Real analyticity of Hausdorff dimension for expanding rational semigroups

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Abstract. We consider the dynamics of expanding semigroups generated by finitely many rational maps on the Riemann sphere. We show that for an analytic family of such semigroups, the Bowen parameter function is real-analytic and plurisubharmonic. Combining this with a result obtained by the first author, we show that if for each semigroup of such an analytic family of expanding semigroups satisfies the open set condition, then the Hausdorff dimension of the Julia set is a real-analytic and plurisubharmonic function of the parameter. Moreover, we provide an extensive collection of examples of analytic families of semigroups satisfying all the above conditions and we analyze in detail the corresponding Bowen's parameters and Hausdorff dimension function.

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1. Introduction

A **rational semigroup** is a semigroup generated by a family of non-constant rational maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition. A polynomial semigroup is a semigroup generated by a family of non-constant polynomial maps on $\hat{\mathbb{C}}$. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([8, 9]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups of Möbius transformations, and by F. Ren's group ([41, 7]), who studied such semigroups from the perspective of random dynamical systems.

The theory of the dynamics of rational semigroups on $\hat{\mathbb{C}}$ has developed in many directions since the 1990s ([8, 41, 9, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 37, 38, 31, 32, 33,

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21, 34, 35]). Since the Julia set $J(G)$ of a rational semigroup generated by finitely many elements $f_1, \dots, f_s$ has **backward self-similarity** i.e.

$$J(G) = f_1^{-1}(J(G)) \cup \cdots \cup f_s^{-1}(J(G)), \tag{1.1}$$

(See [22, 24]), it can be viewed significant generalization and extension of both the theory of iteration of rational maps (see [14]) and conformal iterated function systems (see [12]). The theory of the dynamics of rational semigroups borrows and develops tools from both of these theories. It has also developed its own unique methods, notably the skew product approach (see [24, 25, 27, 28, 31, 32, 33, 35]). We remark that by (1.1), the analysis of the Julia sets of rational semigroups somewhat resembles "backward iterated functions systems", however since each map f_j is not in general injective (critical points), some qualitatively different extra effort in the cases of semigroups is needed.

The theory of the dynamics of rational semigroups is intimately related to that of the random dynamics of rational maps. For the study of random complex dynamics, the reader may consult [5, 3, 4, 2, 1, 6]. The deep relation between these fields (rational semigroups, random complex dynamics, and (backward) IFS) is explained in detail in the subsequent papers ([29, 30, 31, 32, 33, 34, 35]) of the first author.

In this paper, we analyze in detail the Hausdorff dimension of Julia sets of expanding rational semigroups. Our approach utilize the powerful tool of thermodynamic formalism, developed in [27] and a version of Bowen's formula for the Hausdorff dimension of Julia sets, also proved in [27]. We introduce Bowen's parameter as the unique zero of the pressure function. This is an invariant of the generator systems of the semigroup. We then develop a finer analysis of holomorphic families of Perron-Frobenius type operators, and eventually apply Kato-Rellich perturbation theory ([10]) to get real-analyticity of the pressure function, as depending on the complex parameter. Then the Implicit Function Theorem completes the task. Bowen's formula, which is mentioned above, identifies the Hausdorff dimension of the Julia set with Bowen's parameter, whenever in addition the open set condition is satisfied. We thus obtain that under these assumptions the Hausdorff dimension function depends in a real-analytic manner on the parameter. We also show that Bowen's parameter function is real-analytic and plurisubharmonic, even if we do not assume the open set condition. The real analyticity of Hausdorff dimension, or, in a sense, more accurately, of Bowen's parameter, goes back to Ruelle's paper [16], where hyperbolic rational functions were considered. The reader may also consult [42] and [40]. Our approach stems from that in the [40] paper. We develop it and work out techniques to deal with a qualitatively different case of semigroups.

Our article ends with a collection of examples illustrating variety of behavior of the Hausdorff dimension function, Bowen's parameter function (ex. it can be strictly less than 2 or strictly bigger than 2 on an open set of multi-maps), expandingness and the open set condition.

We remark that as illustrated in [30, 35], estimating the Hausdorff dimension of the Julia sets of rational semigroups plays an important role when we investigate random complex dynamics and its associated Markov process on $\hat{\mathbb{C}}$. For example, when we consider the random dynamics of a compact family Γ of polynomials of degree greater than or equal to two, then the function $T_\infty : \hat{\mathbb{C}} \to [0,1]$ of probability of tending to $\infty \in \hat{\mathbb{C}}$ varies only inside the Julia set of rational semigroup generated by Γ, and under some condition, this $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is continuous in $\hat{\mathbb{C}}$. If the Hausdorff dimension of the Julia set is strictly less than two, then it means that $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is a complex version of devil's staircase (Cantor function) ([29, 30, 35]).

2. Preliminaries and the main results

In this section we introduce the notation and basic definitions. We also formulate our main results. Their proofs will be concluded in Section 7.

Throughout the paper, we frequently follow the notation from [24] and [27].

Definition 2.1 ([8, 41, 7, 9]). *A "rational semigroup" G is a semigroup generated by a family of non-constant rational maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition. A "polynomial semigroup" is a semigroup generated by a family of non-constant polynomial maps on $\hat{\mathbb{C}}$. For a rational semigroup G, we set $F(G) := \{z \in \hat{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}$ and $J(G) := \hat{\mathbb{C}} \setminus F(G)$. $F(G)$ is called the* **Fatou set** *of G and $J(G)$ is called the* **Julia set** *of G. If G is generated by a family $\{f_i\}_i$, then we write $G = \langle f_1, f_2, \cdots \rangle$.*

For the study of the dynamics of rational semigroups, see [8, 41, 7, 9, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 37, 38, 31, 32, 33, 21, 34, 35], etc.

Definition 2.2. *For each $s \in \mathbb{N}$, let $\Sigma_s := \{1, \ldots, s\}^{\mathbb{N}}$ be the space of one-sided sequences of s-symbols endowed with the product topology. This is a compact metric space. We denote by* Rat *the set of all non-constant rational maps on $\hat{\mathbb{C}}$ endowed with the topology induced by uniform convergence on $\hat{\mathbb{C}}$. Note that* Rat *has countably many connected components. In addition, each connected component U of* Rat *is an open subset of* Rat *and U has a structure of a finite dimensional complex manifold. For each $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$, we define a map*

$$\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$$

by the formula

$$\tilde{f}(\omega, z) = (\sigma(\omega),\ f_{\omega_1}(z)),$$

where $(\omega, z) \in \Sigma_s \times \hat{\mathbb{C}}$, $\omega = (\omega_1, \omega_2, \ldots)$, and $\sigma : \Sigma_s \to \Sigma_s$ denotes the shift map. This $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ is called the **skew product map** *associated with the multi-map $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$. We denote by $\pi_1 : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s$ the projection onto Σ_s and $\pi_2 : \Sigma_s \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the projection onto $\hat{\mathbb{C}}$. That is, $\pi_1(\omega, z) = \omega$ and $\pi_2(\omega, z) = z$. For each $n \in \mathbb{N}$ and $(\omega, z) \in \Sigma_s \times \hat{\mathbb{C}}$, we put*

$$(\tilde{f}^n)'(\omega, z) := (f_{\omega_n} \circ \cdots \circ f_{\omega_1})'(z).$$

We put $J_\omega(\tilde{f}) := \{z \in \hat{\mathbb{C}} \mid \{f_{\omega_n} \circ \cdots \circ f_{\omega_1}\}_{n \in \mathbb{N}} \text{ is not normal in each neighborhood of } z\}$ for each $\omega \in \Sigma_s$ and we set

$$J(\tilde{f}) := \overline{\cup_{w \in \Sigma_s} \{\omega\} \times J_\omega(\tilde{f})},$$

where the closure is taken in the product space $\Sigma_s \times \hat{\mathbb{C}}$. This is called the **Julia set** *of the skew product map $\tilde{f}$. Moreover, we set $F(\tilde{f}) := (\Sigma_s \times \hat{\mathbb{C}}) \setminus J(\tilde{f})$. Furthermore, we set $\deg(\tilde{f}) := \sum_{j=1}^{s} \deg(f_j)$.*

Remark 2.3. *By definition,* $J(\tilde{f})$ *is compact. Furthermore, if we set* $G = \langle f_1, \dots, f_s \rangle$*, then by* [24, Proposition 3.2]*, we have all of the following:*

1. $J(\tilde{f})$ *is completely invariant under* $\tilde{f}$*;*
2. $\tilde{f}$ *is an open map on* $J(\tilde{f})$*;*
3. *if* $\sharp J(G) \geq 3$ *and* $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp \cup_{g \in G} g^{-1}\{z\} < \infty\}$ *is included in* $F(G)$*, then* $(\tilde{f}, J(\tilde{f}))$ *is topologically exact;*
4. $J(\tilde{f})$ *is equal to the closure of the set of repelling periodic points of* $\tilde{f}$ *if* $\sharp J(G) \geq 3$*, where we say that a periodic point* (ω, z) *of* $\tilde{f}$ *with period* n *is repelling if* $|(\tilde{f}^n)'(\omega, z)| > 1$*; and*
5. $\pi_2(J(\tilde{f})) = J(G)$.

Definition 2.4 ([27])**.** *A finitely generated rational semigroup* $G = \langle f_1, \dots, f_s \rangle$ *is said to be expanding provided that* $J(G) \neq \emptyset$ *and the skew product map* $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *associated with* $f = (f_1, \dots, f_s)$ *is expanding along fibers of the Julia set* $J(\tilde{f})$*, meaning that there exists* $\eta > 1$ *and* $C \in (0, 1]$ *such that for all* $n \geq 1$*,*

$$\inf\{\|(\tilde{f}^n)'(z)\| : z \in J(\tilde{f})\} \geq C\eta^n, \tag{2.1}$$

where we mean in the above formula $\|\cdot\|$ *to denote the absolute value of the spherical derivative.*

Definition 2.5. *Let* G *be a rational semigroup. We set*

$$P(G) := \overline{\cup_{g \in G}\{\text{all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}} \ (\subset \hat{\mathbb{C}})$$

and this is called the **postcritical set** *of* G*. A rational semigroup* G *is said to be* **hyperbolic** *if* $P(G) \subset F(G)$*.*

Remark 2.6. *Let* $G = \langle f_1, \dots, f_s \rangle$ *be a rational semigroup such that there exists an element* $g \in G$ *with* $\deg(g) \geq 2$ *and such that each Möbius transformation in* G *is loxodromic. Then, it was proved in* [23] *that* G *is expanding if and only if* G *is hyperbolic.*

Definition 2.7. *We define*

$$\mathrm{Exp}(s) := \{(f_1, \dots, f_s) \in (\mathrm{Rat})^s \mid \langle f_1, \dots, f_s \rangle \text{ is expanding}\}.$$

We also set $\Sigma_s^* := \cup_{j=1}^{\infty}\{1, \dots, s\}^j$ *(disjoint union). For every* $\omega \in \Sigma_s \cup \Sigma_s^*$ *let* $|\omega|$ *be the length of* ω*. For each* $f = (f_1, \dots, f_s) \in (\mathrm{Rat})^s$ *and each* $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_s^*$*, we put* $f_\omega := f_{\omega_n} \circ \dots \circ f_{\omega_1}$*.*

Then we have the following.

Lemma 2.8. $\mathrm{Exp}(s)$ *is an open subset of* $(\mathrm{Rat})^s$*.*

Proof. Let $f = (f_1, \dots, f_s) \in \mathrm{Exp}(s)$. Then, by (2.1) and the fact $\pi_2(J(\tilde{f})) = J(\langle f_1, \dots, f_s \rangle)$ (Remark 2.3), there exists an $n \in \mathbb{N}$ such that

$$\inf\{\|(f_\omega)'(y)\| : \ \omega \in \Sigma_s^*, |\omega| = n, y \in f_\omega^{-1}(J(\langle f_1, \dots, f_s \rangle))\} \geq 3. \tag{2.2}$$

For each subset A of $\hat{\mathbb{C}}$ and $r > 0$, we denote by $B(A, r)$ the r-neighborhood of A with respect to the spherical distance on $\hat{\mathbb{C}}$. Let $\epsilon > 0$ be any small number. Then, by (2.2), for each $\omega \in \Sigma_s^*$ with $|\omega| = n$,

$$f_\omega^{-1}(B(J(\langle f_1, \ldots, f_s \rangle), \epsilon)) \subset B(J(\langle f_1, \ldots, f_s \rangle), \epsilon/2). \tag{2.3}$$

Hence, there exists a neighborhood U of f in $(\mathrm{Rat})^s$ such that for each $g = (g_1, \ldots, g_s) \in U$ and each ω with $|\omega| = n$,

$$g_\omega^{-1}(B(J(\langle f_1, \ldots, f_s \rangle), \epsilon)) \subset B(J(\langle f_1, \ldots, f_s \rangle), \epsilon) \tag{2.4}$$

and

$$\|(g_\omega)'(y)\| > 3/2 \text{ for each } y \in g_\omega^{-1}(B(J(\langle f_1, \ldots, f_s \rangle), \epsilon)). \tag{2.5}$$

By (2.4), for each $g \in U$, $\langle g_1, \ldots, g_s \rangle$ is normal in $\hat{\mathbb{C}} \setminus \overline{B}(J(\langle f_1, \ldots, f_s \rangle), \epsilon)$. Hence, it follows that for each $g \in U$,

$$J(\langle g_1, \ldots, g_s \rangle) \subset \overline{B}(J(\langle f_1, \ldots, f_s \rangle), \epsilon). \tag{2.6}$$

By (2.5), (2.6), and the fact $\pi_2(J(\tilde{g})) = J(\langle g_1, \ldots, g_s \rangle)$, we obtain that for each $g \in U$, $\langle g_1, \ldots, g_s \rangle$ is expanding. We are done. $\square$

Definition 2.9. *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$ and let $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ be the skew product map associated with $f = (f_1, \ldots, f_s)$. For each $t \in \mathbb{R}$, let $P(t, f)$ be the topological pressure of the potential $\varphi(z) := -t \log \|\tilde{f}'(z)\|$ with respect to the map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$. (For the definition of the topological pressure, see* [15]*.) We denote by $\delta(f)$ the unique zero of $t \mapsto P(t, f)$. (Note that the existence and the uniqueness of the zero of $P(t, f)$ was shown in* [27]*.) This $\delta(f)$ is called the* **Bowen parameter** *of $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$.*

In this paper, we consider the following situation:

Definition 2.10. *Let Λ be a finite dimensional complex manifold. For each $j = 1, \ldots, s$ and each $\lambda \in \Lambda$, suppose that there exists a rational map $f_{\lambda,j} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. For each $\lambda \in \Lambda$, we set $G_\lambda := \langle f_{\lambda,1}, \ldots, f_{\lambda,s} \rangle$. The collection $\{G_\lambda\}_{\lambda \in \Lambda}$ is called an analytic family of rational semigroups if*

(a) *For every $1 \le j \le s$ and every $z \in \mathbb{C}$, $(z, \lambda) \mapsto f_{\lambda,j}(z)$ is a holomorphic map from $\hat{\mathbb{C}} \times \Lambda$ to $\hat{\mathbb{C}}$.*

Furthermore, the collection $\{G_\lambda\}_{\lambda \in \Lambda}$ is called an analytic family of expanding rational semigroups if $\{G_\lambda\}_{\lambda \in \Lambda}$ is an analytic family of rational semigroups and for all $\lambda \in \Lambda$, G_λ is expanding.

Example 2.11. *Let Λ be a connected component of $\mathrm{Exp}(s)$. For each $f = (f_1, \ldots, f_s) \in \Lambda$, let $G_f := \langle f_1, \ldots, f_s \rangle$. Then, by Lemma 2.8, $\{G_f\}_{f \in \Lambda}$ is an analytic family of expanding rational semigroups.*

We will give a large collection of examples of analytic families of expanding rational semigroups in Section 8.

In order to state the main results, we need the following notation.

