Hausdorff measures and packing measures of limit sets of CIFSs of generalized complex continued fractions*†

Kanji INUI‡
Course of Mathematical Science, Department of Human Coexistence,
Graduate School of Human and Environmental Studies, Kyoto University
Yoshida-nihonmatsu-cho, Sakyo-ku, Kyoto, 606-8501, JAPAN
E-mail: inui.kanji.43a@st.kyoto-u.ac.jp

Hiroki SUMI
Course of Mathematical Science, Department of Human Coexistence,
Graduate School of Human and Environmental Studies, Kyoto University
Yoshida-nihonmatsu-cho, Sakyo-ku, Kyoto, 606-8501, JAPAN
E-mail: sumi@math.h.kyoto-u.ac.jp
Homepage: http://www.math.h.kyoto-u.ac.jp/˜sumi/index.html

Abstract

We consider a certain family of CIFSs of the generalized complex continued fractions with a complex parameter space. We show that for each CIFS of the family, the Hausdorff measure of the limit set of the CIFS with respect to the Hausdorff dimension is zero and the packing measure of the limit set of the CIFS with respect to the Hausdorff dimension is positive (main result). This is a new phenomenon of infinite CIFSs which cannot hold in finite CIFSs. We prove the main result by showing some estimates for the unique conformal measure of each CIFS of the family and by using some geometric observations. [1]

1 Introduction

Recent studies of fractal geometry have been developed in many directions. One of the most developed is the study of the limit sets of conformal iterated function systems (for short, CIFS). Indeed, the general theory of limit sets of CIFSs with finitely many mappings (for short, *finite* CIFS) has been well studied (See [1], [4]). For example, there exists the formula on the Hausdorff dimension of the limit sets, and there exist statements which claim that the Hausdorff measure of the limit set of any finite CIFS with respect to the Hausdorff dimension is positive and finite and the packing measure of the limit set with respect to the Hausdorff dimension is also positive and finite (from this, we deduce that the Hausdorff dimension of the limit set of any finite CIFS and the packing dimension of the limit set are the same in general).

On the other hand, studies of limit sets of conformal iterated function systems with infinitely many mappings (for short, *infinite* CIFS) were initiated by Mauldin and Urbański ([4], [5], [6]) and there are many related results on infinite CIFSs with overlaps by Mihailescu and Urbański ([7], [8]). Mauldin and Urbański found a formula on the Hausdorff dimension of limit sets generalizing the above formula on the Hausdorff dimension of limit sets of finite CIFSs. In addition, they found a condition under which the Hausdorff measure of the limit set of the infinite CIFS with respect to the Hausdorff dimension is zero.

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‡corresponding author
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Moreover, Mauldin and Urbański constructed an interesting example of an infinite CIFS which is related to the complex continued fractions in the paper [4]. The construction of the example is the following. Let $X := \{z \in \mathbb{C} \mid |z-1/2| \le 1/2\}$. We call $\hat{S} := \{\hat{\phi}_{(m,n)}(z)\colon X \to X \mid (m,n) \in \mathbb{Z} \times \mathbb{N}\}$ the CIFS of complex continued fractions, where $\mathbb{Z}$ is the set of the integers, $\mathbb{N}$ is the set of the positive integers and

$$\hat{\phi}_{(m,n)}(z) := \frac{1}{z+m+ni} \quad (z \in X).$$

Let $\hat{J}$ be the limit set of $\hat{S}$ (see Definition 2.1) and $\hat{h}$ be the Hausdorff dimension of $\hat{J}$. For each $s \ge 0$, we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and denoted by $\mathcal{P}^s$ the s-dimensional packing measure. For this example, Mauldin and Urbański showed the following theorem.

Theorem 1.1 (D. Mauldin, M. Urbanski (1996))**.** Let S be the CIFS of complex continued fractions. Then, we have that $\mathcal{H}^{\hat{h}}(\hat{J}) = 0$ and $\mathcal{P}^{\hat{h}}(\hat{J}) > 0$.

This is an example of infinite CIFS of which the Hausdorff measure of the limit set with respect to the Hausdorff dimension is zero and the packing measure of the limit set is positive. Note that this is a new phenomenon of infinite CIFSs which cannot hold in finite CIFSs.

It is interesting for us to ask for an infinite CIFS S, how often we have the situation that $\mathcal{H}^{h_S}(J_S) = 0$ and $\mathcal{P}^{h_S}(J_S) > 0$, where we denote by J_S the limit set of S (see Definition 2.1) and h_S the Hausdorff dimension of J_S. We considered the generalization of $\hat{S}$ in our previous paper [2]. That is, we introduced a family of CIFSs of the generalized complex continued fractions $\{S_\tau\}_{\tau \in A_0}$ to present new and interesting examples of infinite CIFSs. Note that $\{S_\tau\}_{\tau \in A_0}$ is a family of CIFSs which has uncountably many elements. The aim of this paper is to generalize Theorem 1.1 and to show that $\mathcal{H}^{h_\tau}(J_\tau) = 0$ and $\mathcal{P}^{h_\tau}(J_\tau) > 0$ for each $\tau \in A_0$ to find new and interesting examples of infinite CIFSs with the phenomenon which cannot hold in finite CIFSs.

The precise statement is the following. Let

$$A_0 := \{\tau = u + iv \in \mathbb{C} \mid u \ge 0 \text{ and } v \ge 1\}$$

and $X := \{z \in \mathbb{C} \mid |z-1/2| \le 1/2\}$. Also, we set $I_\tau := \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{N}\}$ for each $\tau \in A_0$, where $\mathbb{N}$ is the set of the positive integers.

Definition 1.2 (The CIFS of generalized complex continued fractions)**.** For each $\tau \in A_0$, $S_\tau := \{\phi_b\colon X \to X \mid b \in I_\tau\}$ is called the CIFS of generalized complex continued fractions. Here,

$$\phi_b(z) := \frac{1}{z+b} \quad (z \in X).$$

The family $\{S_\tau\}_{\tau \in A_0}$ is called the family of CIFSs of generalized complex continued fractions. For each $\tau \in A_0$, let J_τ be the limit set of the CIFS S_τ (see Definition 2.1) and let h_τ be the Hausdorff dimension of the limit set J_τ.

We remark that this family of CIFSs is a generalization of $\hat{S}$ in some sense. The system S_τ is related to "generalized" complex continued fractions since each point of the limit set of S_τ is of the form

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

for some sequence $(b_1, b_2, b_3, \ldots)$ in I_τ (See Definition 2.1). Note that there are many kinds of general theories for continued fractions and related iterated function systems ([3], [4], [5], [8]).

We now give the main result of this paper.

Theorem 1.3 (Main result)**.** Let $\{S_\tau\}_{\tau \in A_0}$ be the family of CIFSs of generalized complex continued fractions. Then, for each $\tau \in A_0$, we have $\mathcal{H}^{h_\tau}(J_\tau) = 0$ and $\mathcal{P}^{h_\tau}(J_\tau) > 0$.

Remark 1.4. It was shown that for each $\tau \in A_0$, $\overline{J_\tau} \setminus J_\tau$ is at most countable and $h_\tau = \dim_{\mathcal{H}}(\overline{J_\tau})$ ([2]). Thus, for each $\tau \in A_0$, we have $\mathcal{H}^{h_\tau}(\overline{J_\tau}) = \mathcal{H}^{h_\tau}(J_\tau) = 0$. Also, for each $\tau \in A_0$, since the set of attracting fixed points of elements of the semigroup generated by S_τ is dense in J_τ, Theorem 1.1 of [9] implies that $\overline{J_\tau}$ is equal to the Julia set of the rational semigroup generated by $\{\phi_b^{-1} \mid b \in I_\tau\}$.

