Dynamics of postcritically bounded polynomial semigroups I: connected components of the Julia sets *

Hiroki Sumi
Department of Mathematics, Graduate School of Science
Osaka University
1-1, Machikaneyama, Toyonaka, Osaka, 560-0043, Japan
E-mail: sumi@math.sci.osaka-u.ac.jp
http://www.math.sci.osaka-u.ac.jp/∼sumi/

November 3, 2010

Abstract

We investigate the dynamics of semigroups generated by a family of polynomial maps on the Riemann sphere such that the postcritical set in the complex plane is bounded. The Julia set of such a semigroup may not be connected in general. We show that for such a polynomial semigroup, if A and B are two connected components of the Julia set, then one of A and B surrounds the other. From this, it is shown that each connected component of the Fatou set is either simply or doubly connected. Moreover, we show that the Julia set of such a semigroup is uniformly perfect. An upper estimate of the cardinality of the set of all connected components of the Julia set of such a semigroup is given. By using this, we give a criterion for the Julia set to be connected. Moreover, we show that for any $n \in \mathbb{N} \cup \{\aleph_0\}$, there exists a finitely generated polynomial semigroup with bounded planar postcritical set such that the cardinality of the set of all connected components of the Julia set is equal to n. Many new phenomena of polynomial semigroups that do not occur in the usual dynamics of polynomials are found and systematically investigated.

1 Introduction

The theory of complex dynamical systems, which has its origin in the important work of Fatou and Julia in the 1910s, has been investigated by many people and discussed in depth. In particular, since D. Sullivan showed the famous "no wandering domain theorem" using Teichmüller theory in the 1980s, this subject has attracted many researchers from a wide area. For a general reference on complex dynamical systems, see Milnor's textbook [16] or Beardon's textbook [3].

There are several areas in which we deal with generalized notions of classical iteration theory of rational functions. One of them is the theory of dynamics of rational semigroups (semigroups generated by a family of holomorphic maps on the Riemann sphere $\hat{\mathbb{C}}$), and another one is the theory of random dynamics of holomorphic maps on the Riemann sphere.

In this paper, we will discuss the dynamics of rational semigroups.

A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition ([13]). A **polynomial semigroup** is a semigroup generated by a family of non-constant

*Published in Discrete and Continuous Dynamical Sistems Series A, Vol. 29, No. 3, 2011, 1205–1244. 2000 Mathematics Subject Classification. 37F10, 30D05. Keywords: Complex dynamical systems, rational semigroup, polynomial semigroup, random iteration, random complex dynamical systems, Julia set, fractal geometry, iterated function systems, surrounding order.

polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([13, 14]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group([48, 12]), who studied such semigroups from the perspective of random dynamical systems. Moreover, the research on rational semigroups is related to that on "iterated function systems" in fractal geometry. In fact, the Julia set of a rational semigroup generated by a compact family has " backward self-similarity" (cf. Lemma 3.1-2). For other research on rational semigroups, see [19, 20, 21, 47, 22, 24, 44, 43, 45, 46], and [27]–[40].

The research on the dynamics of rational semigroups is also directly related to that on the random dynamics of holomorphic maps. The first study in this direction was by Fornaess and Sibony ([10]), and much research has followed. (See [4, 6, 7, 5, 11, 33, 34, 37, 38, 39, 40].)

We remark that the complex dynamical systems can be used to describe some mathematical models. For example, the behavior of the population of a certain species can be described as the dynamical system of a polynomial $f(z) = az(1-z)$ such that f preserves the unit interval and the postcritical set in the plane is bounded (cf. [8]). It should also be remarked that according to the change of the natural environment, some species have several strategies to survive in the nature. From this point of view, it is very important to consider the random dynamics of such polynomials (see also Example 1.4). For the random dynamics of polynomials on the unit interval, see [26].

We shall give some definitions for the dynamics of rational semigroups:

Definition 1.1 ([13, 12])**.** Let G be a rational semigroup. We set

$$F(G) := \{z \in \hat{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \text{ and } J(G) := \hat{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** of G and $J(G)$ is called the **Julia set** of G. We let $\langle h_1, h_2, \ldots \rangle$ denote the rational semigroup generated by the family $\{h_i\}$. The Julia set of the semigroup generated by a single map g is denoted by $J(g)$.

Definition 1.2.

1. For each rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we set $CV(g) := \{\text{all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}$. Moreover, for each polynomial map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we set $CV^*(g) := CV(g) \setminus \{\infty\}$.

2. Let G be a rational semigroup. We set $P(G) := \overline{\bigcup_{g \in G} CV(g)}$ $(\subset \hat{\mathbb{C}})$. This is called the **postcritical set** of G. Furthermore, for a polynomial semigroup G, we set $P^*(G) := P(G) \setminus \{\infty\}$. This is called the **planar postcritical set** (or **finite postcritical set**) of G. We say that a polynomial semigroup G is **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$.

Remark 1.3. Let G be a rational semigroup generated by a family Λ of rational maps. Then, we have that $P(G) = \overline{\bigcup_{g \in G \cup \{Id\}} g(\bigcup_{h \in \Lambda} CV(h))}$, where Id denotes the identity map on $\hat{\mathbb{C}}$, and that $g(P(G)) \subset P(G)$ for each $g \in G$. From this formula, one can figure out how the set $P(G)$ (resp. $P^*(G)$) spreads in $\hat{\mathbb{C}}$ (resp. $\mathbb{C}$). In fact, in Section 2.6, using the above formula, we present a way to construct examples of postcritically bounded polynomial semigroups (with some additional properties). Moreover, from the above formula, one may, in the finitely generated case, use a computer to see if a polynomial semigroup G is postcritically bounded much in the same way as one verifies the boundedness of the critical orbit for the maps $f_c(z) = z^2 + c$.

Example 1.4. Let $\Lambda := \{h(z) = cz^a(1-z)^b \mid a, b \in \mathbb{N},\ c > 0,\ c(\frac{a}{a+b})^a(\frac{b}{a+b})^b \leq 1\}$ and let G be the polynomial semigroup generated by Λ. Since for each $h \in \Lambda$, $h([0,1]) \subset [0,1]$ and $CV^*(h) \subset [0,1]$, it follows that each subsemigroup H of G is postcritically bounded.

Remark 1.5. It is well-known that for a polynomial g with $\deg(g) \geq 2$, $P^*(\langle g \rangle)$ is bounded in $\mathbb{C}$ if and only if $J(g)$ is connected ([16, Theorem 9.5]).

As mentioned in Remark 1.5, the planar postcritical set is one piece of important information regarding the dynamics of polynomials. Concerning the theory of iteration of quadratic polynomials, we have been investigating the famous "Mandelbrot set".

When investigating the dynamics of polynomial semigroups, it is natural for us to discuss the relationship between the planar postcritical set and the figure of the Julia set. The first question in this regard is:

Question 1.6. Let G be a polynomial semigroup such that each element $g \in G$ is of degree at least two. Is $J(G)$ necessarily connected when $P^*(G)$ is bounded in $\mathbb{C}$?

The answer is **NO.**

Example 1.7 ([47])**.** Let $G = \langle z^3, \frac{z^2}{4} \rangle$. Then $P^*(G) = \{0\}$ (which is bounded in $\mathbb{C}$) and $J(G)$ is disconnected ($J(G)$ is a Cantor set of round circles). Furthermore, according to [31, Theorem 2.4.1], it can be shown that a small perturbation H of G still satisfies that $P^*(H)$ is bounded in $\mathbb{C}$ and that $J(H)$ is disconnected. ($J(H)$ is a Cantor set of quasi-circles with uniform dilatation.)

Question 1.8. What happens if $P^*(G)$ is bounded in $\mathbb{C}$ and $J(G)$ is disconnected?

Problem 1.9. Classify postcritically bounded polynomial semigroups.

In this paper, we show that if G is a postcritically bounded polynomial semigroup with disconnected Julia set, then $\infty \in F(G)$ (cf. Theorem 2.20-1), and for any two connected components of $J(G)$, one of them surrounds the other. This implies that there exists an intrinsic total order " $\leq$ " (called the "surrounding order") in the space $\mathcal{J}_G$ of connected components of $J(G)$, and that every connected component of $F(G)$ is either simply or doubly connected (cf. Theorem 2.7). Moreover, for such a semigroup G, we show that the interior of "the smallest filled-in Julia set" $\hat{K}(G)$ is not empty, and that there exists a maximal element and a minimal element in the space $\mathcal{J}_G$ endowed with the order $\leq$ (cf. Theorem 2.20). From these results, we obtain the result that for a postcritically bounded polynomial semigroup G, the Julia set $J(G)$ is uniformly perfect, even if G is not generated by a compact family of polynomials (cf. Theorem 2.22).

Moreover, we utilize Green's functions with pole at infinity to show that for a postcritically bounded polynomial semigroup G, the cardinality of the set of all connected components of $J(G)$ is less than or equal to that of $J(H)$, where H is the "real affine semigroup" associated with G (cf. Theorem 2.12). From this result, we obtain a sufficient condition for the Julia set of a postcritically bounded polynomial semigroup to be connected (cf. Theorem 2.14). In particular, we show that if a postcritically bounded polynomial semigroup G is generated by a family of quadratic polynomials, then $J(G)$ is connected (cf. Theorem 2.15). The proofs of the results in this and the previous paragraphs are not straightforward. In fact, we first prove (1) that for any two connected components of $J(G)$ that are included in $\mathbb{C}$, one of them surrounds the other; next, using (1) and the theory of Green's functions, we prove (2) that the cardinality of the set of all connected components of $J(G)$ is less than or equal to that of $J(H)$, where H is the associated real affine semigroup; and finally, using (2) and (1), we prove (3) that $\infty \in F(G)$, $\text{int}(\hat{K}(G)) \neq \emptyset$, and other results in the previous paragraph.

Moreover, we show that for any $n \in \mathbb{N} \cup \{\aleph_0\}$, there exists a finitely generated, postcritically bounded, polynomial semigroup G such that the cardinality of the set of all connected components of $J(G)$ is equal to n (cf. Proposition 2.26, Proposition 2.28 and Proposition 2.29). A sufficient condition for the cardinality of the set of all connected components of a Julia set to be equal to $\aleph_0$ is also given (cf. Theorem 2.27). To obtain these results, we use the fact that the map induced by any element of a semigroup on the space of connected components of the Julia set preserves the order $\leq$ (cf. Theorem 2.7). Note that this is in contrast to the dynamics of a single rational map h or a non-elementary Kleinian group, where it is known that either the Julia set is connected, or the Julia set has uncountably many connected components. Furthermore, in Section 2.6 and Section 2.4, we provide a way of constructing examples of postcritically bounded polynomial semigroups with some additional properties (disconnectedness of Julia set, semi-hyperbolicity, hyperbolicity, etc.) (cf. Proposition 2.40, Theorem 2.43, Theorem 2.45). For example, by Proposition 2.40, there exists a 2-generator postcritically bounded polynomial semigroup $G = \langle h_1, h_2 \rangle$ with disconnected Julia set such that h_1 has a Siegel disk.

As we see in Example 1.4 and Section 2.6, it is not difficult to construct many examples, it is not difficult to verify the hypothesis "postcritically bounded", and the class of postcritically bounded polynomial semigroups is very wide.

Throughout the paper, we will see many new phenomena in polynomial semigroups that do not occur in the usual dynamics of polynomials. Moreover, these new phenomena are systematically investigated.

In Section 2, we present the main results of this paper. We give some tools in Section 3. The proofs of the main results are given in Section 4.

There are many applications of the results of postcritically bounded polynomial semigroups in many directions. In the sequel [36], by using the results in this paper, we investigate the fiberwise (sequencewise) and random dynamics of polynomials and the Julia sets. We present a sufficient condition for a fiberwise Julia set to be of measure zero, a sufficient condition for a fiberwise Julia set to be a Jordan curve, a sufficient condition for a fiberwise Julia set to be a quasicircle, and a sufficient condition for a fiberwise Julia set to be a Jordan curve which is not a quasicircle. Moreover, using uniform fiberwise quasiconformal surgery on a fiber bundle, we show that for a $G \in \mathcal{G}_{dis}$, there exist families of uncountably many mutually disjoint quasicircles with uniform dilatation which are parameterized by the Cantor set, densely inside $J(G)$. In the sequel [37], we classify hyperbolic or semi-hyperbolic postcritically bounded compactly generated polynomial semigroups, in terms of the random complex dynamics. It is shown that in one of the classes, for almost every sequence γ, the Julia set J_γ of γ is a Jordan curve but not a quasicircle, the unbounded component of $\hat{\mathbb{C}} \setminus J_\gamma$ is a John domain, and the bounded component of $\mathbb{C} \setminus J_\gamma$ is not a John domain. Moreover, in [37, 36], we find many examples with this phenomenon. Note that this phenomenon does not hold in the usual iteration dynamics of a single polynomial map g with $\deg(g) \geq 2$. In the sequel [38, 42], we investigate the Markov process on $\hat{\mathbb{C}}$ associated with the random dynamics of polynomials and we consider the probability $T_\infty(z)$ of tending to $\infty \in \hat{\mathbb{C}}$ starting with the initial value $z \in \hat{\mathbb{C}}$. Applying many results of this paper, it will be shown in [42] that if the associated polynomial semigroup G is postcritically bounded and the Julia set is disconnected, then the function T_∞ defined on $\hat{\mathbb{C}}$ has many interesting properties which are similar to those of the Cantor function. In fact, under certain conditions, T_∞ is continuous on $\hat{\mathbb{C}}$ and varies precisely on the Julia set, of which Hausdorff dimension is strictly less than two. (For example, if we consider the random dynamics generated by two polynomials $h_1 := g_1^2, h_2 := g_2^2$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4$, then T_∞ is continuous on $\hat{\mathbb{C}}$ and T_∞ varies precisely on the Julia set (Figure 1) of the semigroup generated by h_1, h_2. See [38, 33].) Such a kind of "singular functions on the complex plane" appear very naturally in random dynamics of polynomials, and the results of this paper (for example, the results on the space of all connected components of a Julia set) are the keys to investigating that. (The above results have been announced in [33, 34, 39].)

Moreover, as illustrated before, it is very important for us to recall that the complex dynamics can be applied to describe some mathematical models. For example, the behavior of the population of a certain species can be described as the dynamical systems of a polynomial h such that h preserves the unit interval and the postcritical set in the plane is bounded. When one considers such a model, it is very natural to consider the random dynamics of polynomial with bounded postcritical set in the plane (see Example 1.4).

In the sequel [24], we give some further results on postcritically bounded polynomial semigroups, by using many results in this paper and [36, 37]. Moreover, in the sequel [35], we define a new kind of cohomology theory, in order to investigate the action of finitely generated semigroups (iterated function systems), and we apply it to the study of the dynamics of postcritically bounded finitely generated polynomial semigroups G. In particular, by using this new cohomology theory, we can describe the space $\mathcal{J}_G$ of connected components of Julia sets of G, we can give some estimates on the cardinality of $\mathcal{J}_G$, and we can give a sufficient condition for the cardinality of the space of connected components of the Fatou set of G to be infinity. In [38, 40, 41], we investigate the random complex dynamics and the dynamics of transition operator, by developing the theory of random complex dynamics and that of dynamics of rational semigroups, simultaneously. It is shown that regarding the random dynamics of complex polynomials, generically the chaos of the averaged system disappears due to the cooperation of the generators, even though each map itself in the system has a chaotic part. We call this phenomenon "cooperation principle". Moreover, we see that under certain conditions, in the limit state, complex analogues of singular functions (continuous functions on $\hat{\mathbb{C}}$ which vary only on the Julia set of associated rational semigroup G) naturally appear. The above function T_∞ is a typical example of this complex analogue of singular function.

Acknowledgement: The author thanks R. Stankewitz for many valuable comments.

2 Main results

In this section we present the statements of the main results. Throughout this paper, we deal with semigroups G that might not be generated by a compact family of polynomials. The proofs are given in Section 4.

2.1 Space of connected components of a Julia set, surrounding order

We present some results concerning the connected components of the Julia set of a postcritically bounded polynomial semigroup. The proofs are given in Section 4.1.

The following theorem generalizes [47, Theorem 1].

Theorem 2.1. *Let G be a rational semigroup generated by a family $\{h_\lambda\}_{\lambda\in\Lambda}$. Suppose that there exists a connected component A of $J(G)$ such that $\sharp A > 1$ and $\bigcup_{\lambda\in\Lambda} J(h_\lambda) \subset A$. Moreover, suppose that for any $\lambda \in \Lambda$ such that h_λ is a Möbius transformation of finite order, we have $h_\lambda^{-1}(A) \subset A$. Then, $J(G)$ is connected.*

Definition 2.2. We set $\text{Rat} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the topology induced by uniform convergence on $\hat{\mathbb{C}}$ with respect to the spherical distance. We set $\text{Poly} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant polynomial}\}$ endowed with the relative topology from Rat. Moreover, we set $\text{Poly}_{\deg\geq 2} := \{g \in \text{Poly} \mid \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Remark 2.3. Let $d \geq 1$, $\{p_n\}_{n\in\mathbb{N}}$ a sequence of polynomials of degree d, and p a polynomial. Then, $p_n \to p$ in Poly if and only if the coefficients converge appropriately and p is of degree d.

Definition 2.4. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- each element of G is of degree at least two, and
- $P^*(G)$ is bounded in $\mathbb{C}$, i.e., G is postcritically bounded.

Furthermore, we set $\mathcal{G}_{con} = \{G \in \mathcal{G} \mid J(G) \text{ is connected}\}$ and $\mathcal{G}_{dis} = \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$.

Notation: For a polynomial semigroup G, we denote by $\mathcal{J} = \mathcal{J}_G$ the set of all connected components J of $J(G)$ such that $J \subset \mathbb{C}$. Moreover, we denote by $\hat{\mathcal{J}} = \hat{\mathcal{J}}_G$ the set of all connected components of $J(G)$.

Remark 2.5. If a polynomial semigroup G is generated by a compact set in $\text{Poly}_{\deg\geq 2}$, then $\infty \in F(G)$ and thus $\mathcal{J} = \hat{\mathcal{J}}$.

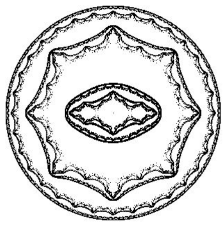
Definition 2.6. For any connected sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" indicates that $K_1 = K_2$, or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Furthermore, "$K_1 < K_2$" indicates $K_1 \leq K_2$ and $K_1 \neq K_2$. Note that "$\leq$" is a partial order in the space of all non-empty compact connected sets in $\mathbb{C}$. This "$\leq$" is called the **surrounding order.**

Theorem 2.7. *Let $G \in \mathcal{G}$ (possibly generated by a non-compact family). Then we have all of the following.*

1. *$(\mathcal{J}, \leq)$ is totally ordered.*
2. *Each connected component of $F(G)$ is either simply or doubly connected.*
3. *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

For the figures of the Julia sets of semigroups $G \in \mathcal{G}_{dis}$, see figure 1 and figure 2.

Figure 1: The Julia set of $G = \langle g_1^2, g_2^2 \rangle$, where $g_1(z) := z^2 - 1$, $g_2(z) := \frac{z^2}{4}$. $G \in \mathcal{G}_{dis}$, G is hyperbolic, and $\sharp(\hat{\mathcal{J}}_G) > \aleph_0$.



2.2 Upper estimates of $\sharp(\hat{\mathcal{J}})$

Next, we present some results on the space $\hat{\mathcal{J}}$ and some results on upper estimates of $\sharp(\hat{\mathcal{J}})$. The proofs are given in Section 4.2 and Section 4.3.

Definition 2.8.

1. For a polynomial g, we denote by $a(g) \in \mathbb{C}$ the coefficient of the highest degree term of g.
2. We set $\text{RA} := \{ax + b \in \mathbb{R}[x] \mid a, b \in \mathbb{R}, a \neq 0\}$ endowed with the topology such that, $a_n x + b_n \to ax + b$ if and only if $a_n \to a$ and $b_n \to b$. The space RA is a semigroup with the semigroup operation being functional composition. Any subsemigroup of RA will be called a *real affine semigroup.* We define a map $\Psi : \text{Poly} \to \text{RA}$ as follows: For a polynomial $g \in \text{Poly}$, we set $\Psi(g)(x) := \deg(g)x + \log|a(g)|$.

   Moreover, for a polynomial semigroup G, we set $\Psi(G) := \{\Psi(g) \mid g \in G\}$ $(\subset \text{RA})$.
3. We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ endowed with the topology such that $\{(r, +\infty]\}_{r \in \mathbb{R}}$ makes a fundamental neighborhood system of $+\infty$, and such that $\{[-\infty, r)\}_{r \in \mathbb{R}}$ makes a fundamental neighborhood system of $-\infty$. For a real affine semigroup H, we set

   $$M(H) := \overline{\{x \in \mathbb{R} \mid \exists h \in H, h(x) = x, |h'(x)| > 1\}} \ (\subset \hat{\mathbb{R}}),$$

   where the closure is taken in the space $\hat{\mathbb{R}}$. Moreover, we denote by $\mathcal{M}_H$ the set of all connected components of $M(H)$.

4. We denote by $\eta : \mathrm{RA} \to \mathrm{Poly}$ the natural embedding defined by $\eta(x \mapsto ax+b) = (z \mapsto az+b)$, where $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

5. We define a map $\Theta : \mathrm{Poly} \to \mathrm{Poly}$ as follows. For a polynomial g, we set $\Theta(g)(z) = a(g)z^{\deg(g)}$. Moreover, for a polynomial semigroup G, we set $\Theta(G) := \{\Theta(g) \mid g \in G\}$.

Remark 2.9.

1. The map $\Psi : \mathrm{Poly} \to \mathrm{RA}$ is a semigroup homomorphism. That is, we have $\Psi(g \circ h) = \Psi(g) \circ \Psi(h)$. Hence, for a polynomial semigroup G, the image $\Psi(G)$ is a real affine semigroup. Similarly, the map $\Theta : \mathrm{Poly} \to \mathrm{Poly}$ is a semigroup homomorphism. Hence, for a polynomial semigroup G, the image $\Theta(G)$ is a polynomial semigroup.

2. The maps $\Psi : \mathrm{Poly} \to \mathrm{RA}$, $\eta : \mathrm{RA} \to \mathrm{Poly}$, and $\Theta : \mathrm{Poly} \to \mathrm{Poly}$ are continuous.

Definition 2.10. For any connected sets M_1 and M_2 in $\hat{\mathbb{R}}$, "$M_1 \leq_r M_2$" indicates that $M_1 = M_2$, or each $(x, y) \in M_1 \times M_2$ satisfies $x < y$. Furthermore, "$M_1 <_r M_2$" indicates $M_1 \leq_r M_2$ and $M_1 \neq M_2$.

Remark 2.11. The above "$\leq_r$" is a partial order in the space of non-empty connected subsets of $\hat{\mathbb{R}}$. Moreover, for each real affine semigroup H, $(\mathcal{M}_H, \leq_r)$ is totally ordered.

