LAMBDA-TOPOLOGY VS. POINTWISE TOPOLOGY

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ABSTRACT. This paper deals with families of conformal iterated function systems (CIFS). The space $\mathrm{CIFS}(X, I)$ of all CIFS, with common seed space X and alphabet I, is successively endowed with the topology of pointwise convergence and the so-called λ-topology.

We show just how bad the topology of pointwise convergence is: although the Hausdorff dimension function is continuous on a dense G_δ-set, it is also discontinuous on a dense subset of $\mathrm{CIFS}(X, I)$. Moreover, all the different types of systems (irregular, critically regular, etc...), have empty interior, have for boundary the whole space, and thus are dense in $\mathrm{CIFS}(X, I)$, which goes against intuition and conception of a natural topology on $\mathrm{CIFS}(X, I)$.

We then prove how good the λ-topology is: Roy and Urbański [8] have previously pointed out that the Hausdorff dimension function is then continuous everywhere on $\mathrm{CIFS}(X, I)$. We go further in this paper. We show that (almost) all the different types of systems have natural topological properties. We also show that, despite not being metrizable (for it does not satisfy the first axiom of countability), the λ-topology makes the space $\mathrm{CIFS}(X, I)$ normal. Moreover, this space has no isolated points. We further prove that the conformal Gibbs measures and invariant Gibbs measures depend continuously on $\Phi \in \mathrm{CIFS}(X, I)$ and on the parameter t of the potential and pressure functions. However, we demonstrate that the coding map and the closure of the limit set are discontinuous on an important subset of $\mathrm{CIFS}(X)$.

1. Introduction

The last 15 years have been a period of extensive study of single conformal iterated function systems (abbreviated to CIFSs). Recently, interest in families of such systems has emerged (see [1], [2], [8] and [9], among others). In [8] Roy and Urbański studied the space $\mathrm{CIFS}(X, I)$ of all CIFSs sharing the same seed space X and the same alphabet I. When I is finite, they endowed $\mathrm{CIFS}(X, I)$ with a natural metric of pointwise convergence (pointwise meaning that corresponding generators are $C^1(X)$-close to one another). They showed that the topological pressure and the Hausdorff dimension functions are then continuous (see Lemma 4.2 and Theorem 4.3 in [8]). When I is infinite, they discovered that these latter are generally not continuous when $\mathrm{CIFS}(X, I)$ is equipped with a "generalized" metric of pointwise convergence (see Theorem 5.2 and Lemma 5.3, as well as the example following these results in [8]). They thereafter introduced a new, weaker topology called λ-topology (see (5.1) in [8]). In that topology, they proved that the topological pressure and the Hausdorff dimension functions are both continuous (see Theorems 5.7 and 5.10 in [8]).

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The aim of this paper is to deepen our understanding of the pointwise and λ- topologies. We will contrast the λ-topology with the pointwise topology, show that the λ-topology is more convenient, and describe this latter further.

In section 2, Preliminaries on Single Iterated Function Systems, we collect the definitions, concepts, and most of the known results concerning single iterated function systems.

In section 3, Preliminaries on Families of Iterated Function Systems, we introduce a new notation for the different types of systems and gather a few simple observations about these types.

In section 4, The Pointwise Topology, we first revisit the question of continuity of the Hausdorff dimension function when the set $\mathrm{CIFS}(X,I)$ is endowed with the pointwise topology and I is infinite. We show that the only points of continuity of the Hausdorff dimension function are those systems whose limit sets have the same Hausdorff dimension as the surrounding Euclidean space in which they live (see Theorem 4.2 and Lemmas 4.3 and 4.4). We further show that this set of continuity points is dense in $\mathrm{CIFS}(X,I)$, but so is its complement (see Lemmas 4.5–4.7). Then we investigate the topological properties of the different types of systems. We show that all these different types have empty interior, have for boundary the whole space, and thus are dense in $\mathrm{CIFS}(X,I)$ (see Proposition 4.8).

Finally, in section 5, The λ-Topology, we study the topological structure of $\mathrm{CIFS}(X,I)$ when it is equipped with the λ-topology. We partially describe the connected and arcwise connected components of this space in Propositions 5.3 and 5.4. This description shows in particular that $\mathrm{CIFS}(X,I)$ has no isolated points. We also prove that this space is normal (see Theorem 5.9). However, it is not metrizable, for it does not even satisfy the first axiom of countability (see Propositions 5.6 and 5.7). We further prove that $\mathrm{CIFS}(X,I)$ is not sequentially compact (see Proposition 5.10). Then, just as we did in the pointwise topology, we investigate the topological properties of the different types of systems. We show that, in contradistinction with the counter-intuitive properties that we observed in the pointwise topology, (almost) all the types exhibit natural properties as one would expect in an appropriate topology (see Propositions 5.11–5.17, inclusively). Despite that the finiteness parameter, the pressure and the Hausdorff dimension functions are continuous when $\mathrm{CIFS}(X)$ is endowed with the λ-topology, we show that the coding map and the closure of the limit set do not depend continuously on the underlying system Φ (see Proposition 5.18 and Corollary 5.19). Finally, we consider the continuity of measures. We prove that the conformal Gibbs measures and the invariant Gibbs measures are continuous functions of the system Φ and the parameter t of the potential (see Theorem 5.20).

2. Preliminaries on Iterated Function Systems

Let us first describe the setting of conformal iterated function systems introduced in [5]. Let I be a countable (finite or infinite) index set (often called alphabet) with at least two elements, and let $\Phi = \{\varphi_i : X \to X \mid i \in I\}$ be a collection of injective contractions of a compact metric space (X, d_X) (sometimes coined seed space) for which there exists a constant $0 < s < 1$ such that $d_X(\varphi_i(x), \varphi_i(y)) \le s\, d_X(x, y)$ for every $x, y \in X$ and for every $i \in I$. Any such collection Φ is called an iterated function system (abbr. IFS). We define the limit set J_Φ of this system as the image of the coding space I^∞ under a coding map π_Φ as follows. Let I^n denote the space of words of length n with letters in I, $I^* := \bigcup_{n \in I\!N} I^n$ be the space of finite words, and I^∞ the space of one-sided infinite words (sequences) of letters in I. For every $\omega \in I^* \cup I^\infty$, we write $|\omega|$ for the length of ω, that is, the unique $n \in I\!N \cup \{\infty\}$ such that $\omega \in I^n$. For $\omega \in I^n$, $n \in I\!N$, let $\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \in I\!N$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2 \dots \omega_n$. Since, given $\omega \in I^\infty$, the diameters of the compact sets $\varphi_{\omega|_n}(X)$, $n \in I\!N$, converge to zero and since these sets form a decreasing family, the set

$$\bigcap_{n=1}^{\infty} \varphi_{\omega|_n}(X)$$

is a singleton, and we denote its element by $\pi_\Phi(\omega)$. This defines the coding map $\pi_\Phi : I^\infty \to X$. Clearly, π_Φ is a continuous function when I^∞ is equipped with the topology generated by the cylinders $[i]_n = \{\omega \in I^\infty : \omega_n = i\}$, $i \in I$, $n \in I\!N$. The main object of our interest will be the limit set

$$J_\Phi = \pi_\Phi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \varphi_{\omega|n}(X).$$

Observe that J_Φ satisfies the natural invariance equality, $J_\Phi = \bigcup_{i \in I} \varphi_i(J_\Phi)$. Note that if I is finite, then J_Φ is compact, and this property usually fails when I is infinite.

An IFS $\Phi = \{\varphi_i : X \to X \,|\, i \in I\}$ is said to satisfy the Open Set Condition (OSC) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. (However, we do not exclude the possibility that $\overline{\varphi_i(U)} \cap \overline{\varphi_j(U)} \neq \emptyset$ for some $i, j \in I$.)

An IFS Φ is called conformal (and thereafter a CIFS) if X is connected, $X = \overline{\mathrm{Int}_{I\!R^d}(X)}$ for some $d \in I\!N$, and the following conditions are satisfied:

(i) Φ satisfies the OSC with $U = \mathrm{Int}_{I\!R^d}(X)$;

(ii) There exists an open connected set V, with $X \subset V \subset I\!R^d$, such that all maps φ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V;

(iii) There exist $\gamma, l > 0$ such that for every $x \in X$ there is an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l;

(iv) There are two constants $L \ge 1$ and $\alpha > 0$ such that

$$\Big| |\varphi_i'(y)| - |\varphi_i'(x)| \Big| \le L \|(\varphi_i')^{-1}\|_V^{-1} \|y - x\|^\alpha$$

for all $x, y \in V$ and all $i \in I$, where $\|\cdot\|_V$ is the supremum norm taken over V.

Remark 2.1. *It has been proved in Proposition 4.2.1 of* [7] *that if $d \ge 2$, then condition (iv) is automatically satisfied with $\alpha = 1$. This condition is also automatically satisfied if $d = 1$ and the set I is finite.*

The following useful fact has been also proved in Lemma 4.2.2 of [7].

Lemma 2.2. *For all $\omega \in I^*$ and all $x, y \in V$ we have that*

$$\Big|\log|\varphi_\omega'(y)| - \log|\varphi_\omega'(x)|\Big| \le L(1-s)^{-1}\|y-x\|^\alpha.$$

As an immediate consequence of this lemma we get the following.

(iv') Bounded Distortion Property (BDP): There exists a constant $K \ge 1$ such that

$$|\varphi_\omega'(y)| \le K|\varphi_\omega'(x)|$$

for every $x, y \in V$ and every $\omega \in I^*$, where $|\varphi_\omega'(x)|$ denotes the norm of the derivative.

As demonstrated in [5], infinite CIFSs, unlike finite ones, may not possess a conformal measure. There are even continued fraction systems which do not admit a conformal measure (see Example 6.5 in [6]). Thus, the infinite systems naturally break into two main types, irregular and regular systems. This dichotomy can be determined from the existence of a conformal measure or, equivalently, the existence of a zero of the topological pressure function. Recall that the topological pressure $\mathrm{P}_\Phi(t)$, $t \ge 0$, is defined as follows. For every $n \in I\!N$, set

$$\mathrm{P}_\Phi^{(n)}(t) = \sum_{\omega \in I^n} \|\varphi_\omega'\|^t,$$

where $\|\cdot\| := \|\cdot\|_X$ is the supremum norm over X. Then

$$\mathrm{P}_\Phi(t) = \lim_{n\to\infty} \frac{1}{n}\log \mathrm{P}_\Phi^{(n)}(t) = \inf_{n \in I\!N} \frac{1}{n}\log \mathrm{P}_\Phi^{(n)}(t).$$

Recall also that the shift map $\sigma : I^* \cup I^\infty \to I^* \cup I^\infty$ is defined for each $\omega \in I^* \cup I^\infty$ as

$$\sigma\Big(\{\omega_n\}_{n=1}^{|\omega|}\Big) = \{\omega_{n+1}\}_{n=1}^{|\omega|-1}.$$

If the function $\zeta_\Phi : I^\infty \to I\!R$ is given by the formula

$$\zeta_\Phi(\omega) = \log|\varphi_{\omega_1}'(\pi(\sigma(\omega)))|,$$

then $\mathrm{P}_\Phi(t) = \mathrm{P}(t\zeta_\Phi)$, where $\mathrm{P}(t\zeta_\Phi)$ is the classical topological pressure of the function $t\zeta_\Phi$ when I is finite (so the space I^∞ is compact), and is understood in the sense of [4] and [7] when I is infinite. The finiteness parameter θ_Φ of the system is defined by $\inf\{t \ge 0 : \mathrm{P}_\Phi^{(1)}(t) < \infty\}$. In [5], it was shown that the topological pressure function P_Φ is non-increasing on $[0,\infty)$, (strictly) decreasing, continuous and convex on $[\theta_\Phi, \infty)$, and $\mathrm{P}_\Phi(d) \le 0$. Of course, $\mathrm{P}_\Phi(0) = \infty$ if and only if I is infinite. The following characterization of the Hausdorff dimension h_Φ of the limit set J_Φ was proved in [5], Theorem 3.15. For every $F \subset I$, we write $\Phi|_F$ for the subsystem $\{\varphi_i\}_{i\in F}$ of Φ.

Theorem 2.3.

$$h_\Phi = \sup\{h_{\Phi|_F} : F \subset I \textit{ is finite }\} = \inf\{t \geq 0 : \mathrm{P}_\Phi(t) \leq 0\}.$$

If $\mathrm{P}_\Phi(t) = 0$, *then* $t = h_\Phi$.

The system Φ was called regular provided there is some $t \geq 0$ such that $\mathrm{P}_\Phi(t) = 0$. It follows from the strict decrease of P_Φ on $[\theta, \infty)$ that such a t is unique. Also, the system is regular if and only if it admits a t-conformal measure. A Borel probability measure m is said to be t-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\varphi_i(A)) = \int_A |\varphi_i'|^t \, dm,$$

and

$$m(\varphi_i(X) \cap \varphi_j(X)) = 0$$

for every pair $i, j \in I$, $i \neq j$.

There are natural subtypes of regular systems. Following [5] still, a system Φ is said to be strongly regular if $0 < \mathrm{P}_\Phi(t) < \infty$ for some $t \geq 0$. As an immediate application of Theorem 2.3 we get the following:

Theorem 2.4. *A system Φ is strongly regular if and only if $h_\Phi > \theta_\Phi$.*

Also, a system $\Phi = \{\varphi_i\}_{i \in I}$ was called hereditarily regular or cofinitely regular provided every nonempty cofinite subsystem $\Phi' = \{\varphi_i\}_{i \in I'}$ (i.e. I' is a cofinite subset of I) is regular. A finite system is clearly cofinitely regular, and it was shown in [5] that an infinite system is cofinitely regular exactly when the pressure is infinite at the finiteness parameter:

Theorem 2.5. *An infinite system Φ is cofinitely regular if and only if* $\mathrm{P}_\Phi(\theta_\Phi) = \infty \Leftrightarrow \mathrm{P}_\Phi^{(1)}(\theta_\Phi) = \infty \Leftrightarrow \{t \geq 0 : \mathrm{P}_\Phi(t) < \infty\} = (\theta_\Phi, \infty) \Leftrightarrow \{t \geq 0 : \mathrm{P}_\Phi^{(1)}(t) < \infty\} = (\theta_\Phi, \infty)$.

Remark that every cofinitely regular system is strongly regular, and every strongly regular system is regular. Finally, recall that critically regular systems are regular systems intuitively located at the threshold between strongly regular and irregular systems:

Definition 2.6. *A system Φ is named critically regular if $P_\Phi(\theta_\Phi) = 0$.*

We will see in section 5 that this intuition is fundamentally correct.

3. Preliminaries on Families of IFSs

When dealing with families of IFSs, we will denote the set of all conformal iterated function systems with phase space X and alphabet I by $\mathrm{CIFS}(X, I)$. Moreover, we will let $\mathrm{SIFS}(X, I)$ represent the subset of $\mathrm{CIFS}(X, I)$ comprising all similarity iterated function systems, that is, systems consisting of similarities only.

Recall that when I is finite, all systems in $\mathrm{CIFS}(X, I)$ are regular. When I is infinite, the classification is more involved. In this case, note that we may assume that $I = \mathbb{N}$ without loss of generality. Henceforth, we thus abbreviate $\mathrm{CIFS}(X, \mathbb{N})$ to $\mathrm{CIFS}(X)$. We will let $\mathrm{IR}(X) \subset \mathrm{CIFS}(X)$ be the subset of irregular systems, while $\mathrm{R}(X) \subset \mathrm{CIFS}(X)$ will represent the subset of regular systems. We will also denote by $\mathrm{CR}(X) \subset \mathrm{R}(X)$ the subset of critically regular systems, by $\mathrm{SR}(X) \subset \mathrm{R}(X)$ the subset of strongly regular systems, and by $\mathrm{CFR}(X) \subset \mathrm{SR}(X)$ the subset of cofinitely regular systems.

