Cooperation principle, stability and bifurcation in random complex dynamics *

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Abstract

We investigate the random dynamics of rational maps and the dynamics of semigroups of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. We show that regarding random complex dynamics of polynomials, generically, the chaos of the averaged system disappears at any point in $\hat{\mathbb{C}}$, due to the automatic cooperation of the generators. We investigate the iteration and spectral properties of transition operators acting on the space of (Hölder) continuous functions on $\hat{\mathbb{C}}$. We also investigate the stability and bifurcation of random complex dynamics. We show that the set of stable systems is open and dense in the space of random dynamical systems of polynomials. Moreover, we prove that for a stable system, there exist only finitely many minimal sets, each minimal set is attracting, and the orbit of a Hölder continuous function on $\hat{\mathbb{C}}$ under the transition operator tends exponentially fast to the finite-dimensional space U of finite linear combinations of unitary eigenvectors of the transition operator. Combining this with the perturbation theory for linear operators, we obtain that for a stable system constructed by a finite family of rational maps, the projection to the space U depends real-analytically on the probability parameters. By taking a partial derivative of the function of probability of tending to a minimal set with respect to a probability parameter, we introduce a complex analogue of the Takagi function, which is a new concept.

1 Introduction

In this paper, we investigate the independent and identically-distributed (i.i.d.) random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of rational semigroups (i.e., semigroups of non-constant rational maps where the semigroup operation is functional composition) on $\hat{\mathbb{C}}$.

One motivation for research in complex dynamical systems is to describe some mathematical models on ethology. For example, the behavior of the population of a certain species can be described by the dynamical system associated with iteration of a polynomial $f(z) = az(1-z)$ (cf. [9]). However, when there is a change in the natural environment, some species have several strategies to survive in nature. From this point of view, it is very natural and important not only to consider the dynamics of iteration, where the same survival strategy (i.e., function) is repeatedly applied, but also to consider random dynamics, where a new strategy might be applied at each time step. Another motivation for research in complex dynamics is Newton's method to find a root of a complex polynomial, which often is expressed as the dynamics of a rational map g on $\hat{\mathbb{C}}$ with $\deg(g) \geq 2$, where $\deg(g)$ denotes the degree of g. We sometimes use computers to analyze such

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dynamics, and since we have some errors at each step of the calculation in the computers, it is quite natural to investigate the random dynamics of rational maps. In various fields, we have many mathematical models which are described by the dynamical systems associated with polynomial or rational maps. For each model, it is natural and important to consider a randomized model, since we always have some kind of noise or random terms in nature. The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([10]). They mainly investigated random dynamics generated by small perturbations of a single rational map. For research on random complex dynamics of quadratic polynomials, see [4, 5, 6, 7, 8, 11]. For research on random dynamics of polynomials (of general degrees), see the author's works [30, 28, 29, 31, 32].

In order to investigate random complex dynamics, it is very natural to study the dynamics of associated rational semigroups. In fact, it is a very powerful tool to investigate random complex dynamics, since random complex dynamics and the dynamics of rational semigroups are related to each other very deeply. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([14]), who were interested in the role of the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([12]), who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ of a finitely generated rational semigroup $G = \langle h_1, \ldots, h_m \rangle$ has "backward self-similarity," i.e., $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ (see [25, Lemma 0.2]), the study of the dynamics of rational semigroups can be regarded as the study of "backward iterated function systems," and also as a generalization of the study of self-similar sets in fractal geometry. For recent work on the dynamics of rational semigroups, see the author's papers [24]–[32], and [22, 23, 33, 34].

In this paper, by combining several results from [31] and many new ideas, we investigate the random complex dynamics and the dynamics of rational semigroups. In the usual iteration dynamics of a single rational map g with $\deg(g) \geq 2$, we always have a non-empty chaotic part, i.e., in the Julia set $J(g)$ of g, which is a perfect set, we have sensitive initial values and dense orbits. Moreover, for any ball B with $B \cap J(g) \neq \emptyset$, $g^n(B)$ expands as $n \to \infty$. Regarding random complex dynamics, it is natural to ask the following question. Do we have a kind of "chaos" in the averaged system? Or do we have no chaos? How do many kinds of maps in the system interact? What can we say about stability and bifurcation? Since the chaotic phenomena hold even for a single rational map, one may expect that in random dynamics of rational maps, most systems would exhibit a great amount of chaos. However, it turns out that this is not true. One of the main purposes of this paper is to prove that for a generic system of random complex dynamics of polynomials, many kinds of maps in the system "automatically" cooperate so that they make the chaos of the averaged system disappear at any point in the phase space, even though the dynamics of each map in the system have a chaotic part (Theorems 1.5, 3.20). We call this phenomenon the "**cooperation principle**". Moreover, we prove that for a generic system, we have a kind of stability (see Theorems 1.7, 3.24). We remark that the chaos disappears in the C^0 "sense", but under certain conditions, the chaos remains in the C^β "sense", where C^β denotes the space of β-Hölder continuous functions with exponent $\beta \in (0, 1)$ (see Remark 1.11).

To introduce the main idea of this paper, we let G be a rational semigroup and denote by $F(G)$ the Fatou set of G, which is defined to be the maximal open subset of $\hat{\mathbb{C}}$ where G is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$. We call $J(G) := \hat{\mathbb{C}} \setminus F(G)$ the Julia set of G. The Julia set is backward invariant under each element $h \in G$, but might not be forward invariant. This is a difficulty of the theory of rational semigroups. Nevertheless, we utilize this as follows. The key to investigating random complex dynamics is to consider the following **kernel Julia set** of G, which is defined by $J_{\ker}(G) = \bigcap_{g \in G} g^{-1}(J(G))$. This is the largest forward invariant subset of $J(G)$ under the action of G. Note that if G is a group or if G is a commutative semigroup, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup G generated by a family of rational maps h with $\deg(h) \geq 2$, it may happen that $\emptyset = J_{\ker}(G) \neq J(G)$.

Let Rat be the space of all non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$, endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical

distance on $\hat{\mathbb{C}}$. Let Rat_+ be the space of all rational maps g with $\deg(g) \geq 2$. Let $\mathcal{P}$ be the space of all polynomial maps g with $\deg(g) \geq 2$. Let τ be a Borel probability measure on Rat with compact support. We consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \mathrm{Rat}$ according to τ. Thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ from x to A is defined as $p(x, A) = \tau(\{g \in \mathrm{Rat} \mid g(x) \in A\})$. Let G_τ be the rational semigroup generated by the support of τ. Let $C(\hat{\mathbb{C}})$ be the space of all complex-valued continuous functions on $\hat{\mathbb{C}}$ endowed with the supremum norm $\| \cdot \|_\infty$. Let M_τ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) = \int \varphi(g(z))d\tau(g)$. This M_τ is called the transition operator of the Markov process induced by τ. For a metric space X, let $\mathfrak{M}_1(X)$ be the space of all Borel probability measures on X endowed with the topology induced by weak convergence (thus $\mu_n \to \mu$ in $\mathfrak{M}_1(X)$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{R}$). Note that if X is a compact metric space, then $\mathfrak{M}_1(X)$ is compact and metrizable. For each $\tau \in \mathfrak{M}_1(X)$, we denote by $\operatorname{supp} \tau$ the topological support of τ. Let $\mathfrak{M}_{1,c}(X)$ be the space of all Borel probability measures τ on X such that $\operatorname{supp} \tau$ is compact. Let $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of M_τ. This M_τ^* can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$ (see Remark 2.14). We define the "Julia set" $J_{meas}(\tau)$ of the dynamics of M_τ^* as the set of all elements $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that for each neighborhood B of μ, $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}}$ is not equicontinuous on B (see Definition 2.11). For each sequence $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$, we denote by J_γ the set of non-equicontinuity of the sequence $\{\gamma_n \circ \cdots \circ \gamma_1\}_{n\in\mathbb{N}}$ with respect to the spherical distance on $\hat{\mathbb{C}}$. This J_γ is called the Julia set of γ. Let $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1((\mathrm{Rat})^{\mathbb{N}})$. For a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we denote by U_τ the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where an eigenvector is said to be unitary if the absolute value of the corresponding eigenvalue is equal to one. Moreover, we set $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$. For a metric space X, we denote by $\mathrm{Cpt}(X)$ the space of all non-empty compact subsets of X endowed with the Hausdorff metric. For a rational semigroup G, we say that a non-empty compact subset L of $\hat{\mathbb{C}}$ is a minimal set for $(G, \hat{\mathbb{C}})$ if L is minimal in $\{C \in \mathrm{Cpt}(\hat{\mathbb{C}}) \mid \forall g \in G, g(C) \subset C\}$ with respect to inclusion. Moreover, we set $\mathrm{Min}(G, \hat{\mathbb{C}}) := \{L \in \mathrm{Cpt}(\hat{\mathbb{C}}) \mid L \text{ is a minimal set for } (G, \hat{\mathbb{C}})\}$. For a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, let $S_\tau := \bigcup_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}})} L$. For a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, let $\Gamma_\tau := \operatorname{supp} \tau (\subset \mathrm{Rat})$. In [31], the following two theorems were obtained.

Theorem 1.1 (Cooperation Principle I, see Theorem 3.14 in [31]). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then $J_{meas}(\tau) = \emptyset$. Moreover, for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, the 2-dimensional Lebesgue measure of J_γ is equal to zero.*

Theorem 1.2 (Cooperation Principle II: Disappearance of Chaos, see Theorem 3.15 in [31]). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then the following (1)(2)(3) hold.*

(1) *There exists a direct sum decomposition $C(\hat{\mathbb{C}}) = U_\tau \oplus \mathcal{B}_{0,\tau}$. Moreover, $1 \leq \dim_{\mathbb{C}} U_\tau < \infty$ and $\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$. Furthermore, each element of U_τ is locally constant on $F(G_\tau)$. Therefore each element of U_τ is a continuous function on $\hat{\mathbb{C}}$ which varies only on the Julia set $J(G_\tau)$.*

(2) *For each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ with the following properties* (a) *and* (b). (a) *For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, there exists a number $\delta = \delta(z, \gamma) > 0$ such that $diam(\gamma_n \circ \cdots \circ \gamma_1(B(z, \delta))) \to 0$ as $n \to \infty$, where diam denotes the diameter with respect to the spherical distance on $\hat{\mathbb{C}}$, and $B(z, \delta)$ denotes the ball with center z and radius δ.* (b) *For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{A}_z$, $d(\gamma_n \circ \cdots \circ \gamma_1(z), S_\tau) \to 0$ as $n \to \infty$.*

(3) *We have $1 \leq \sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.*

Remark 1.3. If $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ and $\Gamma_\tau \cap \mathrm{Rat}_+ \neq \emptyset$, then $\sharp J(G_\tau) = \infty$.

Theorems 1.1 and 1.2 mean that if all the maps in the support of τ cooperate, the chaos of the averaged system disappears, even though the dynamics of each map of the system has a chaotic part. Moreover, Theorems 1.1 and 1.2 describe new phenomena which can hold in random complex dynamics but cannot hold in the usual iteration dynamics of a single $h \in \mathrm{Rat}_+$. For example, for any $h \in \mathrm{Rat}_+$, if we take a point $z \in J(h)$, where $J(h)$ denotes the Julia set of the semigroup generated by h, then the Dirac measure δ_z at z belongs to $J_{meas}(\delta_h)$, and for any ball B with $B \cap J(h) \neq \emptyset$, $h^n(B)$ expands as $n \to \infty$. Moreover, for any $h \in \mathrm{Rat}_+$, we have infinitely many minimal sets (periodic cycles) of h.

Considering these results, we have the following natural question: "When is the kernel Julia set empty?" In order to give several answers to this question, we say that a family $\{g_\lambda\}_{\lambda\in\Lambda}$ of rational (resp. polynomial) maps is a holomorphic family of rational (resp. polynomial) maps if Λ is a finite dimensional complex manifold and the map $(z, \lambda) \mapsto g_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$. In [31], the following result was proved. (Remark. In [31, Lemma 5.34, Definition 3.54-1], Λ should be connected.)

Theorem 1.4 (Cooperation Principle III, see Theorem 1.7 and Lemma 5.34 from [31])**.** *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that for each $z \in \mathbb{C}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of polynomial maps with $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \Gamma_\tau$ such that Λ is connected and $\lambda \mapsto g_\lambda(z)$ is non-constant on Λ. Then $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$ and all statements in Theorems 1.1 and 1.2 hold.*

In this paper, regarding the previous question, we prove the following very strong results. To state the results, we say that a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is **mean stable** if there exist non-empty open subsets U, V of $F(G_\tau)$ and a number $n \in \mathbb{N}$ such that all of the following (I)(II)(III) hold: (I) $\overline{V} \subset U$ and $\overline{U} \subset F(G_\tau)$. (II) For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_\tau^{\mathbb{N}}$, $\gamma_n \circ \cdots \circ \gamma_1(\overline{U}) \subset V$. (III) For each point $z \in \hat{\mathbb{C}}$, there exists an element $g \in G_\tau$ such that $g(z) \in U$. We remark that if $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable, then $J_{\ker}(G_\tau) = \emptyset$. Thus if $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then $J_{\ker}(G_\tau) = \emptyset$ and all statements in Theorems 1.1 and 1.2 hold. Note also that by using [19, Theorem 2.11], it is not so difficult to see that τ is mean stable if and only if $\sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}})) < \infty$ and each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is "attracting", i.e., there exists an open subset W_L of $F(G_\tau)$ with $L \subset W_L$ and an $\epsilon > 0$ such that for each $z \in W_L$ and for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_\tau^{\mathbb{N}}$, $d(\gamma_n \circ \cdots \circ \gamma_1(z), L) \to 0$ and $\mathrm{diam}(\gamma_n \circ \cdots \circ \gamma_1(B(z, \epsilon))) \to 0$ as $n \to \infty$ (see Remark 3.7). Therefore, if $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ is mean stable, then (1) $J_{meas}(\tau) = \emptyset$, (2) for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$, the 2-dimensional Lebesgue measure of J_γ is zero, (3) for each $z \in \hat{\mathbb{C}}$ there exists a Borel subset $\mathcal{C}_z$ of $(\mathrm{Rat}_+)^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ such that for each $\gamma \in \mathcal{C}_z$, $d(\gamma_n \circ \cdots \circ \gamma_1(z), S_\tau) \to 0$ as $n \to \infty$, (4) for $\tilde{\tau}$-a.e. $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat}_+)^{\mathbb{N}}$, for each $z \in \hat{\mathbb{C}} \setminus J_\gamma$, we have $d(\gamma_n \circ \cdots \circ \gamma_1(z), S_\tau) \to 0$ as $n \to \infty$, (5) S_τ is a finite union of "attracting minimal sets", (6)(negativity of Lyapunov exponents) there exists a constant $c < 0$ such that for each $z \in \hat{\mathbb{C}}$ there exists a Borel subset $\mathcal{D}_z$ of $(\mathrm{Rat}_+)^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{D}_z) = 1$ satisfying that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{D}_z$, we have $\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_n \circ \cdots \circ \gamma_1)_z\|_s \leq c$, where $\|D(\gamma_n \circ \cdots \circ \gamma_1)_z\|_s$ denotes the norm of the derivative of $\gamma_n \circ \cdots \circ \gamma_1$ at z with respect to the spherical metric, and (7) for the system generated by τ, there exists a stability (Theorem 1.7). Thus, in terms of averaged systems, the notion "mean stability" of random complex dynamics can be regarded as an analogy of "hyperbolicity" of the usual iteration dynamics of a single rational map. For a metric space (X, d), let $\mathcal{O}$ be the topology of $\mathfrak{M}_{1,c}(X)$ such that $\mu_n \to \mu$ in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$ as $n \to \infty$ if and only if (i) $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{C}$, and (ii) $\Gamma_{\mu_n} \to \Gamma_\mu$ with respect to the Hausdorff metric in the space $\mathrm{Cpt}(X)$. We say that a subset $\mathcal{Y}$ of Rat satisfies condition $(*)$ if $\mathcal{Y}$ is closed in Rat and at least one of the following (1) and (2) holds: (1) for each $(z_0, h_0) \in \hat{\mathbb{C}} \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of rational maps with $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$, such that, $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of λ_0. (2) $\mathcal{Y} \subset \mathcal{P}$ and for each $(z_0, h_0) \in \mathbb{C} \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of polynomial maps with $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$ such that $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of λ_0. For example, Rat, Rat_+, $\mathcal{P}$, and $\{z^d + c \mid c \in \mathbb{C}\}$ $(d \in \mathbb{N}, d \geq 2)$ satisfy condition $(*)$. Under these notations, we prove the following theorem.

Theorem 1.5 (Cooperation Principle IV, Density of Mean Stable Systems, see Theorem 3.20)**.** *Let $\mathcal{Y}$ be a subset of $\mathcal{P}$ satisfying condition $(*)$. Then, we have the following.*

(1) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable$\}$ is open and dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. Moreover, the set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid J_{\ker}(G_\tau) = \emptyset, J(G_\tau) \neq \emptyset\}$ contains $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable$\}$.*

(2) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable, $\sharp\Gamma_\tau < \infty\}$ is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.*

We remark that in the study of iteration of a single rational map, we have a very famous conjecture (HD conjecture, see [18, Conjecture 1.1]) which states that hyperbolic rational maps are dense in the space of rational maps. Theorem 1.5 solves this kind of problem (in terms of averaged systems) in the study of random dynamics of complex polynomials. We also prove the following result.

Theorem 1.6 (see Corollary 3.23)**.** *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. Then, the set*

$$\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable}\} \cup \{\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$$

is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

For the proofs of Theorems 1.5 and 1.6, we need to investigate and classify the minimal sets for $(\langle\Gamma\rangle, \hat{\mathbb{C}})$, where $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$, and $\langle\Gamma\rangle$ denotes the rational semigroup generated by Γ (thus $\langle\Gamma\rangle = \{g_{i_1} \circ \cdots \circ g_{i_n} \mid n \in \mathbb{N}, \forall g_{i_j} \in \Gamma\}$) (Lemmas 3.8,3.16). In particular, it is important to analyze the reason of instability for a non-attracting minimal set.

For each $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, let $T_{L,\tau}$ be the function of probability of tending to L. Namely, for each $z \in \hat{\mathbb{C}}$, we set $T_{L,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}} \mid d(\gamma_n \circ \cdots \circ \gamma_1(z), L) \to 0 \ (n \to \infty)\})$. We set $C(\hat{\mathbb{C}})^* := \{\rho : C(\hat{\mathbb{C}}) \to \mathbb{C} \mid \rho \text{ is linear and continuous}\}$ endowed with the weak*-topology. We prove the following stability result.

Theorem 1.7 (Cooperation Principle V, $\mathcal{O}$-Stability for Mean Stable Systems, see Theorem 3.24)**.** *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be mean stable. Suppose $J(G_\tau) \neq \emptyset$. Then there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ such that all of the following hold.*

(1) *For each $\nu \in \Omega$, ν is mean stable, $\sharp(J(G_\nu)) \geq 3$, and $\sharp(\mathrm{Min}(G_\nu, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}}))$.*

(2) *For each $\nu \in \Omega$, $\dim_{\mathbb{C}}(U_\nu) = \dim_{\mathbb{C}}(U_\tau) \geq 1$.*

(3) *The map $\nu \mapsto \pi_\nu$ and $\nu \mapsto U_\nu$ are continuous on Ω, where $\pi_\nu : C(\hat{\mathbb{C}}) \to U_\nu$ denotes the canonical projection (see Theorem 1.2). More precisely, for each $\nu \in \Omega$, there exists a family $\{\varphi_{j,\nu}\}_{j=1}^q$ of unitary eigenvectors of $M_\nu : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where $q = \dim_{\mathbb{C}}(U_\tau)$, and a finite family $\{\rho_{j,\nu}\}_{j=1}^q$ in $C(\hat{\mathbb{C}})^*$ such that all of the following hold.*

   (a) *$\{\varphi_{j,\nu}\}_{j=1}^q$ is a basis of U_ν.*

   (b) *For each j, $\nu \mapsto \varphi_{j,\nu} \in C(\hat{\mathbb{C}})$ is continuous on Ω.*

   (c) *For each j, $\nu \mapsto \rho_{j,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on Ω.*

   (d) *For each (i,j) and each $\nu \in \Omega$, $\rho_{i,\nu}(\varphi_{j,\nu}) = \delta_{ij}$.*

   (e) *For each $\nu \in \Omega$ and each $\varphi \in C(\hat{\mathbb{C}})$, $\pi_\nu(\varphi) = \sum_{j=1}^q \rho_{j,\nu}(\varphi) \cdot \varphi_{j,\nu}$.*

(4) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, there exists a continuous map $\nu \mapsto L_\nu \in \mathrm{Min}(G_\nu, \hat{\mathbb{C}}) \subset \mathrm{Cpt}(\hat{\mathbb{C}})$ on Ω with respect to the Hausdorff metric such that $L_\tau = L$. Moreover, for each $\nu \in \Omega$, $\{L_\nu\}_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} = \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$. Moreover, for each $\nu \in \Omega$ and for each $L, L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \neq L'$, we have $L_\nu \cap L'_\nu = \emptyset$. Furthermore, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, the map $\nu \mapsto T_{L_\nu,\nu} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is continuous on Ω.*

By applying these results, we give a characterization of mean stability (Theorem 3.25).

We remark that if $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ is mean stable and $\sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}})) > 1$, then the averaged system of τ is stable (Theorem 1.7) and the system also has a kind of variety. Thus such a τ can describe a stable system which does not lose variety. This fact (with Theorems 1.5, 1.1, 1.2) might be useful when we consider mathematical modeling in various fields.

Let $\mathcal{Y}$ be a subset of Rat satisfying $(*)$. Let $\{\mu_t\}_{t\in[0,1]}$ be a continuous family in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. We consider the bifurcation of $\{M_{\mu_t}\}_{t\in[0,1]}$ and $\{G_{\mu_t}\}_{t\in[0,1]}$. We prove the following result.

Theorem 1.8 (Bifurcation: see Theorem 3.26 and Lemmas 3.8, 3.16)**.** *Let $\mathcal{Y}$ be a subset of* Rat_+ *satisfying condition $(*)$. For each $t \in [0,1]$, let μ_t be an element of $\mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that all of the following conditions* (1)–(4) *hold.*

(1) *$t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ is continuous on $[0,1]$.*

(2) *If $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, then $\Gamma_{\mu_{t_1}} \subset \mathrm{int}(\Gamma_{\mu_{t_2}})$ with respect to the topology of $\mathcal{Y}$.*

(3) *$\mathrm{int}(\Gamma_{\mu_0}) \neq \emptyset$ and $F(G_{\mu_1}) \neq \emptyset$.*

(4) *$\sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) \neq \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$.*

Let $B := \{t \in [0,1) \mid \mu_t$ is not mean stable$\}$. Then, we have the following.

(a) *For each $t \in [0,1]$, $J_{\mathrm{ker}}(G_{\mu_t}) = \emptyset$ and $\sharp J(G_{\mu_t}) \geq 3$, and all statements in* [31, Theorem 3.15] *and Theorems 1.1,1.2 (with $\tau = \mu_t$) hold.*

(b) *We have $1 \leq \sharp B \leq \sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) - \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})) < \infty$. Moreover, for each $t \in B$, either* (i) *there exists an element $L \in \mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})$, a point $z \in L$, and an element $g \in \partial\Gamma_{\mu_t} (\subset \mathcal{Y})$ such that $z \in L \cap J(G_{\mu_t})$ and $g(z) \in L \cap J(G_{\mu_t})$, or* (ii) *there exist an element $L \in \mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})$, a point $z \in L$, and finitely many elements $g_1, \ldots, g_r \in \partial\Gamma_{\mu_t}$ such that $L \subset F(G_{\mu_t})$ and z belongs to a Siegel disk or a Hermann ring of $g_r \circ \cdots \circ g_1$.*

(c) *For each $s \in (0,1]$ there exists a number $t_s \in (0,s)$ such that for each $t \in [t_s, s]$, $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_{\mu_s}, \hat{\mathbb{C}}))$.*

In Example 3.27, an example to which we can apply the above theorem is given.

We also investigate the spectral properties of M_τ acting on Hölder continuous functions on $\hat{\mathbb{C}}$ and stability (see subsection 3.2). For each $\alpha \in (0,1)$, let
$C^\alpha(\hat{\mathbb{C}}) := \{\varphi \in C(\hat{\mathbb{C}}) \mid \sup_{x,y\in\hat{\mathbb{C}}, x\neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha < \infty\}$ be the Banach space of all complex-valued α-Hölder continuous functions on $\hat{\mathbb{C}}$ endowed with the α-Hölder norm $\|\cdot\|_\alpha$, where $\|\varphi\|_\alpha := \sup_{z\in\hat{\mathbb{C}}} |\varphi(z)| + \sup_{x,y\in\hat{\mathbb{C}}, x\neq y} |\varphi(x) - \varphi(y)|/d(x,y)^\alpha$ for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$.

Regarding the space U_τ, we prove the following.

Theorem 1.9. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\mathrm{ker}}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, there exists an $\alpha \in (0,1)$ such that $U_\tau \subset C^\alpha(\hat{\mathbb{C}})$. Moreover, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, the function $T_{L,\tau} : \hat{\mathbb{C}} \to [0,1]$ of probability of tending to L belongs to $C^\alpha(\hat{\mathbb{C}})$.*

Thus each element of U_τ has a kind of regularity. For the proof of Theorem 1.9, the result "each element of U_τ is locally constant on $F(G_\tau)$" (Theorem 1.2 (1)) is used.

If $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then by [31, Proposition 3.65], we have $S_\tau \subset F(G_\tau)$. From this point of view, we consider the situation that $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ satisfies $J_{\mathrm{ker}}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Under this situation, we have several very strong results. Note that there exists an example of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau < \infty$ such that $J_{\mathrm{ker}}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, $S_\tau \subset F(G_\tau)$, and τ is not mean stable (see Example 6.3).