Definition 2.12 ([11]). *Let X be a finite dimensional complex manifold. An upper semicontinuous function $u : X \to \mathbb{R}\cup\{-\infty\}$ is said to be plurisubharmonic if for each holomorphic map $\varphi : \mathbb{D} \to X$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the function $u \circ \varphi : \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$ is subharmonic. A function $v : X \to \mathbb{R} \cup \{+\infty\}$ is said to be plurisuperharmonic if $-v : X \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic.*

Definition 2.13. *For any subset A of $\hat{\mathbb{C}}$, we denote by $\mathrm{HD}(A)$ the Hausdorff dimension of A with respect to the spherical distance on $\hat{\mathbb{C}}$.*

Definition 2.14. *Let $f = (f_1, \dots, f_s) \in (\mathrm{Rat})^s$ be an element and let $G = \langle f_1, \dots, f_s\rangle$. Let also U be a non-empty open set in $\hat{\mathbb{C}}$. We say that f (or G) satisfies the open set condition (with U) if $\cup_{j=1}^s f_j^{-1}(U) \subset U$ and $f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset$ for each (i, j) with $i \neq j$. There is a stronger condition. Namely, we say that f (or G) satisfies the separating open set condition (with U) if $\cup_{j=1}^s f_j^{-1}(U) \subset U$ and $f_i^{-1}(\overline{U}) \cap f_j^{-1}(\overline{U}) = \emptyset$ for each (i, j) with $i \neq j$. Furthermore, we say that f (or G) satisfies the strongly separating open set condition (with U) if $\cup_{j=1}^s f_j^{-1}(\overline{U}) \subset U$ and $f_i^{-1}(\overline{U}) \cap f_j^{-1}(\overline{U}) = \emptyset$ for each (i, j) with $i \neq j$.*

The main purpose of this paper is to prove the following results:

Theorem 2.15. (**Theorem A**) *Let Λ be a finite dimensional complex manifold. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of expanding rational semigroups, where $G_\lambda = \langle f_{\lambda,1}, \dots, f_{\lambda,s}\rangle$. For each $\lambda \in \Lambda$, we set $f_\lambda := (f_{\lambda,1}, \dots, f_{\lambda,s}) \in (\mathrm{Rat})^s$. Then, the Bowen parameter function $\lambda \mapsto \delta(f_\lambda)$ defined for all $\lambda \in \Lambda$, is real-analytic. Also, $(\lambda, t) \mapsto P(t, f_\lambda), (\lambda, t) \in \Lambda \times \mathbb{R}$, is real-analytic, $\lambda \mapsto 1/\delta(f_\lambda), \lambda \in \Lambda$, is plurisuperharmonic, $\lambda \mapsto \delta(f_\lambda), \lambda \in \Lambda$, is plurisubharmonic, and $\lambda \mapsto \log \delta(f_\lambda), \lambda \in \Lambda$, is plurisubharmonic. Furthermore, for a fixed $t \in \mathbb{R}$, the function $\lambda \mapsto P(t, f_\lambda), \lambda \in \Lambda$, is plurisubharmonic.*

Theorem 2.16. (**Theorem B**) *Let Λ be a finite dimensional complex manifold. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of expanding rational semigroups, where $G_\lambda = \langle f_{\lambda,1}, \dots, f_{\lambda,s}\rangle$. Suppose that for each $\lambda \in \Lambda$, G_λ satisfies the open set condition i.e., for each $\lambda \in \Lambda$ there exists a non-empty open set U_λ in $\hat{\mathbb{C}}$ such that $\cup_{j=1}^s f_{\lambda,j}^{-1}(U_\lambda) \subset U_\lambda$ and $f_{\lambda,i}^{-1}(U_\lambda) \cap f_{\lambda,j}^{-1}(U_\lambda) = \emptyset$ for each (i, j) with $i \neq j$. Then, the Hausdorff dimension function $\lambda \mapsto \mathrm{HD}(J(G_\lambda)), \lambda \in \Lambda$, is real-analytic. Besides, $\lambda \mapsto 1/\mathrm{HD}(J(G_\lambda)), \lambda \in \Lambda$, is plurisuperharmonic, $\lambda \mapsto \mathrm{HD}(J(G_\lambda)), \lambda \in \Lambda$, is plurisubharmonic, and $\lambda \mapsto \log \mathrm{HD}(J(G_\lambda)), \lambda \in \Lambda$, is plurisubharmonic.*

Remark 2.17. *There exist a number of elements $g = (g_1, \dots, g_s) \in \mathrm{Exp}(s)$ such that the Hausdorff dimension function $f = (f_1, \dots, f_s) \mapsto \mathrm{HD}(J(\langle f_1, \dots, f_s\rangle))$ is not continuous at g. For example, let $g = (z^2, z^2) \in \mathrm{Exp}(2)$ and for each λ with $0 < \lambda \le 1$, let $f_\lambda := (z^2, \lambda z^2)$ and $G_\lambda := \langle z^2, \lambda z^2\rangle$. Then, for each $0 < \lambda < 1$, $J(G_\lambda) = \{z \in \mathbb{C} : 1 \le |z| \le 1/\lambda\}$ which implies $\mathrm{HD}(J(G_\lambda)) = 2$. However, $J(G_1) = J(\langle z^2, z^2\rangle) = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathrm{HD}(J(G_1)) = 1$.*

Remark 2.18.

- *There is a rich collection of finitely generated rational semigroups $G = \langle f_1, \dots, f_s\rangle$ such that $\mathrm{int} J(G) \neq \emptyset$ and $J(G) \neq \hat{\mathbb{C}}$. For example, for each $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$, $J(\langle z^2, \lambda z^2\rangle) = \{z \in \mathbb{C} : 1 \le |z| \le 1/|\lambda|\}$.*

- *If a finitely generated rational semigroup $G = \langle f_1, \dots, f_s\rangle$ satisfies the open set condition with an open set U, then by* [8, Corollary 3.2], $J(G) \subset \overline{U}$. *If, in addition to the above,* $J(G) \neq \overline{U}$, *then* $\mathrm{int}J(G) = \emptyset$ *(See* [25, Proposition 4.3]*).*
- *Suppose a finitely generated rational semigroup $G = \langle f_1, \dots, f_s\rangle$ satisfies the separating open set condition with the set U. Then, by* [8, Corollary 3.2], [23, Theorem 2.3]*, and* [24, Lemma 2.4]*, we have that* $\mathrm{int}J(G) = \emptyset$ *and $J(G)$ is disconnected. In particular, $J(G)$ is a proper disconnected subset of $\overline{U}$.*
- *If a finitely generated expanding rational semigroup $G = \langle f_1, \dots, f_s\rangle$ satisfies the open set condition with the set U and $J(G) \neq \overline{U}$, then by* [28, Theorem 1.25] *and its proof, the Julia set $J(G)$ is porous and* $\mathrm{HD}(J(G)) < 2$. *In particular, if an expanding rational semigroup $G = \langle f_1, \dots, f_s\rangle$ satisfies the separating open set condition with the set U, then $J(G)$ is porous and* $\mathrm{HD}(J(G)) < 2$.

The proofs of Theorem A and Theorem B are given in Section 7. They make use of the thermodynamic formalisms and the perturbation theory for bounded linear operators on Banach spaces.

We give some additional remarks.

Definition 2.19 ([27]). *Let G be a countable rational semigroup. For any $t \geq 0$ and $z \in \hat{\mathbb{C}}$, we set $S_G(z,t) := \sum_{g\in G}\sum_{g(y)=z} \|g'(y)\|^{-t}$, counting multiplicities. We also set $S_G(z) := \inf\{t \geq 0 : S_G(z,t) < \infty\}$ (if no t exists with $S_G(z,t) < \infty$, then we set $S_G(z) := \infty$). Furthermore, we set $s_0(G) := \inf\{S_G(z) : z \in \hat{\mathbb{C}}\}$. This $s_0(G)$ is called the* **critical exponent of the Poincaré series** *of G.*

Definition 2.20 ([27]). *Let $f = (f_1, \dots, f_s) \in (\mathrm{Rat})^s$, $t \geq 0$, and $z \in \hat{\mathbb{C}}$. We put $T_f(z,t) := \sum_{\omega\in\Sigma_s^*}\sum_{f_\omega(y)=z} \|f'_\omega(y)\|^{-t}$, counting multiplicities. Moreover, we set $T_f(z) := \inf\{t \geq 0 : T_f(z,t) < \infty\}$ (if no t exists with $T_f(z,t) < \infty$, then we set $T_f(z) = \infty$). Furthermore, we set $t_0(f) := \inf\{T_f(z) : z \in \hat{\mathbb{C}}\}$. This $t_0(f)$ is called the* **critical exponent of the Poincaré series** *of $f = (f_1, \dots, f_s) \in (\mathrm{Rat})^s$.*

Remark 2.21. *Let $f = (f_1, \dots, f_s) \in (\mathrm{Rat})^s$, $t \geq 0$, $z \in \hat{\mathbb{C}}$ and let $G = \langle f_1, \dots, f_s\rangle$. Then, $S_G(t,z) \leq T_f(t,z), S_G(z) \leq T_f(z)$, and $s_0(G) \leq t_0(f)$. Note that for almost every $f \in (\mathrm{Rat})^s$ with respect to the Lebesgue measure, $G = \langle f_1, \dots, f_s\rangle$ is a free semigroup and so we have $S_G(t,z) = T_f(t,z), S_G(z) = T_f(z)$, and $s_0(G) = t_0(f)$.*

Definition 2.22. *Let G be a rational semigroup. Then, we set*

$$A(G) := \overline{\cup_{g\in G} g(\{z \in \hat{\mathbb{C}} : \exists u \in G, u(z) = z, |u'(z)| < 1\})}.$$

Lemma 2.23. *Let $f = (f_1, \dots, f_s) \in \mathrm{Exp}(s)$. Then $\delta(f) = t_0(f)$.*

Proof. *Let $G = \langle f_1, \dots, f_s\rangle$. By* [27], *$A(G) \cup P(G) \subset F(G)$ and for each $z \in \hat{\mathbb{C}} \setminus (A(G) \cup P(G))$, $\delta(f) = T_f(z)$. Let $z \in A(G) \cup P(G)$. If there exists an $n \in \mathbb{N}$ such that for each $\omega \in \{1, \dots, s\}^n$ and each $y \in f_\omega^{-1}(z)$, we have $y \in \hat{\mathbb{C}} \setminus (A(G) \cup P(G))$, then by the previous argument, $T_f(z) = \delta(f)$. If there exists a strictly increasing sequence $(n_j)_{j=1}^{\infty}$ of positive integers such that for each $j \in \mathbb{N}$, there exists an $\omega \in \{1, \dots, s\}^{n_j}$ and a $y \in f_\omega^{-1}(z)$ with $y \in A(G) \cup P(G)$, then by* [25, Lemma 1.30], *$T_f(z) = \infty$. Thus, we have $\delta(f) = t_0(f)$. We are done.* $\square$

Remark 2.24. *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$ and let $G = \langle f_1, \ldots, f_s \rangle$. Then, by* [27] *and Lemma 2.23, we have* $\mathrm{HD}(J(G)) \leq s_0(G) \leq S_G(z) \leq \delta(f) = T_f(z) = t_0(f)$, *for each $z \in \hat{\mathbb{C}} \setminus (A(G) \cup P(G))$. In addition to the above assumption, if G satisfies the open set condition i.e., if there exists a non-empty open set U in $\hat{\mathbb{C}}$ such that $\cup_{j=1}^{s} f_j^{-1}(U) \subset U$ and $f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset$ for each (i,j) with $i \neq j$, then by* [27],

$$\mathrm{HD}(J(G)) = s_0(G) = S_G(z) = \delta(f) = T_f(z) = t_0(f),$$

for each $z \in \hat{\mathbb{C}} \setminus (A(G) \cup P(G))$.

Remark 2.25. *The Bowen parameter $\delta(f)$ can be strictly larger than two (See* [27, Example 4.14]*). In fact, a small neighborhood U of $(z^2, z^2/4, z^2/3) \in \mathrm{Exp}(3)$ satisfies that for each $f \in U$, $\delta(f) > 2$. For details, see Section 8.*

In the sequel [39], we will give some estimates of $\delta(f)$.

3. Expandingness

In this section, we show that for an element $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$, the skew product map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ associated with $\{f_1, \ldots, f_s\}$ is an expanding map in the sense of Chapter 3 of [15]. We need more notation.

Let $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$ and let $G := \langle f_1, \ldots, f_s \rangle$. For every $n \leq |\omega|$ let $\omega|_n = (\omega_1, \omega_2, \ldots, \omega_n)$. If $\omega \in \Sigma_s^*$, we put

$$[\omega] = \{\tau \in \Sigma_s : \tau|_{|\omega|} = \omega\}.$$

If $\omega, \tau \in \Sigma_s \cup \Sigma_s^*$, $\omega \wedge \tau$ is the longest initial subword common for both ω and τ. Let α be a fixed number with $0 < \alpha < 1/2$. We endow the shift space Σ_s with the metric ρ_α defined as

$$\rho_\alpha(\omega, \tau) = \alpha^{|\omega \wedge \tau|}$$

with the standard convention that $\alpha^\infty = 0$. The metric d_α induces the product topology on Σ_s. Denote the spherical distance on $\hat{\mathbb{C}}$ by $\hat{\rho}$ and equip the product space $\Sigma_s \times \hat{\mathbb{C}}$ with the metric ρ defined as follows.

$$\rho((\omega, x), (\tau, y)) = \max\{\rho_\alpha(\omega, \tau), \hat{\rho}(x, y)\}.$$

Of course ρ induces the product topology on $\Sigma_s \times \hat{\mathbb{C}}$. Using the fiberwise expanding property (2.1), [24, Proposition 3.2] and the expanding property of the shift map $\sigma : \Sigma_s \to \Sigma_s$, it is fairly easy to prove the following.

Theorem 3.1. *Let $G = \langle f_1, \ldots, f_s \rangle$ be an expanding rational semigroup. Let $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ be the skew product associated with $f = (f_1, \ldots, f_s)$. Then, the dynamical system $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ is a topologically exact open distance expanding map (in the sense of Chapter 3 of* [15]*), meaning that*

(a) *The map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ is open.*
(b) *The map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ is Lipschitz continuous.*

(c) *There exists $q \geq 1$ and $\delta > 0$ such that*

$$\rho(\tilde{f}^q(y), \tilde{f}^q(x)) \geq 4\rho(y, x)$$

for all $x, y \in J(\tilde{f})$ with $\rho(x, y) \leq 2\delta$.

(d) *The map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$ is topologically exact.*

In addition q and δ depend only on C, η in (2.1) and the Lipschitz constant of $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$. Note that 2δ is an expansive constant of the map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$; in particular it is an expansive constant for the shift map $\sigma : \Sigma_s \to \Sigma_s$.

4. J-stability

In this section, we construct a conjugacy map $h : J(\tilde{f}) \to J(\tilde{g})$, when $f \in \mathrm{Exp}(s)$ and g is close enough to f. This conjugacy will be used to construct an analytic family of Perron-Frobenius operators.

Define the metric ρ_∞ on $(\mathrm{Rat})^s$ as follows: for any $f = (f_1, \dots, f_s), g = (g_1, \dots, g_s) \in (\mathrm{Rat})^s$, we set

$$\rho_\infty(f, g) := \rho_\infty(\tilde{f}, \tilde{g}) := \sup\{\rho(\tilde{f}(z), \tilde{g}(z)) : z \in \Sigma_s \times \hat{\mathbb{C}}\},$$

where $\tilde{f}$ (resp. $\tilde{g}$) denotes the skew product map associated with $f = (f_1, \dots, f_s)$ (resp. $g = (g_1, \dots, g_s)$). Given a set $D \subset \Sigma_s \times \hat{\mathbb{C}}$ and $r > 0$, we put

$$B(D, r) = \{z \in \Sigma_s \times \hat{\mathbb{C}} : \rho(z, D) < r\}.$$

where

$$\rho(A, B) = \inf\{\rho(a, b) : (a, b) \in A \times B\}.$$

Similarly, given a set $D \subset (\mathrm{Rat})^s$ and $r > 0$, we put

$$B(D, r) := \{f \in (\mathrm{Rat})^s \mid \rho_\infty(f, D) < r\}.$$

Let $\mathrm{Comp}^*(\Sigma_s \times \hat{\mathbb{C}})$ be the set of all non-empty compact (=closed) subsets of $\Sigma_s \times \hat{\mathbb{C}}$ and let $\mathrm{Comp}^*(\hat{\mathbb{C}})$ be the set of all non-empty compact (=closed) subsets of $\hat{\mathbb{C}}$. The Hausdorff metric on ρ_H on $\mathrm{Comp}^*(\Sigma_s \times \hat{\mathbb{C}})$ is defined as follows.

$$\rho_H(A, B) = \inf_{r>0}\{A \subset B(B, r) \ \&\ B \subset B(A, r)\}.$$

The Hausdorff metric $\hat{\rho}_H$ on $\mathrm{Comp}^*(\hat{\mathbb{C}})$ is defined analogously. Let $\Psi : \mathrm{Exp}(s) \to \mathrm{Comp}^*(\hat{\mathbb{C}})$ be the map defined by $\Psi(f_1, \dots, f_s) := J(\langle f_1, \dots, f_s\rangle)$. Then, we have the following.