Lemma 3.1. *Let B be a closed ball in $\mathbb{R}^d$. If S is a contracting similarity of $\mathbb{R}^d$ such that $S(X) \subset B$ and $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(B)$ is such that $\varphi_i(S(X)) \subset S(X)$ for each i, then $S^{-1} \circ \Phi \circ S = \{S^{-1} \circ \varphi_i \circ S\} \in \mathrm{CIFS}(X)$ and has the same pressure function as Φ. In particular, $S^{-1} \circ \Phi \circ S$ belongs to the same type as Φ.*

Proof. Since Φ satisfies the OSC with $\mathrm{Int}(B)$, we have that $S^{-1} \circ \Phi \circ S$ satisfies the OSC with $\mathrm{Int}(X)$. Moreover, since there exists an open connected set V, with $B \subset V \subset \mathbb{R}^d$, such that all the maps φ_i, $i \in \mathbb{N}$, extend to C^1 conformal diffeomorphisms of V into V, we have that $S^{-1}(V)$ is an open connected set with $X \subset S^{-1}(V) \subset \mathbb{R}^d$ and such that all the maps $S^{-1} \circ \varphi_i \circ S$, $i \in \mathbb{N}$, extend to C^1 conformal diffeomorphisms of $S^{-1}(V)$ into $S^{-1}(V)$. Finally, since there are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big| |\varphi_i'(y)| - |\varphi_i'(\tilde{y})| \Big| \leq L \|(\varphi_i')^{-1}\|_V^{-1} |y - \tilde{y}|^\alpha \tag{3.1}$$

for all $i \in \mathbb{N}$ and all $y, \tilde{y} \in B$, the chain rule asserts that $|(S^{-1} \circ \varphi_i \circ S)'(x)| = |\varphi_i'(S(x))|$ for all $x \in S^{-1}(V)$ and thus we have that

$$\begin{aligned} \Big| |(S^{-1} \circ \varphi_i \circ S)'(x)| - |(S^{-1} \circ \varphi_i \circ S)'(\tilde{x})| \Big| &= \Big| |\varphi_i'(S(x))| - |\varphi_i'(S(\tilde{x}))| \Big| \\ &\leq L \|(\varphi_i')^{-1}\|_V^{-1} |S(x) - S(\tilde{x})|^\alpha \\ &\leq L \|((S^{-1} \circ \varphi_i \circ S)')^{-1}\|_{S^{-1}(V)}^{-1} s^\alpha |x - \tilde{x}|^\alpha \end{aligned}$$

for all $i \in \mathbb{N}$ and all $x, \tilde{x} \in S^{-1}(V)$, where s is a ratio for S. This proves that $S^{-1} \circ \Phi \circ S \in \mathrm{CIFS}(X)$.

Regarding the behavior of the pressure function, observe that since $K^{-1}\|\varphi_\omega'\| \leq |(S^{-1} \circ \varphi_\omega \circ S)'(x)| = |\varphi_\omega'(S(x))| \leq \|\varphi_\omega'\|$ for all $x \in X$ and $\omega \in \mathbb{N}^*$, where K is a bounded distortion constant for Φ, we have that $K^{-1}\|\varphi_\omega'\| \leq \|(S^{-1} \circ \varphi_\omega \circ S)'\| \leq \|\varphi_\omega'\|$ and thus for every $n \in \mathbb{N}$

and $t \geq 0$,

$$K^{-t} P_\Phi^{(n)}(t) = K^{-t} \sum_{\omega \in I\!N^n} \|\varphi'_\omega\|^t \leq P^{(n)}_{S^{-1}\circ\Phi\circ S}(t) = \sum_{\omega \in I\!N^n} \|(S^{-1}\circ\varphi_\omega\circ S)'\|^t \leq \sum_{\omega \in I\!N^n} \|\varphi'_\omega\|^t = P_\Phi^{(n)}(t).$$

Then

$$P_{S^{-1}\circ\Phi\circ S}(t) = \lim_{n\to\infty} \frac{1}{n} \log P^{(n)}_{S^{-1}\circ\Phi\circ S}(t) = \lim_{n\to\infty} \frac{1}{n} \log P_\Phi^{(n)}(t) = P_\Phi(t).$$

This shows that $S^{-1} \circ \Phi \circ S \in \mathrm{CIFS}(X)$ and $\Phi \in \mathrm{CIFS}(B)$ have the same pressure function and thus are of the same type. In particular, if $\Phi \in \mathrm{SIFS}(B)$, then $S^{-1} \circ \Phi \circ S \in \mathrm{SIFS}(X)$. ■

From this we make a simple but important observation. Given $0 \leq \theta \leq d$, $P \in I\!R \cup \{\infty\}$ and an open subset U of $\mathrm{Int}(X)$, let $\mathrm{CIFS}(X, U, \theta, P)$ denote the subspace of all systems $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$ with finiteness parameter $\theta_\Phi = \theta$, with $P_\Phi(\theta) = P$ and with $\varphi_i(X) \subset U$ for each i.

Lemma 3.2. *The following statements hold:*

(i) *For each $0 < \theta < d$, each $P \in I\!R \cup \{\infty\}$ and every open subset U of* $\mathrm{Int}(X)$*, we have* $\mathrm{SIFS}(X) \cap \mathrm{CIFS}(X, U, \theta, P) \neq \emptyset$*;*

(ii) *For every open subset U of* $\mathrm{Int}(X)$*, there exists $P(X, U) < 0$ such that for every $P < P(X, U)$ we have* $\mathrm{SIFS}(X) \cap \mathrm{CIFS}(X, U, d, P) \neq \emptyset$*;*

(iii) *There is no system $\Phi \in$* $\mathrm{CIFS}(X)$ *with finiteness parameter $\theta_\Phi = d$ and with $P_\Phi(d) > 0$.*

Proof. The third assertion is obvious. Indeed, if there were such a system Φ, then we would have that $h_\Phi > d$, which is clearly impossible since the system lives in $I\!R^d$ and thus $h_\Phi \leq d$.

Now, let us prove the first two assertions. We will first consider the case in which X is a closed ball B. To establish the first statement, let $0 < \theta < d$ and U an open subset of $\mathrm{Int}(B)$. Let us begin with the case $P = \infty$. One can always find a sequence of positive real numbers $\{r_i\}$ such that the series $\sum_i r_i^t$ is infinite when $t \leq \theta$ while finite when $t > \theta$. Multiplying this sequence by a constant R, we can make $\sum_i (Rr_i)^d$ as small as desired and in particular small enough that one can construct a system $\Phi = \{\varphi_i\} \in \mathrm{SIFS}(B)$ such that each φ_i is a similarity with ratio Rr_i which maps B into U. Then $\Phi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(B, U, \theta, \infty)$.

Now we tackle the case $0 < P < \infty$. Pick two disjoint balls U_1 and U_2 both contained in U. According to the previous paragraph with U replaced by U_1, there exists $\Phi = \{\varphi_i\} \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(B, U_1, \theta, \infty)$. One can also find a sequence of positive real numbers $\{s_i\}$ such that the series $\sum_i s_i^t$ is infinite when $t < \theta$ and finite when $t \geq \theta$. Multiplying this sequence by a constant S, we can make $\sum_i (Ss_i)^d$ small enough that one can build a system $\Psi = \{\psi_i\} \in \mathrm{SIFS}(B)$ such that each ψ_i is a similarity with ratio Ss_i which maps B into U_2. One can further choose S so that $\sum_i (Ss_i)^\theta < 1$. Despite these adjustments, note that $\theta_\Psi = \theta$. Thus, $\Psi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(X, U_2, \theta, p)$ for some $p < 0$. Thereafter define for each $n \in I\!N$ the system $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i \in I\!N}$ by

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_{i-n+1} & \text{if} \quad i \geq n. \end{cases}$$

It follows immediately from the fact that $\Phi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(B, U_1, \theta, \infty)$, $\Psi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(B, U_2, \theta, p)$ and $U_1 \cap U_2 = \emptyset$ that $\Xi^{(n)} \in \mathrm{SIFS}(B)$ for every $n \in I\!N$ and that $\xi_i^{(n)}(B) \subset U$ for every i and n. Moreover, since $\Xi^{(n)}$ and Ψ share all but finitely many generators, we have that $\theta_{\Xi^{(n)}} = \theta_\Psi = \theta$ for every n. Finally, note that

$$e^{P_{\Xi^{(n)}}(\theta)} = \sum_{i \in I\!N} \|(\xi_i^{(n)})'\|^\theta = \sum_{1 \le i < n} \|\varphi_i'\|^\theta + \sum_{j \in I\!N} \|\psi_j'\|^\theta < \infty.$$

Since $\sum_{i \in I\!N} \|\varphi_i'\|^\theta = \infty$, there exists a unique $n \in I\!N$ such that

$$\sum_{1 \le i < n} \|\varphi_i'\|^\theta + \sum_{j \in I\!N} \|\psi_j'\|^\theta \le e^P < \sum_{1 \le i < n+1} \|\varphi_i'\|^\theta + \sum_{j \in I\!N} \|\psi_j'\|^\theta,$$

or equivalently, such that

$$P_{\Xi^{(n)}}(\theta) \le P < P_{\Xi^{(n+1)}}(\theta).$$

If $P_{\Xi^{(n)}}(\theta) = P$, then $\chi := \Xi^{(n)} \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(X, U, \theta, P)$ and is thus the system we have been looking for. If, however, this is not the case, then we have instead the system $\chi = \{\chi_i\}_{i \in I\!N}$ defined by

$$\chi_i = \begin{cases} \xi_i^{(n+1)} & \text{if} \quad i \ne n \\ \tau & \text{if} \quad i = n, \end{cases}$$

where τ is a similarity with ratio $0 < T < \|\varphi_n'\|$ such that $\sum_{1 \le i < n} \|\varphi_i'\|^\theta + \sum_{j \in I\!N} \|\psi_j'\|^\theta + T^\theta = e^P$ and such that $\tau(B) \subset \varphi_n(B)$. Then $\chi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(X, U, \theta, P)$.

Finally, we treat the case $P \le 0$. Take $Q > 0$ and $\chi = \{\chi_i\} \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(X, U, \theta, Q)$. Then one can construct a system $\zeta = \{\zeta_i\} \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(X, U, \theta, P)$ by choosing for each i a similarity ζ_i with ratio $\|\chi_i'\| e^{(P-Q)/\theta}$ such that $\zeta_i(B) \subset \chi_i(B)$. This completes the proof of the first statement in the case $X = B$.

When $\theta = d$, one can always find a sequence of positive real numbers $\{r_i\}$ such that the series $\sum_i r_i^t$ is infinite when $t < d$ while finite when $t = d$. Multiplying this sequence by a constant R, we can make $\sum_i (R r_i)^d$ as small as desired and in particular small enough that one can construct a system $\Phi = \{\varphi_i\} \in \mathrm{SIFS}(B)$ such that each φ_i is a similarity with ratio $R r_i$ which maps B into U. Then $\Phi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(B, U, \theta, p)$ for some $p < 0$. By letting $R \searrow 0$, the second statement clearly holds in the case $X = B$.

Now we consider a general set X, and let U be an open subset of X. Let B be a closed ball in $I\!R^d$ and S a contracting similarity of $I\!R^d$ such that $S(X) \subset B$. Take an open ball $\tilde{B} \subset S(X)$. Conjugating a given system $\Phi \in \mathrm{SIFS}(B) \cap \mathrm{CIFS}(B, \tilde{B}, \theta, P)$ by S, it follows from Lemma 3.1 that $\Psi := S^{-1} \circ \Phi \circ S \in \mathrm{SIFS}(X)$ and that $P_\Psi(t) = P_\Phi(t)$ for every t. By post-composing Ψ with a contracting similarity T of $I\!R^d$ such that $T(X) \subset U$, we conclude that $T \circ \Psi \in \mathrm{SIFS}(X)$ with $P_{T \circ \Psi}(t) = P_\Psi(t) + t \log \|T'\| = P_\Phi(t) + t \log \|T'\|$ and $T \circ \psi_i(X) \subset U$ for each i. Thus, $T \circ \Psi \in \mathrm{SIFS}(X) \cap \mathrm{CIFS}(X, U, \theta, P + \theta \log \|T'\|)$. Each of the first two statements for a general X hence follows from the previous line and its homologue for a ball.
■

4. **The Pointwise Topology**

We now study the set $\mathrm{CIFS}(X)$ equipped with the metric of pointwise convergence, that is, when the distance between Φ and Ψ in $\mathrm{CIFS}(X)$ is given by

$$\rho_\infty(\Phi,\Psi) = \sum_{i=1}^{\infty} 2^{-i} \min\Big\{1, \|\varphi_i - \psi_i\| + \|\varphi_i' - \psi_i'\|\Big\}.$$

Remark 4.1. *It is important to recall that $\|\cdot\| := \|\cdot\|_X$ is the supremum norm over X. In particular, this implies that each term in a sequence $\{\Phi^{(n)}\}$ admits a neighbourhood $V_{\Phi^{(n)}}$ of X (cf. definition of CIFS) and the intersection of these neighbourhoods may not be a neighbourhood of X. A potential consequence of this is that each $\{\Phi^{(n)}\}$ has a minimal constant of bounded distortion $K_{\Phi^{(n)}}$ but these constants may not be bounded.*

Roy and Urbański observed that the Hausdorff dimension function is generally not continuous in the topology induced by ρ_∞ (see the example following Lemma 5.3 in [8]). This raises the question: What are the points of continuity of the Hausdorff dimension function in the pointwise topology?

Theorem 4.2. *$\Phi \in \mathrm{CIFS}(X)$ is a point of continuity of the Hausdorff dimension function $h : \mathrm{CIFS}(X) \to (0,\infty)$ if and only if $h_\Phi = d$. Moreover, the set $\{\Phi : h_\Phi = d\}$ of points of continuity of the Hausdorff dimension function is an uncountable dense G_δ-subset of* $\mathrm{CIFS}(X)$ *(although* $\mathrm{CIFS}(X)$ *is not a complete metric space). Nevertheless, the set $\{\Phi : h_\Phi < d\}$ of points of discontinuity of the Hausdorff dimension function is also an uncountable dense subset of* $\mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. The theorem is a straightforward consequence of the following five lemmas. ■

Lemma 4.3. *If $\Phi \in \mathrm{CIFS}(X)$ is such that $h_\Phi = d$, then Φ is a point of continuity of the Hausdorff dimension function $h : \mathrm{CIFS}(X) \to (0,\infty)$.*

Proof. Since the Hausdorff dimension of the limit set of a system cannot exceed the dimension of the space in which that set resides, we know that the Hausdorff dimension is always less than or equal to d. Thus, the Hausdorff dimension function is upper semi-continuous at every point where it equals d. Moreover, according to Theorem 5.2 in [8], the Hausdorff dimension function is lower semi-continuous in the topology induced by ρ_∞. Consequently, this function is continuous at every $\Phi \in \mathrm{CIFS}(X)$ such that $h_\Phi = d$. ■

The following lemma is the object of the converse result. It relies on the fact that, given any compact subset Y in $\mathbb{R}^d$ with non-empty interior, one can construct a system $\Psi \in \mathrm{SIFS}(Y)$ with any *prescribed* finiteness parameter $0 < \theta \le d$ according to Lemma 3.2.

