

Non-i.i.d. random holomorphic dynamical systems and the generic dichotomy *

Hiroki Sumi[†]

E-mail: sumi@math.h.kyoto-u.ac.jp

<http://www.math.h.kyoto-u.ac.jp/~sumi/index.html>

Takayuki Watanabe[‡]

E-mail: watanabe.takayuki.3m@kyoto-u.ac.jp

January 24, 2022

Abstract

We consider non-i.i.d. random holomorphic dynamical systems whose choice of maps depends on Markovian rules. We show that generically, such a system is mean stable or chaotic with full Julia set. If a system is mean stable, then the Lyapunov exponent is uniformly negative for every initial value and almost every random orbit. Moreover, we consider families of random holomorphic dynamical systems and show that the set of mean stable systems has full measure under certain conditions. The latter is a new result even for i.i.d. random dynamical systems.

1 Introduction

1.1 Background

We consider random dynamical systems (RDSs) of rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. The study of RDS is rapidly growing. The previous works find many new phenomena which cannot happen in deterministic dynamics, which are called noise-induced phenomena or randomness-induced phenomena. For example, chaotic dynamics can be more chaotic if one adds noise, and more surprisingly, chaotic dynamics can be more stable because of noise. The latter phenomena are called noise-induced order. For details on randomness-induced phenomena, the reader is referred to the authors' previous paper [20] and references therein, say [4, 7, 8, 12, 18, 19, 21].

*MSC: 37F10, 37H10. To appear in *Nonlinearity*.

[†]Course of Mathematical Science, Department of Human Coexistence, Graduate School of Human and Environmental Studies, Kyoto University. Yoshida Nihonmatsu-cho, Sakyo-ku, Kyoto, 606-8501, Japan

[‡]Author to whom any correspondence should be addressed. Department of Mathematics, Faculty of Science, Kyoto University. Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

However, most of the previous studies concerned i.i.d. random dynamical systems. It is very natural to generalize the settings and consider non-i.i.d. random dynamical systems. In this paper, we especially treat random dynamical systems with “Markovian rules” whose randomness depends on the past.

Our studies may be applied to the skew products whose base dynamical systems have Markov partitions. We believe that this research will contribute not only toward mathematics but also toward applications to the real world. One motivation for studying dynamical systems is to analyze mathematical models used in the natural or social sciences. Since the environment changes randomly, it is natural to investigate random dynamical systems which describe the time evolution of systems with probabilistic terms. In this sense, it is very important to understand “Markovian” noise since there are a lot of systems whose noise depends on the past.

The authors found a noise-induced phenomenon which can happen in Markov RDSs but cannot happen in i.i.d. RDSs, see [20, Main Result 6]. This exhibits the difference between non-i.i.d. and i.i.d. RDSs and motivates us to study non-i.i.d. RDSs.

In this paper, we show some results regarding noise-induced order which greatly deepen the results in [20]. RDS with Markovian noise is the theme of this paper. In [20], the authors introduced Markov random dynamical systems quite generally but in this paper, we are concerned with rational maps and define such systems as follows. The properties of holomorphic functions allow us to control minimal sets and to study global (random) dynamics. For example, we use Montel’s Theorem and hyperbolic metric to show our results. Let Rat_+ be the space of all rational maps of degree two or more from $\widehat{\mathbb{C}}$ to itself endowed with the metric $\kappa(f, g) := \sup_{z \in \widehat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\widehat{\mathbb{C}}$.

Definition 1.1. Let $m \in \mathbb{N}$. Suppose that m^2 regular Borel measures $(\tau_{ij})_{i,j=1,\dots,m}$ on Rat_+ satisfy $\sum_{j=1}^m \tau_{ij}(\text{Rat}_+) = 1$ for all $i = 1, \dots, m$. We call $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ a *Markov random dynamical system* (MRDS for short). We say that τ is *compactly generated* if $\text{supp } \tau_{ij}$ is compact for each $i, j = 1, \dots, m$.

For a given MRDS $\tau = (\tau_{ij})_{i,j=1,\dots,m}$, we consider the Markov chain on $\widehat{\mathbb{C}} \times \{1, \dots, m\}$ whose transition probability from $(z, i) \in \widehat{\mathbb{C}} \times \{1, \dots, m\}$ to $B \times \{j\}$ is defined by

$$\tau_{ij}(\{f \in \text{Rat}_+; f(z) \in B\}),$$

where B is a Borel subset of $\widehat{\mathbb{C}}$ and $j \in \{1, \dots, m\}$. This (time-homogeneous) transition function defines the one-point motion on $\widehat{\mathbb{C}} \times \{1, \dots, m\}$. We can construct skew-product maps also as a representation of Markov RDSs, see Definition 2.29 of [20] and 2.1.6 Theorem (RDS Corresponding to Markov Chain) of Arnold’s book [1]. For general relation between Markov chain and random mappings, see pp. 53–55 of [1].

The Markov chain induced by $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ describes the following random dynamical system on the phase space $\widehat{\mathbb{C}}$. Fix an initial point $z_0 \in \widehat{\mathbb{C}}$ and choose a vertex $i = 1, \dots, m$ (with some probability if we like). We choose a vertex $i_1 = 1, \dots, m$ with probability $\tau_{ii_1}(\text{Rat}_+) > 0$ and choose a map f_1 according to the probability distribution

$\tau_{ii_1}/\tau_{ii_1}(\text{Rat}_+)$. Repeating this, we randomly choose a vertex i_n and a map f_n for each n -th step. In this paper, we investigate the behavior of random orbits of the form $f_n \circ \cdots \circ f_2 \circ f_1(z_0)$. For the general theory of RDSs, see Arnold's book [1].

By extending the phase space from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}} \times \{1, \dots, m\}$, we can represent MRDSs as naive Markov chains. This simple representation enables us to analyze MRDSs intuitively.

Note that our definition is a generalization of i.i.d. RDS and deterministic dynamics. If $m = 1$ then our definition coincides with i.i.d. RDS on $\widehat{\mathbb{C}}$ induced by $\tau = \tau_{11}$. Besides, if τ_{11} is the Dirac measure δ_f at $f \in \text{Rat}_+$, then our definition treats dynamics of iteration of f essentially.

1.2 Definitions

For an MRDS τ , we define the following set-valued dynamics. We present our results in the next subsection 1.3.

Definition 1.2. Let $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ be an MRDS. We consider the directed graph (V, E) in the following way. We define the vertex set as $V := \{1, 2, \dots, m\}$ and the edge set as

$$E := \{(i, j) \in V \times V; \tau_{ij}(\text{Rat}_+) > 0\}.$$

Define $i : E \rightarrow V$ (resp. $t : E \rightarrow V$) as the projection to the first (resp. second) coordinate and we call $i(e)$ (resp. $t(e)$) the initial (resp. terminal) vertex of $e \in E$. We call (V, E) the associated directed graph of τ . Also, for each $e = (i, j) \in E$, we define $\Gamma_e := \text{supp } \tau_{ij}$. Set $S_\tau := (V, E, (\Gamma_e)_{e \in E})$, which we call the *graph directed Markov system* (GDMS for short) induced by τ . We say that τ is *irreducible* if the associated directed graph (V, E) is (strongly) connected.

Although one may think that our concept is similar to that of [11], ours is completely different from [11]. Mauldin and Urbański are concerned with the limit sets of systems of contracting maps, but in this paper, we discuss the dynamics and the Julia sets of GDMS consisting of rational maps which may have expanding property somewhere in the phase space.

We denote by Poly the set of all polynomial maps of degree two or more. We work on subfamilies of Rat_+ which satisfy the following non-degenerate condition.

Definition 1.3. We say that a non-empty subset X of Rat_+ is non-degenerate if there exist an open subset A of Rat_+ and a closed subset B of Rat_+ such that $X = A \cap B$ and at least one of the following (i) and (ii) holds.

- (i) For each $(f_0, z_0) \in X \times \widehat{\mathbb{C}}$, there exists a holomorphic family $\{g_\theta\}_{\theta \in \Theta}$ of rational maps parametrized by a finite dimensional complex manifold Θ (i.e., $g_\theta \in \text{Rat}_+$ for each $\theta \in \Theta$ and $(z, \theta) \mapsto g_\theta(z)$ is a holomorphic map from $\widehat{\mathbb{C}} \times \Theta$ to $\widehat{\mathbb{C}}$) with $\{g_\theta; \theta \in \Theta\} \subset X$ such that $g_{\theta_0} = f_0$ for some $\theta_0 \in \Theta$ and $\theta \mapsto g_\theta(z_0)$ is non-constant in any neighborhood of θ_0 .

- (ii) $X \subset \text{Poly}$ and for each $(f_0, z_0) \in X \times \mathbb{C}$, there exists a holomorphic family $\{g_\theta\}_{\theta \in \Theta}$ of rational maps parametrized by a complex manifold Θ with $\{g_\theta; \theta \in \Theta\} \subset X$ such that $g_{\theta_0} = f_0$ for some $\theta_0 \in \Theta$ and $\theta \mapsto g_\theta(z_0)$ is non-constant in any neighborhood of θ_0 .

Definition 1.4. Let $X \subset \text{Rat}_+$. Define $\text{MRDS}(X)$ as the space of all irreducible Markov random dynamical systems τ such that the topological support $\text{supp } \tau_e$ is compact and contained in X for each $e \in E$, where E is the set of directed edges of the associated directed graph of τ .

We endow $\text{MRDS}(X)$ with the following topology. A sequence $\{\tau^n\}_{n \in \mathbb{N}}$ in $\text{MRDS}(X)$ converges to $\tau \in \text{MRDS}(X)$ if and only if

- (i) the associated directed graph of τ^n is equal to (V, E) for sufficiently large n , where (V, E) denotes the associated directed graph of τ ,
- (ii) the sequence of compact sets $\{\text{supp } \tau_e^n\}_{n \in \mathbb{N}}$ converges to $\text{supp } \tau_e$ with respect to the Hausdorff metric for each directed edge $e \in E$, and
- (iii) the sequence of measures $\{\tau_e^n\}_{n \in \mathbb{N}}$ converges to τ_e for each $e \in E$ in the weak*-topology.

Definition 1.5. Let $S_\tau = (V, E, (\Gamma_e)_{e \in E})$ be the GDMS induced by an MRDS τ .

- (i) A word $e = (e_1, e_2, \dots, e_N) \in E^N$ with length $N \in \mathbb{N}$ is said to be *admissible* if $t(e_n) = i(e_{n+1})$ for all $n = 1, 2, \dots, N-1$. For this word e , we call $i(e_1)$ (resp. $t(e_N)$) the initial (resp. terminal) vertex of e and we denote it by $i(e)$ (resp. $t(e)$).
- (ii) We set

$$\begin{aligned} H(S_\tau) &:= \{f_N \circ \dots \circ f_2 \circ f_1; \\ &\quad N \in \mathbb{N}, f_n \in \Gamma_{e_n}, t(e_n) = i(e_{n+1}) (\forall n = 1, \dots, N-1)\}, \\ H_i(S_\tau) &:= \{f_N \circ \dots \circ f_2 \circ f_1 \in H(S_\tau); \\ &\quad N \in \mathbb{N}, f_n \in \Gamma_{e_n}, t(e_n) = i(e_{n+1}) (\forall n = 1, \dots, N-1), i = i(e_1)\}, \\ H_i^j(S_\tau) &:= \{f_N \circ \dots \circ f_2 \circ f_1 \in H_i(S_\tau); \\ &\quad N \in \mathbb{N}, f_n \in \Gamma_{e_n}, t(e_n) = i(e_{n+1}) (\forall n = 1, \dots, N-1), i = i(e_1), t(e_N) = j\}. \end{aligned}$$

- (iii) For each $i \in V$, we denote by $F_i(S_\tau)$ the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood U of z in $\widehat{\mathbb{C}}$ such that the family $\bigcup_{j \in V} H_i^j(S_\tau)$ of maps on $\widehat{\mathbb{C}}$ is equicontinuous on U . The set $F_i(S_\tau)$ is called the *Fatou set* of S_τ at the vertex i , and the complement $J_i(S_\tau) := \widehat{\mathbb{C}} \setminus F_i(S_\tau)$ is called the *Julia set* of S_τ at the vertex i .