Lemma 4.1. *The map $\Psi : \mathrm{Exp}(s) \to \mathrm{Comp}^*(\hat{\mathbb{C}})$ is continuous.*

Proof. Let $f = (f_1, \dots, f_s) \in \mathrm{Exp}(s)$. By [8, Theorem 3.1], [24, Lemma 2.3 (g)], *and* [27, Lemma 3.2], *we have*

$$J(\langle f_1, \dots, f_s\rangle) = \overline{\{z \in \hat{\mathbb{C}} : \exists u \in \langle f_1, \dots, f_s\rangle \text{ such that } u(z) = z, |u'(z)| > 1\}}. \tag{4.1}$$

Hence, for any $\epsilon > 0$, there exists a finite set $Q_f = \{\xi_{1,f}, \dots, \xi_{l,f}\}$ such that $J(\langle f_1, \dots, f_s\rangle) \subset B(Q_f, \epsilon/2)$ and such that each $\xi_{j,f}$ is a repelling fixed point of some $u_{j,f} \in \langle f_1, \dots, f_s\rangle$. By Implicit Function Theorem, it follows that there exists an open neighborhood U of f such that for each $g = (g_1, \dots, g_s) \in U$ and each j with $1 \leq j \leq l$, there exists a *repelling fixed point* $\xi_{j,g}$ *of some* $u_{j,g} \in \langle g_1, \dots, g_s \rangle$ *such that* $\hat{\rho}(\xi_{j,f}, \xi_{j,g}) \le \epsilon/2$. *Therefore, setting* $Q_g := \{\xi_{1,g} \dots, \xi_{l,g}\}$, *we obtain that for each* $g \in U$, $J(\langle f_1, \dots, f_s \rangle) \subset B(Q_g, \epsilon) \subset B(J(\langle g_1, \dots, g_s \rangle), \epsilon)$. *Combining this with (2.6) in the proof of Lemma 2.8, it follows that the map* $\Psi : \mathrm{Exp}(s) \to \mathrm{Comp}^*(\hat{\mathbb{C}})$ *is continuous at* f. *We are done.*

□

Remark 4.2. *In [22], some results which are similar to Lemma 2.8 and Lemma 4.1 have been shown, regarding the dynamics of finitely generated hyperbolic rational semigroups having elements of degree greater than or equal to two.*

Proposition 4.3. *The function* $f = (f_1, \dots, f_s) \mapsto J(\tilde{f})$ *from* $\mathrm{Exp}(s)$ *to* $\mathrm{Comp}^*(\Sigma_s \times \hat{\mathbb{C}})$ *is continuous.*

Proof. *Fix* $f = (f_1, \dots, f_s) \in \mathrm{Exp}(s)$. *Now fix* $\varepsilon > 0$. *Since* $J(\tilde{f})$ *is the closure of all repelling periodic points of* $\tilde{f}$, *there exists a finite set* $P = \{\xi_1, \dots, \xi_l\} \subset J(\tilde{f})$ *of repelling periodic points of* $\tilde{f}$ *such that*

$$J(\tilde{f}) \subset B(P, \varepsilon/2). \tag{4.2}$$

Choose $p \ge 1$ *so large that* $\tilde{f}^p$ *fixes every point in* P. *We may assume that for each* $j = 1, \dots, l$, $\|(\tilde{f}^p)'(\xi_j)\| > 8$. *Since all points* $\xi_1, \xi_2, \dots, \xi_l$ *of* P *are repelling, there exists* $0 < r < \varepsilon/2$ *and for every* $1 \le j \le l$ *there exists a continuous inverse branch* $\tilde{f}_{\xi_j}^{-p} : B(\xi_j, 4r) \to \Sigma_s \times \hat{\mathbb{C}}$ *of* $\tilde{f}^p$ *such that* $\tilde{f}_{\xi_j}^{-p}(\xi_j) = \xi_j$ *and*

$$\left\| \left(\tilde{f}_{\xi_j}^{-p} \right)' (z) \right\| \le 1/4 \tag{4.3}$$

for all $z \in B(\xi_j, 4r)$, *where* $(\tilde{f}_{\xi_j}^{-p})'(z) := ((\tilde{f}^p)'(\tilde{f}_{\xi_j}^{-p}(z)))^{-1}$. *Write* $\xi_j = (\overline{\omega}^j, z_j)$, *where* $z_j \in \hat{\mathbb{C}}$ *and* $\overline{\omega}^j \in \Sigma_s$ *is the infinite concatenation of a finite word* ω^j *of length* p. *There then exists a unique injective meromorphic inverse branch* $f_{\omega^j,j}^{-1} : B(z_j, 4r) \to \hat{\mathbb{C}}$ *of* $f_{\omega^j} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *such that*

$$\tilde{f}_{\xi_j}^{-p}(\tau, y) = \left(\omega^j \tau, f_{\omega^j,j}^{-1}(y) \right)$$

for all $y \in B(z_j, 4r)$. *In particular* $f_{\omega^j,j}^{-1}(z_j) = z_j$ *and, by (4.3),*

$$\left\| \left(f_{\omega^j,j}^{-1} \right)' (y) \right\| \le 1/4$$

for all $y \in B(z_j, 4r)$. *Thus, there exists* $\eta_1 > 0$ *such that if* $g = (g_1, \dots g_s) \in B(f, \eta_1) = \{g = (g_1, \dots, g_s) \in \mathrm{Exp}(s) : \rho_\infty(f, g) < \eta_1\}$, *then for every* $1 \le j \le l$ *there exists a meromorphic inverse branch* $g_{\omega^j,j}^{-1} : B(z_j, 2r) \to \hat{\mathbb{C}}$ *of* $g_{\omega^j} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *such that* $g_{\omega^j,j}^{-1}(\overline{B}(z_j, r)) \subset \overline{B}(z_j, r)$ *and, furthermore,* $\|(g_{\omega^j,j}^{-1})'(y)\| \le 1/2$ *for all* $y \in \overline{B}(z_j, r)$. *It therefore follows from the Banach Contraction Principle that there exists* $x_j \in \overline{B}(z_j, r)$, *a unique fixed point of* $g_{\omega^j,j}^{-1} : \overline{B}(z_j, r) \to \overline{B}(z_j, r)$, *and* $\|(g_{\omega^j,j}^{-1})'(x_j)\| \le 1/2$. *Consequently,* $\tilde{g}^p(\overline{\omega}^j, x_j) = (\overline{\omega}^j, x_j)$ *and* $\|(\tilde{g}^p)'(\overline{\omega}^j, x_j)\| \ge 2$. *Hence,* $(\overline{\omega}^j, x_j) \in J(\tilde{g})$. *Since also* $\rho\big((\overline{\omega}^j, x_j), \xi_j\big) = \hat{\rho}(x_j, z_j) < \varepsilon/2$, *using (4.2), we get that*

$$J(\tilde{f}) \subset B(J(\tilde{g}), \varepsilon).$$

In order to prove the "opposite" inclusion (with appropriately smaller η_1) suppose on the contrary that there exists a sequence $(g_n)_{n=1}^\infty = ((g_{n,1},\dots,g_{n,s}))_{n=1}^\infty \subset \mathrm{Exp}(s)$ such that $\lim_{n\to\infty} g_n = f$ and

$$J(\tilde g_n)\cap\Big((\Sigma_s\times\hat{\mathbb{C}})\setminus B(J(\tilde f),\varepsilon)\Big)\neq\emptyset$$

for all $n\ge 1$. For every $n\ge 1$ choose a point z_n belonging to this intersection. Since the space $\Sigma_s\times\hat{\mathbb{C}}$ is compact, passing to a subsequence, we may assume without loss of generality that the sequence $(z_n)_{n=1}^\infty$ converges. Denote its limit by z. Since $z\notin B(J(\tilde f),\varepsilon)$, this point is in $F(\tilde f)$. So, by Lemma 3.13, p.401 in [27], there exists $q\ge 1$ such that $\pi_2(\tilde f^q(z))\in F(\langle f_1,\dots f_s\rangle)$. Applying Lemma 4.1, we therefore conclude there exists $\theta>0$ such that for all $n\ge 1$ large enough

$$\pi_2(\tilde f^q(z))\in B(\pi_2(\tilde f^q(z)),\theta)\subset F(S_n),$$

where $S_n:=\langle g_{n,1},\dots,g_{n,s}\rangle$. Since $\lim_{n\to\infty} z_n = z$, we thus have that

$$\pi_2(\tilde g_n^q(z_n))\in B(\pi_2(\tilde f^q(z)),\theta)\subset F(S_n)$$

for all $n\ge 1$ large enough. On the other hand, $\pi_2(\tilde g_n^q(z_n))\in\pi_2(J(\tilde g_n)) = J(S_n)$. This contradiction finishes the proof. □

Fix now $f=(f_1,\dots,f_s)\in\mathrm{Exp}(s)$. Then there exists $p\ge 1$ such that

$$\|(\tilde f^p)'(z)\|\ge 4 \tag{4.4}$$

for all $z\in J(\tilde f)$. Since $J(\tilde f)$ is compact and the function $z\mapsto\|(\tilde f^p)'(z)\|$ is continuous on $\Sigma_s\times\hat{\mathbb{C}}$, there exists $\theta'>0$ such that

$$\|(\tilde f^p)'(z)\|\ge 3$$

for all $z\in B(J(\tilde f),\theta')$. Combining this and Proposition 4.3, we see that that there exists $\theta''\in(0,\theta']$ such that

$$\|(\tilde g^p)'(z)\|\ge 2$$

for all $g=(g_1,\dots,g_s)\in B(f,\theta'')$ and all $z\in B(J(\tilde g),\theta')$.

Now using the above, in particular Proposition 4.3, fairly straightforward continuity type considerations and [25, Theorem 2.14] *lead to the following.*

Lemma 4.4. *Suppose that $f=(f_1,\dots,f_s)\in\mathrm{Exp}(s)$ and let $p\ge 1$ be given by (4.4). Then there exists a number $\theta=\theta_f^{(1)}>0$ such that the following properties are satisfied.*

(a) *For all $g=(g_1,\dots,g_s)\in B(f,\theta_f^{(1)})$, all $x\in\overline{B}(J(\tilde g),\theta_f^{(1)})$ and all $y\in\overline{B}(x,\theta_f^{(1)})$,*
$$\|(\tilde g^p)'(y)\|\ge 2,\ \ \rho(\tilde g^p(x),\tilde g^p(y))\ge 2\rho(x,y)$$
and $\tilde g^p|_{B(x,\theta_f^{(1)})}$ is one-to-one and $g^p\big(B(x,\theta_f^{(1)})\big)\supset B(\tilde g^p(x),2\theta_f^{(1)})$.

(b) *If $g\in B(f,\theta_f^{(1)})$ and $\{\tilde g^{pn}(x):n\ge 0\}\subset B(J(\tilde g),\theta_f^{(1)})$, then $x\in J(\tilde g)$.*

As a direct consequence of item (a) of this lemma we get the following.

Lemma 4.5. *Suppose that $f=(f_1,\dots,f_s)\in \mathrm{Exp}(s)$ and let $p\ge 1$ be given by (4.4). If $g=(g_1,\dots,g_s)\in B(f,\theta_f^{(1)})$, $n\ge 1$, and $\tilde g^{pk}(x)\in \overline{B}(J(\tilde g),\theta_f^{(1)})$ for all $0\le k\le n$, then there exists a unique continuous inverse branch $\tilde g_x^{-pn}: B\big(\tilde g^{pn}(x),\theta_f^{(1)}\big)\to B(x,\theta_f^{(1)})$ of $\tilde g^{pn}$ sending $\tilde g^{pn}(x)$ to x. In addition, $\tilde g_x^{-pn}$ is Lipschitz continuous with Lipschitz constant $\le 2^{-n}$ and $\tilde g_x^{-pn}\big(B\big(\tilde g^{pn}(x),\theta_f^{(1)}\big)\big)\subset B(x,\theta_f^{(1)})$.*

From now onwards, unless otherwise stated, assume that the integer $p\ge 1$ ascribed to f by (4.4) is equal to 1. We then call f simple.

Recall that a sequence $(x_i)_{i=0}^n\subset \Sigma_s\times\hat{\mathbb{C}}$, $0\le n\le\infty$ is called a γ-pseudoorbit with respect to the map $\tilde f:\Sigma_s\times\hat{\mathbb{C}}\to\Sigma_s\times\hat{\mathbb{C}}$ provided that

$$\rho(\tilde f(x_i),x_{i+1})\le\gamma$$

for all $0\le i\le n-1$. The pseudoorbit $(x_i)_{i=0}^n$ is said to be β-shadowed by a point $x\in\Sigma_s\times\hat{\mathbb{C}}$ provided that

$$\rho(x_i,\tilde f^i(x))\le\beta$$

for all $0\le i\le n$.
We shall prove the following.

Lemma 4.6. *Assume the same as in Lemma 4.4 (and so the same as in Lemma 4.5). Fix $\beta\in(0,\theta_f^{(1)}/2]$. Suppose that $(x_i)_{i=0}^n\subset B(J(\tilde g),\theta_f^{(1)}/4)$ is a β-pseudoorbit for the skew product map $\tilde g$. For every $0\le i\le n-1$, let $y_i=\tilde g_{x_i}^{-1}(x_{i+1})$, which is defined since $x_i,\tilde g(x_i)\in B(J(\tilde g),\theta_f^{(1)})$ and $x_{i+1}\in B\big(\tilde g(x_i),\beta\big)\subset B\big(\tilde g(x_i),\theta_f^{(1)}\big)$. Then for all $0\le i\le n-1$, we have*

(a) *$y_i\in B(x_i,\theta_f^{(1)}/2)$ and $\tilde g(y_i)=x_{i+1}\in B(J(\tilde g),\theta_f^{(1)})$. So, in view of Lemma 4.5, each inverse branch $\tilde g_{y_i}^{-1}: B\big(\tilde g(y_i),\theta_f^{(1)}\big)\to B(y_i,\theta_f^{(1)}/2)$ is well defined.*

(b) *For all $0\le i\le n-1$*

$$\tilde g_{y_i}^{-1}\big(\overline{B}(x_{i+1},\beta)\big)\subset \overline{B}(x_i,\beta)$$

and, consequently, all the compositions

$$\overline{g}_i^{-i}:=\tilde g_{y_0}^{-1}\circ\tilde g_{y_1}^{-1}\circ\cdots\circ\tilde g_{y_{i-1}}^{-1}:\overline{B}(x_i,\beta)\to\Sigma_s\times\hat{\mathbb{C}}$$

are well defined for all $i=1,2,\dots,n$.

(c) *$\big(\overline{g}_i^{-i}(\overline{B}(x_i,\beta))\big)_{i=0}^n$ is a descending sequence of non-empty compact sets.*

(d) *$\bigcap_{i=0}^n\overline{g}_i^{-i}(\overline{B}(x_i,\beta))\ne\emptyset$ and all the elements of this intersection β-shadow the pseudoorbit $(x_i)_{i=0}^n$.*

(e) *If $n=+\infty$, then the intersection in the item (d) is a singleton which belongs to $J(\tilde g)$.*

Proof. Since $(x_i)_{i=0}^n$ is a β-pseudoorbit, $\tilde g(x_i)\in B(x_{i+1},\beta)\subset B(J(\tilde g),\theta_f^{(1)}/2)\subset B(J(\tilde g),\theta_f^{(1)})$. By the Lipschitz part of Lemma 4.5, $y_i=\tilde g_{x_i}^{-1}(x_{i+1})\in B(x_i,\beta/2)\subset B(x_i,\theta_f^{(1)}/2)\subset$

$B(J(\tilde g),\theta_f^{(1)})$ and item (a) is proved. In order to prove item (b) take an arbitrary point $z\in\overline{B}(x_{i+1},\beta)$, $0\le i\le n-1$. Applying the Lipschitz part of Lemma 4.5 again, we get

$$\begin{aligned}\rho\big(\tilde g_{y_i}^{-1}(z),x_i\big)&\le\rho\big(\tilde g_{y_i}^{-1}(z),y_i\big)+\rho(y_i,x_i)\\&=\rho\big(\tilde g_{y_i}^{-1}(z),\tilde g_{y_i}^{-1}(x_{i+1})\big)+\rho\big(\tilde g_{x_i}^{-1}(x_{i+1}),\tilde g_{x_i}^{-1}(\tilde g(x_i))\big)\\&\le 2^{-1}\rho(z,x_{i+1})+2^{-1}\rho(x_{i+1},\tilde g(x_i))\\&\le 2^{-1}\beta+2^{-1}\beta=\beta.\end{aligned}$$

Hence, $\tilde g_{y_i}^{-1}\big(\overline{B}(x_{i+1},\beta)\big)\subset\overline{B}(x_i,\beta)$ and item (b) is proved. Item (c) is now an immediate consequence of (b), and the first part of (d) is an immediate consequence of (c) and compactness of the sets $\overline{g}_i^{-i}(\overline{B}(x_i,\beta))$. Since for every $0\le k\le n$,

$$\tilde g^k\left(\bigcap_{i=0}^n\overline{g}_i^{-i}(\overline{B}(x_i,\beta))\right)\subset\tilde g^k(\overline{g}_k^{-k})(\overline{B}(x_k,\beta))=\overline{B}(x_k,\beta),$$

the second part of item (d) follows. Since $\bigcap_{i=0}^n\overline{g}_i^{-i}(\overline{B}(x_i,\beta))=\overline{g}_n^{-n}(\overline{B}(x_n,\beta))$, it follows from Lemma 4.5 that $\operatorname{diam}\big(\bigcap_{i=0}^n\overline{g}_i^{-i}(\overline{B}(x_i,\beta))\big)\le 2^{-n}\beta$, and the singleton part of item (e) follows. Obviously

$$\overline{g}_n^{-n}(\overline{B}(x_n,\beta))\subset\overline{g}_n^{-n}(\overline{B}(x_n,\theta_f^{(1)}/2))\tag{4.5}$$

and $\overline{B}(x_n,\theta_f^{(1)}/2)\cap J(\tilde g)\ne\emptyset$ as $x_n\in B(J(\tilde g),\theta_f^{(1)}/4)$. Since the set $J(\tilde g)$ is completely invariant under $\tilde g$, we conclude that $J(\tilde g)\cap\overline{g}_n^{-n}(\overline{B}(x_n,\theta_f^{(1)}/2))\ne\emptyset$. Thus $\bigcap_{n=0}^\infty\overline{g}_n^{-n}(\overline{B}(x_n,\theta_f^{(1)}/2))$ is a singleton belonging to $J(\tilde g)$. The second part of item (e) is then concluded by invoking (4.5).

□

As a straightforward consequence of Lemma 4.4 and Lemma 4.5 we get the following.

Proposition 4.7. *Assume the same as in Lemma 4.4. Then for every $g\in B(f,\theta_f^{(1)})$, the number $\theta_f^{(1)}$ is an expansive constant of $\tilde g:J(\tilde g)\to J(\tilde g)$ meaning that if $x,y\in J(\tilde g)$ and $\rho(\tilde g^n(y),\tilde g^n(x))\le\theta_f^{(1)}$ for all $n\ge 0$, then $x=y$.*

Now if y and z β-shadow the same pseudoorbit $(x_n)_{n=0}^\infty$, then $\rho(\tilde g^n(y),\tilde g^n(z))\le 2\beta\le\theta_f^{(1)}$. Thus, as an immediate consequence of Lemma 4.6 and Proposition 4.7, we get the first part of the following.

