

Interaction cohomology of forward or backward self-similar systems*

Hiroki Sumi

Department of Mathematics, Graduate School of Science,
Osaka University

1-1, Machikaneyama, Toyonaka, Osaka, 560-0043, Japan

E-mail: sumi@math.sci.osaka-u.ac.jp

<http://www.math.sci.osaka-u.ac.jp/~sumi/>

Abstract

We investigate the dynamics of forward or backward self-similar systems (iterated function systems) and the topological structure of their invariant sets. We define a new cohomology theory (interaction cohomology) for forward or backward self-similar systems. We show that under certain conditions, the space of connected components of the invariant set is isomorphic to the inverse limit of the spaces of connected components of the realizations of the nerves of finite coverings \mathcal{U} of the invariant set, where each \mathcal{U} consists of (backward) images of the invariant set under elements of finite word length. We give a criterion for the invariant set to be connected. Moreover, we give a sufficient condition for the first cohomology group to have infinite rank. As an application, we obtain many results on the dynamics of semigroups of polynomials. Moreover, we define postunbranched systems and we investigate the interaction cohomology groups of such systems. Many examples are given.

1 Introduction

The theory of iterated function systems has been widely and deeply investigated in fractal geometry ([10, 5, 15, 17, 13, 14]). It deals with systems $\mathfrak{L} = (L, (h_1, \dots, h_m))$, where L is a non-empty compact metric space and $h_j : L \rightarrow L$ is a continuous map for each $j = 1, \dots, m$, such that $L = \bigcup_{j=1}^m h_j(L)$. In this paper, such a system $(L, (h_1, \dots, h_m))$ is called a forward self-similar system (Definition 2.2). For any two forward self-similar systems $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$, a pair $\Lambda = (\alpha, \beta)$, where $\alpha : L_1 \rightarrow L_2$ is a continuous map and $\beta : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ is a map, is called a morphism of \mathfrak{L}_1 to \mathfrak{L}_2 if $g_{\beta(j)} \circ \alpha = \alpha \circ h_j$ on L for each $j = 1, \dots, m$. If Λ is a morphism of \mathfrak{L}_1 to \mathfrak{L}_2 , we write $\Lambda : \mathfrak{L}_1 \rightarrow \mathfrak{L}_2$. If $\Lambda_1 = (\alpha_1, \beta_1) : \mathfrak{L}_1 \rightarrow \mathfrak{L}_2$ and $\Lambda_2 = (\alpha_2, \beta_2) : \mathfrak{L}_2 \rightarrow \mathfrak{L}_3$ are such morphisms, then $\Lambda_2 \circ \Lambda_1 := (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1)$ is a morphism of \mathfrak{L}_1 to \mathfrak{L}_3 . Moreover, for each system $\mathfrak{L} = (L, (h_1, \dots, h_m))$, the morphism $Id_{\mathfrak{L}} = (Id_L, Id) : \mathfrak{L} \rightarrow \mathfrak{L}$ is called the identity morphism. With these notations, we have a category. This is called the category of forward self-similar systems (see Definition 2.3). For any forward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$, the set L is called the invariant set of the system and each h_j is called a generator of the system. In many cases, the invariant set is quite complicated. For example, the Hausdorff dimension of the invariant set may not be an integer ([5, 17]).

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Another famous subject in fractal geometry is the study of Julia sets (where the dynamics are unstable) of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. (For an introduction to complex dynamics, see [1, 18].) The Julia set can be defined for a rational semigroup, i.e., a semigroup of rational maps on $\hat{\mathbb{C}}$ ([11, 8]). For a rational semigroup G , we denote by $F(G)$ the largest open subset of $\hat{\mathbb{C}}$ on which the family of analytic maps G is equicontinuous with respect to the spherical distance. The set $F(G)$ is called the **Fatou set** of G , and the complement $J(G) := \hat{\mathbb{C}} \setminus F(G)$ is called the **Julia set** of G . In [23], it was shown that for a rational semigroup G which is generated by finitely many elements $\{h_1, \dots, h_m\}$, the Julia set $J(G)$ of G satisfies the following backward self-similarity property $J(G) = \bigcup_{j=1}^m h_j^{-1}(J(G))$ (Lemma 2.12). (For additional results on rational semigroups, see [21, 22, 35, 36, 25, 26, 24, 27, 28, 29, 30, 31, 32, 33]. For a software to draw graphics of the Julia sets of rational semigroups, see [4].) We also remark that the study of rational semigroups is directly and deeply related to that of random complex dynamics. (For results on random complex dynamics, see [6, 3, 2, 7, 30, 29, 33, 34].) Based on the above point of view, it is natural to introduce the following “backward self-similar systems.” In this paper, $\mathfrak{L} = (L, (h_1, \dots, h_m))$ is called a backward self-similar system if L is a compact subset of a metric space X , $h_j : X \rightarrow X$ is a continuous map for each $j = 1, \dots, m$, $L = \bigcup_{j=1}^m h_j^{-1}(L)$, and for each $z \in L$ and each j , $h_j^{-1}(\{z\}) \neq \emptyset$ (see Definition 2.4). The category of backward self-similar systems is defined in a similar way to that of forward self-similar systems (Definition 2.5). For a topological manifold M , we investigate how the coordinate neighborhoods overlap to obtain topological or geometric information about M . On the other hand, for the invariant set L of a forward (resp. backward) self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$, we do not have such good coordinate neighborhoods that are homeomorphic to open balls in Euclidian space anymore. However, we have small “copies” (images) $h_{w_1} \cdots h_{w_k}(L)$ (resp. $h_{w_1}^{-1} \cdots h_{w_k}^{-1}(L)$) of L under finite word elements $h_{w_1} \cdots h_{w_k}$. These small copies contain important information on the topology of the invariant set L . For example, we have the following well-known result:

Theorem 1.1 (a weak form of (Theorem 4.6 in [10]) or (Theorem 1.6.2 in [15])).

Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is a contraction. Then, L is connected if and only if for each $i, j \in \{1, \dots, m\}$, there exists a sequence $\{i_t\}_{t=1}^s$ in $\{1, \dots, m\}$ such that $i_1 = i, i_s = j$, and $h_{i_t}(L) \cap h_{i_{t+1}}(L) \neq \emptyset$ for each $t = 1, \dots, s - 1$.

One motivation of this paper is to generalize and further develop the essence of Theorem 1.1. The following is a natural question:

Question 1.2. For a fixed $k \in \mathbb{N}$, we ask in what fashion do the small images $h_{w_1} \cdots h_{w_k}(L)$ (resp. $h_{w_1}^{-1} \cdots h_{w_k}^{-1}(L)$) of L under k -words $h_{w_1} \cdots h_{w_k}$ overlap? How does this vary as k tends to ∞ ?

Here are some other natural questions:

Question 1.3. What can we say about the topological aspects of the invariant set L ? How many connected components does L have? What about the number of connected components of the complement of L when L is embedded in a larger space?

Question 1.4. How can we describe the dynamical complexity of these (forward or backward) self-similar systems? How can we describe the interaction of different kinds of dynamics inside a single (forward or backward) self-similar system? How can we classify the isomorphism classes of forward or backward self-similar systems? How are these questions related to Question 1.2 and 1.3?

These questions are profoundly related to the dynamical behavior of the systems \mathfrak{L} . In this paper, to investigate the above questions, we introduce a new kind of cohomology theory for such systems, which we call “interaction cohomology.” We do this as follows. For each $m \in \mathbb{N}$,

let $\Sigma_m^* := \bigcup_{n=1}^{\infty} \{1, \dots, m\}^n$ (disjoint union). For each $w = (w_1, \dots, w_n) \in \{1, \dots, m\}^n$, we set $|w| := n$ and $\bar{w} := (w_n, \dots, w_1)$. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. For each $w = (w_1, \dots, w_n) \in \Sigma_m^*$, we set $h_w := h_{w_n} \circ \dots \circ h_{w_1}$. Let $\mathcal{U}_k = \mathcal{U}_k(\mathfrak{L})$ be the finite covering of L defined as $\mathcal{U}_k := \{h_{\bar{w}}(L) \mid w \in \Sigma_m^*, |w| = k\}$ (resp. $\mathcal{U}_k := \{h_w^{-1}(L) \mid w \in \Sigma_m^*, |w| = k\}$). Note that for each $k \in \mathbb{N}$, \mathcal{U}_{k+1} is a refinement of \mathcal{U}_k . Let R be a \mathbb{Z} module. Let $N_k = N_k(\mathfrak{L})$ be the nerve of \mathcal{U}_k . Thus N_k is a simplicial complex such that the vertex set is equal to $\{w \in \Sigma_m^* \mid |w| = k\}$ and mutually distinct r elements $w^1, \dots, w^r \in \Sigma_m^*$ with $|w^1| = \dots = |w^r| = k$ make an $(r-1)$ -simplex of N_k if and only if $\bigcap_{j=1}^r h_{\bar{w}^j}(L) \neq \emptyset$ (resp. $\bigcap_{j=1}^r h_{w^j}^{-1}(L) \neq \emptyset$). Let $\varphi_k : N_{k+1} \rightarrow N_k$ be the simplicial map defined as $(w_1, \dots, w_{k+1}) \mapsto (w_1, \dots, w_k)$ for each $(w_1, \dots, w_{k+1}) \in \{1, \dots, m\}^{k+1}$. (For an example of N_k , see Example 2.36, Figure 1, Figure 2.) We consider the cohomology groups $H^r(N_k; R)$. Note that $\{\varphi_k^* : H^r(N_k; R) \rightarrow H^r(N_{k+1}; R)\}_{k \in \mathbb{N}}$ makes a direct system of \mathbb{Z} modules. The interaction cohomology groups $\check{H}^r(\mathfrak{L}; R)$ are defined to be the direct limits $\varinjlim_k H^r(N_k; R)$ (see Definition 2.31, Definition 2.32). Note that $\check{H}^r(\mathfrak{L}; R) \cong \check{H}^r(\varinjlim_k |N_k|; R)$ (see [38]), where for each simplicial complex K , we denote by $|K|$ the realization of K ([20, p.110]). Note also that $\mathfrak{L} \mapsto H^*(N_k(\mathfrak{L}); R)$ and $\mathfrak{L} \mapsto \check{H}^*(\mathfrak{L}; R)$ are contravariant functors from the category of forward (resp. backward) self-similar systems to the category of \mathbb{Z} modules (Remark 2.37). In particular, if $\mathfrak{L}_1 \cong \mathfrak{L}_2$, then $H^*(N_k(\mathfrak{L}_1); R) \cong H^*(N_k(\mathfrak{L}_2); R)$ and $\check{H}^*(\mathfrak{L}_1; R) \cong \check{H}^*(\mathfrak{L}_2; R)$. Thus the isomorphism classes of $H^*(N_k(\mathfrak{L}); R)$ and $\check{H}^*(\mathfrak{L}; R)$ are invariant under the isomorphisms of forward (resp. backward) self-similar systems. We have a natural homomorphism Ψ from the interaction cohomology groups of a system \mathfrak{L} to the Čech cohomology groups $\check{H}^*(L; R)$ of the invariant set L of the system \mathfrak{L} (see Remark 2.41). Note that by the Alexander duality theorem ([20]), for a compact subset K of an oriented n -dimensional manifold X , there exists an isomorphism $\check{H}^p(K; R) \cong H_{n-p}(X, X \setminus K; R)$ (hence if $X = \mathbb{R}^n$ then $\check{H}^p(K; R) \cong \tilde{H}_{n-p-1}(X \setminus K; R)$, where \tilde{H}_* denotes the reduced homology). For a forward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$ such that each $h_j : L \rightarrow L$ is a contraction, Ψ is an isomorphism (see Remark 2.42). However, Ψ is not an isomorphism in general. In fact, Ψ may not even be a monomorphism (see Proposition 3.37). In this paper, we show the following result:

Theorem 1.5 (see Theorems 3.2 and 3.3). *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. Suppose that for each $x \in \{1, \dots, m\}^{\mathbb{N}}$, $\bigcap_{j=1}^{\infty} h_{x_1} \cdots h_{x_j}(L)$ (resp. $\bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(L)$) is connected. Then, we have the following.*

- (1) *There exists a bijection $\text{Con}(L) \cong \varinjlim_k \text{Con}(|N_k|)$, where for each topological space X , we denote by $\text{Con}(X)$ the set of all connected components of X .*
- (2) *L is connected if and only if $|N_1|$ is connected, that is, for each $i, j \in \{1, \dots, m\}$, there exists a sequence $\{i_t\}_{t=1}^s$ in $\{1, \dots, m\}$ such that $i_1 = i, i_s = j$, and $h_{i_t}(L) \cap h_{i_{t+1}}(L) \neq \emptyset$ (resp. $h_{i_t}^{-1}(L) \cap h_{i_{t+1}}^{-1}(L) \neq \emptyset$) for each $t = 1, \dots, s-1$.*
- (3) *Let R be a field. Then, $\sharp \text{Con}(L) < \infty$ if and only if $\dim_R \check{H}^0(\mathfrak{L}; R) < \infty$. If $\sharp \text{Con}(L) < \infty$, then $\Psi : \check{H}^0(\mathfrak{L}; R) \rightarrow \check{H}^0(L; R)$ is an isomorphism.*

Note that Theorem 1.5 (2) generalizes Theorem 1.1. Moreover, note that until now, no research has investigated the space of connected components of the invariant set of such a system; Theorem 1.5 gives us new insight into the topology of the invariant sets of such systems.

Furthermore, a sufficient condition for the rank of the first interaction cohomology groups to be infinite is given (Theorem 3.7, 3.8). More precisely, we show the following result:

Theorem 1.6 (Theorem 3.7). *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system. Let R be a field. We assume all of the following conditions (a), ..., (d):*

- (a) *$|N_1|$ is connected.*

(b) $(h_1^2)^{-1}(L) \cap (\bigcup_{i:i \neq 1} h_i^{-1}(L)) = \emptyset$.

(c) *There exist mutually distinct elements $j_1, j_2, j_3 \in \{1, \dots, m\}$ such that $j_1 = 1$ and such that for each $k = 1, 2, 3$, $h_{j_k}^{-1}(L) \cap h_{j_{k+1}}^{-1}(L) \neq \emptyset$, where $j_4 := j_1$.*

(d) *For each $s, t \in \{1, \dots, m\}$, if $s, t, 1$ are mutually distinct, then $h_1^{-1}(L) \cap h_s^{-1}(L) \cap h_t^{-1}(L) = \emptyset$.*

Then, $\dim_R \check{H}^1(\mathfrak{L}; R) = \infty$.

A similar result is given for forward self-similar systems \mathfrak{L} (Theorem 3.8).

Using Leray's theorem ([9]), we also find a sufficient condition for the natural homomorphism Ψ to be a monomorphism between the first cohomology groups (Lemma 4.8).

The results in the above paragraphs are applied to the study of the dynamics of polynomial semigroups (i.e., semigroups of polynomial maps on $\hat{\mathbb{C}}$). For a polynomial semigroup G , we set $P(G) := \bigcup_{g \in G} \{\text{all critical values of } g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}\}$. We say that a polynomial semigroup G is postcritically bounded if $P(G) \setminus \{\infty\}$ is bounded in \mathbb{C} . For example, if G is generated by a subset of $\{h(z) = cz^a(1-z)^b \mid a, b \in \mathbb{N}, c > 0, c(\frac{a}{a+b})^a(\frac{b}{a+b})^b \leq 1\}$, then G is postcritically bounded (see Remark 3.14 or [31]). Regarding the dynamics of postcritically bounded polynomial semigroups, there are many new and interesting phenomena which cannot hold in the dynamics of a single polynomial ([31, 32, 33, 29]). Combining Theorem 1.5 (Theorem 3.2) with potential theory, we show the following result:

Theorem 1.7 (Theorem 3.17). *Let $m \in \mathbb{N}$ and for each $j = 1, \dots, m$, let $h_j : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a polynomial map with $\deg(h_j) \geq 2$. Let G be the polynomial semigroup generated by $\{h_1, \dots, h_m\}$. Suppose that G is postcritically bounded. Then, for the backward self-similar system $\mathfrak{L} = (J(G), (h_1, \dots, h_m))$, all of the statements (1), (2), and (3) in Theorem 1.5 hold.*

Moreover, combining Theorem 1.6 (Theorem 3.7), Theorem 1.7 (Theorem 3.17), the Riemann-Hurwitz formula ([1, 18]), Leray's theorem ([9]), and the Alexander duality theorem ([20]), we give a sufficient condition for the Fatou set (where the dynamics are stable) of a postcritically bounded polynomial semigroup G to have infinitely many connected components (Theorem 3.19). More precisely, we show the following result:

Theorem 1.8 (Theorem 3.19). *Let $m \in \mathbb{N}$ and for each $j = 1, \dots, m$, let $h_j : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a polynomial map with $\deg(h_j) \geq 2$. Let G be the polynomial semigroup generated by $\{h_1, \dots, h_m\}$. Suppose that G is postcritically bounded. Moreover, regarding the backward self-similar system $\mathfrak{L} = (J(G), (h_1, \dots, h_m))$, suppose that all of the conditions (a), (b), (c), and (d) in the assumptions of Theorem 1.6 hold. Let R be a field. Then, we have that $\dim_R \check{H}^1(\mathfrak{L}; R) = \dim_R \Psi(\check{H}^1(\mathfrak{L}; R)) = \infty$, $\Psi : \check{H}^1(\mathfrak{L}; R) \rightarrow \check{H}^1(J(G); R)$ is a monomorphism, and the Fatou set $F(G)$ of G has infinitely many connected components.*

Moreover, we give an example of a finitely generated postcritically bounded polynomial semigroup $G = \langle h_1, \dots, h_m \rangle$ such that the backward self-similar system $\mathfrak{L} = (J(G), (h_1, \dots, h_m))$ satisfies the assumptions of Theorem 1.8 and the rank of the first interaction cohomology group of \mathfrak{L} is infinite (Proposition 3.20, Figure 8).

Theorem 1.5 and Theorem 1.7 have many applications. In fact, using the connectedness criterion for the Julia set of a postcritically bounded polynomial semigroup (Theorem 1.7), we investigate the space of postcritically bounded polynomial semigroups having 2 generators ([29]). As a result of this investigation, we can obtain numerous results on random complex dynamics. Indeed, letting $T_\infty(z)$ denote the probability of the orbit under a seed value $z \in \hat{\mathbb{C}}$ tending to ∞ under the random walk generated by the application of randomly selected polynomials from the set $\{h_1, h_2\}$, we can show that in some parameter space, the function T_∞ is continuous on $\hat{\mathbb{C}}$ and varies only on the Julia set $J(G)$ of the corresponding polynomial semigroup G generated by $\{h_1, h_2\}$. In this

case, the Julia set $J(G)$ is a very thin fractal set. Moreover, we can show that in some parameter region Λ , the Julia set $J(G)$ has uncountably many connected components, and in the boundary $\partial\Lambda$, the Julia set $J(G)$ is connected. This implies that the function T_∞ on $\hat{\mathbb{C}}$ is a complex analog of the Cantor function or Lebesgue's singular function. (These results have been announced in [29, 30]. See also [34].)

When we investigate a random complex dynamical system, it is important to know the topology of the Julia set and the Fatou set of the associated semigroup. Indeed, setting $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \rightarrow \mathbb{C} \mid \varphi \text{ is continuous}\}$, for a general random complex dynamical system \mathcal{D} , under certain conditions, any unitary eigenvector $\varphi \in C(\hat{\mathbb{C}})$ of the transition operator M of \mathcal{D} is locally constant on the Fatou set of the associated semigroup (see [34]). Thus Theorem 1.8 provides us important information of unitary eigenvectors of M . Moreover, by [34], the space V of all finite linear combinations of unitary eigenvectors of M is finite-dimensional, and for any $\varphi \in C(\hat{\mathbb{C}})$, $\{M^n(\varphi)\}_n$ tends to the finite-dimensional subspace V .

Another area of interest in forward or backward self-similar systems $\mathfrak{L} = (L, (h_1, \dots, h_m))$ is the structure of the cohomology groups $H^r(N_k; R)$ of the nerve N_k of \mathcal{U}_k and the growth rate $g^r(\mathfrak{L})$ of the rank $a_{r,k}$ of $H^r(N_k; R)$ as k tends to ∞ , where R is a field. (See Definition 2.31, Definition 2.32, Definition 3.34.) The above invariants are deeply related to the dynamical complexity of \mathfrak{L} . In section 3.3, we introduce ‘‘postunbranched’’ systems (see Definition 3.22), and we show the following result:

Theorem 1.9 (for the precise statement, see Theorem 3.36). *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Suppose that \mathfrak{L} is postunbranched. When \mathfrak{L} is a forward self-similar system, we assume further that $h_j : L \rightarrow L$ is injective for each $j = 1, \dots, m$. Let R be a field. Then, we have the following.*

(1) For each $r \geq 2$, there exists an exact sequence of R modules:

$$0 \longrightarrow H^r(N_1; R) \longrightarrow \check{H}^r(\mathfrak{L}; R) \longrightarrow \bigoplus_{j=1}^m \check{H}^r(\mathfrak{L}; R) \longrightarrow 0.$$

(2) If $r \geq 2$, or if $r = 1$ and $|N_1|$ is connected, then $a_{r,k+1} = ma_{r,k} + a_{r,1}$ for each $k \in \mathbb{N}$.

(3) $a_{0,k+1} = ma_{0,k} - m + a_{0,1} - a_{1,1} - ma_{1,k} + a_{1,k+1}$ for each $k \in \mathbb{N}$.

(4) $ma_{1,k} \leq a_{1,k+1} \leq ma_{1,k} + a_{1,1}$ and $ma_{0,k} - m + a_{0,1} - a_{1,1} \leq a_{0,k+1} \leq ma_{0,k} - m + a_{0,1}$ for each $k \in \mathbb{N}$.

(5) (a) If $r \geq 1$, then $g^r(\mathfrak{L}) \in \{-\infty, \log m\}$. (b) $g^0(\mathfrak{L}) \in \{0, \log m\}$.

(6) Let $r \geq 1$. Then, $\dim_R \check{H}^r(\mathfrak{L}; R)$ is either 0 or ∞ .

(7) Suppose $m \geq 2$. Then,

$$\dim_R \check{H}^0(\mathfrak{L}; R) \in \{x \in \mathbb{N} \mid a_{0,1} \leq x \leq \frac{1}{m-1}(m - a_{0,1} + a_{1,1})\} \cup \{\infty\}.$$

Moreover, for any $n \in \mathbb{N} \cup \{0\}$, we give an example of a postunbranched backward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_{n+2}))$ such that $L \subset \mathbb{C}$ and the rank of the n -th interaction cohomology group $\check{H}^n(\mathfrak{L}; R)$ of \mathfrak{L} is equal to ∞ (Proposition 3.37). In this case, if $n \geq 2$, the natural homomorphism $\Psi : \check{H}^n(\mathfrak{L}; R) \rightarrow \check{H}^n(L; R)$ is not a monomorphism, since for each $l \geq 2$ the Čech cohomology group $\check{H}^l(L; R)$ of L is equal to zero. For any $n \in \mathbb{N} \cup \{0\}$, we also give an example of a postunbranched forward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_{n+2}))$ such that $L \subset \mathbb{R}^3$, each $h_j : L \rightarrow L$ is injective, and the rank of the n -th interaction cohomology group of \mathfrak{L} is equal to ∞ (Proposition 3.37). In this case, if $n \geq 3$, the natural homomorphism $\Psi : \check{H}^n(\mathfrak{L}; R) \rightarrow \check{H}^n(L; R)$ is not a monomorphism, since for each $l \geq 2$ the Čech cohomology group $\check{H}^l(L; R)$ of L is equal to zero. We

remark that these examples imply that the interaction cohomology groups of $\mathfrak{L} = (L, (h_1, \dots, h_m))$ may contain more (dynamical) information than the Čech cohomology groups of the invariant sets L . Thus interaction cohomology groups of self-similar systems tell us information of dynamical behavior of the systems as well as the topological information of the invariant sets of the systems.

Furthermore, we give many ways to construct examples of postunbranched systems (Lemmas 3.23, 3.24, 3.25, 3.26). From these, we see that if L is one of the Sierpiński gasket, the snowflake, the pentakun, the heptakun, the octakun, and so on ([15]), then there exists a postunbranched forward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$ such that each $h_j : L \rightarrow L$ is an injective contraction (Examples 3.27, 3.28). Moreover, we also see that for each $n \in \mathbb{N}$, any subsystem of an n -th iterate of the above \mathfrak{L} is a postunbranched forward self-similar system (Examples 3.27, 3.28).

We summarize the purpose and the virtue to introduce interaction cohomology groups for the study of self-similar systems $\mathfrak{L} = (L, \{h_1, \dots, h_m\})$ as follows.

- (1) We can get information about the dynamical behavior of the system \mathfrak{L} and the interaction of different maps in the system. The cohomology groups $\check{H}^r(\mathfrak{L}; R)$, the cohomology groups $H^r(N_k; R)$ of the nerve of \mathcal{U}_k , and the growth rate $g^r(\mathfrak{L})$ of the rank $a_{r,k}$ of $H^r(N_k; R)$ are new invariants for the dynamics of self-similar systems. These invariants reflect the dynamical behavior and the complexity of the systems. Under certain conditions, we can show, by using these invariants, that two self-similar systems are not isomorphic, even when we cannot show this by using Čech cohomology groups of the invariant sets of the systems (e.g., Examples 3.40, 3.41, 3.42, 3.43, Proposition 3.37, and the proof of Proposition 3.37). Moreover, the interaction cohomology groups $\check{H}^r(\mathfrak{L}; R)$ are new invariants for the dynamics of finitely generated semigroups of continuous maps. (See Remark 2.38.)
- (2) By using the natural homomorphism $\Psi : \check{H}^r(\mathfrak{L}; R) \rightarrow \check{H}^r(L; R)$, we can get information about the Čech cohomology groups $\check{H}^r(L; R)$ of the invariant sets L (Lemma 4.8, Theorems 3.2, 3.3, 3.7, 3.8, 3.17, 3.19, Proposition 3.20, Theorem 3.36, Proposition 3.38, Examples 3.40, 3.41, 3.42, 3.43, 3.44). By using the Alexander duality theorem, the Čech cohomology groups of L tell us information of the (reduced) homology groups of the complement of L in the bigger space (see Theorem 3.19, Proposition 3.20). Under certain conditions, Ψ is a monomorphism (Lemma 4.8). If \mathfrak{L} is a forward self-similar system and if each $h_j : L \rightarrow L$ is contractive, then $\Psi : \check{H}^r(\mathfrak{L}; R) \rightarrow \check{H}^r(L; R)$ is an isomorphism (Remarks 2.41, 2.42). Moreover, under the same condition, for each $w \in \Sigma_m$ and $x_0 \in L$ such that $x_0 \in h_{w_1} \cdots h_{w_k}(L)$ for each k , the interaction homotopy groups $\check{\pi}_r(\mathfrak{L}, w)$ (see Definition 2.31) of \mathfrak{L} are isomorphic to the shape groups $\check{\pi}_r(L, x_0)$ of the invariant set L (see Remark 2.42). (For the definition of shape groups and shape theory, see [16].)
- (3) Interaction cohomology groups $\check{H}^r(\mathfrak{L}; R)$ and $H^r(N_k; R)$ may have more dynamical information of the systems than the Čech cohomology groups or shape groups of the invariant sets. The natural homomorphism $\Psi : \check{H}^r(\mathfrak{L}; R) \rightarrow \check{H}^r(L; R)$ is not an isomorphism in general. Similarly, $\check{\pi}_r(\mathfrak{L}, w)$ and $\check{\pi}_r(L, x_0)$ ($x_0 \in h_{w_1} \cdots h_{w_k}(L)$ for each $k \in \mathbb{N}$) are not isomorphic in general. It may happen that $\check{\pi}_1(\mathfrak{L}, w)$ is not trivial, $\check{H}_1(\mathfrak{L}; \mathbb{Z}) \neq 0$ and $\check{H}^1(\mathfrak{L}; \mathbb{Z}) \neq 0$, even though $\check{\pi}_r(L, x_0)$, $\check{H}_r(L; \mathbb{Z})$, and $\check{H}^r(L; \mathbb{Z})$ are trivial for all $r \geq 1$ and for all (w, x_0) such that $x_0 \in h_{w_1} \cdots h_{w_k}(L)$ for each $k \in \mathbb{N}$ (see Example 4.9). Moreover, for each $n \geq 4$, there are many examples $\mathfrak{L} = (L, (h_1, \dots, h_m))$ (of postunbranched systems) such that $L \subset \mathbb{R}^3$ and $\check{H}^r(L; R) = 0$ for each $r \geq 3$, but the interaction cohomology group $\check{H}^n(\mathfrak{L}; R)$ is not zero (Proposition 3.37). In these examples, since $\check{H}^*(\mathfrak{L}; R) \cong \check{H}^*(\varprojlim_k |N_k(\mathfrak{L})|; R)$, the above statement means that the dimension of $\varprojlim_k |N_k(\mathfrak{L})|$ is larger than that of L . In other words, the manner of overlapping of small images of L is “more higher-dimensional” than the invariant set L . These phenomena reflect the complexity of the dynamics of the self-similar systems. We remark that as illustrated in Remark 2.14, Remark 2.25, and examples in section 2, it is important to investigate self-similar systems whose generators may not be contractive.