- (iv) Set $\mathbb{F}(S_\tau) := \bigcup_{i \in V} F_i(S_\tau) \times \{i\}$ and $\mathbb{J}(S_\tau) := \bigcup_{i \in V} J_i(S_\tau) \times \{i\}$.

We refer the readers to [20] for examples of Julia sets of GDMSs and MRDSs.

1.3 Main Results

In this subsection, we present the main results (Main Results A-D) of this paper. We first consider mean stable systems.

Definition 1.6. Let $\tau \in \text{MRDS}(\text{Rat}_+)$. We say that τ is *mean stable* if the associated GDMS S_τ satisfies the following. There exist $N \in \mathbb{N}$ and two families of non-empty open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ such that

- (I) $U_i \subset \overline{U_i} \subset W_i \subset \overline{W_i} \subset F_i(S_\tau)$ for each $i \in V$,
- (II) for each admissible word $e = (e_1, \dots, e_N)$ with length N and each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), we have $f_N \circ \dots \circ f_1(\overline{W_{i(e)}}) \subset U_{t(e)}$, and
- (III) for each $z \in \widehat{\mathbb{C}}$ and $i \in V$, there exist $j \in V$ and $h \in H_i^j(S_\tau)$ such that $h(z) \in W_j$.

Fornæss and Sibony proved in [7] that small “random perturbations” of iteration of a single map give examples of mean stable systems, although they do not use the concept of mean stability. In this paper, we show that there exist a lot of mean stable systems in the space of MRDS. See also Example 3.2.

We show that if τ is mean stable, then for every initial value, the sample-wise dynamics is contractive and the Lyapunov exponent is negative almost surely with respect to the probability measure $\tilde{\tau}$ on $(\text{Rat}_+ \times E)^\mathbb{N}$ (Main Results A and B). Here, $\tilde{\tau}$ is the measure naturally associated with the Markov chain on $\widehat{\mathbb{C}} \times \{1, \dots, m\}$ induced by τ . See Definition 3.10 or [20, Lemma 3.4].

Remark 1.7. If an MRDS τ is mean stable, then the kernel Julia set is empty. In other words, for every $i \in V$ and for every $z \in \widehat{\mathbb{C}}$, there exist $j \in V$ and $h \in H_i^j(S_\tau)$ such that $h(z) \notin J_j(S_\tau)$, see [20]. This property implies some interesting results. For instance, almost every random Julia set has Lebesgue measure 0. That is, there exists a Borel set \mathfrak{G} with $\tilde{\tau}(\mathfrak{G}) = 1$ such that for every $(f_n, e_n)_{n=1}^\infty \in \mathfrak{G}$, the Lebesgue measure of the complement of the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood U of z in $\widehat{\mathbb{C}}$ such that the family $\{f_n \circ \dots \circ f_1\}_{n=1}^\infty$ is equicontinuous on U is 0. See [20, Proposition 3.11].

Additionally, for each attractor (attracting minimal set) A , the probability of random orbits tending to A depends continuously on initial points. See [20, Proposition 4.24] and [17]. For the definition of attracting minimal set, see Definition 2.7.

Main Result A (Theorem 3.11). Let $\tau \in \text{MRDS}(\text{Rat}_+)$ be a mean stable system. Then we have all of the following.

- (i) There exists a constant $c \in (0, 1)$ satisfying that for each $z \in \widehat{\mathbb{C}}$, there exists a Borel subset \mathfrak{F} of $(\text{Rat}_+ \times E)^\mathbb{N}$ with $\tilde{\tau}(\mathfrak{F}) = 1$ such that for every $(f_n, e_n)_{n=1}^\infty \in \mathfrak{F}$, there exist $r = r(z, (f_n, e_n)_{n=1}^\infty) > 0$ and $K = K(z, (f_n, e_n)_{n=1}^\infty) > 0$ such that

$$\text{diam } f_n \circ \dots \circ f_1(B(z, r)) \leq Kc^n$$

for every $n \in \mathbb{N}$. Here, we set $\text{diam } A = \inf_{x, y \in A} d(x, y)$ for any $A \subset \widehat{\mathbb{C}}$.

- (ii) For each $z \in \widehat{\mathbb{C}}$, there exists a Borel subset \mathfrak{F} of $(\text{Rat}_+ \times E)^\mathbb{N}$ with $\tilde{\tau}(\mathfrak{F}) = 1$ such that for every $(f_n, e_n)_{n=1}^\infty \in \mathfrak{F}$, there exists an attracting minimal set $(L_i)_{i \in V}$ such that $d(f_n \circ \cdots \circ f_1(z), L_{t(e_n)}) \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) For every $i \in V$, the kernel Julia set $J_{\text{ker}, i}(S_\tau) := \bigcap_{j \in V} \bigcap_{h \in H_i^j(S)} h^{-1}(J_j(S))$ at i is empty.
- (iv) There exists a Borel subset \mathfrak{G} of $(\text{Rat}_+ \times E)^\mathbb{N}$ with $\tilde{\tau}(\mathfrak{G}) = 1$ such that for every $\xi = (f_n, e_n)_{n=1}^\infty \in \mathfrak{G}$, the (2-dimensional) Lebesgue measure of the Julia set J_ξ is zero. Here, the Julia set J_ξ of ξ is the complement of the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood U of z in $\widehat{\mathbb{C}}$ such that the family $\{f_n \circ \cdots \circ f_1\}_{n=1}^\infty$ is equicontinuous on U .
- (v) There exist at most finitely many minimal sets of S_τ . Moreover, each minimal set of S_τ is attracting. Also, for every minimal set $\mathbb{L} = (L_i)_{i \in V}$ of S_τ , we define the function $\mathbb{T}_\mathbb{L}: \widehat{\mathbb{C}} \times V \rightarrow [0, 1]$ by

$$\mathbb{T}_\mathbb{L}(z, i) := \tilde{\tau}_i(\{\xi = (f_n, e_n)_{n \in \mathbb{N}}; d(f_n \circ \cdots \circ f_1(z), L_{t(e_n)}) \rightarrow 0 (n \rightarrow \infty)\})$$

for every point $(z, i) \in \widehat{\mathbb{C}} \times V$. Then $\mathbb{T}_\mathbb{L}$ is continuous on $\widehat{\mathbb{C}} \times V$. Also, averaged function $T_\mathbb{L}(z) := \sum_{i \in V} p_i \mathbb{T}_\mathbb{L}(z, i)$ is continuous on $\widehat{\mathbb{C}}$, where (p_1, \dots, p_m) is the probability vector in Definition 3.10.

- (vi) Suppose τ has exactly ℓ minimal sets $\mathbb{L}_1, \mathbb{L}_2, \dots, \mathbb{L}_\ell$. Then

$$T_{\mathbb{L}_1}(z) + T_{\mathbb{L}_2}(z) + \cdots + T_{\mathbb{L}_\ell}(z) = 1$$

for every $z \in \widehat{\mathbb{C}}$.

Main Result B (Theorem 3.12). Let $\tau \in \text{MRDS}(\text{Rat}_+)$ be a mean stable system. Then there exists $\alpha < 0$ such that the following holds. For each $z \in \widehat{\mathbb{C}}$, there exists a Borel set \mathfrak{F} with $\tilde{\tau}(\mathfrak{F}) = 1$ such that for every $(f_n, e_n)_{n=1}^\infty \in \mathfrak{F}$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|D(f_n \circ \cdots \circ f_1)(z)\| \leq \alpha.$$

Here, $Dg(z)$ denotes the complex differential of a holomorphic map g at z and $\|\cdot\|$ denotes the norm with respect to the spherical metric.

These phenomena cannot happen in deterministic dynamical systems of a single map $f \in \text{Rat}_+$, since, in deterministic dynamical system of a rational map $f \in \text{Rat}_+$, it is well known that the following holds for the Julia set $J(f)$ of f . For every neighborhood O of a point of the Julia set $J(f)$, there exists $N \in \mathbb{N}$ such that $f^{\circ n}(O) \supset J(f)$ for every $n \geq N$. Here, $f^{\circ n}$ denotes the n -th iterate of f . Note that $\text{diam } J(f) > 0$. We refer the readers to Milnor's book [14] or Carleson and Gamelin's book [6] for details. Besides, it was shown by Mañé [10] that the set of all points $z \in \widehat{\mathbb{C}}$ with $\liminf_{n \rightarrow \infty} n^{-1} \log \|Df^{\circ n}(z)\| > 0$ has

positive Hausdorff dimension. In this sense, Main Result A and Main Result B describe noise-induced order.

We next consider how many mean stable systems exist. For our purpose, we give the following definition.

Definition 1.8. Let $X \subset \text{Rat}_+$. Define $\mathcal{A}(X)$ as the set of all $\tau \in \text{MRDS}(X)$ which are mean stable. Further, define $\mathcal{C}(X)$ as the set of all $\tau \in \text{MRDS}(X)$ which satisfy $\mathbb{J}(S_\tau) = \widehat{\mathbb{C}} \times V$ and $\overline{\bigcup_{h \in H_i^j(S_\tau)} \{h(z)\}} = \widehat{\mathbb{C}}$ for each $i, j \in V$ and $z \in \widehat{\mathbb{C}}$.

Note that for each $\tau \in \mathcal{C}(X)$, the set-valued dynamics of S_τ is topologically chaotic, which also describes a noise-induced phenomenon. See Example 3.21. We present the following results regarding the thickness of \mathcal{A} and \mathcal{C} .

Main Result C (Corollary 3.15 and Theorem 3.22). Let $X \subset \text{Rat}_+$. Then the set $\mathcal{A}(X)$ is open in $\text{MRDS}(X)$. Moreover, if X is non-degenerate, then the disjoint union $\mathcal{A}(X) \cup \mathcal{C}(X)$ is dense in the space $\text{MRDS}(X)$.

As a corollary, we have the following result regarding the polynomial dynamics.

Corollary 1.9 (See Corollary 3.18). Let X be a non-degenerate subset of Poly . Then the set $\mathcal{A}(X)$ is open and dense in $\text{MRDS}(X)$. In particular, the set $\mathcal{A}(\text{Poly})$ is open and dense in $\text{MRDS}(\text{Poly})$.

Corollary 1.9 is related to the famous conjecture of Hyperbolic Density (HD conjecture). See McMullen's book [13] for details. The dynamics of iteration of a hyperbolic rational map f has the following properties, which is somewhat similar to the dynamics of a mean stable system. A hyperbolic rational map f is expanding on the Julia set whose area is zero, and every orbit is contracted to an attracting cycle with negative Lyapunov exponent on the Fatou set. The set of all hyperbolic rational maps is conjectured to be open and dense in the space Rat_+ . We solved the RDS version of the HD conjecture in some sense.

