DYNAMICS OF INFINITELY GENERATED NICELY EXPANDING RATIONAL SEMIGROUPS AND THE INDUCING METHOD

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ABSTRACT. We investigate the dynamics of semigroups of rational maps on the Riemann sphere. To establish a fractal theory of the Julia sets of infinitely generated semigroups of rational maps, we introduce a new class of semigroups which we call nicely expanding rational semigroups. More precisely, we prove Bowen's formula for the Hausdorff dimension of the pre-Julia sets, which we also introduce in this paper. We apply our results to the study of the Julia sets of non-hyperbolic rational semigroups. For these results, we do not assume the cone condition, which has been assumed in the study of infinite contracting iterated function systems. Similarly, we show that Bowen's formula holds for the limit set of a contracting conformal iterated function system without the cone condition.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Rat be the set of all non-constant rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. A subsemigroup of Rat with semigroup operation being the functional composition is called a rational semigroup. A semigroup of non-constant polynomial maps is called a polynomial semigroup. The work on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([HM96]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups of Möbius transformations, and by F. Ren's group ([ZR92]), who studied such semigroups from the perspective of random dynamical systems. The theory of the dynamics of rational semigroups on $\widehat{\mathbb{C}}$ has developed in many directions since the 1990s ([HM96, ZR92, Stan12, SS11, SU11a, SU11b], [Sum97] – [Sum13]). We recommend [Stan12] as an introductory article.

Throughout, let I be a topological space. We consider a family $\{f_i : i \in I\}$ of Rat such that $f_i \in$ Rat depends continuously on $i \in I$. We will use $G = \langle f_i : i \in I \rangle$ to denote the rational semigroup generated by $\{f_i : i \in I\}$, i.e., $G = \{f_{i_1} \circ \cdots \circ f_{i_n} : n \in \mathbb{N}, i_1, \ldots, i_n \in I\}$. The Fatou set $F(G)$ and the Julia set $J(G)$ of G are given by

$$F(G) := \left\{ z \in \widehat{\mathbb{C}} : G \text{ is normal in a neighborhood of } z \right\} \quad \text{and} \quad J(G) := \widehat{\mathbb{C}} \setminus F(G).$$

Since the Julia set $J(G)$ of a rational semigroup $G = \langle f_1, \ldots, f_m \rangle := \langle f_i : i \in \{1, \ldots, m\} \rangle$ generated by finitely many elements $f_1, \ldots, f_m$ has backward self-similarity, i.e.,

$$(1.1) \qquad J(G) = f_1^{-1}(J(G)) \cup \cdots \cup f_m^{-1}(J(G)),$$

(see [Sum97, Sum00]), rational semigroups can be viewed as a significant generalization and extension of both the theory of iteration of rational maps (see [Be91, Mil06]) and conformal iterated function systems (see [MU96]). Indeed, because of (1.1), for the analysis of the Julia sets of rational semigroups, we have to consider "backward iterated functions systems", however since each map f_j is not injective and may have critical points in general, we have to deal with critical orbits and some qualitatively different extra effort in the case of semigroups is needed. Also, since one semigroup has many kinds of maps, we have another

difficulty. The theory of the dynamics of rational semigroups borrows and develops tools from both of these theories. It has also developed its own unique methods, notably the skew product approach (see [Sum00]–[Sum06], [Sum10a, Sum11a, Sum13, SU11b]). It is a very exciting problem to estimate the Hausdorff dimension of Julia sets of rational semigroups. Some studies of the Hausdorff dimension of Julia sets of semi-hyperbolic finitely generated rational semigroups were given in [Sum01]–[Sum06], [SU11a, SU11b].

However, there have been no studies on the Hausdorff dimension of Julia sets of infinitely generated expanding rational semigroups $G = \langle f_i : i \in I \rangle$ (see the definition below), or non-semi-hyperbolic rational semigroups. In this paper, we investigate the dynamics of infinitely generated expanding rational semigroups and non-hyperbolic rational semigroups. If I is countable (in this paper, a countable set is a set which is bijective to a subset of $\mathbb{N}$), then I is endowed with the discrete topology. We endow Rat with distance dist_{Rat} given by $\text{dist}_{\text{Rat}}(h_1, h_2) := \sup_{z \in \widehat{\mathbb{C}}} d(h_1(z), h_2(z))$, where d denotes the spherical distance on $\widehat{\mathbb{C}}$. We denote by $C(I, \text{Rat})$ the set of continuous maps from I to Rat.

Let $(f_i)_{i \in I} \in C(I, \text{Rat})$. The skew product associated to the generator system $\{f_i : i \in I\}$ of the rational semigroup $G = \langle f_i : i \in I \rangle$ is given by

$$\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}} \times \widehat{\mathbb{C}}, \quad \tilde{f}(\omega, z) := (\sigma(\omega), f_{\omega_1}(z)),$$

where $\sigma : I^{\mathbb{N}} \to I^{\mathbb{N}}$ denotes the left shift defined by $\sigma(\omega)_i = \omega_{i+1}$, for each $\omega \in I^{\mathbb{N}}$ and $i \in \mathbb{N}$. For $\gamma = (\gamma_i) \in G^{\mathbb{N}}$ we set

$$F_\gamma := \left\{ z \in \widehat{\mathbb{C}} : (\gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1)_{n \in \mathbb{N}} \text{ is normal in a neighborhood of } z \right\} \text{ and } J_\gamma := \widehat{\mathbb{C}} \setminus F_\gamma.$$

Also, for $\omega \in I^{\mathbb{N}}$, we set $\gamma(\omega) := (f_{\omega_i})_{i \in \mathbb{N}}$, $F_\omega := F_{\gamma(\omega)}$ and $J_\omega := J_{\gamma(\omega)}$. We use J^ω to denote the set $\{\omega\} \times J_\omega \subset I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ and we set

$$J(\tilde{f}) := \overline{\bigcup_{\omega \in I^{\mathbb{N}}} J^\omega}, \quad F(\tilde{f}) := \left(I^{\mathbb{N}} \times \widehat{\mathbb{C}} \right) \setminus J(\tilde{f}),$$

where the closure is taken with respect to the product topology on $I^{\mathbb{N}} \times \widehat{\mathbb{C}}$. For a holomorphic map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and $z \in \widehat{\mathbb{C}}$, the norm of the derivative of h at $z \in \widehat{\mathbb{C}}$ with respect to the spherical metric is denoted by $\|h'(z)\|$.

For $n \in \mathbb{N}$ and $(\tau_1, \ldots, \tau_n) \in I^n$, we set $f_{(\tau_1, \ldots, \tau_n)} := f_{\tau_n} \circ f_{\tau_{n-1}} \circ \cdots \circ f_{\tau_1}$. For $\omega \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$, we write $\omega = (\omega_1, \omega_2, \ldots)$ and we set $\omega|_n := (\omega_1, \ldots, \omega_n)$. For each $n \in \mathbb{N}$ and $(\omega, z) \in J(\tilde{f})$, we set $(\tilde{f}^n)'(\omega, z) := (f_{\omega|_n})'(z)$. We say that $\tilde{f}$ is expanding along fibers if $J(\tilde{f}) \neq \varnothing$ and if there exist constants $C > 0$ and $\lambda > 1$ such that for all $n \in \mathbb{N}$,

$$\inf_{(\omega, z) \in J(\tilde{f})} \| (\tilde{f}^n)'(\omega, z) \| \geq C \lambda^n,$$

where $\| (\tilde{f}^n)'(\omega, z) \|$ denotes the norm of the derivative of $f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}$ at z with respect to the spherical metric. G is called expanding with respect to $\{f_i : i \in I\}$ if $\tilde{f}$ is expanding along fibers.

For a rational semigroup G, we say that a subset A of $\widehat{\mathbb{C}}$ is G-forward invariant if $g(A) \subset A$ for each $g \in G$.

Our first main definition is the following.

Definition 1.1. We say that $G = \langle f_i : i \in I \rangle$ is nicely expanding, if G is expanding with respect to $\{f_i : i \in I\}$ and if there exists a non-empty, compact, G-forward invariant set $P_0(G) \subset F(G)$ such that $P(G) \subset P_0(G)$, where $P(G)$ denotes the postcritical set of G given by $P(G) := \bigcup_{g \in G} \{\text{all critical values of } g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}\}$. Here, the closure is taken in $\widehat{\mathbb{C}}$.

Remark. It will follow from Proposition 4.2 below, that the property of a rational semigroup to be nicely expanding is in fact independent of the choice of the generator system. For this reason, we do not refer to the generator system, and we simply say that G is nicely expanding.

We say that a rational semigroup $G = \langle f_i : i \in I \rangle$ (or the system $\{f_i : i \in I\}$) is hyperbolic if $P(G) \subset F(G)$.

Remark. We have $P(G) = \overline{\bigcup_{g \in G \cup \{\text{id}\}} g\left(\bigcup_{i \in I} \text{CV}(f_i)\right)}$, where CV denotes the set of critical values. Thus $P(G)$ is G-forward invariant.

We give some criteria for G to be nicely expanding in Proposition 4.2 and Lemma 4.8. If G is nicely expanding then we are able to control the distortion of inverse branches of maps in G. Note that an expanding rational semigroup is not nicely expanding in general (see Examples 2.1, 2.2).

Regarding the dynamics of infinitely generated nicely expanding rational semigroups, it turns out that we shall work on pre-Julia sets, which is the second main definition of this paper.

Definition 1.2 (Pre-Fatou and pre-Julia set). For a rational semigroup G, the pre-Fatou set $F_{\text{pre}}(G)$ and the pre-Julia set $J_{\text{pre}}(G)$ of G are defined by

$$F_{\text{pre}}(G) := \bigcap_{\gamma \in G^{\mathbb{N}}} F_\gamma \text{ and } J_{\text{pre}}(G) := \widehat{\mathbb{C}} \setminus F_{\text{pre}}(G).$$

Note that if $G = \langle f_i : i \in I \rangle$, then $J_{\text{pre}}(G) = \bigcup_{\omega \in I^{\mathbb{N}}} J_\omega$ and $J_{\text{pre}}(G) = \bigcup_{i \in I} f_i^{-1}(J_{\text{pre}}(G))$. The pre-Julia sets of infinitely generated nicely expanding rational semigroups of this paper correspond to the limit sets of infinite contracting conformal iterated function systems in [MU96] (see Remark 1.6). Roughly speaking, those are the reasons why the pre-Julia sets are the right objects to study and many results (e.g. Bowen's formula) hold for them.

We remark that the pre-Julia set is not necessarily closed in $\widehat{\mathbb{C}}$. In fact, by the density of the repelling fixed points ([HM96], [Sum00, Lemma 2.3 (g)]), we have that, if $\text{card}(J(G)) \geq 3$, then $\overline{J_{\text{pre}}(G)} = J(G)$. However, there are many examples of rational semigroups G, for which $\dim_H(J_{\text{pre}}(G)) < \dim_H(J(G))$ or $J_{\text{pre}}(G) \neq J(G)$ (see Example 2.3), even if we assume that G is nicely expanding.

In order to state the main result of this paper, we define the following critical exponents associated to rational semigroups. If I is countable and $G = \langle f_i : i \in I \rangle$, then the critical exponent $s(G)$ of the Poincaré series of G and the critical exponent $t(I)$ are for each $x \in \widehat{\mathbb{C}}$ given by

$$s(G,x) := \inf\left\{ v \geq 0 : \sum_{g \in G} \sum_{y \in g^{-1}(x)} \|g'(y)\|^{-v} < \infty \right\}, \quad s(G) := \inf\left\{ s(G,x) : x \in \widehat{\mathbb{C}} \right\},$$

$$t(I,x) := \inf\left\{ v \geq 0 : \sum_{n \in \mathbb{N}} \sum_{\omega \in I^n} \sum_{y \in f_\omega^{-1}(x)} \|f_\omega'(y)\|^{-v} < \infty \right\}, \quad t(I) := \inf\left\{ t(I,x) : x \in \widehat{\mathbb{C}} \right\}.$$

Here, the sums $\sum_{y \in g^{-1}(x)}$ and $\sum_{y \in f_\omega^{-1}(x)}$ count the multiplicities, and we set $\inf\{\varnothing\} := \infty$ and $0^{-v} := \infty$, for all $v \geq 0$.

Let $I \subset \mathbb{N}$ be the finite set $\{1, \ldots, n\}$, for some $n \in \mathbb{N}$, or let $I = \mathbb{N}$, endowed with the discrete topology. Let $(f_i)_{i \in I} \in \text{Rat}^I$ and let $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ be the associated skew product. Suppose that $\|f_{\omega_1}'(z)\| \neq 0$, for each $(\omega, z) \in J(\tilde{f})$. We introduce the Gurevič pressure of the geometric potential $\tilde{\varphi} : J(\tilde{f}) \to \mathbb{R}$, $\tilde{\varphi}(\omega, z) := -\log \|f_{\omega_1}'(z)\|$, with respect to the skew product $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$. This notion of topological pressure was introduced in the context of countable Markov shifts by Sarig ([Sar99]) extending the notion of topological entropy due to Gurevič ([Gur69]).

The pressure function of the system $\{f_i : i \in I\}$ is for each $t \in \mathbb{R}$ given by

$$\mathscr{P}(t) := \mathscr{P}(t\tilde{\varphi}, \tilde{f}) := \sup_{K \subset J(\tilde{f}), K \text{ compact}, \tilde{f}(K)=K} \mathscr{P}\left(t\tilde{\varphi}_{|K}, \tilde{f}_{|K}\right).$$

Here, for a continuous function $\psi : X \to \mathbb{R}$ on a compact metric space X, and for a continuous dynamical system $T : X \to X$, we use $\mathscr{P}(\psi, T) := \sup\{h_\mu(T) + \int \psi d\mu\}$ to denote the classical notion of topological pressure introduced by Walters ([Wal75]) following the work of Ruelle ([Rue73]), where the supremum is taken over all T-invariant Borel probability measures on X, and $h_\mu(T)$ refers to the measure-theoretic entropy of the dynamical system (T, μ). Note that the classical pressure is independent of the choice of the metric on X ([Wal82]).

Definition. We say that the rational semigroup $G = \langle f_i : i \in I \rangle$ (or the system $\{f_i : i \in I\}$) satisfies the *open set condition* if there exists a non-empty open set $U \subset \widehat{\mathbb{C}}$ such that $\left(f_i^{-1}(U)\right)_{i \in I}$ consists of mutually disjoint subsets of U.

We refer to [Bow79] (see also [Rue82]) for the by now classical results on the relation between the pressure and the Hausdorff dimension of associated limit sets, which is known as Bowen's formula. We now present the main result of this paper, which establishes Bowen's formula for pre-Julia sets.

Theorem 1.3 (Bowen's formula for pre-Julia sets: see Theorem 6.5, Proposition 6.3)**.** *Let I be a countable set. Let $G = \langle f_i : i \in I \rangle$ be a nicely expanding rational semigroup. Then we have*

$$(1.2) \qquad \dim_H\left(J_{\mathrm{pre}}(G)\right) \le s(G) \le t(I) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\}.$$

If $\{f_i : i \in I\}$ additionally satisfies the open set condition, then all inequalities in (1.2) become equalities.

We apply the above result to the dynamics of non-hyperbolic rational semigroups by using the method of inducing. We deal with critical orbits which do not appear in contracting iterated function systems.

Theorem 1.4 (Inducing method: see Theorem 7.14)**.** *Let I be a countable set and let $G = \langle f_i : i \in I \rangle$ be a rational semigroup. Suppose that there exists a decomposition $I = I_1 \cup I_2$ with $I_2 \ne \varnothing$, such that each of the following (1)–(4) holds for the rational semigroups $G_j := \langle f_i : i \in I_j \rangle$, $j \in \{1,2\}$, and $H := \langle H_0 \rangle$ given by*

$$H_0 := \{f_i : i \in I_2\} \cup \left\{f_i f_{j_1} \ldots f_{j_r} : i \in I_2, r \in \mathbb{N}, (j_1, \ldots, j_r) \in I_1^r\right\}, \quad \langle H_0 \rangle := \langle g : g \in H_0 \rangle.$$

(1) *There exists an H-forward invariant non-empty compact set $L \subset F(H)$ such that $P(G_2) \subset L$ and $f_i(P(G_1)) \subset L$, for each $i \in I_2$.*
(2) $\deg(g) \ge 2$ *for all $g \in G$.*
(3) *There exists a G-forward invariant non-empty compact set $L_0 \subset F(G)$.*
(4) $\{f_i : i \in I\}$ *satisfies the open set condition.*

Then we have that H is nicely expanding (we endow H_0 with the discrete topology), H_0 satisfies the open set condition, $s(H) = s(G)$ and

$$\dim_H\left(J_{\mathrm{pre}}(G)\right) = \max\left\{s(G), \dim_H\left(J_{\mathrm{pre}}(G_1)\right)\right\}.$$

If in addition to the assumptions, we have $\mathrm{card}(I) < \infty$, f_i is a polynomial for each $i \in I_1$, and if there exists a compact G_1-forward invariant subset $K \subset F(G_1)$ such that $f_j(P(f_i)) \subset K$ for all $i, j \in I_1$ with $i \ne j$, then

$$\dim_H(J(G)) = \max\left\{s(G), \max_{i \in I_1}\{\dim_H(J(f_i))\}\right\}.$$

We point out that, even if the semigroup G is finitely generated (e.g. G has two generators), then the inducing method leads to an infinitely generated semigroup H, which is shown to be nicely expanding. This fact is one of the main motivations to develop Bowen's formula for infinitely generated semigroups.

There are many applications of Theorem 1.4 (see Example 2.4, Theorem 1.5, Corollary 7.17, Lemmas 2.6, 2.7). We are interested in the space $\mathscr{A}$ of couples (f_1, f_2) of polynomials with $\deg(f_i) \ge 2$ for each i, for

which the planar postcritical set $P(\langle f_1, f_2 \rangle) \setminus \{\infty\}$ is bounded but the Julia set $J(\langle f_1, f_2 \rangle)$ is disconnected. It is well-known that for a polynomial f with $\deg(f) \geq 2$, the Julia set $J(f)$ of f is connected if and only if $P(f) \setminus \{\infty\}$ is bounded (see [Mil06]). However, the space $\mathscr{A}$ is not empty, and this is a special phenomenon in the dynamics of polynomial semigroups. There have been some studies on the dynamics of the semigroups $\langle f_1, f_2 \rangle$ for elements $(f_1, f_2) \in \overline{\mathscr{A}}$ employing potential theory (see [Sum11b, Sum10a, Sum09, Sum10b, Sum14, SS11]). In this paper, we focus on elements $(f_1, f_2) \in \mathscr{A}$ and some elements $(f_1, f_2) \in \partial\mathscr{A}$. Applying Theorem 1.4, we obtain the following.

Theorem 1.5 (see Corollary 7.17). *Let f_1 and f_2 be polynomials of degree at least two. Let $G = \langle f_1, f_2 \rangle$. Suppose that all of the following hold.*

    (1) *$P(G) \setminus \{\infty\}$ is a bounded subset of $\mathbb{C}$.*
    (2) *$K(f_1) \subset \operatorname{Int}K(f_2)$, where $K(f_i)$ denotes the filled-in Julia set of f_i.*
    (3) *$\{f_1, f_2\}$ satisfies the open set condition with the open set $(\operatorname{Int}K(f_2)) \setminus K(f_1)$.*
    (4) *$\operatorname{CV}(f_2) \setminus \{\infty\} \subset \operatorname{Int}K(f_1)$, where CV denotes the set of critical values.*

Then we have $\dim_H(J(G)) = \max\{s(G), \dim_H(J(f_1))\}$.

Remark. A sufficient condition for $\dim_H(J(f_1)) \leq s(G)$ is that f_1 is a non-recurrent critical point map ([Urb94]) or a Collet-Eckmann map ([Pr98]).

Note that all elements $(f_1, f_2) \in \mathscr{A}$ and some elements $(f_1, f_2) \in \partial\mathscr{A}$ satisfy the assumptions of Theorem 1.5 ([Sum10b, Sum11c, Sum14]). There are many examples of (f_1, f_2) satisfying the assumption of Theorem 1.5 (see Sections 2, 7). For example, for each polynomial f_1 such that $J(f_1)$ is connected and $\operatorname{Int}K(f_1) \neq \varnothing$, there exists a polynomial f_2 such that $(f_1, f_2) \in \mathscr{A}$ ([Sum11b]). Therefore even if f_1 with connected Julia set has a Siegel disk, there exists f_2 such that $(f_1, f_2) \in \mathscr{A}$ and Theorem 1.5 applies to (f_1, f_2). For the Julia sets of the semigroups generated by elements $\{f_1, f_2\}$ with $(f_1, f_2) \in \overline{\mathscr{A}}$, see Figures 2.2 – 2.4.

Remark 1.6. Let $G = \langle f_i : i \in I \rangle$ be a rational semigroup with $G \subset \operatorname{Aut}(\widehat{\mathbb{C}})$, where $\operatorname{Aut}(\widehat{\mathbb{C}})$ denotes the group of Möbius transformations on $\widehat{\mathbb{C}}$. Suppose that G satisfies the open set condition with a bounded connected open set U in $\mathbb{C}$. Suppose also that there exist two bounded open connected subsets V_1, V_2 with $\overline{U} \subset V_j$, $j \in \{1, 2\}$, such that $f_i^{-1}(V_1) \subset V_2$, for each $i \in I$. Suppose further that there exists a constant $0 < s < 1$ such that $|(f_i^{-1})'(z)| \leq s$, for each $z \in \overline{U}$ and for each $i \in I$. Then, G is a nicely expanding rational semigroup and the system $\Phi = \{f_i^{-1} : \overline{U} \to \overline{U}\}_{i \in I}$ is a contracting conformal iterated function system in the sense of [MU96], which does not necessarily satisfy the cone condition. Here, the cone condition refers to the property that there exist $\gamma, l > 0$ such that, for every $z \in \overline{U}$, there exists an open cone C with vertex z, central angle γ and altitude l such that $C \subset U$ ([MU96, (2.7) on page 110]). Moreover, we have that the pre-Julia set of G is equal to the limit set of the system Φ.