Remark 1.5. By the general theories of finite CIFSs, the Hausdorff measure of the limit set of each finite CIFS with respect to the Hausdorff dimension and the packing measure of the limit set with respect to the Hausdorff dimension is positive and finite. However, Theorem 1.3 indicates that for each S_τ of the family of CIFSs of generalized complex continued fractions, which consists of uncountably many elements, the Hausdorff measure of the limit set with respect to the Hausdorff dimension is zero and the packing measure of the limit set with respect to the Hausdorff dimension is positive. This is also a new phenomenon which cannot hold in the finite CIFSs.

Ideas and strategies to prove the main result are the following. To prove $\mathcal{H}^{h_\tau}(J_\tau) = 0$, we use some results from the paper [4] and some results from our previous paper [2]. For example, we use the fact that for each $\tau \in A_0$, S_τ is hereditarily regular (see Lemma 3.3), that for each $\tau \in A_0$, there exists a conformal measure of S_τ (see Proposition 2.7) and that for each $\tau \in A_0$, $1 < h_\tau < 2$ (see Lemma 3.4).

In order to prove the main results, we show that for each $\tau \in A_0$, S_τ satisfies the assumption of the Theorem 2.8. That is, by using Lemma 3.5, we need to show that there exists a sequence $\{r_j\}_{j=1}^{\infty}$ in the set of positive real numbers such that

$$\limsup_{j\to\infty} \frac{m_\tau(B(0, r_j))}{r_j^{h_\tau}} = \infty, \tag{1}$$

where m_τ is the conformal measure of S_τ (see Proposition 2.7).

It is worth pointing out that in order to prove the main result, we use some sharp estimates on the values of the conformal measure. In order to prove the sharp estimates, we have to show another basic estimate at first (see Lemma 3.2). By the properties of the conformal measure (see Proposition 2.7) and by Lemma 3.2, we have that there exists $K_0 \geq 1$ such that for each $b \in I_\tau$, $\phi_b(X) \subset B(0, K_0|b|^{-1})$ and

$$m_\tau(\phi_b(X)) \geq \int_X |\phi_b'|^{h_\tau} \mathrm{d}m_\tau \geq (K_0^{-1}|b|^{-2})^{h_\tau} m_\tau(X) \geq K_0^{-h_\tau}|b|^{-2h_\tau}. \tag{2}$$

Moreover, by using properties of conformal measure (see Proposition 2.7), we have that for all $b, b' \in I_\tau$ with $b \neq b'$, $m_\tau(\phi_b(X) \cap \phi_{b'}(X)) = 0$. Then, for each $\tau \in A_0$, let $N_\tau \in \mathbb{N}$ be large enough and for each $r > 0$, we set $I_\tau(r) := \{b \in I_\tau \mid K_0 r^{-1} < |b| \leq N_\tau K_0 r^{-1}\}$. Note that if $b \in I_\tau(r)$, then $|b|^{-1} > K_0^{-1} N_\tau^{-1} r$ and $m_\tau(\phi_b(X)) \gtrsim r^{2h_\tau}$.

We next show some basic results on the estimate of $|I_\tau(r)|$ (see Lemma 4.2, Proposition 4.3 and Lemma 4.4) by the general theory of linear algebra and elementary geometric observations. By these results, we show that if $r > 0$ is small enough, then $|I_\tau(r)| \gtrsim r^{-2}$ (see inequality (16)). In addition, by the inequality (2), we deduce that for $b \in I_\tau(r)$, we have $B(0, r) \supset B(0, K_0|b|^{-1}) \supset \phi_b(X)$ and if $r > 0$ is small enough, then

$$m_\tau(B(0,r)) \geq \sum_{b\in I_\tau(r)} m_\tau(\phi_b(X)) \gtrsim \sum_{b\in I_\tau(r)} |b|^{-2h_\tau} \gtrsim \sum_{b\in I_\tau(r)} r^{2h_\tau} = |I_\tau(r)|\, r^{2h_\tau} \gtrsim r^{2h_\tau - 2} \tag{3}$$

(see inequality (17)). By the inequality (3), we finally show that if $r > 0$ is small enough,

$$\frac{m_\tau(B(0,r))}{r^{h_\tau}} \gtrsim r^{h_\tau - 2} = \left(\frac{1}{r}\right)^{2-h_\tau}.$$

By Lemma 3.4 (that is, $2 > h_\tau$), we deduce (1).

To prove $\mathcal{P}^{h_\tau}(J_\tau) > 0$, we need to show for each $\tau \in A_0$, S_τ satisfies the assumption of the Theorem 2.9. That is, we need to show that for each $\tau \in A_0$, $J_\tau \cap \mathrm{Int}(X) \neq \emptyset$. By geometric observations and some properties of $\phi_b \in S_\tau$, we obtain that $J_S \cap \mathrm{Int}(X) \neq \emptyset$.

The rest of the paper is organized as follows. In Section 2, we summarize the theory of CIFSs without proofs. In Section 3, we give the proofs of some properties of the CIFS of the generalized complex continued fractions. In Section 4, we prove the main result of this paper.

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2 Conformal iterated function systems

In this section, we recall general settings of CIFSs ([4], [5]).

Definition 2.1 (Conformal iterated function system)**.** Let $X \subset \mathbb{R}^d$ be a non-empty compact and connected set and let I be a finite set or bijective to $\mathbb{N}$. Suppose that I has at least two elements. We say that $S := \{\phi_i \colon X \to X \mid i \in I\}$ is a conformal iterated function system (for short, CIFS) if S satisfies the following conditions.

1. Injectivity: For all $i \in I$, $\phi_i \colon X \to X$ is injective.

2. Uniform Contractivity: There exists $c \in (0,1)$ such that, for all $i \in I$ and $x, y \in X$, the following inequality holds.
$$|\phi_i(x) - \phi_i(y)| \le c|x - y|.$$

3. Cone Condition: For all $x \in \partial X$, there exists an open cone $\mathrm{Con}(x, u, \alpha)$ with a vertex x, a direction u, an altitude $|u|$ and an angle α such that $\mathrm{Con}(x, u, \alpha)$ is a subset of $\mathrm{Int}(X)$.

4. Open Set Condition(OSC): For all $i, j \in I$ $(i \neq j)$, $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ and $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$. Here, $\mathrm{Int}(X)$ denotes the set of interior points of X with respect to the topology in $\mathbb{R}^d$.

5. Bounded Distortion Property(BDP): There exists $K \ge 1$ and an open and connected subset $V \subset \mathbb{R}^d$ with $X \subset V$ such that for all $x, y \in V$ and for all $w \in I^* := \bigcup_{n=1}^{\infty} I^n$, the following inequality holds.
$$|\phi'_w(x)| \le K \cdot |\phi'_w(y)|.$$
Here, for each $n \in \mathbb{N}$ and $w = w_1 w_2 \cdots w_n \in I^n$, we set $\phi_w := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ and $|\phi'_w(x)|$ denotes the norm of the derivative of ϕ_w at $x \in X$ with respect to the Euclidean metric on $\mathbb{R}^d$.

6. Conformality: There exists a positive number ϵ such that for all $i \in I$, ϕ_i extends to a $C^{1+\epsilon}$diffeomorphism on V and ϕ_i is conformal on V, where V is the open and connected subset introduced in 5.