Theorem 1.10 (Cooperation Principle VI, Exponential Rate of Convergence: see Theorem 3.30)**.** *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Then, there exists a constant $\alpha \in (0,1)$, a constant $\lambda \in (0,1)$, and a constant $C > 0$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi\|_\alpha$ for each $n \in \mathbb{N}$.*

For the proof of Theorem 1.10, we need some careful arguments on the hyperbolic metric on each connected component of $F(G_\tau)$.

We remark that in 1983, by numerical experiments, K. Matsumoto and I. Tsuda ([17]) observed that if we add some uniform noise to the dynamical system associated with iteration of a chaotic map on the unit interval $[0,1]$, then under certain conditions, the quantities which represent chaos (e.g., entropy, Lyapunov exponent, etc.) decrease. More precisely, they observed that the entropy decreases and the Lyapunov exponent turns negative. They called this phenomenon "noise-induced order", and many physicists have investigated it by numerical experiments, although there has been only a few mathematical supports for it. In this paper, we deal with not only (i.i.d.) random dynamical systems which are constructed by adding relatively small noises to usual dynamical systems but also more general (i.i.d.) random dynamical systems. In this paper, we study "randomness-induced phenomena" (the phenomena in general random dynamical systems which cannot hold in the usual iteration dynamics) which include "noise-induced phenomena".

Remark 1.11. Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be mean stable and suppose $J(G_\tau) \neq \emptyset$. Then by [31, Theorem 3.15], the chaos of the averaged system of τ disappears (Cooperation Principle II), and by Theorem 1.10, there exists an $\alpha_0 \in (0,1)$ such that for each $\alpha \in (0, \alpha_0)$ the action of $\{M_\tau^n\}_{n\in\mathbb{N}}$ on $C^\alpha(\hat{\mathbb{C}})$ is well-behaved. However, [31, Theorem 3.82] tells us that under certain conditions on a mean stable τ, there exists a $\beta \in (0,1)$ such that any non-constant element $\varphi \in U_\tau$ does not belong to $C^\beta(\hat{\mathbb{C}})$ (note: for the proof of this result, we use the Birkhoff ergodic theorem and potential theory). Hence, there exists an element $\psi \in C^\beta(\hat{\mathbb{C}})$ such that $\|M_\tau^n(\psi)\|_\beta \to \infty$ as $n \to \infty$. Therefore, the action of $\{M_\tau^n\}_{n\in\mathbb{N}}$ on $C^\beta(\hat{\mathbb{C}})$ is not well behaved. In other words, regarding the dynamics of the averaged system of τ, there still exists a kind of chaos (or complexity) in the space $(C^\beta(\hat{\mathbb{C}}), \|\cdot\|_\beta)$ even though there exists no chaos in the space $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$. From this point of view, in the field of random dynamics, we have a kind of gradation or stratification between chaos and non-chaos. It may be nice to investigate and reconsider the chaos theory and mathematical modeling from this point of view.

Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. We now consider the spectrum $\mathrm{Spec}_\alpha(M_\tau)$ of $M_\tau : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$. Note that since the family $\mathrm{supp}\,\tau$ in Rat is uniformly Lipschitz continuous on $\hat{\mathbb{C}}$, we have $M_\tau(C^\alpha(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$ for each $\alpha \in (0,1)$. If τ satisfies the assumptions of Theorem 1.10, then from Theorem 1.10, denoting by $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ the set of unitary eigenvalues of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ (note: by Theorem 1.9, $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}}) \subset \mathrm{Spec}_\alpha(M_\tau)$ for some $\alpha \in (0,1)$), we can show that the distance between $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ and $\mathrm{Spec}_\alpha(M_\tau) \setminus \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ is positive.

Theorem 1.12 (see Theorem 3.31)**.** *Under the assumptions of Theorem 1.10,* $\mathrm{Spec}_\alpha(M_\tau) \subset \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$*, where $\alpha \in (0,1)$ and $\lambda \in (0,1)$ are the constants in Theorem 1.10.*

Combining Theorem 1.12 and perturbation theory for linear operators ([16]), we obtain the following theorem. We remark that even if $g_n \to g$ in Rat, for a $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|\varphi \circ g_n - \varphi \circ g\|_\alpha$ does not tend to zero in general. Thus when we perturb generators $\{h_j\}$ of Γ_τ, we cannot apply perturbation theory for M_τ on $C^\alpha(\hat{\mathbb{C}})$. However, by using the method in the proofs of [33, Lemmas 5.1, 5.2], it is easy to see that for each $\alpha \in (0,1)$, for a fixed generator system $(h_1, \ldots, h_m) \in (\mathrm{Rat})^m$, the map $(p_1, \ldots, p_m) \in \mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \mapsto M_{\sum_{j=1}^m p_j \delta_{h_j}} \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm. Thus we can apply perturbation theory for the above real-analytic family of operators.

Theorem 1.13 (see Theorem 3.32)**.** *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h_1, \ldots, h_m \in \mathrm{Rat}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Suppose that $J_{\ker}(G) = \emptyset, J(G) \neq \emptyset$ and $\cup_{L \in \mathrm{Min}(G, \hat{\mathbb{C}})} L \subset F(G)$. Let $\mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \ldots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \ldots, a_m) \in \mathcal{W}_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Then we have all of the following.*

(1) *For each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ and an open neighborhood of V_b of b in $\mathcal{W}_m$ such that for each $a \in V_b$, we have $LS(\mathcal{U}_{f,\tau_a}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$, $\pi_{\tau_a}(C^\alpha(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$ and $(\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$, and such that the map $a \mapsto (\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic in V_b.*

(2) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Then, for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that the map $a \mapsto T_{L,\tau_a} \in (C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$ is real-analytic in an open neighborhood of b in $\mathcal{W}_m$. Moreover, the map $a \mapsto T_{L,\tau_a} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is real-analytic in $\mathcal{W}_m$. In particular, for each $z \in \hat{\mathbb{C}}$, the map $a \mapsto T_{L,\tau_a}(z)$ is real-analytic in $\mathcal{W}_m$. Furthermore, for any multi-index $n = (n_1, \ldots, n_{m-1}) \in (\mathbb{N} \cup \{0\})^{m-1}$ and for any $b \in \mathcal{W}_m$, the function $z \mapsto [(\frac{\partial}{\partial a_1})^{n_1} \cdots (\frac{\partial}{\partial a_{m-1}})^{n_{m-1}} (T_{L,\tau_a}(z))]|_{a=b}$ is Hölder continuous on $\hat{\mathbb{C}}$ and is locally constant on $F(G)$.*

(3) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and let $b \in \mathcal{W}_m$. For each $i = 1, \ldots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,b}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ and let $\zeta_{i,b}(z) := T_{L,\tau_b}(h_i(z)) - T_{L,\tau_b}(h_m(z))$. Then, $\psi_{i,b}$ is the unique solution of the functional equation $(I - M_{\tau_b})(\psi) = \zeta_{i,b}, \psi|_{S_{\tau_b}} = 0, \psi \in C(\hat{\mathbb{C}})$, where I denotes the identity map. Moreover, there exists a number $\alpha \in (0,1)$ such that $\psi_{i,b} = \sum_{n=0}^\infty M_{\tau_b}^n(\zeta_{i,b})$ in $(C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$.*

Remark 1.14 (see also Example 6.2)**.** (1) By Theorem 1.5-(2), the set of all finite subsets Γ of $\mathcal{P}$ satisfying the assumption "$J_{\ker}(\langle\Gamma\rangle) = \emptyset, J(\langle\Gamma\rangle) \neq \emptyset$ and $\cup_{L \in \mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})} L \subset F(\langle\Gamma\rangle)$" of Theorem 1.13 is dense in $\mathrm{Cpt}(\mathcal{P})$ with respect to the Hausdorff metric. (2) The function T_{L,τ_a} defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of Lebesgue singular functions or the devil's staircase, and the function $z \mapsto \psi_{i,b}(z) = [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ defined on $\hat{\mathbb{C}}$ can be regarded as a **complex analogue of the Takagi function** $\mathcal{T}(x) := \sum_{n=0}^\infty \frac{1}{2^n} \min_{m \in \mathbb{Z}} |2^n x - m|$ where $x \in \mathbb{R}$. (The Takagi function $\mathcal{T}$ has many interesting properties. For example, it is continuous but nowhere differentiable on $\mathbb{R}$. There are many studies on the Takagi function. See [35, 13, 20, 1].) In order to explain the details, let $g_1(x) := 2x, g_2(x) := 2(x-1) + 1$ $(x \in \mathbb{R})$ and let $0 < a < 1$ be a constant. We consider the random dynamical system on $\mathbb{R}$ such that at every step we choose the map g_1 with probability a and the map g_2 with probability $1 - a$. Let $T_{+\infty,a}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then, as the author of this paper pointed out in [31], we can see that for each $a \in (0,1)$ with $a \neq 1/2$, the function $T_{+\infty,a}|_{[0,1]} : [0,1] \to [0,1]$ is equal to **Lebesgue's singular function** L_a with respect to the parameter a. (For the definition of L_a, see [35]. See Figure 1, [31].) The author found that, in a similar way, many singular functions on $\mathbb{R}$ (including **the devil's staircase**) can be regarded as the functions of probability of tending to $+\infty$ with respect to some random dynamical systems on $\mathbb{R}$ ([31, 29]). It is well-known (see [35, 20]) that for each $x \in [0,1]$, $a \mapsto L_a(x)$ is real-analytic in $(0,1)$, and that $x \mapsto (1/2)[\frac{\partial}{\partial a}(L_a(x))]|_{a=1/2}$ is equal to the Takagi function restricted to $[0,1]$ (Figure 1). From this point of view, the function $z \mapsto \psi_{i,b}(z)$ defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of the Takagi function. This is a new concept introduced in this paper. In fact, the author found that by using random dynamical systems and the methods in this paper, we can find many analogues of the devil's staircase, Lebesgue's singular functions and the Takagi function (on $[-\infty, \infty], \hat{\mathbb{C}}$ etc.). For the figure of the graph of $\psi_{i,b}$, see Example 6.2 and Figure 4. Some results on the (non-)differentiability of T_{L,τ_a} were obtained in [31].

In this paper, we present a result on the non-differentiability of the function $\psi_{i,b}(z)$ of Theorem 1.13 at points in $J(G_\tau)$ (Theorem 3.40), which is obtained by the application of the Birkhoff ergodic theorem, potential theory and some results from [31].

Figure 1: From left to right, the graphs of the devil's staircase, Lebesgue's singular function and the Takagi function. The Takagi function is continuous but nowhere differentiable on $\mathbb{R}$.

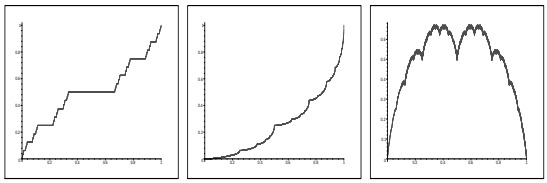


Figure 2: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$. The planar postcritical set of G is bounded in $\mathbb{C}$, $J(G)$ is not connected and G is hyperbolic ([28]). (h_1, h_2) satisfies the open set condition and $\dim_H(J(G)) < 2$ ([33]). Moreover, for each connected component J of $J(G)$, $\exists ! \gamma \in \{h_1, h_2\}^{\mathbb{N}}$ s.t. $J = J_\gamma$. For almost every $\gamma \in \{h_1, h_2\}^{\mathbb{N}}$ with respect to a Bernoulli measure, J_γ is a simple closed curve but not a quasicircle, and the basin A_γ of infinity for the sequence γ is a John domain ([28]).

Figure 3: The graph of $z \mapsto T_{\infty, \tau_{(1/2,1/2)}}(z)$, where, letting (h_1, h_2) be the element in Figure 2, we set $\tau_a := \sum_{j=1}^{2} a_j \delta_{h_j}$ for each $a \in \mathcal{W}_2$. τ_a is mean stable. A devil's coliseum (a complex analogue of the devil's staircase or Lebesgue's singular functions). This function is continuous on $\mathbb{C}$ and the set of varying points is equal to $J(G)$ in Figure 2.

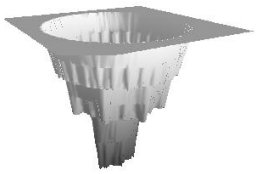
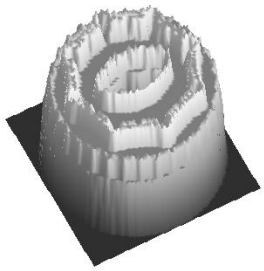


Figure 4: The graph of $z \mapsto [(\partial T_{\infty,\tau_a}(z)/\partial a_1)]|_{a_1=1/2}$, where, τ_a is the element in Figure 3. A **complex analogue of the Takagi function**. This function is continuous on $\mathbb{C}$ and the set of varying points is included in $J(G)$ in Figure 2.



Combining these results, we can say that for a generic $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, the chaos of the averaged system associated with τ disappears, the Lyapunov exponents are negative, $\sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}})) < \infty$, each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is attracting, there exists a stability on U_τ and $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ in a neighborhood of τ in $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$, and there exists an $\alpha \in (0,1)$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $M_\tau^n(\varphi)$ tends to the space U_τ exponentially fast. Note that these phenomena can hold in random complex dynamics but cannot hold in the usual iteration dynamics of a single rational map h with $\deg(h) \geq 2$. We systematically investigate these phenomena and their mechanisms. As the author mentioned in Remark 1.11, these results will stimulate the chaos theory and the mathematical modeling in various fields, and will lead us to a new interesting field. Moreover, these results are related to fractal geometry very deeply.

In section 2, we give some basic notations and definitions. In section 3, we present the main results of this paper. In section 4, we give some basic tools to prove the main results. In section 5, we give the proofs of the main results. In section 6, we present several examples which describe the main results.

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2 Preliminaries

In this section, we give some fundamental notations and definitions.

Notation: Let (X, d) be a metric space, A a subset of X, and $r > 0$. We set $B(A, r) := \{z \in X \mid d(z, A) < r\}$. Moreover, for a subset C of $\mathbb{C}$, we set $D(C, r) := \{z \in \mathbb{C} \mid \inf_{a \in C} |z - a| < r\}$. Moreover, for any topological space Y and for any subset A of Y, we denote by $\mathrm{int}(A)$ the set of all interior points of A. We denote by $\mathrm{Con}(A)$ the set of all connected components of A.

Definition 2.1. Let Y be a metric space. We set $C(Y) := \{\varphi : Y \to \mathbb{C} \mid \varphi \text{ is continuous }\}$. When Y is compact, we endow $C(Y)$ with the supremum norm $\|\cdot\|_\infty$. Moreover, for a subset $\mathcal{F}$ of $C(Y)$, we set $\mathcal{F}_{nc} := \{\varphi \in \mathcal{F} \mid \varphi \text{ is not constant}\}$.

Definition 2.2. A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$ with the semigroup operation being functional composition([14, 12]). A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps. We set $\mathrm{Rat} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Moreover, we set $\mathrm{Rat}_+ := \{h \in \mathrm{Rat} \mid \deg(h) \geq 2\}$ endowed with the relative topology from Rat.

Furthermore, we set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Remark 2.3 ([2])**.** For each $d \in \mathbb{N}$, let $\mathrm{Rat}_d := \{g \in \mathrm{Rat} \mid \deg(g) = d\}$ and for each $d \in \mathbb{N}$ with $d \geq 2$, let $\mathcal{P}_d := \{g \in \mathcal{P} \mid \deg(g) = d\}$. Then for each d, Rat_d (resp. $\mathcal{P}_d$) is a connected component of Rat (resp. $\mathcal{P}$). Moreover, Rat_d (resp. $\mathcal{P}_d$) is open and closed in Rat (resp. $\mathcal{P}$) and is a finite dimensional complex manifold. Furthermore, $h_n \to h$ in $\mathcal{P}$ if and only if $\deg(h_n) = \deg(h)$ for each large n and the coefficients of h_n tend to the coefficients of h appropriately as $n \to \infty$.

Definition 2.4. Let G be a rational semigroup. The **Fatou set** of G is defined to be $F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{g|_U : U \to \hat{\mathbb{C}}\}_{g \in G} \text{ is equicontinuous on } U\}$. (For the definition of equicontinuity, see [2].) The **Julia set** of G is defined to be $J(G) := \hat{\mathbb{C}} \setminus F(G)$. If G is generated by $\{g_i\}_i$, then we write $G = \langle g_1, g_2, \ldots \rangle$. If G is generated by a subset Γ of Rat, then we write $G = \langle \Gamma \rangle$. For finitely many elements $g_1, \ldots, g_m \in \mathrm{Rat}$, we set $F(g_1, \ldots, g_m) := F(\langle g_1, \ldots, g_m \rangle)$ and $J(g_1, \ldots, g_m) := J(\langle g_1, \ldots, g_m \rangle)$. For a subset A of $\hat{\mathbb{C}}$, we set $G(A) := \bigcup_{g \in G} g(A)$ and $G^{-1}(A) := \bigcup_{g \in G} g^{-1}(A)$. We set $G^* := G \cup \{\mathrm{Id}\}$, where Id denotes the identity map.

Lemma 2.5 ([14, 12])**.** *Let G be a rational semigroup. Then, for each $h \in G$, $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that the equality does not hold in general.*

The following is the key to investigating random complex dynamics.

Definition 2.6. Let G be a rational semigroup. We set $J_{\mathrm{ker}}(G) := \bigcap_{g \in G} g^{-1}(J(G))$. This is called the **kernel Julia set** of G.

Remark 2.7. Let G be a rational semigroup. (1) $J_{\mathrm{ker}}(G)$ is a compact subset of $J(G)$. (2) For each $h \in G$, $h(J_{\mathrm{ker}}(G)) \subset J_{\mathrm{ker}}(G)$. (3) If G is a rational semigroup and if $F(G) \neq \emptyset$, then $\mathrm{int}(J_{\mathrm{ker}}(G)) = \emptyset$. (4) If G is generated by a single map or if G is a group, then $J_{\mathrm{ker}}(G) = J(G)$. However, for a general rational semigroup G, it may happen that $\emptyset = J_{\mathrm{ker}}(G) \neq J(G)$ (see [31]).

It is sometimes important to investigate the dynamics of sequences of maps.

Definition 2.8. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}}$ and each $m, n \in \mathbb{N}$ with $m \geq n$, we set $\gamma_{m,n} = \gamma_m \circ \cdots \circ \gamma_n$ and we set

$$F_\gamma := \{z \in \hat{\mathbb{C}} \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{\gamma_{n,1}\}_{n \in \mathbb{N}} \text{ is equicontinuous on } U\}$$

and $J_\gamma := \hat{\mathbb{C}} \setminus F_\gamma$. The set F_γ is called the **Fatou set** of the sequence γ and the set J_γ is called the **Julia set** of the sequence γ.

Remark 2.9. Let $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$. Then by [2, Theorem 2.8.2], $J_\gamma \neq \emptyset$. Moreover, if Γ is a non-empty compact subset of Rat_+ and $\gamma \in \Gamma^{\mathbb{N}}$, then by [25], J_γ is a perfect set and J_γ has uncountably many points.

We now give some notations on random dynamics.

Definition 2.10. For a metric space Y, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on Y endowed with the topology such that $\mu_n \to \mu$ in $\mathfrak{M}_1(Y)$ if and only if for each bounded continuous function $\varphi : Y \to \mathbb{C}$, $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$. Note that if Y is a compact metric space, then $\mathfrak{M}_1(Y)$ is a compact metric space with the metric $d_0(\mu_1, \mu_2) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}{1 + |\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}$, where $\{\phi_j\}_{j \in \mathbb{N}}$ is a dense subset of $C(Y)$. Moreover, for each $\tau \in \mathfrak{M}_1(Y)$, we set $\mathrm{supp}\, \tau := \{z \in Y \mid \forall \text{ neighborhood } U \text{ of } z, \ \tau(U) > 0\}$. Note that $\mathrm{supp}\, \tau$ is a closed subset of Y. Furthermore, we set $\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \mathrm{supp}\, \tau \text{ is compact}\}$.

For a complex Banach space $\mathcal{B}$, we denote by $\mathcal{B}^*$ the space of all continuous complex linear functionals $\rho : \mathcal{B} \to \mathbb{C}$, endowed with the weak* topology.

For any $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, we will consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $g \in \mathrm{Rat}$ according to τ (thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ from x to A is defined as $p(x, A) = \tau(\{g \in \mathrm{Rat} \mid g(x) \in A\})$).

Definition 2.11. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$.

1. We set $\Gamma_\tau := \mathrm{supp}\, \tau$ (thus Γ_τ is a closed subset of Rat). Moreover, we set $X_\tau := (\Gamma_\tau)^{\mathbb{N}}$ $(= \{\gamma = (\gamma_1, \gamma_2, \ldots) \mid \gamma_j \in \Gamma_\tau \ (\forall j)\})$ endowed with the product topology. Furthermore, we set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$. This is the unique Borel probability measure on X_τ such that for each cylinder set $A = A_1 \times \cdots \times A_n \times \Gamma_\tau \times \Gamma_\tau \times \cdots$ in X_τ, $\tilde{\tau}(A) = \prod_{j=1}^{n} \tau(A_j)$. We denote by G_τ the subsemigroup of Rat generated by the subset Γ_τ of Rat.

2. Let M_τ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) := \int_{\Gamma_\tau} \varphi(g(z)) \, d\tau(g)$. M_τ is called the **transition operator** of the Markov process induced by τ. Moreover, let $M_\tau^* : C(\hat{\mathbb{C}})^* \to C(\hat{\mathbb{C}})^*$ be the dual of M_τ, which is defined as $M_\tau^*(\mu)(\varphi) = \mu(M_\tau(\varphi))$ for each $\mu \in C(\hat{\mathbb{C}})^*$ and each $\varphi \in C(\hat{\mathbb{C}})$. Remark: we have $M_\tau^*(\mathfrak{M}_1(\hat{\mathbb{C}})) \subset \mathfrak{M}_1(\hat{\mathbb{C}})$ and for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and each open subset V of $\hat{\mathbb{C}}$, we have $M_\tau^*(\mu)(V) = \int_{\Gamma_\tau} \mu(g^{-1}(V)) \, d\tau(g)$.

3. We denote by $F_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that there exists a neighborhood B of μ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ such that the sequence $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous on B. We set $J_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F_{meas}(\tau)$.

Remark 2.12. Let Γ be a closed subset of Rat. Then there exists a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ such that $\Gamma_\tau = \Gamma$. By using this fact, we sometimes apply the results on random complex dynamics to the study of the dynamics of rational semigroups.

Definition 2.13. Let Y be a compact metric space. Let $\Phi : Y \to \mathfrak{M}_1(Y)$ be the topological embedding defined by: $\Phi(z) := \delta_z$, where δ_z denotes the Dirac measure at z. Using this topological embedding $\Phi : Y \to \mathfrak{M}_1(Y)$, we regard Y as a compact subset of $\mathfrak{M}_1(Y)$.

Remark 2.14. If $h \in \mathrm{Rat}$ and $\tau = \delta_h$, then we have $M_\tau^* \circ \Phi = \Phi \circ h$ on $\hat{\mathbb{C}}$. Moreover, for a general $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, $M_\tau^*(\mu) = \int h_*(\mu) d\tau(h)$ for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Therefore, for a general $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, the map $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$.

Remark 2.15. If $\tau = \delta_h \in \mathfrak{M}_1(\mathrm{Rat}_+)$ with $h \in \mathrm{Rat}_+$, then $J_{meas}(\tau) \neq \emptyset$. In fact, using the embedding $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$, we have $\emptyset \neq \Phi(J(h)) \subset J_{meas}(\tau)$.

The following is an important and interesting object in random dynamics.

Definition 2.16. Let A be a subset of $\hat{\mathbb{C}}$. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$. For each $z \in \hat{\mathbb{C}}$, we set $T_{A,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau \mid d(\gamma_{n,1}(z), A) \to 0 \text{ as } n \to \infty\})$. This is the probability of tending to A starting with the initial value $z \in \hat{\mathbb{C}}$. For any $a \in \hat{\mathbb{C}}$, we set $T_{a,\tau} := T_{\{a\},\tau}$.

Definition 2.17. Let $\mathcal{B}$ be a complex vector space and let $M : \mathcal{B} \to \mathcal{B}$ be a linear operator. Let $\varphi \in \mathcal{B}$ and $a \in \mathbb{C}$ be such that $\varphi \neq 0, |a| = 1$, and $M(\varphi) = a\varphi$. Then we say that φ is a unitary eigenvector of M with respect to a, and we say that a is a unitary eigenvalue.

Definition 2.18. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$. Let K be a non-empty subset of $\hat{\mathbb{C}}$ such that $G_\tau(K) \subset K$. We denote by $\mathcal{U}_{f,\tau}(K)$ the set of all unitary eigenvectors of $M_\tau : C(K) \to C(K)$. Moreover, we denote by $\mathcal{U}_{v,\tau}(K)$ the set of all unitary eigenvalues of $M_\tau : C(K) \to C(K)$. Similarly, we denote by $\mathcal{U}_{f,\tau,*}(K)$ the set of all unitary eigenvectors of $M_\tau^* : C(K)^* \to C(K)^*$, and we denote by $\mathcal{U}_{v,\tau,*}(K)$ the set of all unitary eigenvalues of $M_\tau^* : C(K)^* \to C(K)^*$.

Definition 2.19. Let V be a complex vector space and let A be a subset of V. We set $\mathrm{LS}(A) := \{\sum_{j=1}^{m} a_j v_j \mid a_1, \ldots, a_m \in \mathbb{C}, v_1, \ldots, v_m \in A, m \in \mathbb{N}\}$.

Definition 2.20. Let Y be a topological space and let V be a subset of Y. We denote by $C_V(Y)$ the space of all $\varphi \in C(Y)$ such that for each connected component U of V, there exists a constant $c_U \in \mathbb{C}$ with $\varphi|_U \equiv c_U$.

Definition 2.21. For a topological space Y, we denote by $\mathrm{Cpt}(Y)$ the space of all non-empty compact subsets of Y. If Y is a metric space, we endow $\mathrm{Cpt}(Y)$ with the Hausdorff metric.

