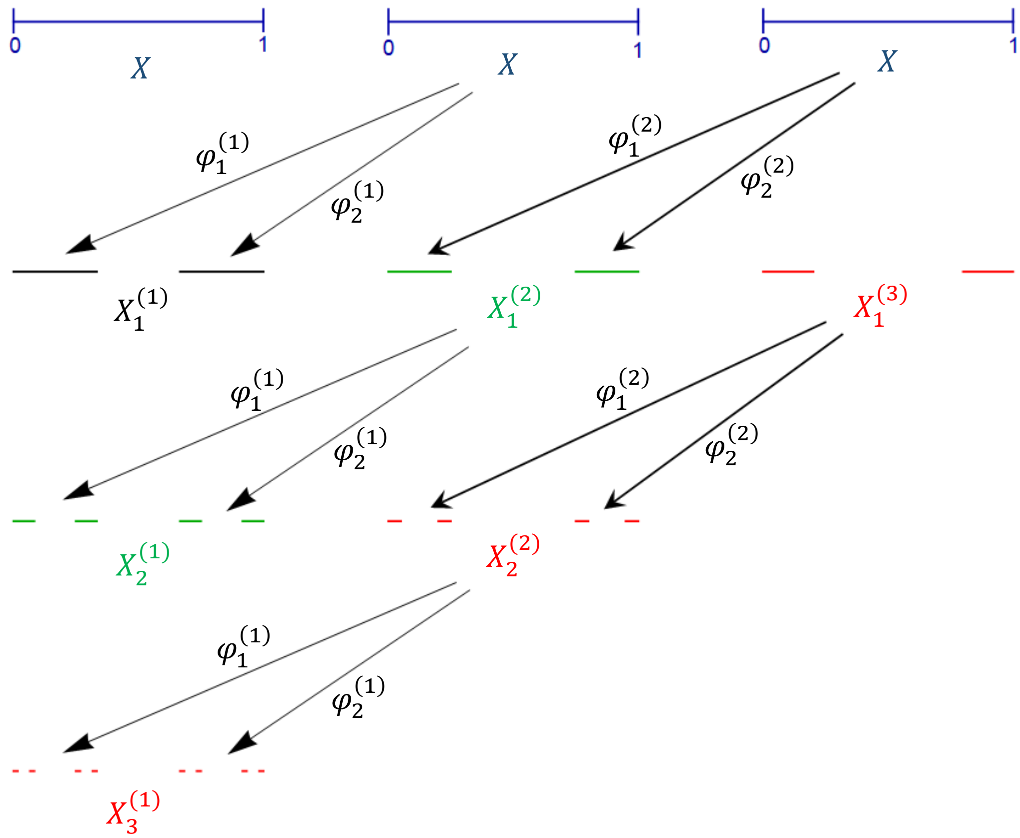

FIGURE 1. Table illustrating Example 1.1 with $a_1 = \frac{1}{3}, a_2 = \frac{1}{4}$, and $a_3 = \frac{1}{5}$. Note that sets in each column decrease down to the corresponding limit set, i.e., for each $j \in \mathbb{N}$ we have $\cap_{k=1}^{\infty} X_k^{(j)} = J^{(j)}$. Also, note that diagonally adjacent sets $X_{k+1}^{(j)}$ and $X_k^{(j+1)}$ are related by the invariance condition (1.1) in Remark 1.4.

Example 1.1. Let $X = [0, 1]$ denote the closed unit interval. Consider a sequence (a_j) such that each $0 < a_j \leq 1/3$, and define maps $\varphi_1^{(j)}(z) = a_j z$ and $\varphi_2^{(j)}(z) = a_j(z-1)+1$. Then the families of maps $\Phi^{(j)} = \{\varphi_1^{(j)}, \varphi_2^{(j)}\}$ define an NIFS. See Figure 1.

Strictly speaking, one has to first establish an open set $U \subseteq \mathbb{C}$ (e.g., $\Delta(0, 10)$) and corresponding compact subset X (e.g., $\overline{\Delta}(0, 9)$) to satisfy the NIFS condition that each $\varphi_i^{(j)}$ maps U into X, and then use Lemma 1.2 to replace X by the forward invariant set $[0, 1]$ without changing the limit set J. However, in later examples we shall leave it to the reader to check that such a procedure can be executed.