Lemma 4.4. *Let $\Phi \in \mathrm{CIFS}(X)$ be a system whose limit set has Hausdorff dimension $h_\Phi < d$. Then Φ is a point of discontinuity of the finiteness parameter function, the Hausdorff dimension function and the pressure function in the pointwise topology.*

Proof. Let $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$ be such that $h_\Phi < d$. Fix $0 < \varepsilon < d - h_\Phi$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset \mathrm{Int}(\varphi_n(X))$. Take $\Psi^{(n)} = \{\psi_i^{(n)}\}_{i\in I\!N} \in \mathrm{CIFS}(S^{(n)}(X))$ such that $\theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$. Then $\Psi^{(n)} \circ S^{(n)} \in \mathrm{CIFS}(X)$ and $\theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$. Now, for each $n \in I\!N$ define the CIFS $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i\in I\!N}$ as

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_i^{(n)} \circ S^{(n)} & \text{if} \quad i \geq n. \end{cases}$$

Noting that $\Phi, \Psi^{(n)} \circ S^{(n)} \in \mathrm{CIFS}(X)$ and $\Psi^{(n)} \circ S^{(n)}(X) \subset \varphi_n(X)$, we deduce that $\Xi^{(n)} \in \mathrm{CIFS}(X)$ for every $n \in I\!N$. Moreover, since $\Xi^{(n)}$ and Φ have the same first $n-1$ generators, we have that $\rho_\infty(\Xi^{(n)}, \Phi) \leq \sum_{i=n}^{\infty} 1/2^i$ and it follows immediately that $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology.

We claim that the finiteness parameter function θ is not upper semi-continuous at Φ. Indeed, for every $n \in I\!N$, we have $\theta_{\Xi^{(n)}} = h_\Phi + \varepsilon$ since $\Xi^{(n)}$ and $\Psi^{(n)} \circ S^{(n)}$ share all but finitely many (the first $n-1$) generators, and thus have the same finiteness parameter. Moreover, as observed earlier, $\theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$. It follows immediately that the finiteness parameter function is not upper semi-continuous at Φ, for $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology, though $\theta_{\Xi^{(n)}} = h_\Phi + \varepsilon \geq \theta_\Phi + \varepsilon > \theta_\Phi$ for all $n \in I\!N$.

In the same vein, the Hausdorff dimension function h is not upper semi-continuous at Φ. Indeed, $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology, while $h_{\Xi^{(n)}} \geq \theta_{\Xi^{(n)}} = h_\Phi + \varepsilon$ for all $n \in I\!N$.

Finally, the pressure function $\chi \mapsto P_\chi(t)$ is not upper semi-continuous at Φ. Indeed, taking $\theta_\Phi < t < h_\Phi + \varepsilon$, we have $P_{\Xi^{(n)}}(t) = \infty$ for all $n \in I\!N$ since $\theta_{\Xi^{(n)}} = h_\Phi + \varepsilon$ for all $n \in I\!N$, whereas $P_\Phi(t) < \infty$. ■

Thus, when $\mathrm{CIFS}(X)$ is endowed with the pointwise topology, the points of continuity of the Hausdorff dimension function are those systems whose limit sets have the same Hausdorff dimension as the Euclidean space in which they live. We will now describe some topological properties of this set.

Lemma 4.5. $\overline{\{\Phi : \theta_\Phi = d\}} = \mathrm{CIFS}(X)$ *in the pointwise topology. In particular, the set* $\{\Phi : h_\Phi = d\}$ *of points of continuity of the Hausdorff dimension function is dense in* $\mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. Let $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset \mathrm{Int}(\varphi_n(X))$. Take $\Psi^{(n)} = \{\psi_i^{(n)}\}_{i\in I\!N} \in \mathrm{CIFS}(S^{(n)}(X))$ such that $\theta_{\Psi^{(n)}} = d$. Then $\Psi^{(n)} \circ S^{(n)} \in \mathrm{CIFS}(X)$ and $\theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}} = d$. Now, for each $n \in I\!N$ define the CIFS $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i\in I\!N}$ as

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_i^{(n)} \circ S^{(n)} & \text{if} \quad i \geq n. \end{cases}$$

Noting that $\Phi, \Psi^{(n)} \circ S^{(n)} \in \mathrm{CIFS}(X)$ and $\Psi^{(n)} \circ S^{(n)}(X) \subset \varphi_n(X)$, we deduce that $\Xi^{(n)} \in \mathrm{CIFS}(X)$ for every $n \in I\!N$. Moreover, since $\Xi^{(n)}$ and Φ have the same first $n-1$ generators, we have that $\rho_\infty(\Xi^{(n)}, \Phi) \leq \sum_{i=n}^{\infty} 1/2^i$ and it follows immediately that $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology. Furthermore, note that for every $n \in I\!N$, we have $\theta_{\Xi^{(n)}} = d$ since $\Xi^{(n)}$ and $\Psi^{(n)} \circ S^{(n)}$ share all but finitely many (the first $n-1$) generators, and thus have the same finiteness parameter. ■

And it follows from the lower semicontinuity of the Hausdorff dimension function and from the above lemma that

Lemma 4.6. *Let $0 < \theta < d$. Then $\{\Phi : h_\Phi \leq \theta\}$ is closed and nowhere dense in* $\mathrm{CIFS}(X)$ *in the pointwise topology. In particular, the set $\{\Phi : h_\Phi = d\}$ of points of continuity of the Hausdorff dimension function is a countable intersection of open dense subsets of* $\mathrm{CIFS}(X)$.

Thus, the set of points of continuity of the Hausdorff dimension function is fairly large. But its complement, the set of points of discontinuity, is also large.

Lemma 4.7. $\overline{\mathrm{CIFS}(X) \backslash \{\Phi : h_\Phi = d\}} = \mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. Let $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset \mathrm{Int}(\varphi_n(X))$. Take $\Psi^{(n)} = \{\psi_i^{(n)}\}_{i \in I\!N} \in \mathrm{CFR}(S^{(n)}(X))$. Then $\Psi^{(n)} \circ S^{(n)} \in \mathrm{CFR}(X)$ and $\theta_{\Psi^{(n)} \circ S^{(n)}} = \theta_{\Psi^{(n)}}$. Now, for each $n \in I\!N$ define the CIFS $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i \in I\!N}$ by

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_i^{(n)} \circ S^{(n)} & \text{if} \quad i \geq n. \end{cases}$$

Then $\Xi^{(n)} \in \mathrm{CFR}(X)$ for every $n \in I\!N$ and $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology. Furthermore, since $\lambda_d(\mathrm{Int}(X) \backslash \cup_{i=1}^{\infty} \xi_i^{(n)}(X)) \geq \lambda_d(\mathrm{Int}(\varphi_{n+1}(X))) > 0$, it follows from Theorem 4.5.10 on page 101 in [7] that $h_{\Xi^{(n)}} < d$. ■

In particular, this shows that the finiteness parameter function, the Hausdorff dimension function and the pressure function are discontinuous on a dense subset of $\mathrm{CIFS}(X)$ when this latter is equipped with the pointwise topology. Recall that, according to [8], these three functions are continuous everywhere on $\mathrm{CIFS}(X)$ when this latter is instead endowed with the λ-topology.

We will now describe the interior and the boundary of each type of systems.

Proposition 4.8. *Let C be any type of systems, that is* $\mathrm{IR}(X)$, $\mathrm{CR}(X)$, $\mathrm{SR}(X) \backslash \mathrm{CFR}(X)$, $\mathrm{CFR}(X)$, *or any union of some but not all of these types. Then* $\mathrm{Int}(C) = \emptyset$ *and* $\partial C = \mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. The proposition can be deduced from the forthcoming four lemmas. ■

We first show that any CIFS is the limit of a sequence of irregular systems in the pointwise topology. That is, $\mathrm{IR}(X)$ is dense in $\mathrm{CIFS}(X)$. This is in stark contrast with the description of this set in the λ-topology, in which, more naturally, $\overline{\mathrm{IR}(X)} = \mathrm{IR}(X) \cup \mathrm{CR}(X)$ as an immediate consequence of Proposition 5.11.

Lemma 4.9. $\overline{\mathrm{IR}(X)} = \mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. By Lemma 4.7, it is sufficient to prove that $\overline{\mathrm{IR}(X)} \supset \mathrm{CIFS}(X) \backslash \{\Phi : h_\Phi = d\}$. To do this, let $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$ be such that $h_\Phi < d$. Fix $0 < \varepsilon < d - h_\Phi$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset \mathrm{Int}(\varphi_n(X))$. Take $\Psi^{(n)} = \{\psi_i^{(n)}\}_{i \in I\!N} \in \mathrm{CIFS}(S^{(n)}(X)) \backslash \mathrm{CFR}(S^{(n)}(X))$ such that $\theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$ and such that the sequence $\{\|(\psi_i^{(n)})'\|\}_{n \in I\!N}$ is decreasing for every $i \in I\!N$. Then $\Psi^{(n)} \circ S^{(n)} \in \mathrm{CIFS}(X) \backslash \mathrm{CFR}(X)$ and $\theta_{\Psi^{(n)} \circ S^{(n)}} = \theta_{\Psi^{(n)}}$. Now, for each $n \in I\!N$ define the CIFS $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i \in I\!N}$ by

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_i^{(n)} \circ S^{(n)} & \text{if} \quad i \geq n. \end{cases}$$

Then $\Xi^{(n)} \in \mathrm{CIFS}(X)$ for every $n \in I\!N$, $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology, and $\theta_{\Xi^{(n)}} = \theta_{\Psi^{(n)} \circ S^{(n)}} = \theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$ for every $n \in I\!N$. We claim that all but finitely many of the $\Xi^{(n)}$'s are irregular. Indeed, since $P_\Phi(h_\Phi + \varepsilon) < 0$ there is $L \in I\!N$ such that $P_\Phi^{(L)}(h_\Phi + \varepsilon) < 1$. Now, for every $n \in I\!N$ we have

$$\begin{aligned}
P_{\Xi^{(n)}}^{(L)}(h_\Phi + \varepsilon) &= \sum_{\omega \in I\!N^L} \|(\xi_\omega^{(n)})'\|^{h_\Phi + \varepsilon} \\
&= \sum_{\omega \in \{1, \dots, n-1\}^L} \|\varphi_\omega'\|^{h_\Phi + \varepsilon} + \sum_{k=0}^{L-1} \sum_{\substack{\omega \in I\!N^L, \\ \omega_j < n \text{ for exactly } k \ \omega_j}} \|(\xi_\omega^{(n)})'\|^{h_\Phi + \varepsilon} \\
&\leq \sum_{\omega \in I\!N^L} \|\varphi_\omega'\|^{h_\Phi + \varepsilon} + \sum_{k=0}^{L-1} \sum_{\substack{\omega \in I\!N^L, \\ \omega_j < n \text{ for exactly } k \ \omega_j}} \prod_{i=1}^{L} \|(\xi_{\omega_i}^{(n)})'\|^{h_\Phi + \varepsilon} \\
&\leq P_\Phi^{(L)}(h_\Phi + \varepsilon) + \sum_{k=0}^{L-1} L^k \left(\sum_{i=1}^{n-1} \|\varphi_i'\|^{h_\Phi + \varepsilon} \right)^k \left(\sum_{i=n}^{\infty} \|(\psi_i^{(n)} \circ S^{(n)})'\|^{h_\Phi + \varepsilon} \right)^{L-k} \\
&\leq P_\Phi^{(L)}(h_\Phi + \varepsilon) + \sum_{k=0}^{L-1} L^k \left(\sum_{i=1}^{\infty} \|\varphi_i'\|^{h_\Phi + \varepsilon} \right)^k \left(\sum_{i=n}^{\infty} \|(\psi_i^{(n)})'\|^{h_\Phi + \varepsilon} \right)^{L-k} \\
&\leq P_\Phi^{(L)}(h_\Phi + \varepsilon) + \sum_{k=0}^{L-1} L^k \left(P_\Phi^{(1)}(h_\Phi + \varepsilon) \right)^k \left(\sum_{i=n}^{\infty} \|(\psi_i^{(1)})'\|^{h_\Phi + \varepsilon} \right)^{L-k},
\end{aligned}$$

where the last inequality follows from the fact that the sequence $\{\|(\psi_i^{(n)})'\|\}_{n \in I\!N}$ is decreasing for every $i \in I\!N$. The right-hand side tends to $P_\Phi^{(L)}(h_\Phi + \varepsilon) < 1$ as $n \to \infty$ since $P_\Phi^{(1)}(h_\Phi + \varepsilon) < \infty$ (because $P_\Phi(h_\Phi + \varepsilon) < \infty$) and $\sum_{i=n}^{\infty} \|(\psi_i^{(1)})'\|^{h_\Phi + \varepsilon} \searrow 0$ as $n \to \infty$ (because $P_{\Psi^{(1)}}(h_\Phi + \varepsilon) =$

$P_{\Psi^{(1)}}(\theta_{\Psi^{(1)}}) < \infty$). It follows that $P^{(L)}_{\Xi^{(n)}}(\theta_{\Xi^{(n)}}) = P^{(L)}_{\Xi^{(n)}}(h_\Phi + \varepsilon) < 1$ for all n large enough, which implies that $P_{\Xi^{(n)}}(\theta_{\Xi^{(n)}}) < 0$ for all n sufficiently large. Therefore $\Xi^{(n)} \in \mathrm{IR}(X)$ for all $n \in I\!N$ large enough and $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology. ■

We will now show that any CIFS is the limit of a sequence of critically regular systems in the pointwise topology. That is, $\mathrm{CR}(X)$ is dense in $\mathrm{CIFS}(X)$. This is in sharp contrast with the description of this set in the λ-topology, in which $\overline{\mathrm{CR}(X)} = \mathrm{CR}(X)$ according to Lemma 5.9(iii) in [8]. Our proof further shows that the finiteness parameter is generally neither lower nor upper semi-continuous in the pointwise topology. Recall that in the λ-topology, the finiteness parameter function is locally constant.