Remark 1.7 (see Section 8). For the results of this paper, the cone condition for the open set in the open set condition is not needed. We will see in Section 7 that there are many examples of semigroups which do not satisfy the cone condition, and for which our results can be applied. For such examples, see Section 7 and Figures 2.3, 2.4. In [MU96, Theorem 3.15] it is proved that, for the Hausdorff dimension of the limit set $J(\Phi)$ of an infinitely generated contracting conformal iterated function system Φ satisfying the cone condition, we have

$$(1.3) \qquad \dim_H J(\Phi) = \inf\{\delta : P(\delta) < 0\} = \sup_{\Phi_F}\{J(\Phi_F)\}.$$

Here, P refers to the associated pressure function, and Φ_F runs over all finitely generated subsystems of Φ. By the methods employed in the proof of Theorem 6.5 of this paper, one can show that (1.3) holds, even if the cone condition is not satisfied. Instead of the cone condition (2.7) in [MU96], we need to assume that $|\phi_i'(x)| \leq s$, for each $x \in X$ in the notation of [MU96]. For the details, see Section 8.

By using Bowen's formula for pre-Julia sets for nicely expanding rational semigroups which is established in this paper, we will investigate the parameter dependence of $\dim_H(J_{\mathrm{pre}}(G))$ for nicely expanding rational semigroups G and $\dim_H(J(S))$ for non-hyperbolic finitely generated rational semigroups S, which is a further interesting task.

Let us briefly comment on the history of the method of inducing. This method was used to investigate invariant measures of non-hyperbolic dynamical systems using ideas of Schweiger ([Sch75]). In [ADU93] the method of inducing is used to develop the ergodic theory of Markov-fibred systems with applications to parabolic rational maps. Using results on the thermodynamic formalism for symbolic dynamical systems with a countable alphabet ([MU96, Sar99]), the method of inducing was used to prove Bowen's formula for parabolic iterated function systems ([MU00], see also [MU03]).

The theory of the dynamics of rational semigroups is intimately related to that of the random dynamics of rational maps. The first study of random complex dynamics was given in [FS91]. For a recent study, see [BBR99, MSU11, Sum11a, Sum10a, Sum10b, Sum11c, Sum13]. The deep relation between these fields (rational semigroups, random complex dynamics, and backward iterated function systems) is explained in detail in the papers ([Sum01]–[Sum14]) of the second author and in [SU11b]. For a random dynamical system generated by a family of polynomial maps on $\widehat{\mathbb{C}}$, let the function $T_\infty : \widehat{\mathbb{C}} \to [0,1]$ be given by the probability of tending to $\infty \in \widehat{\mathbb{C}}$. In [Sum11a, Sum11c] it was shown that under certain conditions, T_∞ is continuous on $\widehat{\mathbb{C}}$ and varies only on the Julia set of the associated rational semigroup (further results were announced in [Sum10b]). For example, there exists a random dynamical system, for which T_∞ is continuous on $\widehat{\mathbb{C}}$ and the set of varying points of T_∞ is equal to the Julia set of Figure 2.2, Figure 2.3 or Figure 2.4, which is a thin fractal set. This function T_∞ is a complex analogue of the devil's staircase (Cantor function) or Lebesgue's singular functions and this is called a "devil's coliseum" (see [Sum11a, Sum11c, Sum13, Sum10b, Sum14]). From this point of view also, it is very interesting and important to investigate the figure, the properties and the dimension of the Julia sets of rational semigroups.

The outline of this paper is as follows. In Section 2, we give various interesting examples which motivate the study in this paper. In Section 3, we give some basic definitions and results on rational semigroups and their associated skew products. In Section 4 we investigate (nicely) expanding rational semigroups. Proposition 4.2 is the key to investigating infinitely generated nicely expanding rational semigroups. In Section 5, we consider the associated skew products systems whose phase spaces are not compact, and we derive basic properties of two notions of topological pressure. We use results on equidistributional measures from [Sum00]. Also, we use some idea similar to the finitely primitive condition ([MU03]) for topological Markov chains with an infinite alphabet. In Section 6, we establish Bowen's formula for pre-Julia sets of (possibly infinitely generated) nicely expanding rational semigroups and we prove the main theorem (Theorem 1.3, Theorem 6.5) of this paper. To verify the lower bound of the Hausdorff dimension in Bowen's formula, we use a reduction to the finitely generated case and we apply [Sum05, Theorem B]. In Section 7, by applying the main theorem to the dynamics of non-hyperbolic rational semigroups and by using the method of inducing, we prove Theorems 1.4, 7.14, and 1.5. In Section 8, we give some remarks on the cone condition which has been assumed in the study of infinite contracting iterated function systems.

Proposition 4.2 is not a simple generalization of finitely generated case. We use a completely new idea based on careful observations on the hyperbolic metric in the proof. Proposition 4.2 is used to prove results in Section 7. In Section 7, we also use some observations on the family $\{J_\omega\}_{\omega \in I^{\mathbb{N}}}$ of fiberwise Julia sets. The ideas in the proof of Proposition 4.2 and Section 7 are new and have not been used in the study of iteration dynamics of holomorphic maps and conformal IFSs so far.

2. EXAMPLES

In this section we give various interesting examples of (nicely) expanding rational semigroups and non-hyperbolic rational semigroups which motivate the study of this paper.

First we give an example of an infinitely generated expanding Möbius semigroup satisfying the open set condition, which is not nicely expanding. Note that this does not happen for finitely generated rational semigroups. We say that $g \in \mathrm{Aut}(\widehat{\mathbb{C}}) \setminus \{\mathrm{id}\}$ is loxodromic if g has two fixed points for which the modulus of the multiplier is not equal to one.

Example 2.1. Let $(a_i) \in \mathbb{D}^{\mathbb{N}}$ and $(b_i) \in (\mathbb{C} \setminus \mathbb{D})^{\mathbb{N}}$ be two sequences of pairwise distinct points, which have a common accumulation point $a_\infty \in \mathbb{S} := \partial(\mathbb{D})$. For each $i \in \mathbb{N}$ we choose a loxodromic Möbius transformation f_i with repelling fixed point a_i and attracting fixed point b_i. Then there exists a sequence $(n_i) \in \mathbb{N}^{\mathbb{N}}$ such that $\{f_i^{n_i} : i \in \mathbb{N}\}$ satisfies the open set condition with respect to $\mathbb{D}$, and such that $\sup_{x \in \overline{\mathbb{D}}} \|(f_i^{-n_i})'(x)\| \leq 1/2$, for each $i \in \mathbb{N}$. Set $G := \langle f_i^{n_i} : i \in \mathbb{N} \rangle$. Clearly, we have that $J(G) \subset \overline{\mathbb{D}}$ and that G is expanding with respect to $\{f_i^{n_i} : i \in \mathbb{N}\}$. However, G is not nicely expanding. To prove this, let P_0 denote a non-empty compact G-forward invariant subset of $F(G)$. Then we have $\{b_n : n \in \mathbb{N}\} \subset P_0$, which implies $a_\infty \in P_0$. Moreover, since a_∞ is an accumulation point of the repelling fixed points (a_n), we have $a_\infty \in J(G)$. Hence, G is not nicely expanding.

The following example shows that an infinitely generated expanding rational semigroup, satisfying the open set condition, is not necessarily hyperbolic. In particular, such a rational semigroup is not nicely expanding. Note that this can not happen for finitely generated rational semigroup (cf. [Sum05, Remark 5]).

Example 2.2. Let $(a_i)_{i \in \mathbb{N}} \in \mathbb{D}^{\mathbb{N}}$ be a sequence of pairwise distinct points in the open unit disc $\mathbb{D}$, such that $(a_i)_{i \in \mathbb{N}}$ has an accumulation point $a_\infty \in \mathbb{S}$. Let $(r_i) \in \mathbb{R}^{\mathbb{N}}$ be a sequence such that the sets $\overline{B(a_i, r_i)}$, $i \in \mathbb{N}$, are pairwise disjoint. There exists a sequence $(b_i) \in \mathbb{R}^{\mathbb{N}}$ such that the quadratic polynomials h_i, given by $h_i(z) := b_i(z - a_i)^2 + a_i$, satisfy $J(h_i) = \partial B(a_i, r_i)$, for each $i \in \mathbb{N}$. Set $V := \mathbb{D} \setminus \bigcup_{i \in \mathbb{N}} \overline{B(a_i, r_i)}$. Observe that there exists a sequence $(n_i) \in \mathbb{N}^{\mathbb{N}}$ such that $\{f_i : i \in \mathbb{N}\}$, given by $f_i := h_i^{n_i}$, satisfies the open set condition with respect to V. Clearly, we have $J(f_i) = J(h_i) = \partial B(a_i, r_i)$. To show that $G := \langle f_i : i \in I \rangle$ is expanding, let $\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ denote the associated skew product. Let $\omega \in I^{\mathbb{N}}$ and $z \in J_\omega$ be given. Since $J(G) \subset \overline{V}$, we have $z \in \overline{V}$ and $f_{\omega_1}(z) \in \overline{V}$. Since $|f'_{\omega_1}(z)| \geq 2$, and since the spherical metric and the Euclidean metric are equivalent on $\overline{V}$, we obtain the G is expanding with respect to $\{f_i : i \in I\}$. Finally, we observe that G is not hyperbolic, because $a_\infty \in J(G) \cap P(G)$.

In the next example, we show that there exists a nicely expanding infinitely generated rational semigroup $G = \langle f_i : i \in I \rangle$, for which $2 = \dim_H(J(G)) = \dim_B(J(G)) > s(G) = t(I) = \dim_H(J_{\mathrm{pre}}(G))$ and $\dim_H(J_{\mathrm{pre}}(G)) < \overline{\dim_B}(J_{\mathrm{pre}}(G)) = \dim_P(J_{\mathrm{pre}}(G)) = \overline{\dim_B}(\overline{J_{\mathrm{pre}}(G)}) = \dim_B(J(G)) = 2$, where $\overline{\dim_B}$ denotes the upper box dimension and $\dim_P$ denotes the packing dimension. The idea is as follows: we put infinitely many repelling fixed points such that the Hausdorff dimension of the closure of the set of repelling fixed points is equal to two. Simultaneously, we can make the multipliers sufficiently large to make the critical exponent close to one.

Example 2.3. Let V denote a bounded open set in $\mathbb{C}$ for which $\dim_H \partial V = 2$. For convenience, suppose that $\overline{\mathbb{D}} \subset V$. Let $(a_i) \in \left(V \setminus \overline{\mathbb{D}}\right)^{\mathbb{N}}$ be a sequence of pairwise distinct points such that $\partial V \subset \overline{\{a_i : i \in \mathbb{N}\}}$. We assume that $\{a_i : i \in \mathbb{N}\}$ is discrete in V. Let $\{f_i : i \in \mathbb{N}\}$ be a generator system such that the following holds for the rational semigroup $G := \langle f_i : i \in \mathbb{N} \rangle$. Let f_1 be given by $f_1(z) = z^d$, for some $d \geq 2$ to be specified later. For each $j \in \mathbb{N}$, $j \geq 2$, let f_j be given by $f_j(z) = \alpha_j(z - a_j) + a_j$, for some sequence $(\alpha_j)_{j=2}^{\infty}$ with $\alpha_j > 1$, such that $\{f_i : i \in \mathbb{N}\}$ satisfies the open set condition with respect to V. We may also assume that $P(G) \cap \overline{V} = P(f_1) = \{0\}$. Since $\{f_i : i \in \mathbb{N}\}$ satisfies the open set condition with respect to V, we have that $J(G) \subset \overline{V}$, which implies that G is hyperbolic. Further, we have that G is nicely expanding by Lemma 4.8. Finally, since the repelling fixed points $\{a_i : i \in \mathbb{N}\}$ are contained in $J(G)$, we have $\dim_H(J(G)) = \dim_B(J(G)) = 2$.

Next, we show that for each $t > 1$ there exist $d \geq 2$ and $(\alpha_j)_{j=2}^{\infty}$ such that $s(G) \leq t$ (thus G depends on t). In order to show it, let $x \in J(G)$. Since the spherical metric and the Euclidean metric are equivalent on $J(G)$, there exists a constant $C > 0$ such that, for each $d \geq 2$,

$$\eta(x) := \sum_{y \in f_1^{-1}(x)} \|f_1'(y)\|^{-t} + \sum_{j=2}^{\infty} \sum_{y \in f_j^{-1}(x)} \|f_j'(y)\|^{-t} \leq C d d^{-t} + C \sum_{j=2}^{\infty} \alpha_j^{-t}.$$

Choose $d \geq 2$ and the sequence $(\alpha_j)_{j=2}^{\infty}$ sufficiently large such that $\sup_{x \in J(G)} \eta(x) < 1$. Hence, we have $s(G) \leq t$. Moreover, by Theorem 1.3, we have $s(G) = t(\mathbb{N}) = \dim_H J_{\mathrm{pre}}(G)$. We have thus shown that $2 = \dim_H(J(G)) = \dim_B(J(G)) > s(G) = t(\mathbb{N}) = \dim_H\left(J_{\mathrm{pre}}(G)\right)$.

We give some examples to which we can apply Theorems 1.4 and 1.5. Recall that the filled-in Julia set $K(g)$ of a polynomial g is defined by

$$K(g) := \left\{z \in \mathbb{C} : (g^n(z))_{n \in \mathbb{N}} \text{ is bounded}\right\}.$$

Example 2.4. Let $I := \{1, \ldots, m + \ell\}$, $m, \ell \in \mathbb{N}$, and let $\{h_i : i \in I\}$ be polynomials of degree at least two. Set $I_2 := \{m+1, \ldots, m+\ell\}$ and suppose that h_i is hyperbolic, for each $i \in I_2$. Suppose that $K(h_i)$ is connected, for each $i \in I$, and that $K(h_i) \cap K(h_j) = \varnothing$, for all $i, j \in I$ with $i \neq j$. Let $R > 0$ such that $K(h_i) \subset B(0, R)$, for each $i \in I$. Then there exists $N \in \mathbb{N}$ such that $h_i^{-N}(\overline{B(0,R)}) \subset B(0,R)$, for each $i \in I$, and that $\{h_i^N : i \in I\}$ satisfies the open set condition with open set $B(0,R)$. Set $f_i := h_i^N$ and consider the rational semigroup $G := \langle f_i : i \in I \rangle$. Set $A := \widehat{\mathbb{C}} \setminus B(0,R)$ and observe that A is G-forward invariant. Since $J(G) = \bigcup_{i \in I} f_i^{-1}(J(G)) \subset \bigcup_{i \in I} f_i^{-1}(\overline{B(0,R)}) \subset B(0,R)$, we have that $A \subset F(G)$. For all $i, j \in I$ with $i \neq j$ we have $f_j(K(f_i)) \subset A$, because $f_j(K(f_j)) \subset B(0,R)$ and $f_i^{-1}(B(0,R)) \cap f_j^{-1}(B(0,R)) = \varnothing$. To see that G satisfies the assumption (1) in Theorem 1.4, we set $L := A \cup P(G_2)$. L is G-forward invariant because $g(P(G_2)) \subset g(\bigcup_{i \in I_2} P(f_i) \cup A) \subset \bigcup_{i \in I_2} P(f_i) \cup A \subset L$. To prove that $P(G_2) \subset F(G)$, it suffices to prove that $P(f_i) \subset F(G)$, for each $i \in I_2$. We observe that $P(f_i) \setminus \{\infty\} \subset \mathrm{Int}(K(f_i))$ because $K(f_i)$ is connected and f_i is hyperbolic, for each $i \in I_2$. Since $g(K(f_i)) \subset A \cup K(f_i)$, for each $g \in G$, it follows from Montel's Theorem that $P(f_i) \subset F(G)$. We have thus shown that G satisfies the assumptions of Theorem 1.4. Hence, we have $\dim_H(J(G)) = \max\left\{s(G), \max_{i \in \{1, \ldots, m\}}\{\dim_H(J(f_i))\}\right\}$. If additionally, each f_i is a non-recurrent critical point map, then $\dim_H(J(G)) = s(G)$ (see Figure 2.1 for an example). Using this formula, the numerical value of the Hausdorff dimension may be computed but we do not do that in this paper. Also, using this formula the parameter-dependence of $\dim_H(J(G))$ may be investigated which is a further interesting task.

Definition 2.5 (PB-D). We say that $G = \langle f_1, f_2 \rangle$ satisfies PB-D, if f_1 and f_2 are polynomials of degree at least two, such that each of the following holds.

   (1) $P(G) \setminus \{\infty\}$ is a bounded subset of $\mathbb{C}$.
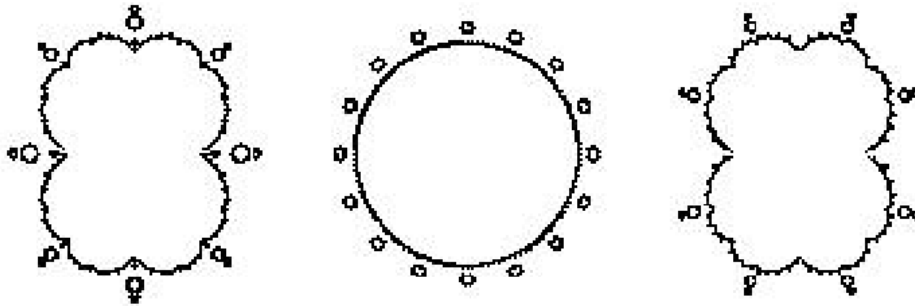   (2) $J(G)$ is disconnected.

FIGURE 2.1. The Julia set of a 3-generator polynomial semigroup G. $G = \langle f_1, f_2, f_3 \rangle$ is given by $f_i = h_i^3$, $i \in \{1, 2, 3\}$, where $h_1(z) = (z+3)^2 + 0.25 - 3$, $h_2(z) = z^2$ and $h_3(z) = (z-3)^2 + 0.25 + 3$. f_1 and f_3 are not hyperbolic, but they are non-recurrent critical point maps. We have $\dim_H(J(G)) = s(G)$.

Lemma 2.6. *If $G = \langle f_1, f_2 \rangle$ satisfies PB-D, then it also satisfies the assumptions of Theorem 1.5 (by renumbering f_1 and f_2 if necessary), and the open set condition is satisfied.*

Proof. [Sum14] or [Sum11c, Proof of Theorem 2.11, Claim 2]. □

Lemma 2.7. *Let f_1 be a polynomial of degree at least two with $\mathrm{Int}(K(f_1)) \neq \varnothing$ such that $K(f_1)$ is connected. Let $b \in \mathrm{Int}(K(f_1))$. Let $d \in \mathbb{N}$ with $d \geq 2$ such that $(\deg(f_1), d) \neq (2, 2)$. Then, there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, setting $f_2 := a(z-b)^d + b$, the polynomial semigroup $G = \langle f_1, f_2 \rangle$ satisfies PB-D.*

Proof. [Sum11b, Proposition 2.40] and Lemma 2.6. □

Remark 2.8. If f_1 is a non-recurrent critical point map or a Collet-Eckmann map, then $\dim_H(J(f_1)) \leq s(\langle f_1 \rangle)$ by ([Urb94]) and ([Pr98]). Thus, if in addition to this assumption, $G = \langle f_1, f_2 \rangle$ satisfies the assumptions of Theorem 1.5, then $\dim_H(J(G)) = s(G)$.

FIGURE 2.2. The Julia set of $G = \langle f_1, f_2 \rangle$, where $f_1(z) = z^2 + e^{2\pi i \sqrt[3]{0.25}}z$, $f_2 = h_2^2$ and $h_2(z) = 0.1z^2$. G satisfies PB-D. f_1 has a Siegel disc with center in 0. We have $\dim_H(J(G)) = \max\{s(G), \dim_H(J(f_1))\}$.
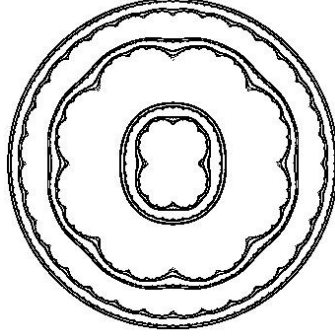
FIGURE 2.3. The Julia set of $G = \langle f_1, f_2 \rangle$ satisfying PB-D, where f_1 is a non-hyperbolic map (but is a non-recurrent critical point map). $J(G)$ is disconnected. The cone condition is not satisfied. We have $\dim_H(J(G)) = s(G) = s(\langle f_2 \circ f_1^r : r \in \mathbb{N} \cup \{0\}\rangle)$.
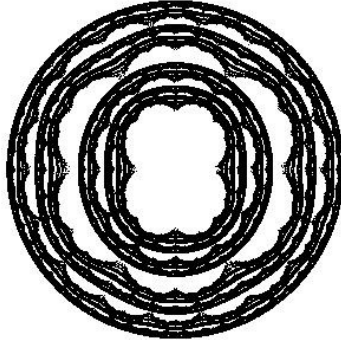

FIGURE 2.4. The Julia set of $G = \langle z^2 + \frac{1}{4}, az^3 \rangle$, where $a \in \mathbb{C}$ is a complex number. G satisfies the assumptions of Theorem 1.5 and $J(G)$ is connected. The cone condition is not satisfied. We have $\dim_H(J(G)) = s(G) = s(\langle f_2 \circ f_1^r : r \in \mathbb{N} \cup \{0\}\rangle)$.

Remark. Regarding Figure 2.4, we remark there exists $a \in \mathbb{C}$ such that for $f_1(z) := z^2 + \frac{1}{4}$ and $f_2(z) := az^3$, we have that $f_1^{-1}(J(G)) \cap f_2^{-1}(J(G)) \neq \varnothing$ (see [Sum14]). Then, it follows from [Sum09, Theorem 1.5, Theorem 1.7] that $J(G)$ is connected. Like this, for each polynomial f_1 of degree at least two with $\mathrm{Int}\, K(f_1) \neq \varnothing$ such that $K(f_1)$ is connected, there are many examples of $G = \langle f_1, f_2 \rangle$ satisfying the assumptions of Theorem 1.5 for which $J(G)$ is connected (see [Sum14, Theorem 2.27 and its proof]).

3. PRELIMINARIES ON RATIONAL SEMIGROUPS AND SKEW PRODUCTS

In this section, we collect some of the basic results on rational semigroups and the associated skew products.