Proposition 4.8. *(shadowing lemma) Assume the same as in Lemma 4.4. Then for every $g\in B(f,\theta_f^{(1)})$ and every $\beta\in(0,\theta_f^{(1)}/2]$, every β-pseudoorbit $(x_n)_{n=0}^\infty\subset B(J(\tilde g),\theta_f^{(1)}/4)$ is β-shadowed by a unique element $x\in J(\tilde g)$. If in addition $\sigma^n(\omega)=\pi_1(x_n)$ for all $n\ge 0$, then $\pi_1(x)=\omega$.*

In order to see the second part of this proposition, just notice that $\rho_\alpha(\sigma^n(\pi_1(x)), \sigma^n(\omega)) \leq \beta \leq \theta_f^{(1)}$ *and* $\theta_f^{(1)}$ *is an expansive constant for the shift map* $\sigma : \Sigma_s \to \Sigma_s$.

By Proposition 4.3 there exist $\theta_f^{(2)} \in (0, \theta_f^{(1)}/3]$ *such that*

$$\rho_H(J(\tilde{g}), J(\tilde{f})) < \theta_f^{(1)}/4 \tag{4.6}$$

whenever $\rho_\infty(g, f) \leq \theta_f^{(2)}$. *We shall prove the following main result of this section.*

Theorem 4.9. *Suppose that* $f \in \mathrm{Exp}(s)$ *is simple. Then for every* $g \in B(f, \theta_f^{(2)})$ *there exists a unique homeomorphism* $h = h_g : J(\tilde{f}) \to J(\tilde{g})$ *with the following properties.*

(a) $\tilde{g} \circ h = h \circ \tilde{f}$,
(b) $\pi_1 \circ h = \pi_1$,
(c) *The homeomorphism* $h : J(\tilde{f}) \to J(\tilde{g})$ *is Hölder continuous with the Hölder exponent* κ_f *and the same Hölder constant* L_f, *which is thus the same for all* $g \in B(f, \theta_f^{(2)})$.
(d) $\rho_{\infty, J(\tilde{f})}(h, \mathrm{Id}) := \sup\{\rho(h(z), Id(z)) : z \in J(\tilde{f})\} \leq \theta_f^{(1)}/2$.

Proof. Fix $g \in B(f, \theta_f^{(2)})$. Fix also $z \in J(\tilde{f})$. Then using (4.6), we get $(\tilde{f}^n(z))_{n=0}^\infty \subset B(J(\tilde{g}), \theta_f^{(1)}/4)$ and

$$\rho\big(\tilde{f}^{n+1}(z), \tilde{g}(\tilde{f}^n(z))\big) = \rho\big(\tilde{f}(\tilde{f}^n(z)), \tilde{g}(\tilde{f}^n(z))\big) \leq \rho_\infty(\tilde{f}, \tilde{g}) < \theta_f^{(2)} \leq \theta_f^{(1)}/3.$$

Therefore, in view of Proposition 4.8 (shadowing lemma) applied with $\beta = \theta_f^{(1)}/3$, there exists a unique element, denote it by $h(z) \in J(\tilde{g})$, that $\theta_f^{(1)}/3$-shadows the pseudoorbit $(\tilde{f}^n(z))_{n=0}^\infty$. In addition $\pi_1(h(z)) = \pi_1(z)$. So, we have defined a map $h : J(\tilde{f}) \to J(\tilde{g})$ such that in particular, the property (b) is satisfied. Also, for every $n \geq 0$,

$$\rho\big(\tilde{g}^n(h(z)), \tilde{f}^n(z)\big) \leq \beta = \theta_f^{(1)}/3, \tag{4.7}$$

and reading it with $n = 0$, we get item (d). Also, reading (4.7) for all $n \geq 1$, in the form $\rho\big(\tilde{g}^{n-1}(\tilde{g}(h(z))), \tilde{f}^{n-1}(\tilde{f}(z))\big) \leq \theta_f^{(1)}/3$, we see that the point $\tilde{g}(h(z))$ $\theta_f^{(1)}/2$-shadows the pseudoorbit $(\tilde{f}^n(\tilde{f}(z)))_{n=0}^\infty$. Thus $\tilde{g}(h(z)) = h(\tilde{f}(z))$ and item (a) is established. In order to prove item (c) fix $L \geq 1$, a Lipschitz constant of $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$. Take $x, y \in J(\tilde{f})$ with $0 < \rho(x, y) < \theta_f^{(1)}/3$. Let $k \geq 0$ be the largest integer such that

$$L^k \rho(x, y) < \theta_f^{(1)}/3. \tag{4.8}$$

Then

$$L^{k+1} \rho(x, y) \geq \theta_f^{(1)}/3. \tag{4.9}$$

It follows from (4.8) that $\rho(\tilde f^j(x),\tilde f^j(y)) < \theta_f^{(1)}/3$ *for all* $j=0,1,2,\ldots,k$. *Hence, invoking (4.7), we get for every* $j=0,1,2,\ldots,k$ *that*

$$\begin{aligned}\rho\big(\tilde g^j(h(x)),\tilde g^j(h(y))\big) &\le \rho\big(\tilde g^j(h(x)),\tilde f^j(x)\big)+\rho\big(\tilde f^j(x),\tilde f^j(y)\big)+\rho\big(\tilde f^j(y),\tilde g^j(h(y))\big)\\ &< \frac13\theta_f^{(1)}+\frac13\theta_f^{(1)}+\frac13\theta_f^{(1)}\\ &=\theta_f^{(1)}.\end{aligned}$$

Hence, Lemma 4.5 and Lemma 4.4 yield

$$\begin{aligned}\rho(h(x),h(y)) &= \rho\big(\tilde g_{h(x)}^{-k}(\tilde g^k(h(x))),\tilde g_{h(x)}^{-k}(\tilde g^k(h(y)))\big)\\ &\le 2^{-k}\rho\big(\tilde g^k(h(x))),\tilde g^k(h(y))\big)\\ &\le 2^{-k}\theta_f^{(1)}.\end{aligned}$$

Since, by (4.9), $2^{-k}=L^{-k\frac{\log 2}{\log L}}\le (3L/\theta_f^{(1)})^{\frac{\log 2}{\log L}}\rho^{\frac{\log 2}{\log L}}(x,y)$, *we therefore conclude that*

$$\rho(h(x),h(y))\le \theta_f^{(1)}(3L/\theta_f^{(1)})^{\frac{\log 2}{\log L}}\rho^{\frac{\log 2}{\log L}}(x,y),$$

and the Hölder continuity in condition (c) is proved. In order to prove that $h:J(\tilde f)\to J(\tilde g)$ *is 1-to-1 suppose that* $x,y\in J(\tilde f)$ *and* $h(x)=h(y)$. *Then we get from (4.7) that* $\rho(\tilde f^n(x),\tilde f^n(y))\le 2\theta_f^{(1)}/3$ *for all* $n\ge 0$, *equality* $x=y$ *follows from Proposition 4.7. So, in order to complete the proof of our theorem, we only need to show that* $h:J(\tilde f)\to J(\tilde g)$ *is surjective. Indeed, fix* $y\in J(\tilde g)$. *Reasoning analogously as in the very beginning of the proof, we get that*

$$(\tilde g^n(y))_{n=0}^\infty\subset B(J(\tilde f),\theta_f^{(1)}/4)\ \text{ and } \rho\big(\tilde g^{n+1}(y),\tilde f(\tilde g^n(y))\big)\le\theta_f^{(1)}/3$$

for all $n\ge 0$. *Therefore, again analogously as in the beginning of the proof, we conclude that there exists a point* $x\in J(\tilde f)$ *that* $\theta_f^{(1)}/3$*-shadows the pseudoorbit* $(\tilde g^n(y))_{n=0}^\infty$. *Writing out what this means, we get that* $\rho(\tilde f^n(x),\tilde g^n(y))<\theta_f^{(1)}/3$. *But this also means that the* $(\theta_f^{(1)}/3)$*-pseudoorbit* $(\tilde f^n(x))_{n=0}^\infty$ *is* $\theta_f^{(1)}/3$*-shadowed (with respect to the map* $\tilde g$ *by the point* y*). So,* $y=h(x)$ *and we are done.* □

5. Perron-Frobenius Operators and Their Regularity Properties

In this section, we investigate some complex analytic families of Perron-Frobenius operators on the Banach space of Hölder continuous functions..

Fix a simple element $f\in \mathrm{Exp}(s)$. *Denote by* C_0 *the Banach space of all complex-valued continuous functions on* $J(\tilde f)$ *endowed with the supremum norm* $\|\cdot\|_\infty$. *Fix* $\alpha>0$. *Given* $\varphi\in C_0$ *let*

$$V_\alpha=\inf\{L\ge 0: |\varphi(y)-\varphi(x)|\le L\rho(y,x)^\alpha \ \text{ for all } \ x,y\in J(\tilde f)\},$$

be the α*-variation of the function* φ *and let*

$$||\varphi||_\alpha = V_\alpha(\varphi)+||\varphi||_\infty.$$

Clearly the space

$$\mathrm{H}_\alpha = \mathrm{H}_\alpha(J(\tilde{f})) = \{\varphi \in C_0 : ||\varphi||_\alpha < \infty\}$$

endowed with the norm $||\cdot||_\alpha$ is a Banach space densely contained in C_0 with respect to the $||\cdot||_\infty$ norm. Each member of H_α is called a Hölder continuous function with exponent α. For each Banach space B, we denote by $L(B)$ the space of bounded operators on B endowed with the operator norm.

Let W be an open subset of $\mathbb{C}^q$ with some $q \geq 1$. Suppose that for every $\lambda \in W$ there is given a function $\zeta_\lambda \in \mathrm{H}_\alpha$. The formula

$$\mathcal{L}_\lambda \varphi(z) = \sum_{x \in \tilde{f}^{-1}(z)} e^{\zeta_\lambda(x)} \varphi(x)$$

defines a bounded linear operator acting on the Banach space C_0, called the Perron-Frobenius operator of the potential ζ_λ. It is straightforward to check that $\mathcal{L}_\lambda$ preserves the Banach space H_α. We shall prove the following.

Lemma 5.1. *If the map $\lambda \mapsto \mathcal{L}_\lambda \in L(\mathrm{H}_\alpha)$, $\lambda \in W$, is continuous and for every $z \in J(\tilde{f})$ the function $\lambda \mapsto \zeta_\lambda(z)$, $\lambda \in W$, is holomorphic, then the map $\lambda \mapsto \mathcal{L}_\lambda \in L(\mathrm{H}_\alpha)$ is holomorphic on W.*

Proof. It follows directly from our assumptions that for every $\varphi \in H_\alpha$ and every $z \in J(\tilde{f})$ the function

$$\lambda \mapsto \mathcal{L}_\lambda \varphi(z) = \sum_{x \in \tilde{f}^{-1}(z)} e^{\zeta_\lambda(x)} \varphi(x) \in \mathbb{C}, \ \lambda \in W,$$

is holomorphic. Fix a one-dimensional round disk $D \subset W$. Let $\gamma \subset D$ be a simple closed curve. Fix $\varphi \in H_\alpha$ and $z \in J(\tilde{f})$. By Cauchy's theorem $\int_\gamma \mathcal{L}_\lambda \varphi(z) d\lambda = 0$. Since the function $\lambda \mapsto \mathcal{L}_\lambda \varphi \in H_\alpha$ is continuous, the integral $\int_\gamma \mathcal{L}_\lambda \varphi d\lambda$ exists and for every $z \in J(\tilde{f})$, we have $(\int_\gamma \mathcal{L}_\lambda \varphi d\lambda)(z) = \int_\gamma \mathcal{L}_\lambda \varphi(z) d\lambda = 0$. Hence, $\int_\gamma \mathcal{L}_\lambda \varphi d\lambda = 0$. Now, since the function $\lambda \mapsto \mathcal{L}_\lambda \in L(H_\alpha)$ is continuous, the integral $\int_\gamma \mathcal{L}_\lambda d\lambda$ exists and for every $\varphi \in H_\alpha$, $(\int_\gamma \mathcal{L}_\lambda d\lambda)(\varphi) = \int_\gamma \mathcal{L}_\lambda \varphi d\lambda = 0$. Thus, $\int_\gamma \mathcal{L}_\lambda d\lambda = 0$ and in view of Morera's theorem, the function $\lambda \mapsto \mathcal{L}_\lambda \in L(H_\alpha)$ is holomorphic in D. So, we are done in virtue of Hartogs Theorem. The proof is complete. □

Notation: For any parameter $\lambda_0 \in \mathbb{C}^q$ and any $R > 0$, we set $B(\lambda_0, R) := \{\lambda \in \mathbb{C}^q : \|\lambda - \lambda_0\| < R\}$.

In order for Lemma 5.1 to be applicable we shall prove the following.

Lemma 5.2. *If for every $\lambda \in W$, we have that $\zeta_\lambda \in \mathrm{H}_\alpha$, the function $\lambda \mapsto \zeta_\lambda \in \mathrm{H}_\alpha$ is continuous and for every $z \in J(\tilde{f})$ the function $\lambda \mapsto \zeta_\lambda(z)$, $\lambda \in W$, is holomorphic, then the map $\lambda \mapsto \mathcal{L}_\lambda \in L(\mathrm{H}_\alpha)$, $\lambda \in W$, is continuous.*

Proof. Fix $\lambda_0 \in W$ and $R > 0$ so small that $B(\lambda_0, R) \subset W$. By our assumptions there exists $M > 0$ such that

$$|\zeta_\lambda(z)| \le M \tag{5.1}$$

for all $z \in J(\tilde f)$ and all $\lambda \in B(\lambda_0, R/2)$. Let

$$\hat M := \sup\left\{\frac{|e^t - 1|}{|t|} : t \in \overline{B}(0, 2M)\right\} < \infty.$$

It suffices to show that the function $\lambda \mapsto \mathcal{L}_\lambda \in L(\mathrm{H}_\alpha)$ is Lipschitz continuous with the same Lipschitz constant separately with respect to each variable. Therefore, taking a complex plane parallel to the coordinate planes, we may assume without loss of generality that $q = 1$. Put $D = B(\lambda_0, R/4)$. It follows from Cauchy's formula that

$$|\dot\zeta_\lambda(z)| \le 16MR^{-1} \tag{5.2}$$

for all $z \in J(\tilde f)$ and all $\lambda \in D$, where $\dot\zeta_\lambda(z) = \frac{d}{d\lambda}\zeta_\lambda(z)$. Hence, for all $\lambda, \lambda' \in D$ and $x \in J(\tilde f)$ we obtain

$$|\zeta_{\lambda'}(x) - \zeta_\lambda(x)| = \left|\int_\lambda^{\lambda'} \dot\zeta_\mu(x)\,d\mu\right| \le 16MR^{-1}|\lambda' - \lambda|.$$

Therefore,

$$\begin{aligned} |e^{\zeta_{\lambda'}(x)} - e^{\zeta_\lambda(x)}| &= \exp(\mathrm{Re}(\zeta_\lambda(x)))\left|\exp\big(\zeta_{\lambda'}(x) - \zeta_\lambda(x)\big) - 1\right| \le \hat M e^M |\zeta_{\lambda'}(x) - \zeta_\lambda(x)| \\ &\le 16M\hat M e^M R^{-1}|\lambda' - \lambda|. \end{aligned} \tag{5.3}$$

Consequently, for all $\varphi \in \mathrm{H}_\alpha$,

$$\begin{aligned} |(\mathcal{L}_{\lambda'} - \mathcal{L}_\lambda)\varphi(z)| = |\mathcal{L}_{\lambda'}\varphi(z) - \mathcal{L}_\lambda\varphi(z)| &= \left|\sum_{x \in \tilde f^{-1}(z)} \varphi(x)\big(e^{\zeta_{\lambda'}(x)} - e^{\zeta_\lambda(x)}\big)\right| \\ &\le ||\varphi||_\infty \sum_{x \in \tilde f^{-1}(z)} |e^{\zeta_{\lambda'}(x)} - e^{\zeta_\lambda(x)}| \\ &\le 16M\hat M e^M R^{-1}\deg(\tilde f)||\varphi||_\infty|\lambda' - \lambda|. \end{aligned}$$

Hence,

$$||(\mathcal{L}_{\lambda'} - \mathcal{L}_\lambda)\varphi||_\infty \le 16M\hat M e^M R^{-1}\deg(\tilde f)|\lambda' - \lambda|||\varphi||_\alpha \tag{5.4}$$

for all $\lambda, \lambda' \in D$ and all $\varphi \in \mathrm{H}_\alpha$. In order to estimate the α-variation of the function $(\mathcal{L}_{\lambda'} - \mathcal{L}_\lambda)\varphi$, let $\lambda' \in D$ be a point and consider the function $\psi : D \times J(\tilde f) \times J(\tilde f) \to \mathbb{C}$ given by the formula

$$\psi(\lambda, w, z) = e^{\zeta_\lambda(w)} - e^{\zeta_\lambda(z)} + e^{\zeta_{\lambda'}(z)} - e^{\zeta_{\lambda'}(w)}.$$