Definition 2.22. Let G be a rational semigroup. Let $Y \in \mathrm{Cpt}(\hat{\mathbb{C}})$ be such that $G(Y) \subset Y$. Let $K \in \mathrm{Cpt}(\hat{\mathbb{C}})$. We say that K is a minimal set for (G, Y) if K is minimal among the space $\{L \in \mathrm{Cpt}(Y) \mid G(L) \subset L\}$ with respect to inclusion. Moreover, we set $\mathrm{Min}(G, Y) := \{K \in \mathrm{Cpt}(Y) \mid K \text{ is a minimal set for } (G, Y)\}$.

Remark 2.23. Let G be a rational semigroup. By Zorn's lemma, it is easy to see that if $K_1 \in \mathrm{Cpt}(\hat{\mathbb{C}})$ and $G(K_1) \subset K_1$, then there exists a $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$ with $K \subset K_1$. Moreover, it is easy to see that for each $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and each $z \in K$, $\overline{G(z)} = K$. In particular, if $K_1, K_2 \in \mathrm{Min}(G, \hat{\mathbb{C}})$ with $K_1 \neq K_2$, then $K_1 \cap K_2 = \emptyset$. Moreover, by the formula $\overline{G(z)} = K$, we obtain that for each $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$, either (1) $\sharp K < \infty$ or (2) K is perfect and $\sharp K > \aleph_0$. Furthermore, it is easy to see that if $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}), G = \langle \Gamma \rangle$, and $K \in \mathrm{Min}(G, \hat{\mathbb{C}})$, then $K = \bigcup_{h \in \Gamma} h(K)$.

Remark 2.24. In [31, Remark 3.9], for the statement "for each $K \in \mathrm{Min}(G, Y)$, either (1) $\sharp K < \infty$ or (2) K is perfect", we should assume that each element $g \in G$ is a finite-to-one map.

Definition 2.25. For each $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we set $S_\tau := \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L$.

In [31], the following result was proved by the author of this paper.

Theorem 2.26 ([31], Cooperation Principle II: Disappearance of Chaos)**.** *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, all of the following statements hold.*

1. *Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$. Then, $\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$, $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \neq \emptyset$ and there exists a direct sum decomposition $C(\hat{\mathbb{C}}) = \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \oplus \mathcal{B}_{0,\tau}$. Moreover, $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C_{F(G_\tau)}(\hat{\mathbb{C}})$ and $1 \leq \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) < \infty$.*
2. *$\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.*
3. *Let $W := \bigcup_{A \in \mathrm{Con}(F(G_\tau)), A \cap S_\tau \neq \emptyset} A$. Then S_τ is compact. Moreover, for each $z \in \hat{\mathbb{C}}$ there exists a Borel measurable subset $\mathcal{C}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ such that for each $\gamma \in \mathcal{C}_z$, there exists an $n \in \mathbb{N}$ with $\gamma_{n,1}(z) \in W$ and $d(\gamma_{m,1}(z), S_\tau) \to 0$ as $m \to \infty$.*

Definition 2.27. Under the assumptions of Theorem 2.26, we denote by $\pi_\tau : C(\hat{\mathbb{C}}) \to \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ the projection determined by the direct sum decomposition $C(\hat{\mathbb{C}}) = \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \oplus \mathcal{B}_{0,\tau}$.

Remark 2.28. Under the assumptions of Theorem 2.26, by the theorem, we have that $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\infty \to 0$ as $n \to \infty$, for each $\varphi \in C(\hat{\mathbb{C}})$.

3 Results

In this section, we present the main results of this paper.

3.1 Stability and bifurcation

In this subsection, we present some results on stability and bifurcation of M_τ or M_τ^*. The proofs of the results are given in subsection 5.1.

Definition 3.1. Let (X, d) be a metric space. Let $\mathcal{O}$ be the topology of $\mathfrak{M}_{1,c}(X)$ such that $\mu_n \to \mu$ in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$ as $n \to \infty$ if and only if (1) $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{C}$, and (2) $\operatorname{supp} \mu_n \to \operatorname{supp} \mu$ with respect to the Hausdorff metric in the space $\mathrm{Cpt}(X)$.

Definition 3.2. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. Let $G = \langle \Gamma \rangle$. We say that G is **mean stable** if there exist non-empty open subsets U, V of $F(G)$ and a number $n \in \mathbb{N}$ such that all of the following hold.

(1) $\overline{V} \subset U$ and $\overline{U} \subset F(G)$.

(2) For each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{U}) \subset V$.

(3) For each point $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ such that $g(z) \in U$.

Note that this definition does not depend on the choice of a compact set Γ which generates G. Moreover, for a $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$, we say that Γ is mean stable if $\langle \Gamma \rangle$ is mean stable. Furthermore, for a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we say that τ is mean stable if G_τ is mean stable.

Remark 3.3. If G is mean stable, then $J_{\ker}(G) = \emptyset$.

Definition 3.4. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ and let $G = \langle \Gamma \rangle$. We say that $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting (for $(G, \hat{\mathbb{C}})$) if there exist non-empty open subsets U, V of $F(G)$ and a number $n \in \mathbb{N}$ such that both of the following hold.

(1) $L \subset V \subset \overline{V} \subset U \subset \overline{U} \subset F(G)$, $\sharp(\hat{\mathbb{C}} \setminus V) \geq 3$.

(2) For each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{U}) \subset V$.

Remark 3.5. If L is attracting for $G = \langle \Gamma \rangle$, then the set U coming from Definition 3.4 satisfies $\sharp(\hat{\mathbb{C}} \setminus U) \geq 3$. Therefore for each connected component of U, we can take the hyperbolic metric. Thus [19, Theorem 2.11] implies that there exist an $n \in \mathbb{N}$ and a constant $0 < c < 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$ and for each connected component W of U, the map $\gamma_{n,1} : W \to W'$, where W' denotes the connected component of U with $\gamma_{n,1}(W) \subset W'$, satisfies $d_h(\gamma_{n,1}(z), \gamma_{n,1}(w)) \leq c d_h(z, w)$ for each $z, w \in U$, where d_h denotes the hyperbolic distance.

Remark 3.6. For each $h \in G$,

$$\sharp\{\text{attracting minimal set for } (G, \hat{\mathbb{C}})\} \leq \sharp\{\text{attracting cycles of } h\} < \infty.$$

For, suppose L is an attracting minimal set for $(G, \hat{\mathbb{C}})$. Then for each $h \in G$, we have $h(L) \subset L \subset F(G)$. Since L is attracting, from [19, Theorem 2.11] it follows that for each $z \in L$, $h^n(z)$ tends to an attracting cycle of h. Thus L contains an attracting cycle of h.

Remark 3.7. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$. Let $G = \langle \Gamma \rangle$. Suppose that $\sharp J(G) \geq 3$. Then [31, Theorem 3.15, Remark 3.61, Proposition 3.65] imply that Γ is mean stable if and only if $\sharp(\mathrm{Min}(G, \hat{\mathbb{C}})) < \infty$ and each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting for $(G, \hat{\mathbb{C}})$. Combining this with Remark 3.6, it follows that Γ is mean stable if and only if each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting for $(G, \hat{\mathbb{C}})$.

We now give a classification of minimal sets.

Lemma 3.8. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Then exactly one of the following holds.*

(1) *L is attracting.*

(2) $L \cap J(G) \neq \emptyset$. *Moreover, for each* $z \in L \cap J(G)$, *there exists an element* $g \in \Gamma$ *with* $g(z) \in L \cap J(G)$.

(3) $L \subset F(G)$ *and there exists an element* $g \in G$ *and an element* $U \in \mathrm{Con}(F(G))$ *with* $L \cap U \neq \emptyset$ *such that* $g(U) \subset U$ *and* U *is a subset of a Siegel disk or a Hermann ring of* g.

Definition 3.9. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$.

- We say that L is J-touching (for $(G, \hat{\mathbb{C}})$) if $L \cap J(G) \neq \emptyset$.
- We say that L is sub-rotative (for $(G, \hat{\mathbb{C}})$) if (3) in Lemma 3.8 holds.

Definition 3.10. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$. Suppose L is J-touching or sub-rotative. Moreover, suppose $L \neq \hat{\mathbb{C}}$. Let $g \in \Gamma$. We say that g is a bifurcation element for (Γ, L) if one of the following statements (1)(2) holds.

(1) L is J-touching and there exists a point $z \in L \cap J(\langle \Gamma \rangle)$ such that $g(z) \in J(\langle \Gamma \rangle)$.

(2) There exist an open subset U of $\hat{\mathbb{C}}$ with $U \cap L \neq \emptyset$ and finitely many elements $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ such that $g \circ \gamma_{n-1} \cdots \circ \gamma_1(U) \subset U$ and U is a subset of a Siegel disk or a Hermann ring of $g \circ \gamma_{n-1} \cdots \circ \gamma_1$.

Furthermore, we say that an element $g \in \Gamma$ is a bifurcation element for Γ if there exists an $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$ such that g is a bifurcation element for (Γ, L).

We now consider families of rational maps.

Definition 3.11. Let Λ be a finite dimensional complex manifold and let $\{g_\lambda\}_{\lambda \in \Lambda}$ be a family of rational maps on $\hat{\mathbb{C}}$. We say that $\{g_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of rational maps if the map $(z, \lambda) \in \hat{\mathbb{C}} \times \Lambda \mapsto g_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$. We say that $\{g_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of polynomials if $\{g_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of rational maps and each g_λ is a polynomial.

Definition 3.12. Let $\mathcal{Y}$ be a subset of Rat and let U be a non-empty open subset of $\hat{\mathbb{C}}$. We say that $\mathcal{Y}$ is strongly U-admissible if for each $(z_0, h_0) \in U \times \mathcal{Y}$, there exists a holomorphic family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational maps with $\bigcup_{\lambda \in \Lambda} \{g_\lambda\} \subset \mathcal{Y}$ and an element $\lambda_0 \in \Lambda$ such that $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(z_0)$ is non-constant in any neighborhood of λ_0.

Example 3.13. Rat_+ is strongly $\hat{\mathbb{C}}$-admissible. $\mathcal{P}$ is strongly $\mathbb{C}$-admissible. Let $f_0 \in \mathcal{P}$. Then $\{f_0 + c \mid c \in \mathbb{C}\}$ is strongly $\mathbb{C}$-admissible.

Definition 3.14. Let $\mathcal{Y}$ be a subset of Rat. We say that $\mathcal{Y}$ satisfies condition $(*)$ if $\mathcal{Y}$ is a closed subset of Rat and at least one of the following (1) and (2) holds. (1): $\mathcal{Y}$ is strongly $\hat{\mathbb{C}}$-admissible. (2) $\mathcal{Y} \subset \mathcal{P}$ and $\mathcal{Y}$ is strongly $\mathbb{C}$-admissible.

Example 3.15. The sets Rat, Rat_+ and $\mathcal{P}$ satisfy $(*)$. For an $h_0 \in \mathcal{P}$, the set $\{h_0 + c \mid c \in \mathbb{C}\}$ is a subset of $\mathcal{P}$ and satisfies $(*)$.

We now present a result on bifurcation elements.

Lemma 3.16. *Let* $\mathcal{Y}$ *be a subset of* Rat_+ *satisfying condition* $(*)$. *Let* $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$ *and let* $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$. *Suppose that* L *is* J*-touching or sub-rotative. Moreover, suppose* $L \neq \hat{\mathbb{C}}$. *Then, there exists a bifurcation element for* (Γ, L). *Moreover, each bifurcation element* $g \in \Gamma$ *for* (Γ, L) *belongs to* $\partial\Gamma$, *where the boundary* $\partial\Gamma$ *of* Γ *is taken in the topological space* $\mathcal{Y}$.

We now present several results on the density of mean stable systems.

Theorem 3.17. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. Let $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$. Suppose that there exists an attracting $L \in \mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})$. Let $\{L_j\}_{j=1}^r$ be the set of attracting minimal sets for $(\langle\Gamma\rangle, \hat{\mathbb{C}})$ such that $L_i \neq L_j$ if $i \neq j$ (Remark: by Remark 3.6, the set of attracting minimal sets is finite). Let $\mathcal{U}$ be a neighborhood of Γ in $\mathrm{Cpt}(\mathcal{Y})$. For each $j = 1, \ldots, r$, let $\mathcal{V}_j$ be a neighborhood of L_j with respect to the Hausdorff metric in $\mathrm{Cpt}(\hat{\mathbb{C}})$. Suppose that $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for each (i, j) with $i \neq j$. Then, there exists an open neighborhood $\mathcal{U}'$ of Γ in $\mathcal{U}$ such that for any element $\Gamma' \in \mathcal{U}'$ satisfying that $\Gamma \subset \mathrm{int}(\Gamma')$ with respect to the topology in $\mathcal{Y}$, both of the following statements hold.*

(1) *$\langle\Gamma'\rangle$ is mean stable and $\sharp\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \sharp\{L' \in \mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) \mid L' \text{ is attracting for } (\langle\Gamma'\rangle, \hat{\mathbb{C}})\} = r$.*

(2) *For each $j = 1, \ldots, r$, there exists a unique element $L'_j \in \mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ with $L'_j \in \mathcal{V}_j$. Moreover, L'_j is attracting for $(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ for each $j = 1, \ldots, r$.*

Remark 3.18. Theorem 3.17 (with [31, Theorem 3.15]) generalizes [10, Theorem 0.1].

Theorem 3.19. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that there exists an attracting $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$. Let $\{L_j\}_{j=1}^r$ be the set of attracting minimal sets for $(G_\tau, \hat{\mathbb{C}})$ such that $L_i \neq L_j$ if $i \neq j$. Let $\mathcal{U}$ be a neighborhood of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. For each $j = 1, \ldots, r$, let $\mathcal{V}_j$ be a neighborhood of L_j with respect to the Hausdorff metric in $\mathrm{Cpt}(\hat{\mathbb{C}})$. Suppose that $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ for each (i, j) with $i \neq j$. Then, there exists an element $\rho \in \mathcal{U}$ with $\sharp\Gamma_\rho < \infty$ such that all of the following hold.*

(1) *G_ρ is mean stable and $\sharp\mathrm{Min}(G_\rho, \hat{\mathbb{C}}) = \sharp\{L' \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) \mid L' \text{ is attracting for } (\Gamma_\rho, \hat{\mathbb{C}})\} = r$.*

(2) *For each $j = 1, \ldots, r$, there exists a unique element $L'_j \in \mathrm{Min}(G_\rho, \hat{\mathbb{C}})$ with $L'_j \in \mathcal{V}_j$. Moreover, L'_j is attracting for $(G_\rho, \hat{\mathbb{C}})$ for each $j = 1, \ldots, r$.*

Theorem 3.20 (Cooperation Principle IV: Density of Mean Stable Systems)**.** *Let $\mathcal{Y}$ be a subset of $\mathcal{P}$ satisfying condition $(*)$. Then, we have the following.*

(1) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable}\}$ is open and dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. Moreover, the set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid J_{\mathrm{ker}}(G_\tau) = \emptyset, J(G_\tau) \neq \emptyset\}$ contains $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable}\}$.*

(2) *The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable, } \sharp\Gamma_\tau < \infty\}$ is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.*

Theorem 3.21. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. Let $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$. Suppose that there exists no attracting minimal set for $(\langle\Gamma\rangle, \hat{\mathbb{C}})$. Then we have the following.*

(1) *For any element $\Gamma' \in \mathrm{Cpt}(\mathrm{Rat})$ such that $\Gamma \subset \mathrm{int}(\Gamma')$ with respect to the topology in $\mathcal{Y}$, we have that $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ and $J(\langle\Gamma'\rangle) = \hat{\mathbb{C}}$.*

(2) *For any neighborhood $\mathcal{U}$ of Γ in $\mathrm{Cpt}(\mathcal{Y})$, there exists an element $\Gamma' \in \mathcal{U}$ with $\Gamma' \supset \Gamma$ such that $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ and $J(\langle\Gamma'\rangle) = \hat{\mathbb{C}}$.*

Corollary 3.22. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that there exists no attracting minimal set for $(G_\tau, \hat{\mathbb{C}})$. Let $\mathcal{U}$ be a neighborhood of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$. Then, there exists an element $\rho \in \mathcal{U}$ such that $\mathrm{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ and $J(G_\rho) = \hat{\mathbb{C}}$.*

Corollary 3.23. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. Then, the set*

$$\{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau \text{ is mean stable } \} \cup \{\rho \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \mathrm{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$$

is dense in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$.

We now present a result on the stability of mean stable systems.

Theorem 3.24 (Cooperation Principle V: $\mathcal{O}$-stability of mean stable systems)**.** *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be mean stable. Suppose $J(G_\tau) \neq \emptyset$. Then there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ such that all of the following statements hold.*

1. *For each $\nu \in \Omega$, ν is mean stable, $\sharp(J(G_\nu)) \geq 3$, and $\sharp(\mathrm{Min}(G_\nu, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}}))$.*

2. *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, there exists a continuous map $\nu \mapsto Q_{L,\nu} \in \mathrm{Cpt}(\hat{\mathbb{C}})$ on Ω with respect to the Hausdorff metric such that $Q_{L,\tau} = L$. Moreover, for each $\nu \in \Omega$, $\{Q_{L,\nu}\}_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} = \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$. Moreover, for each $\nu \in \Omega$ and for each $L, L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \neq L'$, we have $Q_{L,\nu} \cap Q_{L',\nu} = \emptyset$.*

3. *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and $\nu \in \Omega$, let $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$, $\Lambda_{r_L,\nu} := \{h_{r_L} \circ \cdots \circ h_1 \mid h_j \in \Gamma_\nu (\forall j)\}$, and $G_\nu^{r_L} := \langle \Lambda_{r_L,\nu} \rangle$. Let $\{L_j\}_{j=1}^{r_L} = \mathrm{Min}(G_\tau^{r_L}, L)$ (Remark: by [31, Theorem 3.15-12], we have $r_L = \sharp \mathrm{Min}(G_\tau^{r_L}, L)$). Then, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $j = 1, \ldots, r_L$, there exists a continuous map $\nu \mapsto L_{j,\nu} \in \mathrm{Cpt}(\hat{\mathbb{C}})$ with respect to the Hausdorff metric such that, for each $\nu \in \Omega$, $\{L_{j,\nu}\}_{j=1}^{r_L} = \mathrm{Min}(G_\nu^{r_L}, Q_{L,\nu})$ and $L_{i,\nu} \neq L_{j,\nu}$ whenever $i \neq j$. Moreover, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, for each $j = 1, \ldots, r_L$, and for each $\nu \in \Omega$, we have $L_{j+1,\nu} = \bigcup_{h \in \Gamma_\nu} h(L_{j,\nu})$, where $L_{r_L+1,\nu} := L_{1,\nu}$.*

4. *For each $\nu \in \Omega$, $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))) = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} r_L$. For each $\nu \in \Omega$ and for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, we have $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))) = r_L$, $\mathcal{U}_{v,\nu}(Q_{L,\nu}) = \{a_L^i\}_{i=1}^{r_L}$, and $\mathcal{U}_{v,\nu}(\hat{\mathbb{C}}) = \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \{a_L^i\}_{i=1}^{r_L}$, where $a_L := \exp(2\pi i / r_L)$.*

5. *The maps $\nu \mapsto \pi_\nu$ and $\nu \mapsto \mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$ are continuous on Ω. More precisely, for each $\nu \in \Omega$, there exists a finite family $\{\varphi_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ in $\mathcal{U}_{f,\nu}(\hat{\mathbb{C}})$ and a finite family $\{\rho_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ in $C(\hat{\mathbb{C}})^*$ such that all of the following hold.*

   (a) *$\{\varphi_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$ and $\{\rho_{L,i,\nu} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu,*}(\hat{\mathbb{C}}))$.*

   (b) *Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $i = 1, \ldots, r_L$. Let $\nu \in \Omega$. Then $M_\tau(\varphi_{L,i,\nu}) = a_L^i \varphi_{L,i,\nu}$, $\varphi_{L,i,\nu}|_{Q_{L,\nu}} = (\varphi_{L,1,\nu}|_{Q_{L,\nu}})^i$, $\varphi_{L,i,\nu}|_{Q_{L',\nu}} \equiv 0$ for any $L' \in \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ with $L' \neq L$, and $\mathrm{supp}\, \rho_{L,i,\nu} = Q_{L,\nu}$. Moreover, $\{\varphi_{L,i,\nu}|_{Q_{L,\nu}}\}_{i=1}^{r_L}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))$ and $\{\rho_{L,i,\nu}|_{C(Q_{L,\nu})} \mid i = 1, \ldots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu,*}(Q_{L,\nu}))$. In particular, $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))) = r_L$.*

   (c) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $i = 1, \ldots r_L$, the map $\nu \mapsto \varphi_{L,i,\nu} \in C(\hat{\mathbb{C}})$ is continuous on Ω and the map $\nu \mapsto \rho_{j,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on Ω.*

   (d) *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, for each (i, j) and for each $\nu \in \Omega$, $\rho_{L,i,\nu}(\varphi_{L,j,\nu}) = \delta_{ij}$. Moreover, For each $L, L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \neq L'$, for each (i, j), and for each $\nu \in \Omega$, $\rho_{L,i,\nu}(\varphi_{L',j,\nu}) = 0$.*

   (e) *For each $\nu \in \Omega$ and for each $\varphi \in C(\hat{\mathbb{C}})$, $\pi_\nu(\varphi) = \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{i=1}^{r_L} \rho_{L,i,\nu}(\varphi) \cdot \varphi_{L,i,\nu}$.*

6. *For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, the map $\nu \mapsto T_{Q_{L,\nu},\nu} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is continuous on Ω.*

We now present a result on a characterization of mean stability.

Theorem 3.25. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. We consider the following subsets A, B, C, D, E of $\mathfrak{M}_{1,c}(\mathcal{Y})$ which are defined as follows.*

(1) *$A := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{Y}) \mid \tau$ is mean stable$\}$.*

(2) *Let B be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $J_{\ker}(G_\nu) = \emptyset$, and* (b) *$\nu \mapsto \sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on Ω.*

(3) *Let C be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $F(G_\nu) \neq \emptyset$, and* (b) *$\nu \mapsto \sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on Ω.*

(4) *Let D be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that for each $\nu \in \Omega$, $J_{\ker}(G_\nu) = \emptyset$ and $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))) = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$.*

(5) *Let E be the set of $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$ satisfying that for each $\varphi \in C(\hat{\mathbb{C}})$, there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $J_{\ker}(G_\nu) = \emptyset$, and* (b) *the map $\nu \mapsto \pi_\nu(\varphi) \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ defined on Ω is continuous at τ.*

Then, $A = B = C = D = E$.

We now present a result on bifurcation of dynamics of G_τ and M_τ regarding a continuous family of measures τ.

Theorem 3.26. *Let $\mathcal{Y}$ be a subset of Rat_+ satisfying condition $(*)$. For each $t \in [0,1]$, let μ_t be an element of $\mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that all of the following conditions* (1)–(4) *hold.*

(1) *$t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ is continuous on $[0,1]$.*

(2) *If $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, then $\Gamma_{\mu_{t_1}} \subset \mathrm{int}(\Gamma_{\mu_{t_2}})$ with respect to the topology of $\mathcal{Y}$.*

(3) *$\mathrm{int}(\Gamma_{\mu_0}) \neq \emptyset$ with respect to the topology of $\mathcal{Y}$ and $F(G_{\mu_1}) \neq \emptyset$.*

(4) *$\sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) \neq \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$.*

Let $B := \{t \in [0,1) \mid$ there exists a bifurcation element $g \in \Gamma_{\mu_t}$ for $\Gamma_{\mu_t}\}$. Then, we have the following.

(a) *For each $t \in [0,1]$, $J_{\ker}(G_{\mu_t}) = \emptyset$ and $\sharp J(G_{\mu_t}) \geq 3$, and all statements in* [31, Theorem 3.15] *(with $\tau = \mu_t$) hold.*

(b) *We have*
$$1 \leq \sharp B \leq \sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) - \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})) < \infty.$$
Moreover, for each $t \in B$, μ_t is not mean stable. Furthermore, for each $t \in [0,1) \setminus B$, μ_t is mean stable.

(c) *For each $s \in (0,1]$ there exists a number $t_s \in (0,s)$ such that for each $t \in [t_s, s]$, $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_{\mu_s}, \hat{\mathbb{C}}))$.*

Example 3.27. Let c be a point in the interior of the Mandelbrot set $\mathcal{M}$. Suppose $z \mapsto z^2 + c$ is hyperbolic. Let $r_0 > 0$ be a small number such that $\overline{D(c, r_0)} \subset \mathrm{int}(\mathcal{M})$. Let $r_1 > 0$ be a large number such that $D(c, r_1) \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$. For each $t \in [0,1]$, let $\mu_t \in \mathfrak{M}_1(\overline{D(c, (1-t)r_0 + tr_1)})$ be the normalized 2-dimensional Lebesgue measure on $\overline{D(c, (1-t)r_0 + tr_1)}$. Then $\{\mu_t\}_{t \in [0,1]}$ satisfies the conditions (1)–(4) in Theorem 3.26 (for example, $2 = \sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) > \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})) = 1$). Thus
$$\sharp(\{t \in [0,1] \mid \text{there exists a bifurcation element } g \in \Gamma_{\mu_t} \text{ for } \Gamma_{\mu_t}\}) = 1.$$

3.2 Spectral properties of M_τ and stability

In this subsection, we present some results on spectral properties of M_τ acting on the space of Hölder continuous functions on $\hat{\mathbb{C}}$ and the stability. The proofs of the results are given in subsection 5.2.

Definition 3.28. Let $K \in \mathrm{Cpt}(\hat{\mathbb{C}})$. For each $\alpha \in (0,1)$, let
$C^\alpha(K) := \{\varphi \in C(K) \mid \sup_{x,y\in K, x\neq y} |\varphi(x)-\varphi(y)|/d(x,y)^\alpha < \infty\}$ be the Banach space of all complex-valued α-Hölder continuous functions on K endowed with the α-Hölder norm $\|\cdot\|_\alpha$, where $\|\varphi\|_\alpha := \sup_{z\in K}|\varphi(z)| + \sup_{x,y\in K, x\neq y}|\varphi(x)-\varphi(y)|/d(x,y)^\alpha$ for each $\varphi \in C^\alpha(K)$.