Definition 3.1. Let G be a rational semigroup and let $z \in \widehat{\mathbb{C}}$. The backward orbit $G^-(z)$ of z and the set of exceptional points $E(G)$ are defined by $G^-(z) := \bigcup_{g \in G} g^{-1}(z)$ and $E(G) := \left\{ z \in \widehat{\mathbb{C}} : \text{card}(G^-(z)) < \infty \right\}$. We say that a set $A \subset \widehat{\mathbb{C}}$ is G-backward invariant, if $g^{-1}(A) \subset A$, for each $g \in G$.

We refer to [HM96, Sum00] for the fundamental properties of rational semigroups and their Julia sets.

We review some of the basics of the skew product associated to rational semigroups introduced in [Sum00]. We will always assume that I is a topological space. We denote by $\mathscr{S}$ the set of all non-constant polynomial maps on $\widehat{\mathbb{C}}$ endowed with the relative topology inherited from Rat. Note that, for each $d \in \mathbb{N}$, the subspace $\text{Rat}_d := \{ f \in \text{Rat} : \deg(f) = d \}$ of Rat is a connected component of Rat, and Rat_d is an open subset of Rat. Similarly, the subspace $\mathscr{S}_d := \{ f \in \mathscr{S} : \deg(f) = d \}$ of $\mathscr{S}$ is a connected component of $\mathscr{S}$, and $\mathscr{S}_d$ is an open subset of $\mathscr{S}$. A sequence $\{f_n\}$ tends to f in $\mathscr{S}$ if and only if there exists a number $N \in \mathbb{N}$ such that $\deg(f_n) = \deg(f)$, for each $n \geq N$, and if the coefficients of f_n converge to those of f appropriately. For the topology of Rat and $\mathscr{S}$, see [Be91].

Let $\pi_1 : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}}$ and $\pi_{\widehat{\mathbb{C}}} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the canonical projections. For each $F \subset I$, we also set

$$J^F := \bigcup_{\omega \in F^{\mathbb{N}}} J^\omega \text{ and } J_F := \bigcup_{\omega \in F^{\mathbb{N}}} J_\omega.$$

For a finite word $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in I^n$ and an infinite word $\alpha = (\alpha_1, \alpha_2, \ldots) \in I^{\mathbb{N}}$, we set $\omega \alpha = (\omega_1, \ldots, \omega_n, \alpha_1, \alpha_2, \ldots) \in I^{\mathbb{N}}$.

Proposition 3.2. *Let* $(f_i)_{i \in I} \in C(I, \text{Rat})$ *and let* $\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ *be the skew product associated to the generator system* $\{ f_i : i \in I \}$. *Then we have the following.*

(1) $\tilde{f}(J^\omega) = J^{\sigma\omega}$ *and* $(\tilde{f}_{|\pi_1^{-1}(\omega)})^{-1}(J^{\sigma\omega}) = J^\omega$, *for each* $\omega \in I^{\mathbb{N}}$.

(2) $\tilde{f}(J(\tilde{f})) = J(\tilde{f})$, $\tilde{f}^{-1}(J(\tilde{f})) = J(\tilde{f})$, $\tilde{f}(F(\tilde{f})) = F(\tilde{f})$, $\tilde{f}^{-1}(F(\tilde{f})) = F(\tilde{f})$.

(3) Suppose that I is a finite set endowed with the discrete topology. Let $G = \langle f_i : i \in I \rangle$ and suppose that $\text{card}(J(G)) \geq 3$. Then we have $J(\tilde{f}) = \bigcap_{n \in \mathbb{N}_0} \tilde{f}^{-n}(I^{\mathbb{N}} \times J(G))$ and $\pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) = J(G)$. Here, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Proof. The proof of (1) is straightforward and therefore omitted. We give a proof of (2), following [Sum01, Lemma 2.4]. The inclusion $\tilde{f}(J(\tilde{f})) \subset J(\tilde{f})$ is obvious. In order to prove $\tilde{f}^{-1}(J(\tilde{f})) \subset J(\tilde{f})$, let $(\omega, x) \in \tilde{f}^{-1}(J(\tilde{f}))$ be given. Clearly, $(\sigma\omega, f_{\omega_1}(x)) \in J(\tilde{f})$. For each neighborhood U of $\sigma\omega$ and for each neighborhood V of $f_{\omega_1}(x)$, there exists $(\alpha, y) \in U \times V$ such that $y \in J_\alpha$ and $f_{\omega_1}^{-1}(y) \subset J_{\omega_1 \alpha}$. Consequently, for each neighborhood W of (ω, x), we have $W \cap \left(\bigcup_{\rho \in I^{\mathbb{N}}} J^\rho \right) \neq \varnothing$. The assertions on the $F(\tilde{f})$ follow by taking complements. The assertion in (3) is proved in ([Sum00, Proposition 3.2 (b)]). $\square$

Remark. Regarding Proposition 3.2 (3), we remark that in general, we have $J_{\text{pre}}(G) \subset \pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) \subset J(G)$. If G is expanding, then $J_{\text{pre}}(G) = \pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) = \pi_{\widehat{\mathbb{C}}}(\cup_{\omega \in I^{\mathbb{N}}} J^\omega)$ by Lemma 4.1. In particular, if G is a finitely generated expanding rational semigroup, then $J_{\text{pre}}(G) = J(G)$. The inclusion $\cup_{\omega \in I^{\mathbb{N}}} J^\omega \subset J(\tilde{f})$ can be strict, see [Sum10a, Remark 2.8] for an example. Theorem 7.14 provides examples of infinitely generated rational semigroups for which the inclusion $\pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) \subset J(G)$ is strict. Namely, let $G = \langle f_1, f_2 \rangle$ be as in Lemma 2.6. Then $G = \langle f_1, f_2 \rangle$ satisfies the assumptions of Theorem 1.5 (by renumbering f_1, f_2 if necessary). For the infinitely generated rational semigroup $H = \langle f_2 f_1^n : n \in \mathbb{N} \rangle$ we have that $\varnothing \neq J(f_1) \subset J(G) \setminus J_{\text{pre}}(H) = J(H) \setminus J_{\text{pre}}(H)$ by Lemma 7.16, Theorem 7.14 (1) and Lemma 7.5. Further, since H is nicely expanding by Lemma 7.2, we have $J_{\text{pre}}(H) = \pi_{\widehat{\mathbb{C}}}(J(\tilde{f}))$ by Lemma 4.1, where $\tilde{f}$ is the skew product associated to generator system $H_0 = \{ f_2 f_1^r : r \in \mathbb{N} \}$. Thus $\pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) \subsetneqq J(H)$. An example of a nicely expanding rational semigroup for which $\dim_H(J_{\text{pre}}(G)) < \dim_H(J(G))$ is given in Example 2.3. We remark that the pre-Julia

set is a continuous image of a Borel set. In particular, $J_{\mathrm{pre}}(G)$ is a Suslin set and thus universally measurable. For details on Suslin sets we refer to [Fed69, p.65–70].

4. EXPANDING RATIONAL SEMIGROUPS

In this section, we study the dynamics of expanding rational semigroups. For an expanding rational semigroup $G = \langle f_i : i \in I \rangle$ we have that J^I is a closed subset of $I^{\mathbb{N}} \times \widehat{\mathbb{C}}$.

Lemma 4.1. *Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. If $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$, then we have that $J(\tilde{f}) = J^I$. In particular, we have that $\pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) = J_{\mathrm{pre}}(G)$.*

Proof. Let $(\omega, z) \in J(\tilde{f})$. Since G is expanding with respect to $\{f_i : i \in I\}$ we have $\lim_n \|(\tilde{f}^n)'(\omega, z)\| = \infty$. This gives $z \in J_\omega$ and hence, $(\omega, z) \in J^\omega$. $\qquad\square$

Our next aim is to prove the following characterization for a rational semigroup to be nicely expanding.

Proposition 4.2. *Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. For the rational semigroup $G = \langle f_i : i \in I \rangle$, the following statements are equivalent.*

(1) *G is nicely expanding.*
(2) *G is hyperbolic, each element $h \in G \cap \mathrm{Aut}(\widehat{\mathbb{C}})$ is loxodromic, $\mathrm{id} \notin \overline{G \cap \mathrm{Aut}(\widehat{\mathbb{C}})}$ and there exists a non-empty compact G-forward invariant set $P_0(G) \subset F(G)$.*

In order to prove Proposition 4.2 we need the following Lemmas 4.3–4.6. We use the following classification of Möbius transformations. Let $g \in \mathrm{Aut}(\widehat{\mathbb{C}}) \setminus \{\mathrm{id}\}$. We say that g is *elliptic* if g has two fixed points, for which the modulus of the multipliers is equal to one. If g is neither loxodromic nor elliptic, then g is *parabolic*.

Lemma 4.3. *Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. If $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$, then each element $g \in G \cap \mathrm{Aut}(\widehat{\mathbb{C}})$ is loxodromic.*

Proof. Let $g \in G \cap \mathrm{Aut}(\widehat{\mathbb{C}})$ and suppose by way of contradiction that g is not loxodromic. If g is parabolic, then the parabolic fixed point z of g satisfies $z \in J(g)$ and $\|(g^n)'(z)\| = 1$, which contradicts that G is expanding. Now suppose that g is elliptic or the identity map, and let $z_0 \in J_{\mathrm{pre}}(G)$. For each $n \in \mathbb{N}$, set $z_n := g^{-n}(z_0)$ and observe that $z_n \in J_{\mathrm{pre}}(G)$. By conjugating G by a Möbius transformation, we may assume that $z_n \in \mathbb{C}$ for each $n \in \mathbb{N}$ and $g(z) = e^{i\theta}z$, for some $\theta \in \mathbb{R}$. We see that the modulus of $(g^n)'(z_n)$ is equal to one. Letting n tend to infinity, contradicts that G is expanding and completes the proof. $\qquad\square$

Lemma 4.4. *Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. If $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$, then $\mathrm{id} \notin \overline{G \cap \mathrm{Aut}(\widehat{\mathbb{C}})}$, where the closure is taken in $\mathrm{Aut}(\widehat{\mathbb{C}})$.*

Proof. Since G is expanding with respect to $\{f_i : i \in I\}$, there exist $C > 0$ and $\lambda > 1$ such that for all $n \in \mathbb{N}$ we have $\inf_{(\omega, z) \in J(\tilde{f})} \|(\tilde{f}^n)'(\omega, z)\| \geq C\lambda^n$. Suppose by way of contradiction that there exists a sequence $(g_n) \in (G \cap \mathrm{Aut}(\widehat{\mathbb{C}}))^{\mathbb{N}}$ such that $\lim_n \mathrm{dist}_{\mathrm{Rat}}(g_n, \mathrm{id}) = 0$. For each $n \in \mathbb{N}$, let g_n be given by a product of a_n generators in $\{f_i : i \in I\}$, for a sequence $(a_n) \in \mathbb{N}^{\mathbb{N}}$. We may assume without loss of generality that the sequence (a_n) is unbounded. Otherwise, we choose for each $r \in \mathbb{N}$ an element $n_r \in \mathbb{N}$ such that $\sup_{z \in \widehat{\mathbb{C}}} d(g_{n_r}^r(z), z) < r^{-1}$. Then we have $\lim_n \mathrm{dist}_{\mathrm{Rat}}(g_{n_r}^r, \mathrm{id}) = 0$, as r tends to infinity, and $g_{n_r}^r$ is a product of ra_{n_r} generators. By passing to a subsequence, we may assume that $\lim_n a_n = \infty$. Choose an arbitrary

$(\omega, z_0) \in J^I$ and write $g_n = f_{\alpha_{a_n}} \circ \cdots \circ f_{\alpha_1}$, for some $\alpha \in I^{a_n}$. Clearly, $g_n^{-1}(z_0) \in J_{\alpha \omega}$. Consequently, as n tends to infinity,

$$\sup_{z \in \widehat{\mathbb{C}}} \|g_n'(z)\| \geq \|g_n'\left(g_n^{-1}(z_0)\right)\| \geq \inf_{(\tau, y) \in J(\tilde{f})} \|\left(\tilde{f}^{a_n}\right)'(\tau, y)\| \geq C\lambda^{a_n} \to \infty,$$

which is impossible since g_n tends uniformly to the identity on $\widehat{\mathbb{C}}$. This contradiction finishes the proof. $\square$

Lemma 4.5. *Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. Suppose that $G = \langle f_i : i \in I \rangle$ is a rational semigroup such that $\mathrm{card}(J(G)) \leq 2$ and such that each element in $G \cap \mathrm{Aut}(\widehat{\mathbb{C}})$ is loxodromic. Further, assume that there exists a non-empty compact, G-forward invariant subset $P_0(G) \subset F(G)$. Then we have the following.*

(1) $\mathrm{card}(J(G)) = 1$.
(2) *G is expanding with respect to $\{f_i : i \in I\}$ if and only if $\mathrm{id} \notin \overline{G \cap \mathrm{Aut}(\widehat{\mathbb{C}})}$.*
(3) *If I is finite or if G satisfies the open set condition with respect to $\{f_i : i \in I\}$, then G is expanding with respect to $\{f_i : i \in I\}$.*

Proof. It is clear that $G \subset \mathrm{Aut}(\widehat{\mathbb{C}})$ because $\mathrm{card}(J(G)) \leq 2$. Let us start with the proof of (1). Let $g \in G$. It follows from $g^{-1}(J(G)) \subset J(G)$ [HM96, Theorem 2.1] and $\mathrm{card}(J(G)) \leq 2$, that $g(J(G)) = J(G)$. Since g is loxodromic, we have $g(x) = x$, for each $x \in J(G)$. By way of contradiction, suppose that $J(G)$ consists of two points, say $J(G) = \{a, b\}$. We may assume that $\|g'(a)\| > 1$ and $\|g'(b)\| < 1$. Now, for each $z \in P_0(G)$, we have $\lim_n g^n(z) = b$. Since $g(P_0(G)) \subset P_0(G)$, we conclude that $b \in P_0(G) \subset F(G)$, which is a contradiction. Hence, $\mathrm{card}(J(G)) = 1$. For simplicity, we may assume that $J(G) = \{0\}$ in the following.

Next, we turn to the proof of (2). By Lemma 4.4, it remains to show that, if G is not expanding with respect to $\{f_i : i \in I\}$, then $\mathrm{id} \in \overline{G \cap \mathrm{Aut}(\widehat{\mathbb{C}})}$. Let b_i denote the attracting fixed point of f_i. Since $f_i(P_0(G)) \subset P_0(G) \subset F(G)$, we have $b_i \in P_0(G) \subset F(G)$. Since $f_i(0) = 0$, it follows that $\|f_i'(0)\| > 1$ for each $i \in I$. Let $h_i \in \mathrm{Aut}(\widehat{\mathbb{C}})$ be given by $h_i(z) := z/\left(b_i^{-1}z - 1\right)$, and observe that $h_i^{-1} f_i h_i(z) = c_i z$, where c_i denotes the multiplier of f_i at 0. If G is not expanding with respect to $\{f_i : i \in I\}$, then there exists a sequence $(i_n) \in I^{\mathbb{N}}$ tending to infinity such that $\lim_n \|f_{i_n}'(0)\| = 1$. After passing to a subsequence, we may assume that $\lim_n \mathrm{dist}_{\mathrm{Rat}}(h_{i_n}, h) = 0$, for some $h \in \mathrm{Aut}(\widehat{\mathbb{C}})$, which gives that $\lim_n \mathrm{dist}_{\mathrm{Rat}}(f_{i_n}, \mathrm{id}) = 0$, as n tends to infinity. The proof of (2) is complete.

To prove (3), recall that by the proof of (2), we have that G is expanding if $\mathrm{id} \notin \overline{\bigcup_{i \in I} f_i}$. Clearly, if I is finite or if $\{f_i : i \in I\}$ satisfies the open set condition, then $\mathrm{id} \notin \overline{\bigcup_{i \in I} f_i}$. The proof is complete. $\square$

Lemma 4.6. *Let I be a finite set endowed with the discrete topology. Let $G = \langle f_i : i \in I \rangle$ denote a hyperbolic rational semigroup such that each element in $G \cap \mathrm{Aut}(\widehat{\mathbb{C}})$ is loxodromic and there exists a non-empty compact G-forward invariant set $P_0(G) \subset F(G)$. Then G is nicely expanding (see Definition 1.1).*

Proof. If $\mathrm{card}(J(G)) \geq 3$, then we can follow the proof of [Sum98, Theorem 2.6] by replacing $P(G)$ by $P_0(G)$. The remaining case $\mathrm{card}(J(G)) \leq 2$ follows from Lemma 4.5 (3). $\square$

We now give the proof of Proposition 4.2.

Proof of Proposition 4.2. The proof that (1) implies (2) follows from Lemma 4.3 and Lemma 4.4.
We now turn our attention to the proof of the converse implication. Our aim is to show that G is expanding with respect to $\{f_i : i \in I\}$. By Lemma 4.5 (2), we are left to consider the case $\mathrm{card}(J(G)) \geq 3$. Since G is hyperbolic, we may assume that $P(G) \subset P_0(G)$. We denote by $V_1, \ldots, V_r$, $r \in \mathbb{N}$, the finitely many

connected components of $F(G)$ which have non-empty intersection with the non-empty compact set $P_0(G)$. Since $\operatorname{card}(J(G)) \geq 3$, we have that each V_i is a hyperbolic Riemann surface. We denote by d_h the Poincaré metric on V_i and we set $U_i := \{z \in V_i : d_h(z, P_0(G) \cap V_i) < 1\}$. Our main task is to verify the following claim. For $g \in G$ with $g(V_i) \subset V_i$, for some $i \in \{1, \ldots, r\}$, we denote by $\|g'(z)\|_h$ the norm of the derivative of g at z with respect to the Poincaré metric on V_i.

Claim 4.7. *For each $i \in \{1, \ldots, r\}$ there exists $0 < c_i < 1$ such that, for all $g \in G$ satisfying $g(V_i) \subset V_i$, we have $\sup_{z \in U_i} \|g'(z)\|_h \leq c_i$.*

Proof of Claim 4.7. Suppose for a contradiction that the claim is false. Since V_i is hyperbolic and $g : V_i \to V_i$ is holomorphic, it follows by Pick's Theorem ([Mil06, Theorem 2.11]) that $\|g'(z)\|_h \leq 1$, for each $z \in V_i$. Hence, by our assumption, there exist $i \in \{1, \ldots, r\}$ and sequences $(g_n) \in G^{\mathbb{N}}$ and $(z_n) \in U_i^{\mathbb{N}}$ such that $g_n(V_i) \subset V_i$, for each $n \in \mathbb{N}$, and $\lim_n \|g_n'(z_n)\|_h = 1$. We may assume that $\lim_n z_n = z_\infty \in \overline{U_i}$ by passing to a subsequence. Since each family of holomorphic maps between hyperbolic surfaces is normal ([Mil06, Corollary 3.3]), we may assume that there exists a holomorphic map $g_\infty : V_i \to \widehat{\mathbb{C}}$ such that $g_n \rightrightarrows g_\infty$ on V_i, where $\rightrightarrows$ denotes uniform convergence on compact subsets of V_i, and $g_\infty : V_i \to V_i$ is holomorphic. We show that for each g_n there exists a fixed point $w_n \in P_0(G) \cap V_i$. This is clear in the case that the degree of g_n is equal to one by our assumption that each element in $G \cap \operatorname{Aut}(\widehat{\mathbb{C}})$ is loxodromic. We consider the case that the degree of g_n is at least two. Since g_n is hyperbolic, for each $x \in V_i \cap P_0(G)$, the g_n-orbit of x converges to some attracting p-periodic point $w_n \in P_0(G) \cap V_i$ of g_n. Since $g_n(V_i) \subset V_i$, we conclude that $p = 1$. We may assume that $\lim_n w_n = w_\infty \in P_0(G) \cap V_i$. It then follows that $g_\infty(w_\infty) = w_\infty$ and $\|g_\infty'(z_\infty)\|_h = 1$. From the Classification Theorem ([Mil06, Theorem 5.2]) for holomorphic maps between hyperbolic surfaces, we have four possibilities for $g_\infty : V_i \to V_i$, namely, attracting, escape, finite order and irrational rotation. By Pick's Theorem and the fact that $\|g_\infty'(z_\infty)\|_h = 1$ it follows that g_∞ is a local isometry, which implies that $\|g_\infty'(w_\infty)\|_h = 1$. Thus, g_∞ is not attracting. Escape is impossible since we have a fixed point. We conclude that we have finite order or irrational rotation, hence, in every case we have that there exists a sequence $(m_j)_{j \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $g_\infty^{m_j} \rightrightarrows \operatorname{id}_{V_i}$ on V_i. Combining with $g_n \rightrightarrows g_\infty$, we conclude that there exists $(h_n) \in G^{\mathbb{N}}$ such that $h_n \rightrightarrows \operatorname{id}_{V_i}$ on V_i. Let $A := \{z \in \widehat{\mathbb{C}} : (h_n) \text{ is normal in a neighborhood of } z\}$ and let A_0 denote the connected component of A containing V_i. By Vitali's theorem [Be91, Theorem 3.3.3] we conclude that $h_n \rightrightarrows \operatorname{id}_{A_0}$ in A_0. By our assumption that $\operatorname{id} \notin \overline{G \cap \operatorname{Aut}(\widehat{\mathbb{C}})}$, it follows that $A_0 \neq \widehat{\mathbb{C}}$.