Remark 1.5 (Combining Stages). It will be useful later to analyze a limit set of some NIFS Φ by first combining stages. Here we present what this means, in particular, showing that this does not alter the limit set. First, for families of maps $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, we define $\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_n$ to be $\{f_1 \circ f_2 \circ \cdots \circ f_n : f_i \in \Gamma_i\}$.

Given an NIFS $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$ on some (U, X), we can create a new NIFS by combining finite strings of stages as follows. Consider any strictly increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers and define a new NIFS $\widetilde{\Phi}$ by $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(k_1)}$, $\widetilde{\Phi}^{(2)} = \Phi^{(k_1+1)} \circ \cdots \circ \Phi^{(k_2)}$, and, in general for $n > 1$, $\widetilde{\Phi}^{(n)} = \Phi^{(k_{n-1}+1)} \circ \cdots \circ \Phi^{(k_n)}$.

Notice that $\widetilde{\Phi}$ inherits all the defining properties of an NIFS from Φ. Furthermore, $J(\widetilde{\Phi}) = \bigcap_{n=1}^{\infty} X_{k_n} = \bigcap_{k=1}^{\infty} X_k = J(\Phi)$, since the sets X_k are decreasing.

1.2. **Hereditarily non Uniformly Perfect Sets.** We call a doubly connected domain A in $\mathbb{C}$ that can be conformally mapped onto a true (round) annulus $\mathrm{Ann}(w; r, R) = \{z : r < |z - w| < R\}$, for some $0 < r < R$, a *conformal annulus* with the *modulus* of A given by $\mathrm{mod}\ A = \log(R/r)$, noting that R/r is uniquely determined by A (see, e.g., the version of the Riemann mapping theorem for multiply connected domains in [1]).

Definition 1.4. A conformal annulus A is said to *separate* a set $F \subset \mathbb{C}$ if $F \cap A = \emptyset$ and F intersects both components of $\mathbb{C} \setminus A$.

The following well-known lemma (see, e.g., Theorem 2.1 of [5]) often allows one to replace a conformal annulus with an easier to work with round annulus.

Lemma 1.4. *Any conformal annulus $A \subset \mathbb{C}$ of sufficiently large modulus contains an essential true annulus B (i.e., B separates the boundary of A) with* $\mathrm{mod}\ A = \mathrm{mod}\ B + O(1)$. *Since, for any $R > 3r$ and any $w' \in \overline{\Delta}(w, r)$, the true annulus* $\mathrm{Ann}(w'; 2r, R - r)$ *is an essential annulus of* $\mathrm{Ann}(w; r, R)$, *we may choose B to be centered at any given point in the bounded component of* $\mathbb{C} \setminus A$.

Definition 1.5. A compact subset $F \subset \mathbb{C}$ with two or more points is *uniformly perfect* if there exists a uniform upper bound on the modulus of each conformal annulus which separates F.

The concept of hereditarily non uniformly perfect was introduced in [8] and can be thought of as a thinness criterion for sets which is a strong version of failing to be uniformly perfect.

Definition 1.6. A compact set $E \subset \mathbb{C}$ is called *hereditarily non uniformly perfect* (HNUP) if no subset of E is uniformly perfect.

Often a compact set is shown to be HNUP by showing it satisfies the following stronger property of *pointwise thinness*. This is done in several examples in [8, 2, 3], and will be done in the proof of Corollary 1.1.

Definition 1.7. A set $E \subset \mathbb{C}$ is *pointwise thin at* $z \in E$ if there exists a sequence of conformal annuli A_n each of which separates E, has z in the bounded component of its complement, and such that $\operatorname{mod} A_n \to +\infty$ while the Euclidean diameter of A_n tends to zero. A set $E \subset \mathbb{C}$ is called *pointwise thin* when it is pointwise thin at each of its points.

Note that any pointwise thin compact set is HNUP since none of its points can lie in a uniformly perfect subset. Also note that if E is pointwise thin, then $\overline{E}$ is pointwise thin at each point of E (but not necessarily pointwise thin at each point of $\overline{E}$ as the next example illustrates).

Example 1.2 (Closure of pointwise thin is not pointwise thin). The set $E = \{2^{-n} : n \in \mathbb{N}\}$ is trivially pointwise thin, but its closure $\overline{E}$ is not pointwise thin at 0 since the reader can check that the modulus of any round annulus separating $\overline{E}$ and containing 0 must be bounded by $\log 2$.