We set $I^* := \bigcup_{n=1}^{\infty} I^n$. We endow I with the discrete topology, and endow $I^\infty := I^{\mathbb{N}}$ with the product topology. Note that I^∞ is a Polish space. In addition, if I is a finite set, then I^∞ is a compact metrizable space.

Let S be a CIFS. For each $w = w_1 w_2 w_3 \cdots \in I^\infty$, we set $w|_n := w_1 w_2 \cdots w_n \in I^n$ and $\phi_{w|_n} := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$. Then, we have $\bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X)$ is a singleton. We denote it by $\{x_w\}$. The coding map $\pi \colon I^\infty \to X$ of S is defined by $w \mapsto x_w$. Note that $\pi \colon I^\infty \to X$ is continuous. A limit set of S is defined by
$$J_S := \pi(I^\infty) = \bigcup_{w \in I^\infty} \bigcap_{n \in \mathbb{N}} \phi_{w|_n}(X).$$

Note that for all CIFS S, J_S is Borel set in X. For each IFS S, we set $h_S := \dim_{\mathcal{H}} J_S$, where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

For any CIFS S, we define the pressure function of S as follows.

Definition 2.2 (Pressure function)**.** For each $n \in \mathbb{N}$, $[0, \infty]$-valued function ψ_S^n is defined by
$$\psi_S^n(t) := \sum_{w \in I^n} \left(\sup_{z \in X} |\phi'_w(z)| \right)^t \quad (t \ge 0).$$

We set
$$P_S(t) := \lim_{n \to \infty} \frac{1}{n} \log \psi_S^n(t) \in (-\infty, \infty].$$
The function $P_S \colon [0, \infty) \to (-\infty, \infty]$ is called the pressure function of S.

Note that for all $t \ge 0$, $P_S(t)$ exists because of the following proposition.

Proposition 2.3 ([4] Lemma 3.2)**.** For all $m, n \in \mathbb{N}$ and $t \geq 0$, we have $\psi_S^{m+n}(t) \leq \psi_S^m(t)\psi_S^n(t)$. In particular, for all $t \geq 0$, $\log \psi_S^n(t)$ is subadditive with respect to $n \in \mathbb{N}$.

We set $\theta_S := \inf\{t \geq 0 \mid \psi_S^1(t) < \infty\}$. By using the pressure function, we define properties of CIFSs.

Definition 2.4 (Regular, Strongly regular, Hereditarily regular)**.** Let S be a CIFS. We say that S is regular if there exists $t \geq 0$ such that $P_S(t) = 0$. We say that S is strongly regular if there exists $t \geq 0$ such that $P_S(t) \in (0, \infty)$. We say that S is hereditarily regular if for all $I' \subset I$ with $|I \setminus I'| < \infty$, $S' := \{\phi_i \colon X \to X \mid i \in I'\}$ is regular. Here, for any set A, we denote by $|A|$ the cardinality of A.

Note that if a CIFS S is hereditarily regular, then S is strong regular and if S is strong regular, then S is regular.

Definition 2.5. Let S be a CIFS. We write S as $\{\phi_i\}_{i \in I}$. Suppose that I is a countable infinite set. Let $z \in X$ and $\{z_i\}_{i \in I'} \subset X$ with $I' \subset I$ and $|I'| = \infty$. We say that $\lim_{i \in I'} z_i = z$ if for each $\epsilon > 0$, there exists $F' \subset I'$ with $|F'| < \infty$ such that if $i \in I' \setminus F'$, then $|z_i - z| < \epsilon$. We set

$$X_S(\infty) := \{\lim_{i \in I'} z_i \in X \mid {}^{\exists} I' \subset I, {}^{\exists}\{z_i\}_{i \in I'} \text{ s.t. } |I'| = \infty, z_i \in \phi_i(X) \ \ (i \in I')\}.$$

Equivalently, $X_S(\infty)$ is the set of accumulation points of sequences in $\phi_i(X)$, $i \in I$, i.e. limits of infinite sequences from $\phi_i(X)$, $i \in I$.

Mauldin and Urbański showed the following results. Recall that $h_S := \dim_{\mathcal{H}} J_S$, where $\dim_{\mathcal{H}} J_S$ denotes the Hausdorff dimension of the limit set of S.

Theorem 2.6 ([4] Theorem 3.20)**.** Let I be infinite and let S be a CIFS. Then, the following conditions are equivalent.

1. S is hereditarily regular.
2. $\psi_S^1(\theta_S) = \infty$.

Especially, if S is hereditarily regular, then we have $\theta_S < h_S$.

Proposition 2.7 ([4] Lemma 3.13)**.** Let S be a CIFS. If S is regular, then there exists the unique Borel probability measure m_S on X such that the following properties hold.

1. $m_S(J_S) = 1$.
2. For all Borel subset A on X and $i \in I$, $m_S(\phi_i(A)) = \int_A |\phi_i'|^{h_S} \mathrm{d}m_S$.
3. For all $i, j \in I$ with $i \neq j$, $m_S(\phi_i(X) \cap \phi_j(X)) = 0$.

We call m_S the h_S-conformal measure of S.

Theorem 2.8 ([4] Theorem 4.9)**.** Let S be a regular CIFS and m_S be the h_S-conformal measure of S. We set $r_0 := \mathrm{dist}(X, \partial V)$. If there exist a sequence of $\{z_j\}_{j=1}^{\infty}$ in $X_S(\infty)$ and a sequence $\{r_j\}_{j=1}^{\infty}$ in $(0, r_0)$ such that

$$\limsup_{j \to \infty} \frac{m_S(B(z_j, r_j))}{r_j^{h_S}} = \infty,$$

then we have $\mathcal{H}^{h_S}(J_S) = 0$.

Theorem 2.9 ([4] Lemma 4.3)**.** Let S be a regular CIFS. If $J_S \cap \mathrm{Int}(X) \neq \emptyset$, then we have $\mathcal{P}^{h_S}(J_S) > 0$.

3 CIFSs of generalized complex continued fractions

In this section, we prove some properties of the CIFSs of generalized complex continued fractions ([2]). Note that they are important and interesting examples of infinite CIFSs. We introduce some additional notations. For each $\tau \in A_0$, we set $\pi_\tau := \pi_{S_\tau}$, $\theta_\tau := \theta_{S_\tau}$, $\psi_\tau^n(t) := \psi_{S_\tau}^n(t)$ $(t \geq 0, n \in \mathbb{N})$, $P_\tau(t) := P_{S_\tau}(t)$ $(t \geq 0)$ and $X_\tau(\infty) := X_{S_\tau}(\infty)$.

In order to prove the main result, we need the following lemmas 3.1 $\sim$ 3.5 which were shown in [2]. For the readers, we give the proofs.

Lemma 3.1. For all $\tau \in A_0$, S_τ is a CIFS.

Proof. Let $\tau \in A_0$. Firstly, we show that for all $b \in I_\tau$, $\phi_b(X) \subset X$. Let $Y := \{z \in \mathbb{C} | \Re z \geq 1\}$ and let $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Möbius transformation defined by $f(z) := 1/z$. Since $f(0) = \infty$, $f(1) = 1$, $f(1/2 + i/2) = 2/(1+i) = (1-i)$, we have $f(\partial X) = \partial Y \cup \{\infty\}$. Moreover, since f(1/2) = 2, we have $f(X) = Y \cup \{\infty\}$. Thus, $f \colon X \to Y \cup \{\infty\}$ is a homeomorphism. Let $g_b \colon X \to Y$ be the map defined by $g_b(z) := z + b$. We deduce that $\phi_b = f^{-1} \circ g_b$ and $\phi_b(X) \subset f^{-1}(Y) \subset X$. Therefore, we have proved $\phi_b(X) \subset X$.