- (4) For any two self-similar systems $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$, interaction cohomology groups $\check{H}^r(\mathfrak{L}_1; R)$ and $\check{H}^r(\mathfrak{L}_2; R)$ may not be isomorphic even when L_1 and L_2 are homeomorphic (see Example 4.9, Example 4.10, and Remark 4.11).
- (5) For any forward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$ whose generators h_j are contractions, the interaction cohomology groups $\check{H}^r(\mathfrak{L}; R)$ are isomorphic to the Čech cohomology groups $\check{H}^r(L; R)$ (Remark 2.42). Thus in this case the interaction cohomology groups $\check{H}^r(\mathfrak{L}; R)$ are not new invariants. However, even for such systems, $H^r(N_k; R)$ and $g^r(\mathfrak{L})$ are new invariants. Given two forward self-similar systems \mathfrak{L}_1 and \mathfrak{L}_2 whose generators are contractions, $H^r(N_k(\mathfrak{L}_1); R)$ and $H^r(N_k(\mathfrak{L}_2); R)$ may not be isomorphic, and $g^r(\mathfrak{L}_1)$ and $g^r(\mathfrak{L}_2)$ may be different, and these invariants $H^r(N_k; R)$ and g^r may tell us that \mathfrak{L}_1 and \mathfrak{L}_2 are not isomorphic, even when the Čech cohomology groups of their invariants are isomorphic (Examples 3.40, 3.41, 3.42, 3.43).
- (6) Under certain good conditions (e.g., postunbranched condition (Definition 3.22)), $\check{H}^r(\mathfrak{L}; R)$, $H^r(N_k; R)$ and $g^r(\mathfrak{L})$ can be exactly calculated for each $r \geq 0$ (Theorem 3.36). In particular, there are inductive formulae for $H^r(N_k; R)$ with respect to k (with an exception when $r = 1$ and $|N_1|$ is disconnected, in which the situation is more complicated, see Theorem 3.36-9, Proposition 3.38 and Remark 3.39). These results are applicable to many famous examples (e.g., the snowflake, the Sierpiński gasket, the pentakun, the hexakun, the heptakun, the octakun, etc., and any subsystems of their iteration systems, see Examples 3.27, 3.28, 3.40, 3.41, 3.42, 3.43) and many new examples (see Propositions 3.37, 3.38). For one of the keys in the proof of the above results, see the diagrams (19), (20), which come from the long exact sequences of cohomology groups.
- (7) We can apply the results on the interaction cohomology to the self-similar systems whose generators are contractions, to the dynamics of polynomial (rational) maps on the Riemann sphere, and to the random complex dynamics (section 3.2). There are many important examples of forward or backward self-similar systems (section 2). When we investigate the random complex dynamics, we have to see the minimal sets (which are forward invariant) and the Julia sets (which are backward invariant) of the associated semigroups (Lemma 2.24, Remark 2.25). We have many phenomena which can hold in rational semigroups and random complex dynamics, but cannot hold in the usual iteration dynamics of a single polynomial (rational) map (see [31, 33, 34]). By using interaction cohomology, these interesting new phenomena can be systematically investigated (see [29]).
- (8) Given self-similar systems or iterated function systems, we have been investigating the (fractal) dimensions of the invariant sets by using analysis or ergodic theory so far. However, if the small copies overlap heavily, it is very difficult to analyze the fractal dimensions, and there has been no invariant to study or classify the self-similar systems which could be understood well. Interaction cohomology of the systems is a new interest, rather than fractal dimensions of the invariant sets, and can be a new strong research interest in the field of self-similar systems, iterated function systems, the dynamics of semigroups of holomorphic maps, random complex dynamics, and more general semigroup actions. Overlapping of the small copies of the invariant sets is the most difficult point in the study of self-similar systems. Nevertheless, by using the interaction cohomology, we positively study the overlapping of the small copies, rather than avoiding the difficulty. Interaction cohomology provides a new language to investigate self-similar systems. For results when we have overlapping of small copies, see Theorems 3.2, 3.3, 3.7, 3.8, 3.17, 3.19, and Proposition 3.20.

We remark that it is a new idea to use homological theory when we investigate self-similar systems (iterated function systems) and their invariant sets (fractal sets). Using homological theory,

we can introduce many new topological invariants of self-similar systems. Those invariants are naturally and deeply related to the dynamical behavior of the systems and the topological properties of the invariant sets of the systems. Thus, developing the theory of “interaction (co)homology,” we can systematically investigate the dynamics of self-similar systems. The results are applicable to fractal geometry, the dynamics of rational semigroups, and random complex dynamics.

In section 2, we give some basic notations and definitions on forward or backward self-similar systems. In section 3, we present the main results of this paper. We provide some fundamental tools to prove the main results in section 4 and present the proofs of the main results in section 5.

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2 Preliminaries

In this section, we give some fundamental notations and definitions on forward or backward self-similar systems. We sometimes use the notation from [20].

Definition 2.1. If a semigroup G is generated by a family $\{h_1, \dots, h_m\}$ of elements of G , then we write $G = \langle h_1, \dots, h_m \rangle$.

Definition 2.2. Let (L, d) be a non-empty compact metric space. Let $h_j : L \rightarrow L$ ($j = 1, \dots, m$) be a continuous map. We say that $\mathfrak{L} = (L, (h_1, \dots, h_m))$ is a **forward self-similar system** if $L = \bigcup_{j=1}^m h_j(L)$. The set L is called the invariant set of \mathfrak{L} . Each h_j is called a generator of \mathfrak{L} .

Definition 2.3. Let $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$ be any two forward self-similar systems. A pair $\Lambda = (\alpha, \beta)$, where $\alpha : L_1 \rightarrow L_2$ is a continuous map and $\beta : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ is a map, is called a morphism of \mathfrak{L}_1 to \mathfrak{L}_2 if $g_{\beta(j)} \circ \alpha = \alpha \circ h_j$ on L for each $j = 1, \dots, m$. If Λ is a morphism of \mathfrak{L}_1 to \mathfrak{L}_2 , we write $\Lambda : \mathfrak{L}_1 \rightarrow \mathfrak{L}_2$. If $\Lambda_1 = (\alpha_1, \beta_1) : \mathfrak{L}_1 \rightarrow \mathfrak{L}_2$ and $\Lambda_2 = (\alpha_2, \beta_2) : \mathfrak{L}_2 \rightarrow \mathfrak{L}_3$ are two morphisms, then $\Lambda_2 \circ \Lambda_1 := (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1)$ is a morphism of \mathfrak{L}_1 to \mathfrak{L}_3 . Moreover, for each system $\mathfrak{L} = (L, (h_1, \dots, h_m))$, the morphism $Id_{\mathfrak{L}} = (Id_L, Id) : \mathfrak{L} \rightarrow \mathfrak{L}$ is called the identity morphism. With these notations, we have a category. This is called the category of forward self-similar systems.

Definition 2.4. Let X be a metric space. Let $h_j : X \rightarrow X$ ($j = 1, \dots, m$) be a continuous map. Let L be a non-empty compact subset of X . We say that $\mathfrak{L} = (L, (h_1, \dots, h_m))$ is a **backward self-similar system** if (1) $L = \bigcup_{j=1}^m h_j^{-1}(L)$ and (2) for each $z \in L$ and each $j \in \{1, \dots, m\}$, $h_j^{-1}(\{z\}) \neq \emptyset$. The set L is called the invariant set of \mathfrak{L} . Each h_j is called a generator of \mathfrak{L} .

Definition 2.5. Let $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$ be any two backward self-similar systems. A pair $\Lambda = (\alpha, \beta)$, where $\alpha : L_1 \rightarrow L_2$ is a continuous map and $\beta : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ is a map, is called a morphism of \mathfrak{L}_1 to \mathfrak{L}_2 if $\alpha(h_j^{-1}(L_1)) \subset g_{\beta(j)}^{-1}(L_2)$ and $g_{\beta(j)} \circ \alpha = \alpha \circ h_j$ on $h_j^{-1}(L_1)$ for each $j = 1, \dots, m$. If Λ is a morphism of \mathfrak{L}_1 to \mathfrak{L}_2 , we write $\Lambda : \mathfrak{L}_1 \rightarrow \mathfrak{L}_2$. If $\Lambda_1 = (\alpha_1, \beta_1) : \mathfrak{L}_1 \rightarrow \mathfrak{L}_2$ and $\Lambda_2 = (\alpha_2, \beta_2) : \mathfrak{L}_2 \rightarrow \mathfrak{L}_3$ are two morphisms, then $\Lambda_2 \circ \Lambda_1 := (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1)$ is a morphism of \mathfrak{L}_1 to \mathfrak{L}_3 . Moreover, for each system $\mathfrak{L} = (L, (h_1, \dots, h_m))$, the morphism $Id_{\mathfrak{L}} = (Id_L, Id) : \mathfrak{L} \rightarrow \mathfrak{L}$ is called the identity morphism. With these notations, we have a category. This is called the category of backward self-similar systems.

Remark 2.6. Let $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$ be two forward (resp. backward) self-similar systems. By Definition 2.3 and 2.5, \mathfrak{L}_1 is isomorphic to \mathfrak{L}_2 (indicated by $\mathfrak{L}_1 \cong \mathfrak{L}_2$) if and only if $m = n$ and there exists a homeomorphism $\alpha : L_1 \rightarrow L_2$ and a bijection $\tau : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ such that for each $j = 1, \dots, m$, $\alpha h_j = g_{\tau(j)} \alpha$ on L_1 (resp. $\alpha(h_j^{-1}(L_1)) \subset g_{\tau(j)}^{-1}(L_2)$ and $\alpha h_j = g_{\tau(j)} \alpha$ on $h_j^{-1}(L_1)$).

We give several examples of forward or backward self-similar systems.

Definition 2.7. Let (X, d) be a metric space. We say that a map $f : X \rightarrow X$ is contractive (with respect to d) if there exists a number $0 < s < 1$ such that for each $x, y \in X$, $d(f(x), f(y)) \leq sd(x, y)$. A contractive map $f : X \rightarrow X$ is called a contraction.

Definition 2.8. Let (X, d) be a complete metric space. For each $i = 1, \dots, m$, let $h_i : X \rightarrow X$ be a contraction with respect to d . By [15, Theorem 1.1.4], there exists a unique non-empty compact subset M of X such that $(M, (h_1, \dots, h_m))$ is a forward self-similar system. We denote this set M by $M_X(h_1, \dots, h_m)$. The set $M_X(h_1, \dots, h_m)$ is called the **attractor** or **invariant set** of the iterated function system $\{h_1, \dots, h_m\}$ on X .

Definition 2.9. Let X be a compact metric space. Let G be a semigroup of continuous maps on X . We set $F(G) := \{z \in X \mid G \text{ is equicontinuous on a neighborhood of } z\}$. For the definition of equicontinuity, see [1]. The set $F(G)$ is called the **Fatou set** of G . Moreover, we set $J(G) := X \setminus F(G)$. The set $J(G)$ is called the **Julia set** of G . Furthermore, for a continuous map $g : X \rightarrow X$, we set $F(g) := F(\langle g \rangle)$ and $J(g) := J(\langle g \rangle)$.

Remark 2.10. By the definition above, we have that $F(G)$ is open and $J(G)$ is compact.

By the definition above, it is easy to prove that the following Lemmas 2.11 and 2.12 hold.

Lemma 2.11. *Let X be a compact metric space. Let G be a semigroup of continuous maps on X . Suppose that for each $h \in G$, $h : X \rightarrow X$ is an open map. Then, for each $h \in G$, $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$.*

Lemma 2.12. *Let X be a compact metric space. Let G be a semigroup of continuous maps on X . Suppose that G is generated by a finite family $\{h_1, \dots, h_m\}$ of continuous maps on X . Suppose that for each $j = 1, \dots, m$, $h_j : X \rightarrow X$ is an open and surjective map. Moreover, suppose $J(G) \neq \emptyset$. Then, $\mathcal{L} := (J(G), (h_1, \dots, h_m))$ is a backward self-similar system.*

Definition 2.13 ([11, 8]). We denote by $\hat{\mathbb{C}}$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on $\hat{\mathbb{C}}$ with the semigroup operation being the functional composition. A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps on $\hat{\mathbb{C}}$.

Remark 2.14. If a rational semigroup G is generated by $\{h_1, \dots, h_m\}$ and if $J(G) \neq \emptyset$, then by Lemma 2.12, $(J(G), (h_1, \dots, h_m))$ is a backward self-similar system.

Remark 2.15. For each $j = 1, \dots, m$, let $a_j \in \mathbb{C}$ with $|a_j| > 1$ and let $p_j \in \mathbb{C}$. Moreover, let $h_j : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the map defined by $h_j(z) = a_j(z - p_j) + p_j$ for each $z \in \mathbb{C}$. Let $G = \langle h_1, \dots, h_m \rangle$. Then, it is easy to see $\infty \in F(G)$. Hence $\emptyset \neq J(G) \subset \mathbb{C}$. From Lemma 2.12 and [15, Theorem 1.1.4], it follows that $J(G) = M_{\mathbb{C}}(h_1^{-1}, \dots, h_m^{-1})$.

Definition 2.16 ([11]). Let G be a polynomial semigroup. We denote by $K_1(G)$ the set of points $z \in \mathbb{C}$ satisfying that there exists a sequence $\{g_j\}_{j \in \mathbb{N}}$ of mutually distinct elements of G such that $\{g_j(z)\}_{j \in \mathbb{N}}$ is bounded in \mathbb{C} . Moreover, we set $K(G) := \overline{K_1(G)}$, where the closure is taken in $\hat{\mathbb{C}}$. The set $K(G)$ is called the **filled-in Julia set** of G . Furthermore, for a polynomial g , we set $K(g) := K(\langle g \rangle)$.

Remark 2.17. It is easy to see that for each $g \in G$, $g^{-1}(K(G)) \subset K(G)$. Moreover, if a polynomial semigroup G is generated by a finite family $\{h_1, \dots, h_m\}$ and if $K(G) \neq \emptyset$, then $\mathcal{L} = (K(G), (h_1, \dots, h_m))$ is a backward self-similar system ([24, Remark 3]). Furthermore, it is easy to see that if G is generated by finitely many elements $h_j, j = 1, \dots, m$ such that $\deg(h_j) \geq 2$ for each j , then $\emptyset \neq K(G) \subset \mathbb{C}$.

Definition 2.18. For each $m \in \mathbb{N}$, we set $\Sigma_m := \{1, \dots, m\}^{\mathbb{N}}$ endowed with the product topology. Note that Σ_m is a compact metric space. Moreover, we set $\Sigma_m^* := \bigcup_{j=1}^{\infty} \{1, \dots, m\}^j$ (disjoint union). Let X be a space and for each $j = 1, \dots, m$, let $h_j : X \rightarrow X$ be a map. For a finite word $w = (w_1, \dots, w_k) \in \{1, \dots, m\}^k$, we set $h_w := h_{w_k} \circ \dots \circ h_{w_1}$, $\bar{w} = (w_k, w_{k-1}, \dots, w_1)$, and $|w| := k$. For an element $w \in \Sigma_m$, we set $|w| = \infty$. For an element $w \in \Sigma_m \cup \Sigma_m^*$, $|w|$ is called the word length of w . Moreover, for any $w = (w_1, w_2, \dots) \in \Sigma_m \cup \Sigma_m^*$ and any $l \in \mathbb{N}$ with $l \leq |w|$, we set $w|l := (w_1, w_2, \dots, w_l) \in \{1, \dots, m\}^l$. Furthermore, for any $w = (w_1, \dots, w_k) \in \Sigma_m^*$ and any $\tau = (\tau_1, \tau_2, \dots) \in \Sigma_m^* \cup \Sigma_m$, we set $w\tau = (w_1, w_2, \dots, w_k, \tau_1, \tau_2, \dots) \in \Sigma_m^* \cup \Sigma_m$.

Definition 2.19. Let K be a non-empty compact metric space and let $h_j : K \rightarrow K$ be a continuous map for each $j = 1, \dots, m$. We set

$$R_{K,f}(h_1, \dots, h_m) := \bigcap_{n=1}^{\infty} \bigcup_{w \in \Sigma_m^* : |w|=n} h_{\bar{w}}(K).$$

Lemma 2.20. Under Definition 2.19, we have that $R_{K,f}(h_1, \dots, h_m)$ is non-empty and compact, $R_{K,f}(h_1, \dots, h_m) = \bigcup_{w \in \Sigma_m} \bigcap_{k=1}^{\infty} h_{w|k}(K)$, and $\mathfrak{L} := (R_{K,f}(h_1, \dots, h_m), (h_1, \dots, h_m))$ is a forward self-similar system.

Proof. It is easy to see that $R_{K,f}(h_1, \dots, h_m)$ is non-empty and compact. Moreover, it is easy to see that $R_{K,f}(h_1, \dots, h_m) \supset \bigcup_{w \in \Sigma_m} \bigcap_{k=1}^{\infty} h_{w|k}(K)$. To show the opposite inclusion, let $x \in R_{K,f}(h_1, \dots, h_m)$. Then for each $n \in \mathbb{N}$ there exists a word $w^n \in \Sigma_m^*$ with $|w^n| = n$ and a point $y_n \in K$ such that $x = h_{w_1^n} \dots h_{w_n^n}(y_n)$. Then, there exists an infinite word $w^\infty \in \Sigma_m$ and a sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers with $n_k > k$ such that for each $k \in \mathbb{N}$, $w^{n_k}|k = w^\infty|k$. Hence, for each $k \in \mathbb{N}$, $x = h_{w_1^\infty} \dots h_{w_k^\infty} h_{w_{k+1}^{n_k}} \dots h_{w_{n_k}^{n_k}}(y_{n_k})$. Therefore, $x \in \bigcap_{k=1}^{\infty} h_{w|k}(K)$. Thus, we have shown $R_{K,f}(h_1, \dots, h_m) = \bigcup_{w \in \Sigma_m} \bigcap_{k=1}^{\infty} h_{w|k}(K)$. From this formula, it is easy to see that $R_{K,f}(h_1, \dots, h_m) \supset \bigcup_{j=1}^m h_j(R_{K,f}(h_1, \dots, h_m))$. In order to show the opposite inclusion, let $x \in R_{K,f}(h_1, \dots, h_m) = \bigcup_{w \in \Sigma_m} \bigcap_{k=1}^{\infty} h_{w|k}(K)$ be a point. Let $w \in \Sigma_m$ be an element such that $x \in \bigcap_{k=1}^{\infty} h_{w|k}(K)$. Then for each $k \in \mathbb{N}$ with $k \geq 2$, there exists a point $y_k \in h_{w_2} \dots h_{w_k}(K)$ such that $x = h_{w_1}(y_k)$. Since K is a compact metric space, there exists a subsequence $\{y_{k_l}\}_{l \in \mathbb{N}}$ of $\{y_k\}_{k \in \mathbb{N}}$ and a point $y_{k_\infty} \in K$ such that $y_{k_l} \rightarrow y_{k_\infty}$ as $l \rightarrow \infty$. Then, it is easy to see that $y_{k_\infty} \in \bigcap_{i=2}^{\infty} h_{w_i}(K)$. Hence, $x = h_{w_1}(y_{k_\infty}) \in h_{w_1}(\bigcup_{\tau \in \Sigma_m} \bigcap_{k=1}^{\infty} h_{\tau|k}(K))$. Thus, we have proved Lemma 2.20. \square

Definition 2.21. Let X be a metric space and let $h_j : X \rightarrow X$ be a continuous map for each $j = 1, \dots, m$. Let K be a compact subset of X and suppose that for each $z \in K$ and $j = 1, \dots, m$, we have $h_j^{-1}(\{z\}) \neq \emptyset$. Moreover, suppose that $\bigcup_{j=1}^m h_j^{-1}(K) \subset K$. Then we set $R_{K,b}(h_1, \dots, h_m) := \bigcap_{n=1}^{\infty} \bigcup_{w \in \Sigma_m^* : |w|=n} h_w^{-1}(K)$.

Using the argument similar to that in the proof of Lemma 2.20, we can easily prove the following lemma.

Lemma 2.22. Under Definition 2.21, we have that $R_{K,b}(h_1, \dots, h_m)$ is non-empty and compact, $R_{K,b}(h_1, \dots, h_m) = \bigcup_{w \in \Sigma_m} \bigcap_{k=1}^{\infty} h_{w|k}^{-1}(K)$, and $\mathfrak{L} := (R_{K,b}(h_1, \dots, h_m), (h_1, \dots, h_m))$ is a backward self-similar system.

Definition 2.23. Let X be a compact metric space and let G be a semigroup of continuous maps on X . A non-empty compact subset M of X is said to be minimal for (G, X) if M is minimal with respect to the inclusion in the space of all non-empty compact subsets K of X satisfying that for each $g \in G$, $g(K) \subset K$.

Lemma 2.24. Let X be a compact metric space and let G be a semigroup of continuous maps on X . Then, we have the following.

1. Let K be a non-empty compact subset of X such that for each $g \in G$, $g(K) \subset K$. Then, there exists a minimal set L for (G, X) such that $L \subset K$.
2. If, in addition to the assumptions of our lemma, G is generated by a finite family $\{h_1, \dots, h_m\}$ of continuous maps on X , then for any minimal set M for (G, X) , $(M, (h_1, \dots, h_m))$ is a forward self-similar system.

Proof. Statement 1 easily follows from Zorn's lemma. In order to show statement 2, suppose that $G = \langle h_1, \dots, h_m \rangle$ and M is a minimal set for (G, X) . Since M satisfies that $g(M) \subset M$ for each $g \in G$, we have $\bigcup_{j=1}^m h_j(M) \subset M$. Let $K := \bigcup_{j=1}^m h_j(M)$. Since $G = \langle h_1, \dots, h_m \rangle$, we have that for each $g \in G$, $g(K) \subset K$. Thus, by statement 1, there exists a minimal set L for (G, X) such that $L \subset K$. By the minimality of M , it must hold that $L = M$. Hence, $K = M$. Therefore, we have proved statement 2. Thus, we have proved Lemma 2.24. \square

Remark 2.25. It is very important to consider the minimal sets for rational semigroups when we investigate random complex dynamics as well as rational semigroups. In [34], it is shown that for any Borel probability measure τ on the space Rat of non-constant rational maps, if $\text{supp } \tau$ is compact, $\bigcap_{g \in G_\tau} g^{-1}(J(G_\tau)) = \emptyset$ and $J(G_\tau) \neq \emptyset$, where G_τ denotes the rational semigroup generated by $\text{supp } \tau$, then there exist at most finitely many minimal sets K_1, \dots, K_r for $(G_\tau, \hat{\mathbb{C}})$, and for each $z \in \hat{\mathbb{C}}$, for $(\otimes_{j=1}^\infty \tau)$ -a.e. $\gamma = (\gamma_1, \gamma_2, \dots) \in (\text{Rat})^\mathbb{N}$, $d(\gamma_n \cdots \gamma_1(z), \bigcup_{j=1}^r K_j) \rightarrow 0$ as $n \rightarrow \infty$. Note that K_j may meet $J(G_\tau)$, and for a $g \in G_\tau$, $g|_{K_j}$ is neither contractive nor injective in general. Thus it is important to investigate forward self-similar systems whose generators may be neither contractive nor injective.

The above examples give us a natural and strong motivation to investigate forward or backward self-similar systems.

We now give some definitions which we need later.

Definition 2.26. Let $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$ be two forward or backward self-similar systems. We say that \mathfrak{L}_1 is a subsystem of \mathfrak{L}_2 if $L_1 \subset L_2$ and there exists an injection $\tau : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that for each $j = 1, \dots, m$, $h_j = g_{\tau(j)}$.

Definition 2.27. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. A forward (resp. backward) self-similar system $\mathfrak{M} = (L, (g_1, \dots, g_{m^n}))$ is said to be an n -th iterate of \mathfrak{L} if there exists a bijection $\tau : \{1, \dots, m^n\} \rightarrow \{w \in \Sigma_m^* \mid |w| = n\}$ such that for each $j = 1, \dots, m^n$, $g_j = h_{\tau(j)}$.

Definition 2.28. For a topological space X , we denote by $\text{Con}(X)$ the set of all connected components of X .

Definition 2.29. Let X be a space. For any covering $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of X , we denote by $N(\mathcal{U})$ the nerve of \mathcal{U} . By definition, the vertex set of $N(\mathcal{U})$ is equal to Λ .

Definition 2.30. Let \mathcal{S} be an abstract simplicial complex. Moreover, we denote by $|\mathcal{S}|$ the realization (see [20, p.110]). As in [20], we embed the vertex set of \mathcal{S} into $|\mathcal{S}|$.

We now define a new kind of cohomology theory for forward or backward self-similar systems.

Definition 2.31. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system.

1. For each $x = (x_1, x_2, \dots) \in \Sigma_m$, we set $L_x := \bigcap_{j=1}^\infty h_{x_1}^{-1} \cdots h_{x_j}^{-1} L$ ($\neq \emptyset$).
2. For any $k \in \mathbb{N}$, let $\mathcal{U}_k = \mathcal{U}_k(\mathfrak{L})$ be the finite covering of L defined as: $\mathcal{U}_k := \{h_w^{-1}(L)\}_{w \in \Sigma_m^*, |w|=k}$. We denote by N_k or $N_k(\mathfrak{L})$ the nerve $N(\mathcal{U}_k)$ of \mathcal{U}_k . Thus N_k is a simplicial complex such that the vertex set is equal to $\{w \in \Sigma_m^* \mid |w| = k\}$ and mutually distinct r elements $w^1, \dots, w^r \in$

Σ_m^* with $|w^1| = \dots = |w^r| = k$ make an $(r-1)$ -simplex of N_k if and only if $\bigcap_{j=1}^r h_{w^j}^{-1}(L) \neq \emptyset$. Let $\varphi_k : N_{k+1} \rightarrow N_k$ be the simplicial map defined as: $(w_1, \dots, w_{k+1}) \mapsto (w_1, \dots, w_k)$ for each $(w_1, \dots, w_{k+1}) \in \{1, \dots, m\}^{k+1}$. Moreover, for each $k, l \in \mathbb{N}$ with $l > k$, we denote by $\varphi_{l,k} : N_l \rightarrow N_k$ the composition $\varphi_{l-1} \circ \dots \circ \varphi_k$. Then, $\{N_k, \varphi_{l,k}\}_{k,l \in \mathbb{N}, l > k}$ forms an inverse system of simplicial complexes.

3. Let $\{(\varphi_k)_* : \text{Con}(|N_{k+1}|) \rightarrow \text{Con}(|N_k|)\}_{k \in \mathbb{N}}$ be the inverse system induced by $\{(\varphi_k)_*\}_k$.
4. Let R be a \mathbb{Z} module and let $p \in \mathbb{N} \cup \{0\}$. We set $\check{H}^p(L, (h_1, \dots, h_m); R)_k := H^p(N_k; R)$. This is called **the p -th interaction cohomology group of backward self-similar system $\mathcal{L} = (L, (h_1, \dots, h_m))$ at k -th stage with coefficients R** . We sometimes use the notation $\check{H}^p(\mathcal{L}; R)_k$ to denote the above $\check{H}^p(L, (h_1, \dots, h_m); R)_k$. Similarly, we set $\check{H}_p(L, (h_1, \dots, h_m); R)_k := H_p(N_k; R)$. This is called **the p -th interaction homology group of backward self-similar system $\mathcal{L} = (L, (h_1, \dots, h_m))$ at k -th stage with coefficients R** . We sometimes use the notation $\check{H}_p(\mathcal{L}; R)_k$ to denote the above $\check{H}_p(L, (h_1, \dots, h_m); R)_k$.
5. Let R be a \mathbb{Z} module and let $p \in \mathbb{N} \cup \{0\}$. We denote by $\check{H}^p(L, (h_1, \dots, h_m); R)$ the direct limit of the direct system $\{H^p(N_k; R), \varphi_{l,k}^*\}_{l,k \in \mathbb{N}, l > k}$ of \mathbb{Z} modules. This is called the **p -th interaction cohomology group of $\mathcal{L} = (L, (h_1, \dots, h_m))$ with coefficients R** . We sometimes use the notation $\check{H}^p(\mathcal{L}; R)$ in order to denote the above cohomology group $\check{H}^p(L, (h_1, \dots, h_m); R)$.
6. We denote by $\mu_{k,p} : \check{H}^p(\mathcal{L}; R)_k \rightarrow \check{H}^p(\mathcal{L}; R)$ the canonical projection.
7. Similarly, for any \mathbb{Z} module R , we denote by $\check{H}_p(L, (h_1, \dots, h_m); R)$ the inverse limit of the inverse system $\{H_p(N_k; R), (\varphi_{l,k})_*\}_{l,k \in \mathbb{N}, l > k}$ of \mathbb{Z} modules. This is called the **p -th interaction homology group of $\mathcal{L} = (L, (h_1, \dots, h_m))$ with coefficients R** . We sometimes use the notation $\check{H}_p(\mathcal{L}; R)$ in order to denote the above homology group $\check{H}_p(L, (h_1, \dots, h_m); R)$.
8. For each $p \in \mathbb{N}$, $k \in \mathbb{N}$ and $w \in \Sigma_m$, we set $\check{\pi}_p(\mathcal{L}, w)_k := \pi_p(|N_k|, w|k)$ and $\check{\pi}_p(\mathcal{L}, w) := \varprojlim_k \pi_p(|N_k|, w|k)$. We call $\check{\pi}_p(\mathcal{L}, w)_k$ the **p -th interaction homotopy group of \mathcal{L} at k -th stage with base w** and we call $\check{\pi}_p(\mathcal{L}, w)$ the **p -th interaction homotopy group of \mathcal{L} with base w** .

Definition 2.32. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system. For each $x \in \Sigma_m$, we set $L_x := \bigcap_{j=1}^{\infty} h_{x_1} \cdots h_{x_j}(L)$. For any $k \in \mathbb{N}$, let $\mathcal{U}_k = \mathcal{U}_k(\mathcal{L})$ be the finite covering of L defined as: $\mathcal{U}_k := \{h_{\bar{w}}(L)\}_{w \in \Sigma_m^*; |w|=k}$. We denote by N_k or $N_k(\mathcal{L})$ the nerve $N(\mathcal{U}_k)$ of \mathcal{U}_k . Thus N_k is a simplicial complex such that the vertex set is equal to $\{w \in \Sigma_m^* \mid |w| = k\}$ and mutually distinct r elements $w^1, \dots, w^r \in \Sigma_m^*$ with $|w^1| = \dots = |w^r| = k$ make an $(r-1)$ -simplex of N_k if and only if $\bigcap_{j=1}^r h_{w^j}^{-1}(L) \neq \emptyset$. Let $\varphi_k : N_{k+1} \rightarrow N_k$ be the simplicial map defined as: $(w_1, \dots, w_{k+1}) \mapsto (w_1, \dots, w_k)$ for each $(w_1, \dots, w_{k+1}) \in \{1, \dots, m\}^{k+1}$. Moreover, we set $\varphi_{l,k} := \varphi_{l-1} \circ \dots \circ \varphi_k$. We define the p -th interaction cohomology group $\check{H}^p(\mathcal{L}; R)$ and the p -th interaction homology group $\check{H}_p(\mathcal{L}; R)$ as in Definition 2.31. Moreover, we define the p -th interaction homotopy group $\check{\pi}_p(\mathcal{L}, w)_k$ of \mathcal{L} at k -th stage with base $w \in \Sigma_m$, the p -th interaction homotopy group $\check{\pi}_p(\mathcal{L}, w)$ of \mathcal{L} with base w , the p -th interaction cohomology group $\check{H}^p(\mathcal{L}; R)_k$ of \mathcal{L} at k -th stage, and p -th interaction homology group $\check{H}_p(\mathcal{L}; R)_k$ of \mathcal{L} at k -th stage, as in Definition 2.31. Furthermore, we denote by $\mu_{k,p} : \check{H}^p(\mathcal{L}; R)_k \rightarrow \check{H}^p(\mathcal{L}; R)$ the canonical projection.