1.3. **Statements of the Main Results.** In this paper, we prove Theorem 1.1 and Corollary 1.1, regarding conformal NIFS's having the Strong Separation Condition, and Theorem 1.2, regarding analytic NIFS's which do not require the Strong Separation Condition but do require a certain type of separation condition.

Theorem 1.1. *Let Φ be a conformal NIFS on (U, X), with X connected, satisfying the Strong Separation Condition and the following*

Separating Annuli Condition: *there exists a sequence of conformal annuli $\{A_{j_n}\}_{n\in\mathbb{N}}$, where each A_{j_n} and the bounded component of $\mathbb{C} \setminus A_{j_n}$ are in X, such that for all $n \in \mathbb{N}$ the annulus A_{j_n} separates $X_1^{(j_n)}$ where* $\operatorname{mod} A_{j_n} \to \infty$ *as $n \to \infty$.*

For each $n \in \mathbb{N}$, choose $m_n \in I^{(j_n)}$ such that the set $\varphi_{m_n}^{(j_n)}(X)$ is surrounded by A_{j_n} (which can be done since X is connected and A_{j_n} separates $X_1^{(j_n)}$), and fix $\omega = (\omega_1, \omega_2, \dots) \in I^\infty$ such that $\omega_{j_n} = m_n$ for all $n \in \mathbb{N}$. Then, J is pointwise thin at $\pi_\Phi(\omega)$.

Remark 1.6. The Separating Annuli Condition can be visualized in Figure 1 in Example 1.1 by considering annuli A_j of maximum modulus separating the two components in each $X_1^{(j)}$ (in row 1), noting that $\operatorname{mod} A_j \to \infty$ exactly when $a_j \to 0$.

Corollary 1.1. *Let Φ be a conformal NIFS on (U, X), with X connected, satisfying the Strong Separation Condition. Suppose along some subsequence j_n, we have $2 \le \#\Phi^{(j_n)} < \infty$ for all $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$,*

$$b_{j_n} = \min\{\mathrm{dist}(\varphi_i^{(j_n)}(X), \partial X) : i \in I^{(j_n)}\},$$

$$\delta_{j_n} = \min\{\mathrm{dist}(\varphi_a^{(j_n)}(X), \varphi_b^{(j_n)}(X)) : a, b \in I^{(j_n)} \text{ with } a \neq b\}$$

and

$$\eta_{j_n} = \max\{\mathrm{diam}(\varphi_i^{(j_n)}(X)) : i \in I^{(j_n)}\}.$$

Suppose for some $c > 1$, we have $\delta_{j_n} \le c b_{j_n}$ for all $n \in \mathbb{N}$. Further suppose $\frac{\delta_{j_n}}{\eta_{j_n}} \to \infty$ as $n \to \infty$. Then, $J = \pi_\Phi(I^\infty)$ is pointwise thin (and thus HNUP when J is compact).

Remark 1.7. Since each $\delta_{j_n} \le \mathrm{diam}(X)$, we see that we may choose $c = \frac{\mathrm{diam}(X)}{\inf_n\{b_{j_n}\}}$ when $\inf_n\{b_{j_n}\} > 0$.

Remark 1.8. Since each $\delta_{j_n} \le \mathrm{diam}(X)$, we see that for $\frac{\delta_{j_n}}{\eta_{j_n}} \to \infty$ we must have $\eta_{j_n} \to 0$. In such a situation then, Φ cannot satisfy the Derivative Condition from [3] (a key assumption required to prove uniform perfectness of J in Theorem 2.1 of [3]) which states that there exists $\eta > 0$ such that for all $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$ we have $|\varphi'| \ge \eta$ on X. See Remark 5.1 of [3].

Remark 1.9. Corollary 1.1 applies much more generally when we recall that one can combine stages in the manner described in Remark 1.5. Specifically, we may show $J(\Phi)$ is pointwise thin by applying Corollary 1.1 to any $\widetilde{\Phi}$ created by combining stages in Φ. This technique of combining stages is used to analyze Example 4.2 of [3] (see also Example 2.2 in this paper).

