Semi-hyperbolic fibered rational maps and rational semigroups *

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Abstract

This paper is based on the author's previous work [S4]. We consider fiber-preserving complex dynamics on fiber bundles whose fibers are Riemann spheres and whose base spaces are compact metric spaces. In this context, without any assumption on (semi-)hyperbolicity, we show that the fiberwise Julia sets are uniformly perfect. From this result, we show that, for any semigroup G generated by a compact family of rational maps on the Riemann sphere $\overline{\mathbb{C}}$ of degree two or greater, the Julia set of any subsemigroup of G is uniformly perfect.

We define the semi-hyperbolicity of dynamics on fiber bundles and show that, if the dynamics on a fiber bundle is semi-hyperbolic, then the fiberwise Julia sets are porous, and the dynamics is weakly rigid. Moreover, we show that if the dynamics is semi-hyperbolic and the fiberwise maps are polynomials, then under some conditions, the fiberwise basins of infinity are John domains. We also show that the Julia set of a rational semigroup (a semigroup generated by rational maps on $\overline{\mathbb{C}}$) that is semi-hyperbolic, except at perhaps finitely many points in the Julia set, and which satisfies the open set condition, is either porous or equal to the closure of the open set. Furthermore, we derive an upper estimate of the Hausdorff dimension of the Julia set.

1 Introduction

To investigate random one-dimensional complex dynamics, the dynamics of semigroups generated by rational maps on the Riemann sphere $\overline{\mathbb{C}}$ and fiber-preserving holomorphic dynamics on fiber bundles that appear in complex dynamics in several dimensions, we consider the dynamics of fibered rational maps; that is, we consider fiber-preserving complex dynamical systems on fiber bundles

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whose fibers are Riemann spheres and whose base spaces are general compact metric spaces. The notion of the dynamics of fibered rational maps, which is a generalized notion of the "dynamics of the fibered polynomial maps" presented by O. Sester([Se1], [Se2], [Se3]), was introduced by M. Jonsson in [J2]. Researches on the dynamics of semigroups generated by rational maps on the Riemann sphere ([HM1], [HM2], [HM3] [GR], [Bo], [St1], [St2], [St3], [SSS], [S1], [S2], [S3], [S4], [S5], [S6], [S7]), the random iterations of rational functions([FS], [BBR], [Br]) and polynomial skew products on $\mathbb{C}^2$ ([H1], [H2], [J1]) are directly related to this subject. For polynomial skew products (the dynamics of fibered polynomials) whose base spaces are general compact metric spaces, see O.Sester's work [Se1], [Se2] and [Se3]. In [Se3], he investigated the quadratic case in detail. In particular, he developed a combinatorial theory for quadratic fibered polynomials and constructed an abstract space of combinatorics. Moreover, he demonstrated the realizability and rigidity of abstract combinatorics.

For the ergodic theory of random diffeomorphisms, see [K].

For the Hausdorff dimension of the Julia set of a (semi-) hyperbolic rational semigroup, see [S4] and [S7].

For dynamics of semi-hyperbolic transcendental semigroups, see [KS1], [KS2].

In this paper, by applying some of the results in [S4] obtained by the author, we show that semi-hyperbolicity along the fibers of a fibered rational map implies that the fiberwise Julia sets are k-porous, where the constant k does not depend on any points of the base space (hence the upper box dimensions of the fiberwise Julia sets are uniformly less than a constant that is strictly less than 2) (**Theorem 1.16**); that the dynamics have a kind of weak rigidity (**Theorem 1.19**); and that if the fiberwise maps are polynomials with some conditions, then the fiberwise basins of infinity are c-John domains, where c is a constant not depending on any points in the base spaces(**Theorem 1.12**). This generalizes a result from [CJY] and a result from [Se3]. These results are stated in Section 1.1, and the proofs are given in Section 2.5. For research on the semi-hyperbolicity of the usual dynamics of rational functions, see [CJY] and [Ma].