Definition 2.33. Let $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda_1}$ and $\mathcal{B} = \{B_\mu\}_{\mu \in \Lambda_2}$ be two coverings of a topological space L . We say that \mathcal{B} is a refinement of \mathcal{A} if there exists a map $r_{\mathcal{A}, \mathcal{B}} : \Lambda_2 \rightarrow \Lambda_1$ such that $B_\mu \subset A_{r_{\mathcal{A}, \mathcal{B}}(\mu)}$ for each $\mu \in \Lambda_2$. The $r_{\mathcal{A}, \mathcal{B}}$ is called the refining map. If \mathcal{B} is a refinement of \mathcal{A} , we write $\mathcal{A} \preceq \mathcal{B}$.

Remark 2.34. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. Let $r_{\mathcal{U}_k, \mathcal{U}_{k+1}} : \{w \in \Sigma_m^* \mid |w| = k+1\} \rightarrow \{w \in \Sigma_m^* \mid |w| = k\}$ be the map defined by $(w_1, \dots, w_{k+1}) \mapsto$

(w_1, \dots, w_k) . If $w = (w_1, \dots, w_{k+1}) \in \{1, \dots, m\}^{k+1}$, then $h_{\bar{w}}(L) \subset h_{\bar{w|k}}(L)$ (resp. $h_w^{-1}(L) \subset h_{w|k}^{-1}(L)$). Thus for each k , $\mathcal{U}_k \preceq \mathcal{U}_{k+1}$ with refining map $r_{\mathcal{U}_k, \mathcal{U}_{k+1}}$. Moreover, the simplicial map $(r_{\mathcal{U}_k, \mathcal{U}_{k+1}})_* : N_{k+1} \rightarrow N_k$ induced by $r_{\mathcal{U}_k, \mathcal{U}_{k+1}}$ is equal to the simplicial map $\varphi_k : N_{k+1} \rightarrow N_k$.

From the definition of interaction (co)homology groups and the continuity theorem for Čech (co)homology ([38]), it is easy to prove the following lemma.

Lemma 2.35. *Let \mathfrak{L} be a forward or backward self-similar system and let R be a \mathbb{Z} module. Then $\check{H}^p(\mathfrak{L}; R) \cong \check{H}(\varprojlim_k |N_k|; R)$ and $\check{H}_p(\mathfrak{L}; R) \cong \check{H}_p(\varprojlim_k |N_k|; R)$.*

Example 2.36 (Sierpiński gasket). Let $p_1, p_2, p_3 \in \mathbb{C}$ be mutually distinct three points such that $p_1 p_2 p_3$ makes an equilateral triangle. Let $h_i(z) := \frac{1}{2}(z - p_i) + p_i$, for each $i = 1, 2, 3$. Let $L = M_{\mathbb{C}}(h_1, h_2, h_3)$. Then, L is equal to the Sierpiński gasket ([15], see Figure 1). We consider the forward self-similar system $\mathfrak{L} = (L, (h_1, h_2, h_3))$. We see that the set of one-dimensional simplices of N_1 is equal to $\{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ and for each $r \geq 2$, there exists no r -dimensional simplices of N_1 . Moreover, it is easy to see that the set of one-dimensional simplices of N_2 is equal to $\{\{(1, 1), (1, 2)\}, \{(1, 2), (1, 3)\}, \{(1, 3), (1, 1)\}, \{(2, 1), (2, 2)\}, \{(2, 2), (2, 3)\}, \{(2, 3), (2, 1)\}, \{(3, 1), (3, 2)\}, \{(3, 2), (3, 3)\}, \{(3, 3), (3, 1)\}, \{(1, 2), (2, 1)\}, \{(2, 3), (3, 2)\}, \{(3, 1), (1, 3)\}\}$ and for each $r \geq 2$ there exists no r -dimensional simplices of N_2 (see Figure 2). Thus for each \mathbb{Z} module R , $\check{H}^1(\mathfrak{L}; R)_1 = R$, $\check{H}^r(\mathfrak{L}; R)_1 = 0$ ($\forall r \geq 2$), $\check{H}^1(\mathfrak{L}; R)_2 = R^4$, and $\check{H}^r(\mathfrak{L}; R)_2 = 0$ ($\forall r \geq 2$).

Figure 1: The Sierpiński gasket

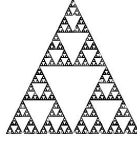
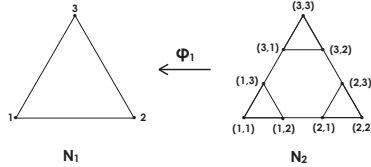


Figure 2: The figure of $\varphi_1 : N_2 \rightarrow N_1$ for the system of the Sierpiński gasket



Remark 2.37. $\mathfrak{L} \mapsto \{N_k(\mathfrak{L}), \varphi_{l,k}\}_{l,k \in \mathbb{N}, l > k}$ is a covariant functor from the category of backward self-similar systems to the category of inverse systems of simplicial complexes. For any \mathbb{Z} module R and any $p \in \mathbb{N} \cup \{0\}$, $\mathfrak{L} \mapsto \{\check{H}_p(\mathfrak{L}; R)_k, (\varphi_{l,k})_*\}_{l,k \in \mathbb{N}, l > k}$ is a covariant functor from the category of backward self-similar systems to the category of inverse systems of \mathbb{Z} modules, $\mathfrak{L} \mapsto \check{H}_p(\mathfrak{L}; R)$ is a covariant functor from the category of backward self-similar systems to the category of \mathbb{Z} modules, $\mathfrak{L} \mapsto \{\check{H}^p(\mathfrak{L}; R)_k, (\varphi_{l,k})^*\}_{l,k \in \mathbb{N}, l > k}$ is a contravariant functor from the category of backward self-similar systems to the category of direct systems of \mathbb{Z} modules, and $\mathfrak{L} \mapsto \check{H}^p(\mathfrak{L}; R)$ is a contravariant functor from the category of backward self-similar systems to the category of \mathbb{Z} modules. Thus the isomorphism classes of $\{N_k(\mathfrak{L}), \varphi_{l,k}\}_{l,k \in \mathbb{N}, l > k}$, $\{\check{H}_p(\mathfrak{L}; R)_k, (\varphi_{l,k})_*\}_{l,k \in \mathbb{N}, l > k}$, $\check{H}_p(\mathfrak{L}; R)$, $\{\check{H}^p(\mathfrak{L}; R)_k, (\varphi_{l,k})^*\}_{l,k \in \mathbb{N}, l > k}$, and $\check{H}^p(\mathfrak{L}; R)$ are invariant under the isomorphisms of backward self-similar systems. The same statements as above hold for forward self-similar systems.

Remark 2.38. Let $\mathfrak{L}_1 = (L, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L, (g_1, \dots, g_n))$ be two backward self-similar systems such that $\langle h_1, \dots, h_m \rangle = \langle g_1, \dots, g_n \rangle$. Then, by the definition of the interaction (co)homology, it is easy to see that there exist isomorphisms $\check{H}^*(\mathfrak{L}_1; R) \cong \check{H}^*(\mathfrak{L}_2; R)$ and $\check{H}_*(\mathfrak{L}_1; R) \cong \check{H}_*(\mathfrak{L}_2; R)$. Similar statement holds for two forward self-similar systems.

Notation: Let (X, d) be a metric space. Let A be a non-empty subset of X . Let $\delta > 0$. We set $B(A, \delta) := \{x \in X \mid d(y, A) < \delta\}$.

Definition 2.39. Let (X, d) be a metric space. Let $\mathcal{A} := \{L_\lambda\}_{\lambda \in \Lambda}$ be a covering of X . For each $\delta > 0$, we set $\mathcal{A}^\delta := \{B(L_\lambda, \delta)\}_{\lambda \in \Lambda}$ and we denote by $\psi_{\mathcal{A}, \delta} : N(\mathcal{A}) \rightarrow N(\mathcal{A}^\delta)$ the simplicial map induced by the refinement $L_\lambda \subset B(L_\lambda, \delta), \lambda \in \Lambda$.

Lemma 2.40. Let (L, d) be a compact metric space. Let $\mathcal{A} = \{L_i\}_{i=1}^r$ be a finite covering of L such that for each $i = 1, \dots, r$, L_i is a non-empty compact subset of L . Then, we have the following.

1. There exists a number $\delta(\mathcal{A}) > 0$ such that for each $0 < \delta < \delta(\mathcal{A})$, $\psi_{\mathcal{A}, \delta} : N(\mathcal{A}) \rightarrow N(\mathcal{A}^\delta)$ is a simplicial isomorphism.
2. Let $\mathcal{B} = \{M_j\}_{j=1}^l$ be another finite covering of L such that for each $j = 1, \dots, l$, M_j is a non-empty compact subset of L . Assume that there exists a map $\beta_{\mathcal{A}, \mathcal{B}} : \{1, \dots, r\} \rightarrow \{1, \dots, l\}$ such that $M_j \subset B_{\beta_{\mathcal{A}, \mathcal{B}}(j)}$ for each $j = 1, \dots, l$. Then, there exists a $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$, we have the following (i), (ii), and (iii): (i) $B(M_j, \delta) \subset B(L_{\beta_{\mathcal{A}, \mathcal{B}}(j)}, \delta)$ ($j = 1, \dots, l$), (ii) the diagram

$$\begin{array}{ccc} N(\mathcal{B}) & \xrightarrow{\psi_{\mathcal{B}, \delta}} & N(\mathcal{B}^\delta) \\ (\beta_{\mathcal{A}, \mathcal{B}})_* \downarrow & & \downarrow (\beta_{\mathcal{A}, \mathcal{B}})_* \\ N(\mathcal{A}) & \xrightarrow{\psi_{\mathcal{A}, \delta}} & N(\mathcal{A}^\delta) \end{array}$$

commutes where $(\beta_{\mathcal{A}, \mathcal{B}})_* : N(\mathcal{B}) \rightarrow N(\mathcal{A})$ and $(\beta_{\mathcal{A}, \mathcal{B}})_* : N(\mathcal{B}^\delta) \rightarrow N(\mathcal{A}^\delta)$ are simplicial maps induced by $\beta_{\mathcal{A}, \mathcal{B}} : \{1, \dots, r\} \rightarrow \{1, \dots, l\}$, and (iii) the simplicial maps $\psi_{\mathcal{B}, \delta} : N(\mathcal{B}) \rightarrow N(\mathcal{B}^\delta)$ and $\psi_{\mathcal{A}, \delta} : N(\mathcal{A}) \rightarrow N(\mathcal{A}^\delta)$ are isomorphisms.

Proof. First, we will show statement 1. If $\bigcap_{i=1}^r L_i \neq \emptyset$, then for any $\delta > 0$, $\psi_{\mathcal{A}, \delta}$ is an isomorphism. Hence we may assume that $\bigcap_{i=1}^r L_i = \emptyset$. Let $(i_1, \dots, i_r) \in \{1, \dots, r\}^r$ be any element with $\bigcap_{t=1}^r L_{i_t} = \emptyset$. Then there exists a $\delta = \delta(i_1, \dots, i_r) > 0$ such that $\bigcap_{t=1}^r B(L_{i_t}, \delta) = \emptyset$. Let

$$\delta(\mathcal{A}) := \min\{\delta(i_1, \dots, i_r) \mid (i_1, \dots, i_r) \in \{1, \dots, r\}^r, \bigcap_{t=1}^r L_{i_t} = \emptyset\}.$$

Then, $\delta(\mathcal{A}) > 0$. Hence, for each $0 < \delta < \delta(\mathcal{A})$, if $\bigcap_{t=1}^r L_{i_t} = \emptyset$, then $\bigcap_{t=1}^r B(L_{i_t}, \delta) = \emptyset$. Therefore, statement 1 holds. Statement 2 follows easily from statement 1. \square

Remark 2.41. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Let R be a \mathbb{Z} module and let $\check{H}^p(L; R)$ be the p -th Čech cohomology group of L with coefficients R . Since $\check{H}^p(L; R) = \varinjlim_{\mathcal{W}} H^p(N(\mathcal{W}); R)$, where \mathcal{W} runs over all open coverings of L , Lemma 2.40 implies that for each $k \in \mathbb{N}$, there exists a homomorphism $\Psi_{\mathcal{U}_k} : H^p(N_k; R) \rightarrow \check{H}^p(L; R)$ induced by $\psi_{\mathcal{U}_k, \delta}$. Using Lemma 2.40 again, $\{\Psi_{\mathcal{U}_k}\}_{k \in \mathbb{N}}$ induces a natural homomorphism

$$\Psi : \check{H}^p(\mathfrak{L}; R) \rightarrow \check{H}^p(L; R). \quad (1)$$

Remark 2.42. Suppose that either (a) $\mathfrak{L} = (L, (h_1, \dots, h_m))$ is a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is a contraction, or (b) $\mathfrak{L} = (L, (h_1, \dots, h_m))$ is a backward self-similar system such that for each $j = 1, \dots, m$, $h_j^{-1} : L \rightarrow L$ is well-defined and

$h_j^{-1} : L \rightarrow L$ is a contraction. Then, for any p and any \mathbb{Z} module R , $\Psi : \check{H}^p(\mathcal{L}; R) \rightarrow \check{H}^p(L; R)$ is an isomorphism. For, given an open covering \mathcal{W} , there exists a $k \in \mathbb{N}$ and a $\delta > 0$ such that $\mathcal{W} \preceq N_k^\delta$. It means that $\{H^p(N_k^\delta; R)\}_{k, \delta}$ is cofinal in $\{H^p(N(\mathcal{W}); R)\}_{\mathcal{W}}$. From Lemma 2.40, it follows that $\Psi : \check{H}^p(\mathcal{L}; R) \rightarrow \check{H}^p(L; R)$ is an isomorphism. Similarly, $\check{H}_p(\mathcal{L}; R)$ and $\check{H}_p(L; R)$ are naturally isomorphic. Moreover, $\tilde{\pi}_r(\mathcal{L}, w)$ and $\tilde{\pi}_r(L, x_0)$ ($x_0 \in h_{w_1} \cdots h_{w_k}(L)$ for each $k \in \mathbb{N}$) are naturally isomorphic, where $\tilde{\pi}_r(L, x_0)$ denotes the r -th shape group of L with base point x_0 . (For the definition of shape groups, see [16].)

However, Ψ is not an isomorphism in general. In fact, Ψ may not be a monomorphism (see Proposition 3.37). Similarly, $\check{H}_p(\mathcal{L}; R)$ and $\check{H}_p(L; R)$ may not be isomorphic in general (Example 4.9). Moreover, $\tilde{\pi}_r(\mathcal{L}, w)$ and $\tilde{\pi}_r(L, x_0)$ ($x_0 \in h_{w_1} \cdots h_{w_k}(L)$ for each $k \in \mathbb{N}$) may not be isomorphic in general (Example 4.9).

3 Main results

In this section, we present the main results of this paper. The proofs of the results are given in section 5.

3.1 General results

In this subsection, we present some general results on the 0-th and the first interaction (co)homology groups of forward or backward self-similar systems. The proofs are given in section 5.1.

We investigate the space of all connected components of an invariant set of a forward or backward self-similar system. This is related to the 0-th interaction (co)homology groups of forward or backward self-similar systems. Note that it is a new point of view to study the above space. As an application, we generalize and further develop the essence of the well-known result (Theorem 1.1) on the necessary and sufficient condition for the invariant sets of the forward self-similar systems to be connected.

Remark 3.1. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. Then $|N_1|$ is connected if and only if for each $i, j \in \{1, \dots, m\}$, there exists a sequence $\{i_t\}_{t=1}^r$ in $\{1, \dots, m\}$ such that $i_1 = i, i_t = j$, and $h_{i_t}(L) \cap h_{i_{t+1}}(L) \neq \emptyset$ (resp. $h_{i_t}^{-1}(L) \cap h_{i_{t+1}}^{-1}(L) \neq \emptyset$) for each $t = 1, \dots, s-1$.

Theorem 3.2. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system such that L_x is connected for each $x \in \Sigma_m$. Let R be a field. Then, we have the following.

1. There exists a bijection: $\Phi : \varprojlim_k \text{Con}(|N_k|) \cong \text{Con}(L)$, where, the map Φ is defined as follows: let $B = (B_k)_k \in \varprojlim_k \text{Con}(|N_k|)$ where $B_k \in \text{Con}(|N_k|)$ and $(\varphi_k)_*(B_{k+1}) = B_k$ for each k . Take a point $x \in \Sigma_m$ such that $(x_1, \dots, x_k) \in B_k$ for each k . Take an element $C \in \text{Con}(L)$ such that $L_x \subset C$. Let $\Phi(B) = C$.
2. L is connected if and only if $|N_1|$ is connected. (See Remark 3.1.)
3. $\sharp \text{Con}(|N_k|) \leq \sharp \text{Con}(|N_{k+1}|)$, for each $k \in \mathbb{N}$. Furthermore, $\{\sharp \text{Con}(|N_k|)\}_{k \in \mathbb{N}}$ is bounded if and only if $\sharp \text{Con}(L) < \infty$. If $\sharp \text{Con}(L) < \infty$, then $\lim_{k \rightarrow \infty} \sharp \text{Con}(|N_k|) = \sharp \text{Con}(L)$.
4. $\dim_R \check{H}^0(\mathcal{L}; R) < \infty$ if and only if $\sharp \text{Con}(L) < \infty$.
5. If $\dim_R \check{H}^0(\mathcal{L}; R) < \infty$, then $\dim_R \check{H}^0(\mathcal{L}; R) = \sharp \text{Con}(L)$ and $\Psi : \check{H}^0(\mathcal{L}; R) \rightarrow \check{H}^0(L; R)$ is an isomorphism.
6. Suppose that $m = 2$ and L is disconnected. Then, $h_1^{-1}(L) \cap h_2^{-1}(L) = \emptyset$, there exists a bijection $\text{Con}(L) \cong \Sigma_2$, and $\sharp \text{Con}(L) > \aleph_0$.

7. Suppose that $m = 3$ and L is disconnected. Then, $\sharp\text{Con}(L) \geq \aleph_0$ and there exists a $j \in \{1, 2, 3\}$ such that $L_{(j)^\infty}$ is a connected component of L , where $(j)^\infty := (j, j, j, \dots) \in \Sigma_3$.

Theorem 3.3. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that L_x is connected for each $x \in \Sigma_m$. Let R be a field. Then, we have the following.

1. There exists a bijection: $\Phi : \varprojlim \text{Con}(|N_k|) \cong \text{Con}(L)$.
2. L is connected if and only if $|N_1|$ is connected. (See Remark 3.1.)
3. $\sharp\text{Con}(|N_k|) \leq \sharp\text{Con}(|N_{k+1}|)$, for each $k \in \mathbb{N}$. Furthermore, $\{\sharp\text{Con}(|N_k|)\}_{k \in \mathbb{N}}$ is bounded if and only if $\sharp\text{Con}(L) < \infty$. If $\sharp\text{Con}(L) < \infty$, then $\lim_{k \rightarrow \infty} \sharp\text{Con}(|N_k|) = \sharp\text{Con}(L)$.
4. $\dim_R \check{H}^0(\mathfrak{L}; R) < \infty$ if and only if $\sharp\text{Con}(L) < \infty$.
5. If $\dim_R \check{H}^0(\mathfrak{L}; R) < \infty$, then $\dim_R \check{H}^0(\mathfrak{L}; R) = \sharp\text{Con}(L)$ and $\Psi : \check{H}^0(\mathfrak{L}; R) \rightarrow \check{H}^0(L; R)$ is an isomorphism.
6. If $m = 2$ and L is disconnected, then $h_1(L) \cap h_2(L) = \emptyset$.
7. If $m = 2$, $h_j : L \rightarrow L$ is injective for each $j = 1, 2$, and L is disconnected, then there exists a bijection $\text{Con}(L) \cong \Sigma_2$ and $\sharp\text{Con}(L) > \aleph_0$.
8. If $m = 3$, $h_j : L \rightarrow L$ is injective for each $j = 1, 2, 3$, and L is disconnected, then $\sharp\text{Con}(L) \geq \aleph_0$ and there exists a $j \in \{1, 2, 3\}$ such that $L_{(j)^\infty}$ is a connected component of L , where $(j)^\infty := (j, j, j, \dots) \in \Sigma_3$.

Remark 3.4. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system. If each $h_j : L \rightarrow L$ is a contraction, then for each $x \in \Sigma_m$, $\sharp L_x = 1$ and L_x is connected.

We now consider the first interaction cohomology groups of forward or backward self-similar systems.

Remark 3.5. Let $\mathfrak{L} := (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. Let $G = \langle h_1, \dots, h_m \rangle$ and let R be a \mathbb{Z} module. If $\bigcap_{g \in G} g(L) \neq \emptyset$ (resp. if $\bigcap_{g \in G} g^{-1}(L) \neq \emptyset$), then, $\check{H}^0(\mathfrak{L}; R) = R$ and $\check{H}^p(\mathfrak{L}; R) = 0$ for each $p \geq 1$. In particular, if there exists a point $z \in L$ such that for each $j = 1, \dots, m$, $h_j(z) = z$, then, $\check{H}^0(\mathfrak{L}; R) = R$ and $\check{H}^p(\mathfrak{L}; R) = 0$ for each $p \geq 1$.

By Remark 3.5, we can find many examples of \mathfrak{L} such that $\check{H}^p(\mathfrak{L}; R) = 0$ for each $p \in \mathbb{N}$ and each \mathbb{Z} module R .

Remark 3.6. For any $n \in \mathbb{N} \cup \{0\}$, we also have many examples of forward or backward self-similar systems $\mathfrak{L} = (L, (h_1, \dots, h_m))$ such that for each field R , $0 < \dim_R \check{H}^n(\mathfrak{L}; R) < \infty$. For example, let M_0 and M_1 be two cubes in \mathbb{R}^{n+1} such that $M_1 \subset \text{int}(M_0)$. Let $L := M_0 \setminus \text{int}(M_1)$. Then, we easily see that there exists a forward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$ such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is an injective contraction. For this \mathfrak{L} , we have $\check{H}^n(\mathfrak{L}; R) \cong \check{H}^n(L; R) = R$.

We give a sufficient condition for the rank of the first interaction cohomology group of a system to be infinite.

Theorem 3.7. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system. Let R be a field. We assume all of the following:

1. $|N_1|$ is connected. (See Remark 3.1.)
2. $(h_1^2)^{-1}(L) \cap (\bigcup_{i:i \neq 1} h_i^{-1}(L)) = \emptyset$.

3. There exist mutually distinct elements $j_1, j_2, j_3 \in \{1, \dots, m\}$ such that $j_1 = 1$ and such that for each $k = 1, 2, 3$, $h_{j_k}^{-1}(L) \cap h_{j_{k+1}}^{-1}(L) \neq \emptyset$, where $j_4 := j_1$.

4. For each $s, t \in \{1, \dots, m\}$, we have the following: if $s, t, 1$ are mutually distinct, then $h_1^{-1}(L) \cap h_s^{-1}(L) \cap h_t^{-1}(L) = \emptyset$.

Then, $\dim_R \check{H}^1(\mathfrak{L}; R) = \infty$.

Theorem 3.8. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is injective. Let R be a field. We assume all of the following:

1. $|N_1|$ is connected. (See Remark 3.1.)

2. $h_1^2(L) \cap (\bigcup_{i:i \neq 1} h_i(L)) = \emptyset$.

3. There exist mutually distinct elements $j_1, j_2, j_3 \in \{1, \dots, m\}$ such that $j_1 = 1$ and such that for each $k = 1, 2, 3$, $h_{j_k}(L) \cap h_{j_{k+1}}(L) \neq \emptyset$, where $j_4 := j_1$.

4. For each $s, t \in \{1, \dots, m\}$, we have the following: if $s, t, 1$ are mutually distinct, then $h_1(L) \cap h_s(L) \cap h_t(L) = \emptyset$.

Then, $\dim_R \check{H}^1(\mathfrak{L}; R) = \infty$.

Corollary 3.9. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that each $h_j : L \rightarrow L$ is injective and such that for each $x \in \Sigma_m$, L_x is connected. Let R be a field. Suppose that the conditions 1,2,3,4 in the assumptions of Theorem 3.8 hold. Then, $\dim_R \check{H}^1(\mathfrak{L}; R) = \dim_R \Psi(\check{H}^1(\mathfrak{L}; R)) = \infty$, $\dim_R \check{H}^1(L; R) = \infty$, and $\Psi : \check{H}^1(\mathfrak{L}; R) \rightarrow \check{H}^1(L; R)$ is a monomorphism.

Remark 3.10. Let K be a non-empty connected compact metric space and let $h_j : K \rightarrow K$ be a continuous map for each $j = 1, \dots, m$. Let $L = R_{K,f}(h_1, \dots, h_m)$ and let $\mathfrak{L} = (L, (h_1, \dots, h_m))$. Regarding the forward self-similar system \mathfrak{L} (cf. Lemma 2.20), suppose that $|N_1|$ is connected. Then, L is connected and L_x is connected for each $x \in \Sigma_m$. For, by Lemma 4.3, which will be proved later, $|N_k|$ is connected for all $k \in \mathbb{N}$, therefore $\bigcup_{|w|=k} h_{\bar{w}}(K)$ is connected for each $k \in \mathbb{N}$. It implies that $L = \bigcap_{k=1}^{\infty} \bigcup_{|w|=k} h_{\bar{w}}(K)$ is connected.

Example 3.11. Let $\mathfrak{L} = (L, (h_1, h_2, h_3))$ be the forward self-similar system in Example 2.36. Then \mathfrak{L} satisfies the assumptions of Corollary 3.9.

3.2 Application to the dynamics of polynomial semigroups

In this subsection, we present some results on the Julia sets of postcritically bounded polynomial semigroups G , which are obtained by applying the results in section 3.1. The proofs of the results are given in section 5.2.

Definition 3.12. For each polynomial map $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, we denote by $CV(g)$ the set of all critical values of the holomorphic map $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. Moreover, for a polynomial semigroup G , we set $P(G) = \overline{\bigcup_{g \in G} CV(g)}$ ($\subset \hat{\mathbb{C}}$). The set $P(G)$ is called the **postcritical set** of G . Moreover, we set $P^*(G) := \overline{P(G)} \setminus \{\infty\}$. The set $P^*(G)$ is called the **planar postcritical set** of G . We say that a polynomial semigroup G is **postcritically bounded** if $P^*(G)$ is bounded in \mathbb{C} .

Definition 3.13. We denote by \mathcal{G} the set of all postcritically bounded polynomial semigroups G such that for each $g \in G$, $\deg(g) \geq 2$. Moreover, we set $\mathcal{G}_{con} := \{G \in \mathcal{G} \mid J(G) \text{ is connected}\}$ and $\mathcal{G}_{dis} := \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$.

Remark 3.14. Let $G = \langle h_1, \dots, h_m \rangle$ be a finitely generated polynomial semigroup. Then, $P(G) = \bigcup_{g \in G \cup \{Id\}} g(\bigcup_{j=1}^m CV(h_j))$ and $g(P(G)) \subset P(G)$ for each $g \in G$. From the above formula, one may use a computer to see if $G \in \mathcal{G}$ much in the same way as one verifies the boundedness of the critical orbit for the maps $f_c(z) = z^2 + c$.

Definition 3.15. We set $\text{Rat} := \{g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \mid g \text{ is a non-constant rational map}\}$ endowed with the topology induced by the uniform convergence on $\hat{\mathbb{C}}$. Moreover, we set $\mathcal{Y} := \{g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \mid g \text{ is a polynomial, } \deg(g) \geq 2\}$ endowed with the relative topology from Rat . Moreover, for each $m \in \mathbb{N}$ we set $\mathcal{Y}_b^m := \{(h_1, \dots, h_m) \in \mathcal{Y}^m \mid \langle h_1, \dots, h_m \rangle \in \mathcal{G}\}$. Furthermore, we set $\mathcal{Y}_{b,con}^m := \{(h_1, \dots, h_m) \in \mathcal{Y}^m \mid \langle h_1, \dots, h_m \rangle \in \mathcal{G}_{con}\}$ and $\mathcal{Y}_{b,dis}^m := \{(h_1, \dots, h_m) \in \mathcal{Y}^m \mid \langle h_1, \dots, h_m \rangle \in \mathcal{G}_{dis}\}$.

Remark 3.16. It is well-known that for a polynomial $g \in \mathcal{Y}$, the semigroup $\langle g \rangle$ belongs to \mathcal{G} if and only if $J(g)$ is connected ([18]). However, for a general polynomial semigroup G , it is not true. For example, $\langle z^3, z^2/4 \rangle$ belongs to \mathcal{G}_{dis} . There are many new phenomena about the dynamics of $G \in \mathcal{G}_{dis}$ which cannot hold in the dynamics of a single polynomial map. For the dynamics of $G \in \mathcal{G}_{dis}$, see [31, 32, 33, 30, 29].

We now present the first main result of this subsection.

Theorem 3.17. Let $G = \langle h_1, \dots, h_m \rangle \in \mathcal{G}$. Then, for the backward self-similar system $\mathcal{L} = (J(G), (h_1, \dots, h_m))$, all of the statements 1, ..., 7 in Theorem 3.2 hold.