We next show that for each $\tau \in A_0$, S_τ satisfies the conditions of Definition 2.1.

1. Injectivity.

Since each ϕ_b is a Möbius transformation, each ϕ_b is injective.

2. Uniform Contractivity.

Let $b = m + n\tau (= m + nu + inv)$ be an element of I_τ and let $z = x + iy$ and $z' = x' + iy'$ be elements of X. We have

$$\begin{aligned} |z+b|^2 &= |x+m+nu+i(y+nv)|^2 \\ &= (x+m+nu)^2 + (y+nv)^2 \geq (0+1+0)^2 + (-1/2+1)^2 = \frac{5}{4}. \end{aligned}$$

Therefore, we deduce that $|z+b| \geq \sqrt{5/4}$. We also deduce that $|z'+b| \geq \sqrt{5/4}$. Finally, we obtain that

$$\begin{aligned} |\phi_b(z) - \phi_b(z')| &= \left| \frac{1}{z+b} - \frac{1}{z'+b} \right| \\ &= \frac{|z-z'|}{|z+b||z'+b|} \leq \left(\sqrt{\frac{4}{5}} \right)^2 |z-z'| = \frac{4}{5}|z-z'|. \end{aligned}$$

Therefore, S_τ is uniformly contractive on X.

3. Cone Condition.

Since X is a closed disk, the Cone Condition is satisfied.

4. Open Set Condition.

Note that $\mathrm{Int}(X) = \{z \in \mathbb{C} | \, |z - 1/2| < 1/2\}$. Let $\tau \in A_0$ and let $b \in I_\tau$. Since $f(\partial X) = \partial Y \cup \{\infty\}$, we deduce that for all $b \in I_\tau$,

$$g_b(\mathrm{Int}(X)) \subset \{z = x + iy \in \mathbb{C} | \; x > 1\} = f(\mathrm{Int}(X)).$$

Moreover, if b and b' are distinct elements, then $g_b(\mathrm{Int}(X))$ and $g_{b'}(\mathrm{Int}(X))$ are disjoint. Therefore, we have that for all $b \in I_\tau$,

$$\phi_b(\mathrm{Int}(X)) = f^{-1} \circ g_b(\mathrm{Int}(X)) \subset f^{-1} \circ f(\mathrm{Int}(X)) = \mathrm{Int}(X).$$

And if b and b' are distinct elements,

$$\phi_b(\mathrm{Int}(X)) \cap \phi_{b'}(\mathrm{Int}(X)) = f^{-1}(g_b(\mathrm{Int}(X)) \cap g_{b'}(\mathrm{Int}(X))) = \emptyset.$$

Therefore, S_τ satisfies the Open Set Condition of S_τ.

5. Bounded Distortion Property.

Let ϵ be a positive real number which is less than 1/12 and let $V' := B(1/2, 1/2 + \epsilon)$ be the open ball with center $1/2$ and radius $1/2+\epsilon$. We set $\tau := u+iv$. Then, for all $(m,n) \in \mathbb{N}^2$ and $z := x+iy \in V'$, we have that

$$\begin{aligned}
|\phi'_{m+n\tau}(z)| &= \frac{1}{|z+m+n\tau|^2} = \frac{1}{(x+m+nu)^2+(y+nv)^2} \\
&\leq \frac{1}{(-\epsilon+1+0)^2+(-1/2-\epsilon+1)^2} \\
&= \frac{1}{2\epsilon^2-3\epsilon+5/4} = \frac{1}{2(\epsilon-3/4)^2+1/8} \\
&\leq \frac{1}{2(1/12-3/4)^2+1/8} = \frac{72}{73} < 1
\end{aligned}$$

For each $z \in V'$, we set

$$z' := \begin{cases} (|z-1/2|-\epsilon)\dfrac{(z-1/2)}{|z-1/2|}+1/2 & (z \notin X) \\ z & (z \in X). \end{cases}$$

Then, we have that $|z-z'| \leq \epsilon$ and $|z'-1/2| < 1/2$. It implies that $z' \in X$. Thus, we obtain that $|\phi_b(z)-\phi_b(z')| \leq (72/73)|z-z'| < \epsilon$ and

$$\left|\phi_b(z)-\frac{1}{2}\right| \leq |\phi_b(z)-\phi_b(z')| + \left|\phi_b(z')-\frac{1}{2}\right| < \frac{1}{2}+\epsilon.$$

It follows that for all $b \in I_\tau$, $\phi_b(V') \subset V'$. In addition, ϕ_b is injective on V' and ϕ_b is holomorphic on $V' := B(1/2, 1/2+\epsilon)$ since ϕ_b is holomorphic on $\mathbb{C} \setminus \{-b\}$.

Let b be an element of I_τ and $r_0 := 1/2+\epsilon$. Let f_b be the function defined by

$$f_b(z) := \frac{(\phi_b(r_0 z+1/2)-\phi_b(1/2))}{r_0\phi'_b(1/2)} \quad (z \in D := \{z \in \mathbb{C} | |z| < 1\}).$$

Note that f_b is holomorphic on D and $f_b(0)=0$ and $f'_b(0)=1$. By using the Koebe distortion theorem (For example, see [6, Theorem 4.1.1]), we deduce that for all $z \in D$,

$$\frac{1-|z|}{(1+|z|)^3} \leq |f_b(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Let $r_1 := (r_0+1/2)/2$. we deduce that there exist $C_1 \geq 1$ and $C_2 \leq 1$ such that for all $z \in B(0, r_1/r_0) (\subset D)$,

$$C_2 \leq \frac{1-|z|}{(1+|z|)^3} \quad \text{and} \quad \frac{1+|z|}{(1-|z|)^3} \leq C_1.$$

Let $C := C_1/C_2$. Then, we have that for all $z, z' \in B(0, r_1/r_0)$

$$\begin{aligned}
\frac{|\phi'_b(r_0 z+1/2)|}{|\phi'_b(1/2)|} &= |f'_b(z)| \leq \frac{1+|z|}{(1-|z|)^3} \\
&\leq C_1 = CC_2 \leq C\frac{1-|z'|}{(1+|z'|)^3} \\
&\leq C|f'_b(z')| \leq C\frac{|\phi'_b(r_0 z'+1/2)|}{|\phi'_b(1/2)|}.
\end{aligned}$$

It follows that for all $z, z' \in B(0, r_1/r_0)$, $|\phi'_b(r_0 z+1/2)| \leq C|\phi'_b(r_0 z'+1/2)|$. Finally, let $V := B(1/2, r_1)$ be the open ball with center $1/2$ and radius r_1. Then, V is an open and connected subset of $\mathbb{C}$ with $X \subset V$ and for all $z, z' \in V$,

$$|\phi'_b(z)| \leq C|\phi'_b(z')|.$$

Therefore, S_τ satisfies the Bounded Distortion Property.

6. Conformality.

Let $\tau \in A_0$ and let $b \in I_\tau$. Since ϕ_b is holomorphic on $\mathbb{C} \setminus \{-b\}$, ϕ_b is C^2 and conformal on V. By the above argument, we have $\phi_b(V) \subset V$. □

For the rest of the paper, let $V := B(1/2, r_1)$, where r_1 is the number in the proof of Lemma 3.1.

Lemma 3.2 (basic inequality)**.** Let $\tau \in A_0$. Then, there exists $K_0 \geq 1$ such that for all $K \geq K_0$ and $b \in I_\tau$, the following properties hold.