The following theorem gives us some upper estimates of $\sharp(\hat{\mathcal{J}}_G)$.

Theorem 2.12.

1. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, we have $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\mathcal{M}_{\Psi(G)})$. More precisely, there exists an injective map $\tilde{\Psi} : \hat{\mathcal{J}}_G \to \mathcal{M}_{\Psi(G)}$ such that if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Psi}(J_1) <_r \tilde{\Psi}(J_2)$.*

2. *If $G \in \mathcal{G}_{dis}$, then we have that $M(\Psi(G)) \subset \mathbb{R}$ and $M(\Psi(G)) = J(\eta(\Psi(G)))$.*

3. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\hat{\mathcal{J}}_{\eta(\Psi(G))})$.*

Corollary 2.13. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, we have $\sharp(\hat{\mathcal{J}}_G) \leq \sharp(\hat{\mathcal{J}}_{\Theta(G)})$. More precisely, there exists an injective map $\tilde{\Theta} : \hat{\mathcal{J}}_G \to \hat{\mathcal{J}}_{\Theta(G)}$ such that if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Theta}(J_1) \in \mathcal{J}_{\Theta(G)}$, $\tilde{\Theta}(J_2) \in \mathcal{J}_{\Theta(G)}$, and $\tilde{\Theta}(J_1) < \tilde{\Theta}(J_2)$.*

The following three theorems give us sufficient conditions for the Julia set of a $G \in \mathcal{G}$ to be connected.

Theorem 2.14. *Let $G = \langle h_1, \ldots, h_m \rangle$ be a finitely generated polynomial semigroup in $\mathcal{G}$. For each $j = 1, \ldots, m$, let a_j be the coefficient of the highest degree term of polynomial h_j. Let $\alpha := \min_{j=1,\ldots,m}\{\frac{-1}{\deg(h_j)-1} \log |a_j|\}$ and $\beta := \max_{j=1,\ldots,m}\{\frac{-1}{\deg(h_j)-1} \log |a_j|\}$. We set $[\alpha, \beta] := \{x \in \mathbb{R} \mid \alpha \leq x \leq \beta\}$. If $[\alpha, \beta] \subset \bigcup_{j=1}^{m} \Psi(h_j)^{-1}([\alpha, \beta])$, then $J(G)$ is connected.*

Theorem 2.15. *Let G be a polynomial semigroup in $\mathcal{G}$ generated by a (possibly non-compact) family of polynomials of degree two. Then, $J(G)$ is connected.*

Theorem 2.16. *Let G be a polynomial semigroup in $\mathcal{G}$ generated by a (possibly non-compact) family $\{h_\lambda\}_{\lambda \in \Lambda}$ of polynomials. Let a_λ be the coefficient of the highest degree term of the polynomial h_λ. Suppose that for any $\lambda, \xi \in \Lambda$, we have $(\deg(h_\xi) - 1) \log |a_\lambda| = (\deg(h_\lambda) - 1) \log |a_\xi|$. Then, $J(G)$ is connected.*

Remark 2.17. In [35], a new cohomology theory for (backward) self-similar systems (iterated function systems) was introduced by the author of this paper. By using this new cohomology theory, for a postcritically bounded finitely generated polynomial semigroup G, we can describe the space of connected components of G and we can give some estimates on $\sharp(\mathcal{J}_G)$ and $\sharp(\mathcal{M}_{\Psi(G)})$.

2.3 Properties of $\mathcal{J}$

In this section, we present some results on $\mathcal{J}$. The proofs are given in Section 4.3.

Definition 2.18. For a polynomial semigroup G, we set

$$\hat{K}(G) := \{z \in \mathbb{C} \mid \bigcup_{g \in G} \{g(z)\} \text{ is bounded in } \mathbb{C}\}$$

and call $\hat{K}(G)$ the **smallest filled-in Julia set** of G. For a polynomial g, we set $K(g) := \hat{K}(\langle g \rangle)$.

Notation: For a set $A \subset \hat{\mathbb{C}}$, we denote by $\text{int}(A)$ the set of all interior points of A.

Proposition 2.19. *Let $G \in \mathcal{G}$. If U is a connected component of $F(G)$ such that $U \cap \hat{K}(G) \neq \emptyset$, then $U \subset \text{int}(\hat{K}(G))$ and U is simply connected. Furthermore, we have $\hat{K}(G) \cap F(G) = \text{int}(\hat{K}(G))$.*

Notation: For a polynomial semigroup G with $\infty \in F(G)$, we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. Moreover, for a polynomial g with $\deg(g) \geq 2$, we set $F_\infty(g) := F_\infty(\langle g \rangle)$.

The following theorem is the key to obtaining further results of postcritically bounded polynomial semigroups in this paper, and those of related random dynamics of polynomials in the sequel [36, 42]. We remark that Theorem 2.20-5 generalizes [47, Theorem 2].

Theorem 2.20. *Let $G \in \mathcal{G}_{dis}$ (possibly generated by a non-compact family). Then, under the above notation, we have the following.*

1. *We have that $\infty \in F(G)$ (thus $\mathcal{J} = \hat{\mathcal{J}}$) and the connected component $F_\infty(G)$ of $F(G)$ containing ∞ is simply connected. Furthermore, the element $J_{\max} = J_{\max}(G) \in \mathcal{J}$ containing $\partial F_\infty(G)$ is the unique element of $\mathcal{J}$ satisfying that $J \leq J_{\max}$ for each $J \in \mathcal{J}$.*

2. *There exists a unique element $J_{\min} = J_{\min}(G) \in \mathcal{J}$ such that $J_{\min} \leq J$ for each element $J \in \mathcal{J}$. Furthermore, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Then, $P^*(G) \subset \hat{K}(G) \subset \mathbb{C} \setminus D$ and $\partial \hat{K}(G) \subset J_{\min}$.*

3. *If G is generated by a family $\{h_\lambda\}_{\lambda \in \Lambda}$, then there exist two elements λ_1 and λ_2 of Λ satisfying:*

   (a) *there exist two elements J_1 and J_2 of $\mathcal{J}$ with $J_1 \neq J_2$ such that $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$;*

   (b) *$J(h_{\lambda_1}) \cap J_{\min} = \emptyset$;*

   (c) *for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ and $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$; and*

   (d) *h_{λ_1} has an attracting fixed point z_1 in $\mathbb{C}$, $\text{int}(K(h_{\lambda_1}))$ consists of only one immediate attracting basin for z_1, and $K(h_{\lambda_2}) \subset \text{int}(K(h_{\lambda_1}))$. Furthermore, $z_1 \in \text{int}(K(h_{\lambda_2}))$.*

4. *For each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have that g has an attracting fixed point z_g in $\mathbb{C}$, $\text{int}(K(g))$ consists of only one immediate attracting basin for z_g, and $J_{\min} \subset \text{int}(K(g))$. Note that it is not necessarily true that $z_g = z_f$ when $g, f \in G$ are such that $J(g) \cap J_{\min} = \emptyset$ and $J(f) \cap J_{\min} = \emptyset$ (see Proposition 2.26).*

5. *We have that $\text{int}(\hat{K}(G)) \neq \emptyset$. Moreover,*

   (a) *$\mathbb{C} \setminus J_{\min}$ is disconnected, $\sharp J \geq 2$ for each $J \in \hat{\mathcal{J}}$, and*

   (b) *for each $g \in G$ with $J(g) \cap J_{\min} = \emptyset$, we have that $J_{\min} < g^*(J_{\min})$, $g^{-1}(J(G)) \cap J_{\min} = \emptyset$, $g(\hat{K}(G) \cup J_{\min}) \subset \text{int}(\hat{K}(G))$, and the unique attracting fixed point z_g of g in $\mathbb{C}$ belongs to $\text{int}(\hat{K}(G))$.*

6. *Let $\mathcal{A}$ be the set of all doubly connected components of $F(G)$. Then, $\bigcup_{A\in\mathcal{A}} A \subset \mathbb{C}$ and $(\mathcal{A}, \leq)$ is totally ordered.*

We present a result on uniform perfectness of the Julia sets of semigroups in $\mathcal{G}$.

Definition 2.21. A compact set K in $\hat{\mathbb{C}}$ is said to be uniformly perfect if $\sharp K \geq 2$ and there exists a constant $C > 0$ such that each annulus A that separates K satisfies that mod $A < C$, where mod A denotes the modulus of A (See the definition in [15]).

Theorem 2.22.

1. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, $J(G)$ is uniformly perfect. Moreover, if $z_0 \in J(G)$ is a superattracting fixed point of an element of G, then $z_0 \in \mathrm{int}(\hat{J}(G))$.*

2. *If $G \in \mathcal{G}$ and $\infty \in J(G)$, then $G \in \mathcal{G}_{con}$ and $\infty \in \mathrm{int}(\hat{J}(G))$.*

3. *Suppose that $G \in \mathcal{G}_{dis}$. Let $z_1 \in J(G) \cap \mathbb{C}$ be a superattracting fixed point of $g \in G$. Then $z_1 \in \mathrm{int}(\hat{J}_{\min})$ and $J(g) \subset J_{\min}$.*

We remark that in [14], it was shown that there exists a rational semigroup G such that $J(G)$ is not uniformly perfect.

We now present results on the Julia sets of subsemigroups of an element of $\mathcal{G}_{dis}$.

Proposition 2.23. *Let $G \in \mathcal{G}_{dis}$ and let $J_1, J_2 \in \mathcal{J} = \mathcal{J}_G$ with $J_1 \leq J_2$. Let A_i be the unbounded component of $\mathbb{C} \setminus J_i$ for each $i = 1, 2$. Then, we have the following.*

1. *Let $Q_1 = \{g \in G \mid \exists J \in \mathcal{J}$ with $J_1 \leq J,\ J(g) \subset J\}$ and let H_1 be the subsemigroup of G generated by Q_1. Then $J(H_1) \subset J_1 \cup A_1$.*

2. *Let $Q_2 = \{g \in G \mid \exists J \in \mathcal{J}$ with $J \leq J_2,\ J(g) \subset J\}$ and let H_2 be the subsemigroup of G generated by Q_2. Then $J(H_2) \subset \mathbb{C} \setminus A_2$.*

3. *Let $Q = \{g \in G \mid \exists J \in \mathcal{J}$ with $J_1 \leq J \leq J_2,\ J(g) \subset J\}$ and let H be the subsemigroup of G generated by Q. Then $J(H) \subset J_1 \cup (A_1 \setminus A_2)$.*

Proposition 2.24. *Let G be a polynomial semigroup generated by a compact subset Γ of $\mathrm{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. Then, there exists an element $h_1 \in \Gamma$ with $J(h_1) \subset J_{\max}$ and there exists an element $h_2 \in \Gamma$ with $J(h_2) \subset J_{\min}$.*

2.4 Finitely generated polynomial semigroups $G \in \mathcal{G}_{dis}$ such that $2 \leq \sharp(\hat{\mathcal{J}}_G) \leq \aleph_0$

In this section, we present some results on various finitely generated polynomial semigroups $G \in \mathcal{G}_{dis}$ such that $2 \leq \sharp(\hat{\mathcal{J}}_G) \leq \aleph_0$. The proofs are given in Section 4.4.

It is well-known that for a rational map g with $\deg(g) \geq 2$, if $J(g)$ is disconnected, then $J(g)$ has uncountably many connected components (See [16]). Moreover, if G is a non-elementary Kleinian group with disconnected Julia set (limit set), then $J(G)$ has uncountably many connected components. However, for general rational semigroups, we have the following examples.

Theorem 2.25. *Let G be a polynomial semigroup in $\mathcal{G}$ generated by a (possibly non-compact) family Γ in $\mathrm{Poly}_{\deg\geq 2}$. Suppose that there exist mutually distinct elements $J_1, \ldots, J_n \in \hat{\mathcal{J}}_G$ such that for each $h \in \Gamma$ and each $j \in \{1, \ldots, n\}$, there exists an element $k \in \{1, \ldots, n\}$ with $h^{-1}(J_j) \cap J_k \neq \emptyset$. Then, we have $\sharp(\hat{\mathcal{J}}_G) = n$.*

Proposition 2.26. *For any $n \in \mathbb{N}$ with $n > 1$, there exists a finitely generated polynomial semigroup $G_n = \langle h_1, \ldots, h_{2n}\rangle$ in $\mathcal{G}$ satisfying $\sharp(\hat{\mathcal{J}}_{G_n}) = n$. In fact, let $0 < \epsilon < \frac{1}{2}$ and we set for each $j = 1, \ldots, n$, $a_j(z) := \frac{1}{j}z^2$ and $\beta_j(z) := \frac{1}{j}(z-\epsilon)^2 + \epsilon$. Then, for any sufficiently large $l \in \mathbb{N}$, there exists an open neighborhood V of $(\alpha_1^l, \ldots, \alpha_n^l, \beta_1^l, \ldots, \beta_n^l)$ in $(\mathrm{Poly})^{2n}$ such that for any $(h_1, \ldots, h_{2n}) \in V$, the semigroup $G = \langle h_1, \ldots, h_{2n}\rangle$ satisfies that $G \in \mathcal{G}$ and $\sharp(\hat{\mathcal{J}}_G) = n$.*

Theorem 2.27. *Let $G = \langle h_1, \ldots, h_m \rangle \in \mathcal{G}_{dis}$ be a polynomial semigroup with $m \geq 3$. Suppose that there exists an element $J_0 \in \hat{\mathcal{J}}$ such that $\bigcup_{j=1}^{m-1} J(h_j) \subset J_0$, and such that for each $j = 1, \ldots, m-1$, we have $h_j^{-1}(J(h_m)) \cap J_0 \neq \emptyset$. Then, we have all of the following.*

1. *$\sharp(\hat{\mathcal{J}}) = \aleph_0$.*
2. *$J_0 = J_{\min}$, or $J_0 = J_{\max}$.*
3. *If $J_0 = J_{\min}$, then $J_{\max} = J(h_m)$, $J(G) = J_{\max} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} (h_m)^{-n}(J_{\min})$, and for any $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$.*
4. *If $J_0 = J_{\max}$, then $J_{\min} = J(h_m)$, $J(G) = J_{\min} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} (h_m)^{-n}(J_{\max})$, and for any $J \in \hat{\mathcal{J}}$ with $J \neq J_{\min}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$.*

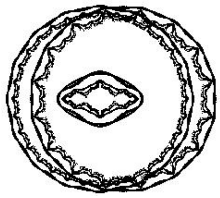
Proposition 2.28. *There exists an open set V in $(\mathrm{Poly}_{\deg \geq 2})^3$ such that for any $(h_1, h_2, h_3) \in V$, $G = \langle h_1, h_2, h_3 \rangle$ satisfies that $G \in \mathcal{G}_{dis}$, $\bigcup_{j=1}^{2} J(h_j) \subset J_{\min}(G)$, $J_{\max}(G) = J(h_3)$, $h_j^{-1}(J(h_3)) \cap J_{\min}(G) \neq \emptyset$ for each $j = 1, 2$, and $\sharp(\hat{\mathcal{J}}_G) = \aleph_0$.*

Proposition 2.29. *There exists a 3-generator polynomial semigroup $G = \langle h_1, h_2, h_3 \rangle$ in $\mathcal{G}_{dis}$ such that $\bigcup_{j=1}^{2} (h_j)^{-1}(J_{\max}(G)) \subset J_{\min}(G)$, $J_{\max}(G) = J(h_3)$, $\sharp(\hat{\mathcal{J}}_G) = \aleph_0$, there exists a superattracting fixed point z_0 of some element of G with $z_0 \in J(G)$, and $\mathrm{int}(J_{\min}(G)) \neq \emptyset$.*

As mentioned before, these results illustrate new phenomena which can hold in the rational semigroups, but cannot hold in the dynamics of a single rational map or Kleinian groups.

For the figure of the Julia set of a 3-generator polynomial semigroup $G \in \mathcal{G}_{dis}$ such that $\sharp\hat{\mathcal{J}}_G = \aleph_0$, see figure 2.

Figure 2: The Julia set of a 3-generator hyperbolic polynomial semigroup $G \in \mathcal{G}_{dis}$ such that $\sharp(\hat{\mathcal{J}}_G) = \aleph_0$.



Remark 2.30. In [35], a new cohomology theory for (backward) self-similar systems (iterated function systems) was introduced by the author of this paper. By using it, for a finitely generated $G \in \mathcal{G}$, we can describe the space $\mathcal{J}_G$ of connected components of $J(G)$, and we can give some estimates on $\sharp(\mathcal{J}_G)$. Moreover, by using this new cohomology, a sufficient condition for the cardinality of the set of all connected components of the Fatou set of a postcritically bounded finitely generated polynomial semigroup G to be infinity was given.

2.5 Hyperbolicity and semi-hyperbolicity

In this section, we present some results on hyperbolicity and semi-hyperbolicity.

Definition 2.31. Let G be a polynomial semigroup generated by a subset Γ of $\mathrm{Poly}_{\deg \geq 2}$. Suppose $G \in \mathcal{G}_{dis}$. Then we set $\Gamma_{\min} := \{h \in \Gamma \mid J(h) \subset J_{\min}\}$, where $J_{\min}$ denotes the unique minimal element in $(\mathcal{J}, \leq)$ in Theorem 2.20-2. Furthermore, if $\Gamma_{\min} \neq \emptyset$, let $G_{\min,\Gamma}$ be the subsemigroup of G that is generated by $\Gamma_{\min}$.

Remark 2.32. Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose $G \in \mathcal{G}_{dis}$. Then, by Proposition 2.24, we have $\Gamma_{\min} \neq \emptyset$ and $\Gamma \setminus \Gamma_{\min} \neq \emptyset$. Moreover, $\Gamma_{\min}$ is a compact subset of Γ. For, if $\{h_n\}_{n\in\mathbb{N}} \subset \Gamma_{\min}$ and $h_n \to h_\infty$ in Γ, then for a repelling periodic point $z_0 \in J(h_\infty)$ of h_∞, we have that $d(z_0, J(h_n)) \to 0$ as $n \to \infty$, which implies that $z_0 \in J_{\min}$ and thus $h_\infty \in \Gamma_{\min}$.

The following Proposition 2.33 means that for a polynomial semigroup $G \in \mathcal{G}_{dis}$ generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$, we rarely have the situation that "$\Gamma \setminus \Gamma_{\min}$ is not compact."

Proposition 2.33. *Let G be a polynomial semigroup generated by a non-empty compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$ and that $\Gamma \setminus \Gamma_{\min}$ is not compact. Then, both of the following statements 1 and 2 hold.*

1. *Let $h \in \Gamma_{\min}$. Then, $J(h) = J_{\min}(G), K(h) = \hat{K}(G)$, and $\text{int}(K(h))$ is a non-empty connected set.*

2. *Either*

   (a) for each $h \in \Gamma_{\min}$, h is hyperbolic and $J(h)$ is a quasicircle; or

   (b) for each $h \in \Gamma_{\min}$, $\text{int}(K(h))$ is an immediate parabolic basin of a parabolic fixed point of h.

Definition 2.34. Let G be a rational semigroup.

1. We say that G is hyperbolic if $P(G) \subset F(G)$.

2. We say that G is semi-hyperbolic if there exists a number $\delta > 0$ and a number $N \in \mathbb{N}$ such that for each $y \in J(G)$ and each $g \in G$, we have $\deg(g : V \to B(y, \delta)) \leq N$ for each connected component V of $g^{-1}(B(y, \delta))$, where $B(y, \delta)$ denotes the ball of radius δ with center y with respect to the spherical distance, and $\deg(g : \cdot \to \cdot)$ denotes the degree of a finite branched covering.

Remark 2.35. There are many nice properties of hyperbolic or semi-hyperbolic rational semigroups. For example, for a finitely generated semi-hyperbolic rational semigroup G , there exists an attractor in the Fatou set ([27, 30]), and the Hausdorff dimension $\dim_H(J(G))$ of the Julia set is less than or equal to the critical exponent $s(G)$ of the Poincaré series of G ([30]). If we assume further the "open set condition", then $\dim_H(J(G)) = s(G)$ ([32, 45]). Moreover, if $G \in \mathcal{G}$ is generated by a compact set Γ and if G is semi-hyperbolic, then for each sequence $\gamma \in \Gamma^{\mathbb{N}}$, the basin of infinity for γ is a John domain and the Julia set of γ is locally connected ([30]). In [37], by using the above result, we classify hyperbolic or semi-hyperbolic postcritically bounded compactly generated polynomial semigroups, in terms of the random complex dynamics. It is shown that in one of the classes, for almost every sequence γ, the Julia set J_γ of γ is a Jordan curve but not a quasicircle, the unbounded component of $\hat{\mathbb{C}} \setminus J_\gamma$ is a John domain, and the bounded component of $\mathbb{C} \setminus J_\gamma$ is not a John domain. Moreover, in [37, 36], we find many examples with this phenomenon. Note that this phenomenon does not hold in the usual iteration dynamics of a single polynomial map g with $\deg(g) \geq 2$.

We now present some results on semi-hyperbolic or hyperbolic polynomial semigroups in $\mathcal{G}_{dis}$. These results are used to construct examples of semi-hyperbolic or hyperbolic polynomial semigroups $G \in \mathcal{G}_{dis}$ (see the proof of Proposition 2.40). Therefore these are important in terms of the sequel [36, 37].

Theorem 2.36. *Let G be a polynomial semigroup generated by a non-empty compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. If $G_{\min,\Gamma}$ is semi-hyperbolic, then G is semi-hyperbolic.*

Theorem 2.37. *Let G be a polynomial semigroup generated by a non-empty compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$. If $G_{\min,\Gamma}$ is hyperbolic and $(\bigcup_{h\in\Gamma\setminus\Gamma_{\min}} CV^*(h)) \cap J_{\min}(G) = \emptyset$, then G is hyperbolic.*

Remark 2.38. In [24], it will be shown that in Theorem 2.37, the condition $(\bigcup_{h\in\Gamma\setminus\Gamma_{\min}} CV^*(h)) \cap J_{\min}(G) = \emptyset$ is necessary. For the figures of the Julia sets of hyperbolic polynomial semigroups $G \in \mathcal{G}_{dis}$, see figure 1 and figure 2.

Proposition 2.39. *Let G be a polynomial semigroup generated by a non-empty compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}_{dis}$ and that $\Gamma \setminus \Gamma_{\min}$ is not compact. Suppose that statement 2a in Theorem 2.33 holds. Then, both of the following statements hold.*

1. *We have that $G_{\min,\Gamma}$ is hyperbolic and G is semi-hyperbolic.*
2. *Suppose further that $(\bigcup_{h\in\Gamma\setminus\Gamma_{\min}} CV^*(h)) \cap J_{\min}(G) = \emptyset$. Then G is hyperbolic.*

2.6 Construction of examples

In this section, we present a way to construct examples of semigroups G in $\mathcal{G}_{dis}$ (with some additional properties). These examples are important in terms of the sequel [36, 37].