Theorem 1.2. *Suppose Φ is an analytic NIFS such that $\overline{J^{(n)}}$, for some integer $n > 1$, is pointwise thin (e.g., when the NIFS given by $\Phi^{(n)}, \Phi^{(n+1)}, \Phi^{(n+2)}, \ldots,$ satisfies the hypotheses of Corollary 1.1 with each $\Phi^{(j)}$ finite). Suppose also that $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(n-1)}$ is finite with $\varphi_a(\overline{J^{(n)}}) \cap \varphi_b(\overline{J^{(n)}}) = \emptyset$ for all distinct $\varphi_a, \varphi_b \in \widetilde{\Phi}^{(1)}$ (e.g., when Φ satisfies the Strong Separation Condition). Then $\overline{J(\Phi)}$ is pointwise thin.*

This paper is organized as follows. Section 2 contains basic results regarding the hyperbolic metric and images of pointwise thin sets under analytic maps, along with some examples to show that our main result generalizes Theorem 4.1(1) of [8] and Example 4.2 of [3]. Section 3 contains applications of Corollary 1.1 to non-autonomous Julia sets along polynomial sequences. Section 4 is then used to prove the Theorem 1.1, Corollary 1.1, and Theorem 1.2.

2. Basic Facts and Examples

The main object of interest to this paper is the *analytic* NIFS. This allows us, via the next result used similarly in [7], to employ the hyperbolic metric in the definition of NIFS. In particular, any sequence $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$, where each $\Phi^{(j)}$ is a collection of non-constant *analytic* functions $(\varphi_i^{(j)} : U \to X)_{i \in I^{(j)}}$, where each function maps the non-empty open connected set $U \subset \mathbb{C}$ into a compact set $X \subset U$, will automatically be uniformly contracting with respect to the hyperbolic metric on U. Note that $U \subseteq \mathbb{C}$ must support a hyperbolic metric since U cannot be the plane or punctured plane else the image of U under a non-constant analytic map would have to be dense in $\mathbb{C}$.

Lemma 2.1. *[Lemma 2.1 of [7]] If the analytic function φ maps an open connected set $U \subset \mathbb{C}$ into a compact set $X \subset U$, then there exists $0 < s < 1$, which depends on U and X only, such that $d(\varphi(z), \varphi(w)) \le s d(z, w)$ for all $z, w \in X$ where d is the hyperbolic metric defined on U.*

The following result follows from Lemma 1.4 and the fact that *locally* non-constant analytic maps are either conformal or behave like $z \mapsto z^k$ for some $k \in \mathbb{N}$, which can distort the modulus of an annulus by at most a factor of k. We leave the details to the reader noting, however, that one may follow the style of argument used to prove Proposition 3.1 of [3].

Proposition 2.1. *Let $f : U \to \mathbb{C}$ be non-constant and analytic on open connected $U \subset \mathbb{C}$. Suppose that $E \subset U$ is pointwise thin at $z \in E$. Then $f(E)$ is pointwise thin at $f(z) \in f(E)$.*

We now present examples to illustrate Corollary 1.1, showing how it generalizes the ad hoc methods of [8] and [3].

Example 2.1. Each set $I_{\bar{a}}$ in Theorem 4.1 of [8] is a limit set of a NIFS suitably chosen as follows. Set $X = [0, 1]$, fix $m \in \{2, 3, \ldots\}$, and choose $0 < a \le \frac{1}{m+1}$. Fix a sequence $\bar{a} = (a_1, a_2, \ldots)$ such that $0 < a_k \le a$ for $k = 1, 2, \ldots$. For each $k \in \mathbb{N}$, set $\Phi^{(k)}$ to be the collection $\{\varphi_1^{(k)}, \ldots, \varphi_m^{(k)}\}$ of linear maps, each with derivative a_k, such that the images $\varphi_1^{(k)}(X), \ldots, \varphi_m^{(k)}(X)$ are m equally spaced subintervals of X with $\varphi_1^{(k)}(X) = [0, a_k]$ and $\varphi_m^{(k)}(X) = [1 - a_k, 1]$. Example 1.1, illustrated in Figure 1, is such an NIFS (with $m = 2$). Each set X_k then coincides with what [8] calls I_k, and consists of m^k basic intervals. And the limit set J then coincides with what [8] calls $I_{\bar{a}}$.

Theorem 4.1(1) of [8] shows that J pointwise thin (and thus HNUP) when $\liminf a_k = 0$. We now show that this also follows from Corollary 1.1. In order to use this corollary we set $U = \Delta(\frac{1}{2}, 0.7)$ and $X = \overline{\Delta}(\frac{1}{2}, 0.6)$, recalling that Lemma 1.2 shows that J is unchanged by this change of X from $[0, 1]$.