Theorem 3.29. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, there exists an $\alpha_0 > 0$ such that for each $\alpha \in (0,\alpha_0)$, $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$. Moreover, for each $\alpha \in (0,\alpha_0)$, there exists a constant $E_\alpha > 0$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, $\|\pi_\tau(\varphi)\|_\alpha \leq E_\alpha\|\varphi\|_\infty$. Furthermore, for each $\alpha \in (0,\alpha_0)$ and for each $L \in \mathrm{Min}(G_\tau,\hat{\mathbb{C}})$, $T_{L,\tau} \in C^\alpha(\hat{\mathbb{C}})$.*

If $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then by [31, Proposition 3.65], we have $S_\tau \subset F(G_\tau)$ (see Definition 2.25). From this point of view, we consider the situation that $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ satisfies $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Under this situation, we have several very strong results. Note that there exists an example of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau < \infty$ such that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, $S_\tau \subset F(G_\tau)$, and τ is not mean stable (see Example 6.3).

Theorem 3.30 (Cooperation Principle VI: Exponential rate of convergence)**.** *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $S_\tau \subset F(G_\tau)$. Let $r := \prod_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}})} \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. Then, there exists a constant $\alpha \in (0,1)$, a constant $\lambda \in (0,1)$, and a constant $C > 0$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, we have all of the following.*

(1) $\|M_\tau^{nr}(\varphi) - \pi_\tau(\varphi)\|_\alpha \leq C\lambda^n\|\varphi - \pi_\tau(\varphi)\|_\alpha$ *for each* $n \in \mathbb{N}$.

(2) $\|M_\tau^{n}(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n\|\varphi - \pi_\tau(\varphi)\|_\alpha$ *for each* $n \in \mathbb{N}$.

(3) $\|M_\tau^{n}(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n\|\varphi\|_\alpha$ *for each* $n \in \mathbb{N}$.

(4) $\|\pi_\tau(\varphi)\|_\alpha \leq C\|\varphi\|_\alpha$.

We now consider the spectrum $\mathrm{Spec}_\alpha(M_\tau)$ of $M_\tau : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$. By Theorem 3.29, $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}}) \subset \mathrm{Spec}_\alpha(M_\tau)$ for some $\alpha \in (0,1)$. From Theorem 3.30, we can show that the distance between $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ and $\mathrm{Spec}_\alpha(M_\tau) \setminus \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ is positive.

Theorem 3.31. *Under the assumptions of Theorem 3.30, we have all of the following.*

(1) $\mathrm{Spec}_\alpha(M_\tau) \subset \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, *where $\alpha \in (0,1)$ and $\lambda \in (0,1)$ are the constants in Theorem 3.30.*

(2) *Let $\zeta \in \mathbb{C} \setminus (\{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}}))$. Then, $(\zeta I - M_\tau)^{-1} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$ is equal to*

$$(\zeta I - M_\tau)|^{-1}_{\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))} \circ \pi_\tau + \sum_{n=0}^{\infty} \frac{M_\tau^n}{\zeta^{n+1}}(I - \pi_\tau),$$

where I denotes the identity on $C^\alpha(\hat{\mathbb{C}})$.

Combining Theorem 3.31 and perturbation theory for linear operators ([16]), we obtain the following. In particular, as we remarked in Remark 1.14, we obtain complex analogues of the Takagi function.

Theorem 3.32. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h_1, \ldots, h_m \in \mathrm{Rat}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Suppose that $J_{\ker}(G) = \emptyset, J(G) \neq \emptyset$ and $\bigcup_{L \in \mathrm{Min}(G, \hat{\mathbb{C}})} L \subset F(G)$. Let $\mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \ldots, a_{m-1}) \in (0,1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \ldots, a_m) \in \mathcal{W}_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Then we have all of the following.*

(1) *For each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ and an open neighborhood V_b of b in $\mathcal{W}_m$ such that for each $a \in V_b$, we have $\mathrm{LS}(\mathcal{U}_{f,\tau_a}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$, $\pi_{\tau_a}(C^\alpha(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$ and $(\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm, and such that the map $a \mapsto (\pi_{\tau_a} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})) \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic in V_b.*

(2) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Then, for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that the map $a \mapsto T_{L,\tau_a} \in (C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$ is real-analytic in an open neighborhood of b in $\mathcal{W}_m$. Moreover, the map $a \mapsto T_{L,\tau_a} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is real-analytic in $\mathcal{W}_m$. In particular, for each $z \in \hat{\mathbb{C}}$, the map $a \mapsto T_{L,\tau_a}(z)$ is real-analytic in $\mathcal{W}_m$. Furthermore, for any multi-index $n = (n_1, \ldots, n_{m-1}) \in (\mathbb{N} \cup \{0\})^{m-1}$ and for any $b \in \mathcal{W}_m$, the function $z \mapsto [(\frac{\partial}{\partial a_1})^{n_1} \cdots (\frac{\partial}{\partial a_{m-1}})^{n_{m-1}} (T_{L,\tau_a}(z))]|_{a=b}$ belongs to $C_{F(G)}(\hat{\mathbb{C}})$.*

(3) *Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and let $b \in \mathcal{W}_m$. For each $i = 1, \ldots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,b}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ and let $\zeta_{i,b}(z) := T_{L,\tau_b}(h_i(z)) - T_{L,\tau_b}(h_m(z))$. Then, $\psi_{i,b}$ is the unique solution of the functional equation $(I - M_{\tau_b})(\psi) = \zeta_{i,b}, \psi|_{S_{\tau_b}} = 0, \psi \in C(\hat{\mathbb{C}})$, where I denotes the identity map. Moreover, there exists a number $\alpha \in (0,1)$ such that $\psi_{i,b} = \sum_{n=0}^\infty M_{\tau_b}^n(\zeta_{i,b})$ in $(C^\alpha(\hat{\mathbb{C}}), \|\cdot\|_\alpha)$.*

We now present a result on the non-differentiability of $\psi_{i,b}$ at points in $J(G_\tau)$. In order to do that, we need several definitions and notations.

Definition 3.33. For a rational semigroup G, we set $P(G) := \overline{\bigcup_{g \in G} \{ \text{ all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$ where the closure is taken in $\hat{\mathbb{C}}$. This is called the postcritical set of G. We say that a rational semigroup G is hyperbolic if $P(G) \subset F(G)$. For a polynomial semigroup G, we set $P^*(G) := P(G) \backslash \{\infty\}$. For a polynomial semigroup G, we set $\hat{K}(G) := \{z \in \mathbb{C} \mid G(z) \text{ is bounded in } \mathbb{C}\}$. Moreover, for each polynomial h, we set $K(h) := \hat{K}(\langle h \rangle)$.

Remark 3.34. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and suppose that $\langle \Gamma \rangle$ is hyperbolic and $J_{\ker}(\langle \Gamma \rangle) = \emptyset$. Then by [31, Propositions 3.63, 3.65], there exists an neighborhood $\mathcal{U}$ of Γ in $\mathrm{Cpt}(\mathrm{Rat})$ such that for each $\Gamma' \in \mathcal{U}$, Γ' is mean stable, $J_{\ker}(\langle \Gamma' \rangle) = \emptyset$, $J(\langle \Gamma' \rangle) \neq \emptyset$ and $\bigcup_{L \in \mathrm{Min}(\langle \Gamma' \rangle, \hat{\mathbb{C}})} L \subset F(\langle \Gamma' \rangle)$.

Definition 3.35. Let $m \in \mathbb{N}$. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$ be an element such that $h_1, \ldots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \ldots, h_m\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the map defined by $f(\gamma, y) = (\sigma(\gamma), \gamma_1(y))$, where $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$ and $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map $((\gamma_1, \gamma_2, \ldots) \mapsto (\gamma_2, \gamma_3, \ldots))$. This map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is called the skew product associated with Γ. Let $\pi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_{\hat{\mathbb{C}}} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the canonical projections. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be an f-invariant Borel probability measure. Let $\mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\}$. For each $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$, we define a function $\tilde{p} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$ by $\tilde{p}(\gamma, y) := p_j$ if $\gamma_1 = h_j$ (where $\gamma = (\gamma_1, \gamma_2, \ldots)$), and we set

$$u(h, p, \mu) := \frac{-(\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \tilde{p}(\gamma, y) \, d\mu(\gamma, y))}{\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \|D(\gamma_1)_y\|_s \, d\mu(\gamma, y)}$$

(when the integral of the denominator converges), where $\|D(\gamma_1)_y\|_s$ denotes the norm of the derivative of γ_1 at y with respect to the spherical metric on $\hat{\mathbb{C}}$.

Definition 3.36. Let $h = (h_1, \dots, h_m) \in \mathcal{P}^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \dots, h_m\}$. For any $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \mathbb{C}$, let $G_\gamma(y) := \lim_{n\to\infty} \frac{1}{\deg(\gamma_{n,1})} \log^+ |\gamma_{n,1}(y)|$, where $\log^+ a := \max\{\log a, 0\}$ for each $a > 0$. By the arguments in [21], for each $\gamma \in \Gamma^{\mathbb{N}}$, $G_\gamma(y)$ exists, G_γ is subharmonic on $\mathbb{C}$, and $G_\gamma|_{A_{\infty,\gamma}}$ is equal to the Green's function on $A_{\infty,\gamma}$ with pole at ∞, where $A_{\infty,\gamma} := \{z \in \hat{\mathbb{C}} \mid \gamma_{n,1}(z) \to \infty \text{ as } n \to \infty\}$. Moreover, $(\gamma, y) \mapsto G_\gamma(y)$ is continuous on $\Gamma^{\mathbb{N}} \times \mathbb{C}$. Let $\mu_\gamma := dd^c G_\gamma$, where $d^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$. Note that by the argument in [15, 21], μ_γ is a Borel probability measure on J_γ such that $\operatorname{supp} \mu_\gamma = J_\gamma$. Furthermore, for each $\gamma \in \Gamma^{\mathbb{N}}$, let $\Omega(\gamma) = \sum_c G_\gamma(c)$, where c runs over all critical points of γ_1 in $\mathbb{C}$, counting multiplicities.

Remark 3.37. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. Let $\Gamma = \{h_1, \dots, h_m\}$ and let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product map associated with Γ. Moreover, let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma)$. Then, there exists a unique f-invariant ergodic Borel probability measure μ on $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ such that $\pi_*(\mu) = \tilde{\tau}$ and $h_\mu(f|\sigma) = \max_{\rho \in \mathfrak{E}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}) : f_*(\rho) = \rho, \pi_*(\rho) = \tilde{\tau}} h_\rho(f|\sigma) = \sum_{j=1}^m p_j \log(\deg(h_j))$, where $h_\rho(f|\sigma)$ denotes the relative metric entropy of (f, ρ) with respect to $(\sigma, \tilde{\tau})$, and $\mathfrak{E}_1(\cdot)$ denotes the space of ergodic measures (see [24]). This μ is called the **maximal relative entropy measure** for f with respect to $(\sigma, \tilde{\tau})$.

Definition 3.38. Let V be a non-empty open subset of $\hat{\mathbb{C}}$. Let $\varphi : V \to \mathbb{C}$ be a function and let $y \in V$ be a point. Suppose that φ is bounded around y. Then we set

$$\mathrm{Höl}(\varphi, y) := \sup\{\beta \in [0, \infty) \mid \limsup_{z \to y, z \neq y} \frac{|\varphi(z) - \varphi(y)|}{d(z, y)^\beta} < \infty\} \in [0, \infty],$$

where d denotes the spherical distance. This is called the **pointwise Hölder exponent of φ at y.**

Remark 3.39. If $\mathrm{Höl}(\varphi, y) < \infty$, then $\mathrm{Höl}(\varphi, y) = \inf\{\beta \in [0, \infty) \mid \limsup_{z \to y, z \neq y} \frac{|\varphi(z) - \varphi(y)|}{d(z, y)^\beta} = \infty\}$. If $\mathrm{Höl}(\varphi, y) < 1$, then φ is non-differentiable at y. If $\mathrm{Höl}(\varphi, y) > 1$, then φ is differentiable at y and the derivative at y is equal to 0. In [31, Definition 3.80], "$\limsup_{z \to y}$" should be replaced by "$\limsup_{z \to y, z \neq y}$" and we should add the following. "If $\{\beta \in [0, \infty) \mid \limsup_{z \to y, z \neq y} \frac{|\varphi(z) - \varphi(y)|}{d(z, y)^\beta} = \infty\} = \emptyset$, then we set $\mathrm{Höl}(\varphi, y) = \infty$."

We now present a result on the non-differentiability of $\psi_{i,b}(z) = [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=b}$ at points in $J(G_\tau)$.

Theorem 3.40. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ and we set $\Gamma := \{h_1, h_2, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $\mathcal{W}_m := \{(a_1, \dots, a_m) \in (0, 1)^m \mid \sum_{j=1}^m a_j = 1\} \cong \{(a_1, \dots, a_{m-1}) \in (0, 1)^{m-1} \mid \sum_{j=1}^{m-1} a_j < 1\}$. For each $a = (a_1, \dots, a_m) \in \mathcal{W}_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j} \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be the maximal relative entropy measure for $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$. Moreover, let $\lambda := (\pi_{\hat{\mathbb{C}}})_*(\mu) \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Suppose that G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i, j) with $i \neq j$. For each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$, for each $i = 1, \dots, m-1$ and for each $z \in \hat{\mathbb{C}}$, let $\psi_{i,p,L}(z) := [\frac{\partial}{\partial a_i}(T_{L,\tau_a}(z))]|_{a=p}$. Then, we have all of the following.*

1. *$G_\tau = G$ is mean stable, $J_{\ker}(G) = \emptyset$, and $S_\tau \subset F(G_\tau)$. Moreover, $0 < \dim_H(J(G)) < 2$, $\operatorname{supp} \lambda = J(G)$, and $\lambda(\{z\}) = 0$ for each $z \in J(G)$.*
2. *Suppose $\sharp \mathrm{Min}(G, \hat{\mathbb{C}}) \neq 1$. Then there exists a Borel subset A of $J(G)$ with $\lambda(A) = 1$ such that for each $z_0 \in A$, for each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and for each $i = 1, \dots, m-1$, exactly one of the following* (a),(b),(c) *holds.*

(a) $\mathrm{Höl}(\psi_{i,p,L}, z_1) = \mathrm{Höl}(\psi_{i,p,L}, z_0) < u(h,p,\mu)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

(b) $\mathrm{Höl}(\psi_{i,p,L}, z_0) = u(h,p,\mu) \leq \mathrm{Höl}(\psi_{i,p,L}, z_1)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

(c) $\mathrm{Höl}(\psi_{i,p,L}, z_1) = u(h,p,\mu) < \mathrm{Höl}(\psi_{i,p,L}, z_0)$ *for each* $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

3. *If* $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$, *then*

$$u(h,p,\mu) = \frac{-(\sum_{j=1}^m p_j \log p_j)}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)}$$

and

$$2 > \dim_H(\lambda) = \frac{\sum_{j=1}^m p_j \log \deg(h_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)} > 0,$$

where $\dim_H(\lambda) := \inf\{\dim_H(A) \mid A$ *is a Borel subset of* $\hat{\mathbb{C}}, \lambda(A) = 1\}$.

4. *Suppose* $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$. *Moreover, suppose that at least one of the following* (a), (b), *and* (c) *holds:* (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) $P^*(G)$ *is bounded in* $\mathbb{C}$. (c) $m = 2$. *Then,* $u(h,p,\mu) < 1$.

4 Tools

In this section, we introduce some fundamental tools to prove the main results.

Let G be a rational semigroup. Then, for each $g \in G$, $g(F(G)) \subset F(G), g^{-1}(J(G)) \subset J(G)$. If G is generated by a compact family Λ of Rat, then $J(G) = \bigcup_{h\in\Lambda} h^{-1}(J(G))$ (this is called the backward self-similarity). If $\sharp J(G) \geq 3$, then $J(G)$ is a perfect set and $J(G)$ is equal to the closure of the set of repelling cycles of elements of G. In particular, $J(G) = \overline{\cup_{g\in G} J(g)}$ if $\sharp J(G) \geq 3$. We set $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp \bigcup_{g\in G} g^{-1}(\{z\}) < \infty\}$. If $\sharp J(G) \geq 3$, then $\sharp E(G) \leq 2$ and for each $z \in J(G) \setminus E(G)$, $J(G) = \overline{\bigcup_{g\in G} g^{-1}(\{z\})}$. If $\sharp J(G) \geq 3$, then $J(G)$ is the smallest set in $\{\emptyset \neq K \subset \hat{\mathbb{C}} \mid K$ is compact, $\forall g \in G, g(K) \subset K\}$ with respect to the inclusion. For more details on these properties of rational semigroups, see [14, 12, 24].

For fundamental tools and lemmas of random complex dynamics, see [31].

5 Proofs

In this section, we give the proofs of the main results.

5.1 Proofs of results in 3.1

In this subsection, we give the proofs of the results in subsection 3.1. We need several lemmas.

Definition 5.1. Let W be an open subset of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus W) \geq 3$ and let $g : W \to W$ be a holomorphic map. Let $\{W_j\}_{j\in J} = \mathrm{Con}(W)$. For each connected component W_j of W, we take the hyperbolic metric ρ_j. For each $z \in W$, we denote by $\|Dg_z\|_h$ the norm of the derivative of g at z which is measured from the hyperbolic metric on the component W_{i_1} of W containing z to that on the component W_{i_2} of W containing $g(z)$. Moreover, for each subset L of W and for each $r \geq 0$, we set $d_h(L,r) := \bigcup_{j\in J}\{z \in W_j \mid d_{\rho_j}(z, L \cap W_j) < r\}$, where $d_{\rho_j}(z, L \cap W_j)$ denotes the distance from z to $L \cap W_j$ with respect to the hyperbolic distance on W_j. Similarly, for each $z \in W$, we denote by $\|Dg_z\|_s$ the norm of the derivative of g at z with respect to the spherical metric on $\hat{\mathbb{C}}$.

Lemma 5.2. *Let* $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ *and let* $G = \langle\Gamma\rangle$. *Let* $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ *be attracting for* $(G, \hat{\mathbb{C}})$. *Let* $W := \bigcup_{A\in \mathrm{Con}(F(G)), A\cap L\neq\emptyset} A$ *and let* W' *be a relative compact open subset of* W *including* L. *Then there exists an open neighborhood* $\mathcal{U}$ *of* Γ *in* $\mathrm{Cpt}(\mathrm{Rat})$ *such that both of the following hold.*

(1) *For each* $\Omega \in \mathcal{U}$, *there exists a unique* $L' \in \mathrm{Min}(\langle \Omega, \hat{\mathbb{C}})$ *with* $L' \subset W'$.

(2) *For each* $\Omega \in \mathcal{U}$, *the above* L' *is attracting for* $(\langle \Omega \rangle, \hat{\mathbb{C}})$.

Proof. Let $\{W_j\}_{j=1}^s = \mathrm{Con}(W)$. For each connected component W_j, we take the hyperbolic metric ρ_j. Let d_{ρ_j} be the distance on W_j induced by ρ_j. For each $r > 0$, we set $d_h(L, r) := \bigcup_{j=1}^s \{z \in W \mid d_{\rho_j}(z, L \cap W_j) < r\}$. Let (U, V, n) be as in Definition 3.4. Then $V \cap W \subset \overline{V \cap W} \subset U \cap W \subset \overline{U \cap W} \subset F(G)$ and $\gamma_{n,1}(\overline{U \cap W}) \subset V \cap W$ for each $\gamma \in \Gamma^{\mathbb{N}}$. Therefore, by [19, Theorem 2.11], there exists a constant $0 < c < 1$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$ and for each $j_1, j_2 \in \{1, \ldots, s\}$, if $\gamma_{n,1}(W_{j_1}) \subset W_{j_2}$, then $d_{\rho_{j_1}}(\gamma_{n,1}(z), \gamma_{n,1}(w)) \leq c d_{\rho_{j_2}}(z, w)$ for each $z, w \in W_{j_1} \cap \overline{W'}$. Thus, replacing n by a larger number if necessary, we may assume that there exists a number $\epsilon_1 \in (0, 1)$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{W'}) \subset d_h(L, \epsilon_1) \subset \overline{d_h(L, \epsilon_1)} \subset W'$. Hence, there exists an open neighborhood $\mathcal{U}$ of Γ in $\mathrm{Cpt}(\mathrm{Rat})$ and a number $\epsilon_2 \in (\epsilon_1, 1)$ such that for each $\Omega \in \mathcal{U}$ and for each $\gamma \in \Omega^{\mathbb{N}}$,

$$\gamma_{n,1}(\overline{W'}) \subset d_h(L, \epsilon_2) \subset \overline{d_h(L, \epsilon_2)} \subset W'. \tag{1}$$

Let $\Omega \in \mathcal{U}$. Setting $\Omega_n := \{\gamma_n \circ \cdots \circ \gamma_1 \mid \gamma_j \in \Omega (\forall j)\}$, we obtain that there exists an element $L_0 \in \mathrm{Min}(\langle \Omega_n \rangle, \hat{\mathbb{C}})$ with $L_0 \subset W'$. Then, for each $g \in \langle \Omega \rangle$, $g(\langle \Omega \rangle(L_0)) \subset \langle \Omega \rangle(L_0)$. Taking $\mathcal{U}$ so small, we may assume that $\langle \Omega \rangle(L_0) \subset W'$. Hence, there exists an element $L' \in \mathrm{Min}(\langle \Omega \rangle, \hat{\mathbb{C}})$ with $L' \subset W'$. From (1) and [19, Theorem 2.11], it follows that there exists no $L'' \in \mathrm{Min}(\langle \Omega \rangle, \hat{\mathbb{C}})$ with $L'' \neq L'$ such that $L'' \subset W'$. Moreover, by (1) and [19, Theorem 2.11] again, we obtain that L' is attracting for $(\langle \Omega \rangle, \hat{\mathbb{C}})$. Thus, we have proved our lemma. □

Lemma 5.3. *Let* $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ *and let* $G = \langle \Gamma \rangle$. *Let* $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ *be attracting for* $(G, \hat{\mathbb{C}})$. *Then* $L = \overline{\{z \in L \mid \exists g \in G \text{ s.t. } g(z) = z, |m(g, z)| < 1\}}$, *where* $m(g, z)$ *denotes the multiplier of* g *at* z.

Proof. Let $z \in L$. Let $U \in \mathrm{Con}(F(G))$ with $z \in U$. Let B be an open neighborhood of z in U. Since $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and since L is attracting, the argument in the proof of Lemma 5.2 implies that there exists an element $g \in G$ such that $\overline{g(B)} \subset B$. Then there exists an attracting fixed point of g in B. Thus the statement of our lemma holds. □

Lemma 5.4. *Let* $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ *and let* $G = \langle \Gamma \rangle$. *Let* $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ *be attracting for* $(G, \hat{\mathbb{C}})$. *Let* $\mathcal{V}$ *be a neighborhood of* L *in the space* $\mathrm{Cpt}(\hat{\mathbb{C}})$. *Then there exists an open neighborhood* $\mathcal{U}$ *of* Γ *in* $\mathrm{Cpt}(\mathrm{Rat})$ *and an open neighborhood* $\mathcal{V}'$ *of* L *in* $\mathrm{Cpt}(\hat{\mathbb{C}})$ *with* $\mathcal{V}' \subset \mathcal{V}$ *such that for each* $\Omega \in \mathcal{U}$, *there exists a unique* $L' \in \mathrm{Min}(\langle \Omega \rangle, \hat{\mathbb{C}})$ *with* $L' \in \mathcal{V}'$. *Moreover, this* L' *is attracting for* $(\langle \Omega \rangle, \hat{\mathbb{C}})$.

Proof. By Lemma 5.2, Lemma 5.3 and Implicit function theorem, the statement of our lemma holds. □

Lemma 5.5. *Let* $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ *and let* $G = \langle \Gamma \rangle$. *Let* $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ *with* $L \subset F(G)$. *Suppose that for each* $g \in G$ *and for each* $U \in \mathrm{Con}(F(G))$ *with* $U \cap L \neq \emptyset$ *and* $g(U) \subset U$, *either* (a) $g \in \mathrm{Rat}_+$ *and* U *is not a subset of a Siegel disk or a Hermann ring of* g *or* (b) $g \in \mathrm{Aut}(\hat{\mathbb{C}})$ *and* g *is loxodromic or parabolic. Then,* L *is attracting for* $(G, \hat{\mathbb{C}})$.

Proof. Let $W := \bigcup_{A \in \mathrm{Con}(F(G)), A \cap L \neq \emptyset} A$ and we take the hyperbolic metric on each connected component of W. Since L is a compact subset of $F(G)$, we have $\sharp \mathrm{Con}(W) < \infty$. Moreover, from assumptions (a) and (b) and [19, Theorem 2.11], we obtain that if $A \in \mathrm{Con}(W)$ and if $\gamma_{n,m}(A) \subset A$, then $\|D(\gamma_{n,m})_z\|_h < 1$ for each $z \in A$. From these arguments, it is easy to see that L is attracting for $(G, \hat{\mathbb{C}})$. □

Proof of Lemma 3.8: Lemma 5.5 implies that if $L \subset F(G)$ and (3) in Lemma 3.8 does not hold, then L is attracting. We now suppose that $L \cap J(G) \neq \emptyset$. Let $z \in L \cap J(G)$. By $J(G) = \cup_{h \in \Gamma} h^{-1}(J(G))$ ([25, Lemma 0.2]), there exists an element $g_0 \in \Gamma$ with $g_0(z) \in J(G)$. Then $g_0(z) \in L \cap J(G)$. Thus we have proved our lemma. □

Lemma 5.6. *Let U be a non-empty open subset of $\hat{\mathbb{C}}$. Let $\mathcal{Y}$ be a closed subset of* Rat. *Suppose that $\mathcal{Y}$ is strongly U-admissible. Let $\Gamma \in \mathrm{Cpt}(\mathcal{Y})$. Let h_0 be an interior point of Γ with respect to the topology in the space $\mathcal{Y}$. Let $K \in \mathrm{Cpt}(U)$. Then, there exists an $\epsilon > 0$ such that for each $z \in K$, $\{h(z) \mid h \in \Gamma\} \supset B(h_0(z), \epsilon)$.*

Proof. Let $w \in K$. Then there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of rational maps with $\bigcup_{\lambda\in\Lambda}\{g_\lambda\} \subset \mathcal{Y}$ and a point $\lambda_0 \in \Lambda$ such that $g_{\lambda_0} = h_0$ and $\lambda \mapsto g_\lambda(w)$ is non-constant in any neighborhood of λ_0. By the argument principle, there exists a $\delta_w > 0$, an $\epsilon_w > 0$ and a neighborhood V_w of λ_0 such that for any $z \in K \cap B(w, \delta_w)$, the map $\Psi_z : \lambda \mapsto g_\lambda(z)$ satisfies that $\Psi_z(V_w) \supset B(h_0(z), \epsilon_w)$. Since K is compact, there exists a finite family $\{B(w_j, \delta_{w_j})\}_{j=1}^s$ in $\{B(w, \delta_w)\}_{w\in K}$ such that $\bigcup_{j=1}^s B(w_j, \delta_{w_j}) \supset K$. From these arguments, the statement of our lemma holds. □

We now prove Lemma 3.16.