Lemma 4.10. $\overline{\mathrm{CR}(X)} = \mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. By Lemma 4.7, it is sufficient to prove that $\overline{\mathrm{CR}(X)} \supset \mathrm{CIFS}(X)\backslash\{\Phi : h_\Phi = d\}$. To do this, let $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$ be such that $h_\Phi < d$. Fix $0 < \varepsilon < d - h_\Phi$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset B_n \subset \varphi_n(X)$, where B_n is an open ball. Take a SIFS $\Psi^{(n)} = \{\psi_i^{(n)}\}_{i\in I\!N} \in \mathrm{SR}(S^{(n)}(X))\backslash\mathrm{CFR}(S^{(n)}(X))$ such that $\theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$ and $\Psi^{(n)} \circ S^{(n)} \in \mathrm{SR}(X)$. Then $\Psi^{(n)} \circ S^{(n)} \in \mathrm{SR}(X)\backslash\mathrm{CFR}(X)$ and $\theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}}$. Now, for each $n \in I\!N$ and $0 < s \le 1$, denote by $m^{(s)}_{x_0}$ the similarity $x \mapsto s(x - x_0) + x_0$ and define the CIFS $\Xi^{(n,s)} = \{\xi_i^{(n,s)}\}_{i\in I\!N}$ by

$$\xi_i^{(n,s)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ m^{(s)}_{x_n} \circ \psi^{(n)}_{i-n+1} \circ S^{(n)} & \text{if} \quad i \ge n, \end{cases}$$

with x_n is the center of the ball B_n. Then $\Xi^{(n,s)} \in \mathrm{CIFS}(X)$, $\Xi^{(n,s)} \to \Phi$ uniformly in s as $n \to \infty$ in the pointwise topology, and $\theta_{\Xi^{(n,s)}} = \theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}} = h_\Phi + \varepsilon$ for every n and s. Observe also that $\Xi^{(n,1)} \in \mathrm{SR}(X)\backslash\mathrm{CFR}(X)$ for each $n \in I\!N$. However, for each $n \in I\!N$ there is $s_n > 0$ such that $\Xi^{(n,s_n)}$ is critically regular (and $\Xi^{(n,s)}$ irregular for every $0 < s < s_n$). Indeed, since $P_\Phi(h_\Phi + \varepsilon) < 0$ there is $L \in I\!N$ such that $P^{(L)}_\Phi(h_\Phi + \varepsilon) < 1$. Now, for every

$n \in I\!N$ and $0 < s \leq 1$ we have

$$\begin{aligned}
P^{(L)}_{\Xi^{(n,s)}}(h_\Phi + \varepsilon) &= \sum_{\omega \in I\!N^L} \|(\xi^{(n,s)}_\omega)'\|^{h_\Phi+\varepsilon} \\
&= \sum_{\omega \in \{1,\dots,n-1\}^L} \|\varphi'_\omega\|^{h_\Phi+\varepsilon} + \sum_{k=0}^{L-1} \sum_{\substack{\omega \in I\!N^L, \\ \omega_j < n \text{ for exactly } k\ \omega_j}} \|(\xi^{(n,s)}_\omega)'\|^{h_\Phi+\varepsilon} \\
&\leq \sum_{\omega \in I\!N^L} \|\varphi'_\omega\|^{h_\Phi+\varepsilon} + \sum_{k=0}^{L-1} \sum_{\substack{\omega \in I\!N^L, \\ \omega_j < n \text{ for exactly } k\ \omega_j}} \prod_{i=1}^{L} \|(\xi^{(n,s)}_{\omega_i})'\|^{h_\Phi+\varepsilon} \\
&\leq P^{(L)}_\Phi(h_\Phi+\varepsilon) + \sum_{k=0}^{L-1} L^k \left(\sum_{i=1}^{n-1} \|\varphi'_i\|^{h_\Phi+\varepsilon}\right)^k \left(\sum_{i=1}^{\infty} \|(\psi^{(n,s)}_i \circ S^{(n)})'\|^{h_\Phi+\varepsilon}\right)^{L-k} \\
&\leq P^{(L)}_\Phi(h_\Phi+\varepsilon) + \sum_{k=0}^{L-1} L^k \left(\sum_{i=1}^{\infty} \|\varphi'_i\|^{h_\Phi+\varepsilon}\right)^k \left(\sum_{i=1}^{\infty} \|(\psi^{(n,s)}_i)'\|^{h_\Phi+\varepsilon}\right)^{L-k} \\
&= P^{(L)}_\Phi(h_\Phi+\varepsilon) + \sum_{k=0}^{L-1} L^k \left(P^{(1)}_\Phi(h_\Phi+\varepsilon)\right)^k \left(s^{h_\Phi+\varepsilon}\sum_{i=1}^{\infty} \|(\psi^{(n)}_i)'\|^{h_\Phi+\varepsilon}\right)^{L-k} \\
&\leq P^{(L)}_\Phi(h_\Phi+\varepsilon) + s^{h_\Phi+\varepsilon}\sum_{k=0}^{L-1} L^k \left(P^{(1)}_\Phi(h_\Phi+\varepsilon)\right)^k \left(P^{(1)}_{\Psi^{(n)}}(h_\Phi+\varepsilon)\right)^{L-k}.
\end{aligned}$$

For any fixed $n \in I\!N$, the right-hand side tends to $P^{(L)}_\Phi(h_\Phi+\varepsilon) < 1$ as $s \searrow 0$ since $P^{(1)}_\Phi(h_\Phi + \varepsilon) < \infty$ and $P^{(1)}_{\Psi^{(n)}}(h_\Phi+\varepsilon) = P^{(1)}_{\Psi^{(n)}}(\theta_{\Psi^{(n)}}) < \infty$. It follows that $P^{(L)}_{\Xi^{(n,s)}}(\theta_{\Xi^{(n,s)}}) = P^{(L)}_{\Xi^{(n,s)}}(h_\Phi+\varepsilon) < 1$ for all s small enough, which implies that $P_{\Xi^{(n,s)}}(\theta_{\Xi^{(n,s)}}) < 0$ for all s small enough. Thus, $\Xi^{(n,s)}$ is irregular for all s sufficiently small, whereas $\Xi^{(n,1)}$ is strongly regular (but not cofinitely regular). It is also clear that the map $s \mapsto \Xi^{(n,s)}$ is continuous on $(0,1]$ when $\mathrm{CIFS}(X)$ is endowed with the λ-topology. Since the pressure function $\star \mapsto P_\star(h_\Phi+\varepsilon)$ is continuous in that topology, there is $s_n > 0$ such that $P_{\Xi^{(n,s_n)}}(\theta_{\Xi^{(n,s_n)}}) = 0$. Then $\Xi^{(n,s_n)} \in \mathrm{CR}(X)$ for all $n \in I\!N$ and $\Xi^{(n,s_n)} \to \Phi$ as $n \to \infty$ in the pointwise topology. ■

We will now show that any CIFS is the limit of a sequence of strongly, but not cofinitely, regular systems in the pointwise topology. That is, $\mathrm{SR}(X)\backslash\mathrm{CFR}(X)$ is dense in $\mathrm{CIFS}(X)$. This is in stark contrast with the description of this set in the λ-topology, in which $\overline{\mathrm{SR}(X)\backslash\mathrm{CFR}(X)} \subset \mathrm{R}(X)\backslash\mathrm{CFR}(X)$. This is an immediate consequence of Proposition 5.13.

Lemma 4.11. $\overline{\mathrm{SR}(X)\backslash\mathrm{CFR}(X)} = \mathrm{CIFS}(X)$ *in the pointwise topology.*

Proof. Let $\Phi = \{\varphi_i\} \in \mathrm{CIFS}(X)$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset \mathrm{Int}(\varphi_n(X))$. Take $\Psi^{(n)} = \{\psi^{(n)}_i\}_{i\in I\!N} \in \mathrm{SR}(S^{(n)}(X))\backslash\mathrm{CFR}(S^{(n)}(X))$ such that $\Psi^{(n)} \circ S^{(n)} \in \mathrm{SR}(X)$. Then $\Psi^{(n)} \circ S^{(n)} \in \mathrm{SR}(X)\backslash\mathrm{CFR}(X)$ and $\theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}}$. Now, for each $n \in I\!N$ define the CIFS $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i\in I\!N}$ by

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_{i-n+1}^{(n)} \circ S^{(n)} & \text{if} \quad i \geq n. \end{cases}$$

Then $\Xi^{(n)} \in \text{CIFS}(X)$ for every $n \in I\!N$, $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology, $\theta_{\Xi^{(n)}} = \theta_{\Psi^{(n)}\circ S^{(n)}} = \theta_{\Psi^{(n)}}$ for every $n \in I\!N$ and $P_{\Xi^{(n)}}(t) \geq P_{\Psi^{(n)}\circ S^{(n)}}(t)$ for every $t \geq 0$. In particular,

$$P_{\Xi^{(n)}}(\theta_{\Xi^{(n)}}) \geq P_{\Psi^{(n)}\circ S^{(n)}}(\theta_{\Psi^{(n)}\circ S^{(n)}}) > 0.$$

Note also that $P_{\Xi^{(n)}}(\theta_{\Xi^{(n)}}) < \infty$ since

$$P_{\Xi^{(n)}}(\theta_{\Xi^{(n)}}) \leq \sum_{i\in I\!N} \|(\xi_i^{(n)})'\|^{\theta_{\Xi^{(n)}}} \leq \sum_{i<n} \|\varphi_i'\|^{\theta_{\Xi^{(n)}}} + \sum_{j\in I\!N} \|(\psi_j^{(n)} \circ S^{(n)})'\|^{\theta_{\Psi^{(n)}\circ S^{(n)}}} < \infty.$$

Therefore $\Xi^{(n)} \in \text{SR}(X)\backslash\text{CFR}(X)$ for all $n \in I\!N$ and $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology. ■

We finally show that any CIFS is the limit of a sequence of cofinitely regular systems in the pointwise topology. That is, CFR(X) is dense in CIFS(X). This is in stark contrast with the description of this set in the λ-topology, in which $\overline{\text{CFR}(X)} = \text{CFR}(X)$, for CFR$(X)$ is a clopen set in the λ-topology according to Lemma 5.9 in [8].

Lemma 4.12. $\overline{\text{CFR}(X)} = \text{CIFS}(X)$ *in the pointwise topology.*

Proof. Let $\Phi = \{\varphi_i\} \in \text{CIFS}(X)$. For each $n \in I\!N$, pick a contracting similarity $S^{(n)}$ of $I\!R^d$ such that $S^{(n)}(X) \subset \text{Int}(\varphi_n(X))$. Take $\Psi^{(n)} = \{\psi_i^{(n)}\}_{i\in I\!N} \in \text{CFR}(S^{(n)}(X))$. Then $\Psi^{(n)} \circ S^{(n)} \in \text{CFR}(X)$ Now, for each $n \in I\!N$ define the CIFS $\Xi^{(n)} = \{\xi_i^{(n)}\}_{i\in I\!N}$ by

$$\xi_i^{(n)} = \begin{cases} \varphi_i & \text{if} \quad i < n \\ \psi_{i-n+1}^{(n)} \circ S^{(n)} & \text{if} \quad i \geq n. \end{cases}$$

Then $\Xi^{(n)} \in \text{CIFS}(X)$ for every $n \in I\!N$, $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology, $\theta_{\Xi^{(n)}} = \theta_{\Psi^{(n)}\circ S^{(n)}}$ for every $n \in I\!N$ and $P_{\Xi^{(n)}}(t) \geq P_{\Psi^{(n)}\circ S^{(n)}}(t)$ for every $t \geq 0$. In particular,

$$P_{\Xi^{(n)}}(\theta_{\Xi^{(n)}}) \geq P_{\Psi^{(n)}\circ S^{(n)}}(\theta_{\Psi^{(n)}\circ S^{(n)}}) = \infty.$$

Therefore $\Xi^{(n)} \in \text{CFR}(X)$ for all $n \in I\!N$ and $\Xi^{(n)} \to \Phi$ as $n \to \infty$ in the pointwise topology. ■

5. **The λ-Topology**

From this point on, we assume that the set CIFS(X) is endowed with the λ-topology. Recall from [8] that a sequence $\{\Phi^{(n)}\}$ converges to Φ in the λ-topology provided that $\{\Phi^{(n)}\}$ converges to Φ in the pointwise topology and that there exist constants $C > 0$ and $N \in I\!N$ such that

$$\Big|\log\|\varphi_i'\| - \log\|(\varphi_i^{(n)})'\|\Big| \leq C \tag{5.1}$$

for all $i \in I\!N$ and all $n \geq N$. A set $F \subset \mathrm{CIFS}(X)$ is declared to be closed if the λ-limit of every λ-converging sequence of points from F belongs to F. The λ-topology defines in this way CIFS(X) as a sequential space. For some purposes, it is more convenient to express this convergence in slightly different terms. For every $\Phi, \Psi \in \mathrm{CIFS}(X)$, define

$$d(\Phi,\Psi) := \sup_{i\in I\!N} \max\left\{ \frac{\|\varphi_i'\|}{\|\psi_i'\|}, \frac{\|\psi_i'\|}{\|\varphi_i'\|} \right\}.$$

Recall that $\|\cdot\| := \|\cdot\|_X$ is the supremum norm over X (cf. Remark 4.1). For every $R \geq 0$, define also

$$D(\Phi,R) = \{\Psi : d(\Phi,\Psi) < R\} \qquad \text{and} \qquad \overline{D}(\Phi,R) = \{\Psi : d(\Phi,\Psi) \leq R\}.$$

In view of these definitions, we may express the convergence of a sequence $\{\Phi^{(n)}\}$ to Φ in the λ-topology by saying that $\{\Phi^{(n)}\}$ converges to Φ in the pointwise topology and that there exist constants $C \geq 1$ and $N \in I\!N$ such that $\Phi^{(n)} \in \overline{D}(\Phi, C)$ for every $n \geq N$.

Now, we define the relation

$$\Phi \sim \Psi \quad \text{if} \quad d(\Phi,\Psi) < \infty.$$

It is easy to see that $\sim$ is an equivalence relation on $\mathrm{CIFS}(X)$. Note that the transitivity of $\sim$ simply follows from the inequality $d(\Phi,\chi) \leq d(\Phi,\Psi)\cdot d(\Psi,\chi)$. Henceforth we denote by $[\Phi]$ the equivalence class of Φ, that is,

$$[\Phi] = \{\Psi : d(\Phi,\Psi) < \infty\}.$$

Finally, observe that the finiteness parameter function is constant on every equivalence class, that is, if $\Phi \sim \Psi$ then $\theta_\Phi = \theta_\Psi$.

The following few results describe the topological properties of the sets $D(\Phi,R)$, $\overline{D}(\Phi,R)$ and $[\Phi]$ for any Φ.

Lemma 5.1. *For every $\Phi \in \mathrm{CIFS}(X)$ and $R \geq 1$, we have that $\mathrm{Int}(\overline{D}(\Phi,R)) = \emptyset$.*

Proof. Let $\Psi \in \mathrm{CIFS}(X)$ and $R \geq 1$. We first prove that $\Psi \notin \mathrm{Int}(\overline{D}(\Psi,R))$. Take a similarity $S : X \to X$ such that $\|S'\| < (KR)^{-1}$, where $K \geq 1$ is a constant of bounded distortion for Ψ. Consider the sequence $\{\Psi^{(n)}\}$ whose generators are

$$\psi_i^{(n)} = \begin{cases} \psi_i & \text{if} \quad i \neq n \\ \psi_i \circ S & \text{if} \quad i = n. \end{cases}$$

For all $n \in I\!N$ we have that $\Psi^{(n)} \in \mathrm{CIFS}(X)$, and clearly $\{\Psi^{(n)}\}$ converges to Ψ in the pointwise topology. Moreover, for every $x \in X$,

$$\frac{|(\psi_n^{(n)})'(x)|}{|\psi_n'(x)|} = \frac{|\psi_n'(S(x))|\cdot\|S'\|}{|\psi_n'(x)|} \in \left[K^{-1}\|S'\|, K\|S'\|\right].$$

Hence $d(\Psi^{(n)},\Psi) \leq K\|S'\|^{-1}$. Thus, $\{\Psi^{(n)}\}$ converges to Ψ in the λ-topology. Furthermore, $d(\Psi^{(n)},\Psi) \geq (K\|S'\|)^{-1} > R$, that is, $\Psi^{(n)} \notin \overline{D}(\Psi,R)$ for every $n \in I\!N$. Hence $\Psi \notin \mathrm{Int}(\overline{D}(\Psi,R))$.