Main Result C is also related to the dichotomy result for the real quadratic family by Lyubich [9], which says “almost every real quadratic map is either regular or stochastic”. Here, regular means hyperbolic and stochastic means that the dynamics has an absolutely continuous invariant probability measure. The latter seems to be similar to the element τ of \mathcal{C} in our context although we do not show that it has an absolutely continuous invariant measure. For real analytic families of unimodal maps, see also the papers [2] by Avila, Lyubich and de Melo and [3] by Avila and Moreira. Compared to these results, we need to control randomness which is possibly large. This large noise causes difficulty for our analysis.

Last but not least, we consider families of MRDSs. We show that for such a family, the set of mean stable systems has full measure under certain conditions.

Main Result D (Theorem 4.6). Suppose $X \subset \text{Rat}_+$ is non-degenerate. Let Λ be a topological space and let m be a σ -finite Borel measure on Λ . Let $I = [a, b)$ be an interval on the real line \mathbb{R} , possibly $I = [a, \infty)$. Suppose $\Phi: \Lambda \times I \rightarrow \text{MRDS}(X)$ satisfies the following three conditions. Denote $\Phi(\lambda, s) = \tau^{\lambda, s}$.

- (i) Φ is continuous and the associated directed graphs (V, E) of $\tau^{\lambda, s}$ are identical for all $(\lambda, s) \in \Lambda \times I$.
- (ii) $\text{supp } \tau_e^{\lambda, s_1} \subset \text{int}(\text{supp } \tau_e^{\lambda, s_2})$ for each $e \in E$, $\lambda \in \Lambda$ and $s_1 < s_2$, where int denotes the set of all interior points with respect to the topological space X .
- (iii) $\tau^{\lambda, s}$ has at least one attracting minimal set for each $(\lambda, s) \in \Lambda \times I$.

For each $s \in I$, we denote by Bif_s the set of all $\lambda \in \Lambda$ satisfying that $\tau^{\lambda, s}$ are not mean stable. Also, for each $\lambda \in \Lambda$, we denote by Bif^λ the set of all $s \in I$ such that $\tau^{\lambda, s}$ are not mean stable. Suppose that there exists $\alpha \in \mathbb{N}$ such that $\#\text{Bif}^\lambda \leq \alpha$ for each $\lambda \in \Lambda$.

Then $m(\text{Bif}_s) = 0$ for all but countably many $s \in I$.

We remark the following corollary holds.

Corollary 1.10 (Corollary 4.7). Let X, Λ, m, I, Φ as in Setting 4.1. Suppose that there exists $d \in \mathbb{N}$ with $d \geq 2$ such that $2 \leq \deg(g) \leq d$ for each $g \in X$. Then there exists $\alpha \in \mathbb{N}$ such that $\#\text{Bif}^\lambda \leq \alpha$ for each $\lambda \in \Lambda$. Hence, $m(\text{Bif}_s) = 0$ for all but countably many $s \in I$.

It is interesting that our result can be applied to the quadratic family $f_c(z) = z^2 + c$, see Example 4.2. On the dynamics of iteration of a single map, Shishikura [16] showed that the bifurcation locus of the quadratic family, namely the boundary of the Mandelbrot set, has Hausdorff dimension 2. However, it is still open whether or not the boundary of the Mandelbrot set has positive area, see [15]. We solved the RDS version of this problem in a general form.

Note that Main Result D is a new result even for i.i.d. systems. If $m = \#V = 1$, then Main Result A, Main Result B and Main Result C coincide with the results for i.i.d. systems, which were shown in [18]. Also, our definition of mean stability coincides with the definition for i.i.d. systems if $m = 1$. Our results and concepts are new for the case $m \neq 1$.

The authors believe that we can work on ergodic properties of mean stable systems in the future. For mean stable i.i.d. systems, the first author proved in [17] that there exist finitely many invariant probability measures (or cycles of probability measures) supported on minimal sets. Moreover, he showed the spectral decomposition for the (dual of) transition operators, the spectral gaps for the transition operators and other measure-theoretic results. The authors believe that we can generalize these results to non-i.i.d. settings.

1.4 Structure of the paper

In Section 2, we define minimal sets of MRDS and give the classification of them. More precisely, a minimal set is one of the three types; it intersects the Julia set, it intersects a rotation domain, or it is *attracting* as defined in Section 2. This is the key to our work. In Section 3, we show the fundamental properties of mean stable systems. In particular,

we explain the relation between mean stable systems and attracting minimal sets. By using these results, we prove Main Results A, B and C. In Section 4, we consider families of MRDS and investigate their bifurcations. Furthermore, we show Main Result D.

Acknowledgment

The authors would like to thank Rich Stankewitz for his valuable comments. They are grateful to the referees for many detailed and constructive comments. The first author is partially supported by JSPS Grant-in-Aid for Scientific Research (B) Grant Number JP 19H01790. The second author is partially supported by JSPS Grant-in-Aid for JSPS Fellows Grant Number JP 19J11045.

2 Classification of minimal sets of Markov RDS

In this section, we consider general graph directed Markov systems consisting of rational maps before analyzing Markov random dynamical systems. In [20], the authors defined more general GDMS regarding continuous self-maps, but we are concerned with rational maps of degree two or more in this paper.

Definition 2.1. Let (V, E) be a directed graph with finite vertices and finite edges, and let Γ_e be a non-empty subset of Rat_+ indexed by a directed edge $e \in E$. We call $S = (V, E, (\Gamma_e)_{e \in E})$ a *graph directed Markov system* (GDMS for short). The symbol $i(e)$ (resp. $t(e)$) denotes the initial (resp. terminal) vertex of each directed edge $e \in E$. We say that $S = (V, E, (\Gamma_e)_{e \in E})$ is *compactly generated* if Γ_e is compact for each $e \in E$.

Definition 2.2. We say that a GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ is *irreducible* if the directed graph (V, E) is strongly connected. We say that MRDS $\tau = (\tau_{ij})_{i,j=1,\dots,m}$ is *irreducible* if the induced GDMS S_τ is irreducible.

In this paper, we usually assume that all GDMSs S and MRDSs τ are irreducible. For each GDMS $S = (V, E, (\Gamma_e)_{e \in E})$, we define admissible words, $H_i^j(S)$, $F_i(S)$, $J_i(S)$, $\mathbb{F}(S)$ and $\mathbb{J}(S)$ similarly as in Definition 1.5 replacing S_τ by S . For instance, the Fatou set $F_i(S)$ at $i \in V$ is the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood U of z in $\widehat{\mathbb{C}}$ such that the family $\bigcup_{j \in V} H_i^j(S)$ of maps on $\widehat{\mathbb{C}}$ is equicontinuous on U , and the complement $J_i(S) := \widehat{\mathbb{C}} \setminus F_i(S)$ is the Julia set of S at the vertex i .

If a GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ is irreducible, then the Julia set $J_i(S)$ contains uncountably many points, see [20, Lemma 4.8]. In particular, the Fatou set $F_i(S)$ admits the hyperbolic metric for each $i \in V$.

In order to analyze dynamical systems, it is useful to investigate minimal sets. For our purpose, we define minimal sets of GDMSs as follows.

Notation 2.3. For a family $\Omega \subset \text{Rat}_+$ and a set $Y \subset \widehat{\mathbb{C}}$, we set

$$\Omega(Y) := \bigcup_{f \in \Omega} f(Y), \quad \Omega^{-1}(Y) := \bigcup_{f \in \Omega} f^{-1}(Y).$$

If $\Omega = \emptyset$, then we set $\Omega(Y) := \emptyset$, $\Omega^{-1}(Y) := \emptyset$.

Definition 2.4. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS and let K_i and L_i be subsets of $\widehat{\mathbb{C}}$ for each $i \in V$. We consider the families $(K_i)_{i \in V}$ and $(L_i)_{i \in V}$ indexed by $i \in V$.

- (i) We say that $(L_i)_{i \in V}$ is *forward S -invariant* if $\Gamma_e(L_{i(e)}) \subset L_{t(e)}$ for all $e \in E$.
- (ii) We write $(K_i)_{i \in V} \subset (L_i)_{i \in V}$ if $K_i \subset L_i$ for each $i \in V$.
- (iii) We say that $(K_i)_{i \in V}$ is a *minimal* set of S if it is a minimal element of the set of all $(L_i)_{i \in V}$ satisfying that L_i is non-empty and compact for each $i \in V$ and $(L_i)_{i \in V}$ is forward S -invariant, with respect to the order \subset .

Also, for any irreducible MRDS τ , any minimal set of S_τ is called a minimal set of τ .

For an irreducible GDMS $S = (V, E, (\Gamma_e)_{e \in E})$, the Fatou set $(F_i(S))_{i \in V}$ is forward S -invariant. For the proof, see [20, Lemma 2.15]. The following can be proved easily.

Lemma 2.5. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS and let $(L_i)_{i \in V}$ be a minimal set of S . Then, for each $k \in V$ and each $z_0 \in L_k$, we have $L_i = \overline{H_k^i(S)(\{z_0\})}$ for each $i \in V$.

Proof. Define $K_i = \overline{H_k^i(S)(\{z_0\})}$. It is easy to prove that $(K_i)_{i \in V}$ is forward S -invariant. Since $(L_i)_{i \in V}$ is forward S -invariant and $z_0 \in L_k$, we have $K_i \subset L_i$ for each $i \in V$. Thus, $L_i = \overline{H_k^i(S)(\{z_0\})}$ for each $i \in V$ by the minimality of $(L_i)_{i \in V}$. \square

Lemma 2.6. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be a compactly generated irreducible GDMS and let $(L_i)_{i \in V}$ be a minimal set. Then $L_j = \bigcup_{t(e)=j} \Gamma_e(L_{i(e)})$ for each $j \in V$.

Proof. Note that $\bigcup_{t(e)=j} \Gamma_e(L_{i(e)})$ is compact since Γ_e is compact and E is finite. Define $K_j = \bigcup_{t(e)=j} \Gamma_e(L_{i(e)})$. It is easy to prove that $(K_i)_{i \in V}$ is forward S -invariant. Since $(L_i)_{i \in V}$ is forward S -invariant, we have $L_j \supset \bigcup_{t(e)=j} \Gamma_e(L_{i(e)})$. Thus, $L_i = \bigcup_{t(e)=j} \Gamma_e(L_{i(e)})$ for each $i \in V$ by the minimality of $(L_i)_{i \in V}$. \square

We now define attracting minimal set, which is one of the most important concepts in this paper.

Definition 2.7. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS and let $(L_i)_{i \in V}$ be a minimal set of S . We say that $(L_i)_{i \in V}$ is an *attracting* minimal set of S if there exist $N \in \mathbb{N}$ and open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ such that

- (i) $L_i \subset U_i \subset \overline{U_i} \subset W_i \subset \overline{W_i} \subset F_i(S)$ for each $i \in V$ and
- (ii) for each admissible word $e = (e_1, \dots, e_N)$ with length N and each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), we have $\overline{f_N \circ \dots \circ f_1(W_{i(e)})} \subset U_{t(e)}$.

Also, for any irreducible MRDS τ , any attracting minimal set of S_τ is called an attracting minimal set of τ .

Attracting minimal sets play a crucial role to figure out the stability of (random) dynamical systems. Regarding minimal sets, we have some equivalent conditions for them to be attracting.