Remark 3.18. It is well known that if G is a semigroup generated by a single $h \in \text{Rat}$ with $\deg(h) \geq 2$ or if G is a non-elementary Kleinian group, then either $J(G)$ is connected or $J(G)$ has uncountably many connected components ([1, 18]). However, even for a finitely generated polynomial semigroup in \mathcal{G} , this is not true any more. In fact, in [31], it was shown that for any positive integer n , there exists an element $(h_1, \dots, h_{2n}) \in \mathcal{Y}_b^{2n}$ such that $\#\text{Con}(J(\langle h_1, \dots, h_{2n} \rangle)) = n$. Moreover, in [31], it was shown that there exists an element $(h_1, h_2, h_3) \in \mathcal{Y}_b^3$ such that $\#\text{Con}(J(\langle h_1, h_2, h_3 \rangle)) = \aleph_0$ (see Figure 4).

Figure 3: The Julia set of $G = \langle g_1^2, g_2^2 \rangle$, where $g_1(z) := z^2 - 1$, $g_2(z) := \frac{z^2}{4}$. $G \in \mathcal{G}_{dis}$ and $\#\text{Con}(J(G)) > \aleph_0$.

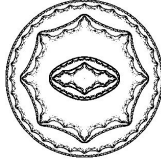
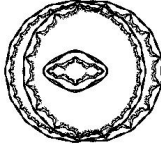


Figure 4: The Julia set of a 3-generator polynomial semigroup $G \in \mathcal{G}_{dis}$ with $\#\text{Con}(J(G)) = \aleph_0$.



By Remark 3.5, for each $m \in \mathbb{N}$, there exists an element $(h_1, \dots, h_m) \in \mathcal{Y}_b^m$ such that setting $G = \langle h_1, \dots, h_m \rangle$, we have $\check{H}^1(J(G), (h_1, \dots, h_m); R) = 0$. We will show that there exists an element $(h_1, \dots, h_4) \in \mathcal{Y}_b^4$ such that setting $G = \langle h_1, \dots, h_4 \rangle$, $\check{H}^1(J(G), (h_1, \dots, h_4); R)$ has infinite rank.

Theorem 3.19. Let $m \in \mathbb{N}$ and let $(h_1, \dots, h_m) \in \mathcal{Y}_b^m$. Let $G = \langle h_1, \dots, h_m \rangle$. Let R be a field. For the backward self-similar system $\mathfrak{L} = (J(G), (h_1, \dots, h_m))$, suppose that all of the conditions 1, 2, 3, 4 in the assumptions of Theorem 3.7 hold. Then, we have the following statements 1, 2, 3.

1. $\dim_R \check{H}^1(\mathfrak{L}; R) = \dim_R \Psi(\check{H}^1(\mathfrak{L}; R)) = \infty$ and $\dim_R \check{H}^1(J(G); R) = \infty$.
2. $\Psi : \check{H}^1(\mathfrak{L}; R) \rightarrow \check{H}^1(J(G); R)$ is a monomorphism.
3. $F(G)$ has infinitely many connected components.

Proposition 3.20. There exists an element $h = (h_1, h_2, h_3, h_4) \in \mathcal{Y}_b^4$ which satisfies the assumptions of Theorem 3.19. In particular, for this h , setting $G = \langle h_1, \dots, h_4 \rangle$, for any field R , we have that $\dim_R \check{H}^1(J(G), (h_1, \dots, h_4); R) = \dim_R \Psi(\check{H}^1(J(G), (h_1, \dots, h_4); R)) = \infty$ and $F(G)$ has infinitely many connected components (see Figure 8).

Problem 3.21 (Open). Let $m \in \mathbb{N}$ with $m \geq 2$. Are there any $(h_1, \dots, h_m) \in \mathcal{Y}_b^m$ such that

$$0 < \dim_R \check{H}^1(J(\langle h_1, \dots, h_m \rangle), (h_1, \dots, h_m); R) < \infty ?$$

3.3 Postunbranched systems

In this subsection, we introduce ‘‘postunbranched systems,’’ and we present some results on the interaction (co)homology groups of such systems. The proofs of the main results are given in section 5.3.

Definition 3.22. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward (resp. backward) self-similar system. For each $(i, j) \in \{1, \dots, m\}^2$ with $i \neq j$, we set $C_{i,j} = C_{i,j}(\mathfrak{L}) := h_i(L) \cap h_j(L)$ (resp. $C_{i,j} = C_{i,j}(\mathfrak{L}) := h_i^{-1}(L) \cap h_j^{-1}(L)$). We say that \mathfrak{L} is **postunbranched** if for any $(i, j) \in \{1, \dots, m\}^2$ such that $i \neq j$ and $C_{i,j} \neq \emptyset$, there exists a unique $x = x(i, j) \in \Sigma_m$ such that

- $h_i^{-1}(C_{i,j}) \subset L_x$ (resp. $h_i(C_{i,j}) \subset L_x$) and
- for each $x' \in \Sigma_m$ with $x' \neq x$, we have $h_i^{-1}(C_{i,j}) \cap L_{x'} = \emptyset$ (resp. $h_i(C_{i,j}) \cap L_{x'} = \emptyset$).

The following Lemmas 3.23, 3.24, 3.25, 3.26 are easy to show from the definition above.

Lemma 3.23. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Suppose that \mathfrak{L} is postunbranched. Then, any subsystem \mathfrak{M} of \mathfrak{L} is postunbranched.

Lemma 3.24. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Suppose that \mathfrak{L} is postunbranched. When \mathfrak{L} is a forward self-similar system, we assume further that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is injective. Then, for each $n \in \mathbb{N}$, an n -th iterate of \mathfrak{L} is postunbranched.

Notation: Let $m \in \mathbb{N}$. For each $j = 1, \dots, m$, we set $(j)^\infty := (j, j, \dots) \in \Sigma_m$.

Lemma 3.25. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system. Suppose that for each $(i, j) \in \{1, \dots, m\}^2$ such that $i \neq j$ and $C_{i,j} \neq \emptyset$, there exists an $r \in \{1, \dots, m\}$ such that $h_i(C_{i,j}) \subset L_{(r)^\infty}$ and $L_{(r)^\infty} \subset (L \setminus \bigcup_{k:k \neq r} h_k^{-1}(L))$. Then, for any $n \in \mathbb{N}$, an n -th iterate of \mathfrak{L} is postunbranched.

Lemma 3.26. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is injective. Suppose that for each $(i, j) \in \{1, \dots, m\}^2$ such that $i \neq j$ and $C_{i,j} \neq \emptyset$, there exists an $r \in \{1, \dots, m\}$ such that $h_i^{-1}(C_{i,j}) \subset L_{(r)^\infty}$ and $L_{(r)^\infty} \subset (L \setminus \bigcup_{k:k \neq r} h_k(L))$. Then, for any $n \in \mathbb{N}$, an n -th iterate of \mathfrak{L} is postunbranched.

From Lemmas 3.23, 3.24, 3.25, 3.26, we can easily obtain many examples of postunbranched systems.

Notation: We denote by $\text{Fix}(f)$ the set of all fixed points of f .

Example 3.27 (Sierpiński gasket). Let $\mathcal{L} = (L, (h_1, h_2, h_3))$ be the forward self-similar system in Example 2.36. Thus L is the Sierpiński gasket. (See Figure 1.) From Figure 1, we see that for each $(i, j) \in \{1, 2, 3\}^2$ such that $i \neq j$ and $C_{i,j} \neq \emptyset$, $h_i^{-1}(C_{i,j}) = \text{Fix}(h_j) \cap L = L_{(j)\infty} \subset (L \setminus \bigcup_{k:k \neq j} h_k(L))$. From Lemmas 3.26 and 3.23, it follows that for any $n \in \mathbb{N}$, if $\mathfrak{M} = (M, (g_1, \dots, g_t))$ is a subsystem of an n -th iterate of \mathcal{L} , then \mathfrak{M} is postunbranched.

Example 3.28 (Pentakun, Snowflake). Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system in [15, Example 3.8.11 (Pentakun)] or [15, Example 3.8.12 (Snowflake)]. Hence L is one of the snowflake, the pentakun, the heptakun, the octakun, and so on. (The definition of the snowflake is as follows: let $p_k = \exp(2k\pi\sqrt{-1}/6)$ for each $k = 1, \dots, 6$ and let $p_7 = 0$. We define $h_k : \mathbb{C} \rightarrow \mathbb{C}$ by $h_k(z) = (z - p_k)/3 + p_k$ for each $k = 1, \dots, 7$. The snowflake is $M_{\mathbb{C}}(h_1, \dots, h_7)$. The definition of the pentakun is as follows: for each $k = 1, \dots, 5$, let $q_k = \exp(2k\pi\sqrt{-1}/5)$. We define $g_k : \mathbb{C} \rightarrow \mathbb{C}$ by $g_k(z) = \frac{3-\sqrt{5}}{2}(z - p_k) + p_k$ for each $k = 1, \dots, 5$. The pentakun is $M_{\mathbb{C}}(g_1, \dots, g_5)$.) Then, looking at Figure 5, it is easy to see that for each $(i, j) \in \{1, \dots, m\}^2$ such that $i \neq j$ and $C_{i,j} \neq \emptyset$, there exists an $r \in \{1, \dots, m\}$ such that $h_i^{-1}(C_{i,j}) = \text{Fix}(h_r) \cap L = L_{(r)\infty} \subset (L \setminus \bigcup_{k:k \neq r} h_k(L))$. From Lemmas 3.26 and 3.23, it follows that for any $n \in \mathbb{N}$, if $\mathfrak{M} = (M, (g_1, \dots, g_t))$ is a subsystem of an n -th iterate of \mathcal{L} , then \mathfrak{M} is postunbranched.

Figure 5: (From left to right) Snowflake, Pentakun



In order to state the main results, we need some definitions.

Definition 3.29. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system and let R be a \mathbb{Z} module. Let $w \in \Sigma_m^*$ with $|w| = l$. Let $k \in \mathbb{N}$ with $k > l$. We denote by $N_{k,w}$ (or $N_{k,w}(\mathcal{L})$) the unique full subcomplex of N_k whose vertex set is equal to $\{wx \mid x \in \{1, \dots, m\}^{k-l}\}$. Moreover, for each $j = 1, \dots, m$, we set $N_{1,j} := \{j\}$ ($\subset N_1$). We denote by $w_* : N_k \rightarrow N_{k+l}$ the simplicial map assigning to each vertex $x = (x_1, \dots, x_k) \in \{1, \dots, m\}^k$ the vertex $wx \in \{1, \dots, m\}^{k+l}$. We denote by $w_* : H_p(N_k; R) \rightarrow H_p(N_{k+l,w}; R)$ the homomorphism induced by the above simplicial map $w_* : N_k \rightarrow N_{k+l,w}$. Moreover, we denote by $w^* : H^p(N_{k+l,w}; R) \rightarrow H^p(N_k; R)$ the homomorphism induced by $w_* : N_k \rightarrow N_{k+l,w}$. Moreover, we denote by $\bigoplus_{j=1}^m (j)_* : \bigoplus_{j=1}^m H_p(N_k; R) \rightarrow \bigoplus_{j=1}^m H_p(N_{k+1,j}; R) \cong H_p(\bigcup_{j=1}^m N_{k+1,j}; R)$ the homomorphism $(\alpha_j)_{j=1}^m \mapsto (j_*(\alpha_j))_{j=1}^m$. Moreover, let $\iota : \bigcup_{j=1}^m N_{k+1,j} \rightarrow N_{k+1}$ be the canonical embedding and let $(\eta_k)_* : \bigoplus_{j=1}^m H_p(N_k; R) \rightarrow H_p(N_{k+1}; R)$ be the composition $\iota_* \circ (\bigoplus_{j=1}^m (j)_*)$. Similarly, we denote by $\bigoplus_{j=1}^m (j)^* : H^p(\bigcup_{j=1}^m N_{k+1,j}; R) \cong \bigoplus_{j=1}^m H^p(N_{k+1,j}; R) \rightarrow \bigoplus_{j=1}^m H^p(N_k; R)$ the homomorphism $(\beta_j)_{j=1}^m \mapsto (j^*\beta_j)_{j=1}^m$. Let $\eta_k^* : H^p(N_{k+1}; R) \rightarrow \bigoplus_{j=1}^m H^p(N_k; R)$ be the composition $(\bigoplus_{j=1}^m (j)^*) \circ \iota^*$.

From this definition, it is easy to see that the following lemma holds.

Lemma 3.30. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. When \mathcal{L} is a forward self-similar system, we assume further that $h_j : L \rightarrow L$ is injective for each j . Let $w \in \Sigma_m^*$ with $|w| = l$. Then, for each $k \in \mathbb{N}$, the simplicial map $w_* : N_k \rightarrow N_{k+l,w}$ is isomorphic.

Definition 3.31. Let $\mathcal{L} := (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system and let R be a \mathbb{Z} module. Let $w \in \Sigma_m^*$ with $|w| = l$ and let $k \in \mathbb{N}$. We denote by $w_* : N_k \rightarrow N_{k+l}$ the simplicial

map assigning to each vertex $x = (x_1, \dots, x_k) \in \{1, \dots, m\}^k$ the vertex $wx \in \{1, \dots, m\}^{k+l}$. We denote by $w_* : H_*(\mathcal{L}; R)_k \rightarrow H_*(\mathcal{L}; R)_{k+l}$ the homomorphism induced by the above simplicial map $w_* : N_k \rightarrow N_{k+l}$. Moreover, we denote by $w^* : H^*(\mathcal{L}; R)_{k+l} \rightarrow H^*(\mathcal{L}; R)_k$ the homomorphism induced by $w_* : N_k \rightarrow N_{k+l}$. Moreover, we denote by $q_w : N_1 \rightarrow N_l$ the constant simplicial map assigning to each vertex $x \in \{1, \dots, m\}$ the vertex w .

From the above definition, it is easy to see that the following lemma holds.

Lemma 3.32. *Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Then, for each $k \in \mathbb{N}$ with $k \geq 2$, we have $\varphi_k j_*(x) = j_* \varphi_{k-1}(x)$ for each $x \in N_k$, and $\varphi_1 j_*(x) = q_j(x)$ for each $x \in N_1$. More generally, let $w \in \Sigma_m^*$ with $|w| = l$. Then, for each $k \in \mathbb{N}$ with $k \geq 2$, we have $\varphi_{l+k-1} w_*(x) = w_* \varphi_{k-1}(x)$ for each $x \in N_k$, and $\varphi_l w_*(x) = q_w(x)$ for each $x \in N_1$.*

Definition 3.33. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system and let R be a \mathbb{Z} module. Let $w = (w_1, \dots, w_l) \in \Sigma_m^*$ with $|w| = l$. We define a homomorphism $w_* : \check{H}_p(\mathcal{L}; R) \rightarrow \check{H}_p(\mathcal{L}; R)$ as follows. Let $a = (a_k) \in \check{H}_p(\mathcal{L}; R) = \varprojlim_k H_p(N_k; R)$ be an element, where for each $k \in \mathbb{N}$, $a_k \in H_p(N_k; R)$ and $(\varphi_k)_*(a_{k+1}) = a_k$. For each $k \in \mathbb{N}$, we set $b_{k+l} := w_*(a_k) \in H_p(N_{k+l}; R)$. Moreover, for each $s \in \mathbb{N}$ with $1 \leq s \leq l$, we set $b_s := (q_{w|s})_*(a_1) \in H_p(N_s; R)$. Then, by Lemma 3.32, $b = (b_t)_{t=1}^\infty$ determines an element in $\check{H}_p(\mathcal{L}; R) = \varprojlim_k \check{H}_p(N_k; R)$. We set $w_*(a) := b$.

Similarly, we define a homomorphism $w^* : \check{H}^p(\mathcal{L}; R) \rightarrow \check{H}^p(\mathcal{L}; R)$ as follows. Let $a \in \check{H}^p(\mathcal{L}; R) = \varinjlim_k H^p(N_k; R)$ be an element. When a is represented by an element $c \in H^p(N_k; R)$ with $k \geq l+1$, we set $c_1 := w^*(c) \in H^p(N_{k-l}; R)$ and let $w^*(a) := \mu_{k-l,p}(c_1) \in \check{H}^p(\mathcal{L}; R)$. When a is represented by an element $c \in H^p(N_k; R)$ with $k \leq l$, we set $c_1 := q_{w|k}^*(c) \in H^p(N_1; R)$ and let $w^*(a) = \mu_{1,p}(c_1) \in \check{H}^p(\mathcal{L}; R)$. By Lemma 3.32, $w^*(a) \in \check{H}^p(\mathcal{L}; R)$ is well defined and independent of the choice of c .

Furthermore, we define a homomorphism $\theta : \check{H}^p(\mathcal{L}; R) \rightarrow \bigoplus_{j=1}^m \check{H}^p(\mathcal{L}; R)$ by $\theta(c) := (j^*(c))_{j=1}^m$.

Definition 3.34. Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Let R be a field and let T be a \mathbb{Z} module. Let $a_{r,k} = a_{r,k}(\mathcal{L}; R) := \dim_R \check{H}^r(\mathcal{L}; R)_k$ for each $r, k \in \mathbb{Z}$ with $r \geq 0, k \geq 1$. Moreover, we set $u^r(\mathcal{L}; R) := \limsup_{k \rightarrow \infty} \frac{1}{k} \log a_{r,k} \in \{-\infty\} \cup [0, +\infty)$ and $l^r(\mathcal{L}; R) := \liminf_{k \rightarrow \infty} \frac{1}{k} \log a_{r,k} \in \{-\infty\} \cup [0, +\infty)$. The quantity $u^r(\mathcal{L}; R)$ is called the r -th **upper cohomological complexity** of \mathcal{L} with coefficients R , and $l^r(\mathcal{L}; R)$ is called the r -th **lower cohomological complexity** of \mathcal{L} with coefficients R . Moreover, let $a_{r,\infty} = a_{r,\infty}(\mathcal{L}; R) := \dim_R \check{H}^r(\mathcal{L}; R)$ and $b_{1,\infty} = b_{1,\infty}(\mathcal{L}; R) := \dim_R \text{Im} \mu_{1,1}$. Moreover, let $S_1 = S_1(\mathcal{L})$ be the CW complex defined by $S_1 := |N_1|/\{1, \dots, m\}$. Moreover, for each $k \in \mathbb{N}$ with $k > 1$, we set $A_k = A_k(\mathcal{L}; T) := \text{Im}((\varphi_{k,1})_* : \check{H}_1(\mathcal{L}; T)_k \rightarrow \check{H}_1(\mathcal{L}; T)_1)$, $B_k = B_k(\mathcal{L}; R) := \text{Im}(\varphi_{k,1}^* : \check{H}^1(\mathcal{L}; R)_1 \rightarrow \check{H}^1(\mathcal{L}; R)_k)$, and $\lambda_k = \lambda_k(\mathcal{L}; R) := \dim_R B_k$.

Remark 3.35. From the above notation, we have $0 \leq a_{r,k} \leq \frac{m^k(m^k-1)\dots(m^k-r)}{(r+1)!}$ and $-\infty \leq l^r(\mathcal{L}; R) \leq u^r(\mathcal{L}; R) \leq (r+1) \log m$. Moreover, by Remark 2.37, it follows that if $\mathcal{L}_1 \cong \mathcal{L}_2$, then $a_{r,k}(\mathcal{L}_1; R) = a_{r,k}(\mathcal{L}_2; R)$, $a_{r,\infty}(\mathcal{L}_1; R) = a_{r,\infty}(\mathcal{L}_2; R)$, $b_{1,\infty}(\mathcal{L}_1; R) = b_{1,\infty}(\mathcal{L}_2; R)$, $l^r(\mathcal{L}_1; R) = l^r(\mathcal{L}_2; R)$, $u^r(\mathcal{L}_1; R) = u^r(\mathcal{L}_2; R)$, $A_k(\mathcal{L}_1; T) \cong A_k(\mathcal{L}_2; T)$, $B_k(\mathcal{L}_1; R) \cong B_k(\mathcal{L}_2; R)$, and $\lambda_k(\mathcal{L}_1; R) = \lambda_k(\mathcal{L}_2; R)$.

We now state one of the main results on the interaction (co)homology groups of postunbranched systems.

Theorem 3.36. *Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. When $\mathcal{L} = (L, (h_1, \dots, h_m))$ is a forward self-similar system, we assume further that $h_j : L \rightarrow L$ is injective for each $j = 1, \dots, m$. Furthermore, let R be a field and let T be a \mathbb{Z} module. Suppose that \mathcal{L} is postunbranched. Then, we have all of the following statements 1, ..., 23.*

1. Let $r \geq 2$ and $k \geq 1$. Then, $a_{r,k+1} = ma_{r,k} + a_{r,1}$ and there exists an exact sequence:

$$0 \longrightarrow \bigoplus_{j=1}^m \check{H}_r(\mathcal{L}; T)_k \xrightarrow{(\eta_k)^*} \check{H}_r(\mathcal{L}; T)_{k+1} \xrightarrow{(\varphi_{k+1,1})^*} \check{H}_r(\mathcal{L}; T)_1 \longrightarrow 0. \quad (2)$$

2. Let $r \geq 2$ and $k \geq 1$. If $\check{H}_r(\mathcal{L}; T)_1 = 0$, then $\check{H}_r(\mathcal{L}; T)_k = \check{H}_r(\mathcal{L}; T) = 0$.

3. Let $r \geq 2$. Then, there exists an exact sequence of R modules:

$$0 \longrightarrow \check{H}^r(\mathcal{L}; R)_1 \xrightarrow{\mu_{1,r}} \check{H}^r(\mathcal{L}; R) \xrightarrow{\theta} \bigoplus_{j=1}^m \check{H}^r(\mathcal{L}; R) \longrightarrow 0. \quad (3)$$

4. Let $r \neq 1$ and $k \geq 1$. Then, $\mu_{k,r} : \check{H}^r(\mathcal{L}; R)_k \rightarrow \check{H}^r(\mathcal{L}; R)$ and $\varphi_k^* : \check{H}^r(\mathcal{L}; R)_k \rightarrow \check{H}^r(\mathcal{L}; R)_{k+1}$ are monomorphisms.

5. Let $r \geq 2$.

(a) If $\check{H}^r(\mathcal{L}; R)_1 = 0$, then for each $k \in \mathbb{N}$, $\check{H}^r(\mathcal{L}; R)_k = 0$ and $\check{H}^r(\mathcal{L}; R) = 0$.

(b) If $\check{H}^r(\mathcal{L}; R)_1 \neq 0$, then $a_{r,\infty} = \infty$.

6. Let $k \in \mathbb{N}$. Then we have the following exact sequences:

$$0 \rightarrow \bigoplus_{j=1}^m \check{H}_1(\mathcal{L}; T)_k \xrightarrow{(\eta_k)^*} \check{H}_1(\mathcal{L}; T)_{k+1} \xrightarrow{(\varphi_{k+1,1})^*} A_{k+1} \rightarrow 0 \quad (4)$$

and

$$0 \rightarrow A_{k+1} \rightarrow H_1(S_1; T) \rightarrow \bigoplus_{j=1}^m \check{H}_0(\mathcal{L}; T)_k \xrightarrow{(\eta_k)^*} \check{H}_0(\mathcal{L}; T)_{k+1} \rightarrow 0. \quad (5)$$

7. Let $k \in \mathbb{N}$. Then we have the following exact sequences of R modules:

$$0 \rightarrow B_{k+1} \rightarrow \check{H}^1(\mathcal{L}; R)_{k+1} \xrightarrow{\eta_k^*} \bigoplus_{j=1}^m \check{H}^1(\mathcal{L}; R)_k \rightarrow 0 \quad (6)$$

and

$$0 \rightarrow \check{H}^0(\mathcal{L}; R)_{k+1} \xrightarrow{\eta_k^*} \bigoplus_{j=1}^m \check{H}^0(\mathcal{L}; R)_k \rightarrow H^1(S_1; R) \rightarrow B_{k+1} \rightarrow 0. \quad (7)$$

8. We have the following exact sequences of R modules:

$$0 \rightarrow \text{Im}\mu_{1,1} \rightarrow \check{H}^1(\mathcal{L}; R) \xrightarrow{\theta} \bigoplus_{j=1}^m \check{H}^1(\mathcal{L}; R) \rightarrow 0 \quad (8)$$

and

$$0 \longrightarrow \check{H}^0(\mathcal{L}; R) \xrightarrow{\theta} \bigoplus_{j=1}^m \check{H}^0(\mathcal{L}; R) \longrightarrow H^1(S_1; R) \longrightarrow \text{Im}\mu_{1,1} \longrightarrow 0. \quad (9)$$

9. Let $k \in \mathbb{N}$. Then, we have that $a_{1,k+1} = ma_{1,k} + \lambda_{k+1}$ and $a_{0,k+1} = ma_{0,k} - m + a_{0,1} - a_{1,1} + \lambda_{k+1}$.

10. For each $k \in \mathbb{N}$, $0 \leq b_{1,\infty} \leq \lambda_{k+2} \leq \lambda_{k+1} \leq \lambda_2 \leq a_{1,1}$. Moreover, there exists a positive integer l such that for each $k \in \mathbb{N}$ with $k \geq l$, $\lambda_k = b_{1,\infty}$.
11. For each $k \in \mathbb{N}$, $a_{0,k+1} = ma_{0,k} - m + a_{0,1} - a_{1,1} - ma_{1,k} + a_{1,k+1}$.
12. For each $k \in \mathbb{N}$, $ma_{1,k} \leq a_{1,k+1} \leq ma_{1,k} + a_{1,1}$.
13. For each $k \in \mathbb{N}$, $ma_{0,k} - m + a_{0,1} - a_{1,1} \leq a_{0,k+1} \leq ma_{0,k} - m + a_{0,1}$.
14. Let $r \geq 1$. Then, either (a) $l^r(\mathfrak{L}; R) = u^r(\mathfrak{L}; R) = -\infty$ or (b) $l^r(\mathfrak{L}; R) = u^r(\mathfrak{L}; R) = \log m$.
15. Either (a) $l^0(\mathfrak{L}; R) = u^0(\mathfrak{L}; R) = 0$ or (b) $l^0(\mathfrak{L}; R) = u^0(\mathfrak{L}; R) = \log m$.
16. Let $r \geq 1$. Then, either $a_{r,\infty} = 0$ or $a_{r,\infty} = \infty$.
17. If $a_{0,\infty} < \infty$, then $m - a_{0,1} + a_{1,1} = (m - 1)a_{0,\infty} + b_{1,\infty}$.
18. If $m \geq 2$ and $\frac{m-a_{0,1}+a_{1,1}}{m-1} \notin \mathbb{N}$, then at least one of $a_{0,\infty}$ and $a_{1,\infty}$ is equal to ∞ .
19. If $m \geq 2$ and there exists an element $k_0 \in \mathbb{N}$ such that $a_{0,k_0} > \frac{1}{m-1}(m - a_{0,1} + a_{1,1})$, then $a_{0,k+1} > a_{0,k}$ for each $k \geq k_0$.
20. If $m \geq 2$, then $a_{0,\infty} \in \{x \in \mathbb{N} \mid a_{0,1} \leq x \leq \frac{1}{m-1}(m - a_{0,1} + a_{1,1})\} \cup \{\infty\}$.
21. If $2 \leq m \leq 6$ and $|N_1|$ is disconnected, then $a_{0,\infty} = \infty$ and L has infinitely many connected components.
22. If $B_2 = 0$, then $\check{H}^1(\mathfrak{L}; R) = 0$.
23. If $|N_1|$ is connected, then we have the following.
 - (a) For each $k \in \mathbb{N}$, we have the following exact sequence:

$$0 \longrightarrow \bigoplus_{j=1}^m \check{H}_1(\mathfrak{L}; T)_k \xrightarrow{(\eta_k)^*} \check{H}_1(\mathfrak{L}; T)_{k+1} \xrightarrow{(\varphi_{k+1,1})^*} \check{H}_1(\mathfrak{L}; T)_1 \longrightarrow 0. \quad (10)$$

- (b) $a_{1,k+1} = ma_{1,k} + a_{1,1}$.
- (c) If $a_{1,1} = 0$, then $a_{1,\infty} = 0$. If $a_{1,1} \neq 0$, then $a_{1,\infty} = \infty$.
- (d) If $\check{H}_1(\mathfrak{L}; \mathbb{Z})_1 = 0$, then, for each $k \in \mathbb{N}$, $\check{H}_1(\mathfrak{L}; T)_k = 0$ and $\check{H}^1(\mathfrak{L}; T)_k = 0$, and $\check{H}_1(\mathfrak{L}; T) = 0$ and $\check{H}^1(\mathfrak{L}; T) = 0$.
- (e) There exists an exact sequence of R modules:

$$0 \longrightarrow \check{H}^1(\mathfrak{L}; R)_1 \xrightarrow{\mu_{1,1}} \check{H}^1(\mathfrak{L}; R) \xrightarrow{\theta} \bigoplus_{j=1}^m \check{H}^1(\mathfrak{L}; R) \longrightarrow 0. \quad (11)$$

We now give some important examples of postunbranched systems.

Proposition 3.37.

1. For each $n \in \mathbb{N} \cup \{0\}$, there exists a postunbranched backward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_{n+2}))$ such that $X = \hat{\mathbb{C}}$, $L \subset \mathbb{C}$, $h_j : X \rightarrow X$ is a topological branched covering for each $j = 1, \dots, n+2$, and $\dim_R \check{H}^n(\mathfrak{L}; R) = \infty$ for each field R . In particular, if $n \geq 2$, then the above \mathfrak{L} satisfies that $\Psi : \check{H}^n(\mathfrak{L}; R) \rightarrow \check{H}^n(L; R)$ is not a monomorphism for each field R .

2. For each $n \in \mathbb{N} \cup \{0\}$, there exists a postunbranched forward self-similar system $\mathcal{L} = (L, (h_1, \dots, h_{n+2}))$ such that $L \subset \mathbb{R}^3$, $h_j : L \rightarrow L$ is injective for each $j = 1, \dots, n+2$, and $\dim_R \check{H}^n(\mathcal{L}; R) = \infty$ for each field R . In particular, if $n \geq 3$, then the above \mathcal{L} satisfies that $\Psi : \check{H}^n(\mathcal{L}; R) \rightarrow \check{H}^n(L; R)$ is not a monomorphism for each field R .