Selecting a subsequence $a_{k_n} \to 0$, the reader can quickly check that $\inf_n\{b_{k_n}\} > 0, \inf_n\{\delta_{k_n}\} > 0$, and $\eta_{k_n} = a_{k_n} \cdot \mathrm{diam}(X) \to 0$, and thus Corollary 1.1 applies (since Φ clearly satisfies the Strong Separation Condition). We also note that when $\liminf a_k = 0$, Corollary 1.1 shows J is pointwise thin even when the strict setup above is considerably relaxed (e.g., the sets $\varphi_1^{(k)}([0,1]), \ldots, \varphi_m^{(k)}([0,1])$ do not need to be *equally spaced* subintervals of $[0,1]$).

Lastly, note that Theorem 4.1(2) of [8] shows that J is uniformly perfect when $\liminf a_k > 0$, which also follows from the more general Theorem 2.1 of [3] as detailed in Example 4.1 of [3].

Example 2.2. We now show how Corollary 1.1 can be applied to Example 4.2 of [3]. Set $f_1(z) = \frac{z}{3}, f_2(z) = \frac{z+2}{3}$ and $f_3(z) = \frac{1}{3}(z - \frac{1}{2}) + \frac{1}{2}$. We fix a sequence of postive integers (l_j), and create NIFS Ψ on (U, X) with $U = \Delta(\frac{1}{2}, 0.7)$ and $X = \overline{\Delta}(\frac{1}{2}, 0.6)$ by stipulating that, for each $k \in \mathbb{N}$, $\Psi^{(k)} = \{f_1 \circ f_3^{l_k}, f_2 \circ f_3^{l_k}\}$. We now show that $\sup l_j = +\infty$ implies $J(\Psi)$ is pointwise thin (noting that use of Theorem 2.1 of [3] as detailed Example 4.2 of [3] shows that $\sup l_j < +\infty$ implies $J(\Psi)$ is uniformly perfect).

Select a subsequence $l_{k_n} \to \infty$. Since the images $f_1 \circ f_3^{l_k}(X) \subseteq f_1(X) \subset \mathrm{Int}(X)$ and $f_2 \circ f_3^{l_k}(X) \subseteq f_2(X) \subset \mathrm{Int}(X)$, the reader can quickly check that Ψ clearly satisfies the Strong Separation Condition and $\inf_n\{b_{k_n}\} > 0, \inf_n\{\delta_{k_n}\} > 0$, and $\eta_{k_n} = \frac{\mathrm{diam}(X)}{3^{l_{k_n}+1}} \to 0$ (since each map in $\Psi^{(k_n)}$ is linear with derivative $\frac{1}{3^{l_{k_n}+1}}$). Hence, Corollary 1.1 applies.

3. Applications to Non-Autonomous Julia Sets

Given a sequence of complex polynomials (P_j), define its *Fatou set* $\mathcal{F} = \mathcal{F}((P_j))$ by

$$\mathcal{F} = \{z \in \overline{\mathbb{C}} : \{P_n \circ \cdots \circ P_2 \circ P_1\}_{n=1}^{\infty} \text{ is a normal family on some neighborhood of } z\}$$

where we take our neighborhoods with respect to the spherical topology on $\overline{\mathbb{C}}$. We then define the *Julia set* $\mathcal{J} = \mathcal{J}((P_j))$ to be the complement $\overline{\mathbb{C}} \setminus \mathcal{F}$.

Theorem 3.1. *Let f be a polynomial on $\mathbb{C}$ of degree at least 2. Suppose f has no critical values in $\overline{\mathbb{D}}$ and that $f^{-1}(\overline{\mathbb{D}}) \subset \mathbb{D}$. Fixing a sequence $a_j \in \mathbb{C}$ with each $|a_j| > 1$, we define polynomials $P_j(z) = a_j f(z)$. Then*

(1) $\mathcal{J}((P_j))$ is uniformly perfect if and only if $\limsup |a_j| < \infty$, and

(2) $\mathcal{J}((P_j))$ is pointwise thin (and HNUP) if and only if $\limsup |a_j| = \infty$.