Lemma 2.8. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be a compactly generated irreducible GDMS and let $(L_i)_{i \in V}$ be a minimal set for S such that $(L_i)_{i \in V} \subset (F_i(S))_{i \in V}$. Let O_i be the finite union of the connected components of $F_i(S)$ each of which intersects L_i . Denote by d_{hyp} the hyperbolic metric on each connected component of O_i for each $i \in V$. Then the following are equivalent.

- (i) $(L_i)_{i \in V}$ is attracting.
- (ii) There exists $N \in \mathbb{N}$ such that for each admissible word $e = (e_1, \dots, e_N)$ with length N and for each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), there exists $c \in (0, 1)$ such that for each connected component U of O_i and for each $x, y \in U$, we have

$$d_{\text{hyp}}(f_N \circ \dots \circ f_1(x), f_N \circ \dots \circ f_1(y)) \leq c d_{\text{hyp}}(x, y).$$

- (iii) The constant c above can be chosen so that c does not depend on neither admissible words e nor maps f_n ; there exist $N \in \mathbb{N}$ and $c' \in (0, 1)$ such that for each admissible word $e = (e_1, \dots, e_N)$ with length N , for each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), for each connected component U of O_i and for each $x, y \in U$, we have

$$d_{\text{hyp}}(f_N \circ \dots \circ f_1(x), f_N \circ \dots \circ f_1(y)) \leq c' d_{\text{hyp}}(x, y).$$

Proof. Statement (iii) immediately implies statement (i), and statement (ii) implies statement (iii) since S is compactly generated.

Suppose that $(L_i)_{i \in V}$ is an attracting minimal set, and we show the statement (ii) holds. Take $N \in \mathbb{N}$, and open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ as in Definition 2.7. For each admissible word $e = (e_1, \dots, e_N)$ with length N and for each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), we have $f_N \circ \dots \circ f_1(W_{i(e)}) \subset U_{t(e)} \subset W_{t(e)}$. Also, $f_N \circ \dots \circ f_1(O_{i(e)}) \subset O_{t(e)}$ since $(F_i(S))_{i \in V}$ is forward S -invariant. It follows that there exists $c \in (0, 1)$ such that for each connected component U of O_i and for each $x, y \in U$, we have

$$d_{\text{hyp}}(f_N \circ \dots \circ f_1(x), f_N \circ \dots \circ f_1(y)) \leq c d_{\text{hyp}}(x, y).$$

Thus, we have completed our proof. □

The following proposition is very important to prove our main results. On (random) dynamics of holomorphic maps, we can classify the minimal sets as follows.

Proposition 2.9. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be a compactly generated irreducible GDMS and let $(L_i)_{i \in V}$ be a minimal set of S . Then $(L_i)_{i \in V}$ satisfies one of the following three conditions.

- (I) The set $(L_i)_{i \in V}$ intersects the Julia set; $L_i \cap J_i(S) \neq \emptyset$ for some $i \in V$.

(II) The set $(L_i)_{i \in V} \subset (F_i(S))_{i \in V}$ and $(L_i)_{i \in V}$ intersects a rotation domain; there exist $i \in V$ and $h \in H_i^i(S)$ such that $L_i \cap D \neq \emptyset$, where D is a connected component of $F_i(S)$ on which h is holomorphically conjugate to an irrational rotation on the unit disk or an annulus.

(III) The set $(L_i)_{i \in V}$ is attracting.

Proof. Suppose that a minimal set $(L_i)_{i \in V}$ is neither of type (I) nor (II), and we show that $(L_i)_{i \in V}$ is of type (III). By our assumption, L_i is contained in the Fatou set $F_i(S)$ at i for each $i \in V$. Let O_i be the finite union of the connected components of $F_i(S)$ each of which intersects L_i and denote by d_{hyp} the hyperbolic metric on each connected component of O_i for each $i \in V$.

We show that statement (ii) of Lemma 2.8 holds. Take a sufficiently large natural number N , say the product of $1 + \#V$, $\#V$ and $1 + N_0$, where

$$N_0 = \max_{i \in V} \{\text{number of the connected components of } O_i\}.$$

Then, for each admissible word $e = (e_1, \dots, e_N)$ with length N and for each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), there exist $1 \leq m \leq n \leq N$ and a connected component A of O_i for some $i \in V$ such that $i(e_m) = t(e_n) = i$ and $f_n \circ \dots \circ f_m(A) \subset A$.

Note that dynamics of $f_n \circ \dots \circ f_m$ on A is well understood and classified as in [14, §5]. Since $L_i \cap A \neq \emptyset$ and $(L_i)_{i \in V}$ is not of type (II), the map $f_n \circ \dots \circ f_m$ is attracting so that there exists $c < 1$ such that $d_{\text{hyp}}(f_n \circ \dots \circ f_m(x), f_n \circ \dots \circ f_m(y)) \leq c d_{\text{hyp}}(x, y)$ for each $x, y \in A$. Thus, for each connected component U of O_i and for each $x, y \in U$, we have $d_{\text{hyp}}(f_N \circ \dots \circ f_1(x), f_N \circ \dots \circ f_1(y)) \leq c d_{\text{hyp}}(x, y)$, and this completes our proof. \square

Definition 2.10. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be a compactly generated irreducible GDMS and let $\mathbb{L} = (L_i)_{i \in V}$ be a minimal set for S . We say that \mathbb{L} is *J-touching* if \mathbb{L} is of type (I) in Proposition 2.9. We say that \mathbb{L} is *sub-rotative* if \mathbb{L} is of type (II) in Proposition 2.9.

In the following, we present examples which possess a J-touching or a sub-rotative minimal set respectively.

Example 2.11. This example is due to [5]. Let V and E be singletons. Define $f_c(z) = z^2 + c$ for $c \in \mathbb{C}$. Let $\Gamma = \{f_c \in \text{Poly}; |c| \leq 1/4\}$. Then GDMS (V, E, Γ) has a minimal set $\overline{D} = \{z \in \mathbb{C}; |z| \leq 1/2\}$ and its Julia set J contains the Julia set of iteration of $f_{1/4}$, which implies $J \ni 1/2$. Hence, the minimal set \overline{D} is J-touching at $z = 1/2$.

Example 2.12. Let V and E be singletons. Let $f(z) = \nu z + z^2$ be a polynomial map which is linearizable at $z = 0$, say $\nu = \exp(2\pi i \xi)$ with the golden ratio ξ [14, §11]. Let $g \in \text{Poly}$ be a map which has an attracting fixed point at $z = 0$, say $g(z) = z^2$. Define $\Gamma = \{f, g\}$. Then the GDMS $S = (V, E, \Gamma)$ has a minimal set $\{0\}$, which is contained in the Fatou set. Since f does not contract the hyperbolic metric near $z = 0$, the minimal set $\{0\}$ is not attracting, and necessarily is sub-rotative.

3 The dichotomy of Markov RDSs

In this section, we discuss the property of mean stable systems and prove Main Results A, B and C. We consider mean stable GDMSs S as follows.

Definition 3.1. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS. We say that S is *mean stable* if there exist $N \in \mathbb{N}$ and two families of non-empty open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ such that

- (I) $U_i \subset \overline{U_i} \subset W_i \subset \overline{W_i} \subset F_i(S)$ for each $i \in V$,
- (II) for each admissible word $e = (e_1, \dots, e_N)$ with length N and for each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$), we have $f_N \circ \dots \circ f_1(\overline{W_{i(e)}}) \subset U_{t(e)}$, and
- (III) for each $z \in \widehat{\mathbb{C}}$ and $i \in V$, there exist $j \in V$ and $h \in H_i^j(S)$ such that $h(z) \in W_j$.

Example 3.2. Let V and E be singletons. Define $f_c(z) = z^2 + c$ for $c \in \mathbb{C}$. Let $\Gamma = \{f_c \in \text{Poly}; |c| \leq \epsilon\}$ for $\epsilon > 0$. Then it is easy to see that the GDMS (V, E, Γ) is mean stable for sufficiently small ϵ . In general, we can show that this GDMS is mean stable if $\epsilon \neq 1/4$. See Remark 4.9.

By Definition 3.1, if an MRDS τ is mean stable, then for every $i \in V$ and for every $z \in \widehat{\mathbb{C}}$, there exist $j \in V$ and $h \in H_i^j(S_\tau)$ such that $h(z) \notin J_j(S_\tau)$. This property implies some interesting results, see [20].

We now show some lemmas concerning relation with the mean stability and minimal sets.

Notation 3.3. Let $X \subset \text{Rat}_+$ and denote by $\text{Cpt}(X)$ the space of all non-empty compact sets of X . We endow $\text{Cpt}(X)$ with the Hausdorff metric.

Lemma 3.4. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS which is mean stable. Then the open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ for S in Definition 3.1 can be chosen such that the two are both forward S -invariant.

Proof. Take $N \in \mathbb{N}$, $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ as in Definition 3.1, which may not be forward S -invariant. For each $i \in V$, define $U'_i = U_i \cup \bigcup \Gamma_{e_\ell} \circ \dots \circ \Gamma_{e_1}(U_{i(e_1)})$ where the union runs over all natural numbers $1 \leq \ell \leq N-1$ and all admissible words (e_1, \dots, e_ℓ) with length ℓ such that $t(e_\ell) = i$. Note that there are at most finitely many numbers of such admissible words. Also, define $W'_i = W_i \cup \bigcup \Gamma_{e_\ell} \circ \dots \circ \Gamma_{e_1}(W_{i(e_1)})$ by a similar way. By the construction, $(U'_i)_{i \in V}$ and $(W'_i)_{i \in V}$ are both forward S -invariant.

We show $(U'_i)_{i \in V}$ and $(W'_i)_{i \in V}$ satisfy the conditions in Definition 3.1. It is trivial that conditions (I) and (III) hold. We show $f_N \circ \dots \circ f_1(\overline{W'_{i(e)}}) \subset U'_{t(e)}$ for each admissible word $e = (e_1, \dots, e_N)$ with length N and each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$). Fix $w \in W'_{i(e)}$. If $w \in W_{i(e)}$, then $f_N \circ \dots \circ f_1(w) \in U_{t(e)} \subset U'_{t(e)}$. If $w \notin W_{i(e)}$, then there exist admissible word $(\epsilon_1, \dots, \epsilon_\ell)$, $g_j \in \Gamma_{\epsilon_j}$ for each $j = 1, \dots, \ell$ and $z \in W_{i(\epsilon_1)}$ such that $w = g_\ell \circ \dots \circ g_1(z)$ and $t(\epsilon_\ell) = i$. Then $f_N \circ \dots \circ f_1(w) = f_N \circ \dots \circ f_{N-\ell+1}(f_{N-\ell} \circ \dots \circ f_1 \circ g_\ell \circ \dots \circ g_1(z))$. The right hand side belongs to $f_N \circ \dots \circ f_{N-\ell+1}(U_{t(\epsilon_{N-\ell})})$, and hence to $U'_{t(e)}$. Thus, the condition (II) holds for $(U'_i)_{i \in V}$ and $(W'_i)_{i \in V}$, and this completes the proof. \square

Lemma 3.5. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS and let $(L_i)_{i \in V}$ be an attracting minimal set of S . Then the open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ in Definition 2.7 can be chosen such that the two are both forward S -invariant.

Proof. The statement can be proved by a similar argument as Lemma 3.4. \square

Lemma 3.6. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS. Then the number of attracting minimal sets of S is finite. More precisely, for each $j \in V$ and $h \in H_j^j(S)$, the number of attracting minimal sets of S is at most the number of attracting cycles of h , and hence at most $2 \deg(h) - 2$.