Proof of Lemma 3.16: Let $G = \langle\Gamma\rangle$. By Lemma 3.8, we have a bifurcation element for (Γ, L). Let $g \in \Gamma$ be a bifurcation element for (Γ, L). Suppose we have $g \in \mathrm{int}(\Gamma)$. We consider the following two cases. Case (1): (L, g) satisfies condition (1) in Definition 3.10. Case (2): (L, g) satisfies condition (2) in Definition 3.10.

We now consider Case (1). Then there exists a point $z \in L \cap J(G)$ such that $g(z) \in J(G)$. Let $\mathcal{U}$ be an open neighborhood of g in $\mathrm{int}(\Gamma)$. Let $A := \{h(z) \mid h \in \mathcal{U}\}$. Then A is an open subset of $\hat{\mathbb{C}}$ and $A \cap J(G) \neq \emptyset$. It follows that $\overline{G(A)} = \hat{\mathbb{C}}$. Since $A \subset L$, we obtain that $L = \hat{\mathbb{C}}$. However, this contradicts our assumption. Therefore, g must belong to $\partial\Gamma$.

We now consider Case (2). Let $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma, U$ be as in condition (2) in Definition 3.10. We set $h = g \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$. We may assume that U is a Siegel disk or Hermann ring of h. Then there exists a biholomorphic map $\zeta : U \to B$, where B is the unit disk or a round annulus, and a $\theta \in \mathbb{R} \setminus \mathbb{Q}$, such that $r_\theta \circ \zeta = \zeta \circ h$ on U, where $r_\theta(z) := e^{2\pi i\theta} z$. Let $z_0 \in L \cap U$ be a point. By Lemma 5.6, it follows that there exists an open subset W of $\hat{\mathbb{C}}$ such that $W \subset G(z_0)$ and $W \cap \partial U \neq \emptyset$. Therefore $J(G) \cap \mathrm{int}(L) \neq \emptyset$. Hence, we obtain $L = \hat{\mathbb{C}}$. However, this contradicts our assumption. Therefore, g must belong to $\partial\Gamma$.

Thus, we have proved Lemma 3.16. □

We now prove Theorem 3.17.

Proof of Theorem 3.17: Let $\mathcal{U}'$ be a small open neighborhood of Γ in $\mathcal{U}$. Let $\Gamma' \in \mathcal{U}'$ be an element such that $\Gamma \subset \mathrm{int}(\Gamma')$ with respect to the topology in the space $\mathcal{Y}$. If $\mathcal{U}'$ is so small, then Lemma 5.4 implies that for each $j = 1, \ldots, r$, there exists a unique element $L'_j \in \mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ with $L'_j \in \mathcal{V}_j$, and this L'_j is attracting for $(\langle\Gamma'\rangle, \hat{\mathbb{C}})$. Taking $\mathcal{U}'$ so small, the inclusion $\Gamma \subset \Gamma'$ and Remark 2.23 imply that for each $j = 1, \ldots, r$, L'_j is the unique element in $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ which contains L_j.

Suppose that there exists an element $L' \in \mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) \setminus \{L'_j\}_{j=1}^r$. Since $\langle\Gamma\rangle(L') \subset L'$, Remark 2.23 implies that there exists a minimal set $K \in \mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})$ such that $K \subset L'$. Since $L_j \subset L'_j$ for each $j = 1, \ldots r$, and since $L' \cap \bigcup_{j=1}^r L'_j = \emptyset$, we obtain that $K \notin \{L_j\}_{j=1}^r$. Hence K is not attracting for $(\langle\Gamma\rangle, \hat{\mathbb{C}})$. Let $g \in \Gamma$ be a bifurcation element for (Γ, K). Then, $g \in \mathrm{int}(\Gamma')$ and g is a bifurcation element for (Γ', L'). However, this contradicts Lemma 3.16. Therefore, $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{L'_j\}_{j=1}^r$. Moreover, from the above arguments and Remark 3.7, it follows that Γ' is mean stable and $\sharp(\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})) = r$. Thus we have proved Theorem 3.17. □

Lemma 5.7. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ be mean stable and suppose $J(\langle\Gamma\rangle)) \neq \emptyset$. Then, there exists an open neighborhood $\mathcal{U}$ of Γ in* Cpt(Rat) *with respect to the Hausdorff metric such that for each $\Gamma' \in \mathcal{U}$, $\langle\Gamma'\rangle$ is mean stable, $\sharp(J(\langle\Gamma'\rangle)) \geq 3$, and $\sharp\mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}}) = \sharp\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})$.*

Proof. Since Γ is mean stable, $J_{\ker}(\langle\Gamma\rangle) = \emptyset$. Combining this with that $J(\langle\Gamma\rangle) \neq \emptyset$ and [31, Theorem 3.15-3], we obtain $\sharp(J(\langle\Gamma\rangle)) \geq 3$. By [14, Theorem 3.1] and [24, Lemma 2.3(f)], the repelling cycles

of elements of $\langle\Gamma\rangle$ is dense in $J(\langle\Gamma\rangle)$. Combining it with implicit function theorem, we obtain that there exists a neighborhood $\mathcal{U}'$ of Γ in $\mathrm{Cpt}(\mathrm{Rat})$ such that for each $\Gamma' \in \mathcal{U}'$, $\sharp(J(\langle\Gamma'\rangle)) \geq 3$.

By [31, Theorem 3.15-6], $\sharp(\mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})) < \infty$. Let $S_\Gamma := \bigcup_{L\in \mathrm{Min}(\langle\Gamma\rangle,\hat{\mathbb{C}})} L$. By [31, Proposition 3.65], $S_\Gamma \subset F(\langle\Gamma\rangle)$. Let $W := \bigcup_{A\in \mathrm{Con}(F(\langle\Gamma\rangle)), A\cap S_\Gamma\neq\emptyset} A$. We use the notation in Definition 5.1 for this W. Let $0 < \epsilon_2 < \epsilon_1$. Since Γ is mean stable, there exists an $n \in \mathbb{N}$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(d_h(S_\Gamma, \epsilon_1)) \subset d_h(S_\Gamma, \epsilon_2)$. Moreover, for each $z \in \hat{\mathbb{C}}$, there exists a map $g_z \in \langle\Gamma\rangle$ such that $g_z(z) \in d_h(S_\Gamma, \epsilon_1)$. Therefore, there exist finitely many points $z_1, \dots z_s$ in $\hat{\mathbb{C}}$ and positive numbers $\delta_1, \dots, \delta_s$ with $\cup_{j=1}^s B(z_j, \delta_j) = \hat{\mathbb{C}}$ such that for each $j = 1, \dots, s$, $\overline{g_{z_j}(B(z_j, \delta_j))} \subset d_h(S_\Gamma, \epsilon_1)$. Let $\epsilon_3 \in (\epsilon_2, \epsilon_1)$. Let $\mathcal{U}(\subset \mathcal{U}')$ be a small neighborhood of Γ in $\mathrm{Cpt}(\mathrm{Rat})$. Then for each $\Gamma' \in \mathcal{U}$ and for each $\gamma \in \Gamma'^{\mathbb{N}}$, $\gamma_{n,1}(d_h(S_\Gamma, \epsilon_1)) \subset d_h(S_\Gamma, \epsilon_3)$. Moreover, for each $\Gamma' \in \mathcal{U}$ and for each $z \in \hat{\mathbb{C}}$, there exists a map $g_{z,\Gamma'} \in \langle\Gamma'\rangle$ such that $g_{z,\Gamma'}(z) \in d_h(S_\Gamma, \epsilon_1)$. Hence, for each $\Gamma' \in \mathcal{U}$, Γ' is mean stable and $\bigcup_{L'\in \mathrm{Min}(\langle\Gamma'\rangle,\hat{\mathbb{C}})} L' \subset d_h(S_\Gamma, \epsilon_1)$. Combining this with Lemma 5.2, and shrinking $\mathcal{U}$ if necessary, we obtain that for each $\Gamma' \in \mathcal{U}$, $\sharp(\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}}))$. □

We now prove Theorem 3.19.

Proof of Theorem 3.19: There exists a sequence $\{\tau_n\}_{n=1}^\infty$ in $\mathfrak{M}_{1,c}(\mathcal{Y})$ with $\sharp\Gamma_{\tau_n} < \infty (\forall n)$ such that $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ as $n \to \infty$. Therefore, by Lemma 5.4, we may assume that $\sharp\Gamma_\tau < \infty$. We write $\tau = \sum_{j=1}^s p_j \delta_{h_j}$, where $\sum_{j=1}^s p_j = 1$, $p_j > 0$ for each j, and $h_j \in \mathcal{Y}$ for each j. By Theorem 3.17, enlarging the support of τ, we obtain an element $\rho' \in \mathcal{U}$ such that statements (1) and (2) in our theorem with ρ being replaced by ρ' hold. Let ρ be a finite measure which is close enough to ρ'. By Lemma 5.7 and Lemma 5.4, we obtain that this ρ has the desired property. Thus we have proved Theorem 3.19. □

We now prove Theorem 3.20.

Proof of Theorem 3.20: Let $\tau \in \mathcal{M}_{1,c}(\mathcal{Y})$. Since Γ_τ is compact in $\mathcal{P}$, we obtain that $\{\infty\}$ is an attracting minimal set for $(G_\tau, \hat{\mathbb{C}})$. By Theorem 3.19 and Lemma 5.7, the statements in our theorem hold. □

We now prove Theorem 3.21.

Proof of Theorem 3.21: Let $\Gamma' \in \mathrm{Cpt}(\mathrm{Rat})$ be an element such that $\Gamma \subset \mathrm{int}(\Gamma')$ with respect to the topology in $\mathcal{Y}$. We now show the following claim.

Claim: $\mathrm{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$.

To prove this claim, suppose this is not true. Then $\mathrm{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}}) \neq \{\hat{\mathbb{C}}\}$. Since there exists no attracting minimal set for $(\langle\Gamma\rangle, \hat{\mathbb{C}})$, from Lemma 3.16 it follows that there exists a bifurcation element $g \in \Gamma$ for Γ. Then $g \in \mathrm{int}(\Gamma')$ and g is a bifurcation element for Γ'. However, this contradicts Lemma 3.16. Thus, we have proved the claim.

Let $h \in \mathrm{int}(\Gamma')$ be an element and let $z \in J(\langle\Gamma'\rangle)$ be a point which is not a critical value of h. Then we obtain that $\mathrm{int}(\langle\Gamma'\rangle^{-1}(\{z\})) \neq \emptyset$. Therefore, $K := \overline{F(\langle\Gamma'\rangle)}$ is not equal to $\hat{\mathbb{C}}$. By Remark 2.23 and the above claim, it follows that $K = \emptyset$. Thus $J(\langle\Gamma'\rangle) = \hat{\mathbb{C}}$. Hence, we have proved statement (1) in our theorem.

Statement (2) in our theorem easily follows from statement (1). □

We now prove Corollary 3.22.

Proof of Corollary 3.22: Let $\epsilon > 0$ be a small number. Let $\{h_{j,\epsilon}\}_{j=1}^\infty$ be a dense countable subset of $\overline{B(\Gamma_\tau, \epsilon)}$ with respect to the topology in $\mathcal{Y}$. Let $\{p_{j,\epsilon}\}_{j=1}^\infty$ be a sequence of positive numbers such that $\sum_{j=1}^\infty p_{j,\epsilon} = 1$. Let $\tau_\epsilon := (1-\epsilon)\tau + \epsilon \sum_{j=1}^\infty p_{j,\epsilon}\delta_{h_{j,\epsilon}}$. Then $\Gamma_\tau \subset \mathrm{int}(\Gamma_{\tau_\epsilon})$ and $\tau_\epsilon \to \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ as $\epsilon \to 0$. Let $\epsilon > 0$ be a small number and let $\rho := \tau_\epsilon$. By Theorem 3.21, this ρ has the desired property. □

We now prove Corollary 3.23.

Proof of Corollary 3.23: Corollary 3.23 easily follows from Theorem 3.19 and Corollary 3.22. □

Definition 5.8. Let $\mathcal{Y}$ be a closed subset of Rat. For each $\tau \in \mathfrak{M}_1(\mathcal{Y})$ and for each $n \in \mathbb{N}$, let $\tau^n := \otimes_{j=1}^n \tau \in \mathfrak{M}_1(\mathcal{Y}^n)$.

The following lemma is easily obtained by some fundamental observations. The proof is left to the readers.

Lemma 5.9. *If $\rho_n \to \rho$ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}^m), \mathcal{O})$ as $n \to \infty$ and if $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ as $n \to \infty$, then $\rho_n \otimes \tau_n \to \rho \otimes \tau$ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}^{m+1}), \mathcal{O})$ as $n \to \infty$. In particular, if $\nu_k \to \tau$ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ as $k \to \infty$, then $\nu_k^m \to \tau^m$ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}^m), \mathcal{O})$ as $k \to \infty$, for each $m \in \mathbb{N}$.*

We now prove Theorem 3.24.

Proof of Theorem 3.24: Statement 1 follows from Lemma 5.7. We now prove statements 2,3,4. Let Ω be a small open neighborhood of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that for each $\nu \in \Omega$, ν is mean stable, $\sharp(J(G_\nu)) \geq 3$ and $\sharp(\mathrm{Min}(G_\nu, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}}))$. For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, let U_L be an open neighborhood of L in $\mathrm{Cpt}(\hat{\mathbb{C}})$ such that $U_L \cap U_{L'} = \emptyset$ if $L, L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and $L \neq L'$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ be an element. Let $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. For each $r \in \mathbb{N}$ and each $\nu \in \Omega$, we set $\Lambda_{r,\nu} := \{h_r \circ \cdots \circ h_1 \mid h_j \in \Gamma_\nu (\forall j)\}$ and set $G_\nu^r := \langle \Lambda_{r,\nu} \rangle$. By [31, Theorem 3.15-12], we have $r_L = \sharp(\mathrm{Min}(G_\tau^{r_L}, L))$. Let $\{L_j\}_{j=1}^{r_L} = \mathrm{Min}(G_\tau^{r_L}, \hat{\mathbb{C}})$. By the proof of Lemma 5.16 in [31], we may assume that for each $j = 1, \ldots, r_L$ and for each $h \in \Gamma_\tau$, $h(L_j) \subset L_{j+1}$, where $L_{r_L+1} := L_1$. Moreover, by Lemma 5.4, shrinking Ω if necessary, we obtain that for each $\nu \in \Omega$, there exists a unique $Q_{L,\nu} \in \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ such that $Q_{L,\nu} \in U_L$. Moreover, by Lemma 5.4 again, we may assume that the map $\nu \mapsto Q_{L,\nu} \in \mathrm{Cpt}(\hat{\mathbb{C}})$ is continuous on Ω. For each $j = 1, \ldots, r_L$, let $V_{L,j} := B(L_j, \epsilon)$ (where $\epsilon > 0$ is a small number) such that $V_{L,i} \cap V_{L,j} = \emptyset$ if $i \neq j$. By Lemma 5.2, shrinking Ω if necessary, there exists a unique element $L_{j,\nu} \in \mathrm{Min}(G_\nu^{r_L}, \hat{\mathbb{C}})$ with $L_{j,\nu} \subset V_{L,j}$. By Lemma 5.4, we may assume that for each j, the map $\nu \mapsto L_{j,\nu}$ is continuous on Ω. Then, $\tilde{L}_{j+1,\nu} := \bigcup_{h \in \Gamma_\nu} h(L_{j,\nu})$ belongs to $\mathrm{Min}(G_\nu^{r_L}, \hat{\mathbb{C}})$ and shrinking Ω if necessary, we obtain $\tilde{L}_{j+1,\nu} \subset V_{L,j+1}$, where $V_{L,r_L+1} := V_{L,1}$. By the uniqueness statement of Lemma 5.2, it follows that for each $j = 1, \ldots, r_L$, we have $\tilde{L}_{j+1,\nu} = L_{j+1,\nu}$, where $L_{r_L+1,\nu} := L_{1,\nu}$. Since $\tilde{Q}_{L,\nu} := \bigcup_{j=1}^{r_L} L_{j,\nu}$ belongs to $\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ and $\tilde{Q}_{L,\nu} \in U_L$ (shrinking Ω if necessary), we obtain that $\tilde{Q}_{L,\nu} = Q_{L,\nu}$. From these arguments, it follows that for each $\nu \in \Omega$, $\mathrm{Min}(G_\nu^{r_L}, Q_{L,\nu}) = \{L_{j,\nu}\}_{j=1}^{r_L}$, $L_{j+1,\nu} = \bigcup_{h \in \Gamma_\nu} h(L_{j,\nu})$, and $\sharp(\mathrm{Min}(G_\nu^{r_L}, Q_{L,\nu})) = r_L$. We now prove the following claim.

Claim 1: For each $\nu \in \Omega$, $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))) \geq r_L$.

To prove this claim, let $a_L := \exp(2\pi i/r_L)$ and let $\psi_i := \sum_{j=1}^{r_L} a_L^{ij} 1_{L_{j,\nu}} \in C(Q_{L,\nu})$. Then $M_\nu(\psi_i) = \sum a_L^{ij} 1_{L_{j-1,\nu}} = a_L^i \sum a_L^{i(j-1)} 1_{L_{j-1,\nu}} = a_L^i \psi_i$, where $L_{0,\nu} := L_{r_L,\nu}$. Hence, the above claim holds.

For each $\nu \in \Omega$, let $\zeta_\nu := \{A \in \mathrm{Con}(F(G_\nu)) \mid A \cap Q_{L,\nu} \neq \emptyset\}$ and let $W_\nu := \bigcup_{A \in \zeta_\nu} A$. Shrinking Ω if necessary, we obtain that for each $A \in \zeta_\tau$, there exists a unique element $\alpha_\nu(A) \in \zeta_\nu$ such that $A \cap \alpha_\nu(A) \neq \emptyset$. It is easy to see that for each $\nu \in \Omega$, $\alpha_\nu : \zeta_\tau \to \zeta_\nu$ is bijective. This α_ν induces a linear isomorphism $\Psi_\nu : C_{W_\nu}(W_\nu) \cong C_{W_\tau}(W_\tau)$. Let $\tilde{M}_\nu : C_{W_\tau}(W_\tau) \to C_{W_\tau}(W_\tau)$ be the linear operator defined by $\tilde{M}_\nu := \Psi_\nu \circ M_\tau \circ \Psi_\nu^{-1}$. Then $\dim_{\mathbb{C}}(C_{W_\tau}(W_\tau)) < \infty$ and $\nu \mapsto (\tilde{M}_\nu : C_{W_\tau}(W_\tau) \to C_{W_\tau}(W_\tau))$ is continuous. Moreover, by [31, Theorem 3.15-8, Theorem 3.15-1], each unitary eigenvalue of $M_\tau : C_{W_\tau}(W_\tau) \to C_{W_\tau}(W_\tau)$ is simple. Therefore, taking Ω small enough, we obtain that the dimension of the space of finite linear combinations of unitary eigenvectors of $\tilde{M}_\nu : C_{W_\tau}(W_\tau) \to C_{W_\tau}(W_\tau)$ is less than or equal to r_L. Combining this with Claim 1 and [31, Theorem 3.15-10, Theorem 3.15-1], we obtain that statement 4 of our theorem holds. By these arguments, statements 2,3, 4 hold.

We now prove statement 5 of our theorem. For each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and each $i = 1, \ldots, r_L$, we set $\tilde{\psi}_{L,i} = \sum_{j=1}^{r_L} a_L^{ij} 1_{L_j} \in C(L)$. Then $M_\tau(\tilde{\psi}_{L,i}) = a_L^i \tilde{\psi}_{L,i}$. By [31, Theorem 3.15-9], there exists a unique element $\varphi_{L,i} \in C(\hat{\mathbb{C}})$ such that $\varphi_{L,i}|_L = \tilde{\psi}_{L,i}$, such that $\varphi_{L,i}|_{L'} = 0$ for any $L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L' \neq L$, and such that $M_\tau(\varphi_{L,i}) = a_L^i \varphi_{L,i}$. Similarly, by using the notation in the previous arguments, for each $\nu \in \Omega$, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, and for each $i = 1, \ldots, r_L$, we set $\tilde{\psi}_{L,i,\nu} := \sum_{j=1}^{r_L} a_L^{ij} 1_{L_{j,\nu}} \in C(Q_{L,\nu})$. By [31, Theorem 3.15-9], there exists a unique element $\varphi_{L,i,\nu} \in C(\hat{\mathbb{C}})$ such that $\varphi_{L,i,\nu}|_{Q_{L,\nu}} = \tilde{\psi}_{L,i,\nu}$, such that $\varphi_{L,i,\nu}|_{Q'} = 0$ for any $Q' \in \mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ with

$Q' \neq Q_{L,\nu}$, and such that $M_\nu(\varphi_{L,i,\nu}) = a_L^i \varphi_{L,i,\nu}$. By statement 4 of our theorem, it follows that $\{\varphi_{L,i,\nu}\}_{L\in \mathrm{Min}(G_\tau,\hat{\mathbb{C}}),i=1,\dots,r_L}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$.

Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $i = 1, \dots, r_L$. We now prove that $\nu \mapsto \varphi_{L,i,\nu} \in C(\hat{\mathbb{C}})$ is continuous on Ω. For simplicity, we prove that $\nu \mapsto \varphi_{L,i,\nu} \in C(\hat{\mathbb{C}})$ is continuous at $\nu = \tau$. In order to do that, let A_j be a relative compact open subset of $\hat{\mathbb{C}}$ such that each connected component of A_j intersects L_j, such that for each $\nu \in \Omega$, $L_{j,\nu} \subset A_j \subset \overline{A_j} \subset F(G_\nu)$, such that $\varphi_{L,i,\nu}|_{A_j} \equiv a_L^{ij}$, and such that $\{\overline{A_j}\}_{j=1}^{r_L}$ are mutually disjoint. For each $j = 1, \dots, r_L$, let A'_j be an open subset of A_j such that $L_j \subset A'_j \subset \overline{A'_j} \subset A_j$. Then there exists a number $s \in \mathbb{N}$ and a neighborhood Ω' of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ such that for each $j = 1, \dots, r_L$, for each $\nu \in \Omega'$, and for each $\gamma \in \Gamma_\nu^{\mathbb{N}}$, $\gamma_{s,1}(A_j) \subset A'_j$. Moreover, for each $K \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $K \neq L$, let B_K and B'_K be two open subsets of $\hat{\mathbb{C}}$ such that $K \subset B'_K \subset \overline{B'_K} \subset B_K \subset \overline{B_K} \subset F(G_\tau)$ and such that each connected component of B_K intersects K. Then shrinking Ω' if necessary, there exists a number $s_K \in \mathbb{N}$ such that for each $\nu \in \Omega'$ and for each $\gamma \in \Gamma_\nu^{\mathbb{N}}$, $\gamma_{s_K,1}(B_K) \subset B'_K$. We may assume that $s_K = s$ for each $K \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $K \neq L$. Let $C := \bigcup_{j=1}^{r_L} A_j \cup \bigcup_{K\in \mathrm{Min}(G_\tau,\hat{\mathbb{C}}),K\neq L} B_K$. Then for each $z \in \hat{\mathbb{C}}$, $\lim_{n\to\infty} \int_{(\mathrm{Rat})^{\mathbb{N}}} 1_C(\gamma_{n,1}(z)) d\tilde{\tau}(\gamma) = 1$. Let $\epsilon \in (0,1)$ be a small number. Let $z \in \hat{\mathbb{C}}$. Then there exists a number $l_z \in \mathbb{N}$ such that $\tau^{l_z}(\{(\gamma_1, \dots, \gamma_{l_z}) \in (\mathrm{Rat})^{l_z} \mid \gamma_{l_z} \circ \cdots \circ \gamma_1(z) \in C\}) \geq 1 - \epsilon$. Hence there exists a compact disk neighborhood U_z of z such that $\tau^{l_z}(\{(\gamma_1, \dots, \gamma_{l_z}) \in (\mathrm{Rat})^{l_z} \mid \gamma_{l_z} \circ \cdots \circ \gamma_1(U_z) \subset C\}) \geq 1 - 2\epsilon$. Let $\{z_k\}_{k=1}^t$ be a finite subset of $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} = \bigcup_{k=1}^t U_{z_k}$. We may assume that there exists an $l \in \mathbb{N}$ such that for each $k = 1, \dots, t$, $l_{z_k} = l$. Taking Ω' so small, we obtain that for each $\nu \in \Omega'$ and for each $k = 1, \dots, t$,

$$\nu^l(\{(\gamma_1, \dots, \gamma_l) \in (\mathrm{Rat})^l \mid \gamma_l \circ \cdots \circ \gamma_1(U_{z_k}) \subset C\}) \geq 1 - 3\epsilon. \tag{2}$$

For each $k = 1 \dots, t$ and for each $j = 1, \dots, r_L$, we set $B_{k,j} := \{(\gamma_1, \dots, \gamma_{sr_L l}) \in (\mathrm{Rat})^{sr_L l} \mid \gamma_{sr_L l} \circ \cdots \circ \gamma_1(U_{z_k}) \subset A_j\}$. We may assume that $\tau^{sr_L l}(\partial B_{k,j}) = 0$ for each k, j. By Lemma 5.9, taking Ω' so small, we obtain that for each $\nu \in \Omega'$, for each $k = 1, \dots, t$, and for each $j = 1, \dots, r_L$,

$$|\nu^{sr_L l}(B_{k,j}) - \tau^{sr_L l}(B_{k,j})| < \epsilon. \tag{3}$$

Let $z \in \hat{\mathbb{C}}$ and let $u \in \{1, \dots, t\}$ be such that $z \in U_{z_u}$. Then for each $\nu \in \Omega'$ and each $i = 1, \dots, r_L$, since $\varphi_{L,i,\nu} \in C_{F(G_\nu))}(\hat{\mathbb{C}})$ ([31, Theorem 3.15-1]), we obtain that

$$\begin{aligned}
&\varphi_{L,i,\nu}(z) = M_\nu^{sr_L l}(\varphi_{L,i,\nu})(z) = \int_{\gamma\in \mathrm{Rat}^{\mathbb{N}}} \varphi_{L,i,\nu}(\gamma_{sr_L l,1}(z)) d\tilde{\nu}(\gamma) \\
&= \int_{\{\gamma\in \mathrm{Rat}^{\mathbb{N}} | \gamma_{sr_L l,1}(U_{z_u}) \subset C\}} \varphi_{L,i,\nu}(\gamma_{sr_L l,1}(z)) d\tilde{\nu}(\gamma) + \int_{\{\gamma\in \mathrm{Rat}^{\mathbb{N}} | \gamma_{sr_L l,1}(U_{z_u}) \not\subset C\}} \varphi_{L,i,\nu}(\gamma_{sr_L l,1}(z)) d\tilde{\nu}(\gamma) \\
&= \sum_{j=1}^{r_L} a_L^{ij} \nu^{sr_L l}(B_{u,j}) + \int_{\{\gamma\in \mathrm{Rat}^{\mathbb{N}} | \gamma_{sr_L l,1}(U_{z_u}) \not\subset C\}} \varphi_{L,i,\nu}(\gamma_{sr_L l,1}(z)) d\tilde{\nu}(\gamma).
\end{aligned}$$

Combining this equation and (2), (3), we obtain $|\varphi_{L,i,\nu}(z) - \varphi_{L,i}(z)| \leq \sum_{j=1}^{r_L} \epsilon + 3\epsilon \cdot 2 = (r_L + 6)\epsilon$. Therefore, $\varphi_{L,i,\nu} \to \varphi_{L,i}$ in $C(\hat{\mathbb{C}})$ as $\nu \to \tau$. From these arguments, we obtain that $\nu \mapsto \varphi_{L,i,\nu}$ is continuous on Ω.