Obviously

$$\psi(\lambda', w, z) = 0. \tag{5.5}$$

Put $\dot\psi(\lambda,w,z)=\frac{\partial}{\partial\lambda}\psi(\lambda,w,z)$. *Applying (5.1) and (5.2), we get*

$$
\begin{aligned}
|\dot\psi(\lambda,w,z)| &= \left|\frac{\partial}{\partial\lambda}e^{\zeta_\lambda(w)}-\frac{\partial}{\partial\lambda}e^{\zeta_\lambda(z)}\right| = \left|e^{\zeta_\lambda(w)}\dot\zeta_\lambda(w)-e^{\zeta_\lambda(z)}\dot\zeta_\lambda(z)\right| \\
&= \left|e^{\zeta_\lambda(w)}\big(\dot\zeta_\lambda(w)-\dot\zeta_\lambda(z)\big)+\dot\zeta_\lambda(z)\big(e^{\zeta_\lambda(w)}-e^{\zeta_\lambda(z)}\big)\right| \\
&\le e^M|\dot\zeta_\lambda(w)-\dot\zeta_\lambda(z)|+16MR^{-1}|e^{\zeta_\lambda(w)}-e^{\zeta_\lambda(z)}|.
\end{aligned}
\tag{5.6}
$$

Since the function $\lambda\mapsto\zeta_\lambda\in \mathrm{H}_\alpha$ *is continuous, we have that* $V=\sup\{V_\alpha(\zeta_\lambda):\lambda\in B(\lambda_0,R/2)\}<+\infty$. *Applying Cauchy's formula, we have for all* $z\in J(\tilde f)$ *and all* $\lambda\in D$ *that*

$$
\begin{aligned}
|\dot\zeta_\lambda(w)-\dot\zeta_\lambda(z)| &= \left|\frac{1}{2\pi i}\int_{\partial B(\lambda_0,R/2)}\frac{\zeta_\xi(w)-\zeta_\xi(z)}{(\xi-\lambda)^2}d\xi\right| \le \frac{16}{2\pi R^2}\int_{\partial B(\lambda_0,R/2)}|\zeta_\xi(w)-\zeta_\xi(z)||d\xi| \\
&\le 16R^{-1}V_\alpha(\zeta_\xi)\rho(w,z)^\alpha \le 16R^{-1}V\rho(w,z)^\alpha.
\end{aligned}
$$

We can therefore continue as follows.

$$
\begin{aligned}
|\dot\psi(\lambda,w,z)| &\le 16R^{-1}Ve^M\rho(w,z)^\alpha+16MR^{-1}e^M\hat M|\zeta_\lambda(w)-\zeta_\lambda(z)| \\
&\le \big(16R^{-1}Ve^M+16MR^{-1}e^M\hat MV\big)\rho(w,z)^\alpha.
\end{aligned}
\tag{5.7}
$$

Put $T=16VR^{-1}e^M(1+M\hat M)$. *Applying (5.7) and using (5.5), we get for all* $(\lambda,w,z)\in D\times J(\tilde f)\times J(\tilde f)$ *that*

$$
\begin{aligned}
|\psi(\lambda,w,z)| &= |\psi(\lambda,w,z)-\psi(\lambda',w,z)| = \left|\int_{\lambda'}^{\lambda}\dot\psi(\mu,w,z)d\mu\right| \\
&\le \int_{\lambda'}^{\lambda}|\dot\psi(\mu,w,z)||d\mu| \le T|\lambda'-\lambda|\rho(w,z)^\alpha.
\end{aligned}
$$

Therefore, using also (5.3), we obtain for all $x,y\in J(\tilde f)$ *with* $\rho(x,y)<\theta_f^{(1)}$ *that*

$$
\begin{aligned}
&|(\mathcal{L}_\lambda-\mathcal{L}_{\lambda'})\varphi(y)-(\mathcal{L}_\lambda-\mathcal{L}_{\lambda'})\varphi(x)| = \\
&= \left|\sum_{z\in\tilde f^{-1}(y)}\big(e^{\zeta_\lambda(z)}-e^{\zeta_{\lambda'}(z)}\big)\varphi(z)-\sum_{w\in\tilde f^{-1}(x)}\big(e^{\zeta_\lambda(w)}-e^{\zeta_{\lambda'}(w)}\big)\varphi(w)\right| \\
&= \Big|\sum\big[\big(e^{\zeta_\lambda(z)}-e^{\zeta_{\lambda'}(z)}\big)(\varphi(z)-\varphi(w)) \\
&\qquad -\big(e^{\zeta_\lambda(w)}-e^{\zeta_\lambda(z)}+e^{\zeta_{\lambda'}(z)}-e^{\zeta_{\lambda'}(w)}\big)\varphi(w)\big]\Big| \\
&= \left|\sum\big(e^{\zeta_\lambda(z)}-e^{\zeta_{\lambda'}(z)}\big)(\varphi(z)-\varphi(w))-\psi(\lambda,w,z)\varphi(w)\right| \\
&\le \sum|e^{\zeta_\lambda(z)}-e^{\zeta_{\lambda'}(z)}|\cdot|\varphi(z)-\varphi(w)|+|\psi(\lambda,w,z)|\cdot|\varphi(w)| \\
&\le 16M\hat Me^MR^{-1}\deg(\tilde f)|\lambda-\lambda'|||\varphi||_\alpha\rho(y,x)^\alpha+T||\varphi||_\infty\deg(\tilde f)|\lambda-\lambda'|\rho(y,x)^\alpha \\
&\le (16M\hat Me^MR^{-1}+T)\deg(\tilde f)|\lambda-\lambda'|||\varphi||_\alpha\rho(y,x)^\alpha.
\end{aligned}
$$

Thus, combining this with (5.4),

$$V_\alpha\big((\mathcal{L}_\lambda - \mathcal{L}_{\lambda'})\varphi\big) \le (32M\hat{M}e^M R^{-1} + T + (\theta_f^{(1)})^\alpha)\deg(\tilde{f})|\lambda - \lambda'|||\varphi||_\alpha,$$

and combining this with (5.4) again, we get

$$||(\mathcal{L}_\lambda - \mathcal{L}_{\lambda'})\varphi||_\alpha \le \deg(\tilde{f})|(48M\hat{M}e^M R^{-1} + T + (\theta_f^{(1)})^\alpha)|\lambda - \lambda'|||\varphi||_\alpha.$$

So,

$$||\mathcal{L}_\lambda - \mathcal{L}_{\lambda'}|| \le \deg(\tilde{f})|(48M\hat{M}e^M R^{-1} + T + (\theta_f^{(1)})^\alpha)|\lambda - \lambda'|.$$

We are done.

□

6. Analytic Extensions of Perron-Frobenius Operators.

In this section, for a given analytic family of expanding rational semigroups, we construct an associated real-analytic family of Perron-Frobenius operators, and then we provide a complex analytic extension of this family, in order to use the results proven in the previous section.

Let us first describe in detail the setting of this section. Λ *is assumed to be an open subset of a finite dimensional complex Banach space (ex.* $\mathbb{C}^d$*).*

Let $\{G_\lambda = \langle f_{\lambda,1}, \dots, f_{\lambda,s}\rangle\}_{\lambda\in\Lambda}$ *be an analytic family of expanding rational semigroups. For every* $\lambda \in \Lambda$ *put* $f_\lambda = (f_{\lambda,1}, \dots, f_{\lambda,s}) \in (\mathrm{Rat})^s$*. Fix* $\lambda_0 \in \Lambda$ *and put* $f = f_{\lambda_0}$*.*

Then we easily obtain the following.

Proposition 6.1. *The map* $\lambda \mapsto f_\lambda$ *is continuous.*

For each $\lambda \in \Lambda$*, let* $\theta_{f_\lambda}^{(i)}$ $(i = 1, 2)$ *be the number for* f_λ *obtained in Section 4. So, for every* $\lambda \in \Lambda$ *there exists* $R_\lambda > 0$ *so small that* $f_\gamma \in B\big(f_\lambda, \frac{1}{2}\min\{\theta_{f_\lambda}^{(1)}, \theta_{f_\lambda}^{(2)}\}\big)$ *whenever* $\gamma \in B(\lambda, R_\lambda)$*. For each* $\lambda \in \Lambda$*, let* $\tilde{f}_\lambda : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *be the skew product map associated with* $f_\lambda = (f_{\lambda,1}, \dots, f_{\lambda,s})$*. We shall prove the following.*

Lemma 6.2. *If* $\lambda_0 \in \Lambda$ *and for every* $\lambda \in B(\lambda_0, R_{\lambda_0})$*,* $h_\lambda : J(\tilde{f}) \to J(\tilde{f}_\lambda)$ *is the unique conjugating homeomorphism coming from Theorem 4.9, then for every* $z \in J(\tilde{f})$ *the map* $\lambda \mapsto \pi_2(h_\lambda(z)) \in \hat{\mathbb{C}}$*,* $\lambda \in B(\lambda_0, R_{\lambda_0})$*, is holomorphic.*

Proof. Conjugating $\langle f_{\lambda_0,1}, \dots, f_{\lambda_0,s}\rangle$ *by a Möbius map, we may assume that* $\pi_2(J(\tilde{f})) = J(\langle f_{\lambda_0,1}, \dots, f_{\lambda_0,s}\rangle) \subset \mathbb{C}$*. By Lemma 4.1, we may assume that* $J(\langle f_{\lambda,1}, \dots, f_{\lambda,s}\rangle) \subset \mathbb{C}$ *for each* $\lambda \in \Lambda$*, in order to prove our lemma.*

Set $R_0 = R_{\lambda_0}$*. Fix a repelling periodic point* $(\omega, x) \in J(\tilde{f})$ *of* $\tilde{f}$*, say of period* $p \ge 1$*. Consider the map* $H(z, \lambda) = \pi_2(\tilde{f}_\lambda^p(\omega, z)) - z$*,* $(z, \lambda) \in \mathbb{C} \times \Lambda$*. Then* $H(x, \lambda_0) = 0$ *and*

$$\frac{\partial}{\partial z}H(z,\lambda)|_{(x,\lambda_0)} = (\tilde{f}^p)'(\omega, x) - 1 \ne 0$$

since $|(\tilde{f}^p)'(\omega,x)|>1$. It therefore follows from the Implicit Function Theorem that there exists $\hat{R}(\omega,x)>0$ and a holomorphic map $u_{\omega,x}:B(\lambda_0,\hat{R}(\omega,x))\to\mathbb{C}$ such that $u_{\omega,x}(\lambda_0)=x$ and $H(u_{\omega,x}(\lambda),\lambda)=0$ for all $\lambda\in B(\lambda_0,\hat{R}(\omega,x))$. But $\pi_2(h_{\lambda_0}(\omega,x))=\pi_2(\omega,x)$ and, because of Theorem 4.9,

$$\begin{aligned}H(\pi_2(h_\lambda(\omega,x)),\lambda)&=\pi_2\circ\tilde{f}^p_\lambda(\omega,\pi_2(h_\lambda(\omega,x)))-\pi_2(h_\lambda(\omega,x))\\&=\pi_2\circ\tilde{f}^p_\lambda(h_\lambda(\omega,x))-\pi_2(h_\lambda(\omega,x))\\&=\pi_2\circ h_\lambda\circ\tilde{f}^p(\omega,x)-\pi_2(h_\lambda(\omega,x))\\&=\pi_2(h_\lambda(\omega,x))-\pi_2(h_\lambda(\omega,x))=0.\end{aligned}$$

Therefore, in view of the uniqueness part of the Implicit Function Theorem, there exists $R(\omega,x)\in(0,\hat{R}(\omega,x)]$ such that $u_{\omega,x}(\lambda)=\pi_2(h_\lambda(\omega,x))$ for all $\lambda\in B(\lambda_0,R(\omega,x))$. In particular, the map $\lambda\mapsto\pi_2(h_\lambda(\omega,x))$, $\lambda\in B(\lambda_0,R(\omega,x))$ is holomorphic. Now suppose that the map $\lambda\mapsto\pi_2(h_\lambda(\omega,x))$ defined on $B(\lambda_0,R_0)$ fails to be holomorphic. Select then a parameter $\lambda_1\in B(\lambda_0,R_0)$ such that the map $\lambda\mapsto\pi_2(h_\lambda(\omega,x))$ fails to be holomorphic at λ_1 but it is holomorphic throughout $B(\lambda_0,||\lambda_1-\lambda_0||)$. Obviously, $||\lambda_1-\lambda_0||\geq R(\omega,x)>0$. Since $h_{\lambda_1}(\omega,x)=\lim_{\lambda\to\lambda_1}h_\lambda(\omega,x)$ (the limit taken throughout $B(\lambda_0,||\lambda_1-\lambda_0||)$), we have that $\tilde{f}^p_{\lambda_1}\big(h_{\lambda_1}(\omega,x)\big)=h_{\lambda_1}(\omega,x)$; also $|(\tilde{f}^p_{\lambda_1})'(h_{\lambda_1}(\omega,x))|>1$ since $f_{\lambda_1}\in\mathrm{Exp}(s)$. Replacing in the above considerations λ_0 by λ_1 and x by $\pi_2(h_{\lambda_1}(\omega,x))$ thus yields that the map $\lambda\mapsto\pi_2\big(h^1_\lambda(h_{\lambda_1}(\omega,x))\big)$ is holomorphic on a neighborhood of λ_1, where $h^1_\lambda:J(\tilde{f}_{\lambda_1})\to J(\tilde{f}_\lambda)$ is the unique conjugating homeomorphism coming from Theorem 4.9. Since $h_\lambda(\omega,x)=h^1_\lambda\circ h_{\lambda_1}(\omega,x)$ on this neighborhood, we thus conclude that the map $\lambda\mapsto\pi_2(h_\lambda(\omega,x))$ is holomorphic on a neighborhood of λ_1, which is a contradiction. We have thus proved the following.

Claim 1. *For every periodic point $\xi\in J(\tilde{f})$ the map $\lambda\mapsto\pi_2(h_\lambda(\xi))\in\hat{\mathbb{C}}$, $\lambda\in B(\lambda_0,R_0)$, is holomorphic.*

Now fix an arbitrary point $z_\infty\in J(\tilde{f})$ and let $(z_n)_{n=1}^\infty$ be a sequence or repelling periodic points of $\tilde{f}$ converging to z_∞. Define the maps $u_n:B(\lambda_0,R_0)\to\hat{\mathbb{C}}$, $n=1,2,\ldots,\infty$, by the formula $u_n(\lambda)=\pi_2(h_\lambda(z_n))$. By Claim 1 all these maps with finite n are holomorphic. It follows from Theorem 4.9(c) that there exists a constant L_{λ_0} and a constant κ_{λ_0} such that for all $1\leq n<\infty$ and all $\lambda\in B(\lambda_0,R_0)$,

$$\hat{\rho}(u_n(\lambda),u_\infty(\lambda))=\hat{\rho}(\pi_2(h_\lambda(z_n)),\pi_2(h_\lambda(z_\infty)))\leq\rho(h_\lambda(z_n),h_\lambda(z_\infty))\leq L_{\lambda_0}\rho(z_n,z_\infty)^{\kappa_{\lambda_0}}.$$

Hence, the sequence $(u_n)_{n=1}^\infty$ of holomorphic maps converges uniformly on $B(\lambda_0,R_0)$ to the map u_∞. So, $u_\infty:B(\lambda_0,R_0)\to\hat{\mathbb{C}}$ is holomorphic, and the proof is complete. □

Remark 6.3. *Theorem 4.9 and Lemma 6.2 imply that for each $\omega\in\Sigma_s$, $J_\omega(\tilde{f}_\lambda)$ moves by a holomorphic motion ([13]).*

Now suppose that our Banach space containing Λ is equal to $\mathbb{C}^d$ with some $d\geq1$. Embed $\mathbb{C}^d$ into $\mathbb{C}^{2d}$ by the formula

$$(x_1+iy_1,x_2+iy_2,\ldots,x_d+iy_d)\mapsto(x_1,y_1,x_2,y_2,\ldots,x_d,y_d).$$

For every $z \in \mathbb{C}^d$ and every $r > 0$ denote by $D_d(z,r)$ the d-dimensional polydisk in $\mathbb{C}^d$ centered at z and with "radius" r. We will need the following lemma, which is of general dynamics independent character.