Proof. Fix $j \in V$ and $h \in H_j^j(S)$. For each attracting minimal set $(L_i)_{i \in V}$ of S , there exist $N \in \mathbb{N}$ and open set W_j such that $L_j \subset W_j$ and $\overline{h^{\circ N}(W_j)} \subset W_j$, where $h^{\circ N}$ denotes N -th iterate of h . It follows that $h^{\circ N}$ has an attracting periodic point a in W_j . For a point $z \in L_j$, the orbit $h^{\circ N \ell}(z)$ accumulates to a as ℓ tends to infinity, and hence $a \in L_j$. Thus, the number of attracting minimal sets is at most the number of attracting cycles of h . \square

Lemma 3.7. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS. If all minimal sets of S are attracting, then S is mean stable.

Proof. Suppose that all minimal sets of S are attracting. By Lemma 3.6, the number of attracting minimal sets is finite. For each attracting minimal set, take a natural number and two kinds of open sets as in Definition 2.7. Define N as the product of these natural numbers and define $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ as the union of these open sets respectively. Then it is easy to see that conditions (I) and (II) of Definition 3.1 holds. We show that condition (III) also holds. For each $z \in \widehat{\mathbb{C}}$ and $i \in V$, define $K_j = \overline{H_i^j(S)(\{z\})}$ for each $j \in V$. Then $(K_j)_{j \in V}$ is forward S -invariant, and it follows from Zorn's lemma that there exists a minimal set $(L_j)_{j \in V}$ such that $(L_j)_{j \in V} \subset (K_j)_{j \in V}$. The minimal set $(L_j)_{j \in V}$ is attracting by our assumption, thus there exists $h \in H_i^j(S)$ such that $h(z) \in W_j$. This completes our proof. \square

Lemma 3.8. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS. If S is mean stable, then all minimal sets of S are attracting.

Proof. For a mean stable S , take $N \in \mathbb{N}$, $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ as in Definition 3.1. We may assume that both $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ are forward S -invariant by Lemma 3.4. Pick any minimal set $(L_i)_{i \in V}$. It suffices to show $L_i \subset U_i$ for each $i \in V$. Fix $k \in V$ and $z \in L_k$, then there exist $j \in V$ and $h \in H_k^j(S)$ such that $h(z) \in U_j$. Since $(L_i)_{i \in V}$ is forward invariant, we have $z_0 = h(z) \in L_j$, and hence $L_i = \overline{H_j^i(S)(\{z_0\})}$ for each $i \in V$ by Lemma 2.5. By Lemma 2.6 and conditions (I) and (II) in Definition 3.1 for $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$, we can show $L_i \subset U_i$ for each $i \in V$. \square

By Lemmas 3.7 and 3.8, we have the following corollary.

Corollary 3.9. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS. Then S is mean stable if and only if all minimal sets of S are attracting.

By the corollary above, we can construct MRDSs which are not mean stable by using Examples 2.11 and 2.12.

We now show that if τ is mean stable, then for every initial value, sample-wise dynamics is contractive and the Lyapunov exponent is negative almost surely with respect to the natural measure $\tilde{\tau}$ associated with τ . We first define the measure $\tilde{\tau}$.

Definition 3.10. For $\tau \in \text{MRDS}(\text{Rat}_+)$, we set $p_{ij} = \tau_{ij}(\text{Rat}_+)$ and set $P = (p_{ij})_{i,j \in V}$. Since τ is irreducible, there exists a unique vector $p = (p_1, \dots, p_m)$ such that $pP = p$, $\sum_{i \in V} p_i = 1$ and $p_i > 0$ for all $i \in V$.

We define the Borel probability measure $\tilde{\tau}$ on $(\text{Rat}_+ \times E)^{\mathbb{N}}$ as follows. For each $i \in V$, let $\tilde{\tau}_i$ be the Borel probability measure on $(\text{Rat}_+ \times E)^{\mathbb{N}}$ such that for any $N \in \mathbb{N}$, for N Borel sets A_n ($n = 1, \dots, N$) of Rat_+ and for $(e_1, \dots, e_N) \in E^N$,

$$\begin{aligned} & \tilde{\tau}_i \left(A'_1 \times \dots \times A'_N \times \prod_{N+1}^{\infty} (\text{Rat}_+ \times E) \right) \\ &= \begin{cases} \tau_{e_1}(A_1) \cdots \tau_{e_N}(A_N), & \text{if } (e_1, \dots, e_N) \text{ is admissible and } i(e_1) = i \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $A'_n = A_n \times \{e_n\}$ ($n = 1, \dots, N$). Then we define $\tilde{\tau}$ as the sum $\sum_{i \in V} p_i \tilde{\tau}_i$.

We now prove Main Results A and B.

Theorem 3.11 (Main Result A). Let $\tau \in \text{MRDS}(\text{Rat}_+)$ be a mean stable system. Then we have all of the following.

- (i) There exists a constant $c \in (0, 1)$ satisfying that for each $z \in \widehat{\mathbb{C}}$, there exists a Borel subset \mathfrak{F} of $(\text{Rat}_+ \times E)^{\mathbb{N}}$ with $\tilde{\tau}(\mathfrak{F}) = 1$ such that for every $(f_n, e_n)_{n=1}^{\infty} \in \mathfrak{F}$, there exist $r = r(z, (f_n, e_n)_{n=1}^{\infty}) > 0$ and $K = K(z, (f_n, e_n)_{n=1}^{\infty}) > 0$ such that

$$\text{diam } f_n \circ \dots \circ f_1(B(z, r)) \leq Kc^n$$

for every $n \in \mathbb{N}$.

- (ii) For each $z \in \widehat{\mathbb{C}}$, there exists a Borel subset \mathfrak{F} of $(\text{Rat}_+ \times E)^{\mathbb{N}}$ with $\tilde{\tau}(\mathfrak{F}) = 1$ such that for every $(f_n, e_n)_{n=1}^{\infty} \in \mathfrak{F}$, there exists an attracting minimal set $(L_i)_{i \in V}$ such that $d(f_n \circ \dots \circ f_1(z), L_{t(e_n)}) \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) For every $i \in V$, the kernel Julia set $J_{\text{ker}, i}(S_\tau) := \bigcap_{j \in V} \bigcap_{h \in H_i^j(S)} h^{-1}(J_j(S))$ at i is empty.
- (iv) There exists a Borel subset \mathfrak{G} of $(\text{Rat}_+ \times E)^{\mathbb{N}}$ with $\tilde{\tau}(\mathfrak{G}) = 1$ such that for every $\xi = (f_n, e_n)_{n=1}^{\infty} \in \mathfrak{G}$, the (2-dimensional) Lebesgue measure of the Julia set J_ξ is zero. Here, the Julia set J_ξ of ξ is the complement of the set of all points $z \in \widehat{\mathbb{C}}$ for which there exists a neighborhood U of z in $\widehat{\mathbb{C}}$ such that the family $\{f_n \circ \dots \circ f_1\}_{n=1}^{\infty}$ is equicontinuous on U .

- (v) There exist at most finitely many minimal sets of S_τ . Moreover, each minimal set of S_τ is attracting. Also, for every minimal set $\mathbb{L} = (L_i)_{i \in V}$, we define the function $\mathbb{T}_\mathbb{L}: \widehat{\mathbb{C}} \times V \rightarrow [0, 1]$ by

$$\mathbb{T}_\mathbb{L}(z, i) := \tilde{\tau}_i(\{\xi = (f_n, e_n)_{n \in \mathbb{N}}; d(f_n \circ \cdots \circ f_1(z), L_{t(e_n)}) \rightarrow 0 (n \rightarrow \infty)\})$$

for every point $(z, i) \in \widehat{\mathbb{C}} \times V$. Then $\mathbb{T}_\mathbb{L}$ is continuous on $\widehat{\mathbb{C}} \times V$. Also, the averaged function $T_\mathbb{L}(z) := \sum_{i \in V} p_i \mathbb{T}_\mathbb{L}(z, i)$ is continuous on $\widehat{\mathbb{C}}$, where (p_1, \dots, p_m) is the probability vector in Definition 3.10.

- (vi) Suppose τ has exactly ℓ minimal sets $\mathbb{L}_1, \mathbb{L}_2, \dots, \mathbb{L}_\ell$. Then

$$T_{\mathbb{L}_1}(z) + T_{\mathbb{L}_2}(z) + \cdots + T_{\mathbb{L}_\ell}(z) = 1$$

for every $z \in \widehat{\mathbb{C}}$.

Proof. The key to the proof of (i) and (ii) is [20, Lemma 3.10]. For the mean stable S_τ , fix a family of open sets $(W_i)_{i \in V}$ and N as in Definition 3.1 and let $z \in \widehat{\mathbb{C}}$. Then, by [20, Lemma 3.10],

$$\mathfrak{E} = \{(f_n, e_n)_{n=1}^\infty \in (\text{Rat}_+ \times E)^\mathbb{N}; f_n \circ \cdots \circ f_1(z) \notin W_{t(e_n)} \text{ for any } n \in \mathbb{N}\}$$

satisfies $\tilde{\tau}(\mathfrak{E}) = 0$, and we define \mathfrak{F} as the complement of \mathfrak{E} .

For every $(f_n, e_n)_{n=1}^\infty \in \mathfrak{F}$, there exists $k \in \mathbb{N}$ such that $f_k \circ \cdots \circ f_1(z) \in W_{t(e_k)}$. Take $r = r(z, (f_n, e_n)_{n=1}^\infty) > 0$ such that $f_k \circ \cdots \circ f_1(B(z, r)) \subset W_{t(e_k)}$. Since dynamics on $(W_i)_{i \in V}$ is uniformly contractive with respect to the hyperbolic metric, there exists $c \in (0, 1)$, which does not depend on z , such that

$$\text{diam}_{\text{hyp}} f_{\ell N+k} \circ \cdots \circ f_1(B(z, r)) \leq c^\ell \text{diam}_{\text{hyp}} f_k \circ \cdots \circ f_1(B(z, r))$$

for every $\ell \in \mathbb{N}$. This completes the proof of (i) since hyperbolic metric and spherical metric are comparable on each compact set. By the similar method, we can show that the statement (ii) is also true.

The statement (iii) is trivial by definition of the kernel Julia set and the mean stability, and it follows from [20, Proposition 3.11] that the 2-dimensional Lebesgue measure of the Julia set J_ξ is zero for $\tilde{\tau}$ -almost every ξ . This completes the proof of (iv).

We next show the statement (v) following [20, Proposition 4.24]. By Corollary 3.9, each minimal set of S_τ is attracting. By Lemma 3.6, it follows that there exist at most finitely many minimal sets of S_τ . For an attracting minimal set $\mathbb{L} = (L_i)_{i \in V}$, for each $i \in V$ we let O_i be the finite union of the connected components of $F_i(S_\tau)$ each of which intersects L_i . Then there exists a continuous function $\phi: \widehat{\mathbb{C}} \times V \rightarrow [0, 1]$ such that $\phi(z, i) = 1$ for every $z \in L_i$ and $\phi(z, i) = 0$ for every $z \notin O_i$. For $\tau = (\tau_{ij})_{i, j \in V}$, we define the transition operator M_τ of τ as follows.

$$M_\tau \psi(z, i) := \sum_{j \in V} \int \psi(f(z), j) d\tau_{ij}(f), \quad (z, i) \in \widehat{\mathbb{C}} \times V.$$

Then, it is easy to show that $\{M_\tau^n \phi\}_{n \in \mathbb{N}}$ converges pointwise to $\mathbb{T}_\mathbb{L}$ on $\widehat{\mathbb{C}} \times V$ as $n \rightarrow \infty$. Since the statement (ii) holds, the family $\{M_\tau^n \phi\}_{n \in \mathbb{N}}$ of continuous maps is equicontinuous on $\widehat{\mathbb{C}} \times V$ by Proposition 3.11, Lemmas 2.39 and 2.38 of [20]. It follows from the Arzelà-Ascoli theorem, the limit $\mathbb{T}_\mathbb{L}$ is also continuous on $\widehat{\mathbb{C}} \times V$. This completes the proof of (v).