Now, suppose that $\mathrm{Int}(\overline{D}(\Phi, R)) \neq \emptyset$ for some $\Phi \in \mathrm{CIFS}(X)$ and some $R \geq 1$. For any $\Psi \in \mathrm{Int}(\overline{D}(\Phi, R))$, we have $\Psi \in \overline{D}(\Phi, R) \subset \overline{D}(\Psi, R\, d(\Psi, \Phi)))$. But, according to the first part, $\Psi \notin \mathrm{Int}(\overline{D}(\Psi, R\, d(\Psi, \Phi))) \supset \mathrm{Int}(\overline{D}(\Phi, R))$. This is a contradiction, and we conclude that $\mathrm{Int}(\overline{D}(\Phi, R)) = \emptyset$. ■

It follows immediately that no set $D(\Phi, R)$, $R \geq 1$, is open. However, all sets $\overline{D}(\Phi, R)$, $R \geq 1$, are closed in $[\Phi]$ and in $\mathrm{CIFS}(X)$.

Lemma 5.2. *For every $\Phi \in \mathrm{CIFS}(X)$ and $R \geq 1$, the set $\overline{D}(\Phi, R)$ is a closed subset of $[\Phi]$.*

Proof. Let $\{\Psi^{(n)}\}$ be a sequence in $\overline{D}(\Phi, R)$ which λ-converges to $\Psi \in \mathrm{CIFS}(X)$. Then, given $i \in I\!N$, we have that

$$\lim_{n\to\infty} \frac{\|\psi_i'\|}{\|(\psi_i^{(n)})'\|} = 1$$

since the sequence converges in the pointwise topology. Observing that

$$\frac{\|\psi_i'\|}{\|\varphi_i'\|} = \frac{\|\psi_i'\|}{\|(\psi_i^{(n)})'\|} \cdot \frac{\|(\psi_i^{(n)})'\|}{\|\varphi_i'\|},$$

we deduce that

$$\frac{\|\psi_i'\|}{\|\varphi_i'\|} \in \left[R^{-1}, R\right].$$

Since this is true for every $i \in I\!N$, we conclude that $\Psi \in \overline{D}(\Phi, R)$. ■

It has already been observed that the space $\mathrm{CIFS}(X)$ is disconnected, for $\mathrm{CFR}(X) \neq \emptyset$ is clopen in the λ-topology according to Lemma 5.9(i) in [8]. It follows immediately from the proposition below that every connected component is contained in some equivalence class.

Proposition 5.3. *For every $\Phi \in \mathrm{CIFS}(X)$, the equivalence class $[\Phi]$ is clopen.*

Proof. Let $\{\Psi^{(n)}\}$ be a sequence in $[\Phi]$ which λ-converges to $\Psi \in \mathrm{CIFS}(X)$. The λ-convergence guarantees that there exist $C \geq 1$ and $N \in I\!N$ such that $\Psi^{(n)} \in \overline{D}(\Phi, C)$ for every $n \geq N$. This implies that $\Psi \sim \Psi^{(n)}$ for every $n \geq N$. Moreover, since $\Psi^{(n)} \in [\Phi]$ for every $n \in I\!N$, we have that $\Psi^{(n)} \sim \Phi$ for every $n \in I\!N$. Consequently, $\Psi \sim \Psi^{(n)} \sim \Phi$ for every $n \geq N$, that is, $\Psi \in [\Phi]$. This shows that $[\Phi]$ is closed.

Now, let $\{\Psi^{(n)}\}$ be a sequence in $\mathrm{CIFS}(X)\backslash[\Phi]$ which λ-converges to $\Psi \in \mathrm{CIFS}(X)$. This implies in particular that $\Psi^{(n)} \sim \Psi$ for all n large enough. If $\Psi \sim \Phi$, then $\Psi^{(n)} \sim \Psi \sim \Phi$ for all n large enough. In other words, if $\Psi \in [\Phi]$, then $\Psi^{(n)} \in [\Phi]$ for all n large enough. This is a contradiction. Thus, $\Psi \notin [\Phi]$. This shows that $\mathrm{CIFS}(X)\backslash[\Phi]$ is closed and therefore $[\Phi]$ is open. ■

Hence, if Φ and Ψ belong to different equivalence classes, that is, if $d(\Phi, \Psi) = \infty$, then Φ and Ψ belong to different connected components. Moreover, since the finiteness parameter function is constant on each equivalence class and for each $0 < \theta \leq d$ there exists a $\Phi \in$

CIFS(X) with $\theta_\Phi = \theta$, the space CIFS(X) has uncountably many connected components. We can say even more about these components.

Proposition 5.4. *If* Int(X) *is star shaped, then for each* $\Phi \in$ CIFS(X) *the arcwise connected component of* Φ *in* $[\Phi]$ *(or in* CIFS(X)*) is non-degenerate.*

Proof. Let $\Phi \in$ CIFS(X). Choose $z \in \mathrm{Int}(X)$ such that Int(X) is star shaped with respect to z, that is, such that $\{(1-t)x + tz : t \in [0,1], x \in \mathrm{Int}(X)\} = \mathrm{Int}(X)$. For each $x \in X$ and $t \in [0,1)$, define $r_t(x) = (1-t)x + tz$. Observe that $r_t(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ for every $t \in [0,1)$. Thereafter define for every $t \in [0,1)$ the CIFS $\Phi^{(t)} = \{\varphi_i^{(t)}\}_{i\in I\!N}$, where $\varphi_1^{(t)} := \varphi_1 \circ r_t$ and $\varphi_i^{(t)} := \varphi_i$ for every $i \geq 2$. Observe that $\Phi^{(0)} = \Phi$. Note also that $(\varphi_1^{(t)})'(x) = (1-t)\varphi_1'(r_t(x))$ for every $x \in X$ and $t \in [0,1)$. It follows easily that the map $t \mapsto \Phi^{(t)}$ is continuous on $[0,1)$ and, in particular, that $\Phi^{(t)}$ λ-converges to Φ as $t \to 0$. Then $\{\Phi^{(t)}\}_{t\in[0,1)}$ is the required arc. ■

The previous result implies in particular that CIFS(X) has no isolated points when Int(X) is star shaped. More generally, for any X, we have the following.

Proposition 5.5. *For each* $\Phi \in$ CIFS(X) *the class* $[\Phi]$ *is non-degenerate.*

Proof. Let $\Phi = \{\varphi_i\} \in$ CIFS(X) and V a neighbourhood of X that makes Φ conformal. Choose a similarity S of $I\!R^d$ such that $S(V) \subset \varphi_1(X)$. Pick $z \in \mathrm{Int}(S(X))$. For each $x \in I\!R^d$ and $t \in [0,1]$, define $r_t(x) = (1-t)x + tz$. Observe that there exists $T \in [0,1]$ such that $r_t(S(V)) \subset S(V)$ for every $t \in [T,1]$. Thereafter define for every $t \in [T,1)$ the CIFS $\Phi^{(t)} = \{\varphi_i^{(t)}\}_{i\in I\!N}$, where $\varphi_1^{(t)} := r_t \circ S$ and $\varphi_i^{(t)} := \varphi_i$ for every $i \geq 2$. Observe that $\Phi^{(t)} \in [\Phi]$ for every $t \in [T,1)$. It is also easy to see that the map $t \mapsto \Phi^{(t)}$ is continuous on $[T,1)$. Then $\{\Phi^{(t)}\}_{t\in[T,1)}$ is the required arc. ■

We will now describe the type of topology the λ-topology constitutes. Obviously, the λ-topology is Hausdorff because it is finer than the pointwise topology, which is metrizable. However, it turns out that the λ-topology is itself not metrizable. This is a straightforward consequence of the following.

Proposition 5.6. *For every* $\Phi \in$ CIFS(X)*, the equivalence class* $[\Phi]$ *is not metrizable.*

Proof. This is an immediate consequence of the following proposition. ■

Proposition 5.7. *Every equivalence class of* $\sim$ *fails to satisfy the first axiom of countability at each of its points.*

Proof. Let $\Phi \in$ CIFS(X). Suppose to the contrary that there exists $\Psi \in [\Phi]$ that has a countable system $\{U_n\}_{n\in I\!N}$ of neighbourhoods in $[\Phi]$. By Lemma 5.1, there exists $\Psi^{(n)} \in U_n \backslash \overline{D}(\Psi, n)$ for every $n \in I\!N$. Since $\Psi^{(n)} \in U_n$, we know that $\{\Psi^{(n)}\}$ converges to Ψ. However, since $\Psi^{(n)} \notin \overline{D}(\Psi, n)$, we deduce that $\{\Psi^{(n)}\}$ cannot converge to Ψ. This contradiction completes the proof. ■

Nonetheless, the space CIFS(X) is normal. In order to prove this, we need the following characterization of closed subsets of equivalence classes.

Proposition 5.8. *Fix a strictly increasing unbounded sequence $\{R_n\}_{n\in I\!N}$ of real numbers larger than or equal to 1. For every $\Phi \in$ CIFS(X), the following statements are equivalent.*

(a) *The set F is a closed subset of $[\Phi]$;*
(b) *For every $R \geq 1$, the set $\overline{D}(\Phi, R) \cap F$ is a closed subset of $[\Phi]$;*
(c) *For every $n \in I\!N$, the set $\overline{D}(\Phi, R_n) \cap F$ is a closed subset of $[\Phi]$.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious. Suppose that (c) holds and that $\{\Psi^{(n)}\}$ is a sequence in F which converges to $\Psi \in [\Phi]$. Then there exist $C \geq 1$ and $N \in I\!N$ such that $d(\Psi^{(n)}, \Psi) \leq C$ for all $n \geq N$. Therefore we have that $d(\Psi^{(n)}, \Phi) \leq d(\Psi^{(n)}, \Psi) \cdot d(\Psi, \Phi) \leq Cd(\Psi, \Phi)$ for all $n \geq N$. Thus, $\Psi^{(n)} \in \overline{D}(\Phi, R_n)$ for all $n \geq N$ so large that $R_n \geq Cd(\Psi, \Phi)$. Hence $\Psi^{(n)} \in \overline{D}(\Phi, R_n) \cap F$ for all n sufficiently large. By (c), we deduce that $\Psi \in \overline{D}(\Phi, R_n) \cap F$ for all n large enough. In particular, $\Psi \in F$ and F is closed. ■

Theorem 5.9. *The space* CIFS(X) *endowed with the λ-topology is normal.*

Proof. We have already observed that CIFS(X) is Hausdorff. Since all the equivalence classes $[\Phi]$, $\Phi \in$ CIFS(X), are clopen, it suffices to show that all the subspaces $[\Phi]$ are normal. So, take $\Phi \in$ CIFS(X) and set $\overline{D}_n := \overline{D}(\Phi, n)$, $n \geq 0$. Fix two disjoint closed subsets A and B of $[\Phi]$. We shall construct by induction two sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ of open subsets of $\overline{D}_n$ (in the relative topology on $\overline{D}_n$) with the following properties:

For all $n \geq 0$,

(a_n) $A \cap \overline{D}_n \subset U_n$ and $B \cap \overline{D}_n \subset V_n$;
(b_n) $\overline{U_n} \subset U_{n+1}$ and $\overline{V_n} \subset V_{n+1}$;
(c_n) $\overline{U_n} \cap \overline{V_n} = \emptyset$.

Set $U_0 = V_0 = \emptyset$. Then (a_0) and (c_0) are trivially satisfied (since $\overline{D}_0 = \emptyset$). For the inductive step, suppose that U_n and V_n, $n \geq 0$, have been constructed in such a way that properties (a_n) and (c_n) hold. We shall construct subsets U_{n+1} and V_{n+1} of $\overline{D}_{n+1}$ so that properties (a_{n+1}), (c_{n+1}) and (b_n) hold. Indeed, both $(A \cap \overline{D}_{n+1}) \cup \overline{U_n}$ and $(B \cap \overline{D}_{n+1}) \cup \overline{V_n}$ are closed subsets of $\overline{D}_{n+1}$. Using in turn the disjointness of A and B, property (c_n), the inclusions $\overline{U_n}, \overline{V_n} \subset \overline{D}_n$, property (a_n), and (c_n) again, we get

$$\begin{aligned}\Big((A\cap\overline{D}_{n+1})\cup\overline{U_n}\Big)\bigcap\Big((B\cap\overline{D}_{n+1})\cup\overline{V_n}\Big) &= (A\cap B\cap\overline{D}_{n+1})\cup(A\cap\overline{D}_{n+1}\cap\overline{V_n})\\ &\qquad\cup(\overline{U_n}\cap B\cap\overline{D}_{n+1})\cup(\overline{U_n}\cap\overline{V_n})\\ &= (A\cap\overline{D}_{n+1}\cap\overline{V_n})\cup(\overline{U_n}\cap B\cap\overline{D}_{n+1})\\ &= (A\cap\overline{D}_n\cap\overline{V_n})\cup(B\cap\overline{D}_n\cap\overline{U_n})\\ &\subset (U_n\cap\overline{V_n})\cup(V_n\cap\overline{U_n})=\emptyset\cup\emptyset=\emptyset.\end{aligned}$$

Since the space $\overline{D}_{n+1}$ is normal (as metrizable), there thus exist two open subsets U_{n+1} and V_{n+1} of $\overline{D}_{n+1}$ (in the relative topology of $\overline{D}_{n+1}$) such that

$$(A\cap\overline{D}_{n+1})\cup\overline{U_n}\subset U_{n+1}, \qquad (B\cap\overline{D}_{n+1})\cup\overline{V_n}\subset V_{n+1}$$

and

$$\overline{U_{n+1}}\cap\overline{V_{n+1}}=\emptyset.$$

So, conditions (a_{n+1}), (b_n), and (c_{n+1}) are readily satisfied, and the inductive construction is complete. Now, set

$$U=\bigcup_{n=0}^{\infty}U_n \qquad \text{and} \qquad V=\bigcup_{n=0}^{\infty}V_n.$$

Conditions (c_n) and (b_n) imply immediately that

$$U\cap V=\emptyset.$$

It follows from (a_n) that

$$A=\bigcup_{n=0}^{\infty}A\cap\overline{D}_n\subset\bigcup_{n=0}^{\infty}U_n=U.$$

Likewise,

$$B\subset V.$$

We are left to show that the sets U and V are open. To do this, fix $k\ge 1$. Using (b_n), we get

$$\overline{D}_k\cap([\Phi]\backslash U)=\overline{D}_k\backslash U=\overline{D}_k\backslash\bigcup_{n=0}^{\infty}U_n=\overline{D}_k\backslash\bigcup_{n=k}^{\infty}U_n=\bigcap_{n=k}^{\infty}\overline{D}_k\backslash U_n=\bigcap_{n=k}^{\infty}\overline{D}_k\cap(\overline{D}_n\backslash U_n).$$

But each set $\overline{D}_n\backslash U_n$ is closed in $\overline{D}_n$, and since for all $n\ge k$ the topology on $\overline{D}_k$ is induced from $\overline{D}_n$, all the sets $\overline{D}_k\cap(\overline{D}_n\backslash U_n)$ are closed in $\overline{D}_k$. Therefore $\overline{D}_k\cap([\Phi]\backslash U)$ is a closed subset of $\overline{D}_k$ and thus it follows from Proposition 5.8 that $[\Phi]\backslash U$ is a closed subset of $[\Phi]$, that is, U is an open subset of $[\Phi]$. Likewise, V is an open subset of $[\Phi]$, and we are done. ■

We now consider the question of compactness of the space CIFS(X).