Remark 3.1. For $a, c \in \mathbb{C}$ with $|c| > 1$ and $|a| - |c| > 1$, one may choose $f(z) = az^2 + c$ in the above theorem. Note then that $|z| \geq 1$ implies $|f(z)| = |az^2 + c| \geq |a| - |c| > 1$, i.e., $f(\mathbb{C} \setminus \mathbb{D}) \subseteq \mathbb{C} \setminus \overline{\mathbb{D}}$, which gives that $f^{-1}(\overline{\mathbb{D}}) \subset \mathbb{D}$. Also, clearly the sole critical value of f is $c \notin \overline{\mathbb{D}}$. Hence applying the above theorem with such an f and a suitable sequence (a_j) with $\limsup |a_j| = \infty$, we can create a simple sequence of polynomials with pointwise thin (and thus HNUP) Julia set without the complicated arguments presented in [2].

Proof. (1) The Julia set of a bounded sequence of polynomials is known to be uniformly perfect (see Theorem 1.21 of [9]).

(2) Suppose $\limsup |a_j| = \infty$, and choose a subsequence a_{j_n} such that $|a_{j_n}| \to \infty$. We complete the proof by showing $\mathcal{J}((P_j))$ is pointwise thin and compact. Calling d the degree of f, we note that f has d well defined inverse branches $f_1, \ldots, f_d$, on some open connected set $U = \Delta(0, 1+\epsilon) \supset \overline{\mathbb{D}}$ since all critical values of f lie outside of $\overline{\mathbb{D}}$. Furthermore, we note that we may choose U such that $f^{-1}(U) \subset \overline{\mathbb{D}}$. Hence, each P_j has d well defined inverse branches on U given by $\varphi_i^{(j)}(z) = f_i(\frac{z}{a_j})$ for $i = 1, \ldots, d$.

For each $j \in \mathbb{N}$, let $\Phi^{(j)} = \{\varphi_1^{(j)} \ldots, \varphi_d^{(j)}\}$ and note that these families form an NIFS Φ on (U, X) where $X = \overline{\mathbb{D}}$. For each j, note that $\varphi_i^{(j)}(X) = f_i(\overline{\Delta}(0, \frac{1}{|a_j|})) \subset f_i(X) \subset \text{Int}(X)$ for $i = 1, \ldots, d$. Hence, Φ satisfies the Strong Separation Condition and, using the notation of Corollary 1.1, we also see that for each $n \in \mathbb{N}$,

$$b_{j_n} \geq b_0 := \min\{\text{dist}(f_i(X), \partial X) : i \in \{1, \ldots, d\}\} > 0,$$

$$\delta_{j_n} \geq \delta_0 := \min\{\text{dist}(f_a(X), f_b(X)) : a, b \in \{1, \ldots, d\} \text{ with } a \neq b\} > 0$$

and

$$\eta_{j_n} = \max\{\text{diam}(\varphi_i^{(j_n)}(X)) : i \in \{1, \ldots, d\}\} = \max\{\text{diam}(f_i(\overline{\Delta}(0, \frac{1}{|a_{j_n}|}))) : i \in \{1, \ldots, d\}\} \to 0.$$

Since $\inf\{b_{j_n}\} > 0$, Corollary 1.1 yields that $J(\Phi)$ is pointwise thin since $\frac{\delta_{j_n}}{\eta_{j_n}} \geq \frac{\delta_0}{\eta_{j_n}} \to \infty$. Further, we note that $J(\Phi)$ is compact since each $I^{(j)}$ is finite.

The result then follows by showing that $\mathcal{J}((P_j)) = J(\Phi)$. Note that $J(\Phi) = \{z \in \mathbb{C} : P_j \circ \cdots \circ P_1(z) \in \overline{\mathbb{D}} \text{ for each } j\}$. Also note that $\mathbb{C} \setminus \overline{\mathbb{D}}$ is forward invariant under each P_j, and so it follows from Montel's Theorem that $\mathbb{C} \setminus J(\Phi) \subseteq \mathcal{F}((P_j))$, i.e., $\mathcal{J}((P_j)) \subseteq J(\Phi)$. Since $J(\Phi)$ is pointwise thin, it is clear that $J(\Phi)$ has no interior. This implies that any $z \in J(\Phi)$, which necessarily has as its orbit contained in the compact subset $f_1(X) \cup \cdots \cup f_d(X)$ of $\mathbb{D}$, must be arbitrarily close to points whose orbits escape $\overline{\mathbb{D}}$. Hence, $J(\Phi) \subseteq \mathcal{J}((P_j))$. □