We have $\partial A_0 \subset J(G) \subset \widehat{\mathbb{C}} \setminus P_0(G)$. Let $x_0 \in \partial A_0$ and choose $s > 0$ such that $B(x_0, s) \subset \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^r U_i$. There exists $y_0 \in A_0$ and $s_0 > 0$ such that $B(y_0, s_0)$ is a relatively compact subset of $B(x_0, s) \cap A_0$. Since $h_n \rightrightarrows \operatorname{id}_{A_0}$, we conclude that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have $h_n(B(y_0, s_0)) \subset B(x_0, s)$. Now choose inverse branches $\gamma_n : B(x_0, s) \to \widehat{\mathbb{C}}$ of h_n, that is, $h_n \circ \gamma_n = \operatorname{id}_{B(x_0, s)}$ such that $\gamma_n(h_n(y_0)) = y_0$. Since $\bigcup_{i=1}^r U_i$ is G-forward invariant, we have that $\gamma_n(B(x_0, s)) \cap \bigcup_{i=1}^r U_i = \varnothing$, which implies that $(\gamma_n)_{n \in \mathbb{N}}$ is normal in $B(x_0, s)$. From this and the equivalence between normality and equicontinuity [Be91, Theorem 3.3.2], we conclude that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for all $n \in \mathbb{N}$ we have $\gamma_n(B(h_n(y_0), \varepsilon_1)) \subset B(y_0, \varepsilon_2) \subset B(y_0, s_0)$. Now suppose that $\gamma_{n_j} \rightrightarrows \gamma_\infty$ on $B(x_0, s)$, for some sequence (n_j) tending to infinity and $\gamma_\infty : B(x_0, s) \to \widehat{\mathbb{C}}$ holomorphic. By [Ah79, p154] we obtain that $\gamma_\infty = \operatorname{id}$ on $B(y_0, \varepsilon_3)$, for some $\varepsilon_3 > 0$ such that $B(y_0, \varepsilon_3) \subset B(h_n(y_0), \varepsilon_1)$ for sufficiently large n. Hence, $\gamma_n \rightrightarrows \operatorname{id}_{B(x_0, s)}$. Combining this with $h_n \circ \gamma_n = \operatorname{id}_{B(x_0, s)}$, we deduce that there is $s_1 < s_0$ such that $h_n(B(x_0, s_1)) \subset B(x_0, s_0)$, for sufficiently large n. Hence, (h_n) is normal in $B(x_0, s_1)$ contradicting the definition of A_0. The claim follows. $\qquad\square$

We now continue the proof of Proposition 4.2. With $r \in \mathbb{N}$ and $U_1, \ldots, U_r$ from above, we set $W := \bigcup_{i=1}^r U_i$. Next, we verify that there exists a compact set $K_1 \subset W$ such that $f_\omega(W) \subset K_1$, for each $\omega \in I^r$. To prove this, note that for each $i \in \{1, \ldots, r\}$ and $\omega \in I^r$, there exist $j, k, q \in \{1, \ldots, r\}$ with $k \leq q$ such that $f_{\omega_{k-1}, \ldots, \omega_1}(U_i) \subset U_j$ and $f_{\omega_q, \ldots, \omega_k}(U_j) \subset U_j$, where we set $f_\varnothing := \operatorname{id}_{\widehat{\mathbb{C}}}$. Now it follows from Claim 4.7

that, for each $j \in \{1, \ldots, r\}$, there exists $0 < c_j < 1$ such that $f_{\omega_q, \ldots, \omega_k}(U_j) \subset \{z \in V_j : d_h(z, P_0(G)) \leq c_j\}$, which is a compact subset of U_j. Hence, by Pick's Theorem and using that $P_0(G)$ is G-forward invariant, we obtain that, for each $i \in \{1, \ldots, r\}$ and $\omega \in I^r$,

$$(4.1) \qquad f_\omega(U_i) \subset \bigcup_{j=1}^{r} \{z \in V_j : d_h(z, P_0(G)) \leq \max\{c_1, \ldots, c_r\}\} := K_1.$$

We have thus shown that there exists a compact set $K_1 \subset W$ such that $f_\omega(W) \subset K_1$, for each $\omega \in I^r$.

The next step is prove the existence of a compact set $K_2 \subset \widehat{\mathbb{C}} \setminus \overline{W}$, such that $f_\omega^{-1}(\widehat{\mathbb{C}} \setminus \overline{W}) \subset K_2$ for each $\omega \in I^r$. To prove this, we verify that $a := \inf_{\omega \in I^r} d(f_\omega^{-1}(\widehat{\mathbb{C}} \setminus \overline{W}), W) > 0$, where $d(A, B) := \inf_{a \in A, b \in B} d(a, b)$. The claim then follows by setting $K_2 := \{z : d(z, W) \geq a\}$. Assume for a contradiction that $a = 0$. Then there exist $(x_n) \in (\widehat{\mathbb{C}} \setminus \overline{W})^{\mathbb{N}}$, $(\omega^n) \in (I^r)^{\mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ with $y_n \in f_{\omega^n}^{-1}(x_n)$, for each $n \in \mathbb{N}$, such that $\lim_n d(y_n, W) = 0$. After passing to a subsequence, we may assume that there exists $y_0 \in \overline{W}$ such that $\lim_n y_n = y_0$. By (4.1), we conclude that $f_{\omega^n}(y_0) \in K_1$ for each $n \in \mathbb{N}$. Since $y_0 \in F(G)$, we may assume that $f_{\omega^n} \rightrightarrows f$, as n tending to infinity, for some holomorphic map f in a neighborhood of y_0. Hence, $f(y_0) \in K_1 \subset W$. On the other hand, we have $f_{\omega^n}(y_n) = x_n \in \widehat{\mathbb{C}} \setminus W$, for each $n \in \mathbb{N}$, which implies that $f(y_0) \in \widehat{\mathbb{C}} \setminus W$. This contradiction shows that $a > 0$.

We now turn to the final step of this proof. We denote by $A_1, \ldots, A_\ell$, $\ell \in \mathbb{N}$, the connected components of $\widehat{\mathbb{C}} \setminus \overline{W}$ for which $A_i \cap J(G) \neq \varnothing$. Clearly, we have $J(G) \subset \bigcup_{i=1}^{\ell} A_i$. Let $z \in J_{\mathrm{pre}}(G)$, $z \in J_\tau$, $\tau \in I^{\mathbb{N}}$, and set $\omega := (\tau_1, \ldots, \tau_r)$. There exist $j_1, j_2 \in \mathbb{N}$ such that $z \in A_{j_1}$ and $f_\omega(z) \in A_{j_2}$. For a holomorphic map $h : S_1 \to S_2$ between hyperbolic Riemann surfaces S_1 and S_2 and $w \in S_1$, we use $\|h'(w)\|_{S_1, S_2}$ to denote the operator norm of the derivative $Dh(w) : T_w S_1 \to T_{h(w)} S_2$ with respect to the norms $\| \cdot \|_{S_1}$ and $\| \cdot \|_{S_2}$ on the tangent spaces TS_1 and TS_2, given by the Poincaré metrics. Now we consider $\|f'_\omega(z)\|_{A_{j_1}, A_{j_2}}$. Let B_{j_2} denote the connected component of $f_\omega^{-1}(A_{j_2})$ which contains z. By the previous step, we have $B_{j_2} \subset A_{j_1} \cap K_2$. For each $j \in \{1, \ldots, \ell\}$ let D_j be an open connected subset of A_j such that $\overline{D_j}$ is compact in A_j and $A_j \cap K_2 \subset D_j$. For domains $\Omega_1 \subset \Omega_2 \subset \widehat{\mathbb{C}}$, let $\iota_{\Omega_1, \Omega_2} : \Omega_1 \to \Omega_2$ denote the inclusion map. Note that, by Pick's Theorem, we have that, for each $j_1, j_2 \in \{1, \ldots, r\}$, there exists a constant $c_{j_1, j_2} := \sup_{z \in A_{j_1} \cap K_2} \|\iota'_{D_{j_1}, A_{j_1}}(z)\|_{D_{j_1}, A_{j_1}} < 1$. Consequently, for each $z \in B_{j_2}$, we have that

$$\|\iota'_{B_{j_2}, A_{j_1}}(z)\|_{B_{j_2}, A_{j_1}} = \|(\iota_{D_{j_1}, A_{j_1}} \circ \iota_{B_{j_2}, D_{j_1}},)'(z)\|_{B_{j_2}, A_{j_1}}$$
$$\leq \|\iota'_{D_{j_1}, A_{j_1}}(z)\|_{D_{j_1}, A_{j_1}} \cdot \|\iota'_{B_{j_2}, D_{j_1}}(z)\|_{B_{j_2}, D_{j_1}} \leq c_{j_1, j_2} \cdot 1.$$

Finally, since $f_\omega : B_{j_2} \to A_{j_2}$ is a covering map, it follows by Pick's Theorem that f_ω is locally a Poincaré isometry, which implies that for $v \in T_z B_{j_2}$ with $v \neq 0$ that $\|Df_\omega(z)(v)\|_{A_{j_2}} / \|v\|_{B_{j_2}} = 1$. Hence, we obtain that for each $v \in T_z A_{j_1}$ with $v \neq 0$,

$$\|f'_\omega(z)\|_{A_{j_1}, A_{j_2}} = \frac{\|Df_\omega(z)(v)\|_{A_{j_2}}}{\|v\|_{A_{j_1}}} = \frac{\|Df_\omega(z)(v)\|_{A_{j_2}}}{\|v\|_{B_{j_2}}} \cdot \frac{\|v\|_{B_{j_2}}}{\|v\|_{A_{j_1}}} = \|\iota'_{B_{j_2}, A_{j_1}}(z)\|_{B_{j_2}, A_{j_1}}^{-1} \geq c_{j_1, j_2}^{-1} > 1.$$

Using the same argument, one verifies that for each $z \in J_{\mathrm{pre}}(G)$, $z \in J_\tau$, $\tau \in I^{\mathbb{N}}$, $\ell \in \mathbb{N}$ with $\ell < r$, and $\rho := (\tau_1, \ldots, \tau_\ell)$, if $z \in A_{j_1}, f_\rho(z) \in A_{j_2}$ then we have $\|f'_\rho(z)\|_{A_{j_1}, A_{j_2}} \geq 1$.

To finish the proof, we remark that the Poincaré metric and the spherical metric are equivalent on the compact subset $J(G) \cap A_i$ of A_i, which proves that there exist a constant $\lambda > 1$ and a constant $C > 0$ such that, for each $(\tau, z) \in J^\tau$, $\tau \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have $\|(\tilde{f}^n)'(\tau, z)\| \geq C\lambda^n$. Finally, continuity of $(\tilde{f}^n)'$ completes the proof of Proposition 4.2. $\qquad \square$

In the following lemma we give a sufficient condition for an infinitely generated rational semigroup to be nicely expanding in terms of the open set condition.

Lemma 4.8. *Let I be a countable set endowed with the discrete topology. Let $G = \langle f_i : i \in I \rangle$ denote a hyperbolic rational semigroup. Suppose that each element in $G \cap \mathrm{Aut}\left(\widehat{\mathbb{C}}\right)$ is loxodromic, there exists a non-empty compact, G-forward invariant subset $P_0(G) \subset F(G)$ and that $\{f_i : i \in I\}$ satisfies the open set condition. Then G is nicely expanding.*

Proof. By Lemma 4.6 we may assume that $I = \mathbb{N}$. By Lemma 4.5 (3), we may assume that $\mathrm{card}(J(G)) \geq 3$. Since G is hyperbolic, we may assume that $P(G) \subset P_0(G)$. Let us first show that we can assume without loss of generality that $\{f_i : i \in I\}$ satisfies the open set condition with respect to an open set V such that $\overline{V} \subset \mathbb{C}$ compact. To prove this, first note that there exists a neighborhood W of $P_0(G)$ in $F(G)$ which is G-forward invariant. By conjugating G with an element of $\mathrm{Aut}\left(\widehat{\mathbb{C}}\right)$ we may also assume that W contains infinity. Now, for any $g \in G$, since $g\left(\overline{W}\right) \subset \overline{W}$ we have that $g^{-1}\left(\widehat{\mathbb{C}} \setminus \overline{W}\right) \subset \widehat{\mathbb{C}} \setminus \overline{W}$. Finally, if G satisfies the open set condition with respect to some open set $V' \subset \widehat{\mathbb{C}}$, then G satisfies the open set condition with respect to $V := V' \cap \left(\widehat{\mathbb{C}} \setminus \overline{W}\right)$ and $\overline{V} \subset \mathbb{C}$.

Our next aim is to verify that

$$(4.2) \qquad \lim_{n \to \infty} \inf_{z \in f_n^{-1}(J(G))} \left\| f_n'(z) \right\| = \infty.$$

Since $\mathrm{card}(J(G)) \geq 3$, the Julia set $J(G)$ is the smallest non-empty compact G-backward invariant subset of $\widehat{\mathbb{C}}$ ([HM96, Sum00]). Hence, $J(G) \subset \overline{V}$. Since $P_0(G) \subset F(G)$, there exists $r_1 > 0$ such that $\overline{B(x, r_1)} \subset \widehat{\mathbb{C}} \setminus P_0(G)$, for all $x \in J(G)$. Using that $J(G)$ is compact we deduce the existence of $r_2 > 0$ with the property that, for each $x \in J(G)$, there exists $y_x \in V$ such that $B(y_x, r_2) \subset B(x, r_1) \cap V$. By the open set condition and Koebe's distortion theorem, it follows that, for each $\varepsilon > 0$, there exists n_0 such that $\|\gamma'(x)\| \leq \varepsilon$, for all $n \geq n_0$, $x \in J(G)$ and for all inverse branches γ of f_n on $B(x, r_1)$. The proof of (4.2) is complete.

We will now verify that $\mathrm{id} \notin \overline{G \cap \mathrm{Aut}\left(\widehat{\mathbb{C}}\right)}$, from which the lemma follows by Proposition 4.2. For the proof, it suffices to show that $H := \left\{ h^{-1} : h \in \mathrm{Aut}\left(\widehat{\mathbb{C}}\right) \cap G \right\}$ is closed in Rat. Let $h \in \overline{H}$ and $(h_n) \in H^{\mathbb{N}}$ with $h_n \rightrightarrows h$ on $\widehat{\mathbb{C}}$ be given, where

$$h_n = f_{\omega_1^n}^{-1} \circ \cdots \circ f_{\omega_{\ell_n}^n}^{-1}, \quad \omega^n = (\omega_1^n, \dots, \omega_{\ell_n}^n) \in I^{\ell_n}, \ell_n \in \mathbb{N}.$$

In order to show that $h \in H$, we will verify that $\sup \ell_n < \infty$ and that there exists a finite set $F \subset I$, such that $\omega^n \in F^{\ell_n}$, for all $n \in \mathbb{N}$. To prove this, we will show that each of the following assumptions (1) and (2) gives a contradiction:

(1) $\sup \ell_n = \infty$ and there exists a finite set $F \subset I$ such that $\omega^n \in F^{\ell_n}$, for all $n \in \mathbb{N}$.
(2) There exists a sequence $(j_n) \in \mathbb{N}^{\mathbb{N}}$ with $j_n \leq \ell_n$ such that $\lim_n \omega_{j_n}^n = \infty$.

Suppose for a contradiction that (1) holds. Set $G_F := \langle f_i : i \in F \rangle$. We may assume that $\mathrm{card}(J(G_F)) \geq 3$. Since G_F is a finitely generated hyperbolic rational semigroup, we have that G_F is expanding with respect to $\{f_i : i \in F\}$ by Lemma 4.6. Since $\sup l_n = \infty$, we have $h' = 0$ on $J(G_F)$. Consequently, since $J(G_F)$ is perfect ([HM96, Sum00]), the identity theorem gives that h is a constant function, which contradicts the continuity of the degree function.

To derive a contradiction from (2), we assume that $\lim_n \omega_{j_n}^n = \infty$. By (4.2) we conclude that

$$(4.3) \qquad \lim_{n \to \infty} \sup_{z \in J(G)} \left\| \left(f_{\omega_{j_n}^n}^{-1} \right)'(z) \right\| = 0.$$

To deduce a contradiction, let W denote a G-forward invariant relatively compact open neighborhood of $P_0(G)$ in $F(G)$. Then we have $g^{-1}\left(\widehat{\mathbb{C}} \setminus \overline{W}\right) \subset \widehat{\mathbb{C}} \setminus \overline{W}$, for each $g \in G$, which implies that H is normal in the neighborhood $\widehat{\mathbb{C}} \setminus \overline{W}$ of $J(G)$. After passing to a subsequence, combining (4.3) with the identity theorem

gives that $f_{\omega|_{j_n}^n}^{-1} \rightrightarrows c_A$ on A, for each connected component A of $\widehat{\mathbb{C}} \setminus \overline{W}$ and some $c_A \in J(G)$. Writing $h_n = r_n f_{\omega|_{j_n}^n}^{-1} s_n$ with $r_n, s_n \in H \cup \{\mathrm{id}\}$, for each $n \in \mathbb{N}$, we may assume that $r_n \rightrightarrows r$ and $s_n \rightrightarrows s$ in the neighborhood $\widehat{\mathbb{C}} \setminus \overline{W}$ of $J(G)$. Consequently, h is a constant function, which gives the desired contradiction. $\qquad\square$

Regarding the dynamics on the Fatou set of a nicely expanding rational semigroup, we prove the following lemma, which will be useful in Section 7.

Lemma 4.9. *Let $G = \langle f_i : i \in I \rangle$ be a nicely expanding rational semigroup with G-forward invariant set $P_0(G)$. Suppose that $\mathrm{card}(J(G)) > 1$. Then, for each $\omega \in I^{\mathbb{N}}$ and $x \in F_\omega$, we have $\lim_n d(f_{\omega|_n}(x), P_0(G)) = 0$ and each limit function of $(f_{\omega|_n})_{n \in \mathbb{N}}$ in a connected neighborhood of x in F_ω is a constant function whose value is in $P_0(G)$.*

Proof. By Lemmas 4.3 and 4.5 we have $\mathrm{card}(J(G)) \geq 3$. Let $V_1, \dots, V_r$, $r \in \mathbb{N}$, denote the connected components of $F(G)$ which meet $P_0(G)$. Let $\omega \in I^{\mathbb{N}}$ and $x \in F_\omega$. Then the family $(f_{\omega|_n})_n$ is normal in a neighborhood of x. Suppose for a contradiction that there exists a subsequence (g_j) of $(f_{\omega|_n})$ which converges to a non-constant map h in a neighborhood of x. Since h is non-constant, it follows from Claim 4.7 in the proof of Proposition 4.2 that $g_j(x) \in \widehat{\mathbb{C}} \setminus \cup_{s=1}^r V_s$, for each j. By the method employed in the final step of the proof of Proposition 4.2, we can show that $\|g_j'(x)\| \to \infty$, as $j \to \infty$, which is a contradiction. Thus, each limit function of $(f_{\omega|_n})$ in a connected neighborhood of x in F_ω is constant. Now suppose that a subsequence (g_j) of $(f_{\omega|_n})$ converges to a constant c in a neighborhood of x. We will show that $c \in P_0(G)$. Otherwise, there exists $\delta > 0$ such that $\overline{B(c, \delta)} \cap P_0(G) = \varnothing$ and, for each large j, there exists a well defined inverse branch $h_j : B(c, \delta) \to \widehat{\mathbb{C}}$ of g_j such that $h_j(g_j(x)) = x$ and $g_j \circ h_j = \mathrm{id}$ on $B(c, \delta)$. Since there exists a G-forward invariant neighborhood W of $P_0(G)$ with $W \cap B(c, \delta) = \varnothing$, we conclude that (h_j) is normal in a neighborhood Ω of x. Hence (h_j) is equicontinuous in Ω. Since $g_j(x) \to c$ as $j \to \infty$, there exist $\delta_0 \in (0, \delta)$ and a relative compact subset Ω_0 of Ω such that for each $j \in \mathbb{N}$, $h_j(B(c, \delta_0)) \subset \Omega_0$. Since $g_j \rightrightarrows c$ on Ω_0 as $j \to \infty$, we have $g_j \circ h_j \to c$ on $B(c, \delta_0)$ as $j \to \infty$. However, this contradicts $g_j \circ h_j = \mathrm{id}$ on $B(c, \delta)$ for each j. Therefore, $\lim_j d(g_j(x), P_0(G)) = 0$ and hence, $\lim_n d(f_{\omega|_n}(x), P_0(G)) = 0$. $\qquad\square$

Finally, we prove some useful facts about the exceptional sets of expanding rational semigroups.