In order to construct $\{\rho_{L,i,\nu}\}$ in statement 5 of our theorem, let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$. By the proof of Lemma 5.16 in [31], for each $j = 1, \dots, r_L$, there exists an element $\omega_{L,j} \in \mathfrak{M}_1(L_j)$ such that for each $\varphi \in C(L_j)$, $M_\tau^{nr_L}(\varphi) \to \omega_{L,j}(\varphi) \cdot 1_{L_j}$ in $C(L_j)$ as $n \to \infty$. We now prove the following claim.

Claim 2. For each $\varphi \in C(\overline{A_j})$, $M_\tau^{nsr_L}(\varphi) \to \omega_{L,j}(\varphi) 1_{\overline{A_j}}$ in $C(\overline{A_j})$ as $n \to \infty$.

To prove this claim, let $\varphi \in C(\overline{A_j})$. Since $\overline{A_j} \subset F(G_\tau)$, $\{M_\tau^{nsr_L}(\varphi)\}_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous on $\overline{A_j}$. Let $z \in \overline{A_j}$ be any point. Let $D_z \in \mathrm{Con}(F(G_\tau))$ with $z \in D_z$ and let $w \in L_j \cap D_z$ be a point. By [31, Theorem 3.15-4], for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{Rat})^{\mathbb{N}}$, $d(\gamma_{nsr_L,1}(z), \gamma_{nsr_L,1}(w)) \to$

0 as $n \to \infty$. Therefore, $|M_\tau^{nsr_L}(\varphi)(z) - M_\tau^{nsr_L}(\varphi)(w)| \to 0$ as $n \to \infty$. From these arguments, it follows that there exists a constant function $\xi : \overline{A_j} \to \mathbb{R}$ such that $M_\tau^{nsr_L}(\varphi) \to \xi$ in $C(\overline{A_j})$ as $n \to \infty$. Thus, we have proved Claim 2.

By using the arguments similar to the above, we obtain that for each $\nu \in \Omega$ and for each $j = 1, \ldots, r_L$, there exists an element $\omega_{L,j,\nu} \in \mathfrak{M}_1(\overline{A_j})$ such that for each $\varphi \in C(L_{j,\nu})$, $M_\nu^{nr_L}(\varphi) \to \omega_{L,j,\nu}(\varphi)1_{L_{j,\nu}}$ in $C(L_{j,\nu})$ as $n \to \infty$, and such that for each $\varphi \in C(\overline{A_j})$, $M_\nu^{nsr_L}(\varphi) \to \omega_{L,j,\nu}(\varphi)1_{\overline{A_j}}$ in $C(\overline{A_j})$ as $n \to \infty$. Since $L_{j,\nu}$ is the unique minimal set for $(G_\nu^{r_L}, \overline{A_j})$ and $L_{j,\nu}$ is attracting for $(G_\nu^{r_L}, \hat{\mathbb{C}})$, we obtain $\operatorname{supp} \omega_{L,j,\nu} = L_{j,\nu}$. For each $\nu \in \Omega$, for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $i = 1, \ldots, r_L$, let $\rho_{L,i,\nu} := \frac{1}{r_L} \sum_{j=1}^{r_L} a_L^{-ij} \omega_{L,j,\nu} \in C(Q_{L,\nu})^* \subset C(\hat{\mathbb{C}})^*$. Then by the proofs of Lemmas 5.16 and 5.14 from [31], we obtain that $M_\nu^*(\rho_{L,i,\nu}) = a_L^i \rho_{L,i,\nu}$, that $\rho_{L,i,\nu}(\varphi_{L,j,\nu}) = \delta_{ij}$, that $\rho_{L,i,\nu}(\varphi_{L',j,\nu}) = 0$ if $L \neq L'$, that $\{\rho_{L,i,\nu}|_{C(Q_{L,\nu})} \mid i = 1, \ldots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu,*}(Q_{L,\nu}))$, that $\{\rho_{L,i,\nu} \mid L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ is a basis of $\mathrm{LS}(\mathcal{U}_{f,\nu,*}(\hat{\mathbb{C}}))$, and that $\pi_\nu(\varphi) = \sum_{L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})} \sum_{i=1}^{r_L} \rho_{L,i,\nu}(\varphi) \cdot \varphi_{L,i,\nu}$ for each $\varphi \in C(\hat{\mathbb{C}})$.

We now prove that for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $i = 1, \ldots, r_L$, the map $\nu \mapsto \rho_{L,i,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on Ω. For simplicity, we prove that $\nu \mapsto \rho_{L,i,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous at $\nu = \tau$. Let $\varphi \in C(\overline{A_j})$. Let $\epsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $\|M_\tau^{nsr_L}(\varphi) - \omega_{L,j}(\varphi)1_{\overline{A_j}}\|_\infty < \epsilon$, where $\|\psi\|_\infty := \sup_{z \in \overline{A_j}} |\psi(z)|$ for each $\psi \in C(\overline{A_j})$. If Ω' is a small open neighborhood of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$, then for each $\nu \in \Omega'$, $\|M_\nu^{nsr_L}(\varphi) - M_\tau^{nsr_L}(\varphi)\|_\infty < \epsilon$. Hence, for each $\nu \in \Omega'$, $\|M_\nu^{nsr_L}(\varphi) - \omega_{L,j}(\varphi)1_{\overline{A_j}}\|_\infty < 2\epsilon$. Therefore, for each $\nu \in \Omega'$ and for each $l \in \mathbb{N}$, $\|M_\nu^{lsr_L}(M_\nu^{nsr_L}(\varphi) - \omega_{L,j}(\varphi)1_{\overline{A_j}})\|_\infty < 2\epsilon$. Thus, $\|M_\nu^{(l+n)sr_L}(\varphi) - \omega_{L,j}(\varphi)1_{\overline{A_j}}\|_\infty < 2\epsilon$. Moreover, $M_\nu^{(l+n)sr_L}(\varphi) \to \omega_{L,j,\nu}(\varphi)1_{\overline{A_j}}$ in $C(\overline{A_j})$ as $l \to \infty$. Hence, we obtain that for each $\nu \in \Omega'$, $|\omega_{L,\nu,j}(\varphi) - \omega_{L,j}(\varphi)| \leq 2\epsilon$. From these arguments, it follows that the map $\nu \mapsto \omega_{L,j,\nu} \in C(\overline{A_j})^* \subset C(\hat{\mathbb{C}})^*$ is continuous at $\nu = \tau$. Therefore, for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $i = 1, \ldots, r_L$, the map $\nu \mapsto \rho_{L,i,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous at $\nu = \tau$. Thus, for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$ and for each $i = 1, \ldots, r_L$, the map $\nu \mapsto \rho_{L,i,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on Ω. Hence, we have proved statement 5 of our theorem.

We now prove statement 6 of our theorem. For each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$, let V_L be an open subset of $F(G_\tau)$ with $L \subset V_L$ such that for each $L, L' \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \neq L'$, $\overline{V_L} \cap \overline{V_{L'}} = \emptyset$. By statement 2 and Lemma 5.4, for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$, there exists a continuous map $\nu \mapsto Q_{L,\nu} \in \operatorname{Cpt}(\hat{\mathbb{C}})$ on Ω with respect to the Hausdorff metric such that $Q_{L,\tau} = L$, such that for each $\nu \in \Omega$, $\{Q_{L,\nu}\}_{L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})} = \operatorname{Min}(G_\nu, \hat{\mathbb{C}})$, and such that for each $\nu \in \Omega$ and for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$, $Q_{L,\nu} \subset V_L$. For each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$, let $\varphi_L : \hat{\mathbb{C}} \to [0,1]$ be a continuous function such that $\varphi_L|_{V_L} \equiv 1$ and $\varphi_L|_{V_{L'}} \equiv 0$ for each $L' \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$ with $L' \neq L$. By [31, Theorem 3.15-15], it follows that for each $z \in \hat{\mathbb{C}}$ and for each $\nu \in \Omega$, $T_{Q_{L,\nu},\nu}(z) = \lim_{n \to \infty} M_\nu^n(\varphi_L)(z)$. Combining this with [31, Theorem 3.14], we obtain $T_{Q_{L,\nu},\nu} = \lim_{n \to \infty} M_\nu^n(\varphi_L)$ in $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$. By [31, Theorem 3.15-6,8,9], for each $\nu \in \Omega$ there exists a number $r \in \mathbb{N}$ such that for each $\psi \in \mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$, $M_\nu^r(\psi) = \psi$. Therefore, for each $\nu \in \Omega$ and for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$, $T_{Q_{L,\nu},\nu} = \lim_{n \to \infty} M_\nu^{nr}(\varphi_L) = \lim_{n \to \infty} M_\nu^{nr}(\varphi_L - \pi_\nu(\varphi_L) + \pi_\nu(\varphi_L)) = \pi_\nu(\varphi_L)$. Combining this with statement 5 of our theorem, it follows that for each $L \in \operatorname{Min}(G_\tau, \hat{\mathbb{C}})$, the map $\nu \mapsto T_{Q_{L,\nu},\nu} \in (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ is continuous on Ω. Thus, we have proved statement 6 of our theorem.

Hence, we have proved Theorem 3.24. □

We now prove Theorem 3.25.

Proof of Theorem 3.25: It is trivial that $B \subset C$. By Theorem 3.24, we obtain that $A \subset B$ and $A \subset D$. In order to show $C \subset A$, let $\tau \in C$. If there exists a non-attracting minimal set for $(G_\tau, \hat{\mathbb{C}})$, or if there exists no attracting minimal set for $(G_\tau, \hat{\mathbb{C}})$, then by Theorem 3.17 and Corollary 3.22, we obtain a contradiction. Hence, $C \subset A$. Therefore, we obtain $A = B = C$. In order to show $D \subset A$, let $\tau \in D$. By [31, Theorem 3.15-10], we have $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) =$

$\sum_{L\in \mathrm{Min}(G_\tau,\hat{\mathbb{C}})} \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. By Corollary 3.22, there exists an attracting minimal set for $(G_\tau, \hat{\mathbb{C}})$. Theorem 3.17 implies that if Ω is a small neighborhood of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$, then for each $\nu \in \Omega$ and for each attracting minimal set L for $(G_\tau, \hat{\mathbb{C}})$, there exists a unique attracting minimal set $Q_{L,\nu}$ for $(G_\nu, \hat{\mathbb{C}})$ which is close to L. By [31, Theorem 3.15-12] and the arguments in the proof of Theorem 3.24, it follows that if Ω is small enough, then for each $\nu \in \Omega$ and for each attracting minimal set L for $(G_\tau, \hat{\mathbb{C}})$, $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(Q_{L,\nu}))) = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. Combining this, Theorem 3.19 and [31, Theorem 3.15-10], we obtain that if there exists a non-attracting minimal set L' for $(G_\tau, \hat{\mathbb{C}})$, then there exists a $\nu' \in \Omega$ such that $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu'}(\hat{\mathbb{C}}))) < \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$. However, this contradicts $\tau \in D$. Therefore, we obtain that each element $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is attracting for $(G_\tau, \hat{\mathbb{C}})$. By Remark 3.7, it follows that $\tau \in A$. Therefore, $D \subset A$.

From these arguments, we obtain $A = B = C = D$.

By Theorem 3.24, we obtain that $A \subset E$. In order to show $E \subset A$, let $\tau \in E$. Suppose that there exists a non-attracting minimal set K for $(G_\tau, \hat{\mathbb{C}})$. Since there exists a neighborhood Ω' of τ such that each $\nu \in \Omega'$ satisfies $J_{\mathrm{ker}}(G_\nu) = \emptyset$, Corollary 3.22 implies that there exists an attracting minimal set for $(G_\tau, \hat{\mathbb{C}})$. Moreover, since $J_{\mathrm{ker}}(G_\tau) = \emptyset$ and $\sharp(J(G_\tau)) \geq 3$, [31, Theorem 3.15-6] implies that $\sharp(\mathrm{Min}(G_\tau, \hat{\mathbb{C}})) < \infty$. Let $\epsilon := \min\{d(z,w) \mid z \in K, w \in L, L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), L \neq K\} > 0$. Let $\varphi \in C(\hat{\mathbb{C}})$ be an element such that $\varphi|_K \equiv 1$ and $\varphi|_{\hat{\mathbb{C}}\setminus B(K,\epsilon/2)} \equiv 0$. Then by [31, Theorem 3.15-13], $\pi_\tau(\varphi) \neq 0$. Since $\tau \in E$, there exists an open neighborhood Ω of τ such that for each $\nu \in \Omega$, $J_{\mathrm{ker}}(G_\nu) = \emptyset$ and such that the map $\nu \mapsto \pi_\nu(\varphi) \in C(\hat{\mathbb{C}})$ defined on Ω is continuous at τ. By Theorem 3.19, for each neighborhood $\mathcal{U}$ of τ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$, there exists an element $\rho \in \mathcal{U} \cap A$ such that each minimal set for $(G_\rho, \hat{\mathbb{C}})$ is included in $\hat{\mathbb{C}} \setminus B(K, \epsilon/2)$. Therefore, by [31, Theorem 3.15-2], $\pi_\rho(\varphi) = 0$. However, this contradicts that the map $\nu \mapsto \pi_\nu(\varphi) \in C(\hat{\mathbb{C}})$ is continuous at τ and that $\pi_\tau(\varphi) \neq 0$. Thus, each element of $\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ is attracting for $(G_\tau, \hat{\mathbb{C}})$. By Remark 3.7, it follows that $\tau \in A$. Hence, we have proved $E \subset A$.

Thus, we have proved Theorem 3.25. □

To prove Theorem 3.26, we need the following.

Lemma 5.10. *Let $\mathcal{Y}$ be a subset of* Rat_+ *satisfying condition* $(*)$. *For each $t \in [0,1]$. let μ_t be an element of $\mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that all conditions* (1)(2)(3) *in Theorem 3.26 are satisfied. Then, statements* (a) *and* (c) *in Theorem 3.26 hold. Moreover, $\sharp\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}}) < \infty$ for each $t \in [0,1]$.*

Proof. Since $F(G_{\mu_1}) \neq \emptyset$ and $G_{\mu_t} \subset G_{\mu_1}$ for each $t \in [0,1]$, we obtain that $F(G_{\mu_t}) \neq \emptyset$. Moreover, we have that for each $t \in [0,1]$, $\mathrm{int}(\Gamma_{\mu_t}) \neq \emptyset$ in the topology of $\mathcal{Y}$. Therefore, [31, Lemma 5.34] implies that for each $t \in [0,1]$, $J_{\mathrm{ker}}(G_{\mu_t}) = \emptyset$. Moreover, since $\mathcal{Y} \subset \mathrm{Rat}_+$, we have that for each $t \in [0,1]$, $\sharp J(G_{\mu_t}) \geq 3$. Thus, by [31, Theorem 3.15], it follows that for each $t \in [0,1]$, all statements (with $\tau = \mu_t$) in [31, Theorem 3.15] hold. In particular, $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) < \infty$ for each $t \in [0,1]$, and statement (a) of Theorem 3.26 holds.

To show that statement (c) of Theorem 3.26, it suffices to show that there exists an element $u \in [0,1)$ such that for each $t \in [u,1]$, $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$. In order to show it, we first note that by Zorn's lemma, we have

$$\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) \geq \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})) \text{ for each } t \in [0,1]. \tag{4}$$

Let $\{L_j\}_{j=1}^r = \mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})$, where $L_i \neq L_j$ for each (i,j) with $i \neq j$. Since $J_{\mathrm{ker}}(G_{\mu_1}) = \emptyset$, there exists an element $w_j \in L_j \cap F(G_{\mu_1})$ for each $j = 1, \ldots, r$. Let $\epsilon > 0$ be a small number such that $W := \bigcup_{j=1}^r B(w_j, \epsilon) \subset F(G_{\mu_1})$. Since $J_{\mathrm{ker}}(G_{\mu_1}) = \emptyset$, [31, Theorem 3.15-7] implies that for each $z \in \hat{\mathbb{C}}$, there exists an element $g_z \in G_{\mu_1}$ and a neighborhood V_z of z in $\hat{\mathbb{C}}$ such that $g_z(\overline{V_z}) \subset W$. Since $\hat{\mathbb{C}}$ is compact, there exist finitely many points $z_1, \ldots, z_n \in \hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} = \bigcup_{j=1}^n V_{z_j}$. Then there exists an element $u \in [0,1)$ such that for each $j = 1, \ldots, n$ and for each $t \in [u,1]$, there exists an element $g_{z_j,t} \in G_{\mu_t}$ with $g_{z_j,t}(\overline{V_j}) \subset W$. Moreover, we have $G_{\mu_t} \subset G_{\mu_1}$ and $F(G_{\mu_1}) \subset F(G_{\mu_t})$ for each $t \in [u,1]$. Applying [31, Theorem 3.15-4] (with $\tau = \mu_t, t \in [u,1]$), it follows that for

each $t \in [u,1]$ and for each $L \in \mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})$, there exists a unique element $L' \in \mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})$ with $L \subset L'$. Therefore $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) \leq \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$ for each $t \in [u,1]$. Combining this with (4), we obtain $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) = \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$ for each $t \in [u,1]$. Thus we have proved our lemma. □

We now prove Theorem 3.26.

Proof of Theorem 3.26: By Lemma 5.10, statements (a) and (c) of our theorem hold and $\sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}})) < \infty$ for each $t \in [0,1]$.

We now prove statement (b). By Lemma 3.8 and Remark 3.7, we obtain that for each $t \in B$, μ_t is not mean stable, and that for each $t \in [0,1) \setminus B$, μ_t is mean stable. Combining this with assumption (4) and Lemma 5.7, we obtain that $B \neq \emptyset$. We now let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$. By assumption (2) and Remark 2.23, for each $L \in \mathrm{Min}(G_{\mu_{t_2}}, \hat{\mathbb{C}})$, there exists an $L' \in \mathrm{Min}(G_{\mu_{t_1}}, \hat{\mathbb{C}})$ with $L' \subset L$. In particular, $\infty > \sharp(\mathrm{Min}(G_{\mu_{t_1}}, \hat{\mathbb{C}})) \geq \sharp(\mathrm{Min}(G_{\mu_{t_2}}, \hat{\mathbb{C}}))$. We now let $t_0 \in [0,1)$ be such that there exists a bifurcation element $g \in \Gamma_{\mu_{t_0}}$ for $\Gamma_{\mu_{t_0}}$. Let $t \in [0,1]$ with $t > t_0$. Then $\Gamma_{\mu_{t_0}} \subset \mathrm{int}(\Gamma_{\mu_t})$. By the above argument, Theorem 3.17, Corollary 3.22 and assumption (3) of our theorem, it follows that $\sharp(\mathrm{Min}(G_{\mu_{t_0}}, \hat{\mathbb{C}})) > \sharp(\mathrm{Min}(G_{\mu_t}, \hat{\mathbb{C}}))$. From these arguments, it follows that $1 \leq \sharp B \leq \sharp(\mathrm{Min}(G_{\mu_0}, \hat{\mathbb{C}})) - \sharp(\mathrm{Min}(G_{\mu_1}, \hat{\mathbb{C}})) < \infty$.

Thus, we have proved Theorem 3.26. □

5.2 Proofs of results in 3.2

In this subsection, we give the proofs of the results in subsection 3.2.

We now prove Theorem 3.29.

Proof of Theorem 3.29: By [31, Theorem 3.15-6,8,9], there exists an $r \in \mathbb{N}$ such that for each $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$, $M_\tau^r(\varphi) = \varphi$. Since $J_{\ker}(G_\tau) = \emptyset$, for each $z \in \hat{\mathbb{C}}$, there exists a map $g_z \in G_\tau$ and a compact disk neighborhood U_z of z in $\hat{\mathbb{C}}$ such that $g_z(U_z) \subset F(G_\tau)$. Since $\hat{\mathbb{C}}$ is compact, there exists a finite family $\{z_j\}_{j=1}^s$ in $\hat{\mathbb{C}}$ such that $\bigcup_{j=1}^s \mathrm{int}(U_{z_j}) = \hat{\mathbb{C}}$. Since $G_\tau(F(G_\tau)) \subset F(G_\tau)$, replacing r by a larger number if necessary, we may assume that for each $j = 1, \ldots, s$, there exists an element $\beta^j = (\beta_1^j, \ldots, \beta_r^j) \in \Gamma_\tau^r$ such that $g_{z_j} = \beta_r^j \circ \cdots \circ \beta_1^j$. For each $j = 1, \ldots, s$, let V_j be a compact neighborhood of β^j in Γ_τ^r such that for each $\zeta = (\zeta_1, \ldots, \zeta_r) \in V_j$, $\zeta_r \cdots \zeta_1(U_{z_j}) \subset F(G_\tau)$. Let $a := \max\{\tau^r(\Gamma_\tau^r \setminus V_j) \mid j = 1, \ldots, s\} \in [0,1)$. Let

$$C_1 := 2 \max\{\max\{\|D(\zeta_r \circ \cdots \circ \zeta_1)_z\|_s \mid (\zeta_1, \ldots, \zeta_r) \in \Gamma_\tau^r, z \in \hat{\mathbb{C}}\}, 1\} \geq 2.$$

Let $\alpha \in [0,1)$ be a number such that $aC_1^\alpha < 1$. Let $C_2 > 0$ be a number such that for each $z \in \hat{\mathbb{C}}$, there exists a $j \in \{1, \ldots, s\}$ with $B(z, C_2) \subset \mathrm{int}(U_{z_j})$. Let $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$. Let $z_0, z \in \hat{\mathbb{C}}$ be two points. If $d(z, z_0) > C_1^{-1} C_2$, then

$$|\varphi(z) - \varphi(z_0)| / d(z, z_0)^\alpha \leq 2\|\varphi\|_\infty \cdot (C_1 C_2^{-1})^\alpha.$$

We now suppose that there exists an $n \in \mathbb{N}$ such that $C_1^{-n-1} C_2 \leq d(z, z_0) \leq C_1^{-n} C_2$. Then, for each $j \in \mathbb{N}$ with $1 \leq j \leq n$ and for each $(\gamma_1, \ldots, \gamma_{rj}) \in \Gamma_\tau^{rj}$, we have $d(\gamma_{rj} \circ \cdots \circ \gamma_1(z), \gamma_{rj} \circ \cdots \circ \gamma_1(z_0)) < C_2$. Let $i_0 \in \{1, \ldots, s\}$ be a number such that $B(z_0, C_2) \subset U_{z_{i_0}}$. Let $A(0) := \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid (\gamma_1, \ldots, \gamma_r) \in V_{i_0}\}$ and $B(0) := \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid (\gamma_1, \ldots, \gamma_r) \notin V_{i_0}\}$. Inductively, for each $j = 1, \ldots, n-1$, let $A(j) := \{\gamma \in B(j-1) \mid \exists i \text{ s.t. } B(\gamma_{rj,1}(z_0), C_2) \subset U_{z_i}, (\gamma_{rj+1}, \ldots, \gamma_{rj+r}) \in V_i\}$ and $B(j) := B(j-1) \setminus A(j)$. Then for each $j = 1, \ldots, n-1$, $\tilde{\tau}(B(j)) \leq a\tilde{\tau}(B(j-1))$. Therefore, $\tilde{\tau}(B(n-1)) \leq a^n$. Moreover, we have $\Gamma_\tau^{\mathbb{N}} = \amalg_{j=0}^{n-1} A(j) \amalg B(n-1)$. Furthermore, by [31, Theorem

3.15-1], $\varphi \in C_{F(G_\tau)}(\hat{\mathbb{C}})$. Thus, we obtain that

$$\begin{aligned}
&|\varphi(z)-\varphi(z_0)| = |M_\tau^{rn}(\varphi)(z) - M_\tau^{rn}(\varphi)(z_0)| \\
\leq &|\sum_{j=0}^{n-1}\int_{A(j)} \varphi(\gamma_{rn,1}(z)) - \varphi(\gamma_{rn,1}(z_0))d\tilde{\tau}(\gamma)| + |\int_{B(n-1)} \varphi(\gamma_{rn,1}(z)) - \varphi(\gamma_{rn,1}(z_0))d\tilde{\tau}(\gamma)| \\
\leq &\int_{B(n-1)} |\varphi(\gamma_{rn,1}(z)) - \varphi(\gamma_{rn,1}(z_0))|d\tilde{\tau}(\gamma) \\
\leq &2a^n\|\varphi\|_\infty \leq a^n(C_1^{n+1}C_2^{-1})^\alpha d(z,z_0)^\alpha 2\|\varphi\|_\infty \leq C_1^\alpha C_2^{-\alpha} 2\|\varphi\|_\infty d(z,z_0)^\alpha.
\end{aligned}$$

From these arguments, it follows that φ belongs to $C^\alpha(\hat{\mathbb{C}})$.