Proposition 2.40. *Let G be a polynomial semigroup generated by a compact subset Γ of $\text{Poly}_{\deg\geq 2}$. Suppose that $G \in \mathcal{G}$ and $\text{int}(\hat{K}(G)) \neq \emptyset$. Let $b \in \text{int}(\hat{K}(G))$. Moreover, let $d \in \mathbb{N}$ be any positive integer such that $d \geq 2$, and such that $(d, \deg(h)) \neq (2,2)$ for each $h \in \Gamma$. Then, there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, there exists a compact neighborhood V of $g_a(z) = a(z-b)^d + b$ in $\text{Poly}_{\deg\geq 2}$ satisfying that for any non-empty subset V' of V, the polynomial semigroup $H_{\Gamma,V'}$ generated by the family $\Gamma \cup V'$ belongs to $\mathcal{G}_{dis}$, $\hat{K}(H_{\Gamma,V'}) = \hat{K}(G)$ and $(\Gamma \cup V')_{\min} \subset \Gamma$. Moreover, in addition to the assumption above, if G is semi-hyperbolic (resp. hyperbolic), then the above $H_{\Gamma,V'}$ is semi-hyperbolic (resp. hyperbolic).*

Remark 2.41. By Proposition 2.40, there exists a 2-generator polynomial semigroup $G = \langle h_1, h_2 \rangle$ in $\mathcal{G}_{dis}$ such that h_1 has a Siegel disk.

Definition 2.42. Let $d \in \mathbb{N}$ with $d \geq 2$. We set $\mathcal{Y}_d := \{h \in \text{Poly} \mid \deg(h) = d\}$ endowed with the relative topology from Poly.

Theorem 2.43. *Let $m \geq 2$ and let $d_2, \ldots, d_m \in \mathbb{N}$ be such that $d_j \geq 2$ for each $j = 2, \ldots, m$. Let $h_1 \in \mathcal{Y}_{d_1}$ with $\text{int}(K(h_1)) \neq \emptyset$ be such that $\langle h_1 \rangle \in \mathcal{G}$. Let $b_2, b_3, \ldots, b_m \in \text{int}(K(h_1))$. Then, both of the following statements hold.*

1. *Suppose that $\langle h_1 \rangle$ is semi-hyperbolic (resp. hyperbolic). Then, there exists a number $c > 0$ such that for each $(a_2, a_3, \ldots, a_m) \in \mathbb{C}^{m-1}$ with $0 < |a_j| < c$ $(j = 2, \ldots, m)$, setting $h_j(z) = a_j(z-b_j)^{d_j} + b_j$ $(j = 2, \ldots, m)$, the polynomial semigroup $G = \langle h_1, \ldots, h_m \rangle$ satisfies that $G \in \mathcal{G}$, $\hat{K}(G) = K(h_1)$ and G is semi-hyperbolic (resp. hyperbolic).*
2. *Suppose that $\langle h_1 \rangle$ is semi-hyperbolic (resp. hyperbolic). Suppose also that either (i) there exists a $j \geq 2$ with $d_j \geq 3$, or (ii) $\deg(h_1) = 3$, $b_2 = \cdots = b_m$. Then, there exist $a_2, a_3, \ldots, a_m > 0$ such that setting $h_j(z) = a_j(z-b_j)^{d_j} + b_j$ $(j = 2, \ldots, m)$, the polynomial semigroup $G = \langle h_1, h_2, \ldots, h_m \rangle$ satisfies that $G \in \mathcal{G}_{dis}$, $\hat{K}(G) = K(h_1)$ and G is semi-hyperbolic (resp. hyperbolic).*

Definition 2.44. Let $m \in \mathbb{N}$. We set

- $\mathcal{H}_m := \{(h_1, \ldots, h_m) \in (\text{Poly}_{\deg\geq 2})^m \mid \langle h_1, \ldots, h_m \rangle$ is hyperbolic$\}$,
- $\mathcal{B}_m := \{(h_1, \ldots, h_m) \in (\text{Poly}_{\deg\geq 2})^m \mid \langle h_1, \ldots, h_m \rangle \in \mathcal{G}\}$, and

- $\mathcal{D}_m := \{(h_1, \ldots, h_m) \in (\text{Poly}_{\deg \geq 2})^m \mid J(\langle h_1, \ldots, h_m \rangle) \text{ is disconnected}\}$.

Moreover, let $\pi_1 : (\text{Poly}_{\deg \geq 2})^m \to \text{Poly}_{\deg \geq 2}$ be the projection defined by $\pi(h_1, \ldots, h_m) = h_1$.

Theorem 2.45. *Under the above notation, all of the following statements hold.*

1. $\mathcal{H}_m, \mathcal{H}_m \cap \mathcal{B}_m$, $\mathcal{H}_m \cap \mathcal{D}_m$, *and* $\mathcal{H}_m \cap \mathcal{B}_m \cap \mathcal{D}_m$ *are open in* $(\text{Poly}_{\deg \geq 2})^m$.
2. *Let* $d_1, \ldots, d_m \in \mathbb{N}$ *be such that* $d_j \geq 2$ *for each* $j = 1, \ldots, m$. *Then,* $\pi_1 : \mathcal{H}_m \cap \mathcal{B}_m \cap (\mathcal{Y}_{d_1} \times \cdots \times \mathcal{Y}_{d_m}) \to \mathcal{H}_1 \cap \mathcal{B}_1 \cap \mathcal{Y}_{d_1}$ *is surjective.*
3. *Let* $d_1, \ldots, d_m \in \mathbb{N}$ *be such that* $d_j \geq 2$ *for each* $j = 1, \ldots, m$ *and such that* $(d_1, \ldots, d_m) \neq (2, 2, \ldots, 2)$. *Then,* $\pi_1 : \mathcal{H}_m \cap \mathcal{B}_m \cap \mathcal{D}_m \cap (\mathcal{Y}_{d_1} \times \cdots \times \mathcal{Y}_{d_m}) \to \mathcal{H}_1 \cap \mathcal{B}_1 \cap \mathcal{Y}_{d_1}$ *is surjective.*

Remark 2.46. Combining Proposition 2.40, Theorem 2.43, and Theorem 2.45, we can construct many examples of semigroups G in $\mathcal{G}$ (or $\mathcal{G}_{dis}$) with some additional properties (semi-hyperbolicity, hyperbolicity, etc.).

3 Tools

To show the main results, we need some tools in this section.

3.1 Fundamental properties of rational semigroups

Notation: For a rational semigroup G, we set $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp(\bigcup_{g \in G} g^{-1}(\{z\})) < \infty\}$. This is called the exceptional set of G.

The following Lemma 3.1 and Theorem 3.2 will be used in the proofs of the main results.

Lemma 3.1 ([13, 12, 29, 27])**.** *Let G be a rational semigroup.*

1. *For each $h \in G$, we have $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*
2. *If $G = \langle h_1, \ldots, h_m \rangle$, then $J(G) = h_1^{-1}(J(G)) \cup \cdots \cup h_m^{-1}(J(G))$. More generally, if G is generated by a compact subset* Γ *of* Rat, *then $J(G) = \bigcup_{h \in \Gamma} h^{-1}(J(G))$. (We call this property of the Julia set of a compactly generated rational semigroup "backward self-similarity.")*
3. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set.*
4. *If $\sharp(J(G)) \geq 3$, then $\sharp(E(G)) \leq 2$.*
5. *If a point z is not in $E(G)$, then $J(G) \subset \overline{\bigcup_{g \in G} g^{-1}(\{z\})}$. In particular if a point z belongs to $J(G) \setminus E(G)$, then $\overline{\bigcup_{g \in G} g^{-1}(\{z\})} = J(G)$.*
6. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest closed backward invariant set containing at least three points. Here we say that a set A is backward invariant under G if for each $g \in G$, $g^{-1}(A) \subset A$.*

Theorem 3.2 ([13, 12, 29])**.** *Let G be a rational semigroup. If $\sharp(J(G)) \geq 3$, then $J(G) = \overline{\{z \in \hat{\mathbb{C}} \mid \exists g \in G,\ g(z) = z,\ |m(g, z)| > 1\}}$, where $m(g, z)$ denotes the multiplier of g at z ([3]). In particular, $J(G) = \overline{\bigcup_{g \in G} J(g)}$.*

Remark 3.3. If a rational semigroup G contains an element g with $\deg(g) \geq 2$, then $\sharp(J(g)) \geq 3$, which implies that $\sharp(J(G)) \geq 3$.

Lemma 3.4. *Let $G = \langle h_1, h_2 \rangle \in \mathcal{G}$. Then, $h_1^{-1}(J(h_2))$ is connected.*

Proof. Since $h_2 \in G \in \mathcal{G}$, $F_\infty(h_2)$ is simply connected. Since $G \in \mathcal{G}$, there exists no finite critical value of h_1 in $F_\infty(h_2)$. By the Riemann-Hurwitz formula, it follows that $h_1^{-1}(F_\infty(h_2))$ is connected and simply connected. Thus $\partial(h_1^{-1}(F_\infty(h_2))) = h_1^{-1}(J(h_2))$ is connected. □

Definition 3.5. Let G be a polynomial semigroup. Let $p \in \mathbb{C}$ and $\epsilon > 0$. We set $\mathcal{F}_{G,p,\epsilon} := \{\alpha : D(p,\epsilon) \to \mathbb{C} \mid \alpha \text{ is a well-defined branch of } g^{-1}, g \in G\}$.

Lemma 3.6. *Let Γ be a non-empty compact subset of $\mathrm{Poly}_{\deg \geq 2}$ and let G be a polynomial semigroup generated by Γ. Let $R > 0, \epsilon > 0$, and*
$\mathcal{F} := \{\alpha \circ \beta : D(0,1) \to \mathbb{C} \mid \beta : D(0,1) \cong D(p,\epsilon),\ \alpha : D(p,\epsilon) \to \mathbb{C},\ \alpha \in \mathcal{F}_{G,p,\epsilon},\ p \in D(0,R)\}$. *Then, $\mathcal{F}$ is normal in $D(0,1)$.*

Proof. Since Γ is a non-empty compact subset of $\mathrm{Poly}_{\deg\geq 2}$, there exists a ball B around ∞ with $B \subset \hat{\mathbb{C}} \setminus D(0, R+\epsilon)$ such that for each $h \in \Gamma$, $h(B) \subset B$. Let $p \in D(0,R)$. Then, for each $\alpha \in \mathcal{F}_{G,p,\epsilon}$, $\alpha(D(p,\epsilon)) \subset \hat{\mathbb{C}} \setminus B$. Hence, $\mathcal{F}$ is normal in $D(0,1)$. □

3.2 A lemma from general topology

Lemma 3.7 ([17])**.** *Let X be a compact metric space and let $f : X \to X$ be a continuous open map. Let A be a compact connected subset of X. Then for each connected component B of $f^{-1}(A)$, we have $f(B) = A$.*

4 Proofs of the main results

In this section, we demonstrate the main results.

4.1 Proofs of results in 2.1

In this section, we demonstrate the results in 2.1.

Proof of Theorem 2.1: First, we show the following:
Claim: For any $\lambda \in \Lambda$, $h_\lambda^{-1}(A) \subset A$.

To show the claim, let $\lambda \in \Lambda$ with $J(h_\lambda) \neq \emptyset$ and let B be a connected component of $h_\lambda^{-1}(A)$. Then by Lemma 3.7, $h_\lambda(B) = A$. Combining this with $h_\lambda^{-1}(J(h_\lambda)) = J(h_\lambda)$, we obtain $B \cap J(h_\lambda) \neq \emptyset$. Hence $B \subset A$. This means that $h_\lambda^{-1}(A) \subset A$ for each $\lambda \in \Lambda$ with $J(h_\lambda) \neq \emptyset$. Next, let $\lambda \in \Lambda$ with $J(h_\lambda) = \emptyset$. Then h_λ is either identity or an elliptic Möbius transformation. By hypothesis and Lemma 3.1-1, we obtain $h_\lambda^{-1}(A) \subset A$. Hence, we have shown the claim.

Combining the above claim with $\sharp A \geq 3$, by Lemma 3.1-6 we obtain $J(G) \subset A$. Hence $J(G) = A$ and $J(G)$ is connected. □

Notation: We denote by d the spherical distance on $\hat{\mathbb{C}}$. Given $A \subset \hat{\mathbb{C}}$ and $z \in \hat{\mathbb{C}}$, we set $d(z,A) := \inf\{d(z,w) \mid w \in A\}$. Given $A \subset \hat{\mathbb{C}}$ and $\epsilon > 0$, we set $B(A,\epsilon) := \{a \in \hat{\mathbb{C}} \mid d(a,A) < \epsilon\}$. Furthermore, given $A \subset \mathbb{C}$, $z \in \mathbb{C}$, and $\epsilon > 0$, we set $d_e(z,A) := \inf\{|z-w| \mid w \in A\}$ and $D(A,\epsilon) := \{a \in \mathbb{C} \mid d_e(a,A) < \epsilon\}$.

We need the following lemmas to prove the main results.

Lemma 4.1. *Let $G \in \mathcal{G}$ and let J be a connected component of $J(G)$, $z_0 \in J$ a point, and $\{g_n\}_{n\in\mathbb{N}}$ a sequence in G such that $d(z_0, J(g_n)) \to 0$ as $n \to \infty$. Then $\sup_{z\in J(g_n)} d(z,J) \to 0$ as $n \to \infty$.*

Proof. Suppose there exists a connected component J' of $J(G)$ with $J' \neq J$ and a subsequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $\min_{z\in J(g_{n_j})} d(z,J') \to 0$ as $j \to \infty$. Since $J(g_{n_j})$ is compact and connected for each j, we may assume, passing to a subsequence, that there exists a non-empty compact connected subset K of $\hat{\mathbb{C}}$ such that $J(g_{n_j}) \to K$ as $j \to \infty$, with respect to the Hausdorff metric. Then $K \cap J \neq \emptyset$ and $K \cap J' \neq \emptyset$. Since $K \subset J(G)$ and K is connected, it contradicts $J' \neq J$. □

Lemma 4.2. *Let $G \in \mathcal{G}$. Then given $J \in \mathcal{J}$ and $\epsilon > 0$, there exists an element $g \in G$ such that $J(g) \subset B(J, \epsilon)$.*

Proof. We take a point $z \in J$. Then, by Theorem 3.2, there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ in G such that $d(z, J(g_n)) \to 0$ as $n \to \infty$. By Lemma 4.1, we conclude that there exists an $n \in \mathbb{N}$ such that $J(g_n) \subset B(J, \epsilon)$. □

Lemma 4.3. *Let G be a polynomial semigroup. Suppose that $J(G)$ is disconnected, and $\infty \in J(G)$. Then, the connected component A of $J(G)$ containing ∞ is equal to $\{\infty\}$.*

Proof. By Lemma 3.7, we obtain $g^{-1}(A) \subset A$ for each $g \in G$. Hence, if $\sharp A \geq 3$, then $J(G) \subset A$, by Lemma 3.1-6. Then $J(G) = A$ and it causes a contradiction, since $J(G)$ is disconnected. □

We now demonstrate Theorem 2.7.

Proof of Theorem 2.7: First, we show statement 1. Suppose the statement is false. Then, there exist elements $J_1, J_2 \in \mathcal{J}$ such that J_2 is included in the unbounded component A_1 of $\mathbb{C} \setminus J_1$, and such that J_1 is included in the unbounded component A_2 of $\mathbb{C} \setminus J_2$. Then we can find an $\epsilon > 0$ such that $\overline{B(J_2, \epsilon)}$ is included in the unbounded component of $\mathbb{C} \setminus \overline{B(J_1, \epsilon)}$, and such that $\overline{B(J_1, \epsilon)}$ is included in the unbounded component of $\mathbb{C} \setminus \overline{B(J_2, \epsilon)}$. By Lemma 4.2, for each $i = 1, 2$, there exists an element $g_i \in G$ such that $J(g_i) \subset B(J_i, \epsilon)$. This implies that $J(g_1) \subset A'_2$ and $J(g_2) \subset A'_1$, where A'_i denotes the unbounded component of $\mathbb{C} \setminus J(g_i)$. Hence we obtain $K(g_2) \subset A'_1$. Let v be a critical value of g_2 in $\mathbb{C}$. Since $P^*(G)$ is bounded in $\mathbb{C}$, we have $v \in K(g_2)$. It implies $v \in A'_1$. Hence $g_1^l(v) \to \infty$ as $l \to \infty$. However, this implies a contradiction since $P^*(G)$ is bounded in $\mathbb{C}$. Hence we have shown statement 1.

Next, we show statement 2. Let F_1 be a connected component of $F(G)$. Suppose that there exist three connected components J_1, J_2 and J_3 of $J(G)$ such that they are mutually disjoint and such that $\partial F_1 \cap J_i \neq \emptyset$ for each $i = 1, 2, 3$. Then, by statement 1, we may assume that we have either (1): $J_i \in \mathcal{J}$ for each $i = 1, 2, 3$ and $J_1 < J_2 < J_3$, or (2): $J_1, J_2 \in \mathcal{J},\ J_1 < J_2$, and $\infty \in J_3$. Each of these cases implies that J_1 is included in a bounded component of $\mathbb{C} \setminus J_2$ and J_3 is included in the unbounded component of $\hat{\mathbb{C}} \setminus J_2$. However, it causes a contradiction, since $\partial F_1 \cap J_i \neq \emptyset$ for each $i = 1, 2, 3$. Hence, we have shown that we have either

Case I: $\sharp\{J : \text{component of } J(G) \mid \partial F_1 \cap J \neq \emptyset\} = 1$ or

Case II: $\sharp\{J : \text{component of } J(G) \mid \partial F_1 \cap J \neq \emptyset\} = 2$.

Suppose that we have Case I. Let J_1 be the connected component of $J(G)$ such that $\partial F_1 \subset J_1$. Let D_1 be the connected component of $\hat{\mathbb{C}} \setminus J_1$ containing F_1. Since $\partial F_1 \subset J_1$, we have $\partial F_1 \cap D_1 = \emptyset$. Hence, we have $F_1 = D_1$. Therefore, F_1 is simply connected.

Suppose that we have Case II. Let J_1 and J_2 be the two connected components of $J(G)$ such that $J_1 \neq J_2$ and $\partial F_1 \subset J_1 \cup J_2$. Let D be the connected component of $\hat{\mathbb{C}} \setminus (J_1 \cup J_2)$ containing F_1. Since $\partial F_1 \subset J_1 \cup J_2$, we have $\partial F_1 \cap D = \emptyset$. Hence, we have $F_1 = D$. Therefore, F_1 is doubly connected. Thus, we have shown statement 2.

We now show statement 3. Let $g \in G$ be an element and J a connected component of $J(G)$. Suppose that $g^{-1}(J)$ is disconnected. Then, by Lemma 3.7, there exist at most finitely many connected components $C_1, \ldots, C_r$ of $g^{-1}(J)$ with $r \geq 2$. Then there exists a positive number ϵ such that denoting by B_j the connected component of $g^{-1}(B(J, \epsilon))$ containing C_j for each $j = 1, \ldots, r$, $\{B_j\}$ are mutually disjoint. By Lemma 3.7, we see that, for each connected component B of $g^{-1}(B(J, \epsilon))$, $g(B) = B(J, \epsilon)$ and $B \cap C_j \neq \emptyset$ for some j. Hence we get that $g^{-1}(B(J, \epsilon)) = \bigcup_{j=1}^r B_j$ (disjoint union) and $g(B_j) = B(J, \epsilon)$ for each j. By Lemma 4.2, there exists an element $h \in G$ such that $J(h) \subset B(J, \epsilon)$. Then it follows that $g^{-1}(J(h)) \cap B_j \neq \emptyset$ for each $j = 1, \ldots, r$. Moreover, we have $g^{-1}(J(h)) \subset g^{-1}(B(J, \epsilon)) = \bigcup_{j=1}^r B_j$. On the other hand, by Lemma 3.4, we have that

$g^{-1}(J(h))$ is connected. This is a contradiction. Hence, we have shown that, for each $g \in G$ and each connected component J of $J(G)$, $g^{-1}(J)$ is connected.

By Lemma 4.3, we get that if $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. Let J_1 and J_2 be two elements of $\mathcal{J}$ such that $J_1 \leq J_2$. Let U_i be the unbounded component of $\mathbb{C} \setminus J_i$, for each $i = 1, 2$. Then $U_2 \subset U_1$. Let $g \in G$ be an element. Then $g^{-1}(U_2) \subset g^{-1}(U_1)$. Since $g^{-1}(U_i)$ is the unbounded connected component of $\mathbb{C} \setminus g^{-1}(J_i)$ for each $i = 1, 2$, it follows that $g^{-1}(J_1) \leq g^{-1}(J_2)$. Hence $g^*(J_1) \leq g^*(J_2)$, otherwise $g^*(J_2) < g^*(J_1)$, and it contradicts $g^{-1}(J_1) \leq g^{-1}(J_2)$. □

4.2 Proofs of results in 2.2

In this section, we prove the results in Section 2.2, except Theorem 2.12-2 and Theorem 2.12-3, which will be proved in Section 4.3.

To demonstrate Theorem 2.12, we need the following lemmas.

Lemma 4.4. *Let G be a polynomial semigroup in $\mathcal{G}_{dis}$. Let $J_1, J_2 \in \hat{\mathcal{J}}$ be two elements with $J_1 \neq J_2$. Then, we have the following.*

1. *If $J_1, J_2 \in \mathcal{J}$ and $J_1 < J_2$, then there exists a doubly connected component A of $F(G)$ such that $J_1 < A < J_2$.*
2. *If $\infty \in J_2$, then there exists a doubly connected component A of $F(G)$ such that $J_1 < A$.*

Proof. First, we show statement 1. Suppose that $J_1, J_2 \in \mathcal{J}$ and $J_1 < J_2$. We set $B = \bigcup_{J \in \mathcal{J}, J_1 \leq J \leq J_2} J$. Then, B is a closed disconnected set. Hence, there exists a multiply connected component A' of $\hat{\mathbb{C}} \setminus B$. Since A' is multiply connected, we have that A' is included in the unbounded component of $\hat{\mathbb{C}} \setminus J_1$, and that A' is included in a bounded component of $\hat{\mathbb{C}} \setminus J_2$. This implies that $A' \cap J(G) = \emptyset$. Let A be the connected component of $F(G)$ such that $A' \subset A$. Since $B \subset J(G)$, we have $F(G) \subset \hat{\mathbb{C}} \setminus B$. Hence, A must be equal to A'. Since A' is multiply connected, Theorem 2.7-2 implies that $A = A'$ is doubly connected. Let J be the connected component $J(G)$ such that $J < A$ and $J \cap \partial A \neq \emptyset$. Then, since $A' = A$ is included in the unbounded component of $\hat{\mathbb{C}} \setminus J_1$, we have that J does not meet any bounded component of $\mathbb{C} \setminus J_1$. Hence, we obtain $J_1 \leq J$, which implies that $J_1 \leq J < A$. Therefore, A is a doubly connected component of $F(G)$ such that $J_1 < A < J_2$. Hence, we have shown statement 1.