1. $\phi_b(V) \subset B(0, K|b|^{-1})$.
2. For each $z \in V$, $K^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq K|b|^{-2}$.

Proof. We use the notations in the proof of Lemma 3.1. Note that $r_1 \in (1/2, 13/24)$. Let $\tau \in A_0$ and $b \in I_\tau$. Since there exists $M \in \mathbb{N}$ such that for all $z \in V = B(1/2, r_1)$ and $b \in I_\tau$, we have that $|b| \leq M|b+z|$, we deduce that

$$|\phi_b(z)| \leq M|b|^{-1}. \tag{4}$$

Note that by using the BDP, there exists $C \geq 1$ such that for each $z \in V$, we have

$$C^{-1}|\phi_b'(0)| \leq |\phi_b'(z)| \leq C|\phi_b'(0)|. \tag{5}$$

We set $K_0 := \max\{C, M\}(\geq 1)$. Let $K \geq K_0$.

By the inequality (4), we deduce that $\phi_b(V) \subset B(0, K|b|^{-1})$. By the inequality (5) and the equality $|\phi_b'(0)| = |b|^{-2}$, we deduce that for each $z \in V$, $K^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq K|b|^{-2}$. Therefore, we have proved our lemma. □

Lemma 3.3. For all $\tau \in A_0$, S_τ is a hereditarily regular CIFS with $\theta_\tau = 1$.

Proof. Let $\tau \in A_0$. For each non-negative integer p, we define $K'(p) := \{b = m + n\tau \in I_\tau \mid (m,n) \in \mathbb{N}^2, m < 2^p, n < 2^p\}$ and $K(p) := K'(p) \setminus K'(p-1)$. Note that for each non-negative integer p, $|K'(p)| = (2^p - 1)^2$. We deduce that for each $p \in \mathbb{N}$, $|K(p)| = |K'(p)| - |K'(p-1)| = (2^p-1)^2 - (2^{p-1}-1)^2 = 3 \cdot 4^{p-1} - 2 \cdot 2^{p-1} = 2^{p-1}(3 \cdot 2^{p-1} - 2)$ and $4^{p-1} \leq |K(p)| \leq 3 \cdot 4^{p-1}$.

Let $b = m + n\tau = m + n(u+iv) \in K(p)$. We consider the following two cases.

(i) If $m \geq 2^{p-1}$ then we have

$$\begin{aligned} |b|^2 &= |m + nu + inv|^2 = (m+nu)^2 + (nv)^2 \\ &\geq (2^{p-1}+u)^2 + v^2 \geq (2^{p-1})^2 + |\tau|^2 = 4^{p-1}\left(1 + \frac{|\tau|^2}{4^{p-1}}\right). \end{aligned}$$

(ii) If $n \geq 2^{p-1}$ then we have

$$|b|^2 = |m + nu + inv|^2 = (m+nu)^2 + (nv)^2 \geq n^2(u^2+v^2) \geq 4^{p-1}|\tau|^2.$$

Then for any $t \geq 0$, we have

$$\begin{aligned} \sum_{b \in I_\tau} |b|^{-2t} = \sum_{p \in \mathbb{N}} \sum_{b \in K(p)} \left\{|b|^2\right\}^{-t} &\leq \sum_{p \in \mathbb{N}} |K(p)| 4^{-t(p-1)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t} \\ &\leq \sum_{p \in \mathbb{N}} 3 \cdot 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t}. \end{aligned}$$

Hence, we deduce that

$$\sum_{b \in I_\tau} |b|^{-2t} \leq 3 \sum_{p \in \mathbb{N}} 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t}. \tag{6}$$

Moreover, by the inequality $|\tau|^2 \geq 1$ and the inequality $1 + \dfrac{|\tau|^2}{4^{p-1}} \geq 1$, we deduce that for all $p \in \mathbb{N}$,

$$3 \cdot 4^{(p-1)(1-t)} \left\{\min\{1 + \frac{|\tau|^2}{4^{p-1}}, |\tau|^2\}\right\}^{-t} \leq 3 \cdot 4^{(p-1)(1-t)}. \tag{7}$$

Also, by the inequality $|b| \leq |m| + |n||\tau| \leq 2^p(1+|\tau|)$, we have

$$\sum_{b \in I_\tau} |b|^{-2t} = \sum_{p \in \mathbb{N}} \sum_{b \in K(p)} \left\{|b|^{-2}\right\}^t \geq \sum_{p \in \mathbb{N}} |K(p)| 4^{-pt} (1+|\tau|)^{-2t}.$$

Thus, we deduce that

$$\sum_{b \in I_\tau} |b|^{-2t} \geq 4^{-1} \sum_{p \in \mathbb{N}} 4^{p(1-t)} (1+|\tau|)^{-2t}. \tag{8}$$

Finally, from Lemma 3.2, the inequality (6) and the inequality (8), it follows that $\psi_\tau^1(1) = \infty$ and if $t > 1$, then $\psi_\tau^1(t) < \infty$. Therefore, we deduce that $\theta_\tau = 1$ and by Theorem 2.6, we obtain that for all $\tau \in A_0$, S_τ is hereditarily regular. Hence, we have proved our lemma. □

Lemma 3.4. Let $\tau \in A_0$. Then we have $1 < h_\tau < 2$.

Proof. Let $\tau \in A_0$. By Theorem 2.6 and Lemma 3.3, we have $1 = \theta_\tau < h_\tau$. We now show that $h_\tau < 2$. We use the notations in the proof of Proposition 3.1. We have

$$\bigcup_{b \in I_\tau} g_b(X) \subset \{z \in \mathbb{C} \mid \Re z \geq 1 \text{ and } \Im z \geq 0\}.$$

Let U_0 be an open ball such that $U_0 \subset \{z \in \mathbb{C} \mid \Re z > 1 \text{ and } \Im z < 0\}$. Since $U_0 \subset Y$, we deduce that $f^{-1}(U_0) \subset f^{-1}(Y) = \mathrm{Int}(X)$. We set $X_1 := \cup_{b \in I_\tau} \phi_b(X)$. Since $U_0 \cap \bigcup_{b \in I_\tau} g_b(X) = \emptyset$, we deduce that $f^{-1}(U_0) \cap X_1 = f^{-1}(U_0 \cap \bigcup_{b \in I_\tau} g_b(X)) = \emptyset$. It follows $\mathrm{Int}(X) \setminus X_1 \supset f^{-1}(U_0)$.

Therefore, we deduce that $\lambda_2(\mathrm{Int}(X) \setminus X_1) > 0$ where, λ_2 is the 2-dimensional Lebesgue measure. By Proposition 4.4 of [4], we obtain that $h_\tau < 2$. Hence, we have proved $1 < h_\tau < 2$. □

Lemma 3.5. Let $\tau \in A_0$. Then, we have that $X_\tau(\infty) = \{0\}$.