Let $\{\rho_j\}_{j=1}^q$ be a basis of $\mathrm{LS}(\mathcal{U}_{f,\tau,*}(\hat{\mathbb{C}}))$ and let $\{\varphi_j\}_{j=1}^q$ be a basis of $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ such that for each $\psi \in C(\hat{\mathbb{C}})$, $\pi_\tau(\psi) = \sum_{j=1}^q \rho_j(\psi)\varphi_j$. Then for each $\psi \in C(\hat{\mathbb{C}})$, $\|\pi_\tau(\psi)\|_\alpha \leq \sum_{j=1}^q |\rho_j(\psi)|\|\varphi_j\|_\alpha \leq (\sum_{j=1}^q \|\rho_j\|_\infty\|\varphi_j\|_\alpha)\|\psi\|_\infty$, where $\|\rho_j\|_\infty$ denotes the operator norm of $\rho_j : (C(\hat{\mathbb{C}}), \|\cdot\|_\infty) \to \mathbb{C}$.

We now let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $\alpha \in (0, \alpha_0)$. By [31, Theorem 3.15-15], $T_{L,\tau} \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$. Thus $T_{L,\tau} \in C^\alpha(\hat{\mathbb{C}})$.

Thus, we have proved Theorem 3.29. □

Remark 5.11. Suppose $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, $J(G_\tau) \neq \emptyset$ and that τ is mean stable. Then by using Theorem 3.24–1 and the method of proof of Theorem 3.29, it is easy to see that there exists an $\alpha \in (0,1)$ and a neighborhood $\mathcal{U}$ of τ in $\mathfrak{M}_{1,c}(\mathrm{Rat})$ such that for each $\nu \in \mathcal{U}$, we have that ν is mean stable and $\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$.

In order to prove Theorem 3.30, we need several lemmas. Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose $J_{\mathrm{ker}}(G_\tau) = \emptyset$ and $\sharp J(G_\tau) \geq 3$. Then all statements in [31, Theorem 3.15] hold. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. By using the notation in the proof of Theorem 3.24, by [31, Theorem 3.15-12], we have $r_L = \sharp(\mathrm{Min}(G_\tau^{r_L}, L))$.

Lemma 5.12. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose $J_{\mathrm{ker}}(G_\tau) = \emptyset$ and $\sharp J(G_\tau) \geq 3$. Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ and let $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. Let $\{L_j\}_{j=1}^{r_L} = \mathrm{Min}(G_\tau^{r_L}, L)$. For each j, let $\{A_i\}_{i\in I_j}$ be the set $\{A \in \mathrm{Con}(F(G_\tau)) \mid A \cap L_j \neq \emptyset\}$. Let $W_{L,j} := \bigcup_{i\in I_j} A_i$. For each $i \in I_j$, we take the hyperbolic metric in A_i. Then, there exists an $m \in \mathbb{N}$ with $r_L|m$ such that for each j and for each $\alpha \in (0,1)$, $\sup_{i\in I_j}\int_{\mathrm{Rat}^{\mathbb{N}}} \sup_{z\in A_i}\{\|D(\gamma_{m,1})_z\|_h^\alpha\}d\tilde{\tau}(\gamma) < 1$.*

Proof. For each $g \in G_\tau^{r_L}$, we have $g(\overline{W_{L,j}}) \subset \overline{W_{L,j}}$. Combining this with [31, Theorem 3.15-7], we obtain that for each $z \in \partial W_{L,j}$, there exists a map $g_z \in G_\tau$ and an open disk neighborhood U_z of z such that $g_z(\overline{U_z}) \subset W_{L,j}$. Then there exists a finite family $\{z_l\}_{l=1}^t$ such that $\partial W_{L,j} \subset \bigcup_{l=1}^t U_{z_l}$. Since $G_\tau^{r_L}(W_{L,j}) \subset W_{L,j}$, we may assume that there exists a $k \in \mathbb{N}$ with $r_L|k$ and a finite family $\{a^l = (\alpha_1^l, \ldots, \alpha_k^l) \in \Gamma_\tau^k\}_{l=1}^t$ such that for each $l = 1, \ldots, t$, $g_{z_l} = \alpha_k^l \circ \cdots \circ \alpha_1^l$. Let $K_0 := (W_{L,j} \setminus \bigcup_{l=1}^t U_{z_l}) \cup \bigcup_{l=1}^t g_{z_l}(U_{z_l})$ and let $\{B_1, \ldots, B_u\}$ be the set $\{A \in \mathrm{Con}(F(G_\tau)) \mid A \cap K_0 \neq \emptyset\}$. By [31, Theorem 3.15-4], for each $v = 1, \ldots u$, there exists an element $h_v \in G_\tau$ such that $\sup_{z\in K_0\cap B_v} \|D(h_v)_z\|_h < 1$. We may assume that there exists a $k' \in \mathbb{N}$ with $k|k'$ such that for each v, the element h_v is a product of k'-elements of Γ_τ. Let $m = 2k'$. Then this m is the desired number. □

Lemma 5.13. *Let $\Lambda \in \mathrm{Cpt}(\mathrm{Rat})$ and let $G = \langle\Lambda\rangle$. Suppose that $\sharp(J(G)) \geq 3$. For each element $A \in \mathrm{Con}(F(G))$, we take the hyperbolic metric in A. Let K be a compact subset of $F(G)$. Then, there exists a positive constant C_K such that for each $g \in G$ and for each $z \in K$, $\|Dg_z\|_s/\|Dg_z\|_h \leq C_K$.*

Proof. By conjugating G by an element of $\mathrm{Aut}(\hat{\mathbb{C}})$, we may assume that $\infty \in J(G)$. For each $U \in \mathrm{Con}(F(G))$, let $\rho_U = \rho_U(z)|dz|$ be the hyperbolic metric on U. Since G is generated by a compact subset of Rat, [22] implies that $J(G)$ is uniformly perfect (for the definition of uniform

perfectness, see [22] and [3]). Therefore, by [3], there exists a constant $C_1 \geq 1$ such that for each $U \in \mathrm{Con}(F(G))$ and for each $z \in U$, $C_1^{-1}\frac{1}{d_e(z,\partial U)} \leq \rho_U(z) \leq C_1\frac{1}{d_e(z,\partial U)}$, where $d_e(z,\partial U) := \inf\{|z-w| \mid w \in \partial U \cap \mathbb{C}\}$. Let $z_0 \in J(G)$ be a point. Let $g \in G$ and let $z \in K$. Let $U, V \in \mathrm{Con}(F(G))$ be such that $z \in U$ and $g(z) \in V$. Then

$$\begin{aligned}\|Dg_z\|_s/\|Dg_z\|_h &= \frac{\sqrt{1+|z|^2}}{\sqrt{1+|g(z)|^2}}\frac{\rho_U(z)}{\rho_V(g(z))} \leq \frac{\sqrt{1+|z|^2}}{\sqrt{1+|g(z)|^2}}C_1^2\frac{d_e(g(z),\partial V)}{d_e(z,\partial U)}\\ &\leq \frac{\sqrt{1+|z|^2}}{\sqrt{1+|g(z)|^2}}C_1^2\frac{|z_0|+|g(z)|}{d_e(z,\partial U)}.\end{aligned}$$

Therefore the statement of our lemma holds. □

Lemma 5.14. *Under the notations and assumptions of Lemma 5.12, let $j \in \{1,\ldots,r_L\}$. For each $\alpha \in (0,1)$, let $\theta_\alpha := \sup_{i\in I_j}\int_{(\mathrm{Rat})^{\mathbb{N}}}\sup_{z\in A_i}\{\|D(\gamma_{m,1})_z\|_h^\alpha\}d\tilde{\tau}(\gamma)(<1)$, where m is the number in Lemma 5.12. Then, we have the following.*

(1) *For each $n \in \mathbb{N}$, $\sup_{i\in I_j}\int_{(\mathrm{Rat})^{\mathbb{N}}}\sup_{z\in A_i}\{\|D(\gamma_{nm,1})_z\|_h^\alpha\} \leq \theta_\alpha^n$.*

(2) *Let $i \in I_j$ and let K be a non-empty compact subset of A_i. Then there exists a constant $\tilde{C}_K \geq 1$ such that for each $\alpha \in (0,1)$, for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$, for each $z,w \in K$, and for each $n \in \mathbb{N}$, $|M_\tau^{nm}(\varphi)(z) - M_\tau^{nm}(\varphi)(w)| \leq \|\varphi\|_\alpha\theta_\alpha^n\tilde{C}_K d(z,w)^\alpha$.*

Proof. Let $i \in I_j$ and let $\alpha \in (0,1)$. Then we have

$$\begin{aligned}&\int_{\Gamma_\tau^{\mathbb{N}}}\sup_{z\in A_i}\|D(\gamma_{nm,1})_z\|_h^\alpha d\tilde{\tau}(\gamma)\\ \leq&\sum_{k\in I_j}\int_{\{\gamma\in\Gamma_\tau^{\mathbb{N}}\mid\gamma_{(n-1)m,1}(A_i)\subset A_k\}}\sup_{z\in A_i}\{\|D(\gamma_{nm,(n-1)m+1})_{\gamma_{(n-1)m,1}(z)}\|_h^\alpha\cdot\|D(\gamma_{(n-1)m,1})_z\|_h^\alpha\}d\tilde{\tau}(\gamma)\\ \leq&\sum_{k\in I_j}\theta_\alpha\int_{\{\gamma\in\Gamma_\tau^{\mathbb{N}}\mid\gamma_{(n-1)m,1}(A_i)\subset A_k\}}\sup_{z\in A_i}\|D(\gamma_{(n-1)m,1})_z\|_h d\tilde{\tau}(\gamma)\\ =&\theta_\alpha\int_{\Gamma_\tau^{\mathbb{N}}}\sup_{z\in A_i}\|D(\gamma_{(n-1)m,1})_z\|_h d\tilde{\tau}(\gamma).\end{aligned}$$

Therefore, statement (1) of our lemma holds.

We now prove statement (2) of our lemma. Let $\tilde{K}$ be a compact subset of A_i such that for each $a,b \in K$, the geodesic arc between a and b with respect to the hyperbolic metric on A_i is included in $\tilde{K}$. Let $C_{\tilde{K}}$ be the number obtained in Lemma 5.13 with $\Lambda = \Gamma_\tau$. Let $\tilde{C}_K := C_{\tilde{K}}$. Let $\alpha \in (0,1)$, $\varphi \in C^\alpha(\hat{\mathbb{C}})$ and let $z,w \in K$. Let $n \in \mathbb{N}$. Then we obtain

$$\begin{aligned}|M_\tau^{nm}(\varphi)(z) - M_\tau^{nm}(\varphi)(w)| &\leq \int_{\Gamma_\tau^{\mathbb{N}}}|\varphi(\gamma_{nm,1}(z)) - \varphi(\gamma_{nm,1}(w))|d\tilde{\tau}(\gamma)\\ &\leq \int_{\Gamma_\tau^{\mathbb{N}}}\|\varphi\|_\alpha d(\gamma_{nm,1}(z),\gamma_{nm,1}(w))^\alpha d\tilde{\tau}(\gamma)\\ &\leq \|\varphi\|_\alpha\int_{\Gamma_\tau^{\mathbb{N}}}\tilde{C}_K^\alpha\sup_{a\in A_i}\{\|D(\gamma_{nm,1})_a\|_h^\alpha\}d(z,w)^\alpha d\tilde{\tau}(\gamma) \leq \|\varphi\|_\alpha\theta_\alpha^n\tilde{C}_K d(z,w)^\alpha.\end{aligned}$$

Therefore, statement (2) of our lemma holds. □

We now prove Theorem 3.30.

Proof of Theorem 3.30: Let $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$. Let $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$. By using the notation in the proof of Theorem 3.24, let $\{L_j\}_{j=1}^{r_L} = \mathrm{Min}(G_\tau^{r_L}, L)$. For each $j \in \{1,\ldots,r_L\}$, let

$W_{L,j} := \bigcup_{A\in \mathrm{Con}(F(G_\tau)):A\cap L_j\neq\emptyset} A$. For each A$\in$ Con$(W_{L,j})$, we take the hyperbolic metric on A. Let $H_j := d_h(L_j, 1)$ be the 1-neighborhood of L_j in $W_{L,j}$ with respect to the hyperbolic metric (see Definition 5.1). Let $\{A_i\}_{i=1}^t = \{A \in \mathrm{Con}(F(G_\tau)) \mid A \cap L_j \neq \emptyset\}$. Let $H_{j,i} := H_j \cap A_i$ and $L_{j,i} := L_j \cap A_i$. By Lemma 5.14, there exists a family $\{D_{0,\alpha}\}_{\alpha\in(0,1)}$ of positive constants, a family $\{D_{1,\alpha}\}_{\alpha\in(0,1)}$ of positive constants, and a family $\{\lambda_{1,\alpha}\}_{\alpha\in(0,1)} \subset (0, 1)$ such that for each $\alpha \in (0, 1)$, for each $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, for each i, for each j, for each $\gamma \in \Gamma_\tau^{\mathbb{N}}$, for each $z, w \in H_{j,i}$, for each $n \in \mathbb{N}$ and for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$,

$$|M_\tau^n(\varphi)(z) - M_\tau^n(\varphi)(w)| \leq D_{0,\alpha}\lambda_{1,\alpha}^n\|\varphi\|_\alpha d(z, w)^\alpha \leq D_{1,\alpha}\lambda_{1,\alpha}^n\|\varphi\|_\alpha. \tag{5}$$

For each subset B of $\hat{\mathbb{C}}$ and for each bounded function $\psi : B \to \mathbb{C}$, we set $\|\psi\|_B := \sup_{z\in B} |\psi(z)|$. For each $i = 1, \ldots, t$, let $x_i \in L_{j,i}$ be a point. Let $\varphi \in C^\alpha(\hat{\mathbb{C}})$. By (5), we obtain $\sup_{z\in H_{j,i}} |M_\tau^{nr_L}(\varphi)(z)-M_\tau^{nr_L}(\varphi)(x_i)| \leq D_{1,\alpha}\lambda_{1,\alpha}^n\|\varphi\|_\alpha$ for each i, j, n. Therefore, for each j and for each $l, n \in \mathbb{N}$,

$$\|M_\tau^{lr_L}(M_\tau^{nr_L}(\varphi) - \sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i) \cdot 1_{H_{j,i}})\|_{H_j} \leq D_{1,\alpha}\lambda_{1,\alpha}^n\|\varphi\|_\alpha. \tag{6}$$

We now consider $M_\tau^{r_L} : C_{H_j}(H_j) \to C_{H_j}(H_j)$. We have $\dim_{\mathbb{C}}(C_{H_j}(H_j)) < \infty$. Moreover, by the argument in the proof of Theorem 3.24, $M_\tau^{r_L} : C_{H_j}(H_j) \to C_{H_j}(H_j)$ has exactly one unitary eigenvalue 1, and has exactly one unitary eigenvector 1_{H_j}. Therefore, there exists a constant $\lambda_2 \in (0, 1)$ and a constant $D_2 > 0$, each of which depends only on τ and does not depend on α and φ, such that for each $l \in \mathbb{N}$,

$$\begin{aligned}&\|M_\tau^{lr_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}}) - \lim_{m\to\infty} M_\tau^{mr_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}})\|_{H_j}\\ \leq& D_2\lambda_2^l\|\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}}\|_{H_j} \leq D_2\lambda_2^l t\|\varphi\|_{\hat{\mathbb{C}}}.\end{aligned} \tag{7}$$

Since λ_2 does not depend on α, we may assume that for each $\alpha \in (0, 1)$, $\lambda_{1,\alpha} \geq \lambda_2$. From (6) and (7), it follows that for each $n \in \mathbb{N}$ and for each $l_1, l_2 \in \mathbb{N}$ with $l_1, l_2 \geq n$,

$$\begin{aligned}&\|M_\tau^{(l_1+n)r_L}(\varphi) - M_\tau^{(l_2+n)r_L}(\varphi)\|_{H_j}\\ \leq&\|M_\tau^{(l_1+n)r_L}(\varphi) - M_\tau^{l_1r_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}})\|_{H_j}\\ &+ \|M_\tau^{l_1r_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}}) - \lim_{m\to\infty} M_\tau^{mr_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}})\|_{H_j}\\ &+ \|\lim_{m\to\infty} M_\tau^{mr_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}}) - M_\tau^{l_2r_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}})\|_{H_j}\\ &+ \|M_\tau^{l_2r_L}(\sum_{i=1}^t M_\tau^{nr_L}(\varphi)(x_i)1_{H_{j,i}}) - M_\tau^{(l_2+n)r_L}(\varphi)\|_{H_j}\\ \leq&2D_{1,\alpha}\lambda_{1,\alpha}^n\|\varphi\|_\alpha + D_2\lambda_2^{l_1}t\|\varphi\|_\alpha + D_2\lambda_2^{l_2}t\|\varphi\|_\alpha \leq (2D_{1,\alpha} + 2D_2t)\lambda_{1,\alpha}^n\|\varphi\|_\alpha.\end{aligned}$$

Letting $l_1 \to \infty$, we obtain that for each $l_2 \in \mathbb{N}$ with $l_2 \geq n$, $\|\pi_\tau(\varphi) - M_\tau^{(l_2+n)r_L}(\varphi)\|_{H_j} \leq (2D_{1,\alpha} + 2D_2t)\lambda_{1,\alpha}^n\|\varphi\|_\alpha$. In particular, for each $n \in \mathbb{N}$, $\|\pi_\tau(\varphi) - M_\tau^{2nr_L}(\varphi)\|_{H_j} \leq (2D_{1,\alpha} + 2D_2t)\lambda_{1,\alpha}^n\|\varphi\|_\alpha$. Therefore, for each $n \in \mathbb{N}$,

$$\|\pi_\tau(\varphi) - M_\tau^{nr_L}(\varphi)\|_{H_j} \leq (2D_{1,\alpha} + 2D_2t)\lambda_{1,\alpha}^{-1/2}(\lambda_{1,\alpha}^{1/2})^n\|M_\tau^{r_L}\|_\alpha\|\varphi\|_\alpha, \tag{8}$$

where $\|M_\tau^{r_L}\|_\alpha$ denotes the operator norm of $M_\tau^{r_L} : C^\alpha(\hat{\mathbb{C}}) \to C^\alpha(\hat{\mathbb{C}})$. Let $U := \bigcup_{L,j} H_j$ and let $r := \prod_L r_L$. From the above arguments, it follows that there exists a family $\{D_{3,\alpha}\}_{\alpha\in(0,1)}$ of positive constants and a family $\{\lambda_{3,\alpha}\}_{\alpha\in(0,1)} \subset (0,1)$ such that for each $\alpha \in (0,1)$, for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$ and for each $n \in \mathbb{N}$,

$$\|\pi_\tau(\varphi) - M_\tau^{rn}(\varphi)\|_{\overline{U}} \leq D_{3,\alpha}\lambda_{3,\alpha}^n \|\varphi\|_\alpha. \tag{9}$$

By [31, Theorem 3.15-5], for each $z \in \hat{\mathbb{C}}$, there exists a map $g_z \in G_\tau$ and a compact disk neighborhood U_z of z such that $g_z(U_z) \subset U$. Since $\hat{\mathbb{C}}$ is compact, there exists a finite family $\{z_j\}_{j=1}^s \subset \hat{\mathbb{C}}$ such that $\bigcup_{j=1}^s \mathrm{int}(U_{z_j}) = \hat{\mathbb{C}}$. Since $G_\tau(U) \subset U$, we may assume that there exists a k such that for each $j = 1, \ldots, s$, there exists an element $\beta^j = (\beta_1^j, \ldots, \beta_k^j) \in \Gamma_\tau^k$ with $g_{z_j} = \beta_k^j \circ \cdots \circ \beta_1^j$. We may also assume that $r|k$. For each $j = 1, \ldots, s$, let V_j be a compact neighborhood of β^j in Γ_τ^k such that for each $\zeta = (\zeta_1, \ldots, \zeta_k) \in V_j$, $\zeta_k \circ \cdots \circ \zeta_1(U_{z_j}) \subset U$. Let $a := \max\{\tau^k(\Gamma_\tau^k \setminus V_j) \mid j = 1, \ldots, s\} \in [0,1)$. Let $C_1 := 2\max\{\max\{\|D(\zeta_k \circ \cdots \circ \zeta_1)_z\|_s \mid (\zeta_1, \ldots, \zeta_k) \in \Gamma_\tau^k, z \in \hat{\mathbb{C}}\}, 1\}$. Let $\alpha_1 \in (0,1)$ be such that $aC_1^{\alpha_1} < 1$ and $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C^{\alpha_1}(\hat{\mathbb{C}})$. Let $C_2 > 0$ be a constant such that for each $z \in \hat{\mathbb{C}}$ there exists a $j \in \{1, \ldots, s\}$ with $B(z, C_2) \subset U_{z_j}$. Let $n \in \mathbb{N}$. Let $z_0 \in \hat{\mathbb{C}}$ be any point. Let $i_0 \in \{1, \ldots, s\}$ be such that $B(z_0, C_2) \in U_{z_{i_0}}$. Let $A(0) := \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid (\gamma_1, \ldots, \gamma_k) \in V_{i_0}\}$ and let $B(0) := \{\gamma \in \Gamma_\tau^{\mathbb{N}} \mid (\gamma_1, \ldots, \gamma_k) \notin V_{i_0}\}$. Inductively, for each $j = 1, \ldots, n-1$, let $A(j) := \{\gamma \in B(j-1) \mid \exists i \text{ s.t. } B(\gamma_{kj,1}(z_0), C_2) \subset U_{z_i}, (\gamma_{kj+1}, \ldots, \gamma_{kj+k}) \in V_i\}$ and let $B(j) := B(j-1) \setminus A(j)$. Then for each $j = 1, \ldots, n-1$, $\tilde{\tau}(B(j)) \leq a\tilde{\tau}(B(j-1)) \leq \cdots \leq a^{j+1}$ and $\tilde{\tau}(A(j)) \leq \tilde{\tau}(B(j-1)) \leq a^j$. Moreover, we have $\Gamma_\tau^{\mathbb{N}} = \amalg_{j=0}^{n-1} A(j) \amalg B(n-1)$. Therefore, we obtain that

$$\begin{aligned}
&|M_\tau^{kn}(\varphi)(z_0) - \pi_\tau(\varphi)(z_0)| = |M_\tau^{kn}(\varphi)(z_0) - M_\tau^{kn}(\pi_\tau(\varphi))(z_0)| \\
\leq &\left|\sum_{j=0}^{n-1} \int_{A(j)} (\varphi(\gamma_{kn,1}(z_0)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))d\tilde{\tau}(\gamma)\right| \\
&+ \left|\int_{B(n-1)} (\varphi(\gamma_{kn,1}(z_0)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))d\tilde{\tau}(\gamma)\right|.
\end{aligned} \tag{10}$$

For each $j = 0, \ldots, n-1$, there exists a Borel subset $A'(j)$ of $\Gamma_\tau^{(k+1)j}$ such that $A(j) = A'(j) \times \Gamma_\tau \times \Gamma_\tau \times \cdots$. Hence, by (9), we obtain that for each $\alpha \in (0,1)$ and for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$,

$$\begin{aligned}
&\left|\int_{A(j)} (\varphi(\gamma_{kn,1}(z_0)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))d\tilde{\tau}(\gamma)\right| \\
=&\left|\int_{A'(j)} (M_\tau^{k(n-j-1)}(\varphi)(\gamma_{k(j+1)} \circ \cdots \circ \gamma_1(z_0)) - \pi_\tau(\varphi)(\gamma_{k(j+1)} \circ \cdots \circ \gamma_1(z_0)))d\tau(\gamma_{k(j+1)}) \cdots d\tau(\gamma_1)\right| \\
\leq &D_{3,\alpha}\lambda_{3,\alpha}^{n-j-1}\|\varphi\|_\alpha \tilde{\tau}(A(j)) \leq D_{3,\alpha}\lambda_{3,\alpha}^{n-j-1}a^j\|\varphi\|_\alpha.
\end{aligned} \tag{11}$$