Lemma 4.10. *Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. Suppose that $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$ and $G \subset \mathrm{Aut}(\widehat{\mathbb{C}})$. Let $G_0 \subset G$ be a subsemigroup such that $\mathrm{card}(J(G_0)) \geq 3$. Then we have $E(G_0) \subset F(G)$.*

Proof. Suppose for a contradiction that there exists $z_0 \in E(G_0) \cap J(G)$. Since $\mathrm{card}(J(G_0)) \geq 3$, it follows from the density of the repelling fixed points in the Julia set and the perfectness of the Julia set ([HM96, Theorem 3.1, Lemma 3.1], [Sum00, Lemma 2.3]) that there exist $z_1 \in J(G_0)$ and $g_1 \in G_0$, such that $z_1 \neq z_0$, $g_1(z_1) = z_1$ and $\|g_1'(z_1)\| > 1$. Furthermore, we have $\mathrm{card}(E(G_0)) \leq 2$ ([HM96, Lemma 3.3], [Sum00, Lemma 2.3]) . Combining with the fact that $g^{-1}(E(G_0)) \subset E(G_0)$ for each $g \in G_0$, we conclude that $g_1^2(z_0) = z_0$. Since G is expanding, we have that g_1 is loxodromic by Lemma 4.3. Thus, z_0 is the attracting fixed point of g_1^2. Let V be a neighborhood of z_0 and let $0 < c < 1$ such that $g_1^2(V) \subset V$ and $\|(g_1^2)'(z)\| < c$, for each $z \in V$. Since the Julia set is perfect and by the density of the repelling fixed points in the Julia set ([HM96, Sum00]), there exists a sequence (a_n) with $a_n \in J_{\mathrm{pre}}(G) \setminus \{z_0\}$ such that $\lim_n a_n = z_0$. Then there exists a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, such that $g_1^{-2n_k}(a_{n_k}) \in V$. Hence, $\lim_k \|(g_1^{2n_k})'(g_1^{-2n_k}(a_{n_k}))\| \leq \lim_k c^{n_k} = 0$. Moreover, write $g_1^2 = f_\alpha$, for some $m \in \mathbb{N}$ and $\alpha \in I^m$, and denote by $\alpha^n := (\alpha \dots \alpha) \in I^{mn}$ the n-fold concatenation of α. Let $(\beta_k) \in I^{\mathbb{N}}$ with $(\beta_k, a_{n_k}) \in J(\tilde{f})$. Then $(\alpha^{n_k} \beta_k, (g_1^{-2n_k})(a_{n_k})) \in J(\tilde{f})$. This contradicts that G is expanding and finishes the proof. $\qquad\square$

Lemma 4.11. *Let I be a topological space and let $(f_i)_{i\in I} \in C(I, \mathrm{Rat})$. Suppose that $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$ and $1 \le \mathrm{card}\,(J(G)) \le 2$. Then we have $\mathrm{card}\,(J(G)) = 1$.*

Proof. Clearly, we have $G \subset \mathrm{Aut}\left(\widehat{\mathbb{C}}\right)$ and each element of G is loxodromic by Lemma 4.3. Now, suppose by way of contradiction that $J(G) = \{a, b\}$. Without loss of generality, we may assume that $a = 0$ and $b = \infty$. Since $J(G)$ is G-backward invariant, we have $g(a) = a$ and $g(b) = b$. Thus, there exists a sequence $(c_i) \in \mathbb{C}^I$ such that $f_i(z) = c_i z$, for each $z \in \widehat{\mathbb{C}}$ and $i \in I$. We may assume that there exists $i_0 \in I$ such that $\|f'_{i_0}(a)\| > 1$. Since G is expanding with respect to $\{f_i : i \in I\}$, there exists a constant $c_0 > 1$ such that $\|f'_i(a)\| = |c_i| \ge c_0 > 1$, for all $i \in I$ and $z \in \mathbb{C}$. Hence, we have $\|f'_i(b)\| \le c_0^{-1} < 1$, for all $i \in I$, which gives that $b \in F(G)$. This contradiction proves the lemma. $\qquad\square$

Lemma 4.12. *Let I be a topological space and let $(f_i)_{i\in I} \in C(I, \mathrm{Rat})$. Suppose that $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$, $\mathrm{card}\,(J(G)) > 1$ and $G \subset \mathrm{Aut}(\widehat{\mathbb{C}})$. Then we have the following.*

(1) $E(G) \subset F(G)$.
(2) *If $E(G) \neq \varnothing$, then we have $g(x) = x$ and $\|g'(x)\| < 1$, for all $g \in G$ and $x \in E(G)$.*

Proof. By Lemma 4.11 we have $\mathrm{card}\,(J(G)) \ge 3$. Hence, the assertion in (1) follows from Lemma 4.10. Further, $\mathrm{card}\,(J(G)) \ge 3$ implies that $\mathrm{card}\,(E(G)) \le 2$ ([HM96, Sum00]). Hence, $g(E(G)) = E(G)$, for each $g \in G$. Since G is expanding, we have that each element $g \in G$ is loxodromic by Lemma 4.3. By the first assertion, we can now conclude that, if $E(G)$ is non-empty, then $\mathrm{card}\,(E(G)) = 1$. Moreover, for each $g \in G$, the element in $E(G)$ is the attracting fixed point of g, which proves (2). $\qquad\square$

5. TOPOLOGICAL PRESSURE

In this section we derive basic properties of two notions of topological pressure associated with the dynamics of rational semigroups. We start with a preparatory lemma. The property derived in this lemma is similar to the finitely primitive condition ([MU03]) for topological Markov chains with an infinite alphabet.

Lemma 5.1. *Let I be a topological space and let $(f_i)_{i\in I} \in C(I, \mathrm{Rat})$. Suppose that either (1) $G = \langle f_i : i \in I \rangle$ is a hyperbolic rational semigroup which contains an element of degree at least two or (2) G is nicely expanding. Then for each finite family $(U_i)_{i \in \{1,\ldots,s\}}$ of non-empty open subsets of $J(G)$, there exists $\ell_0 \in \mathbb{N}$ and a finite set $I_0 \subset I$ such that for each $z \in J(G)$, for each $\ell \in \mathbb{N}$ with $\ell \ge \ell_0$ and for each $i \in \{1,\ldots,s\}$ there is $\omega \in I_0^\ell$ such that $f_\omega^{-1}(z) \cap U_i \neq \varnothing$.*

Proof. Let us first suppose that G satisfies the assumptions in (1). By the density of the repelling fixed points in the Julia set ([HM96, Theorem 3.1]), we have that for each $i \in \{1,\ldots,s\}$ there exists $g \in G$ such that $J(g) \cap U_i \neq \varnothing$. Hence, there exists a finitely generated subsemigroup $G_0 := \langle f_i : i \in I_0 \rangle$, including an element of degree at least two, such that $J(G_0) \cap U_i \neq \varnothing$, for all $i \in \{1,\ldots,s\}$. Since G_0 contains an element of degree at least two, we have $E(G_0) \subset P(G)$. Combining with our assumption that G is hyperbolic, we obtain that $E(G_0) \subset F(G)$. In particular, we have $J(G) = \widehat{\mathbb{C}} \setminus F(G) \subset \widehat{\mathbb{C}} \setminus E(G_0)$.

Our aim is to apply [Sum00, Theorem 4.3] to the finitely generated semigroup G_0. We have seen that $E(G_0) \subset F(G) \subset F(G_0)$. We now verify that $J(G_0) \subset F\left(\{h^{-1} : h \in \mathrm{Aut}(\widehat{\mathbb{C}}) \cap G_0\}\right)$. To prove this, let $V_1, \ldots, V_r$, $r \in \mathbb{N}$, denote the finitely many connected components of $F(G)$ which have non-empty intersection with the compact set $P(G)$. Since G contains an element of degree at least two, we have that each V_i is hyperbolic and we denote by d_h the Poincaré metric on V_i. Set $W_i := \{z \in V_i : d_h(z, P(G) \cap V_i) < 1\}$ and note that $A := \bigcup_{i=1}^r W_i$ is G-forward invariant. Hence, we have $g^{-1}(\widehat{\mathbb{C}} \setminus \overline{A}) \subset \widehat{\mathbb{C}} \setminus \overline{A}$, for each $g \in G$,

which implies that $\widehat{\mathbb{C}} \setminus \overline{A} \subset F\big(\{h^{-1} : h \in \mathrm{Aut}(\widehat{\mathbb{C}}) \cap G_0\}\big)$ by Montel's Theorem. We have thus shown that $J(G_0) \subset J(G) \subset \widehat{\mathbb{C}} \setminus \overline{A} \subset F\big(\{h^{-1} : h \in \mathrm{Aut}(\widehat{\mathbb{C}}) \cap G_0\}\big)$.

Set $K := J(G)$ and observe that K is a compact subset of $\widehat{\mathbb{C}} \setminus E(G_0)$, which is G_0-backward invariant. Let μ denote the Borel probability measure on $J(G_0)$ corresponding to the equidistribution on the generators $\{f_i : i \in I_0\}$ of G_0, which exists by [Sum00, Theorem 4.3] and whose topological support $\mathrm{supp}\,\mu$ is equal to $J(G_0)$. By [Sum00, Theorem 4.3] there exists $\ell_0 \in \mathbb{N}$ such that, for each $z \in J(G)$, for all $\ell \geq \ell_0$ and for each $i \in \{1, \ldots, s\}$, there exists $\omega \in I_0^\ell$ such that $f_\omega^{-1}(z) \cap U_i \neq \varnothing$. The proof of the first case is complete.

Now suppose that G is nicely expanding and $G \subset \mathrm{Aut}(\widehat{\mathbb{C}})$. Since the theorem is obviously true in the case that $\mathrm{card}(J(G)) = 1$, we may assume that $\mathrm{card}(J(G)) \geq 3$ by Lemma 4.11. Choose a finitely generated subsemigroup G_0 of G such that $\mathrm{card}(J(G_0)) \geq 3$. Hence, we have $J(G) \subset \widehat{\mathbb{C}} \setminus E(G_0)$ by Lemma 4.10. By substituting $P(G)$ by $P_0(G)$ in the proof of the first case, we verify that $J(G_0) \subset F\big(\{h^{-1} : h \in \mathrm{Aut}(\widehat{\mathbb{C}}) \cap G_0\}\big)$. Set $\tilde{K} := \pi_{\widehat{\mathbb{C}}}^{-1}(J(G))$. Since $\tilde{K} \subset \pi_{\widehat{\mathbb{C}}}^{-1}\big(\widehat{\mathbb{C}} \setminus E(G_0)\big)$, it follows from [Sum00, Proof of Lemma 4.6] that each unitary eigenvector of the bounded linear operator $\tilde{B} : C(\tilde{K}) \to C(\tilde{K})$, given by $\tilde{B}(\tilde{\varphi})(\tau, z) := \mathrm{card}(I_0)^{-1} \sum_{\omega \in I_0} \tilde{\varphi}\big(\omega\tau, f_\omega^{-1}(z)\big)$, for each $\tilde{\varphi} \in C(\tilde{K})$ and $(\tau, z) \in \tilde{K}$, is a constant function on $\tilde{K}$. Finally, since $J(G_0) \subset F\big(\{h^{-1} : h \in \mathrm{Aut}(\widehat{\mathbb{C}}) \cap G_0\}\big)$, we have that $\big(\tilde{B}^n(\tilde{\varphi})\big)_{n \in \mathbb{N}}$ is a family of equicontinuous functions on $\tilde{K}$. Hence, we have that $\tilde{B}$ is an almost periodic operator by the Arzelà-Ascoli Theorem and the result of [Sum00, Theorem 4.3] follows from the proof of [Sum00, Theorem 4.3]. The rest of the proof runs as in case (1). □

5.1. Bounded distortion property and topological pressure.

Throughout this subsection, let I be a countable set and let $(f_i)_{i \in I} \in \mathrm{Rat}^I$. For each $n \in \mathbb{N}$, $x \in \widehat{\mathbb{C}}$ and $t \in \mathbb{R}$, we define

$$Z_n(I, t, x) := \sum_{\omega \in I^n} \sum_{y \in f_\omega^{-1}(x)} \big\|f_\omega'(y)\big\|^{-t}, \quad P(I, t, x) := \limsup_{n \to \infty} \frac{1}{n} \log Z_n(I, t, x).$$

Here, the sum $\sum_{y \in f_\omega^{-1}(x)}$ counts the multiplicities, and we set $0^{-t} = \infty$ if $t \geq 0$, and we set $0^{-t} = 0$ if $t < 0$. Using Lemma 5.1 the proof of the following lemma is a straightforward application of Koebe's distortion theorem.

Lemma 5.2 (Bounded distortion lemma). *Let I be a countable set and let $(f_i)_{i \in I} \in \mathrm{Rat}^I$. Suppose that either (1) $G = \langle f_i : i \in I \rangle$ is a hyperbolic rational semigroup which contains an element of degree at least two or (2) $G = \langle f_i : i \in I \rangle$ is nicely expanding. Then, for each $t \in \mathbb{R}$, there exist $C > 1$, $\ell \in \mathbb{N}$ and a finite set $I_0 \subset I$, such that for all $I_1 \subset I$ with $I_0 \subset I_1$, for all $n \in \mathbb{N}$ and for all $x, y \in J(G)$ we have*

$$(5.1) \qquad\qquad Z_{n+\ell}(I_1, t, y) \geq C^{-1} Z_n(I_1, t, x).$$

Furthermore, for each $x \in J(G)$ we have that $P(I_1, t, x) < \infty$ if and only if $\sup_{y \in J(G)} Z_1(I_1, t, y) < \infty$. In particular, if $P(I, t, x_0) < \infty$ for some $x_0 \in J(G)$, then there exists $C' > 1$ such that, for all $n \in \mathbb{N}$ and for all $x, y \in J(G)$, we have

$$(5.2) \qquad\qquad (C')^{-1} Z_n(I_1, t, x) \leq Z_n(I_1, t, y) \leq C' Z_n(I_1, t, x) < \infty.$$

Moreover, if $P(I_1, t, x_0) = \infty$ for some $x_0 \in J(G)$, then for all sufficiently large $n \in \mathbb{N}$ and for all $x \in J(G)$ we have $Z_n(I_1, t, x) = \infty$.

Proof. We will verify the lemma under the assumptions given in (1). From this, one can deduce that the lemma holds under the assumptions in (2) by replacing $P(G)$ by $P_0(G)$. Our first aim is to define the finite set $I_0 \subset I$. Since G is hyperbolic, we have $d(J(G), P(G)) > 0$, which allows us to fix some $0 < r < d(J(G), P(G))/2$. Since $J(G)$ is compact, there exist $s \in \mathbb{N}$ and $x_1, \ldots, x_s \in J(G)$, such that $J(G) \subset \bigcup_{j=1}^s B(x_j, r)$. By Lemma 5.1 applied to the open sets $U_i := B(x_i, r)$, for $i \in \{1, \ldots, s\}$, we obtain

that there exist $\ell \in \mathbb{N}$ and a finite set $I_0 \subset I$ such that, for all $j,k \in \{1,\dots,s\}$, there exist $\tau(j,k) \in I_0^\ell$ and $y_{j,k} \in \widehat{\mathbb{C}}$ with the property that $y_{j,k} \in f_{\tau(j,k)}^{-1}(x_k) \cap B(x_j,r)$.

In the following, let I_1 denote an arbitrary subset of I containing I_0. For each $n \in \mathbb{N}$ and for all $j,k \in \{1,\dots,s\}$ we then have that

$$(5.3) \quad Z_{n+\ell}(I_1,t,x_k) \;=\; \sum_{\omega \in I_1^\ell} \sum_{y \in f_\omega^{-1}(x_k)} \left\| f_\omega'(y) \right\|^{-t} Z_n(I_1,t,y) \geq \left\| f_{\tau(j,k)}'(y_{j,k}) \right\|^{-t} Z_n(I_1,t,y_{j,k}).$$

We will now combine the previous estimate with the following consequence of Koebe's distortion theorem. There exists a constant $C_1 = C_1(r)$ such that, for each $m \in \mathbb{N}$, $I' \subset I$ and for all $z,z' \in J(G)$ with $d(z,z') < r$,

$$(5.4) \qquad\qquad C_1^{-1} Z_m(I',t,z') \leq Z_m(I',t,z) \leq C_1 Z_m(I',t,z').$$

Combining (5.3), (5.4) and $d(y_{j,k},x_j) < r$, we obtain $Z_{n+\ell}(I_1,t,x_k) \geq \left\| f_{\tau(j,k)}'(y_{j,k}) \right\|^{-t} C_1^{-1} Z_n(I_1,t,x_j)$. Setting $S := \min_{j,k \in \{1,\dots,s\}} \left\| f_{\tau(j,k)}'(y_{j,k}) \right\|^{-t} > 0$ and combining $J(G) \subset \bigcup_{j=1}^s B(x_j,r)$ with (5.4), we deduce that $Z_{n+\ell}(I_1,t,y) \geq S C_1^{-3} Z_n(I_1,t,x)$, for all $n \in \mathbb{N}$ and for all $x,y \in J(G)$. We have thus shown that (5.1) holds with $C := S^{-1} C_1^3$.

We now turn to the proof of the second assertion of the lemma. Let $x \in J(G)$. First suppose that $P(I_1,t,x) < \infty$. Clearly, there exists $n \geq 2$ such that $Z_{n+\ell}(I_1,t,x) < \infty$. By (5.1) we have $Z_n(I_1,t,y) \leq C Z_{n+\ell}(I_1,t,x) < \infty$, for all $y \in J(G)$. Consequently, fixing one element $a \in I_0$, we have for all $y \in J(G)$,

$$Z_1(I_1,t,y) \Big(\min_{z \in f_a^{-1}(J(G))} \left\| f_a'(z) \right\|^{-t} \Big)^{n-1}$$
$$\leq \sum_{\omega \in I_1} \sum_{z \in f_\omega^{-1}(y)} \left\| f_\omega'(z) \right\|^{-t} Z_{n-1}(I_1,t,z) = Z_n(I_1,t,y) \leq C Z_{n+\ell}(I_1,t,x) < \infty,$$

which proves $\sup_{y \in J(G)} Z_1(I_1,t,y) < \infty$. On the other hand, if $\sup_{y \in J(G)} Z_1(I_1,t,y) < \infty$, then

$$P(I_1,t,x) = \limsup_{n \to \infty} \frac{1}{n} \log Z_n(I_1,t,x) \leq \log \sup_{y \in J(G)} Z_1(I_1,t,y) < \infty,$$

which finishes the proof of the second assertion.

In order to prove (5.2), suppose that $P(I_1,t,x_0) < \infty$, for some $x_0 \in J(G)$, and let $n \in \mathbb{N}$ and $x,y \in J(G)$. By (5.1) we have

$$Z_n(I_1,t,x) \leq C Z_{n+\ell}(I_1,t,y) = C \sum_{\omega \in I_1^n} \sum_{z \in f_\omega^{-1}(y)} \left\| f_\omega'(z) \right\|^{-t} Z_\ell(I_1,t,z) \leq C Z_n(I_1,t,y) \sup_{z \in J(G)} Z_\ell(I_1,t,z).$$

Now, by the second assertion of the lemma, we have $\sup_{z \in J(G)} Z_\ell(I_1,t,z) < \infty$. Hence, the estimates in (5.2) hold with $C' := C \sup_{z \in J(G)} Z_\ell(I_1,t,z)$.

Next, we prove the final assertion of the lemma. Suppose that $P(I_1,t,x_0) = \infty$, for some $x_0 \in J(G)$. By the second assertion of the lemma, we have $\sup_{y \in J(G)} Z_1(I_1,t,y) = \infty$. Let $x \in J(G)$. By (5.1) we have

$$Z_{1+\ell}(I_1,t,x) \geq C^{-1} \sup_{y \in J(G)} Z_1(I_1,t,y) = \infty.$$

Finally, using the estimate $Z_{1+\ell+1}(I_1,t,x) \geq Z_{1+\ell}(I_1,t,x) \min_{z \in f_a^{-1}(J(G))} \left\| f_a'(z) \right\|^{-t} = \infty$, one inductively verifies that $Z_n(I_1,t,x) = \infty$, for all $n \geq 1 + \ell$. The proof is complete. $\qquad\square$

The third assertion in the following proposition shows that an exhaustion principle holds for $P(I,t,x)$. Recall that we say that $\eta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is convex if its epigraph $\mathrm{epi}(\eta) := \left\{ (x,y) \in \mathbb{R}^2 : y \geq \eta(x) \right\}$ is a convex set, and we say that η is closed if $\mathrm{epi}(\eta)$ is closed subset of $\mathbb{R}^2$. Note that $\mathrm{epi}(\eta)$ is closed if and only if η is lower semicontinuous ([Roc70, Theorem 7.1]). The following properties are well-known

for countable topological Markov chains satisfying the finite primitivity condition ([MU03]). The proof of Proposition 5.3 (3) is inspired by ([MU03, Theorem 2.1.5]) .

Proposition 5.3. *Let $G = \langle f_i : i \in I \rangle$ be a rational semigroup. Suppose that either (1) G is a hyperbolic rational semigroup which contains an element of degree at least two or (2) G is nicely expanding. Then, for each $t \in \mathbb{R}$ and for each $x \in J(G)$, the following holds.*

(1) *If $P(I,t,x_0) < \infty$, for some $x_0 \in J(G)$, then $P(I,t,x) = \lim_{n \to \infty} n^{-1} \log Z_n(I,t,x)$.*

(2) *There exists a neighborhood V of $J(G)$, $V \subset \widehat{\mathbb{C}} \setminus P(G)$, such that $z \mapsto P(I,t,z)$ is constant on V.*

(3) $\sup_{F \subset I, \, \mathrm{card}(F) < \infty} P(F,t,x) = P(I,t,x)$

(4) *The map $s \mapsto P(I,s,x)$ is a closed convex function with values in $\mathbb{R} \cup \{\infty\}$.*

(5) *For all $I_1 \subset I$ with $I_0 \subset I_1$, where I_0 is the finite set in Lemma 5.2, we have*

$$P(I_1,t,x) = \inf \left\{ \beta \in \mathbb{R} : \sum_{n \in \mathbb{N}} Z_n(I_1,t,x) \, \mathrm{e}^{-\beta n} < \infty \right\}.$$

Proof. We give the proof under the assumption that G is hyperbolic and contains an element of degree at least two. By replacing $P(G)$ by $P_0(G)$, the proposition can be proved under the assumption that G is nicely expanding.

First note that, for all $m, n \in \mathbb{N}$, we have $Z_{n+m}(I,t,x) = \sum_{\omega \in I^n} \sum_{z \in f_\omega^{-1}(x)} \|f_\omega'(z)\|^{-t} Z_m(I,t,z)$. Since $P(I,t,x_0) < \infty$, Lemma 5.2 (5.2) implies that there exists a constant $C' > 1$ such that

$$\sum_{\omega \in I^n} \sum_{z \in f_\omega^{-1}(x)} \|f_\omega'(z)\|^{-t} Z_m(I,t,z) \leq C' \sum_{\omega \in I^n} \sum_{z \in f_\omega^{-1}(x)} \|f_\omega'(z)\|^{-t} Z_m(I,t,x) = C' Z_n(I,t,x) Z_m(I,t,x).$$

Hence, the sequence $(a_n) \in \mathbb{R}^{\mathbb{N}}$, given by $a_n := \log Z_n(I,t,x)$, $n \in \mathbb{N}$, is almost subadditive in the sense that $a_{n+m} \leq a_n + a_m + \log C'$, for all $n, m \in \mathbb{N}$. Hence, $\lim_n a_n/n$ exists and the proof of (1) is complete.

For the proof of (2), one first observes that $z \mapsto P(I,t,z)$ is constant on $J(G)$ by Lemma 5.2. Since G is hyperbolic, there exists $r > 0$ such that $B(y,r) \subset \widehat{\mathbb{C}} \setminus P(G)$, for each $y \in J(G)$. Set $V := \bigcup_{y \in J(G)} B(y,r)$. An application of Koebe's distortion theorem shows that there exists a constant $C_1 = C_1(r)$ such that $Z_n(I,t,y)C_1 \geq Z_n(I,t,z) \geq C_1^{-1} Z_n(I,t,y)$, for each $y \in J(G)$, $z \in B(y,r)$ and $n \in \mathbb{N}$. It follows that $z \mapsto P(I,t,z)$ is constant on V.