Proof. We first show that for all $\tau \in A_0$, $0 \in X_\tau(\infty)$. We set $I'_\tau := \{m + \tau \in I_\tau \mid m \in \mathbb{N}\} \subset I_\tau$ and $b_m := m + \tau \in I'_\tau$. Then, we have that $|I'_\tau| = \infty$ and since $0 \in X$, $\phi_{b_m}(0) \in \phi_{b_m}(X)$. Let $\epsilon > 0$. Then, there exists $M \in \mathbb{N}$ such that $M > 1/\epsilon$. Let $F_\tau := \{m + \tau \in I_\tau \mid m \in \mathbb{N}, m \leq M\} \subset I'_\tau$. We obtain that $|F_\tau| < \infty$ and if $b_m \in I'_\tau \setminus F_\tau$, then $\phi_{b_m}(0) \in \phi_{b_m}(X)$ and

$$|\phi_{b_m}(0)| = \left|\frac{1}{m+\tau}\right| < \frac{1}{m} < \frac{1}{M} < \epsilon.$$

We next show that for each $\tau \in A_0$, $a \in X_\tau(\infty)$ implies $a = 0$. Suppose that there exists $a \in X_\tau(\infty)$ such that $a \neq 0$. Then, there exist $I'_\tau \subset I_\tau$ and $\{z'_b\}_{b \in I'_\tau}$ such that $|I'_\tau| = \infty$, $z'_b \in \phi_b(X)$ $(b \in I'_\tau)$ and $\lim_{b \in I'_\tau} z'_b = a$. Let $\delta := |a|/2 > 0$. Then, there exists $F'_\tau \subset I'_\tau$ such that $|F'_\tau| < \infty$ and for all $b \in I'_\tau \setminus F'_\tau$, $|z'_b - a| < \delta$. In particular, for all $b \in I'_\tau \setminus F'_\tau$,

$$|z'_b| \geq |a| - |z'_b - a| > \delta. \tag{9}$$

Moreover, for each $z \in X$, $\tau \in A_0$ and $b \in I_\tau$, we write $z := x + yi$, $\tau := u + iv$ and $b := m + n\tau$. Note that

$$\begin{aligned} |z+b|^2 = |x+m+nu+i(y+nv)|^2 &= (x+m+nu)^2 + (y+nv)^2 \\ &\geq (0+m+nu)^2 + (-1/2+nv)^2 \geq m^2 + (n-1/2)^2. \end{aligned}$$

Let $M := 1/\delta$. By using the above inequality, there exists $N_\delta \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $x \in X$, if $m \geq N_\delta$ or $n \geq N_\delta$, then $|z+b| > M = 1/\delta$. In particular, $b \in I_\tau \setminus F_\tau(N_\delta)$ implies that for all $z \in X$, $|\phi_b(z)| < \delta$. Here, $F_\tau(N_\delta) := \{b := m + n\tau \in I_\tau \mid n \leq N_\delta, m \leq N_\delta\}$.

By the inequality (9) and $|F_\tau(N_\delta)| < \infty$, this contradicts that there exist $b \in I'_\tau \setminus (F'_\tau \cup F_\tau(N_\delta))$ and $z'_b \in \phi_b(X)$ such that $|z'_b| > \delta$. Therefore, we have proved that for all $\tau \in A_0$, $X_\tau(\infty) = \{0\}$. □

4 Proof of the main result

In this section, we prove the main result Theorem 1.3. In order to prove Theorem 1.3, we first show a basic estimate for the conformal measure.

Note that for each $\tau \in A_0$, there exists the unique h_{S_τ}-conformal measure m_{S_τ} of S_τ by Proposition 2.7 since for each $\tau \in A_0$, S_τ is hereditarily regular. We set $m_\tau := m_{S_\tau}$.

Lemma 4.1. Let $\tau \in A_0$ and m_τ be the h_{S_τ}-conformal measure of S_τ. Then, there exists $K_0 \geq 1$ such that for each $b \in I_\tau$, we have $\phi_b(X) \subset B(0, K_0|b|^{-1})$ and

$$m_\tau(\phi_b(X)) \geq K_0^{-h_\tau}|b|^{-2h_\tau}.$$

Proof. By Lemma 3.2 with $K = K_0$, we deduce that for all $b \in I_\tau$ and $z \in V$, $\phi_b(V) \subset B(0, K_0|b|^{-1})$ and $K_0^{-1}|b|^{-2} \leq |\phi_b'(z)| \leq K_0|b|^{-2}$. Therefore, we have $\phi_b(X) \subset B(0, K|b|^{-1})$ and

$$m_\tau(\phi_b(X)) = \int_X |\phi_b'|^{h_\tau} \mathrm{d}m_\tau \geq (K_0^{-1}|b|^{-2})^{h_\tau} m_\tau(X) \geq K_0^{-h_\tau}|b|^{-2h_\tau}.$$

Thus, we have proved our lemma. □

We explain the idea of the proof of Theorem 1.3. Recall that $X_\tau(\infty) = \{0\}$. By Lemma 4.1, for sufficiently small $r > 0$, $b \in I_\tau$ and $N > 0$ with $r/N < K_0|b|^{-1} < r$, we have $\phi_b(X) \subset B(0, K_0|b|^{-1}) \subset B(0, r)$ and

$$\frac{m_\tau(B(0,r))}{r^{h_\tau}} \geq \frac{m_\tau(\phi_b(X))}{r^{h_\tau}} \geq \frac{K_0^{-h_\tau}|b|^{-2h_\tau}}{r^{h_\tau}} \geq \frac{K_0^{-h_\tau}}{r^{h_\tau}}\left(\frac{r}{NK_0}\right)^{2h_\tau} \simeq r^{h_\tau}. \tag{10}$$

This inequality (10) does not satisfies the assumption of Theorem 2.8 unfortunately. However, since for all $b, b' \in I_\tau$ with $b \neq b'$, $m_\tau(\phi_b(X) \cap \phi_{b'}(X)) = 0$, we have a sharper estimate on the value of $m_\tau(B(0,r))$. To obtain this estimate, we set

$$I_\tau(r) := \{b \in I_\tau \mid r/N_\tau \leq K_0|b|^{-1} < r\},$$

where N_τ is the number we introduce later. Then, in the proof of Theorem 1.3, we will show that

$$\frac{m_\tau(B(0,r))}{r^{h_\tau}} \geq \sum_{b \in I_\tau(r)} \frac{m_\tau(\phi_b(X))}{r^{h_\tau}} \geq |I_\tau(r)|K_0^{-3h_\tau}N_\tau^{-2h_\tau}r^{h_\tau}. \tag{11}$$

Note that since $I_\tau(r) = \{b \in I_\tau \mid K_0r^{-1} < |b| \leq N_\tau K_0 r^{-1}\}$, we have

$$|I_\tau(r)| \gtrsim r^{-2} \tag{12}$$

intuitively since we have a intuition that the number of the points $b \in I_\tau(r)$ in the slant lattice I_τ is almost the same as the area of $I_\tau(r)$. This estimate will be a key estimate in the proof of Theorem 1.3. After proving Lemma 4.2, Proposition 4.3 and Lemma 4.4, we will rigorously show estimate (12), whose precise statement is given by (16) later.

To prove this intuitive estimate (12) rigorously, we introduce the following notations and prove Lemma 4.2, Proposition 4.3 and Lemma 4.4. We identify $\mathbb{C}$ with $\mathbb{R}^2$, I_τ with $\{{}^t(a,b) \in \mathbb{R}^2 \mid a+ib \in I_\tau\}$ and $\mathbb{N}^2$ with $\{{}^t(m,n) \in \mathbb{R}^2 \mid m, n \in \mathbb{N}\}$, where for any matrix A, we denote by ${}^tA$ the transpose of A. For each $\tau = u + iv \in A_0$, we set

$$E_\tau := \begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix} \quad \text{and} \quad F_\tau := {}^tE_\tau E_\tau = \begin{pmatrix} 1 & u \\ u & |\tau|^2 \end{pmatrix}.$$

Note that $E_\tau \mathbb{N}^2 = I_\tau$, since $\det(E_\tau) = v \neq 0$, E_τ is invertible and by direct calculations, there exist the eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ of F_τ with $\lambda_1 < \lambda_2$. Let $v_1 \in \mathbb{R}^2$ be a eigenvector with respect to λ_1 and $v_2 \in \mathbb{R}^2$ be a eigenvector with respect to λ_2. Note that since F_τ is a symmetric matrix, there exist eigenvectors $v_1 \in \mathbb{R}^2$ and $v_2 \in \mathbb{R}^2$ such that $V_\tau := (v_1, v_2)$ is an orthogonal matrix.