The statement (vi) is a direct consequence of (ii). \square

Theorem 3.12 (Main Result B). Let $\tau \in \text{MRDS}(\text{Rat}_+)$ be a mean stable system. Then there exists $\alpha < 0$ such that the following holds for each $z \in \widehat{\mathbb{C}}$. There exists a Borel set \mathfrak{F} with $\tilde{\tau}(\mathfrak{F}) = 1$ such that for every $(f_n, e_n)_{n=1}^\infty \in \mathfrak{F}$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|D(f_n \circ \cdots \circ f_1)(z)\| \leq \alpha.$$

Here, $Dg(z)$ denotes the complex differential of a map g at z and $\|\cdot\|$ denotes the norm with respect to the spherical metric.

Proof. The statement can be proved by a similar argument as Theorem 3.11. \square

We now investigate perturbations of GDMSs and show that attracting minimal sets are stable under perturbations. For compactly generated GDMS $S = (V, E, (\Gamma_e)_{e \in E})$, we consider another compactly generated GDMS $S' = (V, E, (\Gamma'_e)_{e \in E})$ such that Γ'_e is close to Γ_e with respect to the Hausdorff metric for each $e \in E$.

Lemma 3.13. Let $X \subset \text{Rat}_+$. Let $S = (V, E, (\Gamma_e)_{e \in E})$ be an irreducible GDMS such that $\Gamma_e \in \text{Cpt}(X)$ for each $e \in E$. Suppose that S has an attracting minimal set $(L_i)_{i \in V}$. Take $N \in \mathbb{N}$, open sets $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ for $(L_i)_{i \in V}$ as in Definition 2.7. Also, take an open set G_i , which is close to L_i , such that $L_i \subset G_i \subset \overline{G_i} \subset U_i$ for each $i \in V$.

Then there exists an open neighborhood \mathcal{U}_e of Γ_e in $\text{Cpt}(X)$ for each $e \in E$ such that for each compact set $\Gamma'_e \in \mathcal{U}_e$, GDMS $S' = (V, E, (\Gamma'_e)_{e \in E})$ has a unique minimal set $(L'_i)_{i \in V}$ such that $(L'_i)_{i \in V} \subset (G_i)_{i \in V}$ and $(L'_i)_{i \in V}$ is attracting for S' .

Proof. We may assume that both $(U_i)_{i \in V}$ and $(W_i)_{i \in V}$ are forward S -invariant by Lemma 3.5. By Lemma 2.8, $d(f_n \circ \cdots \circ f_1(x), L_{t(e_n)}) \rightarrow 0$ as $n \rightarrow \infty$ for each infinite admissible word (e_1, e_2, \dots) , $f_n \in \Gamma_{e_n}$ ($n = 1, 2, \dots$) and $x \in \overline{W_{i(e_1)}}$. Thus, taking N sufficiently large, we may assume $\overline{f_N \circ \cdots \circ f_1(W_{i(e)})} \subset G_{t(e)}$ for each admissible word $e = (e_1, \dots, e_N)$ with length N and each $f_n \in \Gamma_{e_n}$ ($n = 1, \dots, N$).

Then there exists an open neighborhood \mathcal{U}_e of Γ_e for each $e \in E$ with respect to the topology of $\text{Cpt}(X)$ such that for each $e \in E$ and $\Gamma'_e \in \mathcal{U}_e$, another GDMS $S' = (V, E, (\Gamma'_e)_{e \in E})$ satisfies the following two conditions.

- (a) $\overline{g_N \circ \cdots \circ g_1(W_{i(e)})} \subset G_{t(e)}$ for each admissible word $e = (e_1, \dots, e_N)$ with length N and each $g_n \in \Gamma'_{e_n}$ ($n = 1, \dots, N$).
- (b) $g_\ell \circ \cdots \circ g_1(\overline{U_{i(e_1)}}) \subset W_{t(e_\ell)}$ for each $1 \leq \ell \leq N-1$, admissible word $e = (e_1, \dots, e_\ell)$ with length ℓ and $g_j \in \Gamma'_{e_j}$ ($j = 1, \dots, \ell$).

Fix $\Gamma'_e \in \mathcal{U}_e$ for each $e \in E$, and define $S' = (V, E, (\Gamma'_e)_{e \in E})$.

For each $i \in V$, we consider $K'_i = \bigcup_{k \in V} H_k^i(S')(L_k)$. Then $(K'_i)_{i \in V}$ is a family of compact sets which is forward S' -invariant, and hence there exists a minimal set $(L'_i)_{i \in V}$ of S' such that $(L'_i)_{i \in V} \subset (K'_i)_{i \in V}$. Also, it follows from (a) and (b) that $(K'_i)_{i \in V} \subset (W_i)_{i \in V}$, and consequently $(L'_i)_{i \in V} \subset (W_i)_{i \in V}$. Using Lemma 2.6 repeatedly, for each $i \in V$, we have $L'_i = \bigcup \Gamma'_{e_N} \circ \cdots \circ \Gamma'_{e_1}(L'_{i(e_1)})$ where the union runs over all admissible words $e = (e_1, \dots, e_N)$ with length N such that $t(e) = i$. It follows from (a) that $(L'_i)_{i \in V} \subset (G_i)_{i \in V}$. Also, by Montel's theorem, we have that $W_i \subset F_i(S')$ for each $i \in V$, and hence $(L'_i)_{i \in V}$ is an attracting minimal set of S' . We now show the uniqueness of the minimal set in $(G_i)_{i \in V}$. Since G_i is close to L_i , we may assume that for every point $p \in G_i$, the S' -orbit tends to the attracting minimal set $(L'_i)_{i \in V}$. Thus, minimal sets in $(G_i)_{i \in V}$ is unique. This completes our proof. \square

Using the lemma above, we can control global stability as follows.

Proposition 3.14. Let $X \subset \text{Rat}_+$. Suppose GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ is irreducible and mean stable with $\Gamma_e \in \text{Cpt}(X)$ for each $e \in E$. Then there exists an open neighborhood \mathcal{U}_e of Γ_e in $\text{Cpt}(X)$ for each $e \in E$ such that for each $\Gamma'_e \in \mathcal{U}_e$, another GDMS $S' = (V, E, (\Gamma'_e)_{e \in E})$ is also mean stable.

Proof. By Lemma 3.8 and Lemma 3.6, S has finitely many minimal sets, which are all attracting. It follows from Lemma 3.13 that there exists an open neighborhood \mathcal{U}_e of Γ_e for each $e \in E$ with respect to the Hausdorff metric such that for each $\Gamma'_e \in \mathcal{U}_e$, another GDMS $S' = (V, E, (\Gamma'_e)_{e \in E})$ satisfies the following. For each attracting minimal set of S , another system S' has the attracting minimal set close to it.

By Lemma 3.7, it suffices to show that S' does not have any minimal set except these attracting ones. For mean stable GDMS S , we take $(W_i)_{i \in V}$ as in Definition 3.1. Then, for each $z \in \widehat{\mathbb{C}}$ and $i \in V$, there exist $j \in V$ and $h \in H_i^j(S)$ such that $h(z) \in W_j$. Choose an open neighborhood O_z of z so that $h(\overline{O_z}) \subset W_j$. Since $\widehat{\mathbb{C}}$ is compact, there exist finitely many O_{z_1}, \dots, O_{z_m} and $h_1, \dots, h_m \in H_i^j(S)$ such that $\widehat{\mathbb{C}} = \bigcup_{\ell=1}^m O_{z_\ell}$ and $h_\ell(\overline{O_{z_\ell}}) \subset W_j$. Taking the open neighborhood \mathcal{U}_e so small if necessary, we may assume that there exist $h'_1, \dots, h'_m \in H_i^j(S')$ such that $h'_\ell(\overline{O_{z_\ell}}) \subset W_j$. Therefore, for each $z \in \widehat{\mathbb{C}}$ and $i \in V$, there exists $h' \in H_i^j(S')$ such that $h'(z) \in W_j$, and hence S' does not have J-touching nor sub-rotative minimal sets. This completes our proof. \square

The following is a consequence of Proposition 3.14.

Corollary 3.15 (A part of Main Result C). Let $X \subset \text{Rat}_+$. Then the set $\mathcal{A}(X)$ of all mean stable systems is open in $\text{MRDS}(X)$.

We now show the key lemma to Main Results C and D. The assumption that X is non-degenerate is essential here. It is easy to check that Rat_+ , Poly and $\{f(z)+c; c \in \mathbb{C}\}$ ($f \in \text{Poly}$) is non-degenerate for example. Note that we see that any set X in the following Example 3.16 does not satisfy the non-degenerate condition.

Example 3.16. Define $g_\theta(z) = \theta z(1-z)$ for each $\theta \in \mathbb{C} \setminus \{0\}$. Then $X = \{g_\theta \in \text{Poly}; \theta \in \mathbb{C} \setminus \{0\}\}$ is NOT non-degenerate since they have a common fixed point $z = 0$. Also, for $P \in \text{Poly}$, define $N_\theta(z) = z - \theta P(z)/P'(z)$ for each $\theta \in \mathbb{C} \setminus \{0\}$. Then $X = \{N_\theta \in \text{Rat}_+; \theta \in \mathbb{C} \setminus \{0\}\}$ is NOT non-degenerate since all N_θ have common fixed points z_0 which satisfy $P(z_0) = 0$. However, we can consider their mean stability in the weak sense. See [19].

Lemma 3.17. Suppose $X \subset \text{Rat}_+$ is non-degenerate. Let $\tau \in \text{MRDS}(X)$. If $S_\tau = (V, E, (\Gamma_e)_{e \in E})$ has an attracting minimal set, then $\tau \in \overline{\mathcal{A}(X)}$ where the closure is taken in the space $\text{MRDS}(X)$ with respect to the topology in Definition 1.4.

Proof. We approximate τ_e by a Borel measure ρ_e on X such that the total measure of τ_e and ρ_e coincide and $\text{supp } \tau_e \subset \text{int}(\text{supp } \rho_e)$ for each $e \in E$, where int denotes the set of all interior points in the space X endowed with the relative topology from Rat_+ . Also, define $\rho_e = 0$ if $\tau_e = 0$. Note that the associated directed graph of ρ is the same as that of τ . Set $\rho = (\rho_e)_{e \in E} \in \text{MRDS}(X)$. We show that $\rho \in \mathcal{A}(X)$ if $\text{supp } \rho_e$ is sufficiently close to $\text{supp } \tau_e$. Let $(L_i)_{i \in V}$ be an attracting minimal set of original system S_τ . By Lemma 3.13, if $\text{supp } \rho_e$ is sufficiently close to $\text{supp } \tau_e$ for each $e \in E$, then there exists an attracting minimal set $(L'_i)_{i \in V}$ of S_ρ . Pick any minimal set $(K'_i)_{i \in V}$ of S_ρ . Since $\text{supp } \tau_e \subset \text{supp } \rho_e$ for each $e \in E$, the set $(K'_i)_{i \in V}$ is forward S_τ -invariant. By Zorn's lemma, there exists a minimal set $(K_i)_{i \in V}$ of S_τ such that $(K_i)_{i \in V} \subset (K'_i)_{i \in V}$.