Let us turn to the proof of (3). Clearly, we have $P(F,t,x) \leq P(I,t,x)$, for each $F \subset I$. Hence, we have $\sup_{F \subset I} P(F,t,x) \leq P(I,t,x)$. For the opposite inequality, we consider two cases. First suppose that $P(I,t,x) < \infty$ for any $x \in J(G)$ and let $\varepsilon > 0$. There exists $n \in \mathbb{N}$, such that $n^{-1} \log Z_n(I,t,x) > P(I,t,x) - \varepsilon$ and $n^{-1} \log(C') < \varepsilon$, where $C' > 1$ is the constant from (5.2) in Lemma 5.2. Let I_0 denote the finite subset of I given by Lemma 5.2. Choose a finite set F with $I_0 \subset F \subset I$ such that $n^{-1} \log Z_n(F,t,x) > P(I,t,x) - 2\varepsilon$. Let $k \in \mathbb{N}$. By (5.2) we have

$$\begin{aligned}
Z_{kn}(F,t,x) &= \sum_{\omega_k \in F^n, y_k \in f_{\omega_k}^{-1}(x)} \|f_{\omega_k}'(y_k)\|^{-t} \sum_{\omega_{k-1} \in F^n, y_{k-1} \in f_{\omega_{k-1}}^{-1}(y_k)} \|f_{\omega_{k-1}}'(y_{k-1})\|^{-t} \quad \cdots \\
&\cdots \sum_{\omega_1 \in F^n, y_1 \in f_{\omega_1}^{-1}(y_2)} \|f_{\omega_1}'(y_1)\|^{-t} \geq (C')^{-k} (Z_n(F,t,x))^k,
\end{aligned}$$

which gives

$$\frac{1}{kn} \log Z_{kn}(F,t,x) \geq -\frac{1}{n} \log C' + \frac{1}{n} \log Z_n(F,t,x) \geq -\varepsilon + P(I,t,x) - 2\varepsilon = P(I,t,x) - 3\varepsilon.$$

We get $P(F,t,x) \geq P(I,t,x) - 3\varepsilon$, because the previous estimate holds for every $k \in \mathbb{N}$. Since ε was chosen to be arbitrary, the proof is complete in the case that $P(I,t,x) < \infty$ for any $x \in J(G)$. Now, we consider

the remaining case $P(I,t,x) = \infty$ for any $x \in J(G)$. Let $N \in \mathbb{N}$. Clearly, there exists $n \in \mathbb{N}$ such that $(n+\ell)^{-1} \log Z_n (I,t,x) > N - \varepsilon$ and such that $(n+\ell)^{-1} \log (C) < \varepsilon$, where $C > 1$ and $\ell \in \mathbb{N}$ are given by (5.1) in Lemma 5.2. Choose a finite set F with $I_0 \subset F \subset I$ such that $(n+\ell)^{-1} \log Z_n (F,t,x) > N - 2\varepsilon$. Let $k \in \mathbb{N}$. By (5.1), we have

$$Z_{k(n+\ell)} (F,t,x) = \sum_{\omega_k \in F^{n+\ell}, y_k \in f_{\omega_k}^{-1}(x)} \left\| f'_{\omega_k} (y_k) \right\|^{-t} \sum_{\omega_{k-1} \in F^{n+\ell}, y_{k-1} \in f_{\omega_{k-1}}^{-1}(y_k)} \left\| f'_{\omega_{k-1}} (y_{k-1}) \right\|^{-t} \cdots$$

$$\cdots \sum_{\omega_1 \in F^{n+\ell}, y_1 \in f_{\omega_1}^{-1}(y_2)} \left\| f'_{\omega_1} (y_1) \right\|^{-t} \geq C^{-k} \left(Z_n (F,t,x) \right)^k,$$

which gives

$$\frac{1}{k(n+\ell)} \log Z_{k(n+\ell)} (F,t,x) \geq -\frac{1}{n+\ell} \log C + \frac{1}{n+\ell} \log Z_n (F,t,x) \geq N - 3\varepsilon.$$

We obtain $P(F,s,x) > N - 3\varepsilon$, and letting N tend to infinity, finishes the proof of (3).

To prove (4), let $x \in J(G)$. If card (I) is finite, then it is standard to deduce (4) from Hölder's inequality. Now, suppose that $I = \mathbb{N}$. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of I such that $F_n \subset F_{n+1}$, for each $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} F_n = I$. For each $n \in \mathbb{N}$, we define $g_n : \mathbb{R} \to \mathbb{R}$, given by $g_n (s) := P(F_n,s,x)$, and $g : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ given by $g(s) := P(I,s,x)$. For each $n \in \mathbb{N}$, g_n is a real-valued convex function. In particular, epi (g_n) is a closed convex subset of $\mathbb{R}^2$. By Proposition 5.3 (3) we have that $g_n(s) \leq g_{n+1}(s)$, for each $s \in \mathbb{R}$ and $n \in \mathbb{N}$, which implies epi $(g) = \bigcap_{n \in \mathbb{N}} \text{epi} (g_n)$. Hence, epi (g) is closed and convex. Since g_n is real-valued and $g_n \leq g$, we have $g(s) > -\infty$, for each $s \in \mathbb{R}$.

Finally, to prove (5), a straightforward modification of the proof of [Jae11, Theorem 3.16] (see also [JKL14]) shows that

$$P(I_1,t,x) = \inf \left\{ \beta \in \mathbb{R} : \limsup_{T \to \infty} \sum_{n \in \mathbb{N}, n > T} Z_n (I_1,t,x) e^{-\beta n} < \infty \right\}.$$

Further, we clearly have that

(5.5) $$\inf \left\{ \beta \in \mathbb{R} : \limsup_{T \to \infty} \sum_{n \in \mathbb{N}, n > T} Z_n (I_1,t,x) e^{-\beta n} < \infty \right\} \leq \inf \left\{ \beta \in \mathbb{R} : \sum_{n \in \mathbb{N}} Z_n (I_1,t,x) e^{-\beta n} < \infty \right\}.$$

Hence, in the case that $P(I_1,t,x) = \infty$, the assertion in (5) follows. If $P(I_1,t,x) < \infty$, then (5.2) gives $Z_n (I_1,t,x) < \infty$, for all $n \in \mathbb{N}$. Hence, equality in (5.5) holds and the assertion in (5) follows. $\square$

5.2. The Gurevič pressure.

Throughout this subsection, let $I \subset \mathbb{N}$ be the finite set $\{1,\ldots,n\}$, for some $n \in \mathbb{N}$, or let $I = \mathbb{N}$, endowed with the discrete topology. Let $(f_i)_{i \in I} \in \text{Rat}^I$ and let $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ be the associated skew product. Suppose that $G = \langle f_i : i \in I \rangle$ is expanding with respect to $\{f_i : i \in I\}$. For each $n \in \mathbb{N}$, we set

$$\mathscr{P}_n (t) := \mathscr{P} \left(t \tilde{\varphi}_{|J\{1,\ldots,n\} \cap I}, \tilde{f}_{|J\{1,\ldots,n\} \cap I} \right).$$

Lemma 5.4. *Suppose that $G = \langle f_i : i \in I \rangle$ is a nicely expanding rational semigroup. Then, for each $t \in \mathbb{R}$ and for each $x \in J(G)$, we have $\mathscr{P}(t) = \lim_{n \to \infty} \mathscr{P}_n (t) = P(I,t,x)$.*

Proof. We may assume that $I = \mathbb{N}$. Let $t \in \mathbb{R}$. Our first aim is to prove that

(5.6) $$\mathscr{P}(t) = \sup_{F \subset I, \text{card}(F) < \infty} \mathscr{P} \left(t \tilde{\varphi}_{|J^F}, \tilde{f}_{|J^F} \right).$$

To prove (5.6), it suffices to verify that, if $K \subset J(\tilde{f})$ is compact and $\tilde{f}(K) = K$, then there exists $F \subset I$ finite such that $K \subset J^F$ and J^F is compact. Let $p_k : I^{\mathbb{N}} \to I$, $p_k(\omega) := \omega_k$, denote the projection on the kth symbol for each $k \in \mathbb{N}$. As the set $p_1(\pi_1(K)) \subset I$ is compact, we have that $F := p_1(\pi_1(K))$ is finite. Using that $\tilde{f}(K) = K$, we conclude that $p_k(\pi_1(K)) = p_1(\pi_1(\tilde{f}^{k-1}(K))) = p_1(\pi_1(K)) = F$, for each $k \in \mathbb{N}$.

Hence, $\pi_1(K) \subset F^{\mathbb{N}}$. Since G is expanding, we have $J(\tilde{f}) = \bigcup_{\omega \in I^{\mathbb{N}}} J^{\omega}$ by Lemma 4.1. Combining this with $\pi_1(K) \subset F^{\mathbb{N}}$, we see that $K \subset J^F$. Since J^I is closed in $I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ by Lemma 4.1, we obtain that $J^F = J^I \cap (F^{\mathbb{N}} \times \widehat{\mathbb{C}})$ is compact, which completes the proof of (5.6). By (5.6) we have $\mathscr{P}(t) = \lim_{n \to \infty} \mathscr{P}_n(t)$.

To prove $\lim_n \mathscr{P}_n(t) = P(I, t, x)$, for $x \in J(G)$, it suffices to consider one point $x_0 \in J(G)$. If F is finite and $x \in J_F$, then it follows from [Sum05, Lemma 3.6 (2) and (4)] that

$$(5.7) \qquad \mathscr{P}\left(t\tilde{\varphi}_{|J^F}, \tilde{f}_{|J^F}\right) = P(F, t, x).$$

Now fix some $x_0 \in J(f_1) \subset \bigcap_{k \in \mathbb{N}} J(\langle f_1, \dots, f_k \rangle)$. By (5.7), we have, for each $n \in \mathbb{N}$, $\mathscr{P}_n(t) = P(\{1, \dots, n\}, t, x_0)$. Letting n tend to infinity, we obtain $\mathscr{P}(t) = \lim_n P(\{1, \dots, n\}, t, x_0)$ by (5.6). By Proposition 5.3 (3) we have $\lim_{n \to \infty} P(\{1, \dots, n\}, t, x_0) = P(I, t, x_0)$, which completes the proof. $\qquad \square$

6. Bowen's Formula for Pre-Julia sets

In this section we prove Bowen's formula for pre-Julia sets, which is one of the main results of this paper.

Definition 6.1 ([Sum98]). Let G be a rational semigroup and let $\delta \geq 0$. A Borel probability measure μ on $\widehat{\mathbb{C}}$ is called δ-subconformal if, for each $g \in G$ and for each Borel set $A \subset \widehat{\mathbb{C}}$, we have

$$\mu(g(A)) \leq \int_A \|g'\|^{\delta} d\mu.$$

Also, we define the following critical exponent

$$u(G) := \inf\{v \geq 0 : \text{ there exists a } v\text{-subconformal measure for } G\}.$$

Lemma 6.2. *Let I be a countable set and let $G = \langle f_i : i \in I \rangle$ be a rational semigroup.*

(1) $\bullet$ *$s(G, x) \leq t(I, x)$, for each $x \in \widehat{\mathbb{C}}$.*
     $\bullet$ *$s(G) \leq t(I)$.*
     $\bullet$ *$u(G) \leq s(G)$.*

(2) *If G is a free semigroup, then for each $x \in \widehat{\mathbb{C}}$,*

$$s(G, x) = t(I, x) \text{ and } s(G) = t(I).$$

(3) *If G is hyperbolic containing an element of degree at least two (resp. nicely expanding), then there exists a neighborhood V of $J(G)$ with $V \subset \widehat{\mathbb{C}} \setminus P(G)$ (resp. $V \subset \widehat{\mathbb{C}} \setminus P_0(G)$) such that $t(I) = t(I, x)$, for each $x \in V$.*

Proof. The assertions in (1) were proved in [Sum98, Theorem 4.2]. The assertions in (2) follow immediately from the definition. In order to prove (3), we verify the following claim.

Claim. *Let $t \geq 0$. If $\sum_{n \in \mathbb{N}} Z_n(I, t, x) = \infty$, for some $x \in J(G)$, then $\sum_{n \in \mathbb{N}} Z_n(I, t, y) = \infty$, for every $y \in \widehat{\mathbb{C}}$.*

Proof of Claim. Suppose that $\sum_{n \in \mathbb{N}} Z_n(I, t, x) = \infty$, for some $x \in J(G)$. We first show that $\sum_{n \in \mathbb{N}} Z_n(I, t, y) = \infty$ for each $y \in J(G)$. Since G is hyperbolic, containing an element of degree at least two (resp. nicely expanding), we can apply Lemma 5.2. If $P(I, t, x) < \infty$, then we have $\sum_{n \in \mathbb{N}} Z_n(I, t, y) = \infty$, for each $y \in J(G)$, because there exists a constant $C' > 1$ such that $Z_n(I, t, y) \geq C'^{-1} Z_n(I, t, x)$. If $P(I, t, x) = \infty$, then it follows from the last assertion in Lemma 5.2 that $Z_n(I, t, y) = \infty$, for each $y \in J(G)$ and for all sufficiently large $n \in \mathbb{N}$. The next step is to show that $\sum_{n \in \mathbb{N}} Z_n(I, t, y) = \infty$ for each y in a neighborhood V of $J(G)$. Since G is hyperbolic (resp. nicely expanding), there exists $r > 0$ such that $B(y, r) \subset \widehat{\mathbb{C}} \setminus P(G)$ (resp. $B(y, r) \subset \widehat{\mathbb{C}} \setminus P_0(G)$), for each $y \in J(G)$. Set $V := \bigcup_{y \in J(G)} B(y, r)$. An application of Koebe's distortion

theorem shows that there exists a constant $C_1 = C_1(r)$ such that $\sum_{n \in \mathbb{N}} Z_n(I,t,z) \geq C_1^{-1} \sum_{n \in \mathbb{N}} Z_n(I,t,y) = \infty$, for each $y \in J(G)$ and $z \in B(y,r)$.

In order to finish the proof of the claim, we will distinguish two cases.

Case (1): $\operatorname{card}(J(G)) > 1$. In this case, we have $\operatorname{card}(J(G)) \geq 3$ by Lemma 4.11. First, let $y \in \widehat{\mathbb{C}} \setminus E(G)$ be given. Then there exists $\omega \in I^*$ such that $f_\omega^{-1}(y) \cap V \neq \varnothing$ ([HM96, Lemma 3.2], [Sum00, Lemma 2.3]). Since $\sum_{n \in \mathbb{N}} Z_n(I,t,z) = \infty$, for each $z \in V$, we conclude that also $\sum_{n \in \mathbb{N}} Z_n(I,t,y) = \infty$. Finally, by Proposition 4.2, each element of $G \cap \operatorname{Aut}(\widehat{\mathbb{C}})$ is loxodromic. Since $\operatorname{card}(J(G)) \geq 3$, we have $\operatorname{card}(E(G)) \leq 2$ ([HM96, Lemma 3.3], [Sum00, Lemma 2.3]). Therefore, by Lemma 4.10 and [Be91, Theorem 4.1.2], for each $y \in E(G)$, there is $g \in G$ such that $g(y) = y$ and $\|g'(y)\| < 1$. Hence, $\sum_{n \in \mathbb{N}} Z_n(I,t,y) = \infty$.

Case (2): $\operatorname{card}(J(G)) = 1$. Let $g \in G$. By Lemma 4.3, we have that g is loxodromic. Let a denote the repelling fixed point of g, and let b denote the attracting fixed point of g. Since $a \in J(G)$, we have that, for each $y \in \widehat{\mathbb{C}} \setminus \{b\}$, there exists $n \in \mathbb{N}$ such that $g^{-n}(z) \in V$. Hence, $\sum_{n \in \mathbb{N}} Z_n(I,t,y) = \infty$. Finally, since $g(b) = b$ and $\|g'(b)\| < 1$, we have $\sum_{n \in \mathbb{N}} Z_n(I,t,b) = \infty$. The proof of the claim is complete. $\qquad\square$

Let us now complete the proof of the lemma. It follows from the claim that $t(I,x) \leq t(I,y)$, for each $x \in J(G)$ and $y \in \widehat{\mathbb{C}}$. In particular, we have $t(I) = t(I,x)$, for every $x \in J(G)$. Again, by Koebe's distortion theorem, letting V be the neighborhood of $J(G)$ in the proof of Claim, we conclude that $t(I) = t(I,x) = t(I,y)$ for all $x,y \in V$, which proves the assertion in (3). Thus we have proved Lemma 6.2. $\qquad\square$

Proposition 6.3. *Let I be a countable set. Let $G = \langle f_i : i \in I \rangle$ denote a nicely expanding rational semigroup. Then $\beta \mapsto \mathscr{P}(\beta)$ is strictly decreasing on $\mathscr{F} := \{x \in \mathbb{R} : \mathscr{P}(x) < \infty\}$, and we have*

$$(6.1) \qquad t(I) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) \leq 0\} = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\} = \sup_{F \subset I, \operatorname{card}(F) < \infty} t(F).$$

Furthermore, we have that $\mathscr{P}(t(I)) \leq 0$.

Proof. We may assume that $I = \{1,\dots,n\}$, for some $n \in \mathbb{N}$, or $I = \mathbb{N}$. Using that G is expanding, it is straightforward to verify that the map $\beta \mapsto \mathscr{P}(\beta)$ is strictly decreasing on $\mathscr{F} := \{\beta \in \mathbb{R} : \mathscr{P}(\beta) < \infty\}$. Let $x \in J(G)$. Since G is nicely expanding, Lemma 5.4 yields $\mathscr{P}(\beta) = P(I,\beta,x)$. By definition of $t(I,x)$ we have

$$\inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) \leq 0\} \leq t(I,x) \leq \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\}.$$

Since $\beta \mapsto \mathscr{P}(\beta)$ is strictly decreasing on $\mathscr{F}$, we have $\inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) \leq 0\} = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\}$, which proves that $t(I,x) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) \leq 0\} = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\}$. By Lemma 6.2 (3) we have $t(I,x) = t(I)$. It remains to show that $t(I) = \sup_{F \subset I, \operatorname{card}(F) < \infty} t(F)$. Clearly, we have $t(F) \leq t(I)$, for each $F \subset I$. Hence, $\sup_{F \subset I, \operatorname{card}(F) < \infty} t(F) \leq t(I)$. For the reversed inequality, let $\varepsilon > 0$. Since $t(I) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) \leq 0\}$, we have $\mathscr{P}(t(I) - \varepsilon) > 0$. Hence, by Lemma 5.4, there exists $n \in \mathbb{N}$ such that $\mathscr{P}_n(t(I) - \varepsilon) > 0$. Therefore, we have that

$$\sup_{F \subset I, \operatorname{card}(F) < \infty} t(F) \geq t(I \cap \{1,\dots,n\}) = \inf\{\beta \in \mathbb{R} : \mathscr{P}_n(\beta) \leq 0\} \geq t(I) - \varepsilon.$$

Letting ε tend to zero, finishes the proof of (6.1).

Next, we will show that $\mathscr{P}(t(I)) \leq 0$. Since $t(I) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) \leq 0\}$ and $\beta \mapsto \mathscr{P}(\beta)$ is a closed function by Proposition 5.3 (4), the claim follows. The proof is complete. $\qquad\square$

In order to state the main theorem of this section, we give the definition of the following subsets of $J_{\text{pre}}(G)$.

Definition 6.4. For a rational semigroup G, we set

$$J_{\mathrm{ur}}(G) := \bigcup_{H \text{ finitely generated subsemigroup of } G} J_{\mathrm{pre}}(H) \text{ and } J_{\mathrm{r}}(G) := \bigcup_{\gamma \in G^{\mathbb{N}}: \exists g \in G: \gamma_i = g \text{ infinitely often}} J_{\gamma}.$$

Remark. If $G = \langle f_i : i \in I \rangle$ is a rational semigroup, where $I = \{1, \ldots, n\}$, for some $n \in \mathbb{N}$, or $I = \mathbb{N}$, then

$$J_{\mathrm{ur}}(G) = \bigcup_{F \subset I: \mathrm{card}(F) < \infty} J_F \text{ and } J_{\mathrm{r}}(G) = \bigcup_{\omega \in I^{\mathbb{N}}: \liminf_i \omega_i < \infty} J_{\omega}.$$

We remark that the subscript *ur* in J_{ur} means uniformly radial, and the subscript *r* in J_{r} means radial.

Theorem 6.5. *For a nicely expanding rational semigroup $G = \langle f_i : i \in I \rangle$, the following holds.*

(1) *The $u(G)$-dimensional outer Hausdorff measure of $J_{\mathrm{pre}}(G)$ is finite. In particular, we have that* $\dim_H \left(J_{\mathrm{pre}}(G) \right) \leq u(G).$

(2) *If I is countable, then* $\dim_H \left(J_{\mathrm{pre}}(G) \right) \leq u(G) \leq s(G) \leq t(I) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\}.$

(3) *If I is countable, and if $\{f_i : i \in I\}$ satisfies the open set condition, then we have*

$$\dim_H (J_{\mathrm{ur}}(G)) = \dim_H (J_{\mathrm{r}}(G)) = \dim_H \left(J_{\mathrm{pre}}(G) \right) = u(G) = s(G) = t(I) = \inf\{\beta \in \mathbb{R} : \mathscr{P}(\beta) < 0\}.$$

Proof. We start with the proof of (1). Let $\delta := u(G)$ and let $(\delta_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_n \delta_n = \delta$, such that $\delta_n > \delta$ for each $n \in \mathbb{N}$, and such that there exists a δ_n-subconformal measure μ_{δ_n} for each $n \in \mathbb{N}$. We may assume that $(\mu_{\delta_n})_{n \in \mathbb{N}}$ converges weakly to a Borel probability measure μ_δ on $\widehat{\mathbb{C}}$. Then the measure μ_δ is δ-subconformal. By [Sum98, Proposition 4.3], Lemmas 4.11, 4.12, [Sum00, Lemma 2.3], and [Be91, Theorem 4.1.2] we have $\mathrm{supp}(\mu_\delta) \supset J(G)$. In order to show that the δ-dimensional Hausdorff measure of $J_{\mathrm{pre}}(G)$ is finite, we will show that there exists a constant $C > 0$ such that, for all $z \in J_{\mathrm{pre}}(G)$,

$$(6.2) \qquad \limsup_{r \to 0} \frac{\mu_\delta(B(z,r))}{r^\delta} \geq C.$$

It then follows from the uniform mass distribution principle ([Fal03, Proposition 4.9 (b)]) and its proof that the δ-dimensional outer Hausdorff measure of $J_{\mathrm{pre}}(G)$ is finite. Hence, $\dim_H \left(J_{\mathrm{pre}}(G) \right) \leq \delta$. In order to prove (6.2) we extend [Sum98, Proof of Theorem 3.4] to our setting. Since G is nicely expanding, we have $d(J(G), P_0(G)) > 0$, and we can fix some $0 < a < d(J(G), P_0(G))/2$. Let $z \in J_{\mathrm{pre}}(G)$. Then there exists $\omega \in I^{\mathbb{N}}$ such that $z \in J_\omega$. Hence, we have $f_{\omega|_n}(z) \in J_{\sigma^n \omega} \subset J_{\mathrm{pre}}(G)$, for each $n \in \mathbb{N}$. We set $g_n := f_{\omega|_n}$, $z_n := g_n(z)$ and we denote by S_n the unique holomorphic branch of g_n^{-1} on $B(z_n, a)$ such that $S_n(g_n(z)) = z$. It follows from Koebe's distortion theorem that there is $c_0 > 1$ such that $c_0^{-1} \leq \|S_n'(x)\| / \|S_n'(y)\| \leq c_0$, for all $z \in J_{\mathrm{pre}}(G)$, $n \in \mathbb{N}$ and for all $x, y \in B(z_n, a)$. We conclude that

$$S_n(B(z_n, a)) \subset B\left(z_0, ac_0 \|S_n'(z_n)\|\right).$$

Since $J(G)$ is compact and since $\mathrm{supp}(\mu_\delta) \supset J(G)$, we have $M(a) := \inf_{z \in J(G)} \mu_\delta(B(z, a)) > 0$. Setting $r_n := ac_0 \|S_n'(z_n)\|$ and using that μ_δ is δ-subconformal, we estimate

$$
\begin{aligned}
\mu_\delta(B(z_0, r_n)) &\geq \mu_\delta(S_n(B(z_n, a))) \geq \int_{B(z_n, a)} \|S_n'\|^\delta d\mu_\delta \\
&\geq c_0^{-\delta} \|S_n'(z_n)\|^\delta \mu_\delta(B(z_n, a)) \geq c_0^{-\delta} (ac_0)^{-\delta} r_n^\delta M(a).
\end{aligned}
$$

Since G is expanding, r_n tends to zero as n tends to infinity. Hence, (6.2) follows with $C := c_0^{-\delta} (ac_0)^{-\delta} M(a)$. The proof of (1) is complete.