By (10) and (11), it follows that

$$\begin{aligned}
&|M_\tau^{kn}(\varphi)(z_0) - \pi_\tau(\varphi)(z_0)| \leq \sum_{j=0}^{n-1} D_{3,\alpha}\lambda_{3,\alpha}^{n-j-1}a^j\|\varphi\|_\alpha + a^n(\|\varphi\|_\infty + \|\pi_\tau(\varphi)\|_\infty) \\
\leq &\left(D_{3,\alpha}n(\max\{\lambda_{3,\alpha}, a\})^{n-1} + a^n(1 + \|\pi_\tau\|_\infty)\right)\|\varphi\|_\alpha,
\end{aligned}$$

where $\|\pi_\tau\|_\infty$ denotes the operator norm of $\pi_\tau : (C(\hat{\mathbb{C}}), \|\cdot\|_\infty) \to (C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$. For each $\alpha \in (0,1)$, let $\zeta_\alpha := \frac{1}{2}(1 + \max\{\lambda_{3,\alpha}, a\}) < 1$. From these arguments, it follows that there exists a family $\{C_{3,\alpha}\}_{\alpha\in(0,1)}$ of positive constants such that for each $\alpha \in (0,1)$, for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$ and for each $n \in \mathbb{N}$,

$$\|M_\tau^{kn}(\varphi) - \pi_\tau(\varphi)\|_\infty \leq C_{3,\alpha}\zeta_\alpha^n\|\varphi\|_\alpha. \tag{12}$$

For the rest of the proof, let $\alpha \in (0, \alpha_1)$. Let $\eta_\alpha := \max\{\lambda_{1,\alpha}, aC_1^\alpha\} \in (0,1)$. Let $z, z_0 \in \hat{\mathbb{C}}$. If $d(z,z_0) \geq C_1^{-1}C_2$, then

$$\frac{|M_\tau^{kn}(\varphi)(z) - M_\tau^{kn}(\varphi)(z_0) - (\pi_\tau(\varphi)(z) - \pi_\tau(\varphi)(z_0))|}{d(z,z_0)^\alpha} \leq 2C_{3,\alpha}\zeta_\alpha^n \|\varphi\|_\alpha (C_1C_2^{-1})^\alpha. \tag{13}$$

We now suppose that there exists an $m \in \mathbb{N}$ such that $C_1^{-m-1}C_2 \leq d(z,z_0) < C_1^{-m}C_2$. Then for each $\gamma \in \Gamma_\tau^{\mathbb{N}}$ and for each $j = 1, \ldots, m$,

$$d(\gamma_{kj,1}(z), \gamma_{kj,1}(z_0)) < C_2. \tag{14}$$

Let $n \in \mathbb{N}$. Let $\tilde{m} := \min\{n, m\}$. Let $i_0 \in \{1, \ldots, s\}$ be such that $B(z_0, C_2) \subset U_{z_{i_0}}$ and let $A(0), B(0), \ldots, A(\tilde{m}-1), B(\tilde{m}-1)$ be as before. Let $\varphi \in C^\alpha(\hat{\mathbb{C}})$ and let $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
&|M_\tau^{kn}(\varphi)(z) - M_\tau^{kn}(\varphi)(z_0) - (\pi_\tau(\varphi)(z) - \pi_\tau(\varphi)(z_0))| \\
\leq &\left| \sum_{j=0}^{\tilde{m}-1} \int_{A(j)} [\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0)) - (\pi_\tau(\varphi)(\gamma_{kn,1}(z)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))] d\tilde{\tau}(\gamma) \right| \\
&+ \left| \int_{B(\tilde{m}-1)} [\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0)) - (\pi_\tau(\varphi)(\gamma_{kn,1}(z)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))] d\tilde{\tau}(\gamma) \right|.
\end{aligned} \tag{15}$$

Let $A'(j)$ be as before. By (5) and (14), we obtain that for each $j = 0, \ldots, \tilde{m}-1$,

$$\begin{aligned}
&\left| \int_{A(j)} [\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0)) - (\pi_\tau(\varphi)(\gamma_{kn,1}(z)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))] d\tilde{\tau}(\gamma) \right| \\
= &\left| \int_{A(j)} (\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0)) d\tilde{\tau}(\gamma) \right| \\
= &\left| \int_{A'(j)} [M_\tau^{k(n-j-1)}(\varphi)(\gamma_{k(j+1),1}(z)) - M_\tau^{k(n-j-1)}(\varphi)(\gamma_{k(j+1),1}(z_0))] d\tau(\gamma_{k(j+1)}) \cdots d\tau(\gamma_1) \right| \\
\leq &\int_{A'(j)} D_{0,\alpha} d(\gamma_{k(j+1),1}(z), \gamma_{k(j+1),1}(z_0))^\alpha \lambda_{1,\alpha}^{n-j-1} \|\varphi\|_\alpha d\tau(\gamma_{k(j+1)}) \cdots d\tau(\gamma_1) \\
\leq &D_{0,\alpha} C_1^{\alpha(j+1)} d(z,z_0)^\alpha \lambda_{1,\alpha}^{n-j-1} a^j \|\varphi\|_\alpha \\
\leq &D_{0,\alpha} C_1^\alpha \eta_\alpha^{n-1} \|\varphi\|_\alpha d(z,z_0)^\alpha.
\end{aligned} \tag{16}$$

Let $B'(\tilde{m}-1)$ be a Borel subset of $\Gamma_\tau^{k\tilde{m}}$ such that $B(\tilde{m}-1) = B'(\tilde{m}-1) \times \Gamma_\tau \times \Gamma_\tau \times \cdots$. We now consider the following two cases. Case (I): $\tilde{m} = m$. Case (II): $\tilde{m} = n$.

Suppose we have Case (I). Then by (12), we obtain that

$$\begin{aligned}
&\left| \int_{B(\tilde{m}-1)} [\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0)) - (\pi_\tau(\varphi)(\gamma_{kn,1}(z)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))] d\tilde{\tau}(\gamma) \right| \\
\leq &\int_{B'(m-1)} |M_\tau^{k(n-m)}(\gamma_{km} \circ \cdots \circ \gamma_1(z)) - \pi_\tau(\varphi)(\gamma_{km} \circ \cdots \circ \gamma_1(z))| d\tau(\gamma_{km}) \cdots d\tau(\gamma_1) \\
&+ \int_{B'(m-1)} |M_\tau^{k(n-m)}(\gamma_{km} \circ \cdots \circ \gamma_1(z_0)) - \pi_\tau(\varphi)(\gamma_{km} \circ \cdots \circ \gamma_1(z_0))| d\tau(\gamma_{km}) \cdots d\tau(\gamma_1) \\
\leq &2C_{3,\alpha}\zeta_\alpha^{n-m} \|\varphi\|_\alpha a^m \leq 2C_{3,\alpha}\zeta_\alpha^{n-m} \|\varphi\|_\alpha a^m \cdot (C_1^{m+1}C_2^{-1} d(z,z_0))^\alpha \\
= &2C_{3,\alpha}\zeta_\alpha^{n-m} (aC_1^\alpha)^m (C_1C_2^{-1})^\alpha \|\varphi\|_\alpha d(z,z_0)^\alpha \leq 2C_{3,\alpha}(C_1C_2^{-1})^\alpha \zeta_\alpha^{n-m} \eta_\alpha^m \|\varphi\|_\alpha d(z,z_0)^\alpha.
\end{aligned} \tag{17}$$

We now suppose we have Case (II). Since $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$, we obtain

$$\begin{aligned}
&\left|\int_{B(\tilde{m}-1)} [\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0)) - (\pi_\tau(\varphi)(\gamma_{kn,1}(z)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0)))] d\tilde{\tau}(\gamma)\right| \\
\leq &\int_{B(n-1)} |\varphi(\gamma_{kn,1}(z)) - \varphi(\gamma_{kn,1}(z_0))| d\tilde{\tau}(\gamma) + \int_{B(n-1)} |\pi_\tau(\varphi)(\gamma_{kn,1}(z)) - \pi_\tau(\varphi)(\gamma_{kn,1}(z_0))| d\tilde{\tau}(\gamma) \\
\leq &C_1^{\alpha n} d(z,z_0)^\alpha a^n \|\varphi\|_\alpha + C_1^{\alpha n} d(z,z_0)^\alpha a^n \|\pi_\tau(\varphi)\|_\alpha \\
\leq &C_1^{\alpha n} a^n (1 + E_\alpha) \|\varphi\|_\alpha d(z,z_0)^\alpha, \qquad (18)
\end{aligned}$$

where E_α denotes the number in Theorem 3.29. Let $\xi_\alpha := \frac{1}{2}(\max\{\zeta_\alpha, \eta_\alpha\} + 1) \in (0,1)$. Combining (15), (16), (17) and (18), it follows that there exists a constant $C_{4,\alpha} > 0$ such that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$,

$$|M_\tau^{kn}(\varphi)(z) - M_\tau^{kn}(\varphi)(z_0) - (\pi_\tau(\varphi)(z) - \pi_\tau(\varphi)(z_0))| \leq C_{4,\alpha} \xi_\alpha^n \|\varphi\|_\alpha d(z,z_0)^\alpha. \qquad (19)$$

Let $C_{5,\alpha} = C_{3,\alpha} + C_{4,\alpha}$. By (12) and (19), we obtain that for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$ and for each $n \in \mathbb{N}$,

$$\|M_\tau^{kn}(\varphi) - \pi_\tau(\varphi)\|_\alpha \leq C_{5,\alpha} \xi_\alpha^n \|\varphi\|_\alpha. \qquad (20)$$

From this, statement (3) of our theorem holds.

Let $\psi \in C^\alpha(\hat{\mathbb{C}})$. Setting $\varphi = \psi - \pi_\tau(\psi)$, by (20), we obtain that statement (2) of our theorem holds. Statement (4) of our theorem follows from Theorem 3.29. Statement (1) follows from statements (2).

Thus, we have proved Theorem 3.30. □

We now prove Theorem 3.31.

Proof of Theorem 3.31: Let $A := \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$. Let $\zeta \in \mathbb{C} \setminus A$. Then by Theorem 3.30, $\sum_{n=0}^\infty \frac{M_\tau^n}{\zeta^{n+1}}(I - \pi_\tau)$ converges in the space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm. Let $\Omega := (\zeta I - M_\tau)|^{-1}_{\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))} \circ \pi_\tau + \sum_{n=0}^\infty \frac{M_\tau^n}{\zeta^{n+1}}(I - \pi_\tau)$. Let $U_\tau := \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$. Then we have

$$\begin{aligned}
(\zeta I - M_\tau) \circ \Omega = &\big((\zeta I - M_\tau)|_{U_\tau} \circ \pi_\tau + (\zeta I - M_\tau)|_{\mathcal{B}_{0,\tau}} \circ (I - \pi_\tau)\big) \\
&\circ \left((\zeta I - M_\tau)|^{-1}_{U_\tau} \circ \pi_\tau + \sum_{n=0}^\infty \frac{M_\tau^n}{\zeta^{n+1}}|_{\mathcal{B}_{0,\tau}}(I - \pi_\tau)\right) \\
= &I|_{U_\tau} \circ \pi_\tau + (\zeta I - M_\tau) \circ (\sum_{n=0}^\infty \frac{M_\tau^n}{\zeta^{n+1}}) \circ (I - \pi_\tau) \\
= &\pi_\tau + (\sum_{n=0}^\infty \frac{M_\tau^n}{\zeta^n} - \sum_{n=0}^\infty \frac{M_\tau^{n+1}}{\zeta^{n+1}}) \circ (I - \pi_\tau) = I.
\end{aligned}$$

Similarly, we have $\Omega \circ (\zeta I - M_\tau) = I$. Therefore, statements (1) and (2) of our theorem hold.

Thus, we have proved Theorem 3.31. □

We now prove Theorem 3.32.

Proof of Theorem 3.32: By using the method in the proofs of [33, Lemmas 5.1, 5.2], we obtain that for each $\alpha \in (0,1)$, the map $a \in \mathcal{W}_m \mapsto M_{\tau_a} \in L(C^\alpha(\hat{\mathbb{C}}))$ is real-analytic, where $L(C^\alpha(\hat{\mathbb{C}}))$ denotes the Banach space of bounded linear operators on $C^\alpha(\hat{\mathbb{C}})$ endowed with the operator norm. Moreover, by using the method in the proof of Theorem 3.29, we can show that for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ and an open neighborhood V_b of b in $\mathcal{W}_m$ such that for each $a \in V_b$, we have $\mathrm{LS}(\mathcal{U}_{f,\tau_a}(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$. In particular, $\pi_{\tau_a}(C^\alpha(\hat{\mathbb{C}})) \subset C^\alpha(\hat{\mathbb{C}})$ for each $a \in V_b$. Statement (1) follows from the above arguments, [31, Theorem 3.15-10], Theorem 3.31 and [16, p368-369, p212]. We now

prove statement (2). For each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$, let $\varphi_L : \hat{\mathbb{C}} \to [0,1]$ be a C^∞ function on $\hat{\mathbb{C}}$ such that $\varphi_L|_L \equiv 1$ and such that for each $L' \in \mathrm{Min}(G, \hat{\mathbb{C}})$ with $L' \neq L$, $\varphi_L|_{L'} \equiv 0$. Then, by [31, Theorem 3.15-15], we have that for each $z \in \hat{\mathbb{C}}$, $T_{L,\tau_a}(z) = \lim_{n\to\infty} M^n_{\tau_a}(\varphi_L)(z)$. Combining this with [31, Theorem 3.14], we obtain $T_{L,\tau_a} = \lim_{n\to\infty} M^n_{\tau_a}(\varphi)$ in $C(\hat{\mathbb{C}})$. By [31, Theorem 3.15-6,8,9], for each $a \in \mathcal{W}_m$, there exists a number $r \in \mathbb{N}$ such that for each $\psi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$, $M^r_{\tau_a}(\psi) = \psi$. Therefore, by [31, Theorem 3.15-1], $T_{L,\tau_a} = \lim_{n\to\infty} M^{nr}_{\tau_a}(\varphi_L) = \lim_{n\to\infty} M^{nr}_{\tau_a}(\varphi_L - \pi_{\tau_a}(\varphi_L) + \pi_{\tau_a}(\varphi_L)) = \pi_{\tau_a}(\varphi_L)$. Combining this with statement (1) of our theorem and [31, Theorem 3.15-1], it is easy to see that statement (2) of our theorem holds.

We now prove statement (3). By taking the partial derivative of $M_{\tau_a}(T_{L,\tau_a}(z)) = T_{L,\tau_a}(z)$ with respect to a_i, it is easy to see that $\psi_{i,b}$ satisfies the functional equation $(I - M_{\tau_b})(\psi_{i,b}) = \zeta_{i,b}, \psi_{i,b}|_{S_{\tau_b}} = 0$. Let $\psi \in C(\hat{\mathbb{C}})$ be a solution of $(I - M_{\tau_b})(\psi) = \zeta_{i,b}, \psi|_{S_{\tau_b}} = 0$. Then for each $n \in \mathbb{N}$,

$$(I - M^n_{\tau_b})(\psi) = \sum_{j=0}^{n-1} M^j_{\tau_b}(\zeta_{i,b}). \tag{21}$$

By the definition of $\zeta_{i,b}$, $\zeta_{i,b}|_{S_{\tau_b}} = 0$. Therefore, by [31, Theorem 3.15-2], $\pi_{\tau_b}(\zeta_{i,b}) = 0$. Thus, denoting by C and λ the constants in Theorem 3.30, we obtain $\|M^n_{\tau_b}(\zeta_{i,b})\|_\alpha \leq C\lambda^n \|\zeta_{i,b}\|_\alpha$. Moreover, since $\psi|_{S_{\tau_b}} = 0$, [31, Theorem 3.15-2] implies $\pi_{\tau_b}(\psi) = 0$. Therefore, $M^n_{\tau_b}(\psi) \to 0$ in $C(\hat{\mathbb{C}})$ as $n \to \infty$. Letting $n \to \infty$ in (21), we obtain that $\psi = \sum_{j=0}^{\infty} M^j_{\tau_b}(\zeta_{i,b})$. Therefore, we have proved statement (3).

Thus, we have proved Theorem 3.32. □

We now prove Theorem 3.40.

Proof of Theorem 3.40: Statements 1,3,4 follow from [31, Theorem 3.82] and its proof. We now prove statement 2. By [31, Theorem 3.82, Theorem 3.15-15], there exists a Borel subset A of $J(G)$ with $\lambda(A) = 1$ such that for each $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ and for each $z \in A$, $\mathrm{Höl}(T_{L,\tau_p}, z) = u(h, p, \mu)$. Let $z_0 \in A$ be a point, let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$, and let $i \in \{1, \ldots, m-1\}$. We consider the following three cases. Case 1: $\mathrm{Höl}(\psi_{i,p,L}, z_0) < u(h, p, \mu)$. Case 2: $\mathrm{Höl}(\psi_{i,p,L}, z_0) = u(h, p, \mu)$. Case 3: $\mathrm{Höl}(\psi_{i,p,L}, z_0) > u(h, p, \mu)$.

Suppose we have Case 1. Let $z_1 \in h_i^{-1}(\{z_0\})$. By the functional equation $(I - M_{\tau_p})(\psi_{i,p,L}) = T_{L,\tau_p} \circ h_i - T_{L,\tau_p} \circ h_m$ (see Theorem 3.32 (3)), [31, Theorem 3.15-1,15], and the assumption $h_k^{-1}(J(G)) \cap h_l^{-1}(J(G)) = \emptyset$ for each (k, l) with $k \neq l$, there exists a neighborhood U of z_1 in $\hat{\mathbb{C}}$ such that for each $z \in U$,

$$\psi_{i,p,L}(z) - \psi_{i,p,L}(z_1) - p_i(\psi_{i,p,L}(h_i(z)) - \psi_{i,p,L}(z_0)) = T_{L,\tau_p}(h_i(z)) - T_{L,\tau_p}(z_0). \tag{22}$$

By equation (22) and the definition of the pointwise Hölder exponent, it is easy to see that $\mathrm{Höl}(\psi_{i,p,L}, z_1) = \mathrm{Höl}(\psi_{i,p,L}, z_0) < u(h, p, \mu)$. We now let $z_1 \in h_m^{-1}(\{z_0\})$. Then by the similar method to the above, we obtain that $\mathrm{Höl}(\psi_{i,p,L}, z_1) = \mathrm{Höl}(\psi_{i,p,L}, z_0) < u(h, p, \mu)$.

We now suppose we have Case 2. By the same method as that in Case 1, we obtain that $\mathrm{Höl}(\psi_{i,p,L}, z_0) = u(h, p, \mu) \leq \mathrm{Höl}(\psi_{i,p,L}, z_1)$ for each $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

We now suppose we have Case 3. By the same method as that in Case 1 again, we obtain that $\mathrm{Höl}(\psi_{i,p,L}, z_1) = u(h, p, \mu) < \mathrm{Höl}(\psi_{i,p,L}, z_0)$ for each $z_1 \in h_i^{-1}(\{z_0\}) \cup h_m^{-1}(\{z_0\})$.

Thus we have proved Theorem 3.40. □

6 Examples

In this section, we give some examples.

Example 6.1 (Proposition 6.1 in [31])**.** Let $f_1 \in \mathcal{P}$. Suppose that $\mathrm{int}(K(f_1))$ is not empty. Let $b \in \mathrm{int}(K(f_1))$ be a point. Let d be a positive integer such that $d \geq 2$. Suppose that $(\deg(f_1), d) \neq (2, 2)$. Then, there exists a number $c > 0$ such that for each $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$, setting $f_\lambda = (f_{\lambda,1}, f_{\lambda,2}) = (f_1, \lambda(z-b)^d + b)$ and $G_\lambda := \langle f_1, f_{\lambda,2} \rangle$, we have all of the following.

(a) f_λ satisfies the open set condition with an open subset U_λ of $\hat{\mathbb{C}}$ (i.e., $f_{\lambda,1}^{-1}(U_\lambda)\cup f_{\lambda,2}^{-1}(U_\lambda)\subset U_\lambda$ and $f_{\lambda,1}^{-1}(U_\lambda)\cap f_{\lambda,2}^{-1}(U_\lambda)=\emptyset$), $f_{\lambda,1}^{-1}(J(G_\lambda))\cap f_{\lambda,2}^{-1}(J(G_\lambda))=\emptyset$, $\mathrm{int}(J(G_\lambda))=\emptyset$, $J_{\mathrm{ker}}(G_\lambda)=\emptyset$, $G_\lambda(K(f_1))\subset K(f_1)\subset \mathrm{int}(K(f_{\lambda,2}))$ and $\emptyset\neq K(f_1)\subset \hat{K}(G_\lambda)$.

(b) If $K(f_1)$ is connected, then $P^*(G_\lambda)$ is bounded in $\mathbb{C}$.

(c) If f_1 is hyperbolic and $K(f_1)$ is connected, then G_λ is hyperbolic, $J(G_\lambda)$ is porous (for the definition of porosity, see [26]), and $\dim_H(J(G_\lambda))<2$.

By Example 6.1, Remark 3.34 and [31, Proposition 6.4], we can obtain many examples of $\tau\in\mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau<\infty$ to which we can apply Theorems 3.24, 3.25, 3.29, 3.30, 3.31, 3.32, 3.40.

Example 6.2 (Devil's coliseum ([31]) and complex analogue of the Takagi function)**.** Let $g_1(z):=z^2-1, g_2(z):=z^2/4, h_1:=g_1^2$, and $h_2:=g_2^2$. Let $G=\langle h_1,h_2\rangle$ and for each $a=(a_1,a_2)\in\mathcal{W}_2:=\{(a_1,a_2)\in(0,1)^2\mid\sum_{j=1}^2 a_j=1\}\cong(0,1)$, let $\tau_a:=\sum_{i=1}^2 a_i\delta_{h_i}$. Then by [31, Example 6.2], setting $A:=K(h_2)\setminus D(0,0.4)$, we have $\overline{D(0,0.4)}\subset\mathrm{int}(K(h_1))$, $h_2(K(h_1))\subset\mathrm{int}(K(h_1))$, $h_1^{-1}(A)\cup h_2^{-1}(A)\subset A$, and $h_1^{-1}(A)\cap h_2^{-1}(A)=\emptyset$. Therefore $h_1^{-1}(J(G))\cap h_2^{-1}(J(G))=\emptyset$ and $\emptyset\neq K(h_1)\subset\hat{K}(G)$. Moreover, G is hyperbolic and mean stable, and for each $a\in\mathcal{W}_2$, we obtain that T_{∞,τ_a} is continuous on $\hat{\mathbb{C}}$ and the set of varying points of T_{∞,τ_a} is equal to $J(G)$. Moreover, by [31] $\dim_H(J(G))<2$ and for each non-empty open subset U of $J(G)$ there exists an uncountable dense subset A_U of U such that for each $z\in A_U$, T_{∞,τ_a} is not differentiable at z. See Figures 2 and 3. The function T_{∞,τ_a} is called a devil's coliseum. It is a complex analogue of the devil's staircase. (Remark: as the author of this paper pointed out in [31], the devil's staircase can be regarded as the function of probability of tending to $+\infty$ regarding the random dynamics on $\mathbb{R}$ such that at every step we choose $h_1(x)=3x$ with probability $1/2$ and we choose $h_2(x)=3(x-1)+1$ with probability $1/2$. For the detail, see [31].) By Theorem 3.32, for each $z\in\hat{\mathbb{C}}$, $a_1\mapsto T_{\infty,\tau_a}(z)$ is real-analytic in $(0,1)$, and for each $b\in\mathcal{W}_2$, $[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a=b}=\sum_{n=0}^\infty M_{\tau_b}^n(\zeta_{1,b})$, where $\zeta_{1,b}(z):=T_{\infty,\tau_b}(h_1(z))-T_{\infty,\tau_b}(h_2(z))$. Moreover, by Theorem 3.32, the function $\psi(z):=[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a=b}$ defined on $\hat{\mathbb{C}}$ is Hölder continuous on $\hat{\mathbb{C}}$ and is locally constant on $F(G)$. As mentioned in Remark 1.14, the function $\psi(z)$ defined on $\hat{\mathbb{C}}$ can be regarded as a complex analogue of the Takagi function. By Theorem 3.40, there exists an uncountable dense subset A of $J(G)$ such that for each $z\in A$, either ψ is not differentiable at z or ψ is not differentiable at each point $w\in h_1^{-1}(\{z\})\cup h_2^{-1}(\{z\})$. For the graph of $[\frac{\partial T_{\infty,\tau_a}(z)}{\partial a_1}]|_{a_1=1/2}$, see Figure 4.

We now give an example of $\tau\in\mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\Gamma_\tau<\infty$ such that $J_{\mathrm{ker}}(G_\tau)=\emptyset$, $J(G_\tau)\neq\emptyset$, $S_\tau\subset F(G_\tau)$ and τ is not mean stable.

Example 6.3. Let $h_1\in\mathcal{P}$ be such that $J(h_1)$ is connected and h_1 has a Siegel disk S. Let $b\in S$ be a point. Let $d\in\mathbb{N}$ be such that $(\deg(h_1),d)\neq(2,2)$. Then by [31, Proposition 6.1] (or [30, Proposition 2.40]) and its proof, there exists a number $c>0$ such that for each $\lambda\in\mathbb{C}$ with $0<|\lambda|<c$, setting $h_2(z):=\lambda(z-b)^d+b$ and $G:=\langle h_1,h_2\rangle$, we have $J_{\mathrm{ker}}(G)=\emptyset$ and $h_2(K(h_1))\subset S\subset\mathrm{int}(K(h_1))\subset\mathrm{int}(K(h_2))$. Then the set of minimal sets for $(G,\hat{\mathbb{C}})$ is $\{\{\infty\},L_0\}$, where L_0 is a compact subset of S $(\subset F(G))$. Let $(p_1,p_2)\in\mathcal{W}_2$ be any element and let $\tau:=\sum_{j=1}^2 p_j\delta_{h_j}$. Then $J_{\mathrm{ker}}(G_\tau)=\emptyset$, $J(G_\tau)\neq\emptyset$, $S_\tau\subset F(G_\tau)$ and τ is not mean stable. In fact, L_0 is sub-rotative. Even though τ is not mean stable, we can apply Theorems 3.29, 3.30, 3.31, 3.32, 3.40 to this τ.

Example 6.4. By [31, Example 6.7], we have an example $\tau\in\mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\mathrm{ker}}(G_\tau)=\emptyset$ and such that there exists a J-touching minimal set for $(G_\tau,\hat{\mathbb{C}})$. This τ is not mean stable but we can apply Theorem 3.29 to this τ.

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