We next show that $(K_i)_{i \in V}$ is attracting for S_τ . Suppose that it is not attracting, then by Proposition 2.9, there are possibly two cases.

(I) For the J-touching case, there exists $i \in V$ and $p \in \widehat{\mathbb{C}}$ such that $p \in K_i \cap J_i(S_\tau)$. By [20, Proposition 2.16], we have that $J_i(S_\tau) = \bigcup_{i(e)=i} \bigcup_{f \in \text{supp } \tau_e} f^{-1}(J_{t(e)}(S_\tau))$. Thus, there exist $e \in E$ and $f \in \text{supp } \tau_e$ such that $f(p) \in J_{t(e)}(S_\tau)$. Since X is non-degenerate, there exists a holomorphic family $\{g_\theta\}_{\theta \in \Theta}$ of rational maps parametrized by a complex manifold Θ with $\{g_\theta; \theta \in \Theta\} \subset X$ such that $g_{\theta_0} = f$ for some $\theta_0 \in \Theta$ and $\theta \mapsto g_\theta(p)$ is non-constant in any neighborhood of θ_0 . Taking Θ so small if necessary, we may assume $\{g_\theta; \theta \in \Theta\} \subset \text{int}(\text{supp } \rho_e)$. Then $f(p) \in J_{t(e)}(S_\rho) \cap \text{int}(K'_{t(e)})$ by the open mapping theorem. By [20, Lemma 2.11] and Montel's theorem, we have that $H_{t(e)}^{t(e)}(S_\rho)$ does not omit three points on $\text{int}(K'_{t(e)})$. However, this contradicts the fact that S_ρ has an attracting minimal set $(L'_i)_{i \in V}$ and another minimal set $(K'_i)_{i \in V}$.

(II) For the sub-rotative case, there exist $i \in V$, $h \in H_i^i(S_\tau)$ and $p \in \widehat{\mathbb{C}}$ such that $p \in K_i \cap D$, where D is a connected component of $F_i(S_\tau)$ on which h is conjugate to an irrational rotation. Since $\text{supp } \tau_e \subset \text{int } \text{supp } \rho_e$ for each $e \in E$, there exists $g \in H_i^i(S_\rho)$ such that $g(p) \in \partial D \subset J_i(S_\tau) \subset J_i(S_\rho)$. Then a similar argument leads to a contradiction as in the case (I).

Consequently, $(K_i)_{i \in V}$ is an attracting minimal set of S_τ . Letting $\text{supp } \rho_e$ sufficiently close to $\text{supp } \tau_e$ for each $e \in E$, we may assume that $(K'_i)_{i \in V}$ is an attracting minimal set of S_ρ by Lemma 3.13. By Lemma 3.6, S_τ has only finitely many attracting sets, and hence all the minimal sets of S_ρ are attracting. Therefore, by Lemma 3.7, S_ρ is mean stable. \square

As a corollary, we have the following result regarding the polynomial dynamics.

Corollary 3.18 (Corollary 1.9). Suppose $X \subset \text{Poly}$ is non-degenerate and satisfies the condition (ii) in Definition 1.3. Then the set $\mathcal{A}(X)$ of all mean stable polynomial dynamics is open and dense in $\text{MRDS}(X)$.

Proof. By Corollary 3.15, $\mathcal{A}(X)$ is open. Since $X \subset \text{Poly}$, each $\tau \in \text{MRDS}(X)$ has the attracting minimal set $(\{\infty\})_{i \in V}$. Thus, $\mathcal{A}(X)$ is dense by Lemma 3.17. \square

We now consider the complement of $\mathcal{A}(X)$.

Definition 3.19. We say that a GDMS $S = (V, E, (\Gamma_e)_{e \in E})$ is *chaotic* if $\mathbb{J}(S) = \widehat{\mathbb{C}} \times V$ and $(\widehat{\mathbb{C}})_{i \in V}$ is a minimal set of S . We say that τ is chaotic if associated GDMS S_τ is chaotic, and define the set $\mathcal{C}(X)$ as the set of all $\tau \in \text{MRDS}(X)$ which are chaotic.

Lemma 3.20. Let $X \subset \text{Rat}_+$ and $\tau \in \text{MRDS}(X)$. If $(\widehat{\mathbb{C}})_{i \in V}$ is a minimal set of $S_\tau = (V, E, (\Gamma_e)_{e \in E})$ and $\text{int}(J_j(S_\tau)) \neq \emptyset$ for some $j \in V$, then $J_i(S_\tau) = \widehat{\mathbb{C}}$ for each $i \in V$, and hence $\tau \in \mathcal{C}(X)$.

Proof. If there exists $k \in V$ such that $J_k(S_\tau) \neq \widehat{\mathbb{C}}$, then $J_i(S_\tau) \neq \widehat{\mathbb{C}}$ for each $i \in V$ by the irreducibility of τ . Thus, there exists a minimal set $(L_i)_{i \in V}$ such that $L_j \subset \overline{F_j(S_\tau)}$ by Zorn's lemma. By the minimality of $(\widehat{\mathbb{C}})_{i \in V}$, we have $\widehat{\mathbb{C}} = L_j$. This contradicts the assumption that $\text{int}J_j(S_\tau) \neq \emptyset$. \square

Example 3.21. Let f be a rational map whose Julia set is $\widehat{\mathbb{C}}$, say a Lattès map [14, §7]. Let $m = 1$ and define $\tau_{11} = \delta_f$, the Dirac measure at f . Then $\tau = (\tau_{11})$ is NOT chaotic since f has periodic cycles as minimal sets. However, it still holds that there exists a dense (actually residual) set R in $\widehat{\mathbb{C}}$ such that for every $z \in R$, the orbit $\{z, f(z), f^{\circ 2}(z), \dots\}$ is dense in $\widehat{\mathbb{C}}$, see [14].

Using this map f , we can construct chaotic systems. We define a probability measure μ as the push-forward of the normalized Lebesgue measure under

$$\{a \in \mathbb{C}; \frac{1}{2} \leq |a| \leq 1\} \ni a \mapsto f_a(z) = (az^2 + a)/(z^2 + 1) \in \text{Rat}_+$$

and define $\tau_{11} = \delta_f/2 + \mu/2$. We now prove that $\tau = (\tau_{11})$ is chaotic. It is enough to show that $\widehat{\mathbb{C}}$ is minimal. For every $z \in \widehat{\mathbb{C}}$, there exists $g \in \text{supp } \mu$ such that $g(z) \in R$, where R is the dense set above. Then $\{g(z), f(g(z)), f^{\circ 2}(g(z)), \dots\}$ is dense in $\widehat{\mathbb{C}}$, hence $\widehat{\mathbb{C}}$ is minimal.

Also, let $m = 2$ and define $\tau = (\tau_{ij})_{i,j=1,2}$ by

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\delta_f & \frac{1}{2}\delta_f \\ \mu & 0 \end{pmatrix}.$$

Then we can show also that $\tau = (\tau_{ij})_{i,j=1,2}$ is chaotic.

We now prove the density of mean stable or chaotic MRDSs, which is a part of Main Result C.

Theorem 3.22 (Main Result C). Suppose $X \subset \text{Rat}_+$ is non-degenerate. Then the union $\mathcal{A}(X) \cup \mathcal{C}(X)$ is dense in the space $\text{MRDS}(X)$.

Proof. Let $\tau \in \text{MRDS}(X)$. If $S_\tau = (V, E, (\Gamma_e)_{e \in E})$ has an attracting minimal set, then $\tau \in \overline{\mathcal{A}(X)}$ by Lemma 3.17. We assume that S_τ has no attracting minimal sets. As in the proof of Lemma 3.17, we approximate τ_e by ρ_e such that $\text{supp } \tau_e \subset \text{int}(\text{supp } \rho_e)$ for each $e \in E$ and define a new system S_ρ . Since $\emptyset \neq J_i(S_\tau) \subset J_i(S_\rho)$ for each $i \in V$, we have $\text{int}(J_j(S_\rho)) \neq \emptyset$ for each $j \in V$. Thus, by Lemma 3.20, it suffices to prove that $(\widehat{\mathbb{C}})_{i \in V}$ is a minimal set of S_ρ .

Take $z_0 \in \widehat{\mathbb{C}}$ and $j \in V$, and we define $K'_i = \overline{H_j^i(S_\rho)(\{z_0\})}$ for each $i \in V$. Since $(K'_i)_{i \in V}$ is forward S_ρ -invariant, there exist a minimal set $(L'_i)_{i \in V}$ of S_ρ and a minimal set $(L_i)_{i \in V}$ of S_τ such that $(L_i)_{i \in V} \subset (L'_i)_{i \in V} \subset (K'_i)_{i \in V}$. Recall that $(L_i)_{i \in V}$ is not attracting for S_τ by our assumption, and hence there are two cases (I) or (II) by Proposition 2.9.

For the J-touching case (I), there exists $i \in V$ such that $L_i \cap J_i(S_\tau) \neq \emptyset$. By a similar argument as the proof of Lemma 3.17, there exists $e \in E$ such that $i(e) = i$ and $J_{t(e)}(S_\rho) \cap \text{int}(L'_{t(e)}) \neq \emptyset$, and hence $H_{t(e)}^{t(e)}(S_\rho)$ does not omit three points on $\text{int}(L'_{t(e)})$. It follows that $(\widehat{\mathbb{C}})_{i \in V}$ is a minimal set of S_ρ . For the sub-rotative case, there exist $i \in V$ and a rotation domain D such that $L_i \cap D \neq \emptyset$. The same idea can be applied to show that $(\widehat{\mathbb{C}})_{i \in V}$ is a minimal set of S_ρ . This completes our proof. \square

4 Bifurcation of MRDSs

In this section, we consider families of MRDSs and their bifurcations.

Setting 4.1. Let Λ be a topological space and let m be a σ -finite Borel measure on Λ , which we consider as a parameter space. Let $I = [a, b)$ be an interval on the real line \mathbb{R} , possibly $I = [a, \infty)$. Suppose $X \subset \text{Rat}_+$ is non-degenerate and suppose $\Phi: \Lambda \times I \rightarrow \text{MRDS}(X)$ satisfies the following three conditions. Denote $\Phi(\lambda, s) = \tau^{\lambda, s}$.

- (i) Φ is continuous and the associated directed graphs (V, E) of $\tau^{\lambda, s}$ are identical for all $(\lambda, s) \in \Lambda \times I$.
- (ii) $\text{supp } \tau_e^{\lambda, s_1} \subset \text{int}(\text{supp } \tau_e^{\lambda, s_2})$ for each $e \in E$, $\lambda \in \Lambda$ and $s_1 < s_2$, where int denotes the set of all interior points with respect to X .
- (iii) $\tau^{\lambda, s}$ has at least one attracting minimal set for each $(\lambda, s) \in \Lambda \times I$.

The essential assumption in Setting 4.1 is (ii), which describes the property that the size of noise increases as s increases. This enables us to control minimal sets.

The most important example of such families is i.i.d. RDS of quadratic polynomial maps. The following is a motivating example, which itself is very interesting.