Lemma 6.4. *For every $M \geq 0$, for every $R > 0$, for every $\lambda^0 \in \mathbb{C}^d$, and for every analytic function $\phi : D_d(\lambda^0, R) \to \mathbb{C}$ bounded in modulus by M there exists an analytic function $\tilde{\phi} : D_{2d}(\lambda^0, R/4) \to \mathbb{C}$ that is bounded in modulus by $4^d M$ and whose restriction to the polydisk $D_d(\lambda^0, R/4)$ coincides with $\mathrm{Re}\phi$, the real part of ϕ.*

Proof. Denote by $\mathbb{N}_0$ the set of all non-negative integers. Write the analytic function $\phi : D_d(\lambda^0, R) \to \mathbb{C}$ in the form of its Taylor series expansion

$$\phi(\lambda_1, \lambda_2, \ldots, \lambda_d) = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha (\lambda_1 - \lambda_1^0)^{\alpha_1} (\lambda_2 - \lambda_2^0)^{\alpha_2} \ldots (\lambda_d - \lambda_d^0)^{\alpha_d}.$$

By Cauchy's estimates we have

$$|a_\alpha| \leq \frac{M}{R^{|\alpha|}} \tag{6.1}$$

for all $\alpha \in \mathbb{N}_0^d$. We have

$$\begin{aligned}
&\mathrm{Re}\phi(\lambda_1, \lambda_2, \ldots, \lambda_d) = \\
&= \sum_{\alpha \in \mathbb{N}_0^d} \mathrm{Re}\Bigg[a_\alpha \left(\sum_{p=0}^{\alpha_1} \binom{\alpha_1}{p} \big(\mathrm{Re}\lambda_1 - \mathrm{Re}\lambda_1^0\big)^p \big(\mathrm{Im}\lambda_1 - \mathrm{Im}\lambda_1^0\big)^{\alpha_1 - p} i^{\alpha_1 - p}\right) \cdot \\
&\qquad \cdot \left(\sum_{p=0}^{\alpha_2} \binom{\alpha_2}{p} \big(\mathrm{Re}\lambda_2 - \mathrm{Re}\lambda_2^0\big)^p \big(\mathrm{Im}\lambda_2 - \mathrm{Im}\lambda_2^0\big)^{\alpha_2 - p} i^{\alpha_2 - p}\right) \cdot \ldots \\
&\qquad \ldots \cdot \left(\sum_{p=0}^{\alpha_d} \binom{\alpha_1}{p} \big(\mathrm{Re}\lambda_d - \mathrm{Re}\lambda_d^0\big)^p \big(\mathrm{Im}\lambda_d - \mathrm{Im}\lambda_d^0\big)^{\alpha_d - p} i^{\alpha_d - p}\right) \Bigg] \\
&= \sum_{\beta \in \mathbb{N}_0^{2d}} \mathrm{Re}\left[a_{\hat{\beta}} \prod_{j=1}^d \binom{\beta_j^{(1)} + \beta_j^{(2)}}{\beta_j^{(1)}} i^{\beta_j^{(2)}} \big(\mathrm{Re}\lambda_j - \mathrm{Re}\lambda_j^0\big)^{\beta_j^{(1)}} \big(\mathrm{Im}\lambda_j - \mathrm{Im}\lambda_j^0\big)^{\beta_j^{(2)}}\right] \\
&= \sum_{\beta \in \mathbb{N}_0^{2d}} \mathrm{Re}\left(a_{\hat{\beta}} \prod_{j=1}^d \binom{\beta_j^{(1)} + \beta_j^{(2)}}{\beta_j^{(1)}} i^{\beta_j^{(2)}}\right) \big(\mathrm{Re}\lambda_j - \mathrm{Re}\lambda_j^0\big)^{\beta_j^{(1)}} \big(\mathrm{Im}\lambda_j - \mathrm{Im}\lambda_j^0\big)^{\beta_j^{(2)}},
\end{aligned}$$

where we wrote $\beta \in \mathbb{N}_0^{2d}$ in the form $\big(\beta_1^{(1)}, \beta_1^{(2)}, \beta_2^{(1)}, \beta_2^{(2)}, \ldots, \beta_d^{(1)}, \beta_d^{(2)}\big)$ and we also put $\hat{\beta} = \big(\beta_1^{(1)} + \beta_1^{(2)}, \beta_2^{(1)} + \beta_2^{(2)}, \ldots, \beta_d^{(1)} + \beta_d^{(2)}\big) \in \mathbb{N}_0^d$. Set

$$c_\beta = \mathrm{Re}\left(a_{\hat{\beta}} \prod_{j=1}^d \binom{\beta_j^{(1)} + \beta_j^{(2)}}{\beta_j^{(1)}} i^{\beta_j^{(2)}}\right).$$

Using (6.1), we get

$$|c_\beta| \le |a_{\hat\beta}| \prod_{j=1}^d \binom{\beta_j^{(1)} + \beta_j^{(2)}}{\beta_j^{(1)}} \le MR^{-|\hat\beta|} \prod_{j=1}^d 2^{\beta_j^{(1)}+\beta_j^{(2)}} = MR^{-|\beta|} 2^{|\beta|}.$$

Thus the formula

$$\tilde\phi(x_1, y_1, x_2, y_2, \ldots, x_d, y_d) = \sum_{\beta\in\mathbb{N}_0^{2d}} c_\beta \prod_{j=1}^d \big(x_j - \mathrm{Re}\lambda_j^0\big)^{\beta_j^{(1)}} \big(y_j - \mathrm{Im}\lambda_j^0\big)^{\beta_j^{(2)}}$$

defines an analytic function on $D_{2d}(\lambda_0, R/4)$ *and*

$$|\tilde\phi(x_1, y_1, x_2, y_2, \ldots, x_d, y_d)| \le 4^d M.$$

Obviously $\tilde\phi|_{D_d(\lambda_0,R/4)} = \mathrm{Re}\phi|_{D_d(\lambda_0,R/4)}$, *and we are done.*

□

Let $\{G_\lambda = \langle f_{\lambda,1}, \ldots, f_{\lambda,s}\rangle\}_{\lambda\in\Lambda}$ *be an analytic family of expanding rational semigroups. Coming back to the dynamical situation (and keeping the assumption that* $\Lambda \subset \mathbb{C}^d$*), we assume that for each* $\lambda \in \Lambda$, $J(G_\lambda) \subset \mathbb{C}$.

Remark 6.5. *Note that in order to prove the main results of this paper, we may assume the above. For, by Lemma 4.1, for a fixed* $\lambda_0 \in \Lambda$, *there exists a neighborhood* B *of* λ_0 *and a Möbius transformation* u *such that for each* $\lambda \in B$, $J(uG_\lambda u^{-1}) \subset \mathbb{C}$, *where* $uG_\lambda u^{-1} := \{ugu^{-1} : g \in G_\lambda\}$.

Fix an element $\lambda_0 \in \Lambda$ *and set* $f = f_{\lambda_0}$, *set* $f_\lambda := (f_{\lambda,1}, \ldots, f_{\lambda,s})$, *etc., and we use the notation which was given in the beginning of this section. For every* $z \in J(\tilde f)$ *consider the function*

$$\lambda \mapsto \psi_z(\lambda) := \frac{\tilde f_\lambda'(h_\lambda(z))}{\tilde f'(z)}, \ \lambda \in D_d(\lambda_0, R_0), \ R_0 := R_{\lambda_0}. \tag{6.2}$$

It follows from Lemma 6.2 that this function is holomorphic. Taking $R_0 > 0$ *and* $\theta_{f_{\lambda_0}}^{(1)} > 0$ *sufficiently small, we have that*

$$\left| \frac{\tilde f_\lambda'(w)}{\tilde f'(z)} - 1 \right| < 1/6$$

whenever $\rho(w, z) < \theta_{f_{\lambda_0}}^{(1)}$ *and* $\lambda \in D_d(\lambda_0, R_0)$. *It therefore follows from Theorem 4.9(d) that*

$$|\psi_z(\lambda) - 1| < 1/5 \tag{6.3}$$

for all $z \in J(\tilde f)$ *and all* $\lambda \in D_d(\lambda_0, R_0)$. *Hence, taking a branch* $\log : \{z \in \mathbb{C} : |z - 1| < 1/5\} \to \mathbb{C}$ *of logarithm with* $\log(1) = 0$, *for every* $z \in J(\tilde f)$ *there exists* $\log\psi_z : D_d(\lambda_0, R_0) \to \mathbb{C}$, *a unique holomorphic branch of logarithm of* ψ_z *sending* λ_0 *to* 0. *It is bounded in modulus by* $1/4$. *Now, in virtue of Lemma 6.4, with* $R_* \in (0, R_0)$, *there exists* $\mathrm{Re}\,\tilde{\log}\,\psi_z : D_{2d}(\lambda_0, R_*) \to \mathbb{C}$, *an analytic extension of* $\mathrm{Re}\log\psi_z : D_d(\lambda_0, R_0) \to \mathbb{R}$ *bounded in modulus by* 4^{d-1}. *Now for all* $(t, \lambda, z) \in \mathbb{C} \times D_{2d}(\lambda_0, R_*) \times J(\tilde f)$, *put*

$$\zeta_{(t,\lambda)}(z) = -t\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda) + t\log|\tilde f'(z)| \tag{6.4}$$

Of course for all $z \in J(\tilde{f})$ all the maps $(t,\lambda) \mapsto \zeta_{(t,\lambda)}(z)$ are holomorphic. Let

$$\kappa := \kappa_{f_{\lambda_0}}$$

coming from Theorem 4.9(c). Aiming to apply Lemma 5.1, we shall prove the following.

Lemma 6.6. *There exists an $\tilde{R}$ $(< R_*)$ such that for every $(t,\lambda) \in \mathbb{C} \times D_{2d}(\lambda_0, \tilde{R})$, the function $\zeta_{(t,\lambda)} : J(\tilde{f}) \to \mathbb{C}$ is in H_κ and the map $(t,\lambda) \mapsto \zeta_{(t,\lambda)} \in \mathrm{H}_\kappa$ is continuous.*

Proof. Note that for all $(t,\lambda) \in \mathbb{C} \times D_{2d}(\lambda_0, R_*)$,

$$||\zeta_{(t,\lambda)}||_\infty \le 4^{d-1}|t| + |t| \cdot || \log |\tilde{f}'|||_\infty. \tag{6.5}$$

Fix $r_1 > 0$ so small that all the maps $\tilde{f}_\lambda$, $\lambda \in D_d(\lambda_0, R_0)$, $1 \le i \le s$, are univalent on all disks centered at points of $J(\tilde{f}_\lambda)$ with radii equal to r_1. Starting with (6.3), applying Lemma 6.2 and making use of item (c) of Theorem 4.9, we deduce that there exists a constant $M_1 \ge 1$ and a constant $0 < R_1 < R_*$ such that

$$|\psi_z(\lambda) - \psi_w(\lambda)| \le M_1 \rho(z,w)^\kappa \tag{6.6}$$

for all $w, z \in J(\tilde{f})$ and all $\lambda \in D_d(\lambda_0, R_1)$. Hence, there exists a constant $M_2 \ge 1$ such that

$$|\log \psi_z(\lambda) - \log \psi_w(\lambda)| \le M_2 \rho(z,w)^\kappa \tag{6.7}$$

for all $w, z \in J(\tilde{f})$ and all $\lambda \in D_d(\lambda_0, R_1)$. Applying Lemma 6.4, we get

$$|\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda) - \mathrm{Re}\,\tilde{\log}\,\psi_w(\lambda)| \le 4^d M_2 \rho(z,w)^\kappa \tag{6.8}$$

for all $w, z \in J(\tilde{f})$ and all $\lambda \in D_{2d}(\lambda_0, R_1/4)$. Thus, there exists a constant $M > 0$ such that

$$|\zeta_{(t,\lambda)}(w) - \zeta_{(t,\lambda)}(z)| \le M|t|\rho(w,z)^\kappa$$

for all $w, z \in J(\tilde{f})$ and all $(t,\lambda) \in \mathbb{C} \times D_{2d}(\lambda_0, R_1/4)$. Along with (6.5) this shows that $\zeta_{(t,\lambda)} \in \mathrm{H}_\kappa$ for all $(t,\lambda) \in \mathbb{C} \times D_{2d}(\lambda_0, R_1/4)$. Now we pass to the continuity part of the proof. Since $\mathrm{Re}\,\tilde{\log}\,\psi_z : D_{2d}(\lambda_0, R_*) \to \mathbb{C}$ is bounded in modulus by 4^{d-1}, taking R_1 small enough, by Cauchy's formula we conclude that for all $z \in J(\tilde{f})$ and all $\lambda, \lambda' \in D_{2d}(\lambda_0, R_1/4)$, we have

$$|\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda') - \mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda)| \le T\|\lambda' - \lambda\|$$

with some universal constant $T > 0$. Consequently, for all $z \in J(\tilde{f})$ and all $\lambda, \lambda' \in D_{2d}(\lambda_0, R_1/4)$ and all $t \in \mathbb{C}$,

$$||\zeta_{(t,\lambda')} - \zeta_{(t,\lambda)}||_\infty \le |t|T\|\lambda' - \lambda\|. \tag{6.9}$$

Also, for all $z \in J(\tilde{f})$, all $\lambda \in D_{2d}(\lambda_0, R_1/4)$ and all $t_0, t \in \mathbb{C}$, we have

$$|\zeta_{(t,\lambda)}(z) - \zeta_{(t_0,\lambda)}(z)| \le |t - t_0||\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda)| + |t - t_0|\|\log|f_0'|\|_\infty \le (4^{d-1} + \|\log|f_0'|\|_\infty)|t - t_0|.$$

As a direct consequence of this inequality and (6.9), we conclude that

$$\begin{aligned} &\forall t_0 \in \mathbb{C}\ \exists L_1 > 0\ \forall \lambda, \lambda' \in D_{2d}(\lambda_0, R_1/4)\ \forall t, t' \in D_1(t_0, 2) \\ &\qquad\qquad ||\zeta_{(t',\lambda')} - \zeta_{(t,\lambda)}||_\infty \le L_1||(t', \lambda') - (t,\lambda)||. \end{aligned} \tag{6.10}$$

In order to cope with the Hölder variation fix $\lambda \in D_{2d}(\lambda_0, R_1/4)$ and $t, t' \in \mathbb{C}$. Then for all $w, z \in J(\tilde{f})$ we have

$$\begin{aligned}
|(\zeta_{(t,\lambda)}(w) - \zeta_{(t',\lambda)}(w)) - (\zeta_{(t,\lambda)}(z) - \zeta_{(t',\lambda)}(z))| = \\
&= \left|(t-t')(\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda) - \mathrm{Re}\,\tilde{\log}\,\psi_w(\lambda))\right| \\
&\le |t-t'||\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda) - \mathrm{Re}\,\tilde{\log}\,\psi_w(\lambda)| \\
&\le 4^d M_2 |t-t'| \rho(w,z)^\kappa.
\end{aligned}$$

Hence

$$V_\kappa\big(\zeta_{(t,\lambda)} - \zeta_{(t',\lambda)}\big) \le 4^d M_2 |t-t'|. \tag{6.11}$$

In order to consider $V_\kappa(\zeta_{(t,\lambda)} - \zeta_{(t,\lambda')})$, by (6.8) and Cauchy's formula, it follows that there exists a constant $M_3 \ge 1$ such that

$$|(\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda) - \mathrm{Re}\,\tilde{\log}\,\psi_w(\lambda)) - (\mathrm{Re}\,\tilde{\log}\,\psi_z(\lambda') - \mathrm{Re}\,\tilde{\log}\,\psi_w(\lambda'))| \le M_3 \rho(z,w)^\kappa \|\lambda - \lambda'\| \tag{6.12}$$

for all $w, z \in J(\tilde{f})$ and all $\lambda, \lambda' \in D_{2d}(\lambda_0, R_1/8)$. Hence, we get for all $t_0 \in \mathbb{C}$ and all $(z, w, t, \lambda, \lambda') \in J(\tilde{f})^2 \times D_1(t_0, 2) \times D_{2d}(\lambda_0, R_1/8)$,

$$|(\zeta_{(t,\lambda)} - \zeta_{(t,\lambda')})(z) - (\zeta_{(t,\lambda)} - \zeta_{(t,\lambda')})(w)| \le (|t_0| + 2) M_3 \rho(z,w)^\kappa \|\lambda - \lambda'\|.$$

So $V_\kappa(\zeta_{(t,\lambda)} - \zeta_{(t,\lambda')}) \le (|t_0|+2) M_3 \|\lambda - \lambda'\|$, and invoking (6.11), we deduce that for any $t_0 \in \mathbb{C}$ there exists a constant $L_2 \ge 1$ such that for any $(t, t', \lambda, \lambda') \in D_1(t_0, 2)^2 \times D_{2d}(\lambda_0, R_1/8)^2$,

$$V_\kappa(\zeta_{(t,\lambda)} - \zeta_{(t',\lambda')}) \le L_2 \|(t, \lambda) - (t', \lambda')\|.$$

And bringing up (6.10), we finally get

$$\|\zeta_{(t,\lambda)} - \zeta_{(t',\lambda')}\|_\kappa \le (L_1 + L_2) \|(t, \lambda) - (t', \lambda')\|$$

for all $(t, t', \lambda, \lambda') \in D_1(t_0, 2)^2 \times D_{2d}(\lambda_0, R_1/8)^2$.

We are done. □

7. Real Analyticity of Bowen parameters and Hausdorff Dimensions

This section is devoted to prove our main results by applying the tools developed in the previous sections.

We now prove Theorem A.

Proof of Theorem A: Let Λ be a finite dimensional complex manifold. Let $\{G_\lambda\}_{\lambda \in \Lambda}$, where $G_\lambda = \langle f_{\lambda,1}, \dots, f_{\lambda s} \rangle$, be an analytic family of expanding rational semigroups. In order to prove Theorem A, as in Remark 6.5, we may assume that Λ is an open subset of $\mathbb{C}^d$ and that for each $\lambda \in \Lambda$, $J(G_\lambda) \subset \mathbb{C}$.