Next, we show statement 2. Suppose that $\infty \in J_2$. We set $B = (\bigcup_{J \in \mathcal{J}, J_1 \leq J} J) \cup J_2$. Then, B is a disconnected closed set. Hence, there exists a multiply connected component A' of $\hat{\mathbb{C}} \setminus B$. By the same method as that of proof of statement 1, we see that A' is equal to a doubly connected component A of $F(G)$ such that $J_1 < A$. Hence, we have shown statement 2. □

Lemma 4.5. *Let H_0 be a real affine semigroup generated by a compact set C in* RA. *Suppose that each element $h \in C$ is of the form $h(x) = b_1(h)x + b_2(h)$, where $b_1(h), b_2(h) \in \mathbb{R}$, $|b_1(h)| > 1$. Then, for any subsemigroup H of H_0, we have $M(H) = J(\eta(H)) \subset \mathbb{R}$.*

Proof. From the assumption, there exists a number $R > 0$ such that for each $h \in C$, $\eta(h)(B(\infty, R)) \subset B(\infty, R)$. Hence, we have $B(\infty, R) \subset F(\eta(H))$, which implies that $J(\eta(H))$ is a bounded subset of $\mathbb{C}$. We consider the following cases:
Case 1: $\sharp(J(\eta(H))) \geq 3$.
Case 2: $\sharp(J(\eta(H))) \leq 2$.

Suppose that we have case 1. Then, from Theorem 3.2, it follows that $M(H) = J(\eta(H)) \subset \mathbb{R}$.

Suppose that we have case 2. Let $b(h)$ be the unique fixed point of $h \in H$ in $\mathbb{R}$. From the hypothesis, we have that for each $h \in H$, $b(h) \in J(\eta(H))$. Since we assume $\sharp(J(\eta(H))) \leq 2$, Lemma 3.1-1 implies that there exists a point $b \in \mathbb{R}$ such that for each $h \in H$, we have $b(h) = b$. Then any element $h \in H$ is of the form $h(x) = c_1(h)(x-b)+c_2(h)$, where $c_1(h), c_2(h) \in \mathbb{R}$, $|c_1(h)| > 1$. Hence, $M(H) = \{b\} \subset J(\eta(H))$. Suppose that there exists a point c in $J(\eta(H)) \setminus \{b\}$. Since

$J(\eta(H))$ is a bounded set of $\mathbb{C}$, and since we have $h^{-1}(J(\eta(H))) \subset J(\eta(H))$ for each $h \in H$ (Lemma 3.1-1), we get that $h^{-1}(c) \in J(\eta(H))\setminus(\{b\}\cup\{c\})$, for each element $h \in H$. This implies that $\sharp(J(\eta(H))) \geq 3$, which is a contradiction. Hence, we must have that $J(\eta(H)) = \{b\} = M(H)$. $\square$

We need the notion of Green's functions, in order to demonstrate Theorem 2.12.

Definition 4.6. Let D be a domain in $\hat{\mathbb{C}}$ with $\infty \in D$. We denote by $\varphi(D, z)$ Green's function on D with pole at ∞. By definition, this is the unique function on $D \cap \mathbb{C}$ with the properties:

1. $\varphi(D, z)$ is harmonic and positive in $D \cap \mathbb{C}$;
2. $\varphi(D, z) - \log |z|$ is bounded in a neighborhood of ∞; and
3. there exists a Borel subset A of ∂D such that the logarithmic capacity of $(\partial D) \setminus A$ is zero and such that for each $\zeta \in A$, we have $\varphi(D, z) \to 0$ as $z \to \zeta$.

Remark 4.7.

1. The limit $\lim_{z\to\infty}(\varphi(D, z) - \log |z|)$ exists and this is called *Robin's constant* of D.
2. If D is a simply connected domain with $\infty \in D$ and $\sharp(\hat{\mathbb{C}} \setminus D) > 1$, then we have $\varphi(D, z) = -\log |\psi(z)|$, where $\psi : D \to \{z \in \mathbb{C} \mid |z| < 1\}$ denotes a biholomorphic map with $\psi(\infty) = 0$.
3. It is well-known that for any $g \in \text{Poly}_{\deg\geq 2}$,
$$\varphi(F_\infty(g), z) = \log |z| + \frac{1}{\deg(g) - 1} \log |a(g)| + o(1) \quad \text{as } z \to \infty. \tag{1}$$
(See [25, p147].) Note that the point $-\frac{1}{\deg(g)-1} \log |a(g)| \in \mathbb{R}$ is the unique fixed point of $\Psi(g)$ in $\mathbb{R}$.

Lemma 4.8. *Let K_1 and K_2 be two non-empty connected compact sets in $\mathbb{C}$ such that $K_1 < K_2$ and $\sharp K_1 \neq 1$. Let A_i denote the unbounded component of $\hat{\mathbb{C}} \setminus K_i$, for each $i = 1, 2$. Then, we have $\lim_{z\to\infty}(\log |z| - \varphi(A_1, z)) < \lim_{z\to\infty}(\log |z| - \varphi(A_2, z))$.*

Proof. The function $\phi(z) := \varphi(A_2, z) - \varphi(A_1, z) = (\log |z| - \varphi(A_1, z)) - (\log |z| - \varphi(A_2, z))$ is harmonic on $A_2 \cap \mathbb{C}$. This ϕ is bounded around ∞. Hence ϕ extends to a harmonic function on A_2. Moreover, since $K_1 < K_2$, we have $\limsup_{z\to\partial A_2} \phi(z) < 0$. From the maximum principle, it follows that $\phi(\infty) < 0$. Therefore, the statement of our lemma holds. $\square$

In order to demonstrate Theorem 2.12-1, we will prove the following lemma. (Theorem 2.12-2 and Theorem 2.12-3 will be proved in Section 4.3.)

Lemma 4.9. *Let G be a polynomial semigroup in $\mathcal{G}$. Then, there exists an injective map $\tilde{\Psi} : \hat{\mathcal{J}}_G \to \mathcal{M}_{\Psi(G)}$ such that:*

1. *if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Psi}(J_1) <_r \tilde{\Psi}(J_2)$;*
2. *if $J \in \hat{\mathcal{J}}_G$ and $\infty \in J$, then $+\infty \in \tilde{\Psi}(J)$; and*
3. *if $J \in \mathcal{J}_G$, then $\tilde{\Psi}(J) \subset \hat{\mathbb{R}} \setminus \{+\infty\}$.*

Proof. We first show the following claim.
Claim 1: In addition to the assumption of Lemma 4.9, if we have $\infty \in F(G)$, then $M(\Psi(G)) \subset \hat{\mathbb{R}} \setminus \{+\infty\}$.

To show this claim, let $R > 0$ be a number such that $J(G) \subset D(0, R)$. Then, for any $g \in G$, we have $K(g) < \partial D(0, R)$. By Lemma 4.8, we get that there exists a constant $C > 0$ such that for each $g \in G$, $\frac{-1}{\deg(g)-1}\log|a(g)| \leq C$. Hence, it follows that $M(\Psi(G)) \subset [-\infty, C]$. Therefore, we have shown Claim 1.

We now prove the statement of the lemma in the case $G \in \mathcal{G}_{con}$. If $\infty \in F(G)$, then claim 1 implies that $M(\Psi(G)) \subset \hat{\mathbb{R}} \setminus \{+\infty\}$ and the statement of the lemma holds. We now suppose $\infty \in J(G)$. We put $L_g := \max_{z\in J(g)}|z|$ for each $g \in G$. Moreover, for each non-empty compact subset E of $\mathbb{C}$, we denote by Cap (E) the logarithmic capacity of E. We remark that $\mathrm{Cap}(E) = \exp(\lim_{z\to\infty}(\log|z| - \varphi(D_E, z)))$, where D_E denotes the connected component of $\hat{\mathbb{C}} \setminus E$ containing ∞. We may assume that $0 \in P^*(G)$. Then, by [1], we have $\mathrm{Cap}(J(g)) \geq \mathrm{Cap}\ ([0, L_g]) \geq L_g/4$ for each $g \in G$. Combining this with $\infty \in J(G)$, Theorem 3.2, and Remark 4.7-3, we obtain $+\infty \in M_{\Psi(G)}$ and defining $\tilde{\Psi}(J(G))$ to be the connected component of $\mathcal{M}_{\Psi(G)}$ containing $+\infty$, the statement of the lemma holds.

We now prove the statement of the lemma in the case $G \in \mathcal{G}_{dis}$. Let $\{J_\lambda\}_{\lambda\in\Lambda}$ be the set $\hat{\mathcal{J}}_G$ of all connected components of $J(G)$. By Lemma 4.2, for each $\lambda \in \Lambda$ and each $n \in \mathbb{N}$, there exists an element $g_{\lambda,n} \in G$ such that

$$J(g_{\lambda,n}) \subset B(J_\lambda, \frac{1}{n}). \tag{2}$$

We have that the fixed point of $\Psi(g_{\lambda,n})$ in $\mathbb{R}$ is equal to $\frac{-1}{\deg(g_{\lambda,n})-1}\log|a(g_{\lambda,n})|$. We may assume that $\frac{-1}{\deg(g_{\lambda,n})-1}\log|a(g_{\lambda,n})| \to \alpha_\lambda$ as $n \to \infty$, where α_λ is an element of $\hat{\mathbb{R}}$. For each $\lambda \in \Lambda$, let $B_\lambda \in \mathcal{M}_{\Psi(G)}$ be the element with $\alpha_\lambda \in B_\lambda$. Let $\tilde{\Psi}(J_\lambda) = B_\lambda$ for each $\lambda \in \Lambda$. We will show the following claim.

Claim 2: If λ, ξ are two elements in Λ with $\lambda \neq \xi$, then $B_\lambda \neq B_\xi$. Moreover, if $J_\lambda, J_\xi \in \mathcal{J}_G$ and $J_\lambda < J_\xi$, then $B_\lambda <_r B_\xi$. Furthermore, if $J_\xi \in \hat{\mathcal{J}}_G$ with $\infty \in J_\xi$, then $+\infty \in B_\xi$.

To show this claim, let λ and ξ be two elements in Λ with $\lambda \neq \xi$. We have the following two cases:

Case 1: $J_\lambda, J_\xi \in \mathcal{J}_G$ and $J_\lambda < J_\xi$.

Case 2: $J_\lambda \in \mathcal{J}_G$ and $\infty \in J_\xi$. (Note: in this case, by Lemma 4.3, we have $J_\xi = \{\infty\}$.)

Suppose that we have case 1. By Lemma 4.4, there exists a doubly connected component A of $F(G)$ such that

$$J_\lambda < A < J_\xi. \tag{3}$$

Let ζ_1 and ζ_2 be two Jordan curves in A such that they are not null-homotopic in A, and such that $\zeta_1 < \zeta_2$. For each $i = 1, 2$, let A_i be the unbounded component of $\hat{\mathbb{C}} \setminus \zeta_i$. Moreover, we set $\beta_i := \lim_{z\to\infty}(\log|z| - \varphi(A_i, z))$, for each $i = 1, 2$. By Lemma 4.8, we have $\beta_1 < \beta_2$. Let $g \in G$ be any element. By (2) and (3), there exists an $m \in \mathbb{N}$ such that $J(g_{\lambda,m}) < \zeta_1$. Since $P^*(G) \subset K(g_{\lambda,m})$, it follows that $P^*(G)$ is included in the bounded component of $\mathbb{C} \setminus \zeta_1$. Hence, we see that

$$\text{either } J(g) < \zeta_1, \text{ or } \zeta_2 < J(g). \tag{4}$$

From Lemma 4.8, it follows that either $\frac{-1}{\deg(g)-1}\log|a(g)| < \beta_1$, or $\beta_2 < \frac{-1}{\deg(g)-1}\log|a(g)|$. This implies that

$$M(\Psi(G)) \subset \hat{\mathbb{R}} \setminus (\beta_1, \beta_2), \tag{5}$$

where $(\beta_1, \beta_2) := \{x \in \mathbb{R} \mid \beta_1 < x < \beta_2\}$. Moreover, combining (2), (3), and (4), we get that there exists a number $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $J(g_{\lambda,n}) < \zeta_1 < \zeta_2 < J(g_{\xi,n})$. From Lemma 4.8, it follows that

$$\frac{-1}{\deg(g_{\lambda,n})-1}\log|a(g_{\lambda,n})| < \beta_1 < \beta_2 < \frac{-1}{\deg(g_{\xi,n})-1}\log|a(g_{\xi,n})|, \tag{6}$$

for each $n \geq n_0$. By (5) and (6), we obtain $B_\lambda <_r B_\xi$.

We now suppose that we have case 2. Then, by Lemma 4.4, there exists a doubly connected component A of $F(G)$ such that $J_\lambda < A$. Continuing the same argument as that of case 1, we obtain $B_\lambda \neq B_\xi$. In order to show $+\infty \in B_\xi$, let R be any number such that $P^*(G) \subset D(0, R)$.

Since $P^*(G) \subset K(g)$ for each $g \in G$, combining it with (2) and Lemma 4.3, we see that there exists an $n_0 = n_0(R)$ such that for each $n \geq n_0$, $D(0,R) < J(g_{\xi,n})$. From Lemma 4.8, it follows that $\frac{-1}{\deg(g_{\xi,n})-1} \log |a(g_{\xi,n})| \to +\infty$. Hence, $+\infty \in B_\xi$. Therefore, we have shown Claim 2.

Combining Claims 1 and 2, the statement of the lemma follows.

Therefore, we have proved Lemma 4.9. □

We now demonstrate Theorem 2.12-1.

Proof of Theorem 2.12-1: From Lemma 4.9, Theorem 2.12-1 follows. □

We now demonstrate Corollary 2.13.

Proof of Corollary 2.13: By Theorem 3.2, we have $J(\Theta(G)) = \overline{\bigcup_{h \in \Theta(G)} J(h)} = \overline{\bigcup_{g \in G} J(\Theta(g))}$, where the closure is taken in $\hat{\mathbb{C}}$. Since $J(\Theta(g)) = \{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}$, we obtain

$$J(\Theta(G)) = \overline{\bigcup_{g \in G} \{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}}, \tag{7}$$

where the closure is taken in $\hat{\mathbb{C}}$. Hence, we see that $\sharp(\hat{\mathcal{J}}_{\Theta(G)})$ is equal to the cardinality of the set of all connected components of $J(\Theta(G)) \cap [0, +\infty]$. Moreover, let $\psi : [0, +\infty] \to \hat{\mathbb{R}}$ be the homeomorphism defined by $\psi(x) := \log(x)$ for $x \in (0, +\infty)$, $\psi(0) := -\infty$, and $\psi(+\infty) = +\infty$. Then, (7) implies that, the map $\psi : [0, \infty] \to \hat{\mathbb{R}}$, maps $J(\Theta(G)) \cap [0, +\infty]$ onto $M(\Psi(\Theta(G)))$. For any $J \in \hat{\mathcal{J}}_{\Theta(G)}$, let $\tilde{\psi}(J) \in \mathcal{M}_{\Psi(\Theta(G))} = \mathcal{M}_{\Psi(G)}$ be the element such that $\psi(J \cap [0, +\infty]) = \tilde{\psi}(J)$. Then, the map $\tilde{\psi} : \hat{\mathcal{J}}_{\Theta(G)} \to \mathcal{M}_{\Psi(\Theta(G))} = \mathcal{M}_{\Psi(G)}$ is a bijection, and moreover, for any $J_1, J_2 \in \mathcal{J}_{\Theta(G)}$, we have that $J_1 < J_2$ if and only if $\tilde{\psi}(J_1) <_r \tilde{\psi}(J_2)$. Furthermore, for any $J \in \hat{\mathcal{J}}_{\Theta(G)}$, $\infty \in J$ if and only if $+\infty \in \tilde{\psi}(J)$. Let $\tilde{\Theta} : \hat{\mathcal{J}}_G \to \hat{\mathcal{J}}_{\Theta(G)}$ be the map defined by $\tilde{\Theta} = \tilde{\psi}^{-1} \circ \tilde{\Psi}$, where $\tilde{\Psi} : \hat{\mathcal{J}}_G \to \mathcal{M}_{\Psi(G)}$ is the map in Lemma 4.9. Then, by Lemma 4.9, $\tilde{\Theta} : \hat{\mathcal{J}}_G \to \hat{\mathcal{J}}_{\Theta(G)}$ is injective, and moreover, if $J_1, J_2 \in \mathcal{J}_G$ and $J_1 < J_2$, then $\tilde{\Theta}(J_1) \in \mathcal{J}_{\Theta(G)}$, $\tilde{\Theta}(J_2) \in \mathcal{J}_{\Theta(G)}$, and $\tilde{\Theta}(J_1) < \tilde{\Theta}(J_2)$.

Thus, we have proved Corollary 2.13. □

We now demonstrate Theorem 2.14.

Proof of Theorem 2.14: We have that for any $j = 1, \ldots, m$, $(\Psi(h_j))^{-1}(x) = \frac{1}{\deg(h_j)}(x - \log |a_j|) = \frac{1}{\deg(h_j)}(x - \frac{-1}{\deg(h_j)-1} \log |a_j|) + \frac{-1}{\deg(h_j)-1} \log |a_j|$, where $x \in \mathbb{R}$. Hence, it is easy to see that $\bigcup_{j=1}^m (\Psi(h_j))^{-1}([\alpha, \beta]) \subset [\alpha, \beta]$. From the assumption, it follows that

$$\bigcup_{j=1}^{m} (\Psi(h_j))^{-1}([\alpha, \beta]) = [\alpha, \beta]. \tag{8}$$

Moreover, by Lemma 3.1-2, we have

$$\bigcup_{j=1}^{m} (\eta(\Psi(h_j)))^{-1}(J(\eta(\Psi(G)))) = J(\eta(\Psi(G))). \tag{9}$$

Furthermore, by Lemma 4.5, $J(\eta(\Psi(G)))$ is a compact subset of $\mathbb{R}$. Applying [9, Theorem 2.6], it follows that $J(\eta(\Psi(G))) = [\alpha, \beta]$. Combined with Lemma 4.5, we obtain $M(\Psi(G)) = [\alpha, \beta]$. Hence, $M(\Psi(G))$ is connected. Therefore, from Theorem 2.12-1, it follows that $J(G)$ is connected. □

We now demonstrate Theorem 2.15.

Proof of Theorem 2.15: Let C be a set of polynomials of degree two such that C generates G. Suppose that $J(G)$ is disconnected. Then, by Theorem 2.1, there exist two elements $h_1, h_2 \in C$ such that the semigroup $H = \langle h_1, h_2 \rangle$ satisfies that $J(H)$ is disconnected. For each $j = 1, 2$, let a_j

be the coefficient of the highest degree term of polynomial h_j. Let $\alpha := \min_{j=1,2}\{\frac{-1}{\deg(h_j)-1}\log|a_j|\}$ and $\beta := \max_{j=1,2}\{\frac{-1}{\deg(h_j)-1}\log|a_j|\}$. Then we have that $\alpha = \min_{j=1,2}\{-\log|a_j|\}$ and $\beta = \max_{j=1,2}\{-\log|a_j|\}$. Since $\Psi(h_j)^{-1}(x) = \frac{1}{2}(x-\log|a_j|) = \frac{1}{2}(x-(-\log|a_j|)) + (-\log|a_j|)$ for each $j=1,2$, we obtain $[\alpha,\beta] = \bigcup_{j=1}^{2}(\Psi(h_j))^{-1}([\alpha,\beta])$. Hence, by Theorem 2.14, it must be true that $J(H)$ is connected. However, this is a contradiction. Therefore, $J(G)$ must be connected. □

We now demonstrate Theorem 2.16.

Proof of Theorem 2.16: For each $\lambda \in \Lambda$, let b_λ be the fixed point of $\Psi(h_\lambda)$ in $\mathbb{R}$. It is easy to see that $b_\lambda = \frac{-1}{\deg(h_\lambda)-1}\log|a_\lambda|$, for each $\lambda \in \Lambda$. From the assumption, it follows that there exists a point $b \in \mathbb{R}$ such that for each $\lambda \in \Lambda$, $b_\lambda = b$. This implies that for any element $g \in G$, the fixed point $b(g) \in \mathbb{R}$ of $\Psi(g)$ in $\mathbb{R}$ is equal to b. Hence, we obtain $M(\Psi(G)) = \{b\}$. Therefore, $M(\Psi(G))$ is connected. From Theorem 2.12-1, it follows that $J(G)$ is connected. □

4.3 Proofs of results in 2.3

In this section, we prove the results in 2.3, Theorem 2.12-2 and Theorem 2.12-3.

In order to demonstrate Theorem 2.20, Theorem 2.12-2, and Theorem 2.12-3, we need the following lemma.

Lemma 4.10. *If $G \in \mathcal{G}_{dis}$, then $\infty \in F(G)$.*

Proof. Suppose that $G \in \mathcal{G}_{dis}$ and $\infty \in J(G)$. We will deduce a contradiction. By Lemma 4.3, the element $J \in \hat{\mathcal{J}}_G$ with $\infty \in J$ satisfies that $J = \{\infty\}$. Hence, by Lemma 4.2, for each $n \in \mathbb{N}$, there exists an element $g_n \in G$ such that $J(g_n) \subset B(\infty, \frac{1}{n})$. Let $R > 0$ be any number which is sufficiently large so that $P^*(G) \subset B(0,R)$. Since we have that $P^*(G) \subset K(g)$ for each $g \in G$, it must hold that there exists a number $n_0 = n_0(R) \in \mathbb{N}$ such that for each $n \geq n_0$, $B(0,R) < J(g_n)$. From Lemma 4.8, it follows that $\lim_{z\to\infty}(\log|z| - \varphi(F_\infty(g_n), z)) \to +\infty$ as $n \to \infty$. Hence, we see that $\frac{-1}{\deg(g_n)-1}\log|a(g_n)| \to +\infty$, as $n \to \infty$. This implies that

$$|a(g_n)|^{-\frac{1}{\deg(g_n)-1}} \to \infty, \text{ as } n \to \infty. \tag{10}$$

Furthermore, by Theorem 2.12-1, we must have that $M(\Psi(G))$ is disconnected.