Theorem 3.36-4 implies that under the assumptions of Theorem 3.36, for each nonnegative integer r with $r \neq 1$, $\mu_{1,r} : \check{H}^r(\mathcal{L}; R)_1 \rightarrow \check{H}^r(\mathcal{L}; R)$ is a monomorphism. However, as illustrated in the following Proposition 3.38, even under the assumptions of Theorem 3.36, if $|N_1|$ is disconnected, $\mu_{1,r} : \check{H}^r(\mathcal{L}; R)_1 \rightarrow \check{H}^r(\mathcal{L}; R)$ is not a monomorphism in general.

Proposition 3.38. *There exists a postunbranched forward self-similar system $\mathcal{L} = (L, (h_1, \dots, h_5))$ such that $L \subset \mathbb{C}$, such that h_j is a contracting similitude on \mathbb{C} (hence $h_j : L \rightarrow L$ is injective) for each $j = 1, \dots, 5$, and such that for each field R , we have $a_{1,1} \neq 0$, $B_2 = 0$, $\check{H}^1(\mathcal{L}; R) \cong \check{H}^1(L; R) = 0$, $|N_1|$ is disconnected, $\mathbb{C} \setminus L$ is connected, and $\mu_{1,1}$ is not injective. See Figure 9.*

Remark 3.39. Proposition 3.38 means that for a postunbranched system $\mathcal{L} = (L, (h_1, \dots, h_m))$, if $|N_1|$ is disconnected, then we need information on not only $\check{H}^1(\mathcal{L}; R)_1$ but also B_2 (or B_k ($k \geq 3$)), to determine $\check{H}^1(\mathcal{L}; R)_k$ ($k \geq 2$) and $\check{H}^1(\mathcal{L}; R)$. This provides us a new problem: ‘‘Investigate B_k of postunbranched systems with disconnected $|N_1|$.’’

Example 3.40 (Sierpiński gasket). Let $\mathcal{L} = (L, (h_1, h_2, h_3))$ be the postunbranched forward self-similar system in Example 2.36. (Hence L is the Sierpiński gasket (Figure 1).) We easily see that $|N_1|$ is connected, the set of all 1-simplexes of N_1 is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and there exists no r -simplex of N_1 , for each $r \geq 2$. Let R be a field. Then we have $\dim_R H^1(N_1; R) = 1$. Hence, by Theorem 3.36, we obtain that for each $k \in \mathbb{N}$, $a_{1,k+1} = 3a_{1,k} + 1$, and that $\dim_R \check{H}^1(\mathcal{L}; R) = \dim_R \check{H}^1(L; R) = \infty$. Combining it with the Alexander duality theorem ([20]), we see that $\hat{\mathbb{C}} \setminus L$ has infinitely many connected components. Note that $\hat{\mathbb{C}} \setminus L = F(\langle h_1^{-1}, h_2^{-1}, h_3^{-1} \rangle)$.

Example 3.41 (Pentakun). Let $\mathcal{L} = (L, (h_1, \dots, h_5))$ be the forward self-similar system in [15, Example 3.8.11]. Hence L is the pentakun (Figure 5). By Example 3.28, \mathcal{L} is postunbranched. Let R be a field. By [15, Example 3.8.11 (Pentakun)] or Figure 5, we get that $|N_1|$ is connected and $\dim_R H^1(N_1; R) = 1$. Hence, by Theorem 3.36, we obtain that for each $k \in \mathbb{N}$, $a_{1,k+1} = 5a_{1,k} + 1$, and that $\dim_R \check{H}^1(\mathcal{L}; R) = \dim_R \check{H}^1(L; R) = \infty$. Combining it with the Alexander duality theorem ([20]), we see that $\hat{\mathbb{C}} \setminus L$ has infinitely many connected components. Note that $\hat{\mathbb{C}} \setminus L = F(\langle h_1^{-1}, \dots, h_5^{-1} \rangle)$.

Example 3.42 (Snowflake). Let $\mathcal{L} = (L, (h_1, \dots, h_7))$ be the forward self-similar system in [15, Example 3.8.12 (Snowflake)]. (Hence L is the snowflake (Figure 5).) By Example 3.28, \mathcal{L} is postunbranched. Let R be a field. By [15, Example 3.8.12 (Snowflake)] or Figure 5, we get that $|N_1|$ is connected and $\dim_R H^1(N_1; R) = 6$. Hence, by Theorem 3.36, we obtain that for each $k \in \mathbb{N}$, $a_{1,k+1} = 7a_{1,k} + 6$, and that $\dim_R \check{H}^1(\mathcal{L}; R) = \dim_R \check{H}^1(L; R) = \infty$. Combining it with the Alexander duality theorem ([20]), we see that $\hat{\mathbb{C}} \setminus L$ has infinitely many connected components. Note that $\hat{\mathbb{C}} \setminus L = F(\langle h_1^{-1}, \dots, h_7^{-1} \rangle)$.

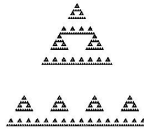
Example 3.43. Let $\mathcal{L} = (L, (h_1, h_2, h_3))$ be the postunbranched forward self-similar system in Example 2.36. (Hence L is the Sierpiński gasket (Figure 1).) Let $g_1 := h_1^2, g_2 := h_1 \circ h_3, g_3 := h_2^2, g_4 := h_2 \circ h_3, g_5 := h_3 \circ h_1, g_6 := h_3 \circ h_2$, and $g_7 := h_3^2$. Let $L' := M_{\mathbb{C}}(g_1, \dots, g_7)$ and let $\mathcal{L}' := (L', (g_1, \dots, g_7))$. For the figure of L' , see Figure 6. By Lemma 3.24 and Lemma 3.23, \mathcal{L}' is postunbranched. It is easy to see that the set of 1-simplexes of $N_1(\mathcal{L}')$ is equal to $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{6, 7\}, \{7, 5\}, \{2, 5\}, \{4, 6\}\}$ and there exists no r -simplex of $N_1(\mathcal{L}')$ for each $r \geq 2$ (cf. Figure 2 and 6). Thus $|N_1(\mathcal{L}')|$ is connected and for each field R , $\check{H}^1(\mathcal{L}'; R)_1 = R$. Hence, by Theorem 3.36, for each $k \in \mathbb{N}$, $a_{1,k+1} = 7a_{1,k} + 1$ and $\dim_R \check{H}^1(\mathcal{L}'; R) = \dim_R \check{H}^1(L'; R) = \infty$. Combining it with the Alexander duality theorem ([20]), $\hat{\mathbb{C}} \setminus L'$ has infinitely many connected components. Note that $\hat{\mathbb{C}} \setminus L' = F(\langle g_1^{-1}, \dots, g_7^{-1} \rangle)$.

Figure 6: The invariant set L' in Example 3.43



Example 3.44. Let $\mathfrak{L} = (L, (h_1, h_2, h_3))$ be the postunbranched forward self-similar system in Example 2.36. (Hence L is the Sierpiński gasket (Figure 1).) Let $g_1 := h_1^2, g_2 := h_1 \circ h_2, g_3 := h_2 \circ h_1, g_4 := h_2^2, g_5 := h_3 \circ h_1, g_6 := h_3 \circ h_2,$ and $g_7 := h_3^2$. Let $L' := M_{\mathbb{C}}(g_1 \dots, g_7)$ and let $\mathfrak{L}' = (L', (g_1, \dots, g_7))$. For the figure of L' , see Figure 7. By Lemma 3.24 and Lemma 3.23, \mathfrak{L}' is postunbranched. It is easy to see that the set of 1-simplexes of $N_1(\mathfrak{L}')$ is equal to $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{5, 6\}, \{6, 7\}, \{7, 5\}\}$ and there exists no r -simplexes of $N_1(\mathfrak{L}')$ for each $r \geq 2$ (cf. Figures 2 and 7). Therefore $|N_1(\mathfrak{L}')|$ is disconnected and $\dim_R \check{H}^1(\mathfrak{L}'; R)_1 = 1$ for each field R . By Theorem 3.36-20 and Remark 2.42, it follows that $\dim_R \check{H}^0(\mathfrak{L}'; R) = \infty$ and L' has infinitely many connected components. Note that $\hat{\mathbb{C}} \setminus L' = F(\langle g_1^{-1}, \dots, g_7^{-1} \rangle)$.

Figure 7: The invariant set L' in Example 3.44



Regarding the postunbranched systems, we have the following lemma.

Lemma 3.45. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a postunbranched forward self-similar system such that for each $j = 1, \dots, m, h_j : L \rightarrow L$ is a contraction. Then, for each $(i, j) \in \{1, \dots, m\}^2$ with $i \neq j, \#C_{i,j} \leq 1$.

Proof. Let $(i, j) \in \{1, \dots, m\}^2$ be any element such that $i \neq j$ and $C_{i,j} \neq \emptyset$. Since \mathfrak{L} is postunbranched, there exists an element $x \in \Sigma_m$ such that $h_i^{-1}(C_{i,j}) \subset L_x$. Since $h_k : L \rightarrow L$ is a contraction for each k , we have that $\#L_x = 1$. Hence $\#C_{i,j} \leq 1$. \square

From Lemma 3.45, it is natural to consider the case $\#C_{i,j} \leq 1$ for each $(i, j) \in \{1, \dots, m\}^2$ with $i \neq j$.

Theorem 3.46. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m, h_j : L \rightarrow L$ is injective. Let T be a \mathbb{Z} module and R a field. Moreover, for each $r \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, let $a_{r,k} := \dim_R \check{H}^r(\mathfrak{L}; R)_k$. Furthermore, let $a_{1,\infty} := \dim_R \check{H}^1(\mathfrak{L}; R)$. Suppose that $\#C_{i,j} \leq 1$ for each (i, j) with $i \neq j$. Then, we have the following.

1. Let $k, r \in \mathbb{N}$ with $r \geq 2$. Then, $\check{H}_r(\mathfrak{L}; T)_k = 0$ and $\check{H}_r(\mathfrak{L}; T) = 0$.

2. For each $k \in \mathbb{N}$, $ma_{1,k} \leq a_{1,k+1}$.
3. If $|N_1|$ is connected and $\check{H}^1(\mathfrak{L}; R) \neq 0$, then $a_{1,\infty} = \infty$.

We present a result on the Čech cohomology groups of the invariant sets of the forward self-similar systems. This is also related to Lemma 3.45.

Proposition 3.47. *Let X be a finite-dimensional topological manifold with a distance. Let L be a non-empty compact subset of X . Let R be a field. Let $m \in \mathbb{N}$ with $m \geq 2$. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system. Suppose that (a) for each $i = 1, 2$, $h_i : L \rightarrow L$ is injective, and (b) for each $(i, j) \in \{1, \dots, m\}^2$ with $i \neq j$, $\dim_T(C_{i,j}) \leq n$, where \dim_T denotes the topological dimension. Then, $\dim_R \check{H}^{n+1}(L; R)$ is either 0 or ∞ .*

4 Tools

In this section, we give some tools to show the main results.

4.1 Fundamental properties of interaction cohomology

In this subsection, we show some fundamental lemmas on the interaction (co)homology groups. We sometimes use the notation from [20].

Definition 4.1. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. For each $k \in \mathbb{N}$, we denote by $\Gamma_k = \Gamma_k(\mathfrak{L})$ the 1-dimensional skeleton of N_k .

Lemma 4.2. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Then, for each $k \in \mathbb{N}$, the simplicial map $\varphi_k : N_{k+1} \rightarrow N_k$ is surjective. That is, for any $r \in \mathbb{N}$, if $x = \{x^1, \dots, x^r\}$ is an $r - 1$ simplex of N_k , then there exists an $r - 1$ simplex $y = \{y^1, \dots, y^r\}$ of N_{k+1} such that $\varphi_k(y) = x$. In particular, $(\varphi_k)_* : \text{Con}(|\Gamma_{k+1}|) \rightarrow \text{Con}(|\Gamma_k|)$ is surjective.*

Proof. We will prove the statement of our lemma when \mathfrak{L} is a backward self-similar system (when \mathfrak{L} is a forward self-similar system, we can prove the statement by using an argument similar to the below). Let $x = \{x^1, \dots, x^r\}$ be an $r - 1$ simplex of N_k , where for each $j = 1, \dots, r$, $x^j = (x_1^j, \dots, x_k^j) \in \{1, \dots, m\}^k$. Then $\bigcap_{j=1}^r h_{x_1^j}^{-1} \cdots h_{x_k^j}^{-1}(L) \neq \emptyset$. Let $z \in \bigcap_{j=1}^r h_{x_1^j}^{-1} \cdots h_{x_k^j}^{-1}(L)$. Then for each $j = 1, \dots, r$, $h_{x_k^j} \cdots h_{x_1^j}(z) \in L = \bigcup_{i=1}^m h_i^{-1}(L)$. Hence, for each $j = 1, \dots, r$, there exists an $x_{k+1}^j \in \{1, \dots, m\}$ such that $h_{x_{k+1}^j} \cdots h_{x_1^j}(z) \in L$. Therefore, $\bigcap_{j=1}^r h_{x_1^j}^{-1} \cdots h_{x_{k+1}^j}^{-1}(L) \neq \emptyset$. Thus, setting $y^j := (x_1^j, \dots, x_{k+1}^j) \in \{1, \dots, m\}^{k+1}$ for each $j = 1, \dots, r$, we have that $y = \{y^1, \dots, y^r\}$ is an $r - 1$ simplex of N_{k+1} such that $\varphi_k(y) = x$. \square

Lemma 4.3. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. If $|\Gamma_1|$ is connected, then, for any $k \in \mathbb{N}$, $|\Gamma_k|$ and $|N_k|$ are connected.*

Proof. We will prove the statement of our lemma when \mathfrak{L} is a backward self-similar system (when \mathfrak{L} is a forward self-similar system, we can prove the statement of our lemma by using an argument similar to the below). First, we show the following claim.

Claim: Let w^1 and w^2 be two elements in $\{1, \dots, m\}^k$ such that $h_{w^1}^{-1}(L) \cap h_{w^2}^{-1}(L) \neq \emptyset$. Then, for any j_1 and j_2 in $\{1, \dots, m\}$, there exists an edge path γ of Γ_{k+1} from $w^1 j_1$ to $w^2 j_2$. (For the definition of edge path, see [20].)

To show this claim, since $L = \bigcup_{j=1}^m h_j^{-1}(L)$, we obtain that there exist w_{k+1}^1 and w_{k+1}^2 in $\{1, \dots, m\}$ such that $h_{w^1}^{-1} h_{w_{k+1}^1}^{-1}(L) \cap h_{w^2}^{-1} h_{w_{k+1}^2}^{-1}(L) \neq \emptyset$. Hence, there exists an edge path α of Γ_{k+1} from $w^1 w_{k+1}^1$ to $w^2 w_{k+1}^2$. Furthermore, since $|\Gamma_1|$ is connected, we have that for each $i = 1, 2$, there

exists an edge path τ_i of Γ_1 from j_i to w_{k+1}^i . Then, for each $i = 1, 2$, there exists an edge path β_i of Γ_{k+1} from $w^i j_i$ to $w^i w_{k+1}^i$. Hence, there exists an edge path of Γ_{k+1} from $w^1 j_1$ to $w^2 j_2$. Therefore, we have shown the above claim.

We now show the statement of our lemma by induction on k . Suppose that $|\Gamma_k|$ is connected. Let x and y be any elements in $\{1, \dots, m\}^{k+1}$. Then, there exists an edge path of Γ_k from $x|k$ and $y|k$. By the above claim, we easily obtain that there exists an edge path of Γ_{k+1} from x and y . Hence, $|\Gamma_{k+1}|$ is connected. Thus, the induction is completed. \square

Definition 4.4. Let K be a simplicial complex and let R be a \mathbb{Z} module. We denote by $C_*(K)$ the oriented chain complex of K ([20, p.159]). Moreover, we set $C_*(K; R) := C_*(K) \otimes R$ and $C^*(K; R) := \text{Hom}(C_*(K), R)$. Similarly, we denote by $\Delta_*(K)$ the ordered chain complex of K ([20, p.170]) and we set $\Delta_*(K; R) := \Delta_*(K) \otimes R$ and $\Delta^*(K; R) := \text{Hom}(\Delta_*(K), R)$. Moreover, for a relative CW complex (X, A) , we denote by $C_*(X, A)$ the chain complex given in [20, p. 475]. Furthermore, we set $C_*(X, A; R) := C_*(X, A) \otimes R$ and $C^*(X, A; R) := \text{Hom}(C_*(X, A), R)$.

Definition 4.5. Let X be a topological space and let R be a \mathbb{Z} module. We regard R as a constant presheaf on X ([20, p. 323]). Moreover, we denote by \hat{R} the completion of the presheaf R ([20, p. 325]). Thus \hat{R} is a sheaf assigning to each non-empty open subset U of X the \mathbb{Z} module of all locally constant functions $a : U \rightarrow R$. Moreover, for an open covering \mathcal{U} of X and a presheaf Γ on X , we denote by $C^*(\mathcal{U}; \Gamma)$ the cochain complex in [20, p. 327] and $H^*(\mathcal{U}; \Gamma)$ its cohomology group. Note that by definition, $\check{H}^*(X; \Gamma) = \varinjlim_{\mathcal{U}} H^*(\mathcal{U}; \Gamma)$ ([20, p. 327]).

Remark 4.6. There is a natural homomorphism $\alpha : R \rightarrow \hat{R}$ such that for each open subset U of X , α assigns $\gamma \in R(U)$ to locally constant function $\hat{\gamma} : U \rightarrow \hat{R}$ with $\hat{\gamma}(a) = \gamma$ for all $a \in U$. (See [20, p. 325].) Thus α induces a natural homomorphism $\alpha_* : C^*(\mathcal{U}; R) \rightarrow C^*(\mathcal{U}; \hat{R})$ for any open covering \mathcal{U} of X .

Lemma 4.7. Let (L, d) be a compact metric space. Let $\mathcal{A} = \{L_i\}_{i=1}^r$ be a finite covering of L such that for each $i = 1, \dots, r$, L_i is a non-empty compact subset of L . Let $\delta(\mathcal{A})$ be the number in Lemma 2.40. Let $0 < \delta < \delta(\mathcal{A})$ and let $\psi_0 : \Delta^*(N(\mathcal{A}); R) \cong \Delta^*(N(\mathcal{A}^\delta); R) \rightarrow C^*(\mathcal{A}^\delta; R)$ be the natural homomorphism. Let $\psi : \Delta^*(N(\mathcal{A}); R) \rightarrow C^*(\mathcal{A}^\delta; \hat{R})$ be the composition $\alpha_* \circ \psi_0$. Moreover, let $\psi_* : H^*(N(\mathcal{A}); R) \cong H^*(\mathcal{A}^\delta; R) \rightarrow H^*(\mathcal{A}^\delta; \hat{R})$ be the homomorphism induced by ψ . Then, we have the following.

1. $\psi_* : H^0(N(\mathcal{A}); R) \cong H^0(\mathcal{A}^\delta; R) \rightarrow H^0(\mathcal{A}^\delta; \hat{R})$ is a monomorphism.
2. In addition to the assumptions of the lemma, suppose that for each $i = 1, \dots, r$, L_i is connected. Then, $\psi_* : H^1(N(\mathcal{A}); R) \cong H^1(\mathcal{A}^\delta; R) \rightarrow H^1(\mathcal{A}^\delta; \hat{R})$ is a monomorphism. Moreover, the natural homomorphism $\Psi_{\mathcal{A}} : H^1(N(\mathcal{A}); R) \cong H^1(\mathcal{A}^\delta; R) \rightarrow \check{H}^1(L; R)$ is monomorphic.

Proof. It is easy to see that statement 1 holds. We now prove statement 2. Let $a = (a_{ij})_{(i,j):L_i \cap L_j \neq \emptyset} \in \Delta^1(N(\mathcal{A}); R)$ be a cocycle, where $a_{ij} : L_i \cap L_j \rightarrow R$ is a constant function for each (i, j) with $L_i \cap L_j \neq \emptyset$. We write $\psi(a)$ as $(b_{ij})_{(i,j):L_i \cap L_j \neq \emptyset}$, where $b_{ij} : B(L_i, \delta) \cap B(L_j, \delta) \rightarrow R$ is a constant function which is an extension of a_{ij} . Suppose that $\psi(a) \in C^1(\mathcal{A}^\delta; \hat{R})$ is a coboundary. Then, there exists an element $(b_i)_{i=1, \dots, r} \in C^0(\mathcal{A}^\delta; \hat{R})$, where each $b_i : B(L_i; \delta) \rightarrow R$ is a locally constant function, such that $b_{ij} = b_j - b_i$ on $B(L_i, \delta) \cap B(L_j, \delta)$. Hence

$$a_{ij} = ((b_j|_{L_j}) - (b_i|_{L_i}))|_{L_i \cap L_j} \text{ on } L_i \cap L_j. \quad (12)$$

Moreover, for each i , since L_i is connected and $b_i : B(L_i, \delta) \rightarrow R$ is locally constant, we have that $b_i|_{L_i} : L_i \rightarrow R$ is constant. Combining it with (12), we obtain that a is a coboundary. Thus, we have proved that $\psi_* : H^1(N(\mathcal{A}); R) \cong H^1(\mathcal{A}^\delta; R) \rightarrow H^1(\mathcal{A}^\delta; \hat{R})$ is a monomorphism. Moreover, by Leray's theorem ([9, Theorem 5 in page 56 and Theorem 11 in page 61]), the natural

homomorphism $H^1(\mathcal{A}^\delta; \hat{R}) \rightarrow \check{H}^1(L; \hat{R})$ is monomorphic. Furthermore, by [20, p.329], the natural homomorphism $\check{H}^1(L; R) \rightarrow \check{H}^1(L, \hat{R})$ is isomorphic. Therefore, the natural homomorphism $\Psi_{\mathcal{A}} : H^1(N(\mathcal{A}); R) \rightarrow \check{H}^1(L; R)$ is monomorphic. Thus, we have proved statement 2. \square

Lemma 4.8. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Let $G = \langle h_1, \dots, h_m \rangle$. Let R be a \mathbb{Z} module. Then, we have the following:*

1. For each $k \in \mathbb{N}$, $\varphi_k^* : \check{H}^0(\mathfrak{L}; R)_k \rightarrow \check{H}^0(\mathfrak{L}; R)_{k+1}$ is a monomorphism. In particular, for each $k \in \mathbb{N}$, the projection map $\mu_{k,0} : \check{H}^0(\mathfrak{L}; R)_k \rightarrow \check{H}^0(\mathfrak{L}; R)$ is injective.
2. $\Psi : \check{H}^0(\mathfrak{L}; R) \rightarrow \check{H}^0(L; R)$ is a monomorphism.
3. Suppose that $|N_1|$ is connected. Then, for each k and each $w \in \Sigma_m$, $(\varphi_k)_* : \check{\pi}_1(\mathfrak{L}, w)_{k+1} \rightarrow \check{\pi}_1(\mathfrak{L}, w)_k$ is an epimorphism and $(\varphi_k)_* : \check{H}_1(\mathfrak{L}; R)_{k+1} \rightarrow \check{H}_1(\mathfrak{L}; R)_k$ is an epimorphism.
4. Suppose that $|N_1|$ is connected. Then, for each k , $\varphi_k^* : \check{H}^1(\mathfrak{L}; R)_k \rightarrow \check{H}^1(\mathfrak{L}; R)_{k+1}$ is a monomorphism and the projection map $\mu_{k,1} : \check{H}^1(\mathfrak{L}; R)_k \rightarrow \check{H}^1(\mathfrak{L}; R)$ is a monomorphism.
5. Suppose that either (a) \mathfrak{L} is a forward self-similar system and L is connected, or (b) \mathfrak{L} is a backward self-similar system such that $g^{-1}(L)$ is connected for each $g \in G$. Then, for each $k \in \mathbb{N}$, the natural homomorphism $\Psi_{\mathcal{U}_k} : H^p(N_k; R) \rightarrow \check{H}^p(L; R)$ in Remark 2.41 is monomorphic, $\Psi : \check{H}^1(\mathfrak{L}; R) \rightarrow \check{H}^1(L; R)$ is a monomorphism, and for each $k \in \mathbb{N}$, $\varphi_k^* : \check{H}^1(\mathfrak{L}; R)_k \rightarrow \check{H}^1(\mathfrak{L}; R)_{k+1}$ is a monomorphism.

Proof. It is easy to see that statement 1 holds. Using Lemma 2.40, it is easy to see that statement 2 holds.

We now prove statements 3 and 4. If $|N_1|$ is connected, then Lemma 4.3 implies that for each $k \in \mathbb{N}$, $|N_k|$ is connected. Let $w \in \Sigma_m$. Let $\zeta \in \pi_1(|N_k|, w)$ be an element. We use the notation in [20]. By [20], there exists a closed edge path $\gamma = \gamma_1 \gamma_2 \cdots \gamma_r$, where each $\gamma_j = (x^j, x^{j+1})$ is an edge of N_k , such that γ represents the element ζ . For each $j = 1, \dots, r+1$, we write x^j as $(x_1^j, \dots, x_m^j) \in \{1, \dots, m\}^k$. By Lemma 4.2, for each $j = 1, \dots, r$ there exists an edge τ_j of N_{k+1} such that $\varphi_k(\tau_j) = \gamma_j$. Then, there exists $y^j, z^j \in \{1, \dots, m\}$ such that the origin of τ_j is equal to $x^j y^j$ and the end of τ_j is equal to $x^{j+1} z^j$. Since we are assuming that $|N_1|$ is connected, for each $j = 2, \dots, r$, there exists an edge path $\beta_j = (v_j^1, v_j^2)(v_j^2, v_j^3) \cdots (v_j^{s_j-1}, v_j^{s_j})$ of N_1 , where each v_j^k is a vertex of N_1 , such that $v_j^1 = z^{j-1}$ and $v_j^{s_j} = y^j$. Similarly, there exists an edge path $\beta_{r+1} = (v_{r+1}^1, v_{r+1}^2) \cdots (v_{r+1}^{s_{r+1}-1}, v_{r+1}^{s_{r+1}})$ of N_1 such that $v_{r+1}^1 = z^r$ and $v_{r+1}^{s_{r+1}} = y^1$. For each $j = 2, \dots, r+1$, let $\delta_j := (x^j v_j^1, x^j v_j^2) \cdots (x^j v_j^{s_j-1}, x^j v_j^{s_j})$. Then, for each $j = 2, \dots, r$, δ_j is an edge path of N_{k+1} from $x^j z^{j-1}$ to $x^j y^j$. Moreover, δ_{r+1} is an edge path of N_{k+1} from $x^{r+1} z^r$ to $x^1 y^1$. Let $\delta := \tau_1 \delta_2 \tau_2 \delta_3 \tau_3 \delta_4 \cdots \tau_r \delta_{r+1}$. Then, δ is a closed edge path of N_{k+1} such that $\varphi_k(\delta) \sim \gamma$. Therefore, $(\varphi_k)_* : \check{\pi}_1(\mathfrak{L}, w)_{k+1} \rightarrow \check{\pi}_1(\mathfrak{L}, w)_k$ is an epimorphism. Moreover, by Lemma 4.3 and [20, p.394], for each $k \in \mathbb{N}$, the natural homomorphism $\check{\pi}_1(\mathfrak{L}, w)_k \rightarrow \check{H}_1(\mathfrak{L}; \mathbb{Z})_k$ is an epimorphism. Therefore, $(\varphi_k)_* : \check{H}_1(\mathfrak{L}; \mathbb{Z})_{k+1} \rightarrow \check{H}_1(\mathfrak{L}; \mathbb{Z})_k$ is an epimorphism. From the universal-coefficient theorem for homology ([20, p.222]), it follows that for any \mathbb{Z} module R , $(\varphi_k)_* : \check{H}_1(\mathfrak{L}; R)_{k+1} \rightarrow \check{H}_1(\mathfrak{L}; R)_k$ is an epimorphism. Similarly, from the universal-coefficient theorem for cohomology ([20, p.243]), it follows that for any \mathbb{Z} module R , $(\varphi_k)_* : \check{H}^1(\mathfrak{L}; R)_k \rightarrow \check{H}^1(\mathfrak{L}; R)_{k+1}$ is a monomorphism. Thus we have proved statement 3 and 4.

Statement 5 follows from Lemma 4.7.

Hence, we have completed the proof of Lemma 4.8. \square

Example 4.9. Let $L = \{a_1, a_2, a_3\}$ be a set where a_j are mutually distinct points. Let $h_1 : L \rightarrow L$ be the map defined by $h_1(a_1) = a_1, h_1(a_2) = h_1(a_3) = a_2$. Similarly, let $h_2 : L \rightarrow L$ be the map defined by $h_2(a_2) = a_2, h_2(a_3) = h_2(a_1) = a_3$. Finally, let $h_3 : L \rightarrow L$ be the map defined by $h_3(a_3) = a_3, h_3(a_1) = h_3(a_2) = a_1$. Then $\mathfrak{L} := (L, (h_1, h_2, h_3))$ is a forward self-similar system. It

is easy to see that the set of one-dimensional simplexes of N_1 is equal to $\{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ and for each $r \geq 2$, there exists no r -dimensional simplex of N_1 . Therefore $|N_1|$ is connected and for each \mathbb{Z} module R , $\check{H}^1(\mathfrak{L}; R)_1 = R \neq 0$. By Lemma 4.8, it follows that $\check{H}_1(\mathfrak{L}; R) \neq 0$, $\check{H}^1(\mathfrak{L}; R) \neq 0$, and $\check{\pi}_1(\mathfrak{L}, w)$ is not trivial for each $w \in \Sigma_m$. However, since L is a finite set, $\check{\pi}_r(L, x)$, $\check{H}^r(L; R)$ and $\check{H}_r(L; R)$ are trivial for each $r \geq 1$ and each $x \in L$. This example means that the interaction cohomology groups of self-similar systems may have more information than the Čech (co)homology groups and the shape groups of the invariant sets of the systems.