In Section 1.2, we present several results on the dynamics of rational semigroups; i.e., on the semigroups generated by rational maps on the Riemann sphere. We show that the Julia set of a finitely generated rational semigroup that is semi-hyperbolic, except at most finitely many points in the Julia set, and that satisfies the "(backward) open set condition" (which is a generalized notion in research on self-similar sets constructed by some similarity transformations) with an open set, is either porous (and hence the upper box dimension is strictly less than 2) or equal to the closure of the open set (**Theorem 1.25**). Furthermore, under the same assumptions, we show that the Hausdorff dimension of the Julia set is less than the infimum of numbers δ, which allows the construction of a "δ-subconformal measure" that is less than the critical exponent of the semigroup (**Theorem 1.28**). The proofs of these results are given in Section 2.6.

A proof of the above results requires the potential theoretic stories (Section 2.3), distortion lemmas for holomorphic proper maps, various results on non-constant limit functions, and the continuity of the fiberwise Julia sets with respect to points in the base spaces (this is an important property and is not easy to show), from [S4]. Note that the upper semicontinuity of fiberwise Julia sets with respect to points in the base spaces does not hold in general.

Furthermore, in Section 1.1, we show that the fiberwise Julia sets of a fibered rational map are C_1-uniformly perfect and that the Hausdorff dimensions are greater than a positive constant C_2, where C_1 and C_2 are constants that do not depend on any points in the base space, if the fiberwise maps are of degree two or greater, even if semi-hyperbolicity is not assumed (**Theorem 1.6**). The proof is given in Section 2.5. This result on uniform perfectness is used to show the result on Johnness. Furthermore, from the uniform perfectness of fiberwise Julia sets, we show that, for any semigroup G generated by a compact family of rational maps on $\overline{\mathbb{C}}$ of degree two or greater, there exists a positive constant C such that the Julia set of any subsemigroup of G is C-uniformly perfect (**Theorem 1.21**). The proof, which is given in Section 2.6, is based on combining the density of repelling fixed points in the Julia set (an application of Ahlfors's five-island theorem) with the potential theory in Section 2.3.

1.1 Results on fibered rational maps

In this section, we state the results on fibered rational maps. The proofs are given in Section 2.5. First, we provide some notation and definitions regarding the dynamics of fibered rational maps.

Definition 1.1. ([J2]) A triplet (π, Y, X) is called a "$\overline{\mathbb{C}}$-bundle" if

1. Y and X are compact metric spaces,
2. $\pi : Y \to X$ is a continuous and surjective map,
3. and there exists an open covering $\{U_i\}$ of X such that, for each i, there exists a homeomorphism $\Phi_i : U_i \times \overline{\mathbb{C}} \to \pi^{-1}(U_i)$ such that $\Phi_i(\{x\} \times \overline{\mathbb{C}}) = \pi^{-1}\{x\}$ and $\Phi_j^{-1} \circ \Phi_i : \{x\} \times \overline{\mathbb{C}} \to \{x\} \times \overline{\mathbb{C}}$ is a Möbius map for each $x \in U_i \cap U_j$, under the identification $\{x\} \times \overline{\mathbb{C}} \cong \overline{\mathbb{C}}$.

Remark 1. By condition 3, each fiber $Y_x := \pi^{-1}\{x\}$ has a complex structure. Furthermore, given $x_0 \in X$, one may find a continuous family $i_x : \overline{\mathbb{C}} \to Y_x$ of homeomorphisms, for x close to x_0. Such a family $\{i_x\}$ is called a "local parameterization". Since X is compact, we may assume throughout this paper that there exists a compact subset M_0 of the set of Möbius transformations of $\overline{\mathbb{C}}$ such that $i_x \circ j_x^{-1} \in M_0$ for any two local parameterizations $\{i_x\}$ and $\{j_x\}$.

Moreover, throughout this paper, we assume the following condition:

- there exists a smooth $(1,1)$-form $\omega_x > 0$ inducing the distance on Y_x from Y, and $x \mapsto \omega_x$ is continuous. That is, if $\{i_x\}$ is a local parameterization, then the pull back $i_x^* \omega_x$ is a positive smooth form on $\overline{\mathbb{C}}$ that depends continuously on x with respect to the C^∞ topology.