We now consider the polynomial semigroup $H = \{z \mapsto |a(g)|z^{\deg(g)} \mid g \in G\} \in \mathcal{G}$. By Theorem 3.2, we have $J(H) = \overline{\bigcup_{h\in H} J(h)}$. Since the Julia set of polynomial $|a(g)|z^{\deg(g)}$ is equal to $\{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}$, it follows that

$$J(H) = \overline{\bigcup_{g\in G}\{z \in \mathbb{C} \mid |z| = |a(g)|^{-\frac{1}{\deg(g)-1}}\}}, \tag{11}$$

where the closure is taken in $\hat{\mathbb{C}}$. Moreover, $J(\Theta(G)) = J(H)$. Combining it with (10), (11), and Corollary 2.13, we see that

$$\infty \in J(H), \text{ and } J(H) \text{ is disconnected.} \tag{12}$$

Let $\psi : [0,+\infty] \to \hat{\mathbb{R}}$ be the homeomorphism as in the proof of Corollary 2.13. By (11), we have

$$\psi(J(H) \cap [0,+\infty]) = M(\Psi(H)) = M(\Psi(G)). \tag{13}$$

Moreover, by Lemma 3.1-1, we have

$$h(F(H) \cap [0,+\infty]) \subset F(H) \cap [0,+\infty], \text{ for each } h \in H. \tag{14}$$

Furthermore, we have that

$$\psi \circ h = \Psi(h) \circ \psi \text{ on } [0, +\infty], \text{ for each } h \in H. \tag{15}$$

Combining (13), (14), and (15), we see that

$$\Psi(h)(\hat{\mathbb{R}} \setminus M(\Psi(H))) \subset (\hat{\mathbb{R}} \setminus M(\Psi(H))), \text{ for each } h \in H. \tag{16}$$

By Lemma 4.3 and (12), we get that the connected component J of $J(H)$ containing ∞ satisfies that

$$J = \{\infty\}. \tag{17}$$

Combined with Lemma 4.2, we see that for each $n \in \mathbb{N}$, there exists an element $h_n \in H$ such that

$$J(h_n) \subset B(\infty, \frac{1}{n}). \tag{18}$$

Combining (11), (13), (17), and (18), we obtain the following claim.
Claim 1: $+\infty$ is a non-isolated point of $M(\Psi(H))$ and the connected component of $M(\Psi(H))$ containing $+\infty$ is equal to $\{+\infty\}$.

Let $h \in H$ be an element. Conjugating G by some linear transformation, we may assume that h is of the form $h(z) = z^s, s \in \mathbb{N}, s > 1$. Hence $\Psi(h)(x) = sx, s > 1$. Since 0 is a fixed point of $\Psi(h)$, we have that $0 \in M(\Psi(H))$. By Claim 1, there exists $c_1, c_2 \in [0, +\infty)$ with $c_1 < c_2$ such that the open interval $I = (c_1, c_2)$ is a connected component of $\hat{\mathbb{R}} \setminus M(\Psi(H))$. We now show the following claim.
Claim 2: Let $Q = (r_1, r_2) \subset (0, +\infty)$ be any connected open interval in $\hat{\mathbb{R}} \setminus M(\Psi(H))$, where $0 \leq r_1 < r_2 < +\infty$. Then, we have $r_2 \leq sr_1$.

To show this claim, suppose that $sr_1 < r_2$. Then, it implies that $\bigcup_{n \in \mathbb{N} \cup \{0\}} \Psi(h)^n(Q) = (r_1, +\infty)$. However, by (16), we have $\bigcup_{n \in \mathbb{N} \cup \{0\}} \Psi(h)^n(Q) \subset \hat{\mathbb{R}} \setminus M(\Psi(H))$, which implies that the connected component Q' of $\hat{\mathbb{R}} \setminus M(\Psi(H))$ containing Q satisfies that $Q' \supset (r_1, +\infty)$. This contradicts Claim 1. Hence, we obtain Claim 2.

By Claim 2, we obtain $c_1 > 0$. Let $c_3 \in (0, c_1)$ be a number so that $c_2 - c_3 > s(c_1 - c_3)$. Since $c_1 \in M(\Psi(H))$, there exists an element $c \in (c_3, c_1]$ and an element $h_1 \in H$ such that $\Psi(h_1)(c) = c$ and $(\Psi(h_1))'(c) > 1$. Since $c_2 - c_3 > s(c_1 - c_3)$, we obtain

$$c_2 - c > s(c_1 - c). \tag{19}$$

Let $t := (\Psi(h_1))'(c) > 1$. Then, for each $n \in \mathbb{N}$, we have $(\Psi(h_1))^n(I) = (t^n(c_1 - c) + c, t^n(c_2 - c) + c)$. From Claim 2 and (16), it follows that $t^n(c_2 - c) + c \leq s(t^n(c_1 - c) + c)$, for each $n \in \mathbb{N}$. Dividing both sides by t^n and then letting $n \to \infty$, we obtain $c_2 - c \leq s(c_1 - c)$. However, this contradicts (19). Hence, we must have that $\infty \in F(G)$. Thus, we have proved Lemma 4.10. $\square$

We now demonstrate Proposition 2.19.
Proof of Proposition 2.19: Let U be a connected component of $F(G)$ with $U \cap \hat{K}(G) \neq \emptyset$. Let $g \in G$ be an element. Then we have $\hat{K}(G) \cap F(G) \subset \text{int}(K(g))$. Since $h(F(G)) \subset F(G)$ and $h(\hat{K}(G) \cap F(G)) \subset \hat{K}(G) \cap F(G)$ for each $h \in G$, it follows that $h(U) \subset \text{int}(K(g))$ for each $h \in G$. Hence $U \subset \text{int}(\hat{K}(G))$. From this, it is easy to see $\hat{K}(G) \cap F(G) = \text{int}(\hat{K}(G))$. By the maximum principle, we see that U is simply connected. $\square$

We now demonstrate Theorem 2.20.
Proof of Theorem 2.20:

First, we show statement 1. By Lemma 4.10, we have that $\infty \in F(G)$. Let $J \in \mathcal{J}$ be an element such that $\partial F_\infty(G) \cap J \neq \emptyset$. Let D be the unbounded component of $\hat{\mathbb{C}} \setminus J$. Then $F_\infty(G) \subset D$ and

D is simply connected. We show $F_\infty(G) = D$. Otherwise, there exists an element $J_1 \in \mathcal{J}$ such that $J_1 \neq J$ and $J_1 \subset D$. By Theorem 2.7-1, we have either $J_1 < J$ or $J < J_1$. Hence, it follows that $J < J_1$ and we have that J is included in a bounded component D_0 of $\mathbb{C} \setminus J_1$. Since $F_\infty(G)$ is included in the unbounded component D_1 of $\hat{\mathbb{C}} \setminus J_1$, it contradicts $\partial F_\infty(G) \cap J \neq \emptyset$. Hence, $F_\infty(G) = D$ and $F_\infty(G)$ is simply connected.

Next, let $J_{\max}$ be the element of $\mathcal{J}$ with $\partial F_\infty(G) \subset J_{\max}$, and suppose that there exists an element $J \in \mathcal{J}$ such that $J_{\max} < J$. Then $J_{\max}$ is included in a bounded component of $\mathbb{C} \setminus J$. On the other hand, $F_\infty(G)$ is included in the unbounded component of $\hat{\mathbb{C}} \setminus J$. Since $\partial F_\infty(G) \subset J_{\max}$, we have a contradiction. Hence, we have shown that $J \leq J_{\max}$ for each $J \in \mathcal{J}$.

Therefore, we have shown statement 1.

Next, we show statement 2. Since $\emptyset \neq P^*(G) \subset \hat{K}(G)$, we have $\hat{K}(G) \neq \emptyset$. By Proposition 2.19, we have $\partial \hat{K}(G) \subset J(G)$. Let J_1 be a connected component of $J(G)$ with $J_1 \cap \partial \hat{K}(G) \neq \emptyset$. By Lemma 4.3, $J_1 \in \mathcal{J}$. Suppose that there exists an element $J \in \mathcal{J}$ such that $J < J_1$. Let $z_0 \in J$ be a point. By Theorem 3.2, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in G such that $d(z_0, J(g_n)) \to 0$ as $n \to \infty$. Then by Lemma 4.1, $\sup_{z\in J(g_n)} d(z, J) \to 0$ as $n \to \infty$. Since J_1 is included in the unbounded component of $\mathbb{C} \setminus J$, it follows that for a large $n \in \mathbb{N}$, J_1 is included in the unbounded component of $\mathbb{C} \setminus J(g_n)$. However, this causes a contradiction, since $J_1 \cap \hat{K}(G) \neq \emptyset$. Hence, by Theorem 2.7-1, it must hold that $J_1 \leq J$ for each $J \in \mathcal{J}$. This argument shows that if J_1 and J_2 are two connected components of $J(G)$ such that $J_i \cap \partial \hat{K}(G) \neq \emptyset$ for each $i = 1, 2$, then $J_1 = J_2$. Hence, we conclude that there exists a unique minimal element $J_{\min}$ in $(\mathcal{J}, \leq)$ and $\partial \hat{K}(G) \subset J_{\min}$.

Next, let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$. Suppose $D \cap \hat{K}(G) \neq \emptyset$. Let $x \in D \cap \hat{K}(G)$ be a point. By Theorem 3.2 and Lemma 4.1, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in G such that $\sup_{z\in J(g_n)} d(z, J_{\min}) \to 0$ as $n \to \infty$. Then, for a large $n \in \mathbb{N}$, x is in the unbounded component of $\mathbb{C} \setminus J(g_n)$. However, this is a contradiction, since $g_n^l(x) \to \infty$ as $l \to \infty$, and $x \in \hat{K}(G)$. Hence, we have shown statement 2.

Next, we show statement 3. By Theorem 2.1, there exist $\lambda_1, \lambda_2 \in \Lambda$ and connected components J_1, J_2 of $J(G)$ such that $J_1 \neq J_2$ and $J(h_{\lambda_i}) \subset J_i$ for each $i = 1, 2$. By Lemma 4.3, we have $J_i \in \mathcal{J}$ for each $i = 1, 2$. Then $J(h_{\lambda_1}) \cap J(h_{\lambda_2}) = \emptyset$. Since $P^*(G)$ is bounded in $\mathbb{C}$, we may assume $J(h_{\lambda_2}) < J(h_{\lambda_1})$. Then we have $K(h_{\lambda_2}) \subset \mathrm{int}(K(h_{\lambda_1}))$ and $J_2 < J_1$. By statement 2, $J_1 \neq J_{\min}$. Hence $J(h_{\lambda_1}) \cap J_{\min} = \emptyset$. Since $P^*(G)$ is bounded in $\mathbb{C}$, we have that $K(h_{\lambda_2})$ is connected. Let U be the connected component of $\mathrm{int}(K(h_{\lambda_1}))$ containing $K(h_{\lambda_2})$. Since $P^*(G) \subset K(h_{\lambda_2})$, it follows that there exists an attracting fixed point z_1 of h_{λ_1} in $K(h_{\lambda_2})$ and U is the immediate attracting basin for z_1 with respect to the dynamics of h_{λ_1}. Furthermore, by Lemma 3.4, $h_{\lambda_1}^{-1}(J(h_{\lambda_2}))$ is connected. Therefore, $h_{\lambda_1}^{-1}(U) = U$. Hence, $\mathrm{int}(K(h_{\lambda_1})) = U$.

Suppose that there exists an $n \in \mathbb{N}$ such that $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) \neq \emptyset$. Then, by Lemma 3.4, $A := \bigcup_{s\geq 0} h_{\lambda_1}^{-ns}(J(h_{\lambda_2}))$ is connected and its closure $\overline{A}$ contains $J(h_{\lambda_1})$. Hence $J(h_{\lambda_1})$ and $J(h_{\lambda_2})$ are included in the same connected component of $J(G)$. This is a contradiction. Therefore, for each $n \in \mathbb{N}$, we have $h_{\lambda_1}^{-n}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$. Similarly, for each $n \in \mathbb{N}$, we have $h_{\lambda_2}^{-n}(J(h_{\lambda_1})) \cap J(h_{\lambda_1}) = \emptyset$. Combining $h_{\lambda_1}^{-1}(J(h_{\lambda_2})) \cap J(h_{\lambda_2}) = \emptyset$ with $z_1 \in K(h_{\lambda_2})$, we obtain $z_1 \in \mathrm{int}(K(h_{\lambda_2}))$. Hence, we have proved statement 3.

We now prove statement 4. Let $g \in G$ be an element with $J(g) \cap J_{\min} = \emptyset$. We show the following:

Claim 2: $J_{\min} < J(g)$.

To show the claim, suppose that $J_{\min}$ is included in the unbounded component U of $\mathbb{C} \setminus J(g)$. Since $\emptyset \neq \partial \hat{K}(G) \subset J_{\min}$, it follows that $\hat{K}(G) \cap U \neq \emptyset$. However, this is a contradiction. Hence, we have shown Claim 2.

Combining Claim 2, Theorem 3.2 and Lemma 4.1, we get that there exists an element $h_1 \in G$ such that $J(h_1) < J(g)$. From an argument which we have used in the proof of statement 3, it follows that g has an attracting fixed point z_g in $\mathbb{C}$ and $\mathrm{int}(K(g))$ consists of only one immediate attracting basin for z_g. Hence, we have shown statement 4.

Next, we show statement 5. Suppose that $\text{int}(\hat{K}(G)) = \emptyset$. We will deduce a contradiction. If $\text{int}(\hat{K}(G)) = \emptyset$, then by Proposition 2.19, we obtain $F(G) \cap \hat{K}(G) = \emptyset$. By statement 3, there exist two elements g_1 and g_2 of G and two elements J_1 and J_2 of $\mathcal{J}$ such that $J_1 \neq J_2$, such that $J(g_i) \subset J_i$ for each $i = 1, 2$, such that g_1 has an attracting fixed point z_0 in $\text{int}(K(g_2))$, and such that $K(g_2) \subset \text{int}(K(g_1))$. Since we assume $F(G) \cap \hat{K}(G) = \emptyset$, we have $z_0 \in P^*(G) \subset \hat{K}(G) \subset J(G)$. Let J be the connected component of $J(G)$ containing z_0. We now show $J = \{z_0\}$. Suppose $\sharp J \geq 2$. Then $J(g_1) \subset \overline{\bigcup_{n\geq 0} g_1^{-n}(J)}$. Moreover, by Theorem 2.7-3, $g_1^{-n}J$ is connected for each $n \in \mathbb{N}$. Since $g_1^{-n}(J) \cap J \neq \emptyset$ for each $n \in \mathbb{N}$, we see that $\overline{\bigcup_{n\geq 0} g_1^{-n}(J)}$ is connected. Combining this with $z_0 \in \text{int}(K(g_2))$, $K(g_2) \subset \text{int}(K(g_1))$, $z_0 \in J$ and $J(g_1) \subset \overline{\bigcup_{n\geq 0} g_1^{-n}(J)}$, we obtain $\overline{\bigcup_{n\geq 0} g_1^{-n}(J)} \cap J(g_2) \neq \emptyset$. Then it follows that $J(g_1)$ and $J(g_2)$ are included in the same connected component of $J(G)$. This is a contradiction. Hence, we have shown $J = \{z_0\}$. By statement 2, we obtain $\{z_0\} = J_{\min} = P^*(G)$. Let $\varphi(z) := \frac{1}{z-z_0}$ and let $\tilde{G} := \{\varphi g \varphi^{-1} \mid g \in G\}$. Then $\tilde{G} \in \mathcal{G}_{dis}$. Moreover, since $z_0 \in J(G)$, we have that $\infty \in J(\tilde{G})$. This contradicts Lemma 4.10. Therefore, we must have that $\text{int}(\hat{K}(G)) \neq \emptyset$.

Since $\partial \hat{K}(G) \subset J_{\min}$ (statement 2) and $\hat{K}(G)$ is bounded, it follows that $\mathbb{C} \setminus J_{\min}$ is disconnected and $\sharp J_{\min} \geq 2$. Hence, $\sharp J \geq 2$ for each $J \in \mathcal{J} = \hat{\mathcal{J}}$. Now, let $g \in G$ be an element with $J(g) \cap J_{\min} = \emptyset$. we show $J_{\min} \neq g^*(J_{\min})$. If $J_{\min} = g^*(J_{\min})$, then $g^{-1}(J_{\min}) \subset J_{\min}$. Since $\sharp J_{\min} \geq 3$, it follows that $J(g) \subset J_{\min}$, which is a contradiction. Hence, $J_{\min} \neq g^*(J_{\min})$, and so $J_{\min} < g^*(J_{\min})$. Combined with Theorem 2.7-3, we obtain $g^{-1}(J(G)) \cap J_{\min} = \emptyset$. Since $g(\hat{K}(G)) \subset \hat{K}(G)$, we have $g(\text{int}(\hat{K}(G))) \subset \text{int}(\hat{K}(G))$. Suppose $g(\partial \hat{K}(G)) \cap \partial \hat{K}(G) \neq \emptyset$. Then, since $\partial \hat{K}(G) \subset J_{\min}$ (statement 2), we obtain $g(J_{\min}) \cap J_{\min} \neq \emptyset$. This implies $g^{-1}(J_{\min}) \cap J_{\min} \neq \emptyset$, which contradicts $g^{-1}(J(G)) \cap J_{\min} = \emptyset$. Hence, it must hold $g(\partial \hat{K}(G)) \subset \text{int}(\hat{K}(G))$, and so $g(\hat{K}(G)) \subset \text{int}(\hat{K}(G))$. Moreover, since $g^{-1}(J(G)) \cap J_{\min} = \emptyset$, we have that $g(J_{\min})$ is a connected subset of $F(G)$. Since $\partial \hat{K}(G) \subset J_{\min}$ and $g(\partial \hat{K}(G)) \subset \text{int}(\hat{K}(G))$, Proposition 2.19 implies that $g(J_{\min})$ must be contained in $\text{int}(\hat{K}(G))$.

By statement 4, g has a unique attracting fixed point z_g in $\mathbb{C}$. Then, $z_g \in P^*(G) \subset \hat{K}(G)$. Hence, $z_g = g(z_g) \in g(\hat{K}(G)) \subset \text{int}(\hat{K}(G))$. Hence, we have shown statement 5.

We now show statement 6. Since $F_\infty(G)$ is simply connected (statement 1), we have $\bigcup_{A \in \mathcal{A}} A \subset \mathbb{C}$. Suppose that there exist two distinct elements A_1 and A_2 in $\mathcal{A}$ such that A_1 is included in the unbounded component of $\mathbb{C} \setminus A_2$, and such that A_2 is included in the unbounded component of $\mathbb{C} \setminus A_1$. For each $i = 1, 2$, let $J_i \in \mathcal{J}$ be the element that intersects the bounded component of $\mathbb{C} \setminus A_i$. Then, $J_1 \neq J_2$. Since $(\mathcal{J}, \leq)$ is totally ordered (Theorem 2.7-1), we may assume that $J_1 < J_2$. Then, it implies that $A_1 < J_2 < A_2$, which is a contradiction. Hence, $(\mathcal{A}, \leq)$ is totally ordered. Therefore, we have proved statement 6.

Thus, we have proved Theorem 2.20. □

We now demonstrate Theorem 2.22.

Proof of Theorem 2.22: First, we show Theorem 2.22-1. If $G \in \mathcal{G}_{con}$, then $J(G)$ is uniformly perfect.

We now suppose that $G \in \mathcal{G}_{dis}$. Let A be an annulus separating $J(G)$. Then A separates $J_{\min}$ and $J_{\max}$. Let D be the unbounded component of $\mathbb{C} \setminus J_{\min}$ and let U be the connected component of $\mathbb{C} \setminus J_{\max}$ containing $J_{\min}$. Then it follows that $A \subset U \cap D$. Since $\sharp J_{\min} > 1$ and $\infty \in F(G)$ (Theorem 2.20), we get that the doubly connected domain $U \cap D$ satisfies mod $(U \cap D) < \infty$. Hence, we obtain mod $A \leq$ mod $(U \cap D) < \infty$. Therefore, $J(G)$ is uniformly perfect.

If a point $z_0 \in J(G)$ is a superattracting fixed point of an element $g \in G$, then, combining uniform perfectness of $J(G)$ and [14, Theorem 4.1], it follows that $z_0 \in \text{int}(J(G))$. Thus, we have shown Theorem 2.22-1.

Next, we show Theorem 2.22-2. If $G \in \mathcal{G}$ and $\infty \in J(G)$, then by Lemma 4.10, we obtain $G \in \mathcal{G}_{con}$. Moreover, Theorem 2.22-1 implies that $\infty \in \text{int}(J(G))$. Therefore, we have shown Theorem 2.22-2.

We now show Theorem 2.22-3. Suppose that $G \in \mathcal{G}_{dis}$. Let $g \in G$ and let $z_1 \in J(G) \cap \mathbb{C}$ with $g(z_1) = z_1$ and $g'(z_1) = 0$. Then, $z_1 \in P^*(G) \subset \hat{K}(G)$. By Theorem 2.20-2, we obtain $z_1 \in J_{\min}$. Moreover, Theorem 2.22-1 implies that $z_1 \in \mathrm{int}(J(G))$. Combining this and $z_1 \in J_{\min}$, we obtain $z_1 \in \mathrm{int}(J_{\min})$. By Theorem 2.20- 5b, we obtain $J(g) \subset J_{\min}$.

Hence, we have shown Theorem 2.22. □

We now demonstrate Theorem 2.12-2.

Proof of Theorem 2.12-2: Suppose $G \in \mathcal{G}_{dis}$. Then, by Lemma 4.10, we obtain $\infty \in F(G)$. Hence, there exists a number $R > 0$ such that for each $g \in G$, $J(g) < \partial B(0, R)$. From Lemma 4.8, it follows that there exists a constant $C_1 > 0$ such that for each $g \in G$, $\frac{-1}{\deg(g)-1} \log |a(g)| < C_1$. This implies that there exists a constant $C_2 \in \mathbb{R}$ such that

$$M(\Psi(G)) \subset [-\infty, C_2]. \tag{20}$$

Moreover, by Theorem 2.20-5, we have that $\mathrm{int}(\hat{K}(G)) \neq \emptyset$. Let B be a closed disc in $\mathrm{int}(\hat{K}(G))$. Then it must hold that for each $g \in G$, $B < J(g)$. Hence, by Lemma 4.8, there exists a constant $C_3 \in \mathbb{R}$ such that for each $g \in G$, $C_3 \leq \frac{-1}{\deg(g)-1} \log |a(g)|$. Therefore, we obtain

$$M(\Psi(G)) \subset [C_3, +\infty]. \tag{21}$$

Combining (20) and (21), we obtain $M(\Psi(G)) \subset \mathbb{R}$. Let C_4 be a large number so that $M(\Psi(G)) \subset D(0, C_4)$. Since for each $g \in G$, the repelling fixed point $-\frac{1}{\deg(g)-1} \log |a(g)|$ of $\eta(\Psi(g))$ belongs to $D(0, C_4) \cap \mathbb{R}$, we see that for each $z \in \mathbb{C} \setminus D(0, C_4)$, $|\eta(\Psi(g))(z)| = |\deg(g)(z - \frac{-1}{\deg(g)-1} \log |a(g)|) + \frac{-1}{\deg(g)-1} \log |a(g)|| \geq \deg(g)C_4 - (\deg(g) - 1)C_4 = C_4$. It follows that $\infty \in F(\eta(\Psi(G)))$. Combining this and Theorem 3.2, we obtain $M(\Psi(G)) = J(\eta(\Psi(G)))$, if $\sharp(J(\eta(\Psi(G)))) \geq 3$.