The assertion in (2) follows by combining (1) with Lemma 6.2 (1) and Proposition 6.3. To prove (3), suppose that G satisfies the open set condition and set $G_n := \langle f_i : i \in I \cap \{1, \ldots, n\} \rangle$, for each $n \in \mathbb{N}$. By [Sum05, Theorem B] we have $\dim_H(J(G_n)) = t(I \cap \{1, \ldots, n\})$, for each $n \in \mathbb{N}$. Since $J(G_n) = J_{\mathrm{ur}}(G_n) \subset J_{\mathrm{ur}}(G) \subset$

$J_r(G) \subset J_{\text{pre}}(G)$, for each $n \in \mathbb{N}$, we obtain that $t(I \cap \{1,\dots,n\}) = \dim_H(J(G_n)) \leq \dim_H(J_{\text{ur}}(G)) \leq \dim_H(J_r(G)) \leq \dim_H(J_{\text{pre}}(G))$. By Proposition 6.3 we have $\lim_n t(I \cap \{1,\dots,n\}) = t(I)$. Combining with the upper bound in (2), finishes the proof of (3). The proof is complete. $\square$

7. APPLICATIONS TO NON-HYPERBOLIC RATIONAL SEMIGROUPS

In this section we apply the results of Section 6 to give estimates for the Hausdorff dimension of the (pre-)Julia sets of non-hyperbolic rational semigroups which possess an inducing structure.

7.1. **General Setting.** Throughout this section we assume that I is countable. For a subset Λ of Rat, we denote by $\langle \Lambda \rangle$ the rational semigroup generated by Λ. Thus $\langle \Lambda \rangle = \langle g : g \in \Lambda \rangle = \{g_1 \circ \cdots \circ g_n : n \in \mathbb{N}, g_j \in \Lambda, j = 1,\dots,n\}$.

Definition 7.1 (Inducing structure). Let $G = \langle f_i : i \in I \rangle$ denote a rational semigroup. Suppose that $\deg(g) \geq 2$ for each $g \in G$. We say that $G = \langle f_i : i \in I \rangle$ has an inducing structure (with respect to $\{I_1, I_2\}$) if there exists a decomposition $I = I_1 \cup I_2$ with $I_2 \neq \varnothing$, such that the following holds for the rational semigroups $G_j := \langle f_i : i \in I_j \rangle$, $j \in \{1,2\}$, and $H := \langle H_0 \rangle$ given by

$$H_0 := \{f_i : i \in I_2\} \cup \{f_i f_{j_1} \dots f_{j_r} : i \in I_2, r \in \mathbb{N}, (j_1,\dots,j_r) \in I_1^r\}.$$

There exists an H-forward invariant non-empty compact set $L \subset F(H)$ such that $P(G_2) \subset L$ and $f_i(P(G_1)) \subset L$, for $i \in I_2$.

In the following, when $G = \langle f_i : i \in I \rangle$ has an inducing structure with respect to $\{I_1, I_2\}$, let H_0 and H be as in Definition 7.1. We endow H_0 with the discrete topology.

Lemma 7.2. *Suppose that $G = \langle f_i : i \in I \rangle$ has an inducing structure. Then H is nicely expanding.*

Proof. By Definition 7.1 we have

$$P(H) = \overline{\bigcup_{h \in H \cup \{\text{id}\}} h \left(\bigcup_{f \in H_0} \text{CV}(f) \right)} \subset \overline{\bigcup_{h \in H \cup \{\text{id}\}} h \left(P(G_2) \cup \bigcup_{i \in I_2} f_i(P(G_1)) \right)} \subset L \subset F(H).$$

Since $P(G_2) \subset P(H)$ and $P(G_2) \neq \varnothing$, we have that H is nicely expanding by Proposition 4.2. $\square$

The proof of the next lemma is straightforward and therefore omitted.

Lemma 7.3. *Suppose that $G = \langle f_i : i \in I \rangle$ has an inducing structure. If $\{f_i : i \in I\}$ satisfies the open set condition with open set U, then H_0 satisfies the open set condition with open set U.*

The following lemma holds for arbitrary finitely generated rational semigroups of degree at least two.

Lemma 7.4. *Let Γ denote a rational semigroup with $\deg(g) \geq 2$, for each $g \in \Gamma$. Let Γ_0 be a finitely generated subsemigroup of Γ. Let Ω denote a subsemigroup of Γ with the property that, for each $g \in \Gamma$, there exists $h \in \Gamma_0$ such that $hg \in \Omega$. Then we have $J(\Gamma) = J(\Omega)$.*

Proof. Clearly have $J(\Omega) \subset J(\Gamma)$, and it remains to show the opposite inclusion. By the density of the repelling fixed points in the Julia set ([HM96, Corollary 3.1]), we have $J(\Gamma) = \overline{\bigcup_{g \in \Gamma} J(g)}$. Hence, it suffices to prove that $J(g) \subset J(\Omega)$ for each $g \in \Gamma$. Let $g \in \Gamma$ and let A be a finite set of generators of $\langle \Gamma_0 \cup \{g\} \rangle$ with $g \in A$. Let $\gamma \in A^{\mathbb{N}}$ be given by $(g,g,\dots)$. For each $n \in \mathbb{N}$, by our assumptions on Ω and Γ, there exists $h_n \in \Gamma_0$ such that $h_n g^n \in \Omega$. For each $n \in \mathbb{N}$, let $\alpha(n) \in A^n$ be given by $(g,g,\dots,g)$. Let $\beta(n) \in A^{m_n}$ be given by $\beta(n)_{m_n} \circ \cdots \circ \beta(n)_1 = h_n$. Further, for each $n \in \mathbb{N}$, we define the sequence $\gamma(n) \in A^{\mathbb{N}}$, given by $\gamma(n) :=$

$(\alpha(n), \beta(n), \alpha(n), \beta(n), \dots) \in A^{\mathbb{N}}$. We observe that $\gamma(n) \to \gamma$ with respect to the product topology on $A^{\mathbb{N}}$, as n tends to infinity. By lower semicontinuity of the Julia set ([Sum06, Proposition 2.2]), we conclude that, for each $z \in J_{\gamma}$, there is a sequence $(y_n) \in \widehat{\mathbb{C}}^{\mathbb{N}}$ with $y_n \in J_{\gamma(n)}$ such that $\lim_n y_n = z$. Consequently, we have $J(g) = J_{\gamma} \subset \overline{\bigcup_{n \in \mathbb{N}} J_{\gamma(n)}} \subset J(\Omega)$, where the last inclusion holds, because we have $J_{\gamma(n)} = J(h_n g^n)$ and $h_n g^n \in \Omega$, for each $n \in \mathbb{N}$. The proof is complete. $\qquad\square$

Lemma 7.5. *Suppose that $G = \langle f_i : i \in I \rangle$ has an inducing structure. Then $J(G) = J(H)$.*

Proof. Let $h \in I_2$ be an element. By Definition 7.1, we have that $hg \in H$, for each $g \in G$. Now, the lemma follows from Lemma 7.4 applied to $\Gamma := G$, $\Gamma_0 := \langle h \rangle$ and $\Omega := H$. $\qquad\square$

Definition 7.6. Let I be a topological space and let $(f_i)_{i \in I} \in C(I, \mathrm{Rat})$. Let $G = \langle f_i : i \in I \rangle$ and let $\tilde{f} : J(\tilde{f}) \to I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ be the associated skew product. For each $K \subset I$ and $\omega \in K^{\mathbb{N}}$, we set $\widehat{J}_{\omega,K} := \pi_{\widehat{\mathbb{C}}}(\overline{J^K} \cap \pi_1^{-1}(\{\omega\}))$, where the closure is taken in $I^{\mathbb{N}} \times \widehat{\mathbb{C}}$.

Lemma 7.7 ([Sum10a], Lemma 3.5). *Let $G = \langle f_i : i \in I \rangle$ be a rational semigroup with $\mathrm{card}(I) < \infty$. Let $K \subset I$ and set $G_K := \langle f_i : i \in K \rangle$. For each $\omega \in K^{\mathbb{N}}$, we have $\widehat{J}_{\omega,K} = \bigcap_{n=1}^{\infty} f_{\omega|_n}^{-1}(J(G_K))$.*

Lemma 7.8. *Let $G = \langle f_i : i \in I \rangle$ be a rational semigroup with $\mathrm{card}(I) < \infty$. Suppose that $\{f_i : i \in I\}$ satisfies the open set condition. Let $I_1 \subset I$ be a non-empty subset and set $G_1 := \langle f_i : i \in I_1 \rangle$. Suppose that $f_i(P(G_1)) \subset F(G)$, for all $i \in I \setminus I_1$, and that there exists a G_1-forward invariant compact subset $L_1 \subset F(G)$. Then we have $\widehat{J}_{\omega,I_1} = \widehat{J}_{\omega,I}$, for each $\omega \in I_1$.*

Proof. Let $\omega \in I_1^{\mathbb{N}}$ and suppose by way of contradiction that there exists $z_{\infty} \in \widehat{J}_{\omega,I} \setminus \widehat{J}_{\omega,I_1}$. Then there exist sequences $(\beta^n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ with $\beta^n \in I^{\mathbb{N}} \setminus I_1^{\mathbb{N}}$ and $z_n \in J_{\beta^n}$, for each $n \in \mathbb{N}$, such that $\lim_n \beta^n = \omega$ and $\lim_n z_n = z_{\infty}$. Since $z_{\infty} \notin \widehat{J}_{\omega,I_1} = \bigcap_{n=1}^{\infty} f_{\omega|_n}^{-1}(J(G_1))$ by Lemma 7.7, there exists $m \in \mathbb{N}$ such that $f_{\omega|_m}(z_{\infty}) \in F(G_1)$. We may assume that $\beta_{|m}^n = \omega_{|m}$, for each $n \in \mathbb{N}$. Define the sequence $(r_n) \in \mathbb{N}^{\mathbb{N}}$, given by $r_n := \min\{k \in \mathbb{N} : \beta_k^n \notin I_1\}$. Clearly, we have $r_n > m$, for each $n \in \mathbb{N}$, and that r_n tends to infinity, because β^n tends to $\omega \in I_1^{\mathbb{N}}$, as n tends to infinity. Since $f_{\omega|_m}(z_{\infty}) \in F(G_1)$ and $\beta_{|r_n-1}^n \in I_1^{r_n-1}$, we have that $(f_{\beta_{|r_n-1}^n})_{n \in \mathbb{N}}$ is normal in a neighborhood of z_{∞} in F_{ω}. Let $(n_j) \in \mathbb{N}^{\mathbb{N}}$ be a sequence tending to infinity, such that the sequence $(g_j) \in G_1^{\mathbb{N}}$, given by $g_j := f_{\beta_{|r_{n_j}-1}^{n_j}}$, converges uniformly in a neighborhood V of z_{∞}. We may also assume that there exists $i_0 \in I \setminus I_1$ such that $\beta_{r_{n_j}}^{n_j} = i_0$, for all $j \in \mathbb{N}$. We will now distinguish two cases.

Case 1: Suppose there exists a constant $c \in \widehat{\mathbb{C}}$, such that $g_j \rightrightarrows c$ on V. Since $g_j(z_{n_j}) \in J(G)$, we have $c \in J(G)$. Hence, we have that $c \notin L_1 \subset F(G)$ and that there exists a G_1-forward invariant neighborhood W of L_1 in $F(G)$, such that $c \notin \overline{W}$. (To take such W, let $\delta_0 > 0$ be a small number such that setting $A = \{z \in F(G) : d(z, L_1) < \delta_0\}$, we have $c \notin \overline{\bigcup_{g \in G_1 \cup \{\mathrm{id}\}} g(A)}$. Let $W := \bigcup_{g \in G_1 \cup \{\mathrm{id}\}} g(A)$.) To prove that $c \in P(G_1)$, suppose on the contrary that $c \notin P(G_1)$. Then there exists $\delta > 0$ such that $\overline{B(c, \delta)} \cap W = \varnothing$ and, for each large j, there exists a well defined inverse branch $h_j : B(c, \delta) \to \widehat{\mathbb{C}}$ of g_j, such that $h_j(g_j(z_{\infty})) = z_{\infty}$ and $g_j \circ h_j = \mathrm{id}$ on $B(c, \delta)$. Since W is G_1-forward invariant, we conclude that $h_j(B(c, \delta)) \cap W = \varnothing$. Hence, we obtain that (h_j) is normal in z_{∞}, which is a contradiction (see the argument in the proof of Lemma 4.9). We have thus shown that $c \in P(G_1)$. Consequently, we have $\lim_j f_{\beta_{|r_{n_j}}^{n_j}}(z_{\infty}) = f_{i_0}(c) \in f_{i_0}(P(G_1)) \subset F(G)$. On the other hand, we have $f_{i_0}(c) = \lim_j f_{\beta_{|r_{n_j}}^{n_j}}(z_{n_j}) \in J(G)$, which gives a contradiction.

Case 2: Suppose there exists a non-constant holomorphic map $\varphi : V \to \widehat{\mathbb{C}}$, such that $g_j \rightrightarrows \varphi$ on V. Suppose that $\{f_i : i \in I\}$ satisfies the open set condition with open set U. Since $f_{i_0}(\varphi(z_{\infty})) = \lim_j f_{\beta_{|r_{n_j}}^{n_j}}(z_{n_j}) \in J(G) \subset$

$\overline{U}$ and $f_{i_0} \circ \varphi$ is non-constant, there exists z in a neighborhood of z_∞, such that $f_{i_0}(\varphi(z)) \in U$. Moreover, there exists $j_0 \in \mathbb{N}$ such that $f_{\beta^{n_j}_{|r_{n_j}}}(z) \in U$, for all $j \geq j_0$. We may also assume that $r_{n_{j+1}} > r_{n_j}$, for all $j \geq j_0$. Hence, for each $j \geq j_0$, we have

$$z \in \left(f^{-1}_{\beta^{n_j}_1} \cdots f^{-1}_{\beta^{n_j}_{r_{n_j}}}(U) \right) \cap \left(f^{-1}_{\beta^{n_{j+1}}_1} \cdots f^{-1}_{\beta^{n_{j+1}}_{r_{n_j}}} \cdots f^{-1}_{\beta^{n_{j+1}}_{r_{n_{j+1}}}}(U) \right) \subset \left(f^{-1}_{\beta^{n_j}_1} \cdots f^{-1}_{\beta^{n_j}_{r_{n_j}}}(U) \right) \cap \left(f^{-1}_{\beta^{n_{j+1}}_1} \cdots f^{-1}_{\beta^{n_{j+1}}_{r_{n_j}}}(U) \right).$$

Since $\{f_i : i \in I\}$ satisfies the open set condition, we conclude that $\beta^{n_j}_1 \ldots \beta^{n_j}_{r_{n_j}} = \beta^{n_{j+1}}_1 \ldots \beta^{n_{j+1}}_{r_{n_j}}$. This is a contradiction, because $\beta^{n_j}_{r_{n_j}} = i_0 \in I \setminus I_1$ and $\beta^{n_{j+1}}_{r_{n_j}} \in I_1$. The proof is complete. $\square$

Lemma 7.9. *Suppose that $G = \langle f_i : i \in I \rangle$ has an inducing structure and that $\mathrm{card}(I) < \infty$. Let $\omega \in I^{\mathbb{N}}$ and suppose that there exists $\gamma \in H_0^{\mathbb{N}}$ and a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, such that $f_{\omega|_{n_k}} = \gamma_k \circ \cdots \circ \gamma_1$, for each $k \in \mathbb{N}$. Then we have $J_\gamma = J_\omega = \widehat{J}_{\omega,I}$.*

Proof. Clearly, we have $J_\gamma \subset J_\omega \subset \widehat{J}_{\omega,I}$. Suppose for a contradiction that there exists $z \in \widehat{J}_{\omega,I} \setminus J_\gamma$. Since H is nicely expanding by Lemma 7.2 , there exists $c \in P(H) \subset F(H)$ and a subsequence (n'_k) of (n_k) tending to infinity, such that $f_{\omega|_{n'_k}} \rightrightarrows c$ in a neighborhood of z by Lemma 4.9. On the other hand, it follows from Lemma 7.7 that $\widehat{J}_{\omega,I} = \bigcap_{k=1}^{\infty} f^{-1}_{\omega|_{n'_k}}(J(G))$. Hence, we have $c = \lim_k f_{\omega|_{n'_k}}(z) \in J(G)$. Since $J(G) = J(H)$ by Lemma 7.5, we get the desired contradiction. $\square$

Lemma 7.10. *Suppose that $G = \langle f_i : i \in I \rangle$ has an inducing structure with respect to $\{I_1, I_2\}$ and that $\mathrm{card}(I) < \infty$. Let $G_1 = \langle f_i : i \in I_1 \rangle$. If there exists a G_1-forward invariant compact set $L_1 \subset F(G)$ and if $\{f_i : i \in I\}$ satisfies the open set condition, then we have*

$$J(G) = J_{\mathrm{pre}}(H) \cup \bigcup_{g \in G} g^{-1}(J(G_1)).$$

Proof. Let $z \in J(G)$. Since $\mathrm{card}(I) < \infty$, there exists $\omega \in I^{\mathbb{N}}$ such that $z \in \widehat{J}_{\omega,I}$ by Proposition 3.2 (3). We now distinguish two cases. If there exists $\ell \in \mathbb{N}$ and $\tau \in I_1^{\mathbb{N}}$ such that $\omega = (\omega_1, \ldots, \omega_\ell, \tau_1, \ldots)$, then $f_{\omega|_\ell}(z) \in \widehat{J}_{\tau,I}$. By Lemma 7.8, we have $\widehat{J}_{\tau,I} = \widehat{J}_{\tau,I_1} \subset J(G_1)$. We have thus shown that $z \in f^{-1}_{\omega|_\ell}(J(G_1))$. If no such ℓ exists, then there exist $\gamma \in H_0^{\mathbb{N}}$ and a sequence $(n_k) \in I^{\mathbb{N}}$ tending to infinity, such that $f_{\omega|_{n_k}} = \gamma_k \circ \cdots \circ \gamma_1$, for each $k \in \mathbb{N}$. Hence, we have $z \in \widehat{J}_{\omega,I} = J_\gamma \subset J_{\mathrm{pre}}(H)$ by Lemma 7.9. $\square$

The following two lemmata give conditions under which we can bound the Hausdorff dimension of the Julia set of a polynomial semigroup from above.

Lemma 7.11. *Let $G = \langle f_i : i \in I \rangle$ be a rational semigroup with $\mathrm{card}(I) < \infty$ such that the following holds.*

(1) *There exists a compact G-forward invariant set $K \subset F(G)$, such that $f_j(P(f_i)) \subset K$, for all $i, j \in I$ with $i \neq j$.*

(2) *f_i is a polynomial of degree at least two, for each $i \in I$.*

Then we have $\dim_H(J_{\mathrm{pre}}(G)) \leq \max\{s(G), \max_{i \in I}\{\dim_H(J(f_i))\}\}$.