Notation: If $(\pi, Y = X \times \overline{\mathbb{C}},\ X)$ is a trivial $\overline{\mathbb{C}}$-bundle, then we denote by $\pi_{\overline{\mathbb{C}}} : Y \to \overline{\mathbb{C}}$ the second projection.

Definition 1.2. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ and $g : X \to X$ be continuous maps. Let f be called a fibered rational map over g (or a rational map fibered over g), if

1. $\pi \circ f = g \circ \pi$

2. $f|_{Y_x} : Y_x \to Y_{g(x)}$ is a rational map, for any $x \in X$. That is, $(i_{g_x})^{-1} \circ f \circ i_x$ is a rational map from $\overline{\mathbb{C}}$ to itself, for any local parameterization i_x at $x \in X$ and $i_{g(x)}$ at $g(x)$.

Notation: If $f : Y \to Y$ is a fibered rational map over $g : X \to X$, then we set $f_x^n = f^n|_{Y_x} : Y_x \to Y_{g^n(x)}$, for any $x \in X$ and $n \in \mathbb{N}$. Moreover, we set $f_x = f|_{Y_x} : Y_x \to Y_{g(x)}$. Furthermore, we set $d_n(x) = \deg(f_x^n)$ and $d(x) = d_1(x)$, for any $x \in X$ and $n \in \mathbb{N}$. Note that the definition of fibered rational map implies that the function $x \mapsto d(x)$ is continuous on X.

Definition 1.3. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Then, for any $x \in X$, we denote by $F_x(f)$(or simply F_x) the set of points $y \in Y_x$ that has a neighborhood U of y in Y_x such that $\{f_x^n\}_{n\in\mathbb{N}}$ is a normal family in U; that is, $y \in F_x$ if and only if the family $Q_x^n = i_{x_n}^{-1} \circ f_x^n \circ i_x$ of rational maps on $\overline{\mathbb{C}}$ ($x_n := g^n(x)$) is normal near $i_x^{-1}(y)$. Note that, by Remark 1, this does not depend on the choices of local parameterizations at x and x_n. Equivalently, F_x is the open subset of Y_x, where the family $\{f_x^n\}$ of mappings from Y_x into Y is locally equicontinuous. We set $J_x(f)$(or simply J_x)$= Y_x \setminus F_x$.

Furthermore, we set $\tilde{J}(f) = \overline{\bigcup_{x\in X} J_x}$, where the closure is taken in the space Y, $\tilde{F}(f) = Y \setminus \tilde{J}(f)$, and $\hat{J}_x(f)$(or simply $\hat{J}_x$)$= \tilde{J}(f) \cap Y_x$, for each $x \in X$.

Remark 2. There exists a fibered rational map $f : Y \to Y$ such that $\bigcup_{x\in X} J_x$ is NOT compact.

Definition 1.4. ([S4]) Let $h_1, \dots, h_m$ be non-constant rational maps. Let $\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$ be the space of one-sided infinite sequences of m symbols, and $\sigma : \Sigma_m \to \Sigma_m$ be the shift map; that is, σ is defined by $\sigma((w_1, w_2, \dots)) = (w_2, w_3, \dots)$. Let X be a compact subset of Σ_m such that $\sigma(X) \subset X$. Let $Y = X \times \overline{\mathbb{C}}$ and $\pi : Y \to X$ be the natural projection. Then (π, Y, X) is a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a map defined by: $f((w, y)) = (\sigma(w),\ h_{w_1}(y))$, where $w = (w_1, w_2, \dots)$. Then, $f : Y \to Y$ is a fibered rational map over $\sigma : X \to X$.

In the above, if $X = \Sigma_m$, then $f : Y \to Y$ is called the **fibered rational map associated with the generator system** $\{h_1, \dots h_m\}$ (of the rational semigroup $G = \langle h_1, \dots, h_m\rangle$). (See Section 1.2 for the definition of rational semigroups.)

Now we present a result on uniform perfectness.