For each $R_1 > 0$ and $R_2 > 0$ with $R_1/\sqrt{\lambda_1} < R_2/\sqrt{\lambda_2}$, we set

$$\begin{aligned} D_1'(\tau, R_1, R_2) &:= \{{}^t(x,y) \in \mathbb{R}^2 \mid R_1^2/\lambda_1 < x^2+y^2 \le R_2^2/\lambda_2\} \quad \text{and} \\ D_2'(R_1, R_2) &:= \{{}^t(x,y) \in \mathbb{R}^2 \mid R_1^2 < x^2+y^2 \le R_2^2\}. \end{aligned}$$

We show the following statement on the annuli $D_1'(\tau, R_1, R_2)$ and $D_2'(R_1, R_2)$.

Lemma 4.2. Let $\tau \in A_0$ and let $R_1 > 0$ and $R_2 > 0$ with $R_1/\sqrt{\lambda_1} < R_2/\sqrt{\lambda_2}$. Then, we have that $E_\tau(D_1'(\tau, R_1, R_2)) \subset D_2'(R_1, R_2)$. In particular, we have that

$$E_\tau(\mathbb{N}^2 \cap D_1'(\tau, R_1, R_2)) \subset I_\tau \cap D_2'(R_1, R_2) \quad \text{and} \quad |\mathbb{N}^2 \cap D_1'(\tau, R_1, R_2)| \le |I_\tau \cap D_2'(R_1, R_2)|.$$

Proof. By the above observation of F_τ, we deduce that

$$F_\tau = V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^tV_\tau.$$

Let ${}^t(x,y) \in D_1'(\tau, R_1, R_2)$. We set $(x', y') := (x,y)\ V_\tau$ and $(v, w) := (x,y)\ {}^tE_\tau$. Note that since V_τ is an orthogonal matrix, we deduce that $(x')^2 + (y')^2 = x^2 + y^2$. Since $\lambda_1 < \lambda_2$, we have

$$\begin{aligned} R_1^2 &< \lambda_1(x^2+y^2) = \lambda_1((x')^2+(y')^2) < \lambda_1(x')^2 + \lambda_2(y')^2 \\ &= (x', y') \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = (x,y)\ V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^tV_\tau \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x,y)\ F_\tau \begin{pmatrix} x \\ y \end{pmatrix} = (x,y)\ {}^tE_\tau E_\tau \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

By the above inequality, we deduce that $R_1^2 < v^2 + w^2$. Also,

$$\begin{aligned} R_2^2 &\ge \lambda_2(x^2+y^2) = \lambda_2((x')^2+(y')^2) \ge \lambda_1(x')^2 + \lambda_2(y')^2 \\ &= (x', y') \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = (x,y)\ V_\tau \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} {}^tV_\tau \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x,y)\ F_\tau \begin{pmatrix} x \\ y \end{pmatrix} = (x,y)\ {}^tE_\tau E_\tau \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

By the above inequality, we deduce that $v^2 + w^2 \le R_2^2$. Therefore, we have proved our lemma. □

For each $R > 0$, we set $I(R) := \{{}^t(m,n) \in \mathbb{N}^2 \mid m^2+n^2 \le R^2\}$.
We give the following estimate on $|I(R)|$.

Proposition 4.3. Let $R > 0$. Then, for each $R \ge 6$,

$$0 < \frac{R^2 - 7R + 7}{2} \le |I(R)| \le R^2.$$

Proof. For each $a \in \mathbb{R}$, we denote by $\lfloor a \rfloor$ the maximum integer of the set $\{n \in \mathbb{Z} \mid n \le a\}$. Let $R \ge 6$. We set $M := \lfloor \sqrt{R^2-1} \rfloor (\ge 1)$. For each $l = 1, \dots, M$, we set $N(l) := \lfloor \sqrt{R^2 - l^2} \rfloor (\ge 1)$. Note that since $M \le \sqrt{R^2-1} < M+1$, we deduce that

$$\sqrt{R^2-1} - 1 < M \le \sqrt{R^2-1}. \tag{13}$$

Also, since $N(l) \le \sqrt{R^2-l^2} < N(l)+1$, we deduce that

$$\sqrt{R^2-l^2} - 1 < N(l) \le \sqrt{R^2-l^2}. \tag{14}$$

Therefore, we deduce that $|I(R)| = \sum_{l=1}^M N(l)$.

By the inequalities (13) and (14), we deduce that

$$|I(R)| \le \sum_{l=1}^M \sqrt{R^2-l^2} \le RM \le R\sqrt{R^2-1} \le R^2.$$

We now show that $|I(R)| \geq (R^2 - 7R + 7)/2$. Since $\sqrt{R^2 - l^2} \geq R - l$ for each $l = 1, \ldots, M$, by the inequalities (13) and (14) again, we deduce that

$$\begin{aligned}|I(R)| &\geq \sum_{l=1}^{M} \left(\sqrt{R^2 - l^2} - 1\right) \geq \sum_{l=1}^{M} (R - l - 1) = M(R-1) - \frac{M(M+1)}{2} \\ &= \frac{M(2R-3) - M^2}{2} \geq \frac{\left(\sqrt{R^2-1} - 1\right)(2R-3) - (R^2 - 1)}{2} \\ &\geq \frac{(R-2)(2R-3) - R^2 + 1}{2} = \frac{R^2 - 7R + 7}{2}.\end{aligned}$$

Thus, we have proved our lemma. □

For each $\tau \in A_0$, we set $N_\tau := \sqrt{2\lambda_2}/\sqrt{\lambda_1} + 1$ (> 2). For each $R > 0$, we set $D_1(\tau, R) := D_1'(\tau, R, N_\tau R)$ and $D_2(\tau, R) := D_2'(R, N_\tau R)$. Note that since $\sqrt{\lambda_2}/\sqrt{\lambda_1} < N_\tau$, we have that $R/\sqrt{\lambda_1} < (N_\tau R)/\sqrt{\lambda_2}$.

We estimate $|\mathbb{N}^2 \cap D_1(\tau, R)|$ from below as follows.

Lemma 4.4. Let $\tau \in A_0$. Then, there exist $R_\tau > 0$ and $L_\tau > 0$ such that for all $R > R_\tau$,

$$|\mathbb{N}^2 \cap D_1(\tau, R)| \geq L_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R.$$

Proof. Let $\tau \in A_0$. We set $L_\tau := N_\tau^2/(2\lambda_2) - 1/\lambda_1$. Note that since $N_\tau > \sqrt{2\lambda_2}/\sqrt{\lambda_1}$, we deduce that $L_\tau > 0$. We set

$$R_\tau := \max\{(6\sqrt{\lambda_2})/N_\tau, 6\sqrt{\lambda_1}\}(> 0).$$

Let $R \geq R_\tau$. Note that $N_\tau R/\sqrt{\lambda_2} \geq 6$, $R/\sqrt{\lambda_1} \geq 6$ and

$$\mathbb{N}^2 \cap D_1(\tau, R) = I\left(\frac{N_\tau R}{\sqrt{\lambda_2}}\right) \setminus I\left(\frac{R}{\sqrt{\lambda_1}}\right). \tag{15}$$

Also, we have $I\left((N_\tau R)/\sqrt{\lambda_2}\right) \supset I\left(R/\sqrt{\lambda_1}\right)$. By (15) and Proposition 4.3, we deduce that

$$\begin{aligned}|\mathbb{N}^2 \cap D_1(\tau, R)| &= \left|I\left(\frac{N_\tau R}{\sqrt{\lambda_2}}\right)\right| - \left|I\left(\frac{R}{\sqrt{\lambda_1}}\right)\right| \\ &\geq \frac{1}{2}\left(\frac{(N_\tau R)^2}{\lambda_2} - 7\frac{N_\tau R}{\sqrt{\lambda_2}} + 7\right) - \frac{R^2}{\lambda_1} > L_\tau R^2 - \frac{7N_\tau}{2\sqrt{\lambda_2}} R.\end{aligned}$$

Therefore, we have proved our lemma. □

By Lemma 4.2, Proposition 4.3 and Lemma 4.4, we now prove the intuitive estimate (12) rigorously.