Suppose that $\sharp(J(\eta(\Psi(G)))) = 2$. Let $g \in G$ be an element and let $b \in \mathbb{R}$ be the unique fixed point of $\Psi(g)$ in $\mathbb{R}$. Then, since $\infty \in F(\eta(\Psi(G)))$, there exists a point $c \in (J(\eta(\Psi(G))) \cap \mathbb{C}) \setminus \{b\}$. By Lemma 3.1-1, $(\eta(\Psi(g)))^{-1}(c) \in J(\eta(\Psi(G))) \setminus \{b, c\}$. This contradicts $\sharp(J(\eta(\Psi(G)))) = 2$. Hence it must hold that $\sharp(J(\eta(\Psi(G)))) \neq 2$.

Suppose that $\sharp(J(\eta(\Psi(G)))) = 1$. Since $M(\Psi(G)) \subset \mathbb{R}$ and $M(\Psi(G)) \cap \mathbb{R} \subset J(\eta(\Psi(G)))$, it follows that $M(\Psi(G)) = J(\eta(\Psi(G)))$.

Therefore, we always have that $M(\Psi(G)) = J(\eta(\Psi(G)))$. Thus, we have proved Theorem 2.12-2. □

We now demonstrate Theorem 2.12-3.

Proof of Theorem 2.12-3: By Theorem 2.12-1 and Theorem 2.12-2, the statement holds. □

We now demonstrate Proposition 2.23.

Proof of Proposition 2.23: First, we show statement 1. Let $g \in Q_1$. We show the following:
Claim 1: For any element $J_3 \in \mathcal{J}$ with $J_1 \leq J_3$, we have $J_1 \leq g^*(J_3)$.
To show this claim, let $J \in \mathcal{J}$ be an element with $J(g) \subset J$. We consider the following two cases;
Case 1: $J \leq J_3$, and
Case 2: $J_1 \leq J_3 \leq J$.

Suppose that we have Case 1. Then, $J_1 \leq J = g^*(J) \leq g^*(J_3)$. Hence, the statement of Claim 1 is true.

Suppose that we have Case 2. If we have $g^*(J_3) < J_3$, then, we have $(g^n)^*(J_3) \leq g^*(J_3) < J_3 \leq J$ for each $n \in \mathbb{N}$. Hence, $\inf\{d(z, J) \mid z \in g^{-n}(J_3), n \in \mathbb{N}\} > 0$. However, since $J(g) \subset J$ and $\sharp J_3 \geq 3$, we obtain a contradiction. Hence, we must have $J_3 \leq g^*(J_3)$, which implies $J_1 \leq J_3 \leq g^*(J_3)$. Hence, we conclude that Claim 1 holds.

Now, let $K_1 := J(G) \cap (J_1 \cup A_1)$. Then, by Claim 1, we obtain $g^{-1}(K_1) \subset K_1$, for each $g \in Q_1$. From Lemma 3.1-6, it follows that $J(H_1) \subset K_1$. Hence, we have shown statement 1.

Next, we show statement 2. Let $g \in Q_2$. Then, by the same method as that of the proof of Claim 1, we obtain the following.
Claim 2: For any element $J_4 \in \mathcal{J}$ with $J_4 \leq J_2$, we have $g^*(J_4) \leq J_2$.

Now, let $K_2 := J(G) \cap (\mathbb{C} \setminus A_2)$. Then, by Claim 2, we obtain $g^{-1}(K_2) \subset K_2$, for each $g \in Q_2$. From Lemma 3.1-6, it follows that $J(H_2) \subset K_2$. Hence, we have shown statement 2.

Next, we show statement 3. By statements 1 and 2, we obtain $J(H) \subset J(H_1) \cap J(H_2) \subset K_1 \cap K_2 \subset (\mathbb{C} \setminus A_2) \cap (J_1 \cup A_1) \subset J_1 \cup (A_1 \setminus A_2)$.

Hence, we have proved Proposition 2.23. □

We now demonstrate Proposition 2.24.
Proof of Proposition 2.24: Suppose that for any $h \in \Gamma$, $J(h) \cap J_{\max} = \emptyset$. Then, since $\sharp J_{\max} \geq 3$ (Theorem 2.20-5a), we get that for any $h \in \Gamma$, $h^{-1}(J_{\max}) \cap J_{\max} = \emptyset$. Combining it with Theorem 2.7-3, it follows that for any $h \in \Gamma$, $h^{-1}(J(G)) \cap J_{\max} = \emptyset$. However, since $J(G) = \bigcup_{h \in \Gamma} h^{-1}(J(G))$ (Lemma 3.1-2), it causes a contradiction. Hence, there must be an element $h_1 \in \Gamma$ such that $J(h_1) \subset J_{\max}$.

By the same method as above, we can show that there exists an element $h_2 \in \Gamma$ such that $J(h_2) \subset J_{\min}$. □

4.4 Proofs of results in 2.4

In this section, we prove the results in 2.4.

We now prove Theorem 2.25.
Proof of Theorem 2.25: Combining the assumption and Theorem 2.7-3, we get that for each $h \in \Gamma$ and each $j \in \{1, \ldots, n\}$, there exists a $k \in \{1, \ldots, n\}$ with $h^{-1}(J_j) \subset J_k$. Hence,

$$h^{-1}(\bigcup_{j=1}^{n} J_j) \subset \bigcup_{j=1}^{n} J_j, \text{ for each } h \in \Gamma. \tag{22}$$

Moreover, by Theorem 2.20-5a, we obtain

$$\sharp(\bigcup_{j=1}^{n} J_j) \geq 3. \tag{23}$$

Combining (22), (23), and Lemma 3.1-6, it follows that $J(G) \subset \bigcup_{j=1}^{n} J_j$. Hence, $J(G) = \bigcup_{j=1}^{n} J_j$. Therefore, we have proved Theorem 2.25. □

We now prove Proposition 2.26.
Proof of Proposition 2.26: Let $n \in \mathbb{N}$ with $n > 1$ and let ϵ be a number with $0 < \epsilon < \frac{1}{2}$. For each $j = 1, \ldots, n$, let $\alpha_j(z) = \frac{1}{j} z^2$ and let $\beta_j(z) = \frac{1}{j}(z - \epsilon)^2 + \epsilon$.

For any large $l \in \mathbb{N}$, there exists an open neighborhood U of $\{0, \epsilon\}$ with $U \subset \{z \mid |z| < 1\}$ and a open neighborhood V of $(\alpha_1^l, \ldots, \alpha_n^l, \beta_1^l, \ldots, \beta_n^l)$ in $(\text{Poly})^{2n}$ such that for each $(h_1, \ldots, h_{2n}) \in V$, we have $\bigcup_{j=1}^{2n} h_j(U) \subset U$ and $\bigcup_{j=1}^{m} C(h_j) \cap \mathbb{C} \subset U$, where $C(h_j)$ denotes the set of all critical points of h_j. Then, by Remark 1.3, for each $(h_1, \ldots, h_{2n}) \in V$, $\langle h_1, \ldots, h_{2n} \rangle \in \mathcal{G}$. If l is large enough and V is so small, then, for each $(h_1, \ldots, h_{2n}) \in V$, the set $I_j := J(h_j) \cup J(h_{j+n})$ is connected, for each $j = 1, \ldots, n$, and we have:

$$(h_i)^{-1}(I_j) \cap I_i \neq \emptyset, (h_{i+n})^{-1}(I_j) \cap I_i \neq \emptyset, \tag{24}$$

for each (i, j). Furthermore, for a closed annulus $A = \{z \mid \frac{1}{2} \leq |z| \leq n+1\}$, if $l \in \mathbb{N}$ is large enough and V is so small, then for each $(h_1, \ldots, h_{2n}) \in V$, $\bigcup_{j=1}^{2n} (h_j)^{-1}(A) \subset \text{int}(A)$ and $\{(h_j)^{-1}(A) \cup (h_{j+n})^{-1}(A)\}_{j=1}^{n}$ are mutually disjoint. Combining it with Lemma 3.1-6 and Lemma 3.1-2, we get that for each $(h_1, \ldots, h_{2n}) \in V$, $J(\langle h_1, \ldots, h_{2n} \rangle) \subset A$ and $\{J_j\}_{j=1}^{n}$ are mutually disjoint, where J_j denotes the connected component of $J(\langle h_1, \ldots, h_{2n} \rangle)$ containing $I_j = J(h_j) \cup J(h_{j+n})$. Combining it with (24) and Theorem 2.25, it follows that for each $(h_1,\dots,h_{2n})\in V$, the polynomial semigroup $G=\langle h_1,\dots,h_{2n}\rangle$ satisfies that $\sharp(\hat{\mathcal{J}}_G)=n$. $\square$

To prove Theorem 2.27, we need the following notation.

Definition 4.11.

1. Let X be a metric space. Let $h_j: X\to X$ $(j=1,\dots,m)$ be a continuous map. Let $G=\langle h_1,\dots,h_m\rangle$ be the semigroup generated by $\{h_j\}$. A non-empty compact subset L of X is said to be a **backward self-similar set with respect to** $\{h_1,\dots,h_m\}$ if (a) $L=\bigcup_{j=1}^m h_j^{-1}(L)$ and (b) $g^{-1}(z)\neq\emptyset$ for each $z\in L$ and $g\in G$. For example, if $G=\langle h_1,\dots,h_m\rangle$ is a finitely generated rational semigroup, then the Julia set $J(G)$ is a backward self-similar set with respect to $\{h_1,\dots,h_m\}$. (See Lemma 3.1-2.)

2. We set $\Sigma_m:=\{1,\dots,m\}^{\mathbb{N}}$. For each $x=(x_1,x_2,\dots,)\in\Sigma_m$, we set $L_x:=\bigcap_{j=1}^\infty h_{x_1}^{-1}\cdots h_{x_j}^{-1}(L)$ $(\neq\emptyset)$.

3. For a finite word $w=(w_1,\dots,w_k)\in\{1\dots,m\}^k$, we set $h_w:=h_{w_k}\circ\cdots\circ h_{w_1}$.

4. Under the notation of [18, page 110–page 115], for any $k\in\mathbb{N}$, let $\Omega_k=\Omega_k(L,\{h_1,\dots,h_m\})$ be the graph (one-dimensional simplicial complex) whose vertex set is $\{1,\dots,m\}^k$ and that satisfies that mutually different $w^1,w^2\in\{1,\dots,m\}^k$ makes a 1-simplex if and only if $\bigcap_{j=1}^2 h_{w^j}^{-1}(L)\neq\emptyset$. Let $\varphi_k:\Omega_{k+1}\to\Omega_k$ be the simplicial map defined by: $(w_1,\dots,w_{k+1})\mapsto(w_1,\dots,w_k)$ for each $(w_1,\dots,w_{k+1})\in\{1,\dots,m\}^{k+1}$. Then $\{\varphi_k:\Omega_{k+1}\to\Omega_k\}_{k\in\mathbb{N}}$ makes an inverse system of simplicial maps. Let $|\Omega_k|$ be the realization ([18]) of Ω_k. As in [18], we embed the vertex set $\{1,\dots,m\}^m$ into $|\Omega_k|$.

5. Let $\mathcal{C}(|\Omega_k)|)$ be the set of all connected components of the realization $|\Omega_k|$ of Ω_k. Let $\{(\varphi_k)_*:\mathcal{C}(|\Omega_{k+1}|)\to\mathcal{C}(|\Omega_k|)\}_{k\in\mathbb{N}}$ be the inverse system induced by $\{\varphi_k\}_k$.

Notation: We fix an $m\in\mathbb{N}$. We set $\mathcal{W}^*:=\bigcup_{k=1}^\infty\{1,\dots,m\}^k$ (disjoint union) and $\tilde{\mathcal{W}}:=\mathcal{W}^*\cup\Sigma_m$ (disjoint union). For an element $x\in\tilde{\mathcal{W}}$, we set $|x|=k$ if $x\in\{1,\dots,m\}^k$, and $|x|=\infty$ if $x\in\Sigma_m$. (This is called the word length of x.) For any $x\in\tilde{\mathcal{W}}$ and any $j\in\mathbb{N}$ with $j\leq|x|$, we set $x|j:=(x_1,\dots,x_j)\in\{1,\dots,m\}^j$. For any $x^1=(x_1^1,\dots,x_p^1)\in\mathcal{W}^*$ and any $x^2=(x_1^2,x_2^2,\dots)\in\tilde{\mathcal{W}}$, we set $x^1x^2:=(x_1^1,\dots,x_p^1,x_1^2,x_2^2,\dots)\in\tilde{\mathcal{W}}$.

To prove Theorem 2.27, we need the following lemmas.

Lemma 4.12. *Let L be a backward self-similar set with respect to $\{h_1,\dots,h_m\}$. Then, for each $k\in\mathbb{N}$, the map $|\varphi_k|:|\Omega_{k+1}|\to|\Omega_k|$ induced from $\varphi_k:\Omega_{k+1}\to\Omega_k$ is surjective. In particular, $(\varphi_k)_*:\mathcal{C}(|\Omega_{k+1}|)\to\mathcal{C}(|\Omega_k|)$ is surjective.*

Proof. Let $x^1,x^2\in\{1,\dots,m\}^k$ and suppose that $\{x^1,x^2\}$ makes a 1-simplex in Ω_k. Then $h_{x^1}^{-1}(L)\cap h_{x^2}^{-1}(L)\neq\emptyset$. Since $L=\bigcup_{j=1}^m h_j^{-1}(L)$, there exist x_{k+1}^1 and x_{k+1}^2 in $\{1,\dots,m\}$ such that $h_{x^1}^{-1}h_{x_{k+1}^1}^{-1}(L)\cap h_{x^2}^{-1}h_{x_{k+1}^2}^{-1}(L)\neq\emptyset$. Hence, $\{x^1x_{k+1}^1,x^2x_{k+1}^2\}$ makes a 1-simplex in Ω_{k+1}. Hence the lemma holds. $\square$

Lemma 4.13. *Let $m\geq 2$ and let L be a backward self-similar set with respect to $\{h_1,\dots,h_m\}$. Suppose that for each j with $j\neq 1$, $h_1^{-1}(L)\cap h_j^{-1}(L)=\emptyset$. For each k, let $C_k\in\mathcal{C}(|\Omega_k|)$ be the element containing $(1,\dots,1)\in\{1,\dots,m\}^k$. Then, we have the following.*

1. *For each $k\in\mathbb{N}$, $C_k=\{(1,\dots,1)\}$.*

2. *For each $k\in\mathbb{N}$, $\sharp(\mathcal{C}(|\Omega_k|))<\sharp(\mathcal{C}(|\Omega_{k+1}|))$.*

3. *L has infinitely many connected components.*

4. *Let $x := (1, 1, 1, \ldots) \in \Sigma_m$ and let $x' \in \Sigma_m$ be an element with $x \neq x'$. Then, for any $y \in L_x$ and $y' \in L_{x'}$, there exists no connected component A of L such that $y \in A$ and $y' \in A$.*

Proof. We show statement 1 by induction on k. We have $C_1 = \{1\}$. Suppose $C_k = \{(1, \ldots, 1)\}$. Let $w \in \{1, \ldots, m\}^{k+1} \cap C_{k+1}$ be any element. Since $(\varphi_k)_*(C_{k+1}) = C_k$, we have $\varphi_k(w) = (1, \ldots, 1) \in \{1, \ldots, m\}^k$. Hence, $w|k = (1, \ldots, 1) \in \{1, \ldots, m\}^k$. Since $h_1^{-1}(L) \cap h_j^{-1}(L) = \emptyset$ for each $j \neq 1$, we obtain $w = (1, \ldots, 1) \in \{1, \ldots, m\}^{k+1}$. Hence, the induction is completed. Therefore, we have shown statement 1.

Since both $(1, \ldots 1, 1) \in \{1, \ldots, m\}^{k+1}$ and $(1, \ldots, 1, 2) \in \{1, \ldots, m\}^{k+1}$ are mapped to $(1, \ldots, 1) \in \{1, \ldots, m\}^k$ under φ_k, by statement 1 and Lemma 4.12, we obtain statement 2. For each $k \in \mathbb{N}$, we have

$$L = \coprod_{C \in \mathcal{C}(|\Omega_k|)} \bigcup_{w \in \{1, \ldots, m\}^k \cap C} h_w^{-1}(L). \tag{25}$$

Hence, by statement 2, we conclude that L has infinitely many connected components.

We now show statement 4. Let $k_0 := \min\{l \in \mathbb{N} \mid x'_l \neq 1\}$. Then, by (25) and statement 1, we get that there exist compact sets B_1 and B_2 in L such that $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = L$, $L_x \subset (h_1^{k_0})^{-1}(L) \subset B_1$, and $L_{x'} \subset h_{x'_1}^{-1} \cdots h_{x'_{k_0}}^{-1}(L) \subset B_2$. Hence, statement 4 holds. □

We now demonstrate Theorem 2.27.

Proof of Theorem 2.27: By Theorem 2.20-1 or Remark 2.5, we have $\hat{\mathcal{J}} = \mathcal{J}$. Let $J_1 \in \hat{\mathcal{J}}$ be the element containing $J(h_m)$. By Theorem 2.1, we must have $J_0 \neq J_1$. Then, by Theorem 2.7-1, we have the following two possibilities.

Case 1. $J_0 < J_1$.

Case 2. $J_1 < J_0$.

Suppose we have case 1. Then, by Proposition 2.24, we have that $J_0 = J_{\min}$ and $J_1 = J_{\max}$. Combining it with the assumption and Theorem 2.7-3, we obtain

$$\bigcup_{j=1}^{m-1} h_j^{-1}(J_{\max}) \subset J_{\min}. \tag{26}$$

By (26) and Theorem 2.7-3, we get

$$\bigcup_{j=1}^{m-1} h_j^{-1}(J(G)) \subset J_{\min}. \tag{27}$$

Moreover, since $J(h_m) \cap J_{\min} = \emptyset$, Theorem 2.20-5b implies that

$$h_m^{-1}(J(G)) \cap J_{\min} = \emptyset. \tag{28}$$

Then, by (27) and (28), we get

$$h_m^{-1}(J(G)) \cap \left(\bigcup_{j=1}^{m-1} h_j^{-1}(J(G)) \right) = \emptyset. \tag{29}$$

We now consider the backward self-similar set $J(G)$ with respect to $\{h_1, \ldots, h_m\}$. By Lemma 3.1-2, we have

$$J(G) = \bigcup_{w \in \Sigma_m} (J(G))_w. \tag{30}$$

By Theorem 2.20-4 and Theorem 2.20-5b, we obtain $(J(G))_{m^\infty} = J(h_m)$, where $m^\infty = (m, m, \ldots) \in \Sigma_m$. Combining this with (29), Lemma 4.13, and (30), we obtain

$$J_{\max} = (J(G))_{m^\infty} = J(h_m). \tag{31}$$

Furthermore, by (29) and Lemma 4.13, we get

$$\sharp(\hat{\mathcal{J}}) \geq \aleph_0. \tag{32}$$

Let $x = (x_1, x_2, \ldots) \in \Sigma_m$ be any element with $x \neq m^\infty$ and let $l := \min\{s \in \mathbb{N} \mid x_s \neq m\}$. Then, by (27), we have

$$(J(G))_x = \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G)) \subset (h_m^{l-1})^{-1}(J_{\min}). \tag{33}$$

Combining (30) with (31) and (33), we obtain

$$J(G) = J_{\max} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} h_m^{-n}(J_{\min}). \tag{34}$$

By (32) and (34), we get $\sharp(\hat{\mathcal{J}}) = \aleph_0$. Moreover, combining (31), (34), Theorem 2.20-4 and Theorem 2.20-5b, we get that for each $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$. Hence, all statements of Theorem 2.27 are true, provided that we have case 1.

We now assume case 2: $J_1 < J_0$. Then, by Proposition 2.24, we have that $J_0 = J_{\max}$ and $J_1 = J_{\min}$. Since $J(h_j) \subset J_0$ for each $j = 1, \ldots, m-1$, and since $J_0 \neq J_{\min}$, Theorem 2.20-5b implies that for each $j = 1, \ldots, m-1$, $h_j(J(h_m)) \subset \text{int}(K(h_m))$. Hence, for each $j = 1, \ldots, m$, $h_j(K(h_m)) \subset K(h_m)$. Therefore, $\text{int}(K(h_m)) \subset F(G)$. Thus, we obtain $(J(G))_{m^\infty} = J(h_m)$. Combining this with the same method as that of case 1, we obtain

$$J_{\min} = (J(G))_{m^\infty} = J(h_m), \tag{35}$$

$$J(G) = J_{\min} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} h_m^{-n}(J_{\max}), \tag{36}$$

and

$$\sharp(\hat{\mathcal{J}}) = \aleph_0. \tag{37}$$

Moreover, by (35) and (36), we get that for each $J \in \hat{\mathcal{J}}$ with $J \neq J_{\min}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$. Hence, we have shown Theorem 2.27. □

We now demonstrate Proposition 2.28.

Proof of Proposition 2.28: Let $0 < \epsilon < \frac{1}{2}$ and let $\alpha_1(z) := z^2, \alpha_2(z) := (z-\epsilon)^2 + \epsilon$, and $\alpha_3(z) := \frac{1}{2}z^2$. If we take a large $l \in \mathbb{N}$, then there exists an open neighborhood U of $\{0, \epsilon\}$ with $U \subset \{|z| < 1\}$ and a neighborhood V of $(\alpha_1^l, \alpha_2^l, \alpha_3^l)$ in $(\text{Poly})^3$ such that for each $(h_1, h_2, h_3) \in V$, we have $\bigcup_{j=1}^3 h_j(U) \subset U$ and $\bigcup_{j=1}^3 C(h_j) \cap \mathbb{C} \subset U$, where $C(h_j)$ denotes the set of all critical points of h_j. Then, by Remark 1.3, for each $(h_1, h_2, h_3) \in V$, $\langle h_1, h_2, h_3 \rangle \in \mathcal{G}$. Moreover, if we take an l large enough and V so small, then for each $(h_1, h_2, h_3) \in V$, we have that:

1. $J(h_1) < J(h_3)$;
2. $J(h_1) \cup J(h_2)$ is connected;
3. $h_i^{-1}(J(h_3)) \cap (J(h_1) \cup J(h_2)) \neq \emptyset$, for each $i = 1, 2$;
4. $\bigcup_{j=1}^3 h_j^{-1}(A) \subset A$, where $A = \{z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq 3\}$; and
5. $h_3^{-1}(A) \cap (\bigcup_{j=1}^2 h_i^{-1}(A)) = \emptyset$.

Combining statements 4 and 5 above, Lemma 3.1-6, and Lemma 3.1-2, we get that for each $(h_1, h_2, h_3) \in V$, $J(\langle h_1, h_2, h_3\rangle) \subset A$ and $J(\langle h_1, h_2, h_3\rangle)$ is disconnected. Hence, for each $(h_1, h_2, h_3) \in V$, we have $\langle h_1, h_2, h_3\rangle \in \mathcal{G}_{dis}$. Combining it with statements 2 and 3 above and Theorem 2.27, it follows that $J(h_1) \cup J(h_2) \subset J_0$ for some $J_0 \in \hat{\mathcal{J}}_{\langle h_1,h_2,h_3\rangle}$, $h_j^{-1}(J(h_3)) \cap J_0 \neq \emptyset$ for each $j = 1, 2$, and $\sharp(\hat{\mathcal{J}}_{\langle h_1,h_2,h_3\rangle}) = \aleph_0$, for each $(h_1, h_2, h_3) \in V$. Since $J(h_1) < J(h_3)$, Theorem 2.27 implies that the connected component J_0 should be equal to $J_{\min}(\langle h_1, h_2, h_3\rangle)$, and that $J_{\max}(\langle h_1, h_2, h_3\rangle) = J(h_3)$.