Example 4.2. Let $X = \{f_c(z) = z^2 + c \in \text{Poly}; c \in \mathbb{C}\}$. Let $\Lambda = \mathbb{C}$ and identify $c \in \mathbb{C}$ with the quadratic polynomial f_c . Let $m = \text{Leb}$ be the Lebesgue measure on c -plane

and $I = [0, \infty)$. We define $\Phi(c, s) = \tau^{c,s}$ for $(c, s) \in \Lambda \times I$ as follows. Define $\tau^{c,0}$ as the Dirac measure at c if $s = 0$, and otherwise define $\tau^{c,s}$ as the Lebesgue measure on the disk $D(c, s)$ centered at c with radius $s > 0$ normalized by $\tau^{c,s}(D(c, s)) = 1$. Regard it as an MRDS whose V and E are both singletons.

Then (X, Λ, m, I, Φ) satisfies the conditions (i), (ii) and (iii) in Setting 4.1.

Example 4.3. Suppose $X \subset \text{Poly}$ is non-degenerate and satisfies the condition (ii) in Definition 1.3 holds. Then the assumption (iii) in Setting 4.1 is satisfied since for every $\tau \in \text{MRDS}(X)$, $(\{\infty\})_{i \in V}$ is an attracting minimal set of S_τ .

We now show our results.

Lemma 4.4. Let X, Λ, m, I, Φ as in Setting 4.1. Then, for any $\lambda \in \Lambda$, the number of minimal sets of $\tau^{\lambda,s}$ does not increase as s increases.

Proof. Fix $s_1 \leq s_2$. For a minimal set $(L'_i)_{i \in V}$ of τ^{λ,s_2} , there exists a minimal set $(L_i)_{i \in V}$ of τ^{λ,s_1} such that $(L_i)_{i \in V} \subset (L'_i)_{i \in V}$ since $\text{supp } \tau_e^{\lambda,s_1} \subset \text{supp } \tau_e^{\lambda,s_2}$ for each $e \in E$. Since minimal sets do not intersect one another, this completes our proof. \square

Lemma 4.5. Let X, Λ, m, I, Φ as in Setting 4.1 and suppose that there exists $d \in \mathbb{N}$ with $d \geq 2$ such that $2 \leq \deg(g) \leq d$ for each $g \in X$. Then there exists $\alpha \in \mathbb{N}$ such that for each $\lambda \in \Lambda$, the number of $s \in I$ such that $\tau^{\lambda,s}$ is not mean stable is at most α .

Proof. For the directed graph (V, E) , fix $i \in V$. Then there exists an admissible word e such that $i(e) = i = t(e)$, whose length is denoted by N . Then, by our assumption, there exists $h \in H_i^i(S_{\tau^{\lambda,a}})$ whose degree is at most d^N . Define $\alpha = 2d^N - 2$ and take $\lambda \in \Lambda$. It follows from the proof of Lemma 3.17 that for each $s_0 \in I$, we have that $\tau^{\lambda,s}$ is mean stable for $s > s_0$ which is sufficiently close to s_0 . Moreover, if τ^{λ,s_0} is not mean stable, then the number of minimal sets of $\tau^{\lambda,s}$ is strictly less than that of τ^{λ,s_0} .

If $s \in (a, b)$ is sufficiently close to a , then $\tau^{\lambda,s}$ is mean stable. For mean stable $\tau^{\lambda,s}$, each minimal set is attracting by Lemma 3.8. By Lemma 3.6, the number of (attracting) minimal sets for $\tau^{\lambda,s}$ is at most α . Since the number of minimal sets strictly decreases at s where $\tau^{\lambda,s}$ is not mean stable, it follows that the number of $s \in I$ such that $\tau^{\lambda,s}$ is not mean stable is less than α . \square

We now prove Main Result D, which asserts the measure-theoretic thickness of mean stable MRDSs.

Theorem 4.6 (Main Result D). Let X, Λ, m, I, Φ as in Setting 4.1. Denote by Bif the set of all $(\lambda, s) \in \Lambda \times I$ satisfying that $\tau^{\lambda,s}$ is not mean stable. Besides, we define the sets Bif^λ and Bif_s as follows.

For each $\lambda \in \Lambda$, we denote by Bif^λ the set of all $s \in I$ satisfying that $\tau^{\lambda,s}$ is not mean stable.

For each $s \in I$, we denote by Bif_s the set of all $\lambda \in \Lambda$ satisfying that $\tau^{\lambda,s}$ is not mean stable.

Suppose that there exists $\alpha \in \mathbb{N}$ such that $\#\text{Bif}^\lambda \leq \alpha$ for each $\lambda \in \Lambda$. Then $m(\text{Bif}_s) = 0$ for all but countably many $s \in I$.

Proof. Since (Λ, m) is σ -finite, we may assume that m is a probability measure. We show by contradiction that $\#\{s \in I; m(\text{Bif}_s) > n^{-1}\} \leq n\alpha$. Suppose that there exist mutually distinct elements $s_1, \dots, s_{n\alpha+1} \in I$ such that $m(\text{Bif}_{s_j}) > n^{-1}$ for every $j = 1, \dots, n\alpha + 1$. Then,

$$\alpha + \frac{1}{n} = (n\alpha + 1) \frac{1}{n} < \sum_{j=1}^{n\alpha+1} m(\text{Bif}_{s_j}) = \int_{\Lambda} \#\{s_j \in \text{Bif}^\lambda; j \in \{1, \dots, n\alpha + 1\}\} dm(\lambda) \leq \alpha.$$

By contradiction, we have $\#\{s \in I; m(\text{Bif}_s) > n^{-1}\} \leq n\alpha$. We now let $I_0 = \bigcup_{n \in \mathbb{N}} \{s \in I; m(\text{Bif}_s) > n^{-1}\}$, which is countable. Then we have $m(\text{Bif}_s) = 0$ for every $s \in I \setminus I_0$. \square

By Lemma 4.5 and Theorem 4.6, we have the following corollary.

Corollary 4.7 (Corollary 1.10). Let X, Λ, m, I, Φ as in Setting 4.1. Suppose that there exists $d \in \mathbb{N}$ with $d \geq 2$ such that $2 \leq \deg(g) \leq d$ for each $g \in X$. Then $m(\text{Bif}_s) = 0$ for all but countably many $s \in I$.

By Corollary 4.7, we have the following corollary regarding the quadratic family.

Corollary 4.8. Let X, Λ, m, I, Φ as in Example 4.2. Then we have

$$\text{Leb}(\{c \in \mathbb{C}; \tau^{c,s} \text{ is not mean stable}\}) = 0$$

for all but countably many $s \in [0, \infty)$.

Remark 4.9. Brück, Büger and Reitz in [5] studied such i.i.d. RDSs. They essentially showed that if the center c satisfies $c = 0$, then the bifurcation occurs at $s^* = 1/4$. More precisely, $\tau^{0,s}$ has two minimal sets including $\{\infty\}$ if $0 < s \leq s^*$, and $\tau^{0,s}$ has the only one attracting minimal set $\{\infty\}$ if $s > s^*$. Hence, $\tau^{0,s}$ is not mean stable if and only if $s = 0$ or $1/4$.

Remark 4.10. Note also that $s^* = 1/4$ is the distance between $c = 0$ and the boundary of the celebrated Mandelbrot set. The Mandelbrot set \mathcal{M} is the set of all parameters $c \in \mathbb{C}$ such that the Julia set of $f_c(z) = z^2 + c$ is connected. In general, it is easy to see that if f_c has an attracting cycle in \mathbb{C} , then there uniquely exists $s^* > 0$ such that $\tau^{c,s}$ is not mean stable if and only if $s = 0$ or s^* . We need to investigate the bifurcation of our quadratic RDS thoroughly in the near future.

References

- [1] Ludwig Arnold, Random dynamical systems. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

- [2] Artur Avila, Mikhail Lyubich and Welington de Melo, Regular or stochastic dynamics in real analytic families of unimodal maps. *Invent. Math.* 154 (2003), no. 3, 451–550.
- [3] Artur Avila and Carlos Gustavo Moreira, Statistical properties of unimodal maps: the quadratic family. *Ann. of Math. (2)* 161 (2005), no. 2, 831-881.
- [4] Rainer Brück and Matthias Bürger, Generalized iteration. *Comput. Methods Funct. Theory* 3 (2003), no. 1-2, 201-252.
- [5] Rainer Brück, Matthias Bürger and Stefan Reitz, Random iterations of polynomials of the form $z^2 + c_n$: connectedness of Julia sets. *Ergod. Th. & Dynam. Sys.* 19 (1999), 1221-1231.
- [6] Lennart Carleson and Theodore Gamelin, *Complex Dynamics*. Springer-Verlag, New York, 1993.
- [7] John Fornæss and Nessim Sibony, Random iterations of rational functions. *Ergod. Th. & Dynam. Sys.* 11 (1991), 687-708.
- [8] Johannes Jaerisch and Hiroki Sumi, Pointwise Hölder exponents of the complex analogues of the Takagi function in random complex dynamics. *Adv. Math.* 313 (2017), 839-874.
- [9] Mikhail Lyubich, Almost every real quadratic map is either regular or stochastic. *Ann. of Math. (2)* 156 (2002), no. 1, 1-78.
- [10] Ricardo Mañé, *The Hausdorff dimension of invariant probabilities of rational maps, Dynamical Systems (Valparaiso, 1986) (Lecture Notes in Mathematics vol 1331) (Berlin: Springer) 86-117, 1988.*
- [11] Daniel Mauldin and Mariusz Urbański, *Graph directed Markov systems: Geometry and dynamics of limit sets. Cambridge Tracts in Mathematics 148, Cambridge University Press, Cambridge, 2003.*
- [12] Volker Mayer, Bartłomiej Skorulski and Mariusz Urbański, *Distance expanding random mappings, thermodynamical formalism, Gibbs measures and fractal geometry. Lecture Notes in Mathematics, 2036, Springer, Heidelberg, 2011.*
- [13] Curtis McMullen, *Complex dynamics and renormalization. Annals of Mathematics Studies, 135. Princeton University Press, 1994.*
- [14] John Milnor, *Dynamics in one complex variable, third edition. Annals of Mathematical Studies, 160. Princeton University Press, 2006.*
- [15] Shunsuke Morosawa, Yasuichiro Nishimura, Masahiko Taniguchi and Tetsuo Ueda, *Holomorphic dynamics. Cambridge Studies in Advanced Mathematics, 66. Cambridge University Press, Cambridge, 2000*

- [16] Mitsuhiro Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *Ann. of Math. (2)* 147 (1998), no. 2, 225-267.
- [17] Hiroki Sumi, Random complex dynamics and semigroups of holomorphic maps. *Proc. Lond. Math. Soc. (3)* 102 (2011), no. 1, 50-112.
- [18] Hiroki Sumi, Cooperation principle, stability and bifurcation in random complex dynamics. *Adv. Math.* 245 (2013), 137-181.
- [19] Hiroki Sumi, Negativity of Lyapunov exponents and convergence of generic random polynomial dynamical systems and random relaxed Newton's methods. *Comm. Math. Phys.* 384 (2021), no. 3, 1513-1583.
- [20] Hiroki Sumi and Takayuki Watanabe, Non-i.i.d. random holomorphic dynamical systems and the probability of tending to infinity. *Nonlinearity* 32 (2019) No. 10, 3742-3771.
- [21] Mariusz Urbański and Anna Zdunik, Regularity of Hausdorff measure function for conformal dynamical systems. *Math. Proc. Cambridge Philos. Soc.* 160 (2016), no. 3, 537-563.