Proof. For each $i \in I$, let $i^\infty = (i, i, i, \ldots) \in I^{\mathbb{N}}$. We will show that

$$\dim_H \left(\bigcup_{\omega \notin \bigcup_{n \in \mathbb{N}_0} \sigma^{-n}(\bigcup_{i \in I} i^\infty)} J_\omega \right) \leq s(G),$$

from which the lemma follows. To prove this, let $\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}} \times \widehat{\mathbb{C}}$ be the skew product associated to $\{f_i : i \in I\}$. We set

$$P(\tilde{f}) := \overline{\bigcup_{n \in \mathbb{N}} \tilde{f}^n(\{(\omega, z) : f'_{\omega_1}(z) = 0\})} \subset I^{\mathbb{N}} \times \widehat{\mathbb{C}},$$

and we first verify that

(7.1) $$J(\tilde{f}) \cap P(\tilde{f}) = \bigcup_{i \in I} \{(i^{\infty}, x) \in J(\tilde{f}) : x \in P(f_i)\}.$$

Let $(\omega, x) \in J(\tilde{f}) \cap P(\tilde{f})$ be given. By (1), there exists $i \in I$ such that $x \in P(f_i)$. Since $f_{\omega|_n}(x) \in J(G)$, for each $n \in \mathbb{N}$, we have $\omega = i^{\infty}$ by (1). Thus (7.1) holds. For each $\omega \notin \bigcup_{n \in \mathbb{N}_0} \sigma^{-n}(\bigcup_{i \in I} i^{\infty})$ and for each $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ such that $J_{\sigma^{n-1}(\omega)} \subset f_i^{-1} f_j^{-1}(J(G))$, for some $i, j \in I$ with $i \neq j$. By (7.1) we then have

$$\min\left\{d(a,b) : a \in J_{\sigma^{n-1}(\omega)}, b \in P(G)\right\} \geq \min\left\{d(a,b) : i,j \in I, i \neq j, a \in f_i^{-1} f_j^{-1}(J(G)), b \in P(G)\right\} > 0.$$

Now, the claim follows from [Sum06, Proposition 2.11 and Proposition 2.20]. □

Lemma 7.12. *Under the hypothesis of Lemma 7.11, suppose that $\{f_i : i \in I\}$ additionally satisfies the open set condition. Then we have $J_{\mathrm{pre}}(G) = J(G)$. In particular, we have*

$$\dim_H(J(G)) \leq \max\{s(G), \max_{i \in I}\{\dim_H(J(f_i))\}\}.$$

Proof. We will show that $\widehat{J}_{\omega, I} = J_{\omega}$, for each $\omega \in I^{\mathbb{N}}$. Suppose for a contradiction that there exists $\omega \in I^{\mathbb{N}}$ and $z \in \widehat{J}_{\omega, I} \setminus J_{\omega}$. Since we have $\widehat{J}_{i^{\infty}, I} = J_{i^{\infty}}$, for each $i \in I$ by Lemma 7.8 applied to $I_1 := \{i\}$, we conclude that there exist $i, j \in I$ with $i \neq j$ and a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, such that $\omega_{n_k+1} = i$ and $\omega_{n_k+2} = j$. We may assume that there exists $g : V \to \widehat{\mathbb{C}}$ in a neighborhood V of z, such that $f_{\omega|_{n_k}} \rightrightarrows g$ on V. We show that g is non-constant. Otherwise, similarly as in the proof of Lemma 7.8 (Case 1), we can show that $g(z) \in P(G)$, which then implies that $\lim_k f_{\omega|_{n_k+2}}(z) = f_j f_i g(z) \in F(G)$. This contradicts that $z \in \widehat{J}_{\omega, I}$. We have thus shown that g is non-constant. We may assume that there exists $\rho \in I^{\mathbb{N}}$ such that $\lim_k \sigma^{n_k}(\omega) = \rho$. Clearly, we have $\rho_1 = i$ and $\rho_2 = j$. Now, it follows from [Sum01, Lemma 2.13] that $(\{\rho\} \times \widehat{J}_{\rho, I}) \cap P(\tilde{f}) \neq \varnothing$, which contradicts (7.1). □

In order to state the main result of this section, let us introduce regularity of the pressure function associated to rational semigroups. We adapt the definitions from [MU03, p.78] in the context of graph directed Markov systems.

Definition 7.13. Let I be a finite or countable set, let $(f_i)_{i \in I} \in \mathrm{Rat}^I$ and let $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ be the associated skew product. Suppose that $\langle f_i : i \in I \rangle$ is nicely expanding and denote by $\mathscr{P}(t)$ the pressure function of the system $\{f_i : i \in I\}$ for $t \in \mathbb{R}$. We say that $\{f_i : i \in I\}$ is regular if there exists $t \geq 0$ such that $\mathscr{P}(t) = 0$. Otherwise, we say that $\{f_i : i \in I\}$ is irregular. If $\{f_i : i \in I\}$ is regular and if there exists $u \in \mathbb{R}$ such that $0 < \mathscr{P}(u) < \infty$, then $\{f_i : i \in I\}$ is called strongly regular, if $\{f_i : i \in I\}$ is regular and no such $u \in \mathbb{R}$ exists, then $\{f_i : i \in I\}$ is called critically regular. Moreover, we set $\Theta(I) := \inf\{\beta \in \mathbb{R} : \sup\{Z_1(I, \beta, x) : x \in J(\langle f_i : i \in I \rangle)\} < \infty\}$.

Theorem 7.14. *Suppose that $G = \langle f_i : i \in I \rangle$ has an inducing structure with respect to $\{I_1, I_2\}$ and that there exists a G-forward invariant compact set $L_0 \subset F(G)$. Let H_0, H, G_1 be as in Definition 7.1. Let H_0 be endowed with the discrete topology. Then, we have the following.*

(1) $J_{\mathrm{pre}}(G) \setminus J_{\mathrm{pre}}(H) = \left(\bigcup_{g \in G \cup \{\mathrm{id}\}} g^{-1}(J_{\mathrm{pre}}(G_1))\right) \setminus J_{\mathrm{pre}}(H).$

(2) *If I is countable, then H is nicely expanding and $\dim_H \left(J_{\mathrm{pre}} \left(H \right) \right) \leq s(H) \leq t(H_0) = \inf\{\beta \in \mathbb{R} :$*
    $\mathscr{P}(\beta) < 0\}$, where $\mathscr{P}$ denotes the pressure function of the system $\{h : h \in H_0\}$.

(3) *If I is countable, and if $\{f_i : i \in I\}$ satisfies the open set condition, then we have*

$$s(G_1) = t(I_1) \leq \Theta(H_0) \leq t(H_0) = s(H) = s(G) = t(I) = \dim_H \left(J_{\mathrm{pre}} \left(H \right) \right)$$

and

$$\dim_H \left(J_{\mathrm{pre}} \left(G \right) \right) = \max \left\{ s(G), \dim_H \left(J_{\mathrm{pre}} \left(G_1 \right) \right) \right\}.$$

If moreover $\mathrm{card}(I) < \infty$, then we have

$$\dim_H \left(J(G) \right) = \max\{s(G), \dim_H(J(G_1))\}.$$

(4) *If $\{f_i : i \in I\}$ satisfies the open set condition, $\mathrm{card}(I) < \infty$, f_i is a polynomial for each $i \in I_1$, and*
    if there exists a compact G_1-forward invariant subset $K \subset F(G_1)$, such that $f_j(P(f_i)) \subset K$ for all
    $i, j \in I_1$ with $i \neq j$, then

$$\dim_H \left(J(G) \right) = \max \left\{ s(G), \max_{i \in I_1}\{\dim_H \left(J(f_i) \right)\} \right\}.$$

(5) *Suppose that I is countable. Then we have all of the following.*
    (a) *$\Theta(H_0) = \inf\{\beta : P(H_0, \beta, x) < \infty\}$, for each $x \in J(H)$.*
    (b) *H_0 is strongly regular if $\Theta(H_0) < t(H_0)$.*
    (c) *H_0 is critically regular or irregular if $\Theta(H_0) = t(H_0)$.*

Proof. To prove the assertion in (1), we first verify that $J_\gamma = J_\omega$, for all $\gamma \in H_0^{\mathbb{N}}$ and $\omega \in I^{\mathbb{N}}$, for which there exists a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, such that $\gamma_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1 = f_{\omega|_{n_k}}$, for each $k \in \mathbb{N}$. Since the inclusion $J_\gamma \subset J_\omega$ is obviously true, we only address the opposite inclusion. Since L_0 is a compact, G-forward invariant subset of $F(G)$, there exists a forward G-invariant neighborhood V of L_0, such that $\overline{V} \subset F(G)$. Now, suppose by way of contradiction that there exists $z \in J_\omega \cap F_\gamma$. Since $z \in J_\omega$ we have $f_{\omega|_{n_k}}(z) \in J(G)$ for each $k \in \mathbb{N}$. Since V is a relatively compact subset of $F(G)$, there exists $\varepsilon > 0$ such that $d\left(V, f_{\omega|_{n_k}}(z)\right) > \varepsilon$, for each $k \in \mathbb{N}$. Combining this with our assumption that $z \in F_\gamma$, we obtain that there exists $\delta > 0$ such that $\gamma_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1 \left(B(z, \delta) \right) \cap V = \varnothing$, for all $k \in \mathbb{N}$. Since V is G-forward invariant, we conclude that $f_{\omega|_n} \left(B(z, \delta) \right) \cap V = \varnothing$, for all $n \in \mathbb{N}$, which implies that $z \in F_\omega$. This contradiction finishes the proof of $J_\gamma = J_\omega$. We now let $z \in J_{\mathrm{pre}}(G) \setminus J_{\mathrm{pre}}(H)$. Then there exists $\omega \in I^{\mathbb{N}}$ such that $z \in J_\omega$. Suppose $\mathrm{card}(\{k \in I : \omega_k \in I_2\}) = \infty$. Then there exist $\gamma \in H_0^{\mathbb{N}}$ and a sequence $(n_k) \in \mathbb{N}^{\mathbb{N}}$ tending to infinity such that $\gamma_k \circ \cdots \circ \gamma_1 = f_{\omega|_{n_k}}$ for each k. By the above observation, we have $z \in J_\omega = J_\gamma \subset J_{\mathrm{pre}}(H)$. However this is a contradiction. Hence $\mathrm{card}(\{k \in I : \omega_k \in I_2\}) < \infty$ and $z \in \cup_{g \in G \cup \{\mathrm{id}\}} g^{-1}(J_{\mathrm{pre}}(G_1))$. Thus the assertion in (1) holds.

The assertion in (2) follows from Lemma 7.2 and Theorem 6.5 (2). To prove (3), first observe that, since $\langle f_i : i \in I \rangle$ satisfies the open set condition, we have that H_0 satisfies the open set condition by Lemma 7.3. Hence, $G = \langle f_i : i \in I \rangle$, $H = \langle H_0 \rangle$ and $G_1 = \langle f_i : i \in I_1 \rangle$ are free semigroups. In particular, we have $s(G_1) = t(I_1)$, $t(H_0) = s(H)$ and $s(G) = t(I)$ by Lemma 6.2 (2). To prove $t(I_1) \leq \Theta(H_0)$, we first note that, for each $\alpha \geq 0$, $j \in I_2$, $x \in \widehat{\mathbb{C}}$ and $y \in f_j^{-1}(x)$,

$$\left\| f_j'(y) \right\|^{-\alpha} \sum_{g \in G_1} \sum_{z \in g^{-1}(y)} \left\| g'(z) \right\|^{-\alpha} \leq Z_1 \left(H_0, \alpha, x \right).$$

For each $x \in J(G_2)$ and $y \in f_j^{-1}(x)$ we have $\left\| f_j'(y) \right\| \neq 0$. Hence, we have

(7.2) $$\sum_{g \in G_1} \sum_{z \in g^{-1}(y)} \left\| g'(z) \right\|^{-\alpha} \leq \left\| f_j'(y) \right\|^{\alpha} Z_1(H_0, \alpha, x).$$

By the definition of $\Theta(H_0)$, we have $Z_1(H_0, \alpha, x) < \infty$, for each $x \in J(G_2) \subset J(H)$, $\varepsilon > 0$ and $\alpha = \Theta(H_0) + \varepsilon$. Hence, we have $\sum_{g \in G_1} \sum_{z \in g^{-1}(y)} \|g'(z)\|^{-\alpha} < \infty$ by (7.2). We have thus shown that $t(I_1) \leq t(I_1, y) \leq \Theta(H_0)$. In order to verify $\Theta(H_0) \leq t(H_0)$, recall that since H is nicely expanding, we have for each $x \in J(H)$ that $t(H_0) = t(H_0, x)$ by Lemma 6.2 (3). Consequently, for each $\varepsilon > 0$ and for each $x \in J(H)$, we have that $\sum_{n \in \mathbb{N}} Z_n(H_0, t(H_0) + \varepsilon, x) < \infty$. In particular, we have $P(H_0, t(H_0) + \varepsilon, x) \leq 0 < \infty$. By Lemma 5.2 we conclude that $\sup_{y \in J(H)} Z_1(H_0, t(H_0) + \varepsilon, y) < \infty$, which proves $\Theta(H_0) \leq t(H_0) + \varepsilon$. Therefore, $\Theta(H_0) \leq t(H_0)$. We now prove $s(H) = s(G)$. It is easy to see that $s(G) \geq s(H)$. In order to show the opposite inequality, let $t > s(H)$. Then by the Claim in the proof of Lemma 6.2 and Lemma 7.5, there exists a point $x_0 \in J(H) = J(G)$ such that $\sum_{n \in \mathbb{N}} Z_n(H_0, t, x_0) < \infty$. Let $h \in \{f_i : i \in I_2\}$ and let $x_1 \in h^{-1}(x_0) \subset J(G) = J(H)$. Then $\sum_{n \in \mathbb{N}} Z_n(I_1, t, x_1) < \infty$. Moreover, we have $\sum_{n \in \mathbb{N}} Z_n(H_0, t, x_1) < \infty$ by the Claim in the proof of Lemma 6.2 and Lemma 7.5. Furthermore, by Lemma 5.2, there exists a constant $C' > 1$ such that $Z_n(H_0, t, x) \leq C' Z_n(H_0, t, x_0)$, for each $x \in J(G)$. Therefore, we have

$$(7.3) \quad \sum_{n \in \mathbb{N}} Z_n(I, t, x_1) = \sum_{n \in \mathbb{N}} Z_n(I_1, t, x_1) + \sum_{n \in \mathbb{N}} Z_n(H_0, t, x_1) + \sum_{g \in G_1} \sum_{a \in g^{-1}(x_1)} \|g'(a)\|^{-t} \sum_{n \in \mathbb{N}} Z_n(H_0, t, a) < \infty.$$

Thus, we have $t > s(G)$, which finishes the proof of $s(G) = s(H)$. That $t(H_0)$ is equal to $\dim_H\left(J_{\text{pre}}(H)\right)$ follows from Theorem 6.5 because H is nicely expanding by Lemma 7.2 and H_0 satisfies the open set condition by Lemma 7.3. Combining this with the fact that the Hausdorff-dimension is σ-stable and that Lipschitz continuous maps do not increase Hausdorff-dimension (we can apply this fact to holomorphic inverse branches of the elements of G defined locally in the complement of the critical values), we obtain that $\dim_H\left(J_{\text{pre}}(G)\right) = \max\{s(G), \dim_H(J_{\text{pre}}(G_1))\}$ by (1). Finally, if $\text{card}(I) < \infty$, then we have $\dim_H(J(G)) = \max\{s(G), \dim_H(J(G_1))\}$ by Lemma 7.10.

The assertion in (4) follows from (3) and Lemma 7.12. Finally, (5a) follows from Lemma 5.2 and the statements in (5b) and (5c) are derived from (5a) and Proposition 6.3. The proof is complete. $\qquad \square$

7.2. Special cases of polynomial semigroups.
In this section, we provide a class of polynomial semigroups which have an inducing structure. Moreover, we can prove further refinements of our main result.

Definition 7.15 (PB-OSC). We say that $G = \langle f_1, f_2 \rangle$ (or the generator system $\{f_1, f_2\}$) satisfies PB-OSC if f_1 and f_2 are polynomials of degree at least two, such that each of the following holds.

(1) $P(G) \setminus \{\infty\}$ is a bounded subset of $\mathbb{C}$.
(2) $K(f_1) \subset \text{Int} K(f_2)$
(3) $\{f_1, f_2\}$ satisfies the open set condition with the open set $(\text{Int} K(f_2)) \setminus K(f_1)$
(4) $\text{CV}(f_2) \setminus \{\infty\} \subset \text{Int} K(f_1)$.

We will frequently make use of the following facts for $G = \langle f_1, f_2 \rangle$ satisfying PB-OSC. For $\omega \in \{1, 2\}^{\mathbb{N}}$, we set $K_\omega = \left\{z \in \mathbb{C} : \left(f_{\omega|_n}(z)\right)_{n \in \mathbb{N}} \text{ is bounded}\right\}$.

By [Sum10a, Lemma 3.6] it follows from (1) that J_ω is connected for each $\omega \in \{1, 2\}^{\mathbb{N}}$. We also have that the corresponding filled Julia set K_ω is connected. Moreover, we have that $\widehat{\mathbb{C}} \setminus K_\omega$ is a connected component of $\widehat{\mathbb{C}} \setminus J_\omega$ and that $\widehat{\mathbb{C}} \setminus K_\omega$ is the basin of attraction of infinity of $\left(g_{\omega|_n}\right)$. By (1), (2) and (3) we have that $J_{2,1,1,\ldots} = f_2^{-1}(J(f_1))$ and $J(f_1)$ are disjoint. So, by [Sum10a, Lemma 3.9] we have that either $f_2^{-1}(J(f_1))$ surrounds $J(f_1)$ or that $J(f_1)$ surrounds $f_2^{-1}(J(f_1))$, where, for two compact connected subsets K_1 and K_2 of $\mathbb{C}$, we say that K_1 surrounds K_2 if K_2 is included in a bounded component of $\mathbb{C} \setminus K_1$. The following argument shows that $J(f_1)$ does not surround $f_2^{-1}(J(f_1))$: Otherwise, we have $f_2^{-1}(J(f_1)) \subset \text{Int} K(f_1)$, which implies that $f_2^{-1}(K(f_1)) \subset K(f_1)$ (here we use that $f_2^{-1}(K(f_1))$ containing $f_2^{-1}(J(f_1))$ is connected and that $\widehat{\mathbb{C}} \setminus K(f_1)$ is a connected component of $\widehat{\mathbb{C}} \setminus J(f_1)$). However, $f_2^{-1}(K(f_1)) \subset K(f_1)$ implies that

$J(f_2) \subset K(f_1)$ contradicting (2). We have thus shown that $f_2^{-1}(J(f_1))$ surrounds $J(f_1)$. Consequently, we have that $f_2(J(f_1)) \subset K(f_1)$, so $f_2(K(f_1)) \subset K(f_1)$. Since $f_2(J(f_1)) \cap J(f_1) = \varnothing$, it follows that $f_2(K(f_1)) \subset \operatorname{Int} K(f_1)$. Combining with the fact that $f_1(\operatorname{Int} K(f_1)) \subset \operatorname{Int} K(f_1)$, we obtain that $\operatorname{Int} K(f_1) \subset F(G)$ by Montel's Theorem. We have thus shown that $f_2(K(f_1)) \subset \operatorname{Int} K(f_1) \subset F(G)$.

Our first observation is that PB-OSC implies the existence of an inducing structure.

Lemma 7.16. *If $G = \langle f_1, f_2 \rangle$ satisfies PB-OSC, then $\{f_1, f_2\}$ has an inducing structure with respect to $I_1 = \{1\}$ and $I_2 = \{2\}$. Moreover, there exists a G-forward invariant compact subset of $F(G)$. Furthermore, we have $J(G) = J_{\mathrm{pre}}(G)$.*

Proof. We verify that $G = \langle f_1, f_2 \rangle$ has an inducing structure. Let $G_i = \langle f_i \rangle$, for $i = 1, 2$, let $H_0 := \{f_2\} \cup \{f_2 \circ f_1^r : r \in \mathbb{N}\}$ and let $H = \langle H_0 \rangle$. Set $L := f_2(K(f_1)) \cup \operatorname{CV}(f_2)$ and note that we have shown above that L is an H-forward invariant compact subset of $F(G)$. Moreover, we have shown that $P(H) \subset L$, which implies that $P(G_2) \subset P(H) \subset L$. Furthermore, we have that $\operatorname{CV}(f_1) \setminus \{\infty\} \subset K(f_1)$. Hence, $f_2\left(\overline{\bigcup_{n \in \mathbb{N}} f_1^n(\operatorname{CV}(f_1) \setminus \{\infty\})}\right) \subset f_2(K(f_1)) \subset L$. Thus, $f_2(P(G_1)) \subset L$. We have thus shown that G has an inducing structure. To finish the proof, note that since G is a finitely generated polynomial semigroup, we have $\infty \in F(G)$ and that $\{\infty\}$ is G-forward invariant. Consequently, by Lemma 7.10, we have

$$J(G) \subset J_{\mathrm{pre}}(H) \cup \bigcup_{g \in G} g^{-1}(J(f_1)) \subset J_{\mathrm{pre}}(G).$$

$\square$

The main result of this section is the following corollary of Theorem 7.14.

Corollary 7.17. *If $G = \langle f_1, f_2 \rangle$ satisfies PB-OSC, then we have $\dim_H(J(G)) = \max\{s(G), \dim_H(J(f_1))\}$. Moreover, all assertions in (1)(2)(3)(5) of Theorem 7.14 hold, where $I_1 = \{1\}, I_2 = \{2\}$.*

8. REMARKS ON THE CONE CONDITION

We comment on the cone condition used in the context of conformal iterated function systems ([MU96]).

Remark 8.1. For the results of this paper, the cone condition is not needed. We have seen in Section 2 that there are many examples of rational semigroups which do not satisfy the cone condition, and for which our results can be applied.

In [MU96, Theorem 3.15] it is proved that, for the Hausdorff dimension of the limit set $J(\Phi)$ of an infinitely generated conformal iterated function system Φ satisfying the cone condition, we have

$$(8.1) \qquad \dim_H J(\Phi) = \inf\{\delta : P(\delta) < 0\} = \sup_{\Phi_F}\{J(\Phi_F)\}.$$

Here, P refers to the associated pressure function and Φ_F runs over all finitely generated subsystems of Φ.

Remark 8.2. By the methods employed in the proof of Theorem 6.5, one can show that (8.1) holds, even if the cone condition is not satisfied. Instead of the cone condition (2.7) in [MU96], we need to assume that $|\phi_i'(x)| \leq s$, for each $x \in X$ in the notation of [MU96]. Since the upper bound for the Hausdorff dimension is straightforward, we only comment on the lower bound of the Hausdorff dimension. Since the pressure satisfies an exhaustion principle (see [MU03, Theorem 2.15] or Proposition 5.3 (3)) it suffices to verify (8.1) for finitely generated conformal iterated function systems, which can be obtained by extending the proof of [Fal03, Theorem 4.3] via the bounded distortion property of the conformal iterated function system. For finitely generated expanding rational semigroups, the dimension formula in (8.1) was proved in [Sum05].

9. Acknowledgements

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