Fix $\lambda_0 \in \Lambda$ and we use the notations f, f_λ, etc., which were given in Section 6. Consider the family of potentials $\phi_{(t,\lambda)} : J(\tilde{f}) \to \mathbb{R}$, $(t, \lambda) \in \mathbb{R} \times D_d(\lambda_0, R_0)$ given by the formula

$$\phi_{(t,\lambda)}(w) = -t \log |\tilde{f}_\lambda'(h_\lambda(w))|.$$

Let $\mathrm{P}_\lambda(t)$ *be the topological pressure of the potential* $\phi_{(t,\lambda)}$ *with respect the map* $\tilde{f}: J(\tilde{f}) \to J(\tilde{f})$*, and let* $\mathcal{L}_{(t,\lambda)}: C(J(\tilde{f})) \to C(J(\tilde{f}))$ *be the Perron-Frobenius operator defined by the formula*

$$\mathcal{L}_{(t,\lambda)}\varphi(z) = \sum_{x \in \tilde{f}^{-1}(z)} e^{\phi_{(t,\lambda)}(x)}\varphi(x) = \sum_{x \in \tilde{f}^{-1}(z)} |\tilde{f}'_\lambda(h_\lambda(x))|^{-t}\varphi(x). \tag{7.1}$$

Note that $\mathrm{P}_\lambda(t) = \tilde{\mathrm{P}}_\lambda(t)$*, where* $\tilde{\mathrm{P}}_\lambda(t)$ *is the topological pressure of the potential* $\tilde{\phi}_{(t,\lambda)}(w) = -t\log|\tilde{f}'_\lambda(w)|$ *with respect to the map* $\tilde{f}_\lambda: J(\tilde{f}_\lambda) \to J(\tilde{f}_\lambda)$*. Since we are assuming* $J(G_\lambda) \subset \mathbb{C}$ *and the Euclidian metric and the spherical metric are comparable on a compact subset of* $\mathbb{C}$*, we see that* $\tilde{\mathrm{P}}_\lambda(t)$ *is equal to the topological pressure* $P(t, f_\lambda)$ *of potential* $-t\log\|\tilde{f}'_\lambda\|$ *with respect to the map* $\tilde{f}_\lambda: J(\tilde{f}_\lambda) \to J(\tilde{f}_\lambda)$*, where* $\|\cdot\|$ *denotes the norm of the derivative with respect to the spherical metric on* $\hat{\mathbb{C}}$*. Hence, by* [27]*, for every* $\lambda \in D_d(\lambda_0, R_0)$*, the function* $t \mapsto \mathrm{P}_\lambda(t)$ *is continuous and strictly decreasing from* $+\infty$ *to* $-\infty$*, and has a unique zero* $\delta(f_\lambda)$*, which is the Bowen parameter of* $f_\lambda = (f_{\lambda,1}, \dots, f_{\lambda,s})$*.*

Note that $\phi_{(t,\lambda)} = \zeta_{(t,\lambda)}$ *for all* $(t,\lambda) \in \mathbb{R} \times D_d(\lambda_0, R_*)$*, where* $\zeta_{(t,\lambda)}$ *are the potentials defined by (6.4). In particular the formula*

$$\mathcal{L}_{(t,\lambda)}\varphi(z) = \sum_{x \in \tilde{f}^{-1}(z)} e^{\zeta_{(t,\lambda)}(x)}\varphi(x)$$

extends (7.1) to $\mathbb{C} \times D_{2d}(\lambda_0, R_*)$*. In view of Lemma 6.6, analyticity of the maps* $(t,\lambda) \mapsto \zeta_{(t,\lambda)}(z)$*, Lemma 5.2 and Lemma 5.1, the function* $(t,\lambda) \mapsto \mathcal{L}_{(t,\lambda)} \in L(\mathrm{H}_\kappa)$*,* $(t,\lambda) \in \mathbb{C} \times D_{2d}(\lambda_0, \tilde{R})$*, is holomorphic. One of the central facts (see* [15]*) of the thermodynamic formalism of distance expanding mappings and Hölder continuous potentials, adapted to our setting, is that* $e^{\mathrm{P}_\lambda(t)}$ *is a simple isolated eigenvalue of the operator* $\mathcal{L}_{(t,\lambda)}: \mathrm{H}_\kappa \to \mathrm{H}_\kappa$ *for all* $(t,\lambda) \in \mathbb{R} \times D_d(\lambda_0, \tilde{R})$*. It therefore follows from the theory of perturbations for linear operators (*[10]*, Kato-Rellich Theorem, p212, p368-370) that given* $t_0 \in \mathbb{R}$ *there exists* $R \in (0, \tilde{R})$ *and an analytic function* $\gamma: D_1(t_0, R) \times D_{2d}(\lambda_0, R) \to \mathbb{C}$ *such that for every* $(t,\lambda) \in D_1(t_0, R) \times D_{2d}(\lambda_0, R)$*,* $\gamma(t,\lambda)$ *is a simple isolated eigenvalue of the operator* $\mathcal{L}_{(t,\lambda)}: \mathrm{H}_\kappa \to \mathrm{H}_\kappa$ *and* $\gamma(t_0, \lambda_0) = e^{\mathrm{P}_{\lambda_0}(t_0)}$*. Using continuity of the function* $(t,\lambda) \mapsto \mathrm{P}_\lambda(t)$*,* $(t,\lambda) \in (t_0 - R, t_0 + R) \times D_d(\lambda_0, R)$*, we easily deduce, decreasing* $R > 0$ *if necessary, that* $\gamma(t,\lambda) = e^{\mathrm{P}_\lambda(t)}$ *for all* $(t,\lambda) \in (t_0 - R, t_0 + R) \times D_d(\lambda_0, R)$*. Thus we obtain the following:*
Claim: *The function* $(t,\lambda) \mapsto \mathrm{P}_\lambda(t)$*,* $(t,\lambda) \in (t_0 - R, t_0 + R) \times D_d(\lambda_0, R)$*, is real-analytic.*

Since

$$\frac{\partial \mathrm{P}_\lambda(t)}{\partial t} = -\int \log|(\tilde{f}_\lambda)' \circ h_\lambda| d\mu_{(t,\lambda)} < 0, \tag{7.2}$$

where $\mu_{(t,\lambda)}$ *is the Gibbs (equilibrium) state of the potential* $\phi_{(t,\lambda)}$ *with respect the map* $\tilde{f}: J(\tilde{f}) \to J(\tilde{f})$ *(see* [15, Chapter 4, Theorem 4.6.5]*), applying the Implicit Function Theorem, we get that* $\lambda \mapsto \delta(f_\lambda)$ *is real-analytic.*

We now prove the rest of Theorem A. We may assume that Λ *is an open subset of* $\mathbb{C}$*. Moreover, as before, we may assume that* $J(G_\lambda) \subset \mathbb{C}$ *for each* $\lambda \in \Lambda$*. Let* $\lambda_0 \in \Lambda$ *be a point. We now prove that* $\lambda \mapsto 1/\delta(f_\lambda)$ *is (pluri)superharmonic around* λ_0*. If* $s = 1$ *and* $\deg(f_{\lambda_0,1}) = 1$*, then* $\delta(f_\lambda) \equiv 0$ *around* λ_0*. Hence, we may assume that either (1)*$s > 1$ *or (2)*$s = 1$ *and* $\deg(f_{\lambda_0,1}) > 1$*. In both cases, we have* $\delta(f_{\lambda_0}) > 0$*. Hence, we may assume that* *for each $\lambda \in \Lambda$, $\delta(f_\lambda) > 0$. We set $f = f_{\lambda_0}$ and use the same notations as before. By the variational principle, we have that for each λ,*

$$0 = P_\lambda(\delta(f_\lambda)) = \sup_{\mu \in M(\tilde{f})} \left\{ h_\mu(\tilde{f}) - \delta(f_\lambda) \int_{J(\tilde{f})} \log |(\tilde{f}_\lambda)' \circ h_\lambda| d\mu \right\}, \tag{7.3}$$

where $M(\tilde{f})$ denotes the set of all $\tilde{f}$-invariant Borel probability measures on $J(\tilde{f})$. From this formula, we obtain

$$\delta(f_\lambda) = \sup_{\mu \in M(\tilde{f})} \frac{h_\mu(\tilde{f})}{\int_{J(\tilde{f})} \log |(\tilde{f}_\lambda)' \circ h_\lambda| d\mu}. \tag{7.4}$$

So,

$$\frac{1}{\delta(f_\lambda)} = \inf_{\mu \in M(\tilde{f})} \frac{\int_{J(\tilde{f})} \log |(\tilde{f}_\lambda)' \circ h_\lambda| d\mu}{h_\mu(\tilde{f})}. \tag{7.5}$$

Therefore, the function $\lambda \mapsto 1/\delta(f_\lambda)$ is an infimum of a family of positive (pluri)harmonic functions. Hence, $\lambda \mapsto 1/\delta(f_\lambda)$ is (pluri)superharmonic. Thus, we have proved that $\lambda \mapsto 1/\delta(f_\lambda)$ is plurisuperharmonic.

Since the functions $x \mapsto 1/x$ and $x \mapsto -\log x$ are decreasing convex functions in $(0, \infty)$, Jensen's inequality implies that $\lambda \mapsto \delta(f_\lambda)$ and $\lambda \mapsto \log \delta(f_\lambda)$ are plurisubharmonic.

Finally, let $t \in \mathbb{R}$ be a fixed number. By the variational principle, we have

$$P_\lambda(t) = \sup_{\mu \in M(\tilde{f})} \{h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |(\tilde{f}_\lambda)' \circ h_\lambda| \ d\mu\}.$$

Hence the function $\lambda \mapsto P_\lambda(t)$ is equal to the supremum of a family of pluriharmonic functions of $\lambda \in \Lambda$. Therefore, $\lambda \mapsto P_\lambda(t)$ is plurisubharmonic. We are done. □

Corollary 7.1. *Under the assumption of Theorem A, for each $\lambda \in \Lambda$, let μ_λ be the maximal entropy measure of $\tilde{f}_\lambda : J(\tilde{f}_\lambda) \to J(\tilde{f}_\lambda)$ and let τ_λ be the Gibbs (equilibrium) state of the potential $-\delta(f_\lambda) \log \|\tilde{f}_\lambda'\|$ with respect to the map $\tilde{f}_\lambda : J(\tilde{f}_\lambda) \to J(\tilde{f}_\lambda)$. Then, the functions $\lambda \mapsto \int_{J(\tilde{f}_\lambda)} \log \|\tilde{f}_\lambda'\| d\mu_\lambda$, $\lambda \mapsto \int_{J(\tilde{f}_\lambda)} \log \|\tilde{f}_\lambda'\| d\tau_\lambda$, $\lambda \mapsto h_{\tau_\lambda}(\tilde{f}_\lambda)$, where $\lambda \in \Lambda$, are real-analytic.*

Proof. As in the proof of Theorem A, we may assume that for each $\lambda \in \Lambda$, $J(G_\lambda) \subset \mathbb{C}$. We use the same notation as that in the proof of Theorem A. As in the proof of Theorem A, we have that $\mathrm{P}_\lambda(t) = \tilde{\mathrm{P}}_\lambda(t) = P(t, f_\lambda)$ and that $(t, \lambda) \mapsto P(t, f_\lambda)$ is real-analytic. Since $\lambda \mapsto \delta(f_\lambda)$ is real-analytic, the formula (7.2) and the equation $\delta(f_\lambda) = h_{\tau_\lambda}(\tilde{f}_\lambda) / \int_{J(\tilde{f}_\lambda)} \log \|\tilde{f}_\lambda'\| d\tau_\lambda$ imply that the statement of the theorem holds. We are done.

□

Corollary 7.2. *Under the assumption of Theorem A, suppose $\Lambda \subset \mathbb{C}$. Let $\varphi(\lambda) := \delta(f_\lambda)$. Then, $\varphi \triangle \varphi \geq 2|\nabla \varphi|^2$ in Λ.*

Proof. As in the proof of Theorem A, we may assume that $\varphi(\lambda) > 0$ for each $\lambda \in \Lambda$. Since φ is real-analytic and $1/\varphi$ is superharmonic, the inequality $\triangle(1/\varphi) \leq 0$ implies the inequality of the corollary. $\square$

Corollary 7.3. *Under the assumption of Theorem A, suppose Λ is connected and let $\varphi(\lambda) = \delta(f_\lambda)$. If φ is pluriharmonic in a non-empty open subset of Λ, then φ is constant in Λ.*

Proof. If φ is pluriharmonic in a non-empty subdomain U of Λ, then Corollary 7.2 implies that φ is constant in U. Since φ is real-analytic, it follows that φ is constant in Λ. $\square$

We now prove Theorem B.

Proof of Theorem B: The first author has proved in [27] that if $G = \langle f_1, \dots, f_s\rangle$ is an expanding rational semigroup satisfying the open set condition i.e. there exists a non-empty open subset U of $\hat{\mathbb{C}}$ such that $\cup_{j=1}^s f_j^{-1}(U) \subset U$ and such that for each (i, j) with $i \neq j$, $f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset$, then $\mathrm{HD}(J(G)) = \delta(f)$, where $f = (f_1, \dots, f_s)$. As a direct consequence of Theorem A, the statement of Theorem B holds. $\square$

8. Examples

Throughout this section, we provide an extensive collection of classes of examples of analytic family of semigroups satisfying all the hypothesis of Theorem A and Theorem B and we analyze in detail the corresponding Bowen parameter or Hausdorff dimension function.

First, we give some examples of analytic families of expanding rational semigroups satisfying the open set condition.

Example 8.1. *Let $d_1, d_2 \in \mathbb{N}, \geq 2$ with $(d_1, d_2) \neq (2, 2)$. Let $a \in \mathbb{C}$ with $0 < |a| < 1$. Then, setting $g = (z^{d_1}, az^{d_2})$, we see that the semigroup $G = \langle g_1, g_2\rangle$ is hyperbolic, and hence expanding. There exists also an open neighborhood U of $\{z \in \mathbb{C} : 1 \leq |z| \leq 1/|a|^{\frac{1}{d_2-1}}\}$ such that $g_1^{-1}(\overline{U}) \cup g_2^{-1}(\overline{U}) \subset U$ and $g_1^{-1}\overline{U}) \cap g_2^{-1}(\overline{U}) = \emptyset$. Hence, small perturbation of g satisfies the same property. Therefore, if we take a small neighborhood of V of g in $(\mathrm{Rat})^2$, then setting $G_f := \langle f_1, f_2\rangle$ for each $f \in V$, we have that $\{G_f\}_{f\in V}$ is an analytic family of expanding rational semigroups and for each $f \in V$, the semigroup G_f satisfies the strongly separating open set condition with the set U. By Remark 2.18, for each $f \in V$, $\delta(f) = \mathrm{HD}(J(G_f)) < 2$.*

Proposition 8.2. *(See [31, 33]) Let f_1 be a hyperbolic polynomial with $\deg(f_1) \geq 2$ such that $J(f_1)$ is connected. Let $K(f_1)$ be the filled-in Julia set of f_1 and let $b \in \mathrm{int}K(f_1)$ be a point. Let d be a positive integer such that $d \geq 2$. Suppose that $(\deg(f_1), d) \neq (2, 2)$. Then, there exists a number $c > 0$ such that for each $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$, setting $f_\lambda = (f_{\lambda,1}, f_{\lambda,2}) = (f_1, \lambda(z - b)^d + b)$, we have that $f_\lambda \in \mathrm{Exp}(2)$, f_λ satisfies the separating open set condition with an open set U_λ, $J(\langle f_{\lambda,1}, f_{\lambda,2}\rangle)$ is porous, $\mathrm{HD}(J(\langle f_{\lambda 1.}, f_{\lambda,2}\rangle)) = \delta(f_\lambda) < 2$, and $P(\langle f_{\lambda,1}, f_{\lambda,2}\rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.*

Proof. *We will follow the argument in* [31, 33]. *Conjugating f_1 by a Möbius transformation, we may assume that $b=0$ and the coefficient of the highest degree term of f_1 is equal to* 1.

For each $r>0$, we denote by $D(0,r)$ the Euclidean disc with radius r and center 0. *Let $r>0$ be a number such that $\overline{D(0,r)} \subset \mathrm{int}K(f_1)$. We set $d_1 := \deg(f_1)$. Let $\alpha > 0$ be a number. Since $d \geq 2$ and $(d,d_1) \neq (2,2)$, it is easy to see that $(\frac{r}{\alpha})^{\frac{1}{d}} > 2\left(2(\frac{1}{\alpha})^{\frac{1}{d-1}}\right)^{\frac{1}{d_1}}$ if and only if*

$$\log \alpha < \frac{d(d-1)d_1}{d+d_1-d_1 d}(\log 2 - \frac{1}{d_1}\log\frac{1}{2} - \frac{1}{d}\log r). \tag{8.1}$$

We set

$$c_0 := \exp\left(\frac{d(d-1)d_1}{d+d_1-d_1 d}(\log 2 - \frac{1}{d_1}\log\frac{1}{2} - \frac{1}{d}\log r)\right) \in (0,\infty). \tag{8.2}$$

Let $0 < c < c_0$ be a small number and let $\lambda \in \mathbb{C}$ be a number with $0 < |\lambda| < c$. Put $f_{\lambda,2}(z) = \lambda z^d$. Then, we obtain $K(f_{\lambda,2}) = \{z \in \mathbb{C} \mid |z| \leq (\frac{1}{|\lambda|})^{\frac{1}{d-1}}\}$ and

$$f_{\lambda,2}^{-1}(\{z \in \mathbb{C} \mid |z| = r\}) = \{z \in \mathbb{C} \mid |z| = (\frac{r}{|\lambda|})^{\frac{1}{d}}\}.$$

Let $D_\lambda := \overline{D(0, 2(\frac{1}{|\lambda|})^{\frac{1}{d-1}})}$. Since $f_1(z) = z^{d_1}(1+o(1))$ $(z \to \infty)$, it follows that if c is small enough, then for any $\lambda \in \mathbb{C}$ with $0 < |\lambda| < c$,

$$f_1^{-1}(D_\lambda) \subset \left\{ z \in \mathbb{C} \mid |z| \leq 2\left(2(\frac{1}{|\lambda|})^{\frac{1}{d-1}}\right)^{\frac{1}{d_1}} \right\}.$$

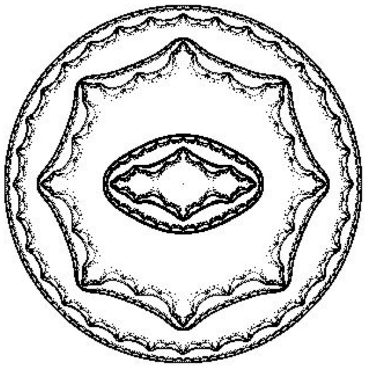
This implies that

$$f_1^{-1}(D_\lambda) \subset f_{\lambda,2}^{-1}(\{z \in \mathbb{C} \mid |z| < r\}). \tag{8.3}$$