Thus, we have proved Proposition 2.28. □

We now show Proposition 2.29.

Proof of Proposition 2.29: In fact, we show the following claim:

Claim: There exists a polynomial semigroup $G = \langle h_1, h_2, h_3\rangle$ in $\mathcal{G}$ such that all of the following hold.

1. $\sharp(\hat{\mathcal{J}}) = \aleph_0$.
2. $J_{\min} \supset J(h_1) \cup J(h_2)$ and there exists a superattracting fixed point z_0 of h_1 with $z_0 \in \text{int}(J_{\min})$.
3. $J_{\max} = J(h_3)$.
4. There exists a sequence $\{n_j\}_{j\in\mathbb{N}}$ of positive integers such that $\hat{\mathcal{J}} = \{J_{\min}\} \cup \{J_j \mid j \in \mathbb{N}\}$, where J_j denotes the element of $\hat{\mathcal{J}}$ with $h_3^{-n_j}(J_{\min}) \subset J_j$.
5. For any $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j\in\mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z\in C_j} d(z, J) \to 0$ as $j \to \infty$.
6. G is sub-hyperbolic: i.e., $\sharp(P(G) \cap J(G)) < \infty$ and $P(G) \cap F(G)$ is compact.

To show the claim, let $g_1(z)$ be the second iterate of $z \mapsto z^2 - 1$. Let g_2 be a polynomial such that $J(g_2) = \{z \mid |z| = 1\}$ and $g_2(-1) = -1$. Then, we have $g_1(\sqrt{-1}) = 3 \in \hat{\mathbb{C}} \setminus K(g_1)$. Take a large, positive integer m_1, and let $a := g_1^{m_1}(\sqrt{-1})$. Then,

$$J(\langle g_1^{m_1}, g_2\rangle) \subset \{z \mid |z| < a\}. \tag{38}$$

Furthermore, since $a > \frac{1}{2} + \frac{\sqrt{5}}{2}$, we have

$$\overline{(g_1^{m_1})^{-1}(\{z \mid |z| < a\})} \subset \{z \mid |z| < a\}. \tag{39}$$

Let g_3 be a polynomial such that $J(g_3) = \{z \mid |z| = a\}$. Since -1 is a superattracting fixed point of $g_1^{m_1}$ and it belongs to $J(g_2)$, by [14, Theorem 4.1], we see that for any $m \in \mathbb{N}$,

$$-1 \in \text{int}(J(\langle g_1^{m_1}, g_2^m\rangle)). \tag{40}$$

Since $J(g_2) \cap \text{int}(K(g_1^{m_1})) \neq \emptyset$ and $J(g_2) \cap F_\infty(g_1^{m_1}) \neq \emptyset$, we can take an $m_2 \in \mathbb{N}$ such that

$$(g_2^{m_2})^{-1}(\{z \mid |z| = a\}) \cap J(\langle g_1^{m_1}, g_2^{m_2}\rangle) \neq \emptyset \tag{41}$$

and

$$\overline{(g_2^{m_2})^{-1}(\{z \mid |z| < a\})} \subset \{z \mid |z| < a\}. \tag{42}$$

Take a small $r > 0$ such that

$$\text{for each } j = 1, 2, 3,\ g_j(\{z \mid |z| \leq r\}) \subset \{z \mid |z| < r\}. \tag{43}$$

Take an m_3 such that

$$(g_3^{m_3})^{-1}(\{z \mid |z| = r\}) \cap (\bigcup_{j=1}^{2} (g_j^{m_j})^{-1}(\{z \mid |z| \leq a\})) = \emptyset \tag{44}$$

and

$$g_3^{m_3}(-1) \in \{z \mid |z| < r\}. \tag{45}$$

Let $K := \{z \mid r \le |z| \le a\}$. Then, by (39), (42), (43) and (44), we have

$$(g_j^{m_j})^{-1}(K) \subset K, \text{ for } j = 1, 2, 3, \text{and } (g_3^{m_3})^{-1}(K) \cap (\bigcup_{j=1}^{2} (g_j^{m_j})^{-1}(K)) = \emptyset. \tag{46}$$

Let $h_j := g_j^{m_j}$, for each $j = 1, 2, 3$, and let $G = \langle h_1, h_2, h_3 \rangle$. Then, by (46) and Lemma 3.1-6, we obtain:

$$J(G) \subset K \text{ and } h_3^{-1}(J(G)) \cap (\bigcup_{j=1}^{2} h_j^{-1}(J(G))) = \emptyset. \tag{47}$$

Combining it with Lemma 3.1-2, it follows that $J(G)$ is disconnected. Furthermore, combining (43) and (45), we see $G \in \mathcal{G}$, $P(G) \cap J(G) = \{-1\}$, and that $P(G) \cap F(G)$ is compact. By Proposition 2.24, there exists a $j \in \{1, 2, 3\}$ with $J(h_j) \subset J_{\min}$. Since $J(G) \subset K \subset \{z \mid |z| \le a\}$ and $J(h_3) = \{z \mid |z| = a\}$, we have

$$J(h_3) \subset J_{\max}. \tag{48}$$

Hence, either $J(h_1) \subset J_{\min}$ or $J(h_2) \subset J_{\min}$. Since $J(h_1) \cup J(h_2)$ is connected, it follows that

$$J(h_1) \cup J(h_2) \subset J_{\min}. \tag{49}$$

Combining this with Theorem 2.7-3, we have $h_j^{-1}(J_{\min}) \subset J_{\min}$, for each $j = 1, 2$. Hence,

$$J(\langle h_1, h_2 \rangle) \subset J_{\min}. \tag{50}$$

Since $\sqrt{-1} \in J(h_2)$ and $h_1(\sqrt{-1}) = a \in J(h_3)$, we obtain

$$h_1^{-1}(J(h_3)) \cap J_{\min} \neq \emptyset. \tag{51}$$

Similarly, by (41) and (50), we obtain

$$h_2^{-1}(J(h_3)) \cap J_{\min} \neq \emptyset. \tag{52}$$

Combining (48), (51), (52), and Theorem 2.27, we obtain $\sharp(\hat{\mathcal{J}}) = \aleph_0$, $J_{\max} = J(h_3)$, $J(G) = J_{\max} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} h_3^{-n}(J_{\min})$, and that for any $J \in \hat{\mathcal{J}}$ with $J \neq J_{\max}$, there exists no sequence $\{C_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of $\hat{\mathcal{J}}$ such that $\min_{z \in C_j} d(z, J) \to 0$ as $j \to \infty$.

Moreover, by (40) and (50) (or by Theorem 2.22-3), the superattracting fixed point -1 of h_1 belongs to $\text{int}(J_{\min})$.

Hence, we have shown the claim.

Therefore, we have proved Proposition 2.29. $\square$

4.5 Proofs of results in 2.5

In this section, we prove the results in section 2.5.

We now demonstrate Proposition 2.33.

Proof of Proposition 2.33: Since $\Gamma \setminus \Gamma_{\min}$ is not compact, there exists a sequence $\{h_j\}_{j \in \mathbb{N}}$ in $\Gamma \setminus \Gamma_{\min}$ and an element $h_\infty \in \Gamma_{\min}$ such that $h_j \to h_\infty$ as $j \to \infty$. By Theorem 2.20-5b, for each $j \in \mathbb{N}$, $h_j(K(h_\infty))$ is included in a connected component U_j of $\text{int}(\hat{K}(G))$. Let $z_1 \in \text{int}(\hat{K}(G))$ ($\subset \text{int}(K(h_\infty))$) be a point. Then, $h_\infty(z_1) \in \text{int}(\hat{K}(G))$ and $h_j(z_1) \to h_\infty(z_1)$ as $j \to \infty$. Hence, we may assume that there exists a connected component U of $\text{int}(\hat{K}(G))$ such that for each $j \in \mathbb{N}$, $h_j(K(h_\infty)) \subset U$. Therefore, $K(h_\infty) = h_\infty(K(h_\infty)) \subset \overline{U}$. Since $\overline{U} \subset K(h_\infty)$, we obtain $K(h_\infty) = \overline{U}$. Since $U \subset \text{int}(K(h_\infty)) \subset \overline{U}$ and U is connected, it follows that $\text{int}(K(h_\infty))$ is connected. Moreover, we have $U \subset \text{int}(K(h_\infty)) \subset \text{int}(\overline{U}) \subset \text{int}(\hat{K}(G))$. Thus,

$$\text{int}(K(h_\infty)) = U. \tag{53}$$

Furthermore, since

$$J(h_\infty) < J(h_j) \text{ for each } j \in \mathbb{N}, \tag{54}$$

and $h_j \to h_\infty$ as $j \to \infty$, we obtain

$$J(h_j) \to J(h_\infty) \text{ as } j \to \infty, \tag{55}$$

with respect to the Hausdorff metric. Combining that $h_j \in \Gamma \setminus \Gamma_{\min}$ for each $j \in \mathbb{N}$ with Theorem 2.20-4, (53), (54), and (55), we see that for each $h \in \Gamma_{\min}$, $K(h) = K(h_\infty)$. Combining it with (53), (54) and (55), it follows that statement 1 in Proposition 2.33 holds. To prove statement 2, let $h \in \Gamma_{\min}$. Aplying the Riemann-Hurwitz formula to $h : \text{int}(K(h)) \to \text{int}(K(h))$, we obtain that each finite critical point of h belongs to $\text{int}(K(h))$. If h is hyperbolic, then by using quasiconformal surgery ([3]), we can see that statement 2a holds. If h is not hyperbolic, then statement 2b holds.

Thus we have proved Proposition 2.33. $\square$

To demonstrate Theorem 2.36, we need the following.

Lemma 4.14. *Let G be a polynomial semigroup generated by a non-empty compact set Γ in* $\text{Poly}_{\deg \geq 2}$. *Suppose that $G \in \mathcal{G}_{dis}$. Then, we have $\hat{K}(G_{\min,\Gamma}) = \hat{K}(G)$.*

Proof. Since $G_{\min,\Gamma} \subset G$, we have $\hat{K}(G) \subset \hat{K}(G_{\min,\Gamma})$. Moreover, it is easy to see $\hat{K}(G_{\min,\Gamma}) = \bigcap_{g \in G_{\min,\Gamma}} K(g)$. Let $g \in G_{\min,\Gamma}$ and $h \in \Gamma \setminus \Gamma_{\min}$. For each $\alpha \in \Gamma_{\min}$, we have $\alpha^{-1}(J_{\min}(G)) \subset J_{\min}(G)$. Since $\sharp(J_{\min}(G)) \geq 3$ (Theorem 2.20-5a), Lemma 3.1-6 implies that $J(g) \subset J_{\min}(G)$. Hence, from Theorem 2.20-5b, it follows that

$$h(J(g)) \subset \text{int}(\hat{K}(G)) \subset \text{int}(\hat{K}(g)). \tag{56}$$

Since $J(g)$ is connected and each connected component of $\text{int}(K(g))$ is simply connected, the above (56) implies that $h(K(g)) \subset K(g)$. Hence, we obtain $h(\hat{K}(G_{\min,\Gamma})) = h(\bigcap_{g \in G_{\min,\Gamma}} K(g)) \subset \bigcap_{g \in G_{\min,\Gamma}} K(g) = \hat{K}(G_{\min,\Gamma})$. Combined with that $\alpha(\hat{K}(G_{\min,\Gamma})) \subset \hat{K}(G_{\min,\Gamma})$ for each $\alpha \in \Gamma_{\min}$, it follows that for each $\beta \in G$, $\beta(\hat{K}(G_{\min,\Gamma})) \subset \hat{K}(G_{\min,\Gamma})$. Therefore, we obtain $\hat{K}(G_{\min,\Gamma}) \subset \hat{K}(G)$. Thus, it follows that $\hat{K}(G_{\min,\Gamma}) = \hat{K}(G)$. $\square$

Definition 4.15. Let G be a rational semigroup and N a positive integer. We denote by $SH_N(G)$ the set of points $z \in \hat{\mathbb{C}}$ satisfying that there exists a positive number δ such that for each $g \in G$, $\deg(g : V \to B(z, \delta)) \leq N$, for each connected component V of $g^{-1}(B(z, \delta))$. Moreover, we set $UH(G) := \hat{\mathbb{C}} \setminus \bigcup_{N \in \mathbb{N}} SH_N(G)$.

Lemma 4.16. *Let G be a polynomial semigroup generated by a compact subset Γ of* $\text{Poly}_{\deg \geq 2}$. *Suppose that $G \in \mathcal{G}_{dis}$ and that $\Gamma \setminus \Gamma_{\min}$ is not compact. Moreover, suppose that (a) in Proposition 2.33-2 holds. Then, there exists an open neighborhood $\mathcal{U}$ of $\Gamma_{\min}$ in Γ and an open set U in $\text{int}(\hat{K}(G))$ with $\overline{U} \subset \text{int}(\hat{K}(G))$ such that:*

1. *$\bigcup_{h \in \mathcal{U}} h(U) \subset U$;*
2. *$\bigcup_{h \in \mathcal{U}} CV^*(h) \subset U$, and*
3. *denoting by H the polynomial semigroup generated by $\mathcal{U}$, we have that $P^*(H) \subset \text{int}(\hat{K}(G)) \subset F(H)$ and that H is hyperbolic.*

Proof. Let $h_0 \in \Gamma_{\min}$ be an element. Let $\mathcal{E} := \{\psi(z) = az+b \mid a, b \in \mathbb{C}, |a| = 1, \psi(J(h_0)) = J(h_0)\}$. Then, by [2], $\mathcal{E}$ is compact in Poly. Moreover, by [2], we have the following two claims:
Claim 1: If $J(h_0)$ is a round circle with the center b_0 and radius r, then $\mathcal{E} = \{\psi(z) = a(z-b_0)+b_0 \mid |a| = r\}$.
Claim 2: If $J(h_0)$ is not a round circle, then $\sharp \mathcal{E} < \infty$.

Let z_0 be the unique attracting fixed point of h_0 in $\mathbb{C}$. Let $g \in G_{\min,\Gamma}$. By [2], for each $n \in \mathbb{N}$, there exists an $\psi_n \in \mathcal{E}$ such that $h_0^n g = \psi_n g h_0^n$. Hence, for each $n \in \mathbb{N}$, $h_0^n g(z_0) = \psi_n g h_0^n(z_0) = \psi_n g(z_0)$. Combining it with Claim 1 and Claim 2, it follows that there exists an $n \in \mathbb{N}$ such that $h_0^n(g(z_0)) = z_0$. For this n, $g(z_0) = \psi_n^{-1}(h_0^n(g(z_0))) = \psi_n^{-1}(z_0) \in \bigcup_{\psi\in\mathcal{E}} \psi(z_0)$. Combining it with Claim 1 and Claim 2 again, we see that the set $C := \overline{\bigcup_{g\in G_{\min,\Gamma}} \{g(z_0)\}}$ is a compact subset of $\mathrm{int}(\hat{K}(G))$. Let d_H be the hyperbolic distance on $\mathrm{int}(\hat{K}(G))$. Let $R > 0$ be a large number such that setting $U := \{z \in \mathrm{int}(\hat{K}(G)) \mid \min_{a\in C} d_H(z, a) < R\}$, we have $\bigcup_{h\in\Gamma_{\min}} CV^*(h) \subset U$. Then, for each $h \in \Gamma_{\min}$, $\overline{h(U)} \subset U$. Therefore, there exists an open neighborhood $\mathcal{U}$ of $\Gamma_{\min}$ in Γ such that $\bigcup_{h\in\mathcal{U}} h(U) \subset U$, and such that $\bigcup_{h\in\mathcal{U}} CV^*(h) \subset U$. Let H be the polynomial semigroup generated by $\mathcal{U}$. From the above argument, we obtain $P^*(H) = \overline{\bigcup_{g\in H} CV^*(g)} \subset \overline{\bigcup_{g\in H\cup\{Id\}} g\left(\bigcup_{h\in\mathcal{U}} CV^*(h)\right)} \subset \overline{\bigcup_{g\in H\cup\{Id\}} g(U)} \subset \overline{U} \subset \mathrm{int}(\hat{K}(G)) \subset F(H)$. Hence, H is hyperbolic. Thus, we have proved Lemma 4.16. $\square$

We now demonstrate Theorem 2.36.

Proof of Theorem 2.36: Suppose that $G_{\min,\Gamma}$ is semi-hyperbolic. We will consider the following two cases:
Case 1: $\Gamma \setminus \Gamma_{\min}$ is compact.
Case 2: $\Gamma \setminus \Gamma_{\min}$ is not compact.

Suppose that we have Case 1. Since $UH(G_{\min,\Gamma}) \subset P(G_{\min,\Gamma})$, $G_{\min,\Gamma} \in \mathcal{G}$, and $G_{\min,\Gamma}$ is semi-hyperbolic, we obtain $UH(G_{\min,\Gamma}) \cap \mathbb{C} \subset F(G_{\min,\Gamma}) \cap \hat{K}(G_{\min,\Gamma}) = \mathrm{int}(\hat{K}(G_{\min,\Gamma}))$. By Lemma 4.14, we have $\hat{K}(G_{\min,\Gamma}) = \hat{K}(G)$. Hence, we obtain

$$UH(G_{\min,\Gamma}) \cap \mathbb{C} \subset \ \mathrm{int}(\hat{K}(G)) \subset \mathbb{C} \setminus J_{\min}(G). \tag{57}$$

Therefore, there exists a positive integer N and a positive number δ such that for each $z \in J_{\min}(G)$ and each $h \in G_{\min,\Gamma}$, we have

$$\deg(h : V \to D(z, \delta)) \leq N, \tag{58}$$

for each connected component V of $h^{-1}(D(z, \delta))$. Moreover, combining Theorem 2.20-5b and Theorem 2.20-2, we obtain $\bigcup_{\alpha\in\Gamma\setminus\Gamma_{\min}} \alpha^{-1}(J_{\min}(G)) \cap P^*(G) = \emptyset$. Hence, there exists a number δ_1 such that for each $z \in \bigcup_{\alpha\in\Gamma\setminus\Gamma_{\min}} \alpha^{-1}(J_{\min}(G))$ and each $\beta \in G \cup \{Id\}$,

$$\deg(\beta : W \to D(z, \delta_1)) = 1, \tag{59}$$

for each connected component W of $\beta^{-1}(D(z, \delta_1))$. For this δ_1, there exists a number $\delta_2 > 0$ such that for each $z \in J_{\min}(G)$ and each $\alpha \in \Gamma \setminus \Gamma_{\min}$,

$$\mathrm{diam}\, B \leq \delta_1,\ \deg(\alpha : B \to D(z, \delta_2)) \leq \max\{\deg(\alpha) \mid \alpha \in \Gamma \setminus \Gamma_{\min}\} \tag{60}$$

for each connected component B of $\alpha^{-1}(D(z, \delta_2))$. Furthermore, by [27, Lemma 1.10] (or [28]), we have that there exists a constant $0 < c < 1$ such that for each $z \in J_{\min}(G)$, each $h \in G_{\min,\Gamma} \cup \{Id\}$, and each connected component V of $h^{-1}(D(z, c\delta))$,

$$\mathrm{diam}\, V \leq \delta_2. \tag{61}$$

Let $g \in G$ be any element.

Suppose that $g \in G_{\min,\Gamma}$. Then, by (58), for each $z \in J_{\min}(G)$, we have $\deg(g : V \to D(z, c\delta)) \leq N$, for each connected component V of $g^{-1}(D(z, c\delta))$.

Suppose that g is of the form $g = h \circ \alpha \circ g_0$, where $h \in G_{\min,\Gamma} \cup \{Id\}$, $\alpha \in \Gamma \setminus \Gamma_{\min}$, and $g_0 \in G \cup \{Id\}$. Then, combining (59), (60), and (61), we get that for each $z \in J_{\min}(G)$, $\deg(g : W \to D(z, c\delta)) \leq N \cdot \max\{\deg(\alpha) \mid \alpha \in \Gamma \setminus \Gamma_{\min}\}$, for each connected component W of $g^{-1}(D(z, c\delta))$.

From the above argument, we see that $J_{\min}(G) \subset SH_{N'}(G)$, where $N' := N \cdot \max\{\deg(\alpha) \mid \alpha \in \Gamma \setminus \Gamma_{\min}\}$. Moreover, by Theorem 2.20-2, we see that for any point $z \in J(G) \setminus J_{\min}(G)$, $z \in SH_1(G)$. Hence, we have shown that $J(G) \subset \hat{\mathbb{C}} \setminus UH(G)$. Therefore, G is semi-hyperbolic, provided that we have Case 1.

We now suppose that we have Case 2. Then, by Proposition 2.33, we have that for each $h \in \Gamma_{\min}$, $K(h) = \hat{K}(G)$ and $\text{int}(K(h))$ is non-empty and connected. Moreover, for each $h \in \Gamma_{\min}$, $\text{int}(K(h))$ is an immediate basin of an attracting fixed point $z_h \in \mathbb{C}$. Let $\mathcal{U}$ be the open neighborhood of $\Gamma_{\min}$ in Γ as in Lemma 4.16. Denoting by H the polynomial semigroup generated by $\mathcal{U}$, we have $P^*(H) \subset \text{int}(\hat{K}(G))$. Therefore, there exists a number $\delta > 0$ such that

$$D(J(G), \delta) \subset \mathbb{C} \setminus P(H). \tag{62}$$

Moreover, combining Theorem 2.20-5b and that $\Gamma \setminus \mathcal{U}$ is compact, we see that there exists a number $\epsilon > 0$ such that

$$\overline{\bigcup_{\alpha \in \Gamma \setminus \mathcal{U}} \alpha^{-1}(D(J_{\min}(G), \epsilon))} \subset A_0, \tag{63}$$

where A_0 denotes the unbounded component of $\mathbb{C} \setminus J_{\min}(G)$. Combining it with Theorem 2.20-2, it follows that there exists a number $\delta_1 > 0$ such that

$$D\left(\bigcup_{\alpha \in \Gamma \setminus \mathcal{U}} \alpha^{-1}(D(J_{\min}(G), \epsilon)),\ \delta_1\right) \subset \mathbb{C} \setminus P(G). \tag{64}$$

For this δ_1, there exists a number $\delta_2 > 0$ such that for each $\alpha \in \Gamma \setminus \mathcal{U}$ and each $x \in D(J_{\min}(G), \epsilon)$,

$$\text{diam } B \leq \delta_1,\ \deg(\alpha : B \to D(x, \delta_2)) \leq \max\{\deg(\beta) \mid \beta \in \Gamma \setminus \mathcal{U}\} \tag{65}$$

for each connected component B of $\alpha^{-1}(D(x, \delta_2))$. By Lemma 3.6 and (62), there exists a constant $c > 0$ such that for each $h \in H$ and each $z \in J_{\min}(G)$,

$$\text{diam } V \leq \min\{\delta_2, \epsilon\}, \tag{66}$$

for each connected component V of $h^{-1}(D(z, c\delta))$. Let $z \in J_{\min}(G)$ and $g \in G$. We will show that $z \in \mathbb{C} \setminus UH(G)$.