Example 4.10. Let $L = \{a_1, a_2, a_3\}$ be a set where a_j are mutually distinct points. For each $j = 1, 2, 3$, let $g_j : L \rightarrow L$ be the map defined by $g_j(L) = \{a_j\}$. Then $\mathfrak{L}' = (L, (g_1, g_2, g_3))$ is a forward self-similar system. It is easy to see that for each $r \geq 1$, there exists no r -dimensional simplexes of N_1 . Moreover, since each g_j is a contraction and L is a finite set, it follows that $\check{H}^r(\mathfrak{L}'; R)_k$, $\check{H}^r(\mathfrak{L}'; R)$, $\check{H}_r(\mathfrak{L}'; R)_k$, $\check{H}_r(\mathfrak{L}'; R)_k$, and $\check{\pi}_r(\mathfrak{L}'; w)$ are trivial, for all $r \geq 1$, $k \geq 1$, $w \in \Sigma_3$, and \mathbb{Z} modules R .

Remark 4.11. Example 4.9 and Example 4.10 mean that for any two self-similar systems $\mathfrak{L}_1 = (L_1, (h_1, \dots, h_m))$ and $\mathfrak{L}_2 = (L_2, (g_1, \dots, g_n))$, interaction cohomology groups $\check{H}^r(\mathfrak{L}_1; R)$ and $\check{H}^r(\mathfrak{L}_2; R)$ may not be isomorphic even when L_1 and L_2 are homeomorphic.

4.2 Fundamental properties of rational semigroups

We give some fundamental properties of rational semigroups. Let G be a rational semigroup. We set $E(G) := \{z \in \hat{\mathbb{C}} \mid \#\bigcup_{g \in G} g^{-1}(z) < \infty\}$. This is called the **exceptional set** of G . If $z \in \hat{\mathbb{C}} \setminus E(G)$, then $J(G) \subset \overline{\bigcup_{g \in G} g^{-1}(\{z\})}$. In particular if $z \in J(G) \setminus E(G)$, then $\overline{\bigcup_{g \in G} g^{-1}(\{z\})} = J(G)$. If $\#J(G) \geq 3$, then $J(G)$ is a perfect set, $\#E(G) \leq 2$, $J(G)$ is the smallest in $\{K \subset \hat{\mathbb{C}} \mid K : \text{compact}, \#K \geq 3, \text{ and } g^{-1}(K) \subset K \text{ for each } g \in G\}$, and

$$J(G) = \overline{\{z \in \hat{\mathbb{C}} \mid \exists g \in G \text{ s.t. } g(z) = z \text{ and } |g'(z)| > 1\}} = \bigcup_{g \in G} J(g).$$

For the proofs of these results, see [11, 8] and [25, Lemma 2.3].

4.3 Fiberwise (Wordwise) dynamics

In this subsection, we give some notations and fundamental properties of skew products related to finitely generated rational semigroups.

Definition 4.12 ([26, 25]). Let $G = \langle h_1, \dots, h_m \rangle$ be a finitely generated rational semigroup. We define a map $\sigma : \Sigma_m \rightarrow \Sigma_m$ by: $\sigma(x_1, x_2, \dots) := (x_2, x_3, \dots)$. This is called the **shift map** on Σ_m . Moreover, we define a map $f : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m \times \hat{\mathbb{C}}$ by: $(x, y) \mapsto (\sigma(x), h_{x_1}(y))$, where $x = (x_1, x_2, \dots)$. This is called the **skew product associated with the multi-map** $(h_1, \dots, h_m) \in (\text{Rat})^m$. Let $\pi : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m$ and $\pi_{\hat{\mathbb{C}}} : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the projections. For each $x \in \Sigma_m$ and each $n \in \mathbb{N}$, we set $f_x^n := f^n|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \rightarrow \pi^{-1}(\{\sigma^n(x)\}) \subset \Sigma_m \times \hat{\mathbb{C}}$ and $f_{x,n} := f_{x,n} \circ \dots \circ f_{x_1}$. Moreover, we denote by $F_x(f)$ the set of all points $y \in \hat{\mathbb{C}}$ which has a neighborhood U in $\hat{\mathbb{C}}$ such that $\{f_{x,n} : U \rightarrow \hat{\mathbb{C}}\}_{n \in \mathbb{N}}$ is normal on U . Moreover, we set $J_x(f) := \hat{\mathbb{C}} \setminus F_x(f)$. Furthermore, we set $F^x(f) := \{x\} \times F_x(f)$ and $J^x(f) := \{x\} \times J_x(f)$. We set $\tilde{J}(f) := \overline{\bigcup_{x \in \Sigma_m} J^x(f)}$, where the closure is taken in the product space $\Sigma_m \times \hat{\mathbb{C}}$. Moreover, for each $x \in \Sigma_m$, we set $\hat{J}^x(f) := \pi^{-1}(\{x\}) \cap \tilde{J}(f)$ and $\hat{J}_x(f) := \pi_{\hat{\mathbb{C}}}(\hat{J}^x(f))$. Furthermore, we set $\tilde{F}(f) := (\Sigma_m \times \hat{\mathbb{C}}) \setminus \tilde{J}(f)$.

Remark 4.13. (See [26, Lemma 2.4].) $\tilde{J}(f)$, $J_x(f)$, $J^x(f)$, $\hat{J}_x(f)$, and $\hat{J}^x(f)$ are compact. We have that $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$, $f^{-1}J^{\sigma(x)}(f) = J^x(f)$, $f^{-1}\hat{J}^{\sigma(x)}(f) = \hat{J}^x(f)$, and $\hat{J}_x(f) \supset J_x(f)$. However, the equality $\hat{J}_x(f) = J_x(f)$ does not hold in general. (This is one of the difficulties when we investigate the dynamics of rational semigroups or random complex dynamics.)

Remark 4.14 ([12, 26]). (**Lower semicontinuity of $x \mapsto J_x(f)$**) Suppose that $\deg(h_j) \geq 2$ for each $j = 1, \dots, m$. Then, for each $x \in \Sigma_m$, $J_x(f)$ is a non-empty perfect set. Furthermore, $x \mapsto J_x(f)$ is **lower semicontinuous**, that is, for any point $y \in J_x(f)$ and any sequence $\{x^n\}_{n \in \mathbb{N}}$ in Σ_m with $x^n \rightarrow x$, there exists a sequence $\{y^n\}_{n \in \mathbb{N}}$ in $\hat{\mathbb{C}}$ with $y^n \in J_{x^n}(f)$ ($\forall n$) such that $y^n \rightarrow y$. The above result was shown by using the potential theory. We remark that $x \mapsto J_x(f)$ is not continuous with respect to the Hausdorff topology in general.

Lemma 4.15. *Let $(h_1, \dots, h_m) \in (\text{Rat})^m$ and let $f : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m \times \hat{\mathbb{C}}$ be the skew product associated with (h_1, \dots, h_m) . Let $G = \langle h_1, \dots, h_m \rangle$. Suppose $\sharp J(G) \geq 3$. Then, $\pi_{\hat{\mathbb{C}}}(J(f)) = J(G)$ and for each $x = (x_1, x_2, \dots) \in \Sigma_m$, $\hat{J}_x(f) = \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$.*

Proof. Since $J_x(f) \subset J(G)$ for each $x \in \Sigma_m$, we have $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) \subset J(G)$. By [11, Corollary 3.1] (see also [25, Lemma 2.3 (g)]), we have $J(G) = \overline{\bigcup_{g \in G} J(g)}$. Since $\bigcup_{g \in G} J(g) \subset \pi_{\hat{\mathbb{C}}}(\tilde{J}(f))$, we obtain $J(G) \subset \pi_{\hat{\mathbb{C}}}(\tilde{J}(f))$. Therefore, we obtain $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) = J(G)$.

We now show the latter statement. Let $x = (x_1, x_2, \dots) \in \Sigma_m$. By [26, Lemma 2.4], we see that for each $j \in \mathbb{N}$, $h_{x_j} \cdots h_{x_1}(\hat{J}_x(f)) = \hat{J}_{\sigma^j(x)}(f) \subset J(G)$. Hence, $\hat{J}_x(f) \subset \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$. Suppose that there exists a point $(x, y) \in \Sigma_m \times \hat{\mathbb{C}}$ such that $y \in \left(\bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G)) \right) \setminus \hat{J}_x(f)$. Then, we have $(x, y) \in (\Sigma_m \times \hat{\mathbb{C}}) \setminus \tilde{J}(f)$. Hence, there exists a neighborhood U of x in Σ_m and a neighborhood V of y in $\hat{\mathbb{C}}$ such that $U \times V \subset \tilde{F}(f)$. Then, there exists an $n \in \mathbb{N}$ such that $\{w \in \Sigma_m \mid w_j = x_j, j = 1, \dots, n\} \subset U$. Combining it with [26, Lemma 2.4], we obtain $\tilde{F}(f) \supset f^n(U \times V) \supset \Sigma_m \times \{f_{x,n}(y)\}$. Moreover, since we have $f_{x,n}(y) \in J(G) = \pi_{\hat{\mathbb{C}}}(\tilde{J}(f))$, we get that there exists an element $x' \in \Sigma_m$ such that $(x', f_{x,n}(y)) \in \tilde{J}(f)$. However, it contradicts $(x', f_{x,n}(y)) \in \Sigma_m \times \{f_{x,n}(y)\} \subset \tilde{F}(f)$. Hence, we obtain $\hat{J}_x(f) = \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$. \square

Definition 4.16. Let h_1, \dots, h_m be polynomials and let $f : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m \times \hat{\mathbb{C}}$ be the skew product associated with (h_1, \dots, h_m) . For each $x \in \Sigma_m$, we set $K_x(f) := \{y \in \mathbb{C} \mid \{f_{x,n}(y)\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{C}\}$ and $A_x(f) := \{y \in \hat{\mathbb{C}} \mid f_{x,n}(y) \rightarrow \infty \text{ as } n \rightarrow \infty\}$.

By using the method in [1, 18], the following Lemma 4.17 is easy to show and we omit the proof.

Lemma 4.17. *Let $h_1, \dots, h_m \in \mathcal{Y}$ and let $G := \langle h_1, \dots, h_m \rangle$. Let $f : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m \times \hat{\mathbb{C}}$ be the skew product associated with (h_1, \dots, h_m) . Then, $\infty \in F(G)$ and for each $x \in \Sigma_m$, we have that $\infty \in F_x(f)$, $J_x(f) = \partial K_x(f) = \partial A_x(f)$, and $A_x(f)$ is the connected component of $F_x(f)$ containing ∞ .*

Lemma 4.18. *Let $h_1, \dots, h_m \in \mathcal{Y}$ and let $f : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m \times \hat{\mathbb{C}}$ be the skew product associated with (h_1, \dots, h_m) . Let $G = \langle h_1, \dots, h_m \rangle$. Then, the following (1),(2),(3) are equivalent. (1) $G \in \mathcal{G}$. (2) For each $x \in X$, $J_x(f)$ is connected. (3) For each $x \in X$, $\hat{J}_x(f)$ is connected.*

Proof. First, we show (1) \Rightarrow (2). Suppose that (1) holds. Let $R > 0$ be a number such that for each $x \in X$, $B := \{y \in \hat{\mathbb{C}} \mid |y| > R\} \subset A_x(f)$ and $\overline{f_{x,1}(B)} \subset B$. Then, for each $x \in X$, we have $A_x(f) = \bigcup_{n \in \mathbb{N}} (f_{x,n})^{-1}(B)$ and $(f_{x,n})^{-1}(B) \subset (f_{x,n+1})^{-1}(B)$, for each $n \in \mathbb{N}$. Furthermore, since we assume (1), we see that for each $n \in \mathbb{N}$, $(f_{x,n})^{-1}(B)$ is a simply connected domain, by the Riemann-Hurwitz formula ([1, 18]). Hence, for each $x \in X$, $A_x(f)$ is a simply connected domain. Since $\partial A_x(f) = J_x(f)$ for each $x \in X$, we conclude that for each $x \in X$, $J_x(f)$ is connected. Hence, we have shown (1) \Rightarrow (2).

Next, we show (2) \Rightarrow (3). Suppose that (2) holds. Let $z_1 \in \hat{J}_x(f)$ and $z_2 \in J_x(f)$ be two points. Let $\{x^n\}_{n \in \mathbb{N}}$ be a sequence in Σ_m such that $x^n \rightarrow x$ as $n \rightarrow \infty$, and such that $d(z_1, J_{x^n}(f)) \rightarrow 0$ as $n \rightarrow \infty$. We may assume that there exists a non-empty compact set K in $\hat{\mathbb{C}}$ such that $J_{x^n}(f) \rightarrow K$ as $n \rightarrow \infty$, with respect to the Hausdorff topology in the space of non-empty compact subsets of

$\hat{\mathbb{C}}$. Since we assume (2), K is connected. By Remark 4.14, we have $d(z_2, J_{x^n}(f)) \rightarrow 0$ as $n \rightarrow \infty$. Hence, $z_i \in K$ for each $i = 1, 2$. Therefore, denoting by J the connected component of $\hat{J}_x(f)$ containing K , z_1 and z_2 belong to the same connected component J of $\hat{J}_x(f)$. Thus, we have shown (2) \Rightarrow (3).

Next, we show (3) \Rightarrow (1). Suppose that (3) holds. It is easy to see that $A_x(f) \cap \hat{J}_x(f) = \emptyset$ for each $x \in X$. Hence, $A_x(f)$ is a connected component of $\hat{\mathbb{C}} \setminus \hat{J}_x(f)$. Since we assume (3), we have that for each $x \in X$, $A_x(f)$ is a simply connected domain. Since $(f_{x,1})^{-1}(A_{g(x)}(f)) = A_x(f)$ for each $x \in \Sigma_m$, the Riemann-Hurwitz formula implies that for each $x \in X$, there exists no critical point of $f_{x,1}$ in $A_x(f) \cap \mathbb{C}$. Therefore, we obtain (1). Thus, we have shown (3) \Rightarrow (1). \square

Corollary 4.19. *Let $G = \langle h_1, \dots, h_m \rangle \in \mathcal{G}$. Let $f : \Sigma_m \times \hat{\mathbb{C}} \rightarrow \Sigma_m \times \hat{\mathbb{C}}$ be the skew product associated with (h_1, \dots, h_m) . Then, for each $x \in \Sigma_m$, the following sets $J_x(f)$, $\hat{J}_x(f)$, and $\bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$ are connected.*

Proof. From Lemma 4.15 and Lemma 4.18, the statement of the corollary easily follows. \square

4.4 Dynamics of postcritically bounded polynomial semigroups

We show a lemma on the dynamics of polynomial semigroups in \mathcal{G} .

Lemma 4.20. *Let $G \in \mathcal{G}$. Suppose that $J(G)$ is connected. Then, for any element $g \in G$, $g^{-1}(J(G))$ is connected.*

Proof. Let $g \in G$. Since $G \in \mathcal{G}$, we have that $J(g)$ is a non-empty connected subset of $J(G)$. Let $J \in \text{Con}(g^{-1}(J(G)))$ be any element. By [19] or [1, Lemma 5.7.2], we have that $g(J) = J(G)$. Since $g^{-1}(J(g)) = J(g)$, it follows that $J \cap J(g) \neq \emptyset$. Hence $J(g) \subset J$. Since this holds for any $J \in \text{Con}(g^{-1}(J(G)))$, $g^{-1}(J(G))$ is connected. \square

Remark 4.21. For further results on the dynamics of $G \in \mathcal{G}$, see [31, 32, 33, 30, 29].

5 Proofs of results

In this section, we give the proofs of the main results in section 3.

5.1 Proofs of results in section 3.1

In this subsection, we give the proofs of the results in section 3.1. We need some lemmas.

Definition 5.1. For each $j \in \{1, \dots, m\}$ and each $k \in \mathbb{N}$, we set $(j)^k := (j, j, \dots, j) \in \{1, \dots, m\}^k$.

Lemma 5.2. *Let $m \geq 2$ and let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system. Suppose that for each j with $j \neq 1$, $h_1^{-1}(L) \cap h_j^{-1}(L) = \emptyset$. For each k , let $C_k \in \text{Con}(|\Gamma_k|)$ be the element containing $(1)^k \in \{1, \dots, m\}^k$. Then, we have the following.*

1. For each $k \in \mathbb{N}$, $C_k = \{(1)^k\}$.
2. For each $k \in \mathbb{N}$, $\sharp(\text{Con}(|\Gamma_k|)) < \sharp(\text{Con}(|\Gamma_{k+1}|))$.
3. L has infinitely many connected components.
4. Let $x := (1)^\infty \in \Sigma_m$ and let $x' \in \Sigma_m$ be an element with $x \neq x'$. Then, for any $y \in L_x$ and $y' \in L_{x'}$, there exists no connected component A of L such that $y \in A$ and $y' \in A$.

Proof. We show statement 1 by induction on k . We have $C_1 = \{1\}$. Suppose $C_k = \{(1)^k\}$. Let $w \in \{1, \dots, m\}^{k+1} \cap C_{k+1}$ be any element. Since $(\varphi_k)_*(C_{k+1}) = C_k$, we have $\varphi_k(w) = (1)^k$. Hence, $w|k = (1)^k$. Since $h_1^{-1}(L) \cap h_j^{-1}(L) = \emptyset$ for each $j \neq 1$, we obtain $w = (1)^{k+1}$. Hence, the induction is completed. Therefore, we have shown statement 1.

Since both $(1)^{k+1} \in \{1, \dots, m\}^{k+1}$ and $(1)^k 2 \in \{1, \dots, m\}^{k+1}$ are mapped to $(1)^k$ by φ_k , combining statement 1 and Lemma 4.2, we obtain statement 2. For each $k \in \mathbb{N}$, we have

$$L = \coprod_{C \in \text{Con}(|\Gamma_k|)} \bigcup_{w \in \{1, \dots, m\}^k \cap C} h_w^{-1}(L). \quad (13)$$

Hence, by statement 2, we obtain that L has infinitely many connected components.

We now show statement 4. Let $k_0 := \min\{l \in \mathbb{N} \mid x'_l \neq 1\}$. Then, by (13) and statement 1, we obtain that there exist compact sets B_1 and B_2 such that $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = L$, $L_x \subset (h_1^{k_0})^{-1}(L) \subset B_1$, and $L_{x'} \subset h_{x'_1}^{-1} \cdots h_{x'_{k_0}}^{-1}(L) \subset B_2$. Hence, statement 4 holds. \square

By an argument similar to that of the proof of Lemma 5.2, we can prove the following.

Lemma 5.3. *Let $m \geq 2$ and let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is injective. Suppose that for each j with $j \neq 1$, $h_1(L) \cap h_j(L) = \emptyset$. For each k , let $C_k \in \text{Con}(|\Gamma_k|)$ be the element containing $(1)^k \in \{1, \dots, m\}^k$. Then, all of the statements 1–4 in Lemma 5.2 hold.*

To prove Theorem 3.2, we need the following lemma.

Lemma 5.4. *Under the assumptions of Theorem 3.2, let M_1, \dots, M_r be mutually disjoint non-empty compact subsets of L with $L = \bigcup_{i=1}^r M_i$. Then there exists a number $l_0 \in \mathbb{N}$ such that for each $x \in \Sigma_m$ and each $l \in \mathbb{N}$ with $l \geq l_0$, there exists a number $i = i(x, l) \in \{1, \dots, r\}$ with $h_{x|l}^{-1}(L) \subset M_i$.*

Proof. Suppose that the statement is not true. Then for each $n \in \mathbb{N}$, there exist an element $w^n \in \Sigma_m$, an $l(n) > n$, and elements $i_{1,n}, i_{2,n} \in \{1, \dots, r\}$ with $M_{i_{1,n}} \neq M_{i_{2,n}}$, such that $(h_{w^n|l(n)})^{-1}(L) \cap M_i \neq \emptyset$, for each $i = i_{1,n}, i_{2,n}$. Since Σ_m is compact, we may assume that there exists an element $w \in \Sigma_m$ such that for each $n \in \mathbb{N}$, $w^n|l(n) = (w|n)\alpha_n$ for some $\alpha_n \in \Sigma_m^*$.

Then, we have $h_{w|n}^{-1} h_{\alpha_n}^{-1}(L) \cap M_i \neq \emptyset$, for each $i = i_{1,n}, i_{2,n}$. Hence, $h_{w|n}^{-1}(L) \cap M_i \neq \emptyset$, for each $i = i_{1,n}, i_{2,n}$. Since $h_{w|n}^{-1}(L) \rightarrow L_w$ as $n \rightarrow \infty$ with respect to the Hausdorff topology and L_w is connected (the assumption), we obtain a contradiction. \square

By an argument similar to that of the proof of Lemma 5.5, we can prove the following.

Lemma 5.5. *Under the assumptions of Theorem 3.3, let M_1, \dots, M_r be mutually disjoint non-empty compact subsets of L with $L = \bigcup_{i=1}^r M_i$. Then, there exists a number $l_0 \in \mathbb{N}$ such that for each $x \in \Sigma_m$ and each $l \in \mathbb{N}$ with $l \geq l_0$, there exists a number $i = i(x, l) \in \{1, \dots, r\}$ with $h_{x|l}^{-1}(L) \subset M_i$.*

We now demonstrate Theorem 3.2.

Proof of Theorem 3.2: Step 1: First, we show the following:

Claim 1: Let $B = (B_k)_k \in \varprojlim \text{Con}(|\Gamma_k|)$ where $B_k \in \text{Con}(|\Gamma_k|)$ and $(\varphi_k)_*(B_{k+1}) = B_k$ for each k . Take a point $x \in \Sigma_m$ such that $x|k \in B_k$ for each k . Take an element $C_x \in \text{Con}(L)$ such that $L_x \subset C_x$. Then, C_x does not depend on the choice of $x \in \Sigma_m$ such that $x|k \in B_k$ for each k . Hence, the map $\Phi : B \mapsto C_x$ is well-defined.

To show Claim 1, suppose that there exist $x \in \Sigma_m$ and $y \in \Sigma_m$ such that $x|k, y|k \in B_k$ for each $k \in \mathbb{N}$ and such that there exist mutually different connected components J_1 and J_2 of L with $L_x \subset J_1$ and $L_y \subset J_2$. By the ‘‘Cut Wire Theorem’’ in [19], there exist mutually disjoint

compact subsets M_1 and M_2 of L such that $J_i \subset M_i$ for each $i = 1, 2$. We apply Lemma 5.4 to the disjoint union $L = M_1 \cup M_2$ and let l_0 be the number in the lemma. Then, we have $h_{x|l_0}^{-1}(L) \subset M_1$, $h_{y|l_0}^{-1}(L) \subset M_2$, and $L = \bigcup_{|w|=l_0} h_w^{-1}(L) = \prod_{i=1}^2 \bigcup_{h_w^{-1}(L) \subset M_i, |w|=l_0} h_w^{-1}(L)$. This implies that $x|l_0$ and $y|l_0$ do not belong to the same connected component of $|\Gamma_{l_0}|$. This is a contradiction. Hence, we have shown Claim 1.

Step 2: Next, we show the following:

Claim 2: $\Phi : \varprojlim \text{Con}(|\Gamma_k|) \rightarrow \text{Con}(L)$ is bijective.

To show Claim 2, since $L = \bigcup_{j=1}^m h_j^{-1}(L)$, we have $L = \bigcup_{x \in \Sigma_m} L_x$. Hence, Φ is surjective. To show that Φ is injective, let $B = (B_k)$ and $B' = (B'_k)$ be distinct elements in $\varprojlim \text{Con}(|\Gamma_k|)$, let $x \in \Sigma_m$ be such that $x|k \in B_k$ for each $k \in \mathbb{N}$, and let $y \in \Sigma_m$ be such that $y|k \in B'_k$ for each $k \in \mathbb{N}$. Then, there exists a $k \in \mathbb{N}$ with $B_k \neq B'_k$. Combining this with $L = \prod_{C \in \text{Con}(|\Gamma_k|)} \bigcup_{w \in \Sigma_m^* \cap C, |w|=k} h_w^{-1}(L)$, which follows from $L = \bigcup_{w \in \Sigma_m^*, |w|=k} h_w^{-1}(L)$, we obtain that there exist two compact subsets K_1 and K_2 of L such that $L = K_1 \amalg K_2$, $L_x \subset h_{x|k}^{-1}(L) \subset K_1$, and $L_y \subset h_{y|k}^{-1}(L) \subset K_2$. Hence, $\Phi(B) \neq \Phi(B')$. Therefore, Φ is injective.

Step 3: We now show statement 2. Since $L = \bigcup_{j=1}^m h_j^{-1}(L)$, it is easy to see that if L is connected, then $|\Gamma_1|$ is connected. Conversely, suppose that $|\Gamma_1|$ is connected. Then, by Lemma 4.3, we obtain that for each $k \in \mathbb{N}$, $|\Gamma_k|$ is connected. From statement 1, it follows that L is connected. Hence, we have shown statement 2.

Step 4: Statement 3 follows from statement 1 and Lemma 4.2. Statement 4 and 5 easily follow from statement 3.

Step 5: We now show statement 6. If $m = 2$ and L is disconnected, then by statement 2, we have $h_1^{-1}(L) \cap h_2^{-1}(L) = \emptyset$. Combining this with statement 1, we obtain $\text{Con}(L) \cong \{1, 2\}^{\mathbb{N}}$.

Step 6: We now show statement 7. Suppose that $m = 3$ and L is disconnected. By statement 2, we may assume $h_1^{-1}(L) \cap h_2^{-1}(L) = h_1^{-1}(L) \cap h_3^{-1}(L) = \emptyset$. By Lemma 5.2, we obtain that L has infinitely many connected components and that $L_{(1)^\infty}$ is a connected component of L .

Thus we have completed the proof of Theorem 3.2. \square

We now prove Theorem 3.3.

Proof of Theorem 3.3: The statements of the theorem easily follow from the argument of the proof of Theorem 3.2, Lemma 5.5, and Lemma 5.3. \square

In order to prove Theorem 3.7, we need the following notations and lemmas.

Lemma 5.6. *Under the assumptions of Theorem 3.7 or Theorem 3.8, let $k \in \mathbb{N}$. Then, for any simplex s of N_k with $(1)^k \in s$, the dimension $\dim s$ of s is less than or equal to 1.*

Proof. We will show the conclusion of our lemma for a backward self-similar system $\mathfrak{L} = (L, (h_1, \dots, h_m))$ satisfying the assumptions of Theorem 3.7 (using an argument similar to the below, we can show the conclusion of our lemma for a forward self-similar system \mathfrak{L} satisfying the assumptions of Theorem 3.8). We will show the conclusion of our lemma by induction on $k \in \mathbb{N}$. If $k = 1$, then, assumption 4 of Theorem 3.7 implies that for any simplex s of N_1 with $1 \in s$, we have $\dim s \leq 1$. Let $l \in \mathbb{N}$ and we now suppose that for any simplex s of N_l with $(1)^l \in s$, we have $\dim s \leq 1$. Then, Lemma 3.30 implies that for any simplex s of $N_{l+1,1}$ with $(1)^{l+1} \in s$, we have $\dim s \leq 1$. Moreover, by assumption 2 of Theorem 3.7, we have $(h_1^r)^{-1}(L) \cap (\bigcup_{i \neq 1} h_i^{-1}(L)) = \emptyset$ for each $r \geq 2$. Hence, it follows that for any $i \in \{1, \dots, m\}$ with $i \neq 1$ and any $w \in \Sigma_m^*$ with $|w| = l$, $\{(1)^{l+1}, iw\}$ is not a simplex of N_{l+1} . Therefore, for any simplex s of N_{l+1} with $(1)^{l+1} \in s$, we have $\dim s \leq 1$. Thus, the induction is completed. \square

Definition 5.7. Let S be a simplicial complex and let $\tau = (v_1, v_2)(v_2, v_3) \cdots (v_{n-1}, v_n)$ be an edge path of S . We denote by $|\tau|$ the curve in $|S|$ which is induced by τ in the way as in [20, p.136].

Definition 5.8. Let \mathcal{L} be a forward or backward self-similar system, let $k \in \mathbb{N}$, and let $w \in \Sigma_m^*$. Then for any edge path $\tau = (v_1, v_2)(v_2, v_3) \cdots (v_{n-1}, v_n)$ of N_k , we denote by $w_*(\tau)$ the edge path $(wv_1, wv_2)(wv_2, wv_3) \cdots (wv_{n-1}, wv_n)$ of $N_{k+|w|}$.