Hence, setting $U_\lambda := \mathrm{int}K(f_{\lambda,2}) \setminus K(f_1)$, $f_\lambda = (f_1, f_{\lambda,2})$ satisfies the separating open set condition with U_λ. Therefore, setting $G_\lambda := \langle f_1, f_{\lambda,2}\rangle$, we have $J(G_\lambda) \subset \overline{U_\lambda} \subset K(f_{\lambda,2}) \setminus \mathrm{int}K(f_1)$. In particular, $\mathrm{int}K(f_1) \cup (\hat{\mathbb{C}} \setminus K(f_{\lambda,2})) \subset F(G_\lambda)$. Furthermore, (8.3) implies that $f_{\lambda,2}(K(f_1)) \subset \mathrm{int}K(f_1)$. Thus, we have $P(G_\lambda)\setminus\{\infty\} = \overline{\bigcup_{g \in G_\lambda \cup \{Id\}} g(CV^*(f_1) \cup CV^*(f_{\lambda,2}))} \subset \mathrm{int}K(f_1) \subset F(G_\lambda)$, where $CV^*(\cdot)$ denotes the set of all critical values in $\mathbb{C}$. Hence, G_λ is expanding and $P(G_\lambda) \setminus \{\infty\}$ is bounded in $\mathbb{C}$. By Theorem B and Remark 2.18, we obtain that for each λ with $0 < |\lambda| < c$, $J(G_\lambda)$ is porous and $\mathrm{HD}(J(G_\lambda)) = \delta(f_\lambda) < 2$. We are done. □

Example 8.3 ([33])**.** *Let $h_1(z) = z^2/4, h_2(z) = z^2 - 1, f_1 := h_1^2, f_2 := h_2^2$, and $f := (f_1, f_2)$. Let $G = \langle f_1, f_2\rangle$. Then it is easy to see that $f_1(K(f_2)) \subset \mathrm{int}(K(f_2))$ and $P(G) \setminus \{\infty\} \subset \mathrm{int}(K(f_2))$. Hence, we have $P(G) \subset F(G)$, which implies that $f \in \mathrm{Exp}(2)$. Moreover, it is easy to see that f satisfies the strongly separating open set condition with an open set U (letting U be an open neighborhood of $K(f_1) \setminus \mathrm{int}(K(f_2))$). Thus there exists an open neighborhood V of f in $(\mathrm{Rat})^2$ such that for each $g \in V$, we have that $g \in \mathrm{Exp}(2)$, g satisfies the strongly separating open set condition with U, $P(\langle g_1, g_2\rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, and $\mathrm{HD}(J(\langle g_1, g_2\rangle)) = \delta(g) < 2$. See Figure 1 for the Julia set of $\langle f_1, f_2\rangle$.*

FIGURE 1. The Julia set of $\langle f_1, f_2\rangle$, where $h_1(z) := z^2/4,\ h_2(z) := z^2 - 1,\ f_1 := h_1^2,\ f_2 := h_2^2$.



Example 8.4. *For each $j = 1, 2$, let γ_j be a hyperbolic polynomial such that $\deg(\gamma_j) \geq 2$ and $J(\gamma_j)$ is connected. Suppose that $K(\gamma_1) \cap K(\gamma_2) = \emptyset$, where $K(\cdot)$ denotes the filled-in Julia set. Let $R > 0$ be a large number such that $B(0, R) \supset K(\gamma_1) \cup K(\gamma_2)$. Then, there exists a large positive integer n such that with $U := B(0, R)$,*

$$\gamma_1^{-n}(\overline{U}) \cup \gamma_2^{-n}(\overline{U}) \subset U \text{ and } \gamma_1^{-n}(\overline{U}) \cap \gamma_2^{-n}(\overline{U}) = \emptyset. \tag{8.4}$$

Thus, setting $g = (g_1, g_2) := (\gamma_1^n, \gamma_2^n)$, there exists an open neighborhood V of g in $(\mathrm{Rat})^2$ such that each $f = (f_1, f_2) \in V$ satisfies the strongly separating open set condition with U. Since $K(g_j) \subset U$ for each $j = 1, 2$, (8.4) implies that setting $W := \mathrm{int}K(g_1) \cup \mathrm{int}K(g_2) \cup (\hat{\mathbb{C}} \setminus \overline{U})$, we have that for each $j = 1, 2$, $g_j(W) \subset W$. Hence, $W \subset F(\langle g_1, g_2\rangle)$. Combining this, $CV^*(g_j) \subset K(g_j) \subset U$ where $CV^*(g_j)$ denotes the set of critical values of g_j in $\mathbb{C}$, and (8.4), we obtain that $P(\langle g_1, g_2\rangle) \subset F(\langle g_1, g_2\rangle)$. Therefore, $g = (g_1, g_2) \in \mathrm{Exp}(2)$. Thus, if we take the above V small enough, it follows that for each $f = (f_1, f_2) \in V$, we have that $f \in \mathrm{Exp}(2)$ and f satisfies the strongly separating open set condition. In particular, for each $f \in V$, $\mathrm{HD}(J(\langle f_1, f_2\rangle)) = \delta(f) < 2$ and $J(\langle f_1, f_2\rangle)$ is porous.

Now we describe an example of analytic family $\{G_\lambda\}_{\lambda\in\Lambda}$ of expanding rational semigroups, where $G_\lambda = \langle f_{\lambda,1}, \ldots, f_{\lambda,s}\rangle$, $f_\lambda = (f_{\lambda,1}, \ldots, f_{\lambda,s})$, such that each G_λ satisfies the open set condition with U_λ but does not satisfy the separating open set condition with any open subset of $\hat{\mathbb{C}}$, and such that for each λ, $\mathrm{HD}(J(G_\lambda)) = \delta(f_\lambda) < 2$.

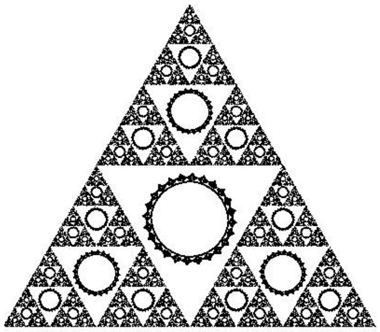
Example 8.5. *(See [27, Example 6.2]) Let p_1, p_2, and $p_3 \in \mathbb{C}$ be mutually distinct points that form an equilateral triangle. Let U be the interior part of the triangle. Let $\gamma_j(z) = 2(z - p_j) + p_j$, for each $j = 1, 2, 3$. Then $J(\langle\gamma_1, \gamma_2, \gamma_3\rangle)$ is the Sierpiński gasket, which is connected. Hence $(\gamma_1, \gamma_2, \gamma_3)$ satisfies the open set condition with U but fails to satisfy the separating open set condition with any open subset of $\hat{\mathbb{C}}$. Let x be the barycenter of the equilateral triangle $p_1p_2p_3$ and let $r > 0$ be a small number such that $D(x, r) \subset U \setminus \cup_{j=1}^3 \gamma_j^{-1}(\overline{U})$, where $D(x, r)$ denotes the Euclidean disc with center x and radius r. Let γ_4 be a polynomial such that $J(\gamma_4) = \partial D(x, r)$. Let u be a large positive integer such that $\gamma_4^{-u}(\overline{U}) \subset U \setminus \cup_{j=1}^3 \gamma_j^{-1}(\overline{U})$. Set $\alpha := \gamma_4^u$. Then, there exists a neighborhood V of α in Rat such that for each $\beta \in V$, $f = (\gamma_1, \gamma_2, \gamma_3, \beta)$ satisfies the open set condition with U. Let $G_\beta = \langle\gamma_1, \gamma_2, \gamma_3, \beta\rangle$, for each*

$\beta \in V$. If we take u large enough, then we may assume that $P(G_\alpha) \subset F(G_\alpha)$. Therefore, G_α is expanding. Thus, if we take the above V small enough, it follows that for each $\beta \in V$, we have that G_β is expanding, G_β satisfies the open set condition with U, and G_β fails to satisfy the separating open set condition with any open subset of $\hat{\mathbb{C}}$. also note that for each $\beta \in V$, $\cup_{j=1}^3 \gamma_j^{-1}(\overline{U}) \cup \beta^{-1}(\overline{U})$ is a proper subset of $\overline{U}$. Combining it with [24, Lemma 2.4] *and* [28, Theorem 1.25], *we obtain that for each $\beta \in V$, $J(G_\beta)$ is porous and $\mathrm{HD}(J(G_\beta)) < 2$. Thus, for each $\beta \in V$, we have*

$$\mathrm{HD}(J(G_\beta)) = \delta(\gamma_1, \gamma_2, \gamma_3, \beta) < 2.$$

See Figure 2 for the Julia set of G_α.

FIGURE 2. The Julia set of $G_\alpha = \langle \gamma_1, \gamma_2, \gamma_3, \alpha \rangle$.



Remark 8.6. *In the sequel* [36] *(announced in* [30]*), we will see that there are plenty of parameters $f = (f_1, f_2) \in \mathrm{Exp}(2)$ such that f satisfies all of the following conditions: (1)f_1 and f_2 are polynomials of degree greater than or equal to two, (2)f satisfies the open set condition, (3)$P(\langle f_1, f_2 \rangle) \setminus \{\infty\}$ is bounded in $\mathbb{C}$, (4)$\mathrm{HD}(J(\langle f_1, f_2 \rangle)) < 2$, and (5)$J(\langle f_1, f_2 \rangle)$ is connected.*

We give an example of analytic family $\{G_\lambda\}_{\lambda \in \Lambda}$ of expanding rational semigroups satisfying the open set condition and $\delta(f_\lambda) = \mathrm{HD}(J(G_\lambda)) = 2$, where $G_\lambda = \langle f_{\lambda,1}, \ldots, f_{\lambda,s} \rangle$, $f_\lambda := (f_{\lambda,1}, \ldots, f_{\lambda,s})$, $\Lambda = \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$.

Example 8.7. *Let $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ and let $f_\lambda = (f_{\lambda,1}, f_{\lambda,2}) = (z^2, \lambda z^2) \in (\mathrm{Rat})^2$. Let $G_\lambda = \langle f_{\lambda,1}, f_{\lambda,2} \rangle$. Then, $P(G_\lambda) = \{0, \infty\} \subset F(G_\lambda)$ and therefore G_λ is expanding. Let $A_\lambda = \{z \in \mathbb{C} : 1 < |z| < 1/|\lambda|\}$. Then, we have $f_{\lambda,1}^{-1}(A_\lambda) \cup f_{\lambda,2}^{-1}(A_\lambda) \subset A_\lambda$ and $f_{\lambda,1}^{-1}(A_\lambda) \cap f_{\lambda,2}^{-1}(A_\lambda) = \emptyset$. Hence, G_λ satisfies the open set condition. Therefore, by* [27], *we have $\delta(f_\lambda) = \mathrm{HD}(J(G_\lambda))$. Since the point 1 belongs to the Julia set of $f_1 = f_{\lambda,1}$, we have $1 \in J(G_\lambda)$. Moreover, we easily obtain that $\cup_{g \in G_\lambda} g^{-1}(1)$ is dense in $\overline{A_\lambda}$. Hence, by* [8, Lemma 3.2], *it follows that $J(G_\lambda) = \overline{A_\lambda}$. Thus, $\delta(f_\lambda) = \mathrm{HD}(J(G_\lambda)) = 2$.*

We give another example of analytic family of expanding rational semigroups such that $\mathrm{HD}(J(\langle f_{\lambda,1}, \ldots, f_{\lambda s} \rangle)) \le \delta(f_\lambda) < 2$.

Example 8.8. *Let $d_1, d_2, \dots, d_s \in \mathbb{N}$ such that $d_j \geq 2$ for each $j = 1, \dots, s$. Let $g = (g_1, \dots, g_s) = (z^{d_1}, \dots, z^{d_s}) \in (\mathrm{Rat})^s$. Let $L_t : C(J(\tilde{g})) \to C(J(\tilde{g}))$ be the Perron Frobenius operator defined by the formula*

$$L_t\varphi(z) = \sum_{\tilde{g}(y)=z} |\tilde{g}'(y)|^{-t}\varphi(y).$$

Then, we have $L_t 1 \equiv (\sum_{j=1}^{s} \frac{1}{d_j^{t-1}})1$, where 1 denotes the constant function taking the value 1. Hence, setting $\beta(t) = \sum_{j=1}^{s} \frac{1}{d_j^{t-1}}$, we have

$$\beta(t) = e^{P(t,g)}. \tag{8.5}$$

Thus, $\beta(\delta(g)) = 1$. We now assume that $\sum_{j=1}^{s} \frac{1}{d_j} < 1$. Then, since the function $t \mapsto \beta(t)$ is strictly decreasing, we obtain that $\delta(g) < 2$. Since $f \mapsto \delta(f)$ is continuous around g, it follows from [27] that there exists an open neighborhood U of g in $(\mathrm{Rat})^s$ and an $\epsilon > 0$ such that for each $f \in U$, $\mathrm{HD}(\langle f_1, \dots, f_s\rangle) \leq \delta(f) \leq 2 - \epsilon$. In particular, for each $f \in U$, $\mathrm{int}J(\langle f_1, \dots, f_s\rangle)) = \emptyset$. By Remark 2.21 and Remark 2.24, for almost every $f \in U$ with respect to the Lebesgue measure, $s_0(\langle f_1, \dots, f_s\rangle) = t_0(f) = \delta(f) \leq 2 - \epsilon < 2$.

Let us now provide several sufficient conditions for f to satisfy $\delta(f) > 2$.

Example 8.9. *Using the same notation as that in Example 8.8, suppose there exists an integer m such that $d_1 = \cdots = d_s = m$. Then, by (8.5), we obtain $\delta(g) = 1 + \frac{\log s}{\log m}$. Since the function $f \mapsto \delta(f)$ is continuous and plurisubharmonic around g, it follows that for each open neighborhood U of g in $(\mathrm{Rat})^s$, there exists a non-empty open subset V of $U \setminus \{g\}$ such that for each $f \in V$,*

$$\delta(f) \geq 1 + \frac{\log s}{\log m}. \tag{8.6}$$

We now assume that $s > m$. Then, from the equality $\delta(g) = 1 + \frac{\log s}{\log m}$ and the continuity of $f \mapsto \delta(f)$ around g, it follows that for each ϵ with $0 < \epsilon < \frac{\log s}{\log m} - 1$, there exists an open neighborhood W of g in $(\mathrm{Rat})^s$ such that for each $f \in W$, $\delta(f) \geq 1 + \frac{\log s}{\log m} - \epsilon > 2$. In particular, for each $f \in W$, f does not satisfy the open set condition. Moreover, by Remark 2.21 and Remark 2.24, for almost every $f \in W$ with respect to the Lebesgue measure, we have $s_0(\langle f_1, \dots, f_s\rangle) = t_0(f) = \delta(f) \geq 1 + \frac{\log s}{\log m} - \epsilon > 2$. Note that for a fixed m, $1 + \frac{\log s}{\log m} \to \infty$ as $s \to \infty$. Thus, the functions $f \mapsto \delta(f)$, $f \mapsto t_0(f)$, and $f \mapsto s_0(\langle f_1, \dots, f_s\rangle)$, where $f \in \mathrm{Exp}(s)$, are unbounded, if s runs over all positive integers.

Proposition 8.10. *Let $g = (g_1, \dots, g_s) \in \mathrm{Exp}(s)$ and let $G = \langle g_1, \dots, g_s\rangle$. Let m_2 be the 2-dimensional Lebesgue measure. Suppose that there exists a couple (i, j) with $i \neq j$ such that $m_2(g_i^{-1}(J(G)) \cap g_j^{-1}(J(G))) > 0$. Then, $\delta(g) > 2$. In addition, for each $0 < \epsilon < \delta(f) - 2$, there exists an open neighborhood U of g in $(\mathrm{Rat})^s$ such that for each $f \in U$, $\delta(f) \geq \delta(g) - \epsilon > 2$.*

Proof. By the assumption, [27] implies that $\delta(g) \geq \mathrm{HD}(J(G)) = 2$. Suppose $\delta(g) = 2$. Then, by [27, Proposition 4.13], we obtain a contradiction. Thus, $\delta(g) > 2$. Since the *function $f \mapsto \delta(f)$ is continuous around g, the rest of the statement of the proposition holds. We are done.* □

Example 8.11. *Let $g = (g_1, g_2, g_3) = (z^2, z^2/4, z^2/3) \in (\mathrm{Rat})^3$ and let $G := \langle g_1, g_2, g_3\rangle$. Then, $P(G) = \{0, \infty\} \subset F(G)$ and so G is expanding and $J(G) = \{z \in \mathbb{C} : 1 \leq |z| \leq 4\}$. By* [27, Example 4.14], *(or by Proposition 8.10), we have that $\delta(g) > 2$. Since the function $f \mapsto \delta(f)$ is continuous around g, it follows that for each ϵ with $0 < \epsilon < \delta(g) - 2$, there exists an open neighborhood U of g in $(\mathrm{Rat})^3$ such that for each $f \in U$, $\delta(f) \geq \delta(g) - \epsilon > 2$. Moreover, by Remark 2.21 and Remark 2.24, for almost every $f \in U$ with respect to the Lebesgue measure, we have $s_0(\langle f_1, f_2, f_3\rangle) = t_0(f) = \delta(f) \geq \delta(g) - \epsilon > 2$.*

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