Proposition 5.10. *The space* CIFS(X) *endowed with the λ-topology is not sequentially compact.*

Proof. Let $\Phi \in \mathrm{CIFS}(X)$. For each $n \in I\!N$ let $S^{(n)}$ be a similarity such that $S^{(n)}(X) \subset \mathrm{Int}(\varphi_n(X))$ and $|(S^{(n)})'| \leq (1/n)\|\varphi_n'\|$. For each $n \geq N$, define $\Phi^{(n)} = \{\varphi_i\}_{i=1}^{n-1} \cup \{S^{(n)}\} \cup \{\varphi_i\}_{i=n+1}^{\infty}$. Then $\Phi^{(n)} \to \Phi$ in the pointwise topology as $n \to \infty$. However, every subsequence of $\{\Phi^{(n)}\}$ diverges in the λ-topology. Indeed, if a subsequence converged in the λ-topology, then it would converge to Φ. But for every n we have $d(\Phi^{(n)}, \Phi) \geq n$. ■

We now study the boundaries of the different types of CIFSs that live on X. Let us begin with the irregular systems.

Proposition 5.11. $\partial(\mathrm{IR}(X)) = \partial(\mathrm{R}(X)) = \mathrm{CR}(X)$ *in the λ-topology.*

Proof. Observe first that $\partial(\mathrm{IR}(X)) = \partial(\mathrm{R}(X))$, for $\mathrm{IR}(X) = \mathrm{CIFS}(X)\backslash \mathrm{R}(X)$.

Now we show that $\partial(\mathrm{IR}(X)) \subset \mathrm{CR}(X)$. By Lemma 5.9(iv) from [8], we know that $\mathrm{IR}(X)$ is open in the λ-topology. Thus, $\partial(\mathrm{IR}(X)) \cap \mathrm{IR}(X) = \emptyset$. Moreover, according to Lemma 5.9(i)+(ii) from [8], we have that $\mathrm{SR}(X)$ is open. Thus, $\partial(\mathrm{IR}(X)) \cap \mathrm{SR}(X) = \emptyset$. Hence $\partial(\mathrm{IR}(X)) \cap (\mathrm{IR}(X) \cup \mathrm{SR}(X)) = \emptyset$, and therefore $\partial(\mathrm{IR}(X)) \subset \mathrm{CR}(X)$.

To prove the opposite inclusion, let $\Phi \in \mathrm{CR}(X)$. Choose a similarity $S : I\!R^d \to I\!R^d$ such that $S(X) \subset \mathrm{Int}(X)$ and whose similarity ratio $A < K^{-2}/2$, where K is a distortion constant for Φ. For every $i \in I\!N$, set $\psi_i = \varphi_i \circ S$. Notice that

$$K^{-1}A\|\varphi_i'\| \leq \|\psi_i'\| \leq A\|\varphi_i'\|. \tag{5.2}$$

For each $n \in I\!N$, define $\Phi^{(n)} = \{\varphi_i\}_{i=1}^{n-1} \cup \{\psi_n\} \cup \{\varphi_i\}_{i=n+1}^{\infty}$. Il follows immediately from (5.2) that $\Phi^{(n)} \to \Phi$ as $n \to \infty$ in the λ-topology and that $\theta(\Phi^{(n)}) = \theta(\Phi) =: \theta > 0$. We claim that all the $\Phi^{(n)}$'s are irregular systems. Suppose for a contradiction that $P_n := P_{\Phi^{(n)}}(\theta) \geq 0$ for some $n \in I\!N$. Let μ be the σ-invariant Gibbs state on $I\!N^\infty$ for the potential $\omega \mapsto \theta \log|\varphi_{\omega_1}'(\pi_\Phi(\sigma\omega))|$, $\omega \in I\!N^\infty$, where $\pi_\Phi : I\!N^\infty \to X$ is the projection onto the limit set induced by the system Φ. Likewise, let μ_n be the σ-invariant Gibbs state on $I\!N^\infty$ for the potential $\omega \mapsto \theta \log|(\varphi_{\omega_1}^{(n)})'(\pi_{\Phi^{(n)}}(\sigma\omega))|$, $\omega \in I\!N^\infty$, where $\pi_{\Phi^{(n)}} : I\!N^\infty \to X$ is the projection onto the limit set induced by the system $\Phi^{(n)}$. From the Gibbs property there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu([\omega])}{\|\varphi_\omega'\|^\theta} \leq C \quad \text{ and } \quad C^{-1} \leq \frac{\mu_n([\omega])}{\|(\varphi_\omega^{(n)})'\|^\theta e^{-P_n|\omega|}} \leq C \tag{5.3}$$

for every $\omega \in I\!N^*$. Fix an arbitrary such ω. Let $k = \#\{1 \leq j \leq |\omega| : \omega_j = n\}$. Then there is a unique concatenation

$$\omega = \overline{\omega}^{(1)} \star n^{k_1} \star \overline{\omega}^{(2)} \star n^{k_2} \star \ldots \star \overline{\omega}^{(l)} \star n^{k_l} \star \overline{\omega}^{(l+1)},$$

such that $\overline{\omega}^{(1)}, \overline{\omega}^{(2)}, \ldots, \overline{\omega}^{(l)}, \overline{\omega}^{(l+1)} \in (I\!N\backslash\{n\})^*$, and where $|\overline{\omega}^{(2)}|, |\overline{\omega}^{(3)}|, \ldots, |\overline{\omega}^{(l)}| \geq 1$ ($\overline{\omega}^{(1)}$ and $\overline{\omega}^{(l+1)}$ can be empty), $k_1, k_2, \ldots, k_l \geq 1$ and $k_1 + k_2 + \ldots + k_l = k$ ($l \leq k$). Set $\omega^{(j)} := \overline{\omega}^{(j)} \star n^{k_j}$ for all $1 \leq j \leq l$, and $\omega^{(l+1)} := \overline{\omega}^{(l+1)}$. Using the bounded distortion property and (5.2), we get

$$\begin{aligned}
\|(\varphi_\omega^{(n)})'\| &\leq \|\varphi'_{\overline{\omega}^{(1)}}\| \cdot \|\psi'_n\|^{k_1} \cdot \|\varphi'_{\overline{\omega}^{(2)}}\| \cdot \|\psi'_n\|^{k_2} \cdot \ldots \cdot \|\varphi'_{\overline{\omega}^{(l)}}\| \cdot \|\psi'_n\|^{k_l} \cdot \|\varphi'_{\overline{\omega}^{(l+1)}}\| \\
&\leq \|\varphi'_{\overline{\omega}^{(1)}}\| \cdot (A\|\varphi'_n\|)^{k_1} \cdot \|\varphi'_{\overline{\omega}^{(2)}}\| \cdot (A\|\varphi'_n\|)^{k_2} \cdot \ldots \cdot \|\varphi'_{\overline{\omega}^{(l)}}\| \cdot (A\|\varphi'_n\|)^{k_l} \cdot \|\varphi'_{\overline{\omega}^{(l+1)}}\| \\
&= A^k(\|\varphi'_{\overline{\omega}^{(1)}}\| \cdot \|\varphi'_n\|^{k_1}) \cdot (\|\varphi'_{\overline{\omega}^{(2)}}\| \cdot \|\varphi'_n\|^{k_2}) \cdot \ldots \cdot (\|\varphi'_{\overline{\omega}^{(l)}}\| \cdot \|\varphi'_n\|^{k_l}) \cdot \|\varphi'_{\overline{\omega}^{(l+1)}}\| \\
&\leq A^k(K^{k_1}\|\varphi'_{\omega^{(1)}}\|) \cdot (K^{k_2}\|\varphi'_{\omega^{(2)}}\|) \cdot \ldots \cdot (K^{k_l}\|\varphi'_{\omega^{(l)}}\|) \cdot \|\varphi'_{\omega^{(l+1)}}\| \\
&= A^k\, K^k \|\varphi'_{\omega^{(1)}}\| \cdot \|\varphi'_{\omega^{(2)}}\| \cdot \ldots \cdot \|\varphi'_{\omega^{(l)}}\| \cdot \|\varphi'_{\omega^{(l+1)}}\| \\
&\leq A^k\, K^{k+l} \|(\varphi_{\omega^{(1)}} \circ \varphi_{\omega^{(2)}} \circ \ldots \circ \varphi_{\omega^{(l)}} \circ \varphi_{\omega^{(l+1)}})'\| \\
&= A^k\, K^{k+l}\|\varphi'_\omega\| \leq A^k\, K^{2k}\|\varphi'_\omega\| = (AK^2)^k\|\varphi'_\omega\| \leq 2^{-k}\|\varphi'_\omega\|,
\end{aligned}$$

where the last inequality follows from the fact that $A < K^{-2}/2$.

Applying (5.3), we deduce that

$$\mu_n([\omega]) \leq C\|(\varphi_\omega^{(n)})'\|^\theta e^{-P_n|\omega|} \leq C\|(\varphi_\omega^{(n)})'\|^\theta \leq C \cdot 2^{-k\theta}\|\varphi'_\omega\|^\theta \leq C^2 \cdot 2^{-k\theta}\mu([\omega]). \tag{5.4}$$

Hence the measure μ_n is absolutely continuous with respect to μ, and since these two measures are ergodic (as Gibbs states of Hölder continuous potentials) with respect to the shift map $\sigma : I\!N^\infty \to I\!N^\infty$, they must be equal, that is, $\mu_n = \mu$. But if $\omega \in I\!N^*$ is a word with k so large that $C^2 \cdot 2^{-k\theta} < 1$, then, in view of (5.4), we have $\mu_n([\omega]) < \mu([\omega])$, which implies that $\mu_n \neq \mu$. This contradiction completes the proof. ■

As a straightforward corollary, we obtain the following.

Corollary 5.12. $\partial(\mathrm{CR}(X)) = \mathrm{CR}(X)$.

Proof. The previous proposition implies that $\mathrm{Int}(\mathrm{CR}(X)) = \emptyset$, for if this were not the case then $\partial(\mathrm{IR}(X)) \neq \mathrm{CR}(X)$. ■

We now turn our attention to $\mathrm{SR}(X)$.

Proposition 5.13. $\partial(\mathrm{SR}(X)) = \partial(\mathrm{SR}(X)\backslash\mathrm{CFR}(X)) \subset \mathrm{CR}(X)$.

Proof. We first prove that $\partial(\mathrm{SR}(X)) \subset \mathrm{CR}(X)$. Since $\mathrm{SR}(X)$ is open, we have $\partial(\mathrm{SR}(X)) \cap \mathrm{SR}(X) = \emptyset$. Since $\mathrm{IR}(X)$ is open, we also have $\partial(\mathrm{SR}(X)) \cap \mathrm{IR}(X) = \emptyset$. Therefore $\partial(\mathrm{SR}(X)) \cap (\mathrm{IR}(X) \cup \mathrm{SR}(X)) = \emptyset$ and hence $\partial(\mathrm{SR}(X)) \subset \mathrm{CR}(X)$.

Regarding the equality $\partial(\mathrm{SR}(X)) = \partial(\mathrm{SR}(X)\backslash\mathrm{CFR}(X))$, observe that this latter follows from the topological lemma below and the fact that $\mathrm{CFR}(X)$ is clopen in the λ-topology according to Lemma 5.9(i) from [8].■

Lemma 5.14. *Let Z be a topological space, $S \subset Z$ a set and $T \subset S$ a clopen set in Z. Then $\partial S = \partial(S\backslash T)$.*

Proof. This follows readily from the characterization of a clopen set T as having no boundary, that is, $\partial T = \partial(Z\backslash T) = \emptyset$. ■

However, notice that the reverse inclusion $\mathrm{CR}(X) \subset \partial(\mathrm{SR}(X))$ does not hold in general.

Proposition 5.15. *If $\Phi \in \mathrm{CR}(X) \cup \mathrm{IR}(X)$, $\theta_\Phi = d$ and $\Psi \sim \Phi$, then $\Psi \in \mathrm{CR}(X) \cup \mathrm{IR}(X)$.*

Proof. Suppose for a contradiction that $\Psi \in \mathrm{SR}(X)$. Then $d = \theta_\Phi = \theta_\Psi < h_\Psi \leq d$. This is a contradiction. ■

As an immediate consequence of this proposition, we obtain the following.

Corollary 5.16. *If $\Phi \in \mathrm{CR}(X)$ and $\theta_\Phi = d$, then $\Phi \notin \overline{\mathrm{SR}(X)}$. In particular, $\Phi \notin \partial(\mathrm{SR}(X))$.*

A system with the property required in the hypothesis of Corollary 5.16 was described in Example 5.2.5 on page 141 of [7], and thus the inclusion $\mathrm{CR}(X) \subset \partial(\mathrm{SR}(X))$ does not hold in general.

The following result shows that a "large" subset of $\mathrm{CR}(X)$ is contained in $\partial(\mathrm{SR}(X))$. This subset is determined by two conditions: (1) that all the systems consist of similarities only and (2) that they satisfy a separation condition. Note that Corollary 5.16 guarantees that the following result is not true without this separation condition. Observe also that the separation condition in question is weaker than the more common super strong separation condition, which states that for every $i \in I\!N$

$$\varphi_i(X) \cap \overline{\bigcup_{j \neq i} \varphi_j(X)} = \emptyset.$$

Proposition 5.17. *Let $\Phi \in \mathrm{CR}(X) \cap \mathrm{SIFS}(X)$ be such that there exists some $m \in I\!N$ for which*

$$\varphi_m(X) \subset \mathrm{Int}(X) \quad \textit{and} \quad \varphi_m(X) \cap \overline{\bigcup_{j \neq m} \varphi_j(X)} = \emptyset.$$

Then $\Phi \in \partial(\mathrm{SR}(X) \backslash \mathrm{CFR}(X)) = \partial(\mathrm{SR}(X))$.

Proof. Let $\Phi \in \mathrm{CR}(X) \cap \mathrm{SIFS}(X)$ be such that there exists $m \in I\!N$ for which $\varphi_m(X) \cap \overline{\cup_{j \neq m} \varphi_j(X)} = \emptyset$. Then we can find a sequence of similarities $\{\psi_n\}$ which converges to φ_m in $C^1(X)$, which is such that $\psi_n(X) \subset X$, $\psi_n(X) \cap \overline{\cup_{j \neq m} \varphi_j(X)} = \emptyset$ and such that each ψ_n has a larger similarity ratio than that of φ_m (in other terms, $\|\psi_n'\| > \|\varphi_m'\|$ for each $n \in I\!N$). For each $n \in I\!N$, define

$$\Phi^{(n)} = \{\varphi_i\}_{i=1}^{m-1} \cup \{\psi_n\} \cup \{\varphi_i\}_{i=m+1}^{\infty}.$$

Clearly, $\Phi^{(n)} \in \mathrm{SIFS}(X)$ since $\Phi^{(n)}$ satisfies the OSC. Indeed, $\Phi^{(n)}$ is the same as Φ except for its m-th generator which is replaced by ψ_n, Φ satisfies the OSC and $\psi_n(\mathrm{Int}(X)) \cap \cup_{j \neq m} \varphi_j(\mathrm{Int}(X)) = \emptyset$. Moreover, $\Phi^{(n)} \to \Phi$ as $n \to \infty$ in the λ-topology since $\Phi^{(n)}$ and Φ share the same generators except the m-th one and $\{\psi_n\}_{n \in I\!N}$ converges to φ_m in $C^1(X)$.