Definition 1.5. Let C be a positive number. Let K be a closed subset of $\overline{\mathbb{C}}$. We say that K is **C-uniformly perfect** if, $\sharp K \geq 2$ and for any doubly connected domain A in $\overline{\mathbb{C}}$ such that A separates K; i.e., such that $A \subset \overline{\mathbb{C}} \setminus K$ and both of the two connected components of $\overline{\mathbb{C}} \setminus A$ have non-empty intersections with K, mod A (the modulus of A; for the definition, see [LV]) is less than C.

Theorem 1.6. (**Uniform perfectness**) *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$ such that $d(x) \geq 2$, for any $x \in X$. (In this theorem, we do NOT assume semi-hyperbolicity.) Then, we have the following.*

1. *There exists a positive constant C such that, for any $x \in X$, J_x and $\hat{J}_x$ are C-uniformly perfect. Furthermore, there exists a positive constant C_1 such that for each $x \in X$, we have diam $J_x \geq C_1$, with respect to the*

distance in Y_x induced by ω_x. Moreover, there exists a positive constant C_2 such that, for each $x \in X$, the Hausdorff dimension $\dim_H(J_x)$ of J_x, with respect to the distance on Y_x induced by ω_x, satisfies the condition that $\dim_H(J_x) \geq C_2$. (Note that C_1 and C_2 do not depend on $x \in X$.)

2. *Suppose further that $f(\tilde{F}(f)) \subset \tilde{F}(f)$(for example, assume that $g : X \to X$ is an open map). If a point $z \in Y$ satisfies $f^n_{\pi(z)}(z) = z$ and $(f^n_{\pi(z)})'(z) = 0$ for some $n \in \mathbb{N}$ and $z \in \hat{J}_{\pi(z)}$, then z belongs to the interior of $\hat{J}_{\pi(z)}$ with respect to the topology of $Y_{\pi(z)}$.*

Theorem 1.6 is now used to show a result on Johnness (Theorem 1.12) and Theorem 1.21.

Example 1.7. Let $z_0 \in \overline{\mathbb{C}}$ be a point. Let h_1 and h_2 be two rational maps on $\overline{\mathbb{C}}$ of degree two or greater. Let $f : \Sigma_2 \times \overline{\mathbb{C}} \to \Sigma_2 \times \overline{\mathbb{C}}$ be the fibered rational map associated with the generator system $\{h_1, h_2\}$. Suppose that z_0 is a superattracting fixed point of h_1 and is a repelling fixed point of h_2. Then, it can easily be seen that $z_0 \in \hat{J}_x$, where $x = (1, 1, \dots) \in \Sigma_2$. Since the shift map $\sigma : \Sigma_2 \to \Sigma_2$ is an open map, by Theorem 1.6 it follows that z_0 belongs to the interior of $\hat{J}_x$.

Remark 3. Theorem 1.6 generalizes Theorem 3.1 in [Br].

Remark 4. Uniform perfectness implies many useful properties ([BP],[Po],[Su]). This terminology was introduced in [Po]. For a survey on uniform perfectness, see [Su]. We now consider the following:

1. In [BP], it was shown that a closed subset K of $\overline{\mathbb{C}}$ with $\sharp K \geq 3$ is C-uniformly perfect if and only if there exists a constant δ such that, for any component U of $\overline{\mathbb{C}} \setminus K$,
$$\lambda_U(z) > \delta / \mathrm{dist}(z, \partial U), \tag{1}$$
where $\lambda_U(z)$ denotes the density of the hyperbolic metric of U at z and $\mathrm{dist}(z, \partial U)$ denotes the Euclidian distance of the point z from the set ∂U. (If K is bounded and U is the unbounded component of $\overline{\mathbb{C}} \setminus K$, then inequality (1) will hold only for all $z \in U$ sufficiently close to K.) In the above discussion, δ depends only on C, and C depends only on δ. This fact will be used to show Theorem 1.12. Detailed inequalities regarding the relationships among δ, C and other invariants are presented in [Su].

2. If a closed