Lemma 5.9. *Under the assumptions of Theorem 3.7 or Theorem 3.8, let τ be the closed edge path $(1, j_2)(j_2, j_3)(j_3, 1)$ of N_1 . Moreover, let $\gamma \in H_1(|N_1|; R)$ be the element induced by the closed curve $|\tau|$ in $|N_1|$. Then, for each $k \in \mathbb{N}$, the element $((1)^k)_*(\gamma) \in H_1(|N_{k+1}|; R)$ is not zero.*

Proof. For each $k \in \mathbb{N}$, let M_k be the unique full subcomplex of N_k whose vertex set is equal to $\{1, \dots, m\}^k \setminus \{(1)^k\}$. Moreover, let P_k be the set of all 1-simplexes e of N_{k+1} such that $(1)^{k+1} \in e$, $(1)^k j_2 \notin e$, and $(1)^k j_3 \notin e$. Furthermore, let $Q_k = \bigcup_{e \in P_k} e$. Note that Q_k is a subcomplex of N_{k+1} . Lemma 5.6 implies that for each $k \in \mathbb{N}$, $|N_{k+1}| = |((1)^k)_*(\tau)| \cup |Q_k| \cup |M_{k+1}|$. Moreover, $(|((1)^k)_*(\tau)| \cup |Q_k|) \cap |M_{k+1}| = |((1)^k j_2, (1)^k j_3)| \cup \bigcup_{e \in P_k} \{e_0\}$, where for each $e \in P_k$, e_0 denotes the vertex of e which is not equal to $(1)^{k+1}$. In particular, each connected component of $(|((1)^k)_*(\tau)| \cup |Q_k|) \cap |M_{k+1}|$ is contractible. Using the Mayer-Vietoris sequence of $\{|((1)^k)_*(\tau)| \cup |Q_k|, |M_{k+1}|\}$, we obtain the following exact sequence:

$$0 = H_1(|((1)^k)_*(\tau)| \cup |Q_k| \cap |M_{k+1}|; R) \rightarrow H_1(|((1)^k)_*(\tau)| \cup |Q_k|; R) \oplus H_1(|M_{k+1}|; R) \rightarrow H_1(|N_{k+1}|; R). \quad (14)$$

Let $u_1 : |((1)^k)_*(\tau)| \rightarrow |((1)^k)_*(\tau)| \cup |Q_k|$, $u_2 : |((1)^k)_*(\tau)| \cup |Q_k| \rightarrow |N_{k+1}|$, and $u_3 : |((1)^k)_*(\tau)| \rightarrow |N_{k+1}|$ be the inclusion maps. Then, $u_3 = u_2 \circ u_1$. Moreover, $(u_1)_* : H_1(|((1)^k)_*(\tau)|; R) \rightarrow H_1(|((1)^k)_*(\tau)| \cup |Q_k|; R)$ is an isomorphism. Furthermore, $((1)^k)_*(\gamma) = (u_3)_*(a)$ in $H_1(|N_{k+1}|; R)$, where a is a generator in $H_1(|((1)^k)_*(\tau)|; R)$. From these arguments, it follows that the element $((1)^k)_*(\gamma) \in H_1(|N_{k+1}|; R)$ is not zero. Thus, we have proved the lemma. \square

Lemma 5.10. *Under the assumptions of Theorem 3.7 or Theorem 3.8, we have that for each $k \in \mathbb{N}$, $\dim_R \check{H}^1(\mathcal{L}; R)_k = \dim_R \check{H}_1(\mathcal{L}; R)_k < \dim_R \check{H}^1(\mathcal{L}; R)_{k+1} = \dim_R \check{H}_1(\mathcal{L}; R)_{k+1}$.*

Proof. We use the notation in Lemma 5.9. By Lemma 5.9, we have that for each $k \in \mathbb{N}$, $((1)^k)_*(\gamma) \in H_1(|N_{k+1}|; R)$ is not zero. Moreover, by Lemma 3.32, we have that for each $k \in \mathbb{N}$, $(\varphi_k)_*((1)^k)_*(\gamma) = 0$. Hence $(\varphi_k)_* : H_1(|N_{k+1}|; R) \rightarrow H_1(|N_k|; R)$ is not a monomorphism. Furthermore, by assumption 1 of Theorem 3.7 and Theorem 3.8 and Lemma 4.8-3, we have that $(\varphi_k)_* : H_1(|N_{k+1}|; R) \rightarrow H_1(|N_k|; R)$ is an epimorphism. It follows that for each $k \in \mathbb{N}$, $\dim_R H_1(|N_k|; R) < \dim_R H_1(|N_{k+1}|; R)$. We are done. \square

We now prove Theorem 3.7 and Theorem 3.8.

Proof of Theorem 3.7 and Theorem 3.8: By the assumption 1 of Theorem 3.7 and Theorem 3.8 and Lemma 4.8-4, the projection map $\mu_{k,1} : \check{H}^1(\mathcal{L}; R)_k \rightarrow \check{H}^1(\mathcal{L}; R)$ is injective for each $k \in \mathbb{N}$. Combining it and Lemma 5.10, we obtain that $\dim_R \check{H}^1(\mathcal{L}; R) = \infty$. Thus, we have proved Theorem 3.7 and Theorem 3.8. \square

We now prove Corollary 3.9.

Proof of Corollary 3.9: Since $|N_1|$ is connected and L_x is connected for each $x \in \Sigma_m$, Theorem 3.3 implies that L is connected. Thus for each $w \in \Sigma_m^*$, $h_{\bar{w}}(L)$ is connected. Combining it with Lemma 4.8-5 and Theorem 3.8, we obtain that the statement our corollary holds. \square

5.2 Proofs of results in section 3.2

In this subsection, we give the proofs of the results in subsection 3.2.

We now prove Theorem 3.17.

Proof of Theorem 3.17 : From Theorem 3.2 and Corollary 4.19, the statement of the theorem follows. \square

We now prove Theorem 3.19.

Proof of Theorem 3.19: By Theorem 3.17, $J(G)$ is connected. Combining it with Lemma 4.20, we obtain that for each $g \in G$, $g^{-1}(J(G))$ is connected. By Lemma 4.8-5, it follows that $\Psi : \check{H}^1(\mathcal{L}; R) \rightarrow \check{H}^1(J(G), R)$ is a monomorphism. Moreover, by Theorem 3.7, we obtain $\dim_R \check{H}^1(\mathcal{L}; R) = \infty$. Hence, $\dim_R \Psi(\check{H}^1(\mathcal{L}; R)) = \infty$. Therefore, $\dim_R \check{H}^1(J(G); R) = \infty$. By the Alexander duality theorem (see [20, p.296]), we have $\check{H}^1(J(G); R) \cong \check{H}_0(\hat{\mathbb{C}} \setminus J(G); R)$, where \check{H}_0 denotes the 0-th reduced homology. Hence, $F(G) = \hat{\mathbb{C}} \setminus J(G)$ has infinitely many connected components. \square

We now prove Proposition 3.20.

Proof of Proposition 3.20: Let $a \in \mathbb{R}$ with $1 < a \leq 5$. Let $h_1(z) = \frac{1}{a^2}z^3$ and $h_2(z) = z^2$. Then $J(h_1) = \{z \in \mathbb{C} \mid |z| = a\}$, $J(h_2) = \{z \in \mathbb{C} \mid |z| = 1\}$, $h_1^{-1}(J(h_2)) = \{z \in \mathbb{C} \mid |z| = a^{2/3}\}$, and $h_2^{-1}(J(h_1)) = \{z \in \mathbb{C} \mid |z| = a^{1/2}\}$. Let $c_1 := (a^{2/3} - a^{1/2})/2$. Let g_3 be a polynomial such that $J(g_3) = \{z \in \mathbb{C} \mid |z - c_1| = a^{2/3} - c_1\}$ and let g_4 be a polynomial such that $J(g_4) = \{z \in \mathbb{C} \mid |z + c_1| = a^{2/3} - c_1\}$. Take a sufficiently large $n \in \mathbb{N}$ and let $h_3 = g_3^n$ and $h_4 = g_4^n$. Let $G = \langle h_1, h_2, h_3, h_4 \rangle$ and let $K := \{z \in \mathbb{C} \mid 1 \leq |z| \leq a\}$. Then, taking a sufficiently large n , we have $\bigcup_{i=1}^4 h_i^{-1}(K) \subset K$. Therefore, by [11, Corollary 3.2], $J(G) \subset K$. (For the figure of $J(G)$, see Figure 8.) Moreover, we can show that $G \in \mathcal{G}$, the set of all 1-simplexes of N_1 is equal to $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, and there exists no r -simplex S of N_1 for each $r \geq 2$. Taking a sufficiently large n again, it is easy to show that $\mathfrak{L} = (J(G), (h_1, h_2, h_3, h_4))$ satisfies all of the conditions 1,...,4 in the assumptions of Theorem 3.7. From Theorem 3.19, it follows that $\dim_R \check{H}^1(J(G), (h_1, \dots, h_4); R) = \dim_R \Psi(\check{H}^1(J(G), (h_1, \dots, h_4); R)) = \infty$ and $F(G)$ has infinitely many connected components. Thus we have completed the proof. \square

Figure 8: The Julia set of G in Proposition 3.20.



5.3 Proofs of results in section 3.3

In this subsection, we prove the results in section 3.3. We need some lemmas.

Lemma 5.11. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Suppose that \mathfrak{L} is postunbranched. Let $r \in \mathbb{N}$. Then, for each r -simplex e of N_1 , there exists a unique r -simplex e_{k+1} of N_{k+1} such that $\varphi_{k+1,1}(e_{k+1}) = e$.*

Proof. We will show the conclusion of our lemma when \mathfrak{L} is a backward self-similar system (we can show the conclusion of our lemma when \mathfrak{L} is a forward self-similar system by using an argument similar to the below). The existence of e_{k+1} follows from Lemma 4.2. We now prove the uniqueness. Case 1: $r = 1$. Let $e = \{i_1, j_1\}$ be a 1-simplex of N_1 . Then $C_{i_1, j_1} = h_{i_1}^{-1}(L) \cap h_{j_1}^{-1}(L) \neq \emptyset$. Since \mathfrak{L} is postunbranched, there exists a unique $x \in \Sigma_m$ such that $h_{i_1}(C_{i_1, j_1}) \subset L_x$ and such that for each

$x' \in \Sigma_m$ with $x' \neq x$, $h_{i_1}(C_{i_1, j_1}) \cap L_{x'} = \emptyset$. Let $e_{k+1} = \{(i_1, \dots, i_{k+1}), (j_1, \dots, j_{k+1})\}$ be a 1-simplex of N_{k+1} such that $\varphi_{k+1,1}(e_{k+1}) = e$. We will show that (i_2, \dots, i_{k+1}) and (j_2, \dots, j_{k+1}) are uniquely determined by the element (i_1, j_1) . Since e_{k+1} is a 1-simplex of N_{k+1} , we have $h_{i_1}^{-1} \dots h_{i_{k+1}}^{-1}(L) \cap h_{j_1}^{-1} \dots h_{j_{k+1}}^{-1}(L) \neq \emptyset$. Let $z \in h_{i_1}^{-1} \dots h_{i_{k+1}}^{-1}(L) \cap h_{j_1}^{-1} \dots h_{j_{k+1}}^{-1}(L)$ be a point. Then

$$h_{i_1}(z) \in h_{i_2}^{-1} \dots h_{i_{k+1}}^{-1}(L). \quad (15)$$

Moreover, since $z \in h_{i_1}^{-1} \dots h_{i_{k+1}}^{-1}(L) \cap h_{j_1}^{-1} \dots h_{j_{k+1}}^{-1}(L) \subset C_{i_1, j_1}$, we have $h_{i_1}(z) \in h_{i_1}(C_{i_1, j_1})$ and for each $x' \in \Sigma_m$ with $x' \neq x$, $h_{i_1}(z) \notin L_{x'}$. Furthermore, since $L = \bigcup_{y \in \Sigma_m} L_y$, (15) implies that there exists an element $y = (y_1, y_2, \dots) \in \Sigma_m$ such that $h_{i_1}(z) \in h_{i_2}^{-1} \dots h_{i_{k+1}}^{-1}(L_y)$. Let $y' = (i_2, i_3, \dots, i_{k+1}, y_1, y_2, \dots) \in \Sigma_m$. Then $h_{i_1}(z) \in L_{y'}$. From the above arguments, it follows that $y' = x$. Therefore, $(i_2, \dots, i_{k+1}) = (x_1, \dots, x_k)$. Thus, (i_2, \dots, i_{k+1}) is uniquely determined by (i_1, j_1) . Similarly, we can show that (j_2, \dots, j_{k+1}) is uniquely determined by (i_1, j_1) . Hence, e_{k+1} is uniquely determined by e .

Case 2: $r \geq 2$. The uniqueness immediately follows from Case 1.

Thus, we have proved Lemma 5.11. \square

Definition 5.12. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. For each $k \in \mathbb{N}$, we denote by S_k (or $S_k(\mathfrak{L})$) the CW complex $|N_k|/|\bigcup_{j=1}^m N_{k,j}|$. Furthermore, we denote by $p_k : (|N_k|, |\bigcup_{j=1}^m N_{k,j}|) \rightarrow (S_k, *)$ the canonical projection. Moreover, for all $l, k \in \mathbb{N}$ with $l > k$, we denote by $\tilde{\varphi}_{l,k} : S_l \rightarrow S_k$ the cellular map such that the following commutes.

$$\begin{array}{ccc} (|N_l|, |\bigcup_{j=1}^m N_{l,j}|) & \xrightarrow{p_l} & (S_l, *) \\ \downarrow |\varphi_{l,k}| & & \downarrow \tilde{\varphi}_{l,k} \\ (|N_k|, |\bigcup_{j=1}^m N_{k,j}|) & \xrightarrow{p_k} & (S_k, *) \end{array} \quad (16)$$

Lemma 5.13. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Suppose that \mathfrak{L} is postunbranched. Let R be a \mathbb{Z} module. Then, we have the following.

1. For each $k \in \mathbb{N}$, the cellular map $\tilde{\varphi}_{k+1,1} : (S_{k+1}, *) \rightarrow (S_1, *)$ is a cellular isomorphism and a homeomorphism. In particular, $\tilde{\varphi}_{k+1,1}$ induces isomorphisms on homology and cohomology groups with coefficients R .
2. For each $k \in \mathbb{N}$, $|\varphi_{k+1,1}| : (|N_{k+1}|, |\bigcup_{j=1}^m N_{k+1,j}|) \rightarrow (|N_1|, \{1, \dots, m\})$ induces isomorphisms on homology and cohomology groups with coefficient R .

Proof. From Lemma 5.11, statement 1 follows. Since p_k induces isomorphisms on homology and cohomology groups, statement 2 follows from statement 1. Thus, we have proved Lemma 5.13. \square

Lemma 5.14. Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward or backward self-similar system. Suppose that \mathfrak{L} is postunbranched. Let R be a \mathbb{Z} module. Let $r \in \mathbb{N}$ and $k \in \mathbb{N}$. Then, the connecting homomorphism $\partial_* : H_{r+1}(|N_k|, |\bigcup_{j=1}^m N_{k,j}|; R) \rightarrow H_r(|\bigcup_{j=1}^m N_{k,j}|; R)$ of the homology sequence of the pair $(|N_k|, |\bigcup_{j=1}^m N_{k,j}|)$ is the zero map.

Proof. For each $k \in \mathbb{N}$, let $\alpha_k : (N_k, *) \rightarrow (N_k, \bigcup_{j=1}^m N_{k,j})$ be the canonical embedding. Moreover, for each $k \in \mathbb{N}$, let $\gamma_k : (|N_k|, *) \rightarrow (S_k, *)$ be the canonical projection. Then, for each $k \in \mathbb{N}$, $(p_k)_* : H_{r+1}(|N_k|, |\bigcup_{j=1}^m N_{k,j}|; R) \rightarrow H_{r+1}(|S_k|, *; R)$ is an isomorphism, and the following diagram commutes.

$$\begin{array}{ccc} H_{r+1}(|N_k|, *; R) & \xrightarrow{(\gamma_k)_*} & H_{r+1}(S_k, *; R) \\ (\alpha_k)_* \downarrow & & \downarrow Id \\ H_{r+1}(|N_k|, |\bigcup_{j=1}^m N_{k,j}|; R) & \xrightarrow{(p_k)_*} & H_{r+1}(S_k, *; R) \end{array} \quad (17)$$

Hence, we have only to prove that for each $k > 1$, $(\gamma_k)_* : H_{r+1}(|N_k|, *; R) \rightarrow H_{r+1}(S_k, *; R)$ is an epimorphism (if $k = 1$, then it is easy to see that $\text{Im } \partial_* = 0$). In order to do that, let $a = \sum_{i=1}^t a_i d_i \in C_{r+1}(S_k, *; R)$ be a cycle, where for each i , $a_i \in R$ and d_i is an oriented $(r+1)$ -cell. For each i , let $d'_i := \tilde{\varphi}_{k,1}(d_i)$. Then, by Lemma 5.13-1, d'_i is an $(r+1)$ -cell of S_1 . Let d''_i be an oriented $(r+1)$ -cell of $|N_1|$ such that $\gamma_1(d''_i) = d'_i$. Let e''_i be the oriented $(r+1)$ -simplex of N_1 which induces d''_i . Then, by Lemma 5.11, there exists a unique oriented $(r+1)$ -simplex \tilde{e}_i of N_k such that $\varphi_{k,1}(\tilde{e}_i) = e''_i$. Let \tilde{d}_i be the oriented $(r+1)$ -cell of $|N_k|$ which corresponds \tilde{e}_i . Let $c := \sum_{i=1}^t a_i \tilde{d}_i \in C_{r+1}(|N_k|, *; R)$. Then we have $(\gamma_k)_*(c) = a$. We shall prove the following claim: Claim: $\tilde{a} = \sum_{i=1}^t a_i \tilde{e}_i \in C_{r+1}(N_k; R)$ is a cycle.

In order to prove the claim, let $\{i_1 w_1^i, i_2 w_2^i, \dots, i_{r+2} w_{r+2}^i\}$ be the set of vertices of \tilde{e}_i , where $i_s \in \{1, \dots, m\}$ and $w_s^i \in \{1, \dots, m\}^{k-1}$ for each $s = 1, \dots, r+2$. Then, since $\varphi_{k,1}(\tilde{e}_i) = e''_i$, we have that the elements i_1, i_2, \dots, i_{r+2} are mutually distinct. Moreover, we have

$$(\gamma_k)_*(\partial(c)) = (\gamma_k)_*(\partial(\sum_{i=1}^t a_i \tilde{d}_i)) = \partial(\sum_{i=1}^t a_i d_i) = 0. \quad (18)$$

We now suppose that $\partial(\sum_{i=1}^t a_i \tilde{e}_i) = \sum_{j=1}^\beta b_j e_j \neq 0$, where for each $j = 1, \dots, \beta$, e_j is an oriented r -simplex of N_k such that $\{e_1, \dots, e_\beta\}$ is linearly independent, and $b_j \in R$ with $b_j \neq 0$ for each j . We will deduce a contradiction. Let $\{j_1 u_1^j, j_2 u_2^j, \dots, j_{r+1} u_{r+1}^j\}$ be the set of all vertices of e_j , where $j_v \in \{1, \dots, m\}$ and $u_v^j \in \{1, \dots, m\}^{k-1}$ for each $v = 1, \dots, r+1$. Then, since the elements i_1, i_2, \dots, i_{r+2} are mutually distinct, we have that the elements j_1, j_2, \dots, j_{r+1} are mutually distinct. In particular, denoting by c_j the oriented $(r+1)$ -cell of $|N_k|$ which corresponds e_j , we have that $\gamma_k(c_j)$ is an oriented $(r+1)$ -cell for each j . Moreover, since $\{e_1, \dots, e_\beta\}$ is linearly independent, $\{\gamma_k(c_1), \dots, \gamma_k(c_\beta)\}$ is linearly independent. Hence, $(\gamma_k)_*(\partial(c)) = (\gamma_k)_*(\sum_{j=1}^\beta b_j c_j) \neq 0$. However, it contradicts (18). Therefore, $\partial(\sum_{i=1}^t a_i \tilde{e}_i) = 0$. Thus, we have proved the claim.

Since $(\gamma_k)_*(c) = a$, the above claim implies that $(\gamma_k)_* : H_{r+1}(|N_k|, *; R) \rightarrow H_{r+1}(S_k, *; R)$ is an epimorphism. Thus, we have proved Lemma 5.14. \square

Before proving Theorem 3.36, we state one of the main idea in the proof. One of the keys to proving Theorem 3.36 is the (co)homology sequence of the pair $(|N_k|, |\bigcup_{j=1}^m N_{k,j}|)$. By the cohomology sequence of the above pair, we obtain the following commutative diagram of the cohomology groups (with coefficients R):

$$\begin{array}{ccccccc} \longrightarrow & \tilde{H}^r(S_k) & \longrightarrow & H^r(N_k) & \xrightarrow{\eta_{k-1}^*} & \bigoplus_{j=1}^m H^r(N_{k-1}) & \longrightarrow & \tilde{H}^{r+1}(S_k) & \longrightarrow \\ & \downarrow \tilde{\varphi}_{k+1,k}^* & & \downarrow \varphi_{k+1,k}^* & & \downarrow \bigoplus_{j=1}^m \varphi_{k,k-1}^* & & \downarrow \tilde{\varphi}_{k+1,k}^* & \\ \longrightarrow & \tilde{H}^r(S_{k+1}) & \longrightarrow & H^r(N_{k+1}) & \xrightarrow{\eta_k^*} & \bigoplus_{j=1}^m H^r(N_k) & \longrightarrow & \tilde{H}^{r+1}(S_{k+1}) & \longrightarrow \end{array} \quad (19)$$

in which each row is an exact sequence of groups. By (19), we obtain the following commutative diagram of the cohomology groups (with coefficients R):

$$\begin{array}{ccccccc} \longrightarrow & \tilde{H}^r(S_1) & \longrightarrow & H^r(N_1) & \longrightarrow & \bigoplus_{j=1}^m H^r(\{j\}) & \longrightarrow & \tilde{H}^{r+1}(S_1) & \longrightarrow \\ & \downarrow \tilde{\varphi}_{k+1,1}^* & & \downarrow \varphi_{k+1,1}^* & & \downarrow \bigoplus_{j=1}^m \varphi_{k,1}^* & & \downarrow \tilde{\varphi}_{k+1,1}^* & \\ \longrightarrow & \tilde{H}^r(S_{k+1}) & \longrightarrow & H^r(N_{k+1}) & \xrightarrow{\eta_k^*} & \bigoplus_{j=1}^m H^r(N_k) & \longrightarrow & \tilde{H}^{r+1}(S_{k+1}) & \longrightarrow \\ & \downarrow & & \downarrow \mu_{k+1,r} & & \downarrow \bigoplus_{j=1}^m \mu_{k,r}^* & & \downarrow & \\ \longrightarrow & \varinjlim_k \tilde{H}^r(S_k) & \longrightarrow & \tilde{H}^r(\mathcal{L}) & \xrightarrow{\theta} & \bigoplus_{j=1}^m \tilde{H}^r(\mathcal{L}) & \longrightarrow & \varinjlim_k \tilde{H}^{r+1}(S_k) & \longrightarrow \end{array} \quad (20)$$

in which each row is an exact sequence of groups, and $\varinjlim_k \tilde{H}^*(S_k)$ denotes the direct limit of $\{\tilde{H}^*(S_k), \tilde{\varphi}_{l,k}^*\}_{l,k \in \mathbb{N}, l > k}$.

Proof of Theorem 3.36: Under the assumptions of Theorem 3.36, let $r, k \in \mathbb{N}$. Using the homology sequence of the pair $(|N_k|, |\bigcup_{j=1}^m N_{k,j}|)$, we have the following exact sequence:

$$\begin{aligned} \tilde{H}_{r+1}(S_k; T) \xrightarrow{\alpha_1} H_r(|\bigcup_{j=1}^m N_{k,j}|; T) \xrightarrow{\alpha_2} H_r(|N_k|; T) \xrightarrow{\alpha_3} \tilde{H}_r(S_k; T) \\ \xrightarrow{\alpha_4} H_{r-1}(|\bigcup_{j=1}^m N_{k,j}|; T) \xrightarrow{\alpha_5} H_{r-1}(|N_k|; T) \xrightarrow{\alpha_6} \tilde{H}_{r-1}(S_k; T), \end{aligned} \quad (21)$$

where for each j , α_j denotes some homomorphism. Moreover, by Lemma 5.13-1, we have

$$\tilde{H}_r(S_k; T) \xrightarrow{(\tilde{\varphi}_{k,1})^*} \tilde{H}_r(S_1; T) \text{ and } \tilde{H}_{r-1}(S_k; T) \xrightarrow{(\tilde{\varphi}_{k,1})^*} \tilde{H}_{r-1}(S_1; T). \quad (22)$$

Furthermore, by Lemma 5.14, we have that

$$\text{Im}(\alpha_1) = 0. \quad (23)$$

We now prove statement 1. Let $r, k \geq 2$. By Lemma 5.14, we have

$$\text{Im}(\alpha_1) = \text{Im}(\alpha_4) = 0. \quad (24)$$

Moreover, we have the following commutative diagram:

$$\begin{array}{ccccc} H_r(|\bigcup_{j=1}^m N_{k,j}|; T) & \xrightarrow{\alpha_2} & H_r(|N_k|; T) & \xrightarrow{\alpha_3} & \tilde{H}_r(S_k; T) \\ \downarrow & & (\varphi_{k,1})_* \downarrow & & \downarrow (\tilde{\varphi}_{k,1})_* \\ 0 & \longrightarrow & H_r(|N_1|; T) & \xrightarrow{(\gamma_1)_*} & \tilde{H}_r(S_1; T) \end{array} \quad (25)$$

and $(\gamma_1)_* : H_r(|N_1|; T) \rightarrow \tilde{H}_r(S_1; T)$ is an isomorphism, where $\gamma_1 : |N_1| \rightarrow S_1$ denotes the canonical projection. Combining (21), (22), (24), (25), and Lemma 3.30, we obtain the following exact sequence:

$$0 \longrightarrow \bigoplus_{j=1}^m H_r(N_{k-1}; T) \xrightarrow{(\eta_{k-1})^*} H_r(N_k; T) \xrightarrow{(\varphi_{k,1})^*} H_r(N_1; T) \longrightarrow 0. \quad (26)$$

By (26), we obtain $a_{r,k} = ma_{r,k-1} + a_{r,1}$. Thus, we have proved statement 1. Statement 2 follows easily from statement 1.

We now prove statement 3. Let $r \geq 2$. By (26), for each $k \in \mathbb{N}$, we have the following exact sequence of cohomology groups:

$$0 \longrightarrow \check{H}^r(N_1; R) \xrightarrow{(\varphi_{k+1,1})^*} \check{H}^r(N_{k+1}; R) \xrightarrow{\eta_k^*} \bigoplus_{j=1}^m \check{H}^r(N_k; R) \longrightarrow 0. \quad (27)$$

Taking the direct limit of (27) with respect to k , we obtain the exact sequence (3). Thus, we have proved statement 3.

We now prove statement 4. If $r = 0$, then from Lemma 4.8, $\mu_{k,r}$ and φ_k^* are monomorphisms. Let $r \geq 2$ and $k \geq 1$. Let $\mathfrak{L}_k = (L, (g_1, \dots, g_m^k))$ be a k -th iterate of \mathfrak{L} . Then, there exist

isomorphisms $\zeta_1 : \check{H}^r(\mathcal{L}; R)_k \cong \check{H}^r(\mathcal{L}_k; R)_1$ and $\zeta_2 : \check{H}^r(\mathcal{L}; R) \cong \check{H}^r(\mathcal{L}_k; R)$ such that the following diagram commutes:

$$\begin{array}{ccc} \check{H}^r(\mathcal{L}; R)_k & \xrightarrow{\mu_{k,r}} & \check{H}^r(\mathcal{L}; R) \\ \zeta_1 \downarrow & & \downarrow \zeta_2 \\ \check{H}^r(\mathcal{L}_k; R)_1 & \xrightarrow{\mu_{1,r}} & \check{H}^r(\mathcal{L}_k; R). \end{array} \quad (28)$$

Moreover, by Lemma 3.24, \mathcal{L}_k is postunbranched. Combining it with statement 3, we obtain that $\mu_{1,r} : \check{H}^r(\mathcal{L}_k; R)_1 \rightarrow \check{H}^r(\mathcal{L}_k; R)$ is a monomorphism. Hence, $\mu_{k,r} : \check{H}^r(\mathcal{L}; R)_k \rightarrow \check{H}^r(\mathcal{L}; R)$ is a monomorphism. Therefore, statement 4 follows.

Statement 5 easily follows from statement 1 and statement 3.

We now prove statement 6. By (21), (22), (23), and Lemma 3.30, we obtain the following commutative diagram of homology groups (with coefficients T):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{j=1}^m H_1(N_k) & \xrightarrow{(\eta_k)_*} & H_1(N_{k+1}) & \longrightarrow & \tilde{H}_1(S_1) \\ \downarrow & & \downarrow & & \downarrow (\varphi_{k+1,1})_* & & \downarrow Id \\ 0 & \longrightarrow & 0 & \longrightarrow & H_1(N_1) & \longrightarrow & \tilde{H}_1(S_1) \\ & & \longrightarrow & \bigoplus_{j=1}^m H_0(N_k) & \xrightarrow{(\eta_k)_*} & H_0(N_{k+1}) & \longrightarrow 0 \\ & & & \downarrow & & \downarrow (\varphi_{k+1,1})_* & \downarrow \\ & & & \bigoplus_{j=1}^m H_0(\{j\}) & \longrightarrow & H_0(N_1) & \longrightarrow 0 \end{array} \quad (29)$$

in which each row is an exact sequence of groups. By (29), it is easy to see that statement 6 holds.

We now prove statement 7 and statement 8. By the cohomology sequence of the pair $(|N_k|, |\bigcup_{j=1}^m N_{k,j}|)$, (19), (20), (22), (23), and Lemma 3.30, for each $k \in \mathbb{N}$, we have the following commutative diagram of cohomology groups (with coefficients R):

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(N_1) & \longrightarrow & \bigoplus_{j=1}^m H^0(\{j\}) & \longrightarrow & \tilde{H}^1(S_1) \\ \downarrow & & \downarrow \varphi_{k+1,1}^* & & \downarrow & & \downarrow Id \\ 0 & \longrightarrow & H^0(N_{k+1}) & \xrightarrow{\eta_k^*} & \bigoplus_{j=1}^m H^0(N_k) & \longrightarrow & \tilde{H}^1(S_1) \\ \downarrow & & \downarrow \mu_{k+1,0} & & \downarrow \bigoplus_{j=1}^m \mu_{k,0} & & \downarrow Id \\ 0 & \longrightarrow & \check{H}^0(\mathcal{L}) & \xrightarrow{\theta} & \bigoplus_{j=1}^m \check{H}^0(\mathcal{L}) & \longrightarrow & \tilde{H}^1(S_1) \\ & & \longrightarrow & H^1(N_1) & \longrightarrow & 0 & \longrightarrow 0 \\ & & & \downarrow \varphi_{k+1,1}^* & & \downarrow & \downarrow \\ & & & \longrightarrow & H^1(N_{k+1}) & \xrightarrow{\eta_k^*} & \bigoplus_{j=1}^m H^1(N_k) & \longrightarrow 0 \\ & & & & \downarrow \mu_{k+1,1} & & \downarrow \bigoplus_{j=1}^m \mu_{k,1} & \downarrow \\ & & & & \longrightarrow & \check{H}^1(\mathcal{L}) & \xrightarrow{\theta} & \bigoplus_{j=1}^m \check{H}^1(\mathcal{L}) & \longrightarrow 0 \end{array} \quad (30)$$

in which each row is an exact sequence of groups. By (30), it is easy to see that statement 7 and statement 8 hold. Thus, we have proved statement 7 and statement 8.

We now prove statement 9. By (30), we have the following exact sequence:

$$0 \rightarrow H^0(N_1; R) \rightarrow H^0(\{1, \dots, m\}; R) \rightarrow \tilde{H}^1(S_1; R) \rightarrow H^1(N_1; R) \rightarrow 0. \quad (31)$$

Hence we have $\dim_R H^1(S_1; R) = m - a_{0,1} + a_{1,1}$. Combining it with the exact sequences (6) and (7), we can easily obtain that statement 9 holds. Thus we have proved statement 9.