Corollary 3.1. *Let f be a polynomial on $\mathbb{C}$ of degree at least 2. Suppose f has no critical values in $\overline{\mathbb{D}}$ and that $f^{-1}(\overline{\mathbb{D}}) \subset \mathbb{D}$. Let τ be a probability measure on $\mathbb{C} \setminus \overline{\mathbb{D}}$ with unbounded support. Then for* *almost all sequences* $(a_j) \in \prod_{j=1}^{\infty}(\mathbb{C} \setminus \overline{\mathbb{D}})$ *with respect to* $\tilde{\tau} = \bigotimes_{j=1}^{\infty} \tau$*, the maps* $P_j = a_j \cdot f$ *define a sequence of polynomials whose Julia set* $\mathcal{J}((P_j))$ *is pointwise thin.*

Proof. For $N \in \mathbb{N}$, set $B_N = \{(a_j) : |a_j| \leq N \text{ for all } j\}$ and note that since τ has unbounded support, $\tilde{\tau}(B_N) = 0$ by the law of large numbers. Hence, $\tilde{\tau}(\cup_{N \in \mathbb{N}} B_N) = 0$, i.e., the set of bounded sequences has $\tilde{\tau}$-measure zero. The result then follows from Theorem 3.1. □

4. Proof of the Main Results

In this section we prove the Theorem 1.1, Corollary 1.1, and Theorem 1.2.

Proof of Theorem 1.1. Note that since the NIFS Φ is conformal and both the annulus A_{j_n} and its bounded complementary component lie inside $X \subset U$, we see that $\pi_\Phi(\omega) \in \varphi_{\omega_1 \cdots \omega_{j_n - 1}}(\varphi_{m_n}^{(j_n)}(X))$ (see Remark 1.3) is surrounded by the conformal annulus $A'_{j_n} = \varphi_{\omega_1 \cdots \omega_{j_n-1}}(A_{j_n})$, which separates $\varphi_{\omega_1 \cdots \omega_{j_n-1}}(X_1^{(j_n)}) \subseteq X_{j_n}^{(1)}$. See Figure 2. We claim that $A'_{j_n} \cap X_{j_n}^{(1)} = \emptyset$, from which it follows that A'_{j_n} separates $X_{j_n}^{(1)}$, and thus separates J. Since mod $A_{j'_n} =$ mod $A_{j_n} \to \infty$ with $\operatorname{diam}(A'_{j_n}) \to 0$, we see that J is pointwise thin at $\pi_\Phi(\omega)$.

To prove the claim, suppose towards a contradiction that A'_{j_n} meets

$$\begin{aligned} X_{j_n}^{(1)} = \bigcup_{\omega^* \in I^{j_n}} \varphi_{\omega^*}(X) &= \bigcup_{\omega_1^* \cdots \omega_{j_n-1}^* \in I^{j_n - 1}} \bigcup_{\omega_{j_n}^* \in I^{(j_n)}} \varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(\varphi_{\omega_{j_n}^*}(X)) \\ &= \bigcup_{\omega_1^* \cdots \omega_{j_n-1}^* \in I^{j_n - 1}} \varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X_1^{(j_n)}). \end{aligned}$$

Hence, A'_{j_n} meets $\varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X_1^{(j_n)})$ for some $\omega_1^* \cdots \omega_{j_n-1}^* \in I^{j_n-1}$. Note that $\omega_1^* \cdots \omega_{j_n-1}^* \neq \omega_1 \cdots \omega_{j_n-1}$ since A'_{j_n} separates $\varphi_{\omega_1 \cdots \omega_{j_n-1}}(X_1^{(j_n)})$. However, since $X_1^{(j_n)} \subseteq X$, $A_{j_n} \subseteq X$, and $\varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X) \cap \varphi_{\omega_1 \cdots \omega_{j_n-1}}(X) = \emptyset$ by the strong separation condition, we see that $\varphi_{\omega_1^* \cdots \omega_{j_n-1}^*}(X_1^{(j_n)})$ cannot meet $A'_{j_n} = \varphi_{\omega_1 \cdots \omega_{j_n-1}}(A_{j_n})$, which is a contradiction. □