Suppose that $g \in H$. Then, (62) implies that for each connected component V of $g^{-1}(D(z, c\delta))$, $\deg(g : V \to D(z, c\delta)) = 1$.

Suppose that g is of the form $g = h \circ \alpha \circ g_0$, where $h \in H \cup \{Id\}, \alpha \in \Gamma \setminus \mathcal{U}, g_0 \in G \cup \{Id\}$. Let W be a connected component of $g^{-1}(D(z, c\delta))$ and let $W_1 := g_0(W)$ and $V := \alpha(W_1)$. Let z_1 be the point such that $\{z_1\} = V \cap h^{-1}(\{z\})$. If $z_1 \in \mathbb{C} \setminus D(J_{\min}(G), \epsilon)$, then, by (66) and Theorem 2.20-2, $V \subset D(z_1, \epsilon) \subset \mathbb{C} \setminus P(G)$. Hence, $\deg(\alpha \circ g_0 : W \to V) = 1$, which implies that $\deg(g : W \to D(z, c\delta)) = 1$. If $z_1 \in D(J_{\min}(G), \epsilon)$, then by (66), $V \subset D(z_1, \delta_2)$. Combining it with (64) and (65), we obtain $\deg(\alpha \circ g_0 : W \to V) = \deg(\alpha : W_1 \to V) \leq \max\{\deg(\beta) \mid \beta \in \Gamma \setminus \mathcal{U}\}$. Therefore, $\deg(g : W \to D(z, c\delta)) \leq \max\{\deg(\beta) \mid \beta \in \Gamma \setminus \mathcal{U}\}$. Thus, $J_{\min}(G) \subset \mathbb{C} \setminus UH(G)$.

Moreover, Theorem 2.20-2 implies that $J(G) \setminus J_{\min}(G) \subset \mathbb{C} \setminus P(G) \subset \mathbb{C} \setminus UH(G)$. Therefore, $J(G) \subset \mathbb{C} \setminus UH(G)$, which implies that G is semi-hyperbolic.

Thus, we have proved Theorem 2.36. □

We now demonstrate Theorem 2.37.

Proof of Theorem 2.37: We use the same argument as that in the proof of Theorem 2.36, but we modify it as follows:

1. In (57), we replace $UH(G_{\min,\Gamma}) \cap \mathbb{C}$ by $P^*(G_{\min,\Gamma})$.
2. In (58), we replace N by 1.
3. We replace (60) by the following (60)' $\operatorname{diam} B \leq \delta_1, \ \deg(\alpha : B \to D(z, \delta_2)) = 1$.
4. We replace (65) by the following (65)' $\operatorname{diam} B \leq \delta_1, \ \deg(\alpha : B \to D(x, \delta_2)) = 1$. (We take the number $\epsilon > 0$ so small.)

With these modifications, it is easy to see that G is hyperbolic.

Thus, we have proved Theorem 2.37. □

We now prove Proposition 2.39.

Proof of Proposition 2.39: Combining Lemma 4.16 and Theorems 2.36, 2.37, it is easy to see that Proposition 2.39 holds.

4.6 Proofs of results in 2.6

In this section, we prove the results in 2.6.

We now demonstrate Proposition 2.40.

Proof of Proposition 2.40: Conjugating G by $z \mapsto z + b$, we may assume that $b = 0$. For each $h \in \Gamma$, we set $a_h := a(h)$ and $d_h := \deg(h)$. Let $r > 0$ be a number such that $\overline{D(0, r)} \subset \operatorname{int}(\hat{K}(G))$.

Let $h \in \Gamma$ and let $\alpha > 0$ be a number. Since $d \geq 2$ and $(d, d_h) \neq (2, 2)$, it is easy to see that $(\frac{r}{\alpha})^{\frac{1}{d}} > 2\left(\frac{2}{|a_h|}(\frac{1}{\alpha})^{\frac{1}{d-1}}\right)^{\frac{1}{d_h}}$ if and only if

$$\log \alpha < \frac{d(d-1)d_h}{d + d_h - d_h d}(\log 2 - \frac{1}{d_h} \log \frac{|a_h|}{2} - \frac{1}{d} \log r). \tag{67}$$

We set

$$c_0 := \min_{h \in \Gamma} \exp\left(\frac{d(d-1)d_h}{d + d_h - d_h d}(\log 2 - \frac{1}{d_h} \log \frac{|a_h|}{2} - \frac{1}{d} \log r)\right) \in (0, \infty). \tag{68}$$

Let $0 < c < c_0$ be a small number and let $a \in \mathbb{C}$ be a number with $0 < |a| < c$. Let $g_a(z) = az^d$. Then, we obtain $K(g_a) = \{z \in \mathbb{C} \mid |z| \leq (\frac{1}{|a|})^{\frac{1}{d-1}}\}$ and $g_a^{-1}(\{z \in \mathbb{C} \mid |z| = r\}) = \{z \in \mathbb{C} \mid |z| = (\frac{r}{|a|})^{\frac{1}{d}}\}$. Let $D_a := \overline{D(0, 2(\frac{1}{|a|})^{\frac{1}{d-1}})}$. Since $h(z) = a_h z^{d_h}(1 + o(1))$ $(z \to \infty)$ uniformly on Γ, it follows that if c is small enough, then for any $a \in \mathbb{C}$ with $0 < |a| < c$ and for any $h \in \Gamma$, $h^{-1}(D_a) \subset \left\{z \in \mathbb{C} \mid |z| \leq 2\left(\frac{2}{|a_h|}(\frac{1}{|a|})^{\frac{1}{d-1}}\right)^{\frac{1}{d_h}}\right\}$. This implies that for each $h \in \Gamma$,

$$h^{-1}(D_a) \subset g_a^{-1}(\{z \in \mathbb{C} \mid |z| < r\}). \tag{69}$$

Moreover, if c is small enough, then for any $a \in \mathbb{C}$ with $0 < |a| < c$ and any $h \in \Gamma$,

$$\hat{K}(G) \subset g_a^{-1}(\{z \in \mathbb{C} \mid |z| < r\}), \ \overline{h(\hat{\mathbb{C}} \setminus D_a)} \subset \hat{\mathbb{C}} \setminus D_a. \tag{70}$$

Let $a \in \mathbb{C}$ with $0 < |a| < c$. By (69) and (70), there exists a compact neighborhood V of g_a in $\text{Poly}_{\deg \geq 2}$, such that

$$\hat{K}(G) \cup \bigcup_{h \in \Gamma} h^{-1}(D_a) \subset \operatorname{int}\left(\bigcap_{g \in V} g^{-1}(\{z \in \mathbb{C} \mid |z| < r\})\right), \text{ and} \tag{71}$$

$$\bigcup_{h \in \Gamma \cup V} \overline{h(\hat{\mathbb{C}} \setminus D_a)} \subset \hat{\mathbb{C}} \setminus D_a, \tag{72}$$

which implies that

$$\mathrm{int}(\hat{K}(G)) \cup (\hat{\mathbb{C}} \setminus D_a) \subset F(H_{\Gamma,V}), \tag{73}$$

where $H_{\Gamma,V}$ denotes the polynomial semigroup generated by the family $\Gamma \cup V$.

By (71), we obtain that for any non-empty subset V' of V,

$$\hat{K}(G) = \hat{K}(H_{\Gamma,V'}), \tag{74}$$

where $H_{\Gamma,V'}$ denotes the polynomial semigroup generated by the family $\Gamma \cup V'$. If the compact neighborhood V of g_a is so small, then

$$\bigcup_{g\in V} CV^*(g) \subset \mathrm{int}(\hat{K}(G)). \tag{75}$$

Since $P^*(G) \subset \hat{K}(G)$, combining it with (74) and (75), we get that for any non-empty subset V' of V, $P^*(H_{\Gamma,V'}) \subset \hat{K}(H_{\Gamma,V'})$. Therefore, for any non-empty subset V' of V, $H_{\Gamma,V'} \in \mathcal{G}$.

We now show that for any non-empty subset V' of V, $J(H_{\Gamma,V'})$ is disconnected and $(\Gamma \cup V')_{\min} \subset \Gamma$. Let

$$U := \left(\mathrm{int}(\bigcap_{g\in V} g^{-1}(\{z \in \mathbb{C} \mid |z| < r\}))\right) \setminus \bigcup_{h\in\Gamma} h^{-1}(D_a).$$

Then, for any $h \in \Gamma$,

$$h(U) \subset \hat{\mathbb{C}} \setminus D_a. \tag{76}$$

Moreover, for any $g \in V$, $g(U) \subset \mathrm{int}(\hat{K}(G))$. Combining it with (73), (76), and Lemma 3.1-2, it follows that $U \subset F(H_{\Gamma,V})$. If the neighborhood V of g_a is so small, then there exists an annulus A in U such that for any $g \in V$, A separates $J(g)$ and $\bigcup_{h\in\Gamma} h^{-1}(J(g))$. Hence, it follows that for any non-empty subset V' of V, the polynomial semigroup $H_{\Gamma,V'}$ generated by the family $\Gamma \cup V'$ satisfies that $J(H_{\Gamma,V'})$ is disconnected and $(\Gamma \cup V')_{\min} \subset \Gamma$.

We now suppose that in addition to the assumption, G is semi-hyperbolic. Let V' be any non-empty subset of V. Since $(\Gamma \cup \overline{V'})_{\min} \subset \Gamma$, Theorem 2.36 implies that the above $H_{\Gamma,V'}$ is semi-hyperbolic.

We now suppose that in addition to the assumption, G is hyperbolic. Let V' be any non-empty subset of V. By (74) and (75), we have

$$\bigcup_{g\in\Gamma\cup\overline{V'}} CV^*(g) \subset \mathrm{int}(\hat{K}(H_{\Gamma,\overline{V'}})). \tag{77}$$

Since $(\Gamma \cup \overline{V'})_{\min} \subset \Gamma$, combining it with (77) and Theorem 2.37, we obtain that $H_{\Gamma,V'}$ is hyperbolic.

Thus, we have proved Proposition 2.40. $\square$

We now demonstrate Theorem 2.43.

Proof of Theorem 2.43: First, we show 1. Let $r > 0$ be a number such that $D(b_j, 2r) \subset \mathrm{int}(K(h_1))$ for each $j = 1, \dots, m$. If we take $c > 0$ so small, then for each $(a_2, \dots, a_m) \in \mathbb{C}^{m-1}$ such that $0 < |a_j| < c$ for each $j = 2, \dots, m$, setting $h_j(z) = a_j(z - b_j)^{d_j} + b_j$ $(j = 2, \dots, m)$, we have

$$h_j(K(h_1)) \subset D(b_j, r) \subset \mathrm{int}(K(h_1)) \ (j = 2, \dots, m). \tag{78}$$

Hence, $K(h_1) = \hat{K}(G)$, where $G = \langle h_1, \dots, h_m \rangle$. Moreover, by (78), we have $P^*(G) \subset K(h_1)$. Hence, $G \in \mathcal{G}$.

If $\langle h_1 \rangle$ is semi-hyperbolic, then using the same method as that of Case 1 in the proof of Theorem 2.36, we obtain that G is semi-hyperbolic.

We now suppose that $\langle h_1 \rangle$ is hyperbolic. By (78), we have $\bigcup_{j=2}^m CV^*(h_j) \subset \mathrm{int}(\hat{K}(G))$. Combining it with the same method as that in the proof of Theorem 2.37, we obtain that G is hyperbolic. Hence, we have proved statement 1.

We now show statement 2. Suppose we have case (i). We may assume $d_m \geq 3$. Then, by statement 1, there exists an element $a > 0$ such that setting $h_j(z) = a(z - b_j)^{d_j} + b_j$ ($j = 2, \ldots, m-1$), $G_0 = \langle h_1, \ldots, h_{m-1} \rangle$ satisfies that $G_0 \in \mathcal{G}$ and $\hat{K}(G_0) = K(h_1)$ and if $\langle h_1 \rangle$ is semi-hyperbolic (resp. hyperbolic), then G_0 is semi-hyperbolic (resp. hyperbolic). Combining it with Proposition 2.40, it follows that there exists an $a_m > 0$ such that setting $h_m(z) = a_m(z - b_m)^{d_m} + b_m$, $G = \langle h_1, \ldots, h_m \rangle$ satisfies that $G \in \mathcal{G}_{dis}$ and $\hat{K}(G) = \hat{K}(G_0) = K(h_1)$ and if G_0 is semi-hyperbolic (resp. hyperbolic), then G is semi-hyperbolic (resp. hyperbolic).

Suppose now we have case (ii). Then by Proposition 2.40, there exists an $a_2 > 0$ such that setting $h_j(z) = a_2(z - b_j)^2 + b_j$ ($j = 2, \ldots, m$), $G = \langle h_1, \ldots, h_m \rangle = \langle h_1, h_2 \rangle$ satisfies that $G \in \mathcal{G}_{dis}$ and $\hat{K}(G) = K(h_1)$ and if $\langle h_1 \rangle$ is semi-hyperbolic (resp. hyperbolic), then G is semi-hyperbolic (resp. hyperbolic).

Thus, we have proved Theorem 2.43. □

We now demonstrate Theorem 2.45.

Proof of Theorem 2.45: Statements 2 and 3 follow from Theorem 2.43.

We now show statement 1. By [31, Theorem 2.4.1], $\mathcal{H}_m$ and $\mathcal{H}_m \cap \mathcal{D}_m$ are open.

We now show that $\mathcal{H}_m \cap \mathcal{B}_m$ is open. In order to do that, let $(h_1, \ldots, h_m) \in \mathcal{H}_m \cap \mathcal{B}_m$. Let $\epsilon > 0$ such that $D(P^*(\langle h_1, \ldots, h_m \rangle),\ 3\epsilon) \subset F(\langle h_1, \ldots, h_m \rangle)$. By [27, Theorem 1.35], there exists an $n \in \mathbb{N}$ such that for each $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$,

$$h_{i_n} \cdots h_{i_1}(D(P^*(\langle h_1, \ldots, h_m \rangle),\ 2\epsilon)) \subset D(P^*(\langle h_1, \ldots, h_m \rangle),\ \epsilon/2).$$

Hence, there exists a neighborhood U of $(h_1, \ldots, h_m)$ in $(\mathrm{Poly}_{\deg \geq 2})^m$ such that for each $(g_1, \ldots, g_m) \in U$ and each $(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$,

$$g_{i_n} \cdots g_{i_1}(D(P^*(\langle h_1, \ldots, h_m \rangle),\ 2\epsilon)) \subset D(P^*(\langle h_1, \ldots, h_m \rangle),\ \epsilon).$$

If U is small, then for each $(g_1, \ldots, g_m) \in U$, $\bigcup_{j=1}^m CV^*(g_j) \subset D(P^*(\langle h_1, \ldots, h_m \rangle),\ \epsilon)$. Hence, if U is small enough, then for each $(g_1, \ldots, g_m) \in U$, $P^*(\langle g_1, \ldots, g_m \rangle) \subset D(P^*(\langle h_1, \ldots, h_m \rangle), \epsilon)$. Hence, for each $(g_1, \ldots, g_m) \in U$, $\langle g_1, \ldots, g_m \rangle \in \mathcal{G}$. Therefore, $\mathcal{H}_m \cap \mathcal{B}_m$ is open.

Thus, statement 1 holds.

Thus, we have proved Theorem 2.45. □

References

[1] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Dusseldorf-Johannesburg, 1973.

[2] A. F. Beardon, *Symmetries of Julia sets*, Bull. London Math. Soc. **22** (1990), 576–582.

[3] A. F. Beardon, *Iteration of Rational Functions*, Graduate Text of Mathematics 132, Springer-Verlag, 1991.

[4] R. Brück, *Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$*, Pacific J. Math., **198** (2001), no. 2, 347–372.

[5] R. Brück, M. Büger and S. Reitz, *Random iterations of polynomials of the form $z^2 + c_n$: Connectedness of Julia sets*, Ergodic Theory Dynam. Systems, **19**, (1999), No.5, 1221–1231.

[6] M. Büger, *Self-similarity of Julia sets of the composition of polynomials*, Ergodic Theory Dynam. Systems, **17** (1997), 1289–1297.

[7] M. Büger, *On the composition of polynomials of the form $z^2 + c_n$*, Math. Ann. 310 (1998), no. 4, 661–683.

[8] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Perseus Books, 1989.

[9] K. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, 1997.

[10] J. E. Fornaess and N. Sibony, *Random iterations of rational functions*, Ergodic Theory Dynam. Systems, **11**(1991), 687–708.

[11] Z. Gong, W. Qiu and Y. Li, *Connectedness of Julia sets for a quadratic random dynamical system*, Ergodic Theory Dynam. Systems, (2003), **23**, 1807-1815.

[12] Z. Gong and F. Ren, *A random dynamical system formed by infinitely many functions,* Journal of Fudan University, **35**, 1996, 387–392.

[13] A. Hinkkanen and G. J. Martin, *The Dynamics of Semigroups of Rational Functions I*, Proc. London Math. Soc. (3)**73**(1996), 358–384.

[14] A. Hinkkanen, G. J. Martin, *Julia Sets of Rational Semigroups*, Math. Z. **222**, 1996, no.2, 161–169.

[15] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the plane,* Springer-Verlag, 1973.

[16] J. Milnor, *Dynamics in One Complex Variable* (Third Edition), Annals of Mathematical Studies, Number 160, Princeton University Press, 2006.

[17] S. B. Nadler, *Continuum Theory: An introduction*, Marcel Dekker, 1992.

[18] E. H. Spanier, *Algebraic topology*, Springer-Verlag, New York-Berlin, 1981.

[19] R. Stankewitz, *Completely invariant Julia sets of polynomial semigroups,* Proc. Amer. Math. Soc., **127**, (1999), No. 10, 2889–2898.

[20] R. Stankewitz, *Completely invariant sets of normality for rational semigroups*, Complex Variables Theory Appl., Vol. 40.(2000), 199–210.

[21] R. Stankewitz, *Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's* , Proc. Amer. Math. Soc. **128**, (2000), No. 9, 2569–2575.

[22] R. Stankewitz, T. Sugawa, and H. Sumi, *Some counterexamples in dynamics of rational semigroups*, Annales Academiae Scientiarum Fennicae Mathematica Vol. 29, 2004, 357–366.

[23] R. Stankewitz and H. Sumi, *Structure of Julia sets of polynomial semigroups with bounded finite postcritical set*, Appl. Math. Comput. 187 (2007), no. 1, 479–488. (Proceedings paper of a conference. This is not a full paper.)

[24] R. Stankewitz and H. Sumi, *Dynamical properties and structure of Julia sets of postcritically bounded polynomial semigroups*, to appear in Trans. Amer. Math. Soc., http://arxiv.org/abs/0708.3187.

[25] N. Steinmetz, *Rational Iteration*, de Gruyter Studies in Mathematics 16, Walter de Gruyter, 1993.

[26] D. Steinsaltz, *Random logistic maps and Lyapunov exponents*, Indag. Mathem., N. S., 12 (4), 557–584, 2001.

[27] H. Sumi, *Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*, Ergodic Theory Dynam. Systems, (2001), **21**, 563–603.

[28] H. Sumi, *A correction to the proof of a lemma in 'Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products'*, Ergodic Theory Dynam. Systems, (2001), **21**, 1275–1276.

[29] H. Sumi, *Skew product maps related to finitely generated rational semigroups,* Nonlinearity, **13**, (2000), 995–1019.

[30] H. Sumi, *Semi-hyperbolic fibered rational maps and rational semigroups*, Ergodic Theory Dynam. Systems, (2006), **26**, 893–922.

[31] H. Sumi, *On Dynamics of Hyperbolic Rational Semigroups,* Journal of Mathematics of Kyoto University, Vol. 37, No. 4, 1997, 717–733.

[32] H. Sumi, *Dimensions of Julia sets of expanding rational semigroups*, Kodai Mathematical Journal, Vol. 28, No. 2, 2005, pp390–422. (See also http://arxiv.org/abs/math.DS/0405522.)

[33] H. Sumi, *Random dynamics of polynomials and devil's-staircase-like functions in the complex plane*, Applied Mathematics and Computation 187 (2007) pp489-500. (Proceedings paper of a conference. This is not a full paper.)

[34] H. Sumi, *The space of postcritically bounded 2-generator polynomial semigroups with hyperbolicity*, RIMS Kokyuroku 1494, 62–86, 2006. (Proceedings paper of a conference. This is not a full paper.)

[35] H. Sumi, *Interaction cohomology of forward or backward self-similar systems*, Adv. Math., 222 (2009) no.3, 729-781.

[36] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups II: fiberwise dynamics and the Julia sets*, preprint, http://arxiv.org/abs/1007.0613.

[37] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups III: classification of semi-hyperbolic semigroups and random Julia sets which are Jordan curves but not quasicircles*, to appear in Ergodic Theory Dynam. Systems, http://arxiv.org/abs/0811.4536.

[38] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, to appear in Proc. London Math. Soc., http://arxiv.org/abs/0812.4483.

[39] H. Sumi, *Rational semigroups, random complex dynamics and singular functions on the complex plane*, survey article, to appear in Sugaku Expositions.

[40] H. Sumi, *Cooperation principle in random complex dynamics and singular functions on the complex plane*, to appear in RIMS Kokyuroku. (proceedings paper.)

[41] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, preprint 2010, http://arxiv.org/abs/1008.3995.

[42] H. Sumi, in preparation.

[43] H. Sumi and M. Urbański, *The equilibrium states for semigroups of rational maps*, Monatsh. Math., 156 (2009), No. 4, 371-390.

[44] H. Sumi and M. Urbański, *Real analyticity of Hausdorff dimension for expanding rational semigroups*, Ergodic Theory Dynam. Systems, (2010), Vol. 30, No. 2, 601-633.

[45] H. Sumi and M. Urbański, *Measures and dimensions of Julia sets of semi-hyperbolic rational semigroups*, to appear in Discrete and Continuous Dynamical Systems Ser. A, http://arxiv.org/abs/0811.1809.

[46] H. Sumi and M. Urbański, *Bowen parameter and Hausdorff dimension for expanding rational semigroups*, preprint 2009, http://arxiv.org/abs/0911.3727.

[47] Y. Sun and C-C. Yang. *On the connectivity of the Julia set of a finitely generated rational semigroup*, Proc. Amer. Math. Soc., Vol. 130, No. 1, 49–52, 2001.

[48] W. Zhou, F. Ren, The Julia sets of the random iteration of rational functions, Chinese Sci. Bull., **37**(12), 1992, 969–971.

The author's E-mail address: sumi@math.sci.osaka-u.ac.jp