We claim that all $\Phi^{(n)}$'s are strongly regular, though not cofinitely regular. First, observe that $\theta_{\Phi^{(n)}} = \theta_\Phi$ since $\Phi^{(n)}$ and Φ share all but finitely many generators. Moreover, $\Phi^{(n)}$ is strongly regular for all $n \in I\!N$ since

$$\begin{aligned} P_{\Phi^{(n)}}(\theta_{\Phi^{(n)}}) = P_{\Phi^{(n)}}(\theta_\Phi) &= \log \sum_{i=1}^{\infty} \|(\varphi_i^{(n)})'\|^{\theta_\Phi} \\ &= \log \left[\sum_{i=1}^{m-1} \|\varphi_i'\|^{\theta_\Phi} + \|\psi_n'\|^{\theta_\Phi} + \sum_{i=m+1}^{\infty} \|\varphi_i'\|^{\theta_\Phi} \right] \\ &> \log \left[\sum_{i=1}^{m-1} \|\varphi_i'\|^{\theta_\Phi} + \|\varphi_m'\|^{\theta_\Phi} + \sum_{i=m+1}^{\infty} \|\varphi_i'\|^{\theta_\Phi} \right] \\ &= P_\Phi(\theta_\Phi) \\ &= 0. \end{aligned}$$

Finally, $\Phi^{(n)}$ is not cofinitely regular for every $n \in I\!N$ since

$$\begin{aligned} P_{\Phi^{(n)}}(\theta_{\Phi^{(n)}}) = P_{\Phi^{(n)}}(\theta_\Phi) &= \log \sum_{i=1}^{\infty} \|(\varphi_i^{(n)})'\|^{\theta_\Phi} \\ &= \log \left[\sum_{i=1}^{m-1} \|\varphi_i'\|^{\theta_\Phi} + \|\psi_n'\|^{\theta_\Phi} + \sum_{i=m+1}^{\infty} \|\varphi_i'\|^{\theta_\Phi} \right] \\ &= \log \left[\sum_{i=1}^{\infty} \|\varphi_i'\|^{\theta_\Phi} + \|\psi_n'\|^{\theta_\Phi} - \|\varphi_m'\|^{\theta_\Phi} \right] \\ &= \log \left[\exp(P_\Phi(\theta_\Phi)) + \|\psi_n'\|^{\theta_\Phi} - \|\varphi_m'\|^{\theta_\Phi} \right] \\ &< \infty. \end{aligned}$$

■

This completes our study of the boundaries of the diverse types of systems. Note that the question whether the previous result holds for any $\Phi \in \mathrm{CR}(X)\backslash\mathrm{SIFS}(X)$ remains open.

We have earlier reminded the reader that the λ-topology ensures that the finiteness parameter function, the pressure function and the Hausdorff dimension function are continuous. However, we will see in the forthcoming two results that this topology is not fine enough to guarantee that the coding map and the closure of the limit set depend continuously on the system Φ.

Proposition 5.18. *Equipping the space* $\mathrm{CIFS}(X)$ *with the* λ*-topology and the space* $\kappa(X)$ *of all compact subsets of* X *with the standard Hausdorff metric* ρ_H*, the map*

$$\begin{array}{cccc} \overline{J_\cdot} & : & \mathrm{CIFS}(X) & \to & \kappa(X) \\ & & \Phi & \mapsto & \overline{J_\Phi} \end{array}$$

is discontinuous at every $\Phi \in \mathrm{CIFS}(X)$ *such that* $X\backslash\overline{\cup_{n\in I\!N}\varphi_n(X)} \neq \emptyset$.

Proof. Let $\Phi \in \mathrm{CIFS}(X)$ be such that $X \backslash \overline{\cup_{n\in I\!N}\varphi_n(X)} \neq \emptyset$. Pick a ball $B \subset X \backslash \overline{\cup_{n\in I\!N}\varphi_n(X)}$. Let $\frac{1}{2}B$ denote the ball concentric with B and of radius half that of B. Finally, let V be an open neighbourhood of X such that all φ_n's extend to C^1 conformal diffeomorphisms of V into V. Since $\lim_{n\to\infty} \|\varphi_n'\| = 0$, there exists $N \in I\!N$ such that for every $n \geq N$ there is a similarity $S_n : V \to \frac{1}{2}B$ with $|S_n'| = \|\varphi_n'\|$. For each $n \geq N$, define $\Phi^{(n)} = \{\varphi_i\}_{i=1}^{n-1} \cup \{S_n\} \cup \{\varphi_i\}_{i=n+1}^{\infty}$.

Then $\Phi^{(n)} \to \Phi$ in the λ-topology as $n \to \infty$. Indeed, it is clear that $\Phi^{(n)} \to \Phi$ in the pointwise topology as $n \to \infty$, and $d(\Phi^{(n)}, \Phi) = 1$ for every $n \geq N$.

However, $\rho_H(\overline{J_{\Phi^{(n)}}}, \overline{J_\Phi}) \not\to 0$ as $n \to \infty$. Indeed, $\overline{J_\Phi} \subset \overline{\cup_{n\in I\!N}\varphi_n(X)}$. Nonetheless, for every $n \geq N$ we have $\varphi_n^{(n)}(X) = S_n(X) \subset \frac{1}{2}B$, and hence $J_{\Phi^{(n)}} \cap \frac{1}{2}B \neq \emptyset$. Thus, $\rho_H(\overline{J_{\Phi^{(n)}}}, \overline{J_\Phi}) \geq \frac{1}{4}\mathrm{diam}(B) > 0$ for every $n \geq N$. ■

Corollary 5.19. *Endowing the space* $\mathrm{CIFS}(X)$ *with the λ-topology and the space* $C(I\!N^{I\!N}, X)$ *of all continuous maps from* $I\!N^{I\!N}$ *into* X *with the standard supremum norm, the map*

$$\begin{array}{cccc} \pi & : & \mathrm{CIFS}(X) & \to & C(I\!N^{I\!N}, X) \\ & & \Phi & \mapsto & \pi_\Phi \end{array}$$

is discontinuous at every $\Phi \in \mathrm{CIFS}(X)$ *such that* $X \backslash \overline{\cup_{n\in I\!N}\varphi_n(X)} \neq \emptyset$.

Proof. Let $\Phi \in \mathrm{CIFS}(X)$ be such that $X \backslash \overline{\cup_{n\in I\!N}\varphi_n(X)} \neq \emptyset$. Suppose for a contradiction that π is continuous at Φ, and let $\{\Phi^{(n)}\}$ be a sequence in $\mathrm{CIFS}(X)$ such that $\Phi^{(n)} \to \Phi$ in the λ-topology. Then, for every $\varepsilon > 0$ there exists $N \in I\!N$ such that for each $n \geq N$ we have $\|\pi_{\Phi^{(n)}} - \pi_\Phi\|_\infty \leq \varepsilon$. This means that $|\pi_{\Phi^{(n)}}(\omega) - \pi_\Phi(\omega)| \leq \varepsilon$ for every $\omega \in I\!N^{I\!N}$. But then $\rho_H(\overline{J_{\Phi^{(n)}}}, \overline{J_\Phi}) \leq \varepsilon$. Since $\varepsilon > 0$ and the sequence $\{\Phi^{(n)}\}$ were chosen arbitrarily, Lemma 3.3 in [8] asserts that the map $\overline{J_\cdot}$ is continuous at Φ. This contradicts Proposition 5.18, and we are done. ■

Finally, we turn our attention to the continuity of measures. For every $\Phi \in \mathrm{CIFS}(X)$, let

$$\mathrm{Fin}(\Phi) = \Big\{t \geq 0 : P_\Phi(t) < \infty\Big\} = \Big\{t \geq 0 : P_\Phi^{(1)}(t) < \infty\Big\}.$$

Note that if $d(\Phi, \Psi) < \infty$, then $\mathrm{Fin}(\Phi) = \mathrm{Fin}(\Psi)$. For every $t \in \mathrm{Fin}(\Phi)$, let $m_{\Phi,t}$ be the corresponding conformal (Gibbs geometric) measure and $\mu_{\Phi,t}$ the corresponding Gibbs invariant measure. Our main result is the following.

Theorem 5.20. *Let* $\Phi \in \mathrm{CIFS}(X)$*, and suppose that the sequence* $\{\Phi^{(n)}\}$ *converges to* Φ *in the λ-topology on* $\mathrm{CIFS}(X)$ *with all* $\Phi^{(n)}$*'s admitting a common neighbourhood* V *of* X*. If* $t_n \to t$*, where* $t_n \in Fin(\Phi)$ *for all* $n \in I\!N$ *and* $t \in Fin(\Phi)$*, then* $m_{\Phi^{(n)},t_n} \to m_{\Phi,t}$ *weakly and* $\mu_{\Phi^{(n)},t_n} \to \mu_{\Phi,t}$ *weakly.*

We shall first prove the following property of the pressure function. This result partially generalizes Theorem 5.6 in [8].

Lemma 5.21. *Let* $\Phi \in \mathrm{CIFS}(X)$*, and suppose that the sequence* $\{\Phi^{(n)}\}$ *converges to* Φ *in the λ-topology on* $\mathrm{CIFS}(X)$*. If* $t_n \to t$*, where* $t_n \in Fin(\Phi)$ *for all* $n \in I\!N$ *and* $t \in Fin(\Phi)$*, then for every* $k \in I\!N$*, we have* $\lim_{n\to\infty} P_{\Phi^{(n)}}^{(k)}(t_n) = P_\Phi^{(k)}(t)$.

Proof. The proof being inspired from that of Theorem 5.6 in [8], we just give some guidelines. Fix $k \in I\!N$. Let $\varepsilon > 0$. Define $t_{\min} := \min\{t, \inf\{t_n : n \in I\!N\}\}$ and $t_{\max} := \max\{t, \sup\{t_n : n \in I\!N\}\}$. Since $t_n \in \mathrm{Fin}(\Phi)$ for all $n \in I\!N$, $t \in \mathrm{Fin}(\Phi)$ and $t_n \to t$ as $n \to \infty$, we have $[t_{\min}, t_{\max}] \subset \mathrm{Fin}(\Phi)$. Moreover, since $\{\Phi^{(n)}\}$ converges to Φ in the λ-topology, it follows from condition (5.1) that there exist $C > 0$ and $M \in I\!N$ such that

$$e^{-C\tilde{t}} \leq \frac{\|(\varphi_i^{(n)})'\|^{\tilde{t}}}{\|\varphi_i'\|^{\tilde{t}}} \leq e^{C\tilde{t}}$$

for all $\tilde{t} \in I\!R$, all $i \in I\!N$ and all $n \geq M$. Since $P_\Phi^{(k)}(t_{\min}) < \infty$, there exists a finite set $G \subset I^k$ such that

$$\sum_{\omega \in I^k \setminus G} \|\varphi_\omega'\|^{t_{\min}} < e^{-kCt_{\max}} K^{-kt_{\max}} \frac{\varepsilon}{2},$$

where $K = K_\Phi$ is a constant of bounded distortion for the CIFS Φ. By Lemma 5.1 in [8] there is $N \in I\!N$ such that for every $n \geq N$,

$$\left| \sum_{\omega \in G} \|(\varphi_\omega^{(n)})'\|^{t_n} - \sum_{\omega \in G} \|\varphi_\omega'\|^{t} \right| < \frac{\varepsilon}{2}.$$

On the one hand, it follows that for every $n \geq \max\{N, M\}$

$$\begin{aligned} P_{\Phi^{(n)}}^{(k)}(t_n) &\leq \sum_{\omega \in G} \|(\varphi_\omega^{(n)})'\|^{t_n} + e^{kCt_n} K^{kt_n} \sum_{\omega \in I^k \setminus G} \|\varphi_\omega'\|^{t_n} \\ &\leq \sum_{\omega \in G} \|\varphi_\omega'\|^{t} + \frac{\varepsilon}{2} + e^{kCt_{\max}} K^{kt_{\max}} \sum_{\omega \in I^k \setminus G} \|\varphi_\omega'\|^{t_{\min}} < \sum_{\omega \in G} \|\varphi_\omega'\|^{t} + \varepsilon < P_\Phi^{(k)}(t) + \varepsilon. \end{aligned}$$

On the other hand, for every $n \geq \max\{N, M\}$

$$P_{\Phi^{(n)}}^{(k)}(t_n) > \sum_{\omega \in G} \|(\varphi_\omega^{(n)})'\|^{t_n} > \sum_{\omega \in G} \|\varphi_\omega'\|^{t} - \frac{\varepsilon}{2} \geq \sum_{\omega \in I^k} \|\varphi_\omega'\|^{t} - \sum_{\omega \in I^k \setminus G} \|\varphi_\omega'\|^{t_{\min}} - \frac{\varepsilon}{2} \geq P_\Phi^{(k)}(t) - \varepsilon.$$

Consequently, for every $n \geq \max\{N, M\}$

$$\left| P_{\Phi^{(n)}}^{(k)}(t_n) - P_\Phi^{(k)}(t) \right| < \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we have thus shown that $\lim_{n\to\infty} P_{\Phi^{(n)}}^{(k)}(t_n) = P_\Phi^{(k)}(t)$. ■

We then obtain the following, which partly extends Theorem 5.7 in [8].

Lemma 5.22. *Let $\Phi \in \mathrm{CIFS}(X)$, and suppose that the sequence $\{\Phi^{(n)}\}$ converges to Φ in the λ-topology on $\mathrm{CIFS}(X)$. If $t_n \to t$, where $t_n \in Fin(\Phi)$ for all $n \in I\!N$ and $t \in Fin(\Phi)$, then $\lim_{n\to\infty} P_{\Phi^{(n)}}(t_n) = P_\Phi(t)$.*

Proof. Let $\varepsilon > 0$. Choose $k \in I\!N$ such that

$$\frac{1}{k} \log P_\Phi^{(k)}(t) < P_\Phi(t) + \frac{\varepsilon}{2}.$$

Using Lemma 5.21, pick $N \in I\!N$ such that

$$\left|\frac{1}{k}\log P^{(k)}_{\Phi^{(n)}}(t_n) - \frac{1}{k}\log P^{(k)}_{\Phi}(t)\right| < \frac{\varepsilon}{2}$$

for every $n \geq N$. It follows that for every such n,

$$P_{\Phi^{(n)}}(t_n) \leq \frac{1}{k}\log P^{(k)}_{\Phi^{(n)}}(t_n) < \frac{1}{k}\log P^{(k)}_{\Phi}(t) + \frac{\varepsilon}{2} < P_\Phi(t) + \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we have thus shown that $\limsup_{n\to\infty} P_{\Phi^{(n)}}(t_n) \leq P_\Phi(t)$.

Now, let $t^* > t$. Since $t_n \to t$ as $n \to \infty$, there is $N \in I\!N$ such that $t_n \leq t^*$ for all $n \geq N$. Let F be a finite subset of $I\!N$. Based on Theorem 2.1.5 from [7] and on Lemma 4.2 from [8], and since the pressure function for every system is a non-increasing function, we have that

$$\liminf_{n\to\infty} P_{\Phi^{(n)}}(t_n) \geq \liminf_{n\to\infty} P_{\Phi^{(n)},F}(t_n) \geq \liminf_{n\to\infty} P_{\Phi^{(n)},F}(t^*) = P_{\Phi,F}(t^*).$$

Since this is true for every finite subset F of $I\!N$, we deduce from Theorem 2.1.5 in [7] that

$$\liminf_{n\to\infty} P_{\Phi^{(n)}}(t_n) \geq \sup_{F\subset I\!N, F\text{finite}} P_{\Phi,F}(t^*) = P_\Phi(t^*).$$

Since this is true for every $t^* > t$, and since $t \in \text{Fin}(\Phi)$ and P_Φ is continuous and decreasing on $\text{Fin}(\Phi)$, we conclude that

$$\liminf_{n\to\infty} P_{\Phi^{(n)}}(t_n) \geq \sup_{t^*>t} P_\Phi(t^*) = P_\Phi(t) \geq \limsup_{n\to\infty} P_{\Phi^{(n)}}(t_n).$$

■

Now consider the following normalized linear operators acting continuously on $C(X)$:

$$\mathcal{L}_t g(x) = \sum_{i\in I} e^{-P_\Phi(t)}|\varphi_i'(x)|^t g(\varphi_i(x))$$

and

$$\mathcal{L}_{n,t_n} g(x) = \sum_{i\in I} e^{-P_{\Phi^{(n)}}(t_n)}|(\varphi_i^{(n)})'(x)|^{t_n} g(\varphi_i^{(n)}(x)).$$

Moreover, recall that by definition the operator norm is

$$\|\mathcal{L}_t\|_\infty := \sup\{\|\mathcal{L}_t g\|_\infty : g \in C(X), \|g\|_\infty \leq 1\}.$$

Recall also that $I = I\!N$. We have the following result.