Proof of Corollary 1.1. Pick an arbitrary $\omega \in I^\infty$. For each n, choose some $z_n \in \varphi_{\omega_{j_n}}^{(j_n)}(X)$, and define $A_{j_n} = \operatorname{Ann}(z_n; \eta_{j_n}, \frac{\delta_{j_n}}{c})$, which by definition of η_{j_n} must surround $\varphi_{\omega_{j_n}}^{(j_n)}(X)$. Hence by definition of δ_{j_n}, the annulus A_{j_n} must separate $X_1^{(j_n)}$. Lastly, since $\frac{\delta_{j_n}}{c} \leq b_{j_n} \leq \operatorname{dist}(\varphi_{\omega_{j_n}}^{(j_n)}(X), \partial X)$, we see that $\Delta(z_n, \frac{\delta_{j_n}}{c}) \subseteq X$. Thus by Theorem 1.1, noting that mod $A_{j_n} = \log \frac{\delta_{j_n}}{c \eta_{j_n}} \to \infty$, we see that J is pointwise thin at $\pi_\Phi(\omega)$. The proof is then complete by noting that since Φ satisfies the Strong Separation Condition, Lemma 1.1 implies $J = \pi_\Phi(I^\infty)$. □

Proof of Theorem 1.2. Consider the analytic NIFS $\widetilde{\Phi}$ given by $\widetilde{\Phi}^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(n-1)}$ and $\widetilde{\Phi}^{(j)} = \Phi^{(j+n-2)}$ for each $j > 1$. Hence, by Remark 1.5, we see that $J(\Phi) = J(\widetilde{\Phi})$. By Proposition 2.1, for

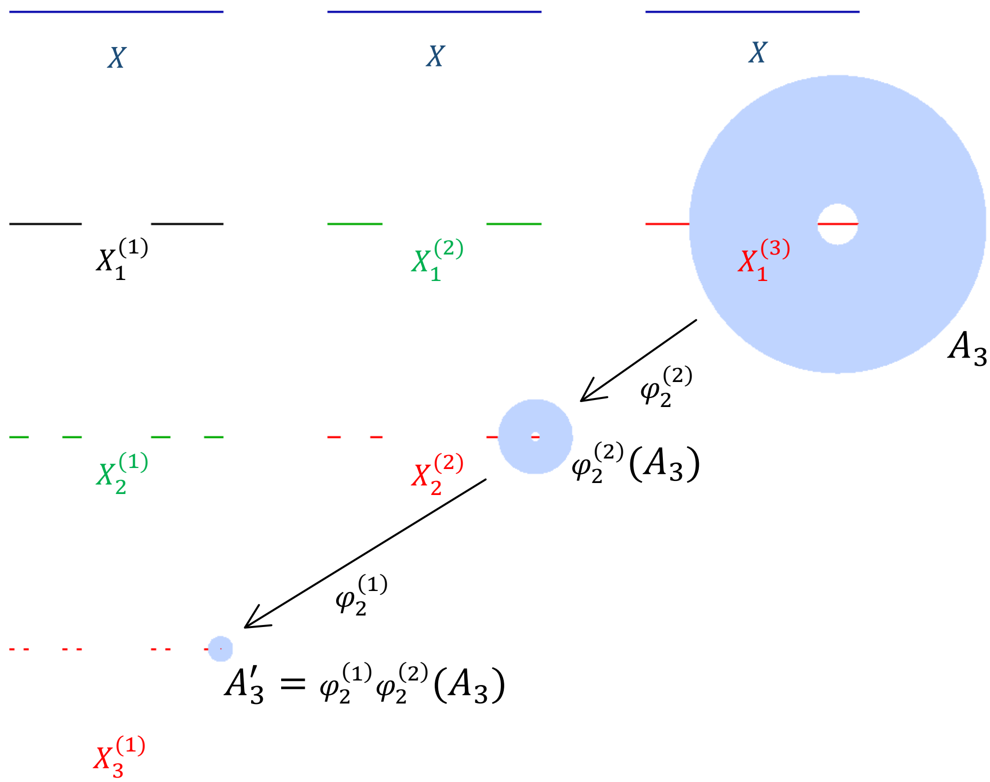

FIGURE 2. Table illustrating the proof of Theorem 1.1 using the system of Example 1.1.

each $\varphi \in \widetilde{\Phi}^{(1)}$ the set $\phi\left(\overline{J^{(n)}}\right)$ is pointwise thin. Lemma 1.3 gives that $\overline{J^{(1)}} = \bigcup_{\varphi \in \widetilde{\Phi}^{(1)}} \varphi\left(\overline{J^{(n)}}\right)$, and the result follows since the finite disjoint union of compact pointwise thin sets is pointwise thin. $\square$

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