Statement 10 easily follows from the definition of λ_k and $b_{1,\infty}$.

Statement 11 easily follows from statement 9.

Statement 12 and statement 13 easily follow from statement 9 and statement 10.

Statement 14 easily follows from statement 1 and statement 12.

Statement 15 easily follows from statement 13.

We now prove statement 16. By statements 3 and 8, for each $r \in \mathbb{N}$ with $r \geq 2$ there exists an exact sequence $\check{H}^r(\mathcal{L}; R) \rightarrow \bigoplus_{j=1}^m \check{H}^r(\mathcal{L}; R) \rightarrow 0$. Hence, if $m > 1$, then either $a_{r,\infty} = 0$ or $a_{r,\infty} = \infty$. If $m = 1$, then obviously we have $a_{r,\infty} = 0$. Thus, we have proved statement 16.

We now prove statement 17. Suppose $a_{0,\infty} < \infty$. Since $a_{0,\infty} = \lim_{k \rightarrow \infty} a_{0,k}$, statement 9 and statement 10 imply that $a_{0,\infty} = ma_{0,\infty} - m + a_{0,1} - a_{1,1} + b_{1,\infty}$. Therefore, statement 17 follows.

Statement 18 easily follows from statement 16 and statement 17.

We now prove statement 19. Suppose that there exists an element $k_0 \in \mathbb{N}$ such that $a_{0,k_0} > \frac{1}{m-1}(m - a_{0,1} + a_{1,1})$. We will show that $a_{0,k+1} > a_{0,k}$ for each $k \geq k_0$, by using induction on $k \geq k_0$. For the first step, by statement 13 and the assumption $a_{0,k_0} > \frac{1}{m-1}(m - a_{0,1} + a_{1,1})$, we have $a_{0,k_0+1} - a_{0,k_0} \geq (m-1)a_{0,k_0} - m + a_{0,1} - a_{1,1} > 0$. We now suppose that $a_{0,k+1} > a_{0,k}$ for each $k \in \{k_0, k_0+1, k_0+2, \dots, t\}$. Then, by statement 13, we have $a_{0,t+2} - a_{0,t+1} \geq (m-1)a_{0,t+1} - m + a_{0,1} - a_{1,1} \geq (m-1)a_{0,k_0} - m + a_{0,1} - a_{1,1} > 0$. Therefore, inductive step is completed. Thus, we have proved statement 19.

Statement 20 follows easily from statement 19 (or from statement 17 and statement 10).

We now prove statement 21. Suppose $2 \leq m \leq 6$ and $|N_1|$ is disconnected. In order to show $a_{0,\infty} = \infty$, by Lemmas 5.2 and 5.3 we may assume that each connected component of $|N_1|$ has at least two vertices. Then it is easy to see that $\frac{m-a_{0,1}+a_{1,1}}{m-1} < 2$. Therefore statement 20 implies that $a_{0,\infty} = \infty$. By Lemma 4.8-2, it follows that L has infinitely many connected components. Thus we have proved statement 21.

We now prove statement 22. Suppose that $B_2 = 0$. Let $C_k := \text{Im}(\varphi_k^* : \check{H}^1(\mathcal{L}; R)_k \rightarrow \check{H}^1(\mathcal{L}; R)_{k+1})$. We will show the following claim:

Claim: For each $k \in \mathbb{N}$, $C_k = 0$.

To prove the claim, we will use the induction on k . Since $B_2 = 0$, we have $C_1 = 0$. Moreover, since $B_2 = 0$, the exact sequence (6) implies that for each $k \in \mathbb{N}$, $\eta_k^* : \check{H}^1(\mathcal{L}; R)_{k+1} \rightarrow \bigoplus_{j=1}^m \check{H}^1(\mathcal{L}; R)_k$ is an isomorphism. Furthermore, for each $k \in \mathbb{N}$, we have the following commutative diagram.

$$\begin{array}{ccc} \check{H}^1(\mathcal{L}; R)_{k+1} & \xrightarrow{\eta_k^*} & \bigoplus_{j=1}^m \check{H}^1(\mathcal{L}; R)_k \\ \varphi_{k+1}^* \downarrow & & \downarrow \bigoplus_{j=1}^m \varphi_k^* \\ \check{H}^1(\mathcal{L}; R)_{k+2} & \xrightarrow{\eta_{k+1}^*} & \bigoplus_{j=1}^m \check{H}^1(\mathcal{L}; R)_{k+1}. \end{array} \quad (32)$$

Hence, if we assume $C_k = 0$, then $C_{k+1} = 0$. Therefore, the induction is completed. Thus, we have proved the claim.

From the above claim, it is easy to see that $\check{H}^1(\mathcal{L}; R) = 0$. Hence, we have proved statement 22.

We now prove statement 23. For each $j = 1, \dots, m$, let $c_j : N_k \rightarrow \{j\}$ be the constant map. By (29), we have the following commutative diagram of homology groups (with coefficients T):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{j=1}^m H_1(N_k) & \xrightarrow{(\eta_k)^*} & H_1(N_{k+1}) & \longrightarrow & \check{H}_1(S_1) & \longrightarrow & \bigoplus_{j=1}^m H_0(N_k) \\ \downarrow & & \downarrow & & \downarrow (\varphi_{k+1,1})^* & & \downarrow Id & & \downarrow \bigoplus_{j=1}^m (c_j)^* \\ 0 & \longrightarrow & 0 & \longrightarrow & H_1(N_1) & \longrightarrow & \check{H}_1(S_1) & \longrightarrow & \bigoplus_{j=1}^m H_0(\{j\}) \end{array} \quad (33)$$

in which each row is an exact sequence of groups.

Suppose that $|N_1|$ is connected. Then, by Lemma 4.3, $|N_k|$ is connected for each $k \in \mathbb{N}$. Hence, $\bigoplus_{j=1}^m (c_j)_* : \bigoplus_{j=1}^m H_0(N_k; T) \rightarrow \bigoplus_{j=1}^m H_0(\{j\}; T)$ is an isomorphism. Combining it with (33), the five lemma implies that $(\varphi_{k+1,1})_* : H_1(N_{k+1}; T) \rightarrow H_1(N_1; T)$ is an epimorphism. Combining it with (4), we obtain the exact sequence (10) in statement 23a. Hence, we have proved statement 23a. Statement 23b easily follows from statement 23a. We now prove statement 23c. If $a_{1,1} = 0$, then statement 23b implies that for each $k \in \mathbb{N}$, $a_{1,k} = 0$. Therefore, $a_{1,\infty} = 0$. If $a_{1,1} \neq 0$, then statement 23b implies that $a_{1,k} \rightarrow \infty$ as $k \rightarrow \infty$. From Lemma 4.8-4, it follows that $a_{1,\infty} = \infty$. Therefore, we have proved statement 23c. Statement 23d follows from statement 23a and the universal-coefficient theorem. We now prove statement 23e. By the exact sequence (10), for each $k \in \mathbb{N}$ we have the following exact sequence:

$$0 \longrightarrow H^1(N_1; R) \xrightarrow{(\varphi_{k+1,1})_*} H^1(N_{k+1}; R) \xrightarrow{\eta_k^*} \bigoplus_{j=1}^m H^1(N_k; R) \longrightarrow 0. \quad (34)$$

Taking the direct limit of (34) with respect to k , we obtain the exact sequence (11).

Therefore, we have proved statement 23.

Thus, we have proved Theorem 3.36. \square

We now prove Proposition 3.37.

Proof of Proposition 3.37: We first prove statement 1. Let $K := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$. It is easy to see that there exists a finite family $\{h_1, \dots, h_{n+2}\}$ of topological branched covering maps on $\hat{\mathbb{C}}$ with the following properties:

1. $h_j^{-1}(K) \subset K$ for each $j = 1, \dots, n+2$;
2. $h_i^{-1}(\text{int}(K)) \cap h_j^{-1}(\text{int}(K)) = \emptyset$ for each (i, j) with $i \neq j$;
3. $h_1^{-1}(K) \cap \{z \in \mathbb{C} \mid |z| = 2\} = \emptyset$ and $h_2^{-1}(K) \cap \{z \in \mathbb{C} \mid |z| = 1\} = \emptyset$;
4. $h_j^{-1}(K) \subset \text{int}(K)$ for each $j = 3, \dots, n+2$;
5. $h_j|_{\{z \in \mathbb{C} \mid |z| = j\}} = \text{Id}$ for each $j = 1, 2$;
6. $h_1(h_1^{-1}(K) \cap h_k^{-1}(K)) \subset \{z \in \mathbb{C} \mid |z| = 2\}$ for each $k \in \{1, \dots, n+2\}$ with $k \neq 1$;
7. $h_2(h_2^{-1}(K) \cap h_k^{-1}(K)) \subset \{z \in \mathbb{C} \mid |z| = 1\}$ for each $k \in \{1, \dots, n+2\}$ with $k \neq 2$;
8. $h_j(h_j^{-1}(K) \cap h_k^{-1}(K)) \subset \{z \in \mathbb{C} \mid |z| = 2\}$ for each $j, k \in \{1, \dots, n+2\}$ with $j \neq k$ and $j \geq 3$; and
9. $\bigcap_{i=1}^{n+2} h_i^{-1}(\partial K) = \emptyset$ and for each $j = 1, \dots, n+2$, $\bigcap_{i \in \{1, \dots, n+2\} \setminus \{j\}} h_i^{-1}(\partial K) \neq \emptyset$.

Let $L := R_{K,b}(h_1, \dots, h_{n+2})$ and let $\mathfrak{L} := (L, (h_1, \dots, h_{n+2}))$. Then, by Lemma 2.22, \mathfrak{L} is a backward self-similar system. From properties 1, 3, 4, and 5, we have that for each $j = 1, 2$, $\{z \in \mathbb{C} \mid |z| = j\} \subset L_{(j)^\infty} \setminus L_x$ for any $x \in \Sigma_{n+2}$ with $x \neq (j)^\infty$. Combining it with properties 6, 7, and 8, it follows that \mathfrak{L} is postunbranched. Moreover, since $\partial K = \bigcup_{j=1}^2 \{z \in \mathbb{C} \mid |z| = j\} \subset L$, properties 2 and 9 imply that $\bigcap_{i=1}^{n+2} h_i^{-1}(L) = \emptyset$ and for each $j = 1, \dots, n+2$, $\bigcap_{i \in \{1, \dots, n+2\} \setminus \{j\}} h_i^{-1}(L) \neq \emptyset$. Hence $H^n(N_1; R) = R$ for each field R . From Theorem 3.36-5, it follows that $\dim_R H^n(\mathfrak{L}; R) = \infty$ for each field R . Thus, we have proved statement 1 of Proposition 3.37.

We now prove statement 2 of Proposition 3.37. Let $K' := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\} \times [0, 1] \subset \mathbb{R}^3$. We can construct a finite family $\{h_j\}_{j=1}^{n+2}$ of continuous and injective maps on K' satisfying properties similar to the above properties 1, ..., 9 (with “-1” removed). Let $L := R_{K',f}(h_1, \dots, h_{n+2})$ and let $\mathfrak{L} := (L, (h_1, \dots, h_{n+2}))$. Then, by the argument similar to that in the previous paragraph, we

obtain that \mathcal{L} is postunbranched, $H^n(N_1; R) = R$ for each field R , and $\dim_R \check{H}^n(\mathcal{L}; R) = \infty$ for each field R . Thus, we have proved statement 2 of Proposition 3.37. \square

We now prove Proposition 3.38.

Proof of Proposition 3.38: Let $p_1, p_2, p_3 \in \mathbb{C}$ be mutually distinct three points such that $p_1 p_2 p_3$ makes an equilateral triangle. For each $j = 1, 2, 3$, let $g_j(z) = \frac{1}{2}(z - p_j) + p_j$. Let $h_1 := g_1^2, h_2 := g_2^2, h_3 := g_3^2, h_4 := g_3 \circ g_1$, and $h_5 := g_3 \circ g_2$. Let $L := M_{\mathbb{C}}(h_1, \dots, h_5)$ and let $\mathcal{L} := (L, (h_1, \dots, h_5))$ (see Figure 9). Then \mathcal{L} is a forward self-similar system. By Example 3.27, \mathcal{L} is postunbranched. Since $p_j \in L$ for each $j = 1, 2, 3$, we have that $h_3(L) \cap h_4(L) \neq \emptyset, h_4(L) \cap h_5(L) \neq \emptyset, h_5(L) \cap h_3(L) \neq \emptyset$, and $h_3(L) \cap h_4(L) \cap h_5(L) = \emptyset$. Moreover, it is easy to see that for each $r \in \mathbb{N}$ with $r \geq 2$, there exists no r -simplex of $N_1 = N_1(\mathcal{L})$. Hence $\tau = (3, 4)(4, 5)(5, 3)$ is a closed edge path of $N_1 = N_1(\mathcal{L})$ which induces a non-trivial element of $H_1(N_1; R)$ for each field R . Hence $\check{H}^1(\mathcal{L}; R)_1 \neq 0$. However, considering N_2 , it is easy to see that $\text{Im}((\varphi_1)_* : H_1(N_2; R) \rightarrow H_1(N_1; R)) = 0$. Hence, $B_2 = 0$. From Theorem 3.36-22, it follows that $\check{H}^1(\mathcal{L}; R) = 0$. Moreover, since $\check{H}^1(\mathcal{L}; R)_1 \neq 0$, we obtain that $\mu_{1,1} : \check{H}^1(\mathcal{L}; R)_1 \rightarrow \check{H}^1(\mathcal{L}; R)$ is not injective. Furthermore, since each $h_j : L \rightarrow L$ is a contraction, we have that $\Psi : \check{H}^1(\mathcal{L}; R) \rightarrow \check{H}^1(L; R)$ is an isomorphism. Hence $\check{H}^1(L; R) = 0$. From the Alexander duality theorem ([20]), it follows that $\mathbb{C} \setminus L$ is connected. Thus, we have proved Proposition 3.38. \square

Figure 9: The invariant set of a postunbranched system such that $\mu_{1,1}$ is not injective.



In order to prove Theorem 3.46, we need several lemmas.

Lemma 5.15. *Let $\mathcal{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is injective. Suppose that $\sharp C_{i,j} \leq 1$ for each (i, j) with $i \neq j$. Then, for each $r \geq 2$, each $k \geq 1$ and each \mathbb{Z} module T , we have $\check{H}_r(\mathcal{L}; T)_k = \check{H}_r(\mathcal{L}; T) = 0$.*

Proof. Let $a = \sum_{i=1}^t a_i d_i \in C_r(N_k; T)$ be a cycle, where for each i , $a_i \in T$ and d_i is an oriented r -simplex. We may assume that $\{d_1, \dots, d_t\}$ is linearly independent. Let Ω be the graph such that the vertex set is equal to $\{d_1, \dots, d_t\}$ and such that $\{d_i, d_j\}$ is an edge if and only if there exists a 1-simplex e of N_k with $|e| \subset |d_i| \cap |d_j|$. Let $\{\Omega_1, \Omega_2, \dots, \Omega_p\}$ be the set of all connected components of $|\Omega|$. Then we have $\sum_{i=1}^t a_i d_i = \sum_{l=1}^p \sum_{d_i \in \Omega_l} a_i d_i$. We now show the following claim:

Claim 1: For each l , $\partial(\sum_{d_i \in \Omega_l} a_i d_i) = 0$ in $C_{r-1}(N_k; T)$.

In order to show claim 1, suppose that there exists an l such that $\partial(\sum_{d_i \in \Omega_l} a_i d_i) = \sum_{j=1}^{\beta} b_j e_j \neq 0$, where e_j is an oriented $r-1$ simplex of N_k for each j , $\{e_1, \dots, e_{\beta}\}$ is linearly independent, and $b_j \in T$ with $b_j \neq 0$ for each j . Since $\partial(\sum_{i=1}^t a_i d_i) = 0$, there exists an l' with $l' \neq l$ and an element $d_q \in \Omega_{l'}$ such that $|d_q| \supset |e_1|$. However, it implies that $d_q \in \Omega_l$ and this is a contradiction since $\Omega_l \cap \Omega_{l'} = \emptyset$. Hence, we have proved claim 1.

We now prove the following claim:

Claim 2: Let $l \in \{1, \dots, p\}$ be a number. Let $\{v_0, \dots, v_s\}$ be the union $\bigcup_{d_i \in \Omega_l} \{\text{all vertices of } d_i\}$. Then, $M_l := \{v_0, \dots, v_s\}$ is an s -simplex of N_k .

In order to prove claim 2, let $d_i, d_j \in \Omega_l$ be two elements such that there exists a 1-simplex $e = \{u_1, u_2\}$ of N_k with $|d_i| \cap |d_j| \supset |e|$, where $u_1, u_2 \in \{1, \dots, m\}^k$. Let $\{w_0, \dots, w_r\}$ be the set of all vertices of d_i and let $\{w'_0, \dots, w'_r\}$ be the set of all vertices of d_j . Then we have

$$\emptyset \neq \bigcap_{j=0}^r h_{\overline{w_j}}(L) \subset h_{\overline{u_1}}(L) \cap h_{\overline{u_2}}(L) \quad \text{and} \quad \emptyset \neq \bigcap_{j=0}^r h_{\overline{w'_j}}(L) \subset h_{\overline{u_1}}(L) \cap h_{\overline{u_2}}(L). \quad (35)$$

Since $\#C_{i,j} \leq 1$ for each (i, j) with $i \neq j$ and $h_j : L \rightarrow L$ is injective for each j , we have $\#(h_{\overline{u_1}}(L) \cap h_{\overline{u_2}}(L)) \leq 1$. Combining it with (35), it follows that there exists a point $z \in L$ such that $\bigcap_{j=0}^r h_{\overline{w_j}}(L) = \bigcap_{j=0}^r h_{\overline{w'_j}}(L) = \{z\}$. The above argument implies that $\bigcap_{j=0}^s h_{\overline{v_j}}(L) = \{z\}$. Hence, $M_l = \{v_0, \dots, v_s\}$ is an s -simplex of N_k . Therefore, we have proved claim 2.

By Claim 2, we obtain that for each l , $\sum_{d_i \in \Omega_l} a_i d_i \in C_r(M_l; T)$. Combining it with claim 1, we get that for each l , $\sum_{d_i \in \Omega_l} a_i d_i$ is a cycle of $C_r(M_l; T)$. Since $H_r(M_l; T) = 0$, it follows that for each l , $\sum_{d_i \in \Omega_l} a_i d_i$ is a boundary element of $C_r(N_k; T)$. Hence, we get that $H_r(N_k; T) = 0$. Therefore, we have proved Lemma 5.15. \square

By the same method, we can prove the following lemma.

Lemma 5.16. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system. Suppose that $\#C_{i,j} \leq 1$ for each (i, j) with $i \neq j$. Let T be a \mathbb{Z} module. Then, for each $r \in \mathbb{N}$ with $r \geq 2$, we have $\tilde{H}_r(\mathfrak{L}; T)_1 = 0$.*

Lemma 5.17. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a forward self-similar system such that for each $j = 1, \dots, m$, $h_j : L \rightarrow L$ is injective. Let T be a \mathbb{Z} module. Suppose that $\#C_{i,j} \leq 1$ for each (i, j) with $i \neq j$. Then, for each $k \in \mathbb{N}$, $H_2(S_k; T) = 0$.*

Proof. Let $a = \sum_{i=1}^r a_i d_i \in C_2(S_k; T)$ be a cycle, where for each i , $a_i \in T$ and d_i is an oriented 2-cell of S_k . We will show that a is a boundary. Let $\gamma_k : |N_k| \rightarrow S_k$ be the canonical projection. For each i , let \tilde{d}_i be an oriented 2-simplex of N_k such that $\gamma_k(|\tilde{d}_i|) = d_i$. Let Ω be the graph such that the vertex set is equal to $\{d_1, \dots, d_r\}$ and such that $\{d_i, d_j\}$ is an edge of Ω if and only if there exists an 1-cell e of S_k such that $d_i \cap d_j \supset e$. Let $\{\Omega_1, \dots, \Omega_p\}$ be the set of all connected components of $|\Omega|$. Then we have $a = \sum_{l=1}^p \sum_{d_i \in \Omega_l} a_i d_i$. We now prove the following claim.

Claim 1: For each l , $\partial(\sum_{d_i \in \Omega_l} a_i d_i) = 0$ in $C_1(S_k; T)$.

In order to prove claim 1, suppose that the statement is false. Then, there exists an l such that $\partial(\sum_{d_i \in \Omega_l} a_i d_i) = \sum_{j=1}^\beta b_j e_j \neq 0$, where for each j , $b_j \in T$ and e_j is an oriented 1-cell of S_k such that $\{e_1, \dots, e_\beta\}$ is linearly independent. Since $\partial(\sum_{i=1}^r a_i d_i) = 0$, it follows that there exists an l' with $l' \neq l$ and an element $d_i \in \Omega_{l'}$ such that $d_i \supset e_1$. It implies that $d_i \in \Omega_l$. However, this is a contradiction, since $l' \neq l$. Therefore, we have proved claim 1.

We now prove the following claim.

Claim 2: For each l , there exists an $s \in \mathbb{N}$ with $s \geq 2$ and an s -simplex M of N_k such that $\bigcup_{d_i \in \Omega_l} d_i \subset \gamma_k(|M|)$.

In order to prove claim 2, let $d_i \in \Omega_l$ be an element. Let $d_j \in \Omega_l$ be another element such that $\{d_i, d_j\}$ is an edge of Ω_l . Then there exist four vertices w_1, w_2, w_3, w_4 of N_k such that the set of vertices of \tilde{d}_i is equal to $\{w_1, w_2, w_3\}$ and the set of vertices of \tilde{d}_j is equal to $\{w_2, w_3, w_4\}$. We have $\bigcap_{j=1}^3 h_{\overline{w_j}}(L) \neq \emptyset$ and $\bigcap_{j=2}^4 h_{\overline{w_j}}(L) \neq \emptyset$. Since $\#C_{i,j} \leq 1$ for each (i, j) with $i \neq j$ and $h_j : L \rightarrow L$ is injective for each j , there exists a point $z \in L$ such that $h_{\overline{w_2}}(L) \cap h_{\overline{w_3}}(L) = \{z\}$. Therefore, $\bigcap_{j=1}^4 h_{\overline{w_j}}(L) = \{z\}$. This argument implies that denoting by $\{v_1, \dots, v_s\}$ the set of all vertices of $\bigcup_{d_i \in \Omega_l} \tilde{d}_i$, we have $\bigcap_{j=1}^s h_{\overline{v_j}}(L) = \{z\}$. Let $M = \{v_1, \dots, v_s\}$. Then M is an s -simplex of N_k and $\bigcup_{d_i \in \Omega_l} d_i \subset \gamma_k(|M|)$. Thus, we have proved claim 2.

Since $\gamma_k(|M|)$ is a subcomplex of S_k and $\sum_{d_i \in \Omega_l} a_i d_i$ is a cycle of $C_2(S_k; T)$, we obtain that $\sum_{d_i \in \Omega_l} a_i d_i$ is a cycle of $C_2(\gamma_k(|M|); T)$. We now prove the following claim.

Claim 3: $H_2(\gamma_k(|M|); T) = 0$.

In order to prove claim 3, let $\tilde{\gamma}_k : |M|/(|M \cap \bigcup_{j=1}^m N_{k,j}|) \rightarrow \gamma_k(|M|)$ be the cellular map induced by γ_k . Then, $\tilde{\gamma}_k$ is a homeomorphism. Moreover, we have the following homology sequence of the pair $(|M|, |\bigcup_{j=1}^m N_{k,j}|)$:

$$\cdots \rightarrow H_2(|M|; T) \rightarrow H_2(|M|/|M \cap \bigcup_{j=1}^m N_{k,j}|; T) \rightarrow H_1(|M \cap \bigcup_{j=1}^m N_{k,j}|; T) \rightarrow \cdots \quad (36)$$

Since M is an s -simplex, $H_2(|M|; T) = 0$. Moreover,

$$H_1(|M \cap \bigcup_{j=1}^m N_{k,j}|; T) \cong \bigoplus_{j=1}^m H_1(|M \cap N_{k,j}|; T).$$

Let $\{u_1, \dots, u_t\}$ be the set of all vertices of $M \cap N_{k,j}$. Then, $u = \{u_1, \dots, u_t\}$ is a $(t-1)$ -simplex of M . Since M is a subcomplex of N_k , we obtain that u is a simplex of N_k . Moreover, since each u_j is a vertex of $N_{k,j}$, it follows that u is a simplex of $N_{k,j}$. Therefore, u is a simplex of $M \cap N_{k,j}$. Hence, $H_1(|M \cap N_{k,j}|; T) = 0$. Combining these arguments, we obtain that $H_2(\gamma_k(|M|); T) = 0$. Thus, we have proved claim 3.

By claim 3, the cycle $\sum_{d_i \in \Omega_l} a_i d_i \in C_2(\gamma_k(|M|); T)$ is a boundary element of $C_2(\gamma_k(|M|); T)$. Therefore, $\sum_{d_i \in \Omega_l} a_i d_i$ is a boundary element of $C_2(S_k; T)$. Hence, $a = \sum_{j=1}^r a_j d_j$ is a boundary element of $C_2(S_k; T)$. Thus, we have proved Lemma 5.17. \square

By the same method, we can prove the following lemma.

Lemma 5.18. *Let $\mathfrak{L} = (L, (h_1, \dots, h_m))$ be a backward self-similar system. Suppose that $\sharp C_{i,j} \leq 1$ for each (i, j) with $i \neq j$. Let T be any \mathbb{Z} module. Then, $H_2(S_1; T) = 0$.*

We now prove Theorem 3.46.

Proof of Theorem 3.46: From Lemma 5.15, statement 1 follows.

We now prove statement 2. Let $k \in \mathbb{N}$. By the homology sequence of the pair $(|N_{k+1}|, |\bigcup_{j=1}^m N_{k+1,j}|)$, we have the following exact sequence:

$$\cdots \rightarrow H_2(S_{k+1}; T) \rightarrow H_1(\bigcup_{j=1}^m N_{k+1,j}; T) \rightarrow H_1(N_{k+1}; T) \rightarrow \cdots \quad (37)$$

By Lemma 5.17, we have $H_2(S_{k+1}; T) = 0$. Moreover, $H_1(\bigcup_{j=1}^m N_{k+1,j}; T) \cong \bigoplus_{j=1}^m H_1(N_{k+1,j}; T)$. Therefore, it follows that $ma_{1,k} \leq a_{1,k+1}$. Thus, we have proved statement 2.

We now prove statement 3. Suppose $|N_1|$ is connected and $\check{H}^1(\mathfrak{L}; R) \neq 0$. Then, there exists a $k \in \mathbb{N}$ such that $a_{1,k} \neq 0$. From statement 2, it follows that $\lim_{k \rightarrow \infty} a_{1,k} = \infty$. By Lemma 4.8-4, we obtain that $a_{1,\infty} = \infty$. Therefore, we have proved statement 3.

Thus, we have proved Theorem 3.46. \square

We now prove Proposition 3.47.

Proof of Proposition 3.47: For each $i = 1, 2$, let U_i be an open neighborhood of $h_i(L)$. Then, by the Mayer-Vietoris sequence, we have the following exact sequence:

$$\cdots \rightarrow H^{n+1}(U_1 \cup U_2; R) \rightarrow H^{n+1}(U_1; R) \oplus H^{n+1}(U_2; R) \rightarrow H^{n+1}(U_1 \cap U_2; R) \rightarrow \cdots \quad (38)$$

We take the direct limit \varinjlim_{U_1, U_2} of this sequence, where U_i runs over all open neighborhoods of $h_i(L)$. Then, by [20, p 341, Corollary 9 and p 334, Corollary 8], we obtain the following exact sequence:

$$\check{H}^{n+1}(h_1(L) \cup h_2(L); R) \rightarrow \check{H}^{n+1}(h_1(L); R) \oplus \check{H}^{n+1}(h_2(L); R) \rightarrow \check{H}^{n+1}(C_{1,2}; R) \rightarrow \cdots \quad (39)$$

By the assumption, we have $\check{H}^{n+1}(C_{1,2}; R) = 0$. Similarly, we obtain the following exact sequence:

$$\begin{aligned} \check{H}^{n+1}\left(\bigcup_{j=1}^3 h_j(L); R\right) &\rightarrow \check{H}^{n+1}\left(\bigcup_{j=1}^2 h_j(L); R\right) \oplus \check{H}^{n+1}(h_3(L); R) \\ &\rightarrow \check{H}^{n+1}\left(\left(\bigcup_{j=1}^2 h_j(L)\right) \cap h_3(L); R\right) \rightarrow \cdots \end{aligned} \quad (40)$$

By the assumption, we have $\check{H}^{n+1}\left(\left(\bigcup_{j=1}^2 h_j(L)\right) \cap h_3(L); R\right) = \check{H}^{n+1}(C_{1,3} \cup C_{2,3}; R) = 0$. From these arguments, it follows that there exists an exact sequence:

$$\check{H}^{n+1}\left(\bigcup_{j=1}^3 h_j(L); R\right) \rightarrow \bigoplus_{j=1}^3 \check{H}^{n+1}(h_j(L); R) \rightarrow 0. \quad (41)$$

Continuing this method, we obtain the following exact sequence:

$$\check{H}^{n+1}\left(\bigcup_{j=1}^m h_j(L); R\right) \rightarrow \bigoplus_{j=1}^m \check{H}^{n+1}(h_j(L); R) \rightarrow 0. \quad (42)$$

Since for each $j = 1, 2$, $h_j : L \rightarrow h_j(L)$ is a homeomorphism, we obtain the following exact sequence:

$$\check{H}^{n+1}(L; R) \rightarrow \check{H}^{n+1}(L; R) \oplus \check{H}^{n+1}(L; R) \oplus \bigoplus_{j=3}^m \check{H}^{n+1}(h_j(L); R) \rightarrow 0. \quad (43)$$

From this exact sequence, it follows that either $\check{H}^{n+1}(L; R) = 0$ or $\dim_R \check{H}^{n+1}(L; R) = \infty$. Thus, we have proved Proposition 3.47. \square

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