Lemma 5.23. *Let $\Phi \in \text{CIFS}(X)$, and suppose that the sequence $\{\Phi^{(n)}\}$ converges to Φ in the λ-topology on $\text{CIFS}(X)$. If $t_n \to t$, where $t_n \in Fin(\Phi)$ for all $n \in I\!N$ and $t \in Fin(\Phi)$, then $\lim_{n\to\infty} \|\mathcal{L}_{n,t_n} - \mathcal{L}_t\|_\infty = 0$.*

Proof. As previously, let $t_{\min} = \min\{t, \inf\{t_n : n \in I\!N\}\}$ and $t_{\max} = \max\{t, \sup\{t_n : n \in I\!N\}\}$, and let $P = \min\{P_\Phi(t), \inf\{P_{\Phi^{(n)}}(t_n) : n \in I\!N\}\}$. Recall that $[t_{\min}, t_{\max}] \subset \text{Fin}(\Phi)$.

Observe also that $P < \infty$ since $P_\Phi(t) < \infty$. Fix $0 < \varepsilon < 1$. Then there exists a finite set $I_\varepsilon \subset I$ such that

$$e^{-P} \sum_{i \in I \setminus I_\varepsilon} C^l \|\varphi_i'\|^{t_{\min}} < \frac{\varepsilon}{4},$$

where $C \geq 1$ is a constant arising from the hypothesis that $\Phi^{(n)} \to \Phi$ in the λ-topology. Hence,

$$\sum_{i \in I \setminus I_\varepsilon} e^{-P_\Phi(t)} \|\varphi_i'\|^t, \sum_{i \in I \setminus I_\varepsilon} e^{-P_{\Phi^{(n)}}(t_n)} \|(\varphi_i^{(n)})'\|^{t_n} < \frac{\varepsilon}{4} \tag{5.5}$$

for every $n \in I\!N$. Now, for every $g \in C(X)$ and every $x \in X$, we get

$$\begin{aligned}
\Big| \sum_{i \in I_\varepsilon} & e^{-P_{\Phi^{(n)}}(t_n)} |(\varphi_i^{(n)})'(x)|^{t_n} g(\varphi_i^{(n)}(x)) - \sum_{i \in I_\varepsilon} e^{-P_\Phi(t)} |\varphi_i'(x)|^t g(\varphi_i(x)) \Big| \\
&\leq \sum_{i \in I_\varepsilon} e^{-P_\Phi(t)} |\varphi_i'(x)|^t |g(\varphi_i^{(n)}(x)) - g(\varphi_i(x))| \\
&+ \sum_{i \in I_\varepsilon} |g(\varphi_i^{(n)}(x))| \left| e^{-P_{\Phi^{(n)}}(t_n)} |(\varphi_i^{(n)})'(x)|^{t_n} - e^{-P_\Phi(t)} |\varphi_i'(x)|^t \right| \\
&\leq e^{-P_\Phi(t)} \sum_{i \in I_\varepsilon} \|\varphi_i'\|^t |g(\varphi_i^{(n)}(x)) - g(\varphi_i(x))| \\
&+ \|g\|_\infty \sum_{i \in I_\varepsilon} \left| e^{-P_{\Phi^{(n)}}(t_n)} |(\varphi_i^{(n)})'(x)|^{t_n} - e^{-P_\Phi(t)} |\varphi_i'(x)|^t \right|.
\end{aligned}$$

Since $\Phi^{(n)} \to \Phi$ pointwise and since the set I_ε is finite, taking $n \in I\!N$ large enough, the first summand will be bounded above by $\frac{\varepsilon}{4}\|g\|_\infty$; recalling that $\lim_{n\to\infty} P_{\Phi^{(n)}}(t_n) = P_\Phi(t)$ according to Lemma 5.22 and $\lim_{n\to\infty} t_n = t$, the second summand will also be bounded by $\frac{\varepsilon}{4}\|g\|_\infty$ for all n sufficiently large. Furthermore, taking into account (5.5), we deduce that

$$\begin{aligned}
\left| \mathcal{L}_{n,t_n} g(x) - \mathcal{L}_t g(x) \right| &\leq \frac{\varepsilon}{2}\|g\|_\infty + \|g\|_\infty \sum_{i \in I \setminus I_\varepsilon} e^{-P_{\Phi^{(n)}}(t_n)} \|(\varphi_i^{(n)})'\|^{t_n} + \|g\|_\infty \sum_{i \in I \setminus I_\varepsilon} e^{-P_\Phi(t)} \|\varphi_i'\|^t \\
&\leq \frac{\varepsilon}{2}\|g\|_\infty + \frac{\varepsilon}{4}\|g\|_\infty + \frac{\varepsilon}{4}\|g\|_\infty = \varepsilon\|g\|_\infty.
\end{aligned}$$

Hence $\|\mathcal{L}_{n,t_n} g - \mathcal{L}_t g\|_\infty \leq \varepsilon \|g\|_\infty$. Since g was chosen arbitrarily in $C(X)$, we conclude that $\|\mathcal{L}_{n,t_n} - \mathcal{L}_t\|_\infty \leq \varepsilon$. Since ε was chosen arbitrarily, we are done. ■

Proof of Theorem 5.20. In order to allege notation, set $m_t = m_{\Phi,t}$, and $m_n = m_{\Phi^{(n)},t_n}$ for every $n \in I\!N$. Do similarly for the measures μ and the operators $\mathcal{L}$. Let ν be an arbitrary weak accumulation point of the sequence $\{m_n\}$, say $\nu = \lim_{j\to\infty} m_{n_j}$. In virtue of Lemma 5.23 we know that for every $g \in C(X)$

$$\lim_{j\to\infty} \mathcal{L}_{n_j}^* m_{n_j}(g) = \lim_{j\to\infty} m_{n_j}(\mathcal{L}_{n_j} g) = \nu(\mathcal{L}_t g) = \mathcal{L}_t^* \nu(g).$$

This means that the sequence $\{\mathcal{L}_{n_j}^* m_{n_j}\}$ converges weakly to $\mathcal{L}_t^*\nu$. But $\mathcal{L}_{n_j}^* m_{n_j} = m_{n_j}$ and $\{m_{n_j}\}$ converges weakly to ν. Thus, $\mathcal{L}_t^*\nu = \nu$. It then follows from the uniqueness part of Theorem 3.2.3 in [7] that $\nu = m_t$. Thus, the set of weak accumulation points of the sequence $\{m_n\}$ consists solely of m_t, meaning that $\{m_n\}$ converges to m_t weakly.

Now, let us consider invariant measures. We shall show the following.

Claim. *The β-Hölder norms of the normalized positive fixed points ρ_n (i.e. with $m_n(\rho_n) = 1$) of the Perron-Frobenius operators $\mathcal{L}_n$ are uniformly bounded.*

Indeed, starting from Theorem 2.4.3 and Lemma 2.4.1 in [7] or, in fact, their "downstairs" equivalents on X, we need to show that the constants Q and C are uniformly bounded above in n. According to the last line in the proof of Lemma 2.4.1, we have $C = Q(M\log(T(f))+1)$, and thus we need to show that Q, M and $T(f)$ are uniformly bounded, where $f = \zeta_{\Phi^{(n)}}$. From the line following inequality (2.23) in the proof of Lemma 2.4.1, we observe that $M \geq 1$ will be uniformly bounded provided that $T(f)$ is. Thus, we only need to establish that Q and $T(f)$ are uniformly bounded. But Q is a ratio bounding constant for the Gibbs state of f, and simplifying the proof of Theorem 2.3.3 to the case of a full-shift we see that Q can be taken as $\max\{T(f), T(f)^{-1}\}$. Therefore it remains to prove that $T(f)$ is uniformly bounded away from 0 and ∞. By definition, $T(f) = \exp(V_\alpha(f)/(e^\alpha - 1))$ (still in [7], look at the bottom of page 26 for the definition of $T(f)$ and the top of page 19 for $V_\alpha(f)$). In our case, $f = \zeta_{\Phi^{(n)}}$ is Lipschitz for all n and thus $\alpha \equiv 1$ and $V_\alpha(f)$ is the Lipschitz constant for f. Since $\alpha \equiv 1$ and $V_\alpha(f) \geq 0$ for all n, we have $T(f) \geq 1$ for all n. On the other hand, the uniform upper bound on $V_\alpha(f)$ is a consequence of Theorems 4.1.2 and 4.1.3. Indeed, note the presence of the constants K_3 and K_4 in these two theorems. These constants generally depend on the neighbourhood $V_{\Phi^{(n)}}$. Since all $\Phi^{(n)}$'s admit a common such neighbourhood V, we deduce that the constants K_3 and K_4 are uniformly bounded in n. Therefore $V_\alpha(f)$, and hence $T(f)$, is uniformly bounded above. The claim is proved.

In virtue of this claim, it follows from Arzela-Ascoli's Theorem that each subsequence of the sequence $\{\rho_n\}$ has a converging subsequence in the supremum norm. It then directly follows from Lemma 5.23 that the limit $\hat{\rho}$ of each such subsequence is a non-negative fixed point of $\mathcal{L}_t$. Since, as we already know, $\{m_n\}$ converges weakly to m_t, we have $m_t(\hat{\rho}) = 1$. Thus, $\hat{\rho}m_t$ is an invariant Borel probability measure which is absolutely continuous with respect to m_t. Hence $\hat{\rho}m_t = \rho_t m_t$, meaning that $\hat{\rho} = \rho_t$. Therefore $\{\rho_n\}$ converges to ρ_t in the supremum norm on $C(X)$. Consequently, for every $g \in C(X)$ the sequence $\{g\rho_n\}$ converges uniformly to $g\rho_t$. Thus,

$$\lim_{n\to\infty}\int_X g\,d\mu_n = \lim_{n\to\infty}\int_X g\rho_n\,dm_n = \int_X g\rho_t\,dm_t = \int_X g\,d\mu_t.$$

This means that $\{\mu_n\}$ converges weakly to μ_t. We are done. ■

Remark 5.24. *In the case $d \geq 3$, Theorem 5.20 remains valid when replacing the existence of a common V by the weaker assumption that the centers of inversion of the generators of all the systems $\Phi^{(n)} = \{\varphi_i^{(n)}\}_{i\in\mathbb{N}}$, $n \in \mathbb{N}$, remain uniformly away from X. That is, if $a_i^{(n)}$ is the center of inversion of $\varphi_i^{(n)}$ (if $\varphi_i^{(n)}$ is a similarity, then we declare $a_i^{(n)} = \infty$) and if $A = \cup_{i,n}\{a_i^{(n)}\}$, then it is sufficient to assume that* $\mathrm{dist}(A, X) > 0$.

As an immediate consequence of this theorem and Theorem 5.10 in [8], we obtain the following.

Corollary 5.25. *If $\Phi^{(n)} \to \Phi$ in the λ-topology with all $\Phi^{(n)}$'s admitting a common neighbourhood V, then $m_{\Phi^{(n)},h_{\Phi^{(n)}}} \to m_{\Phi,h_\Phi}$ weakly and $\mu_{\Phi^{(n)},h_{\Phi^{(n)}}} \to \mu_{\Phi,h_\Phi}$ weakly.*

Remark 5.26. *Note that we do not assume in the above corollary that $P_\Phi(h_\Phi) = 0$ or $P_{\Phi^{(n)}}(h_{\Phi^{(n)}}) = 0$ for any n.*

Remark 5.27. *In* [8], *we should have assumed the space X to be connected and to satisfy $X = \overline{\mathrm{Int}(X)}$. Moreover, there are two instances in which we failed to take into account the distortion created by the generators of the systems:*

1- Lemma 4.1 in [8] *holds when X is convex. However, when this is not the case, the following, slightly weaker result is a consequence of the local bounded distortion of the generators of the systems. The proof of this result goes along similar lines to those in* [8].

Lemma. The coding map $\pi : \mathrm{CIFS}(X, I) \to C(I^\infty, X)$ is continuous. Moreover, given $\Phi \in \mathrm{CIFS}(X)$, for each $1 < D < \|\Phi'\|^{-1}$ there is $\varepsilon > 0$ such that

$$\|\pi_\Psi - \pi_\Phi\| \leq \frac{1}{1 - D\|\Phi'\|}\|\Psi - \Phi\| \leq \frac{1}{1 - D\|\Phi'\|}\rho(\Psi, \Phi)$$

for all $\Psi \in B(\Phi, \varepsilon)$.

Note that Lemma 4.2 in [8] *remains valid. One only needs to use in the proof the above amended form of Lemma 4.1 instead. Thus, all the following results, as announced in the paper, hold.*

2- The inequality in the first part of the proof of Lemma 5.1 in [8] *should be replaced by*

$$\begin{aligned}
\left|\varphi_\omega^n(x) - \varphi_\omega(x)\right| &= \left|\varphi_{\omega_1}^n(\varphi_{\sigma\omega}^n(x)) - \varphi_{\omega_1}(\varphi_{\sigma\omega}(x))\right| \\
&\leq \left|\varphi_{\omega_1}^n(\varphi_{\sigma\omega}^n(x)) - \varphi_{\omega_1}(\varphi_{\sigma\omega}^n(x))\right| + \left|\varphi_{\omega_1}(\varphi_{\sigma\omega}^n(x)) - \varphi_{\omega_1}(\varphi_{\sigma\omega}(x))\right| \\
&\leq \|\varphi_{\omega_1}^n - \varphi_{\omega_1}\| + K_\Phi\|\varphi_{\omega_1}'\| \cdot \|\varphi_{\sigma\omega}^n - \varphi_{\sigma\omega}\|,
\end{aligned}$$

where K_Φ is a distortion constant for Φ.

References

[1] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
[2] L. Baribeau and M. Roy, *Analytic multifunctions, holomorphic motions and Hausdorff dimension in IFSs*, Monatsh. Math. **147** (2006), no. 3, 199-217.
[3] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, 1990.
[4] P. Hanus, R.D. Mauldin and M. Urbański, *Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems*, Acta Math. Hungarica **96** (2002), 27–98.
[5] R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. **73** (3) (1996), no. 1, 105–154.
[6] R. D. Mauldin, M. Urbański, *Conformal iterated function systems with applications to the geometry of continued fractions*, Trans. Amer. Math. Soc. **351** (1999), 4995-5025.
[7] R. D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets,* Cambridge 2003.
[8] M. Roy, M. Urbański, *Regularity properties of Hausdorff dimension in infinite conformal IFSs*, Ergod. Th. & Dynam. Sys. **25** (2005), no. 6, 1961–1983.
[9] M. Roy, M. Urbański, *Real analyticity of Hausdorff dimension for higher dimensional graph directed Markov systems*, to appear in Math. Z.

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