Rigorous proof of the estimate (12). Let $\tau \in A_0$. We set $r_\tau := K_0 R_\tau^{-1}(> 0)$ and $M_\tau := (7N_\tau)/(2\sqrt{\lambda_2})$. We show that for all $r \in (0, r_\tau]$,

$$|I_\tau(r)| = |\{b \in I_\tau \mid r/N_\tau \leq K_0|b|^{-1} < r\}| \geq L_\tau K_0^2 r^{-2} - M_\tau K_0 r^{-1}. \tag{16}$$

Let $r \in (0, r_\tau]$. We set $R := K_0 r^{-1}$. Note that $r \leq r_\tau$ if and only if $R \geq R_\tau$. Recall that $I_\tau(r) := \{b \in I_\tau \mid r/N_\tau \leq K_0|b|^{-1} < r\}$ and

$$I_\tau(r) = \{b \in I_\tau \mid K_0 r^{-1} < |b| \leq N_\tau K_0 r^{-1}\} = I_\tau \cap D_2'(K_0 r^{-1}, N_\tau K_0 r^{-1}).$$

Recall that $R := K_0 r^{-1}$, $D_1(\tau, R) := D_1'(\tau, R, N_\tau R)$ and $M_\tau := (7N_\tau)/(2\sqrt{\lambda_2})$. By Lemmas 4.2 and 4.4, it follows that

$$\begin{aligned}|I_\tau(r)| &= |I_\tau \cap D_2'(K_0 r^{-1}, N_\tau K_0 r^{-1})| = |I_\tau \cap D_2'(R, N_\tau R)| \geq |\mathbb{N}^2 \cap D_1'(\tau, R, N_\tau R)| \\ &= |\mathbb{N}^2 \cap D_1(\tau, R)| \geq L_\tau R^2 - M_\tau R = L_\tau K_0^2 r^{-2} - M_\tau K_0 r^{-1}.\end{aligned}$$

Thus, we have proved the inequality (16). □

We now give the proof of the main result Theorem 1.3.

Proof of Theorem 1.3. Let $\tau \in A_0$. Recall that there exists the unique h_{S_τ}-conformal measure m_τ of S_τ. We set $r_\tau := K_0 R_\tau^{-1} (> 0)$ and $M_\tau := (7N_\tau)/(2\sqrt{\lambda_2})$.

We first show that for all $r \in (0, r_\tau]$,

$$m_\tau(B(0,r)) \geq L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 2} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 1}. \tag{17}$$

By Lemma 4.1 and the definition of $I_\tau(r)$, we have that for all $b \in I_\tau(r)$, $\phi_b(X) \subset B(0, K_0|b|^{-1}) \subset B(0,r)$. It follows that

$$\bigcup_{b \in I_\tau(r)} \phi_b(X) \subset B(0,r). \tag{18}$$

In addtion, if $b, b' \in I_\tau$ with $b \neq b'$, then $m_\tau(\phi_b(X) \cap \phi_{b'}(X)) = 0$ by the definition of the conformal measure (Proposition 2.7). Thus, by inclusion (18) and Lemma 4.1, it follows that

$$\begin{aligned} m_\tau(B(0,r)) &\geq m_\tau\left(\bigcup_{b \in I_\tau(r)} \phi_b(X)\right) = \sum_{b \in I_\tau(r)} m_\tau(\phi_b(X)) \geq \sum_{b \in I_\tau(r)} K_0^{-h_\tau} |b|^{-2h_\tau} \\ &\geq \sum_{b \in I_\tau(r)} K_0^{-h_\tau} \left(\frac{r}{N_\tau K_0}\right)^{2h_\tau} = |I_\tau(r)| K_0^{-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau}. \end{aligned}$$

By the inequality (16), we obtain that

$$m_\tau(B(0,r)) \geq L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 2} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r^{2h_\tau - 1}.$$

Thus, we have proved inequality (17).

We now show that $\mathcal{H}^{h_\tau}(J_\tau) = 0$. For each $j \in \mathbb{N}$, we set $z_j := 0$ and $r_j := r_\tau / j$ $(\in (0, r_\tau])$. Note that $\{r_j\}_{j \in \mathbb{N}}$ is a sequence in the set of positive real numbers and by Lemma 3.5, $\{z_j\}_{j \in \mathbb{N}}$ is a sequence in $X_\tau(\infty)$. Thus, by the inequality (17), we deduce that for each $j \in \mathbb{N}$,

$$\begin{aligned} &\frac{m_\tau(B(z_j, r_j))}{r_j^{h_\tau}} = \frac{m_\tau(B(0, r_j))}{r_j^{h_\tau}} \\ &\geq L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r_j^{h_\tau - 2} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r_j^{h_\tau - 1} \\ &= L_\tau K_0^{2-3h_\tau} N_\tau^{-2h_\tau} r_\tau^{h_\tau - 2} j^{2-h_\tau} - M_\tau K_0^{1-3h_\tau} N_\tau^{-2h_\tau} r_\tau^{h_\tau - 1} \left(\frac{1}{j}\right)^{h_\tau - 1}. \end{aligned}$$

By Lemma 3.4, we have that $2 - h_\tau > 0$ and $h_\tau - 1 > 0$. It follows that

$$\limsup_{j \to \infty} \frac{m_\tau(B(z_j, r_j))}{r_j^{h_\tau}} = \infty.$$

By Theorem 2.8, we obtain that $\mathcal{H}^{h_\tau}(J_\tau) = 0$.

We finally show that $\mathcal{P}^{h_\tau}(J_\tau) > 0$. Let $\tau = u + iv \in A_0$. We set $b_2 := 2 + \tau \in I_\tau$. We use some notations in Lemma 3.1. For any $z = x + iy \in X$,

$$\begin{aligned} g_{b_2}(z) &= z + (2 + \tau) \\ &= (x + 2 + u) + i(y + v) \in \{z \in \mathbb{C} \mid \Re z > 1\} = \mathrm{Int}(Y). \end{aligned}$$

Since $f(\partial X) = \partial Y \cup \{\infty\}$ and $f \colon X \to Y \cup \{\infty\}$ is bijective, we have

$$\phi_{b_2}(X) = (f^{-1} \circ g_{b_2})(X) \subset \mathrm{Int}(X).$$

Therefore, we obtain that $J_\tau \cap \mathrm{Int}(X) \neq \emptyset$. Since S_τ is hereditarily regular and $J_\tau \cap \mathrm{Int}(X) \neq \emptyset$, we deduce that $\mathcal{P}^{h_\tau}(J_\tau) > 0$ by Theorem 2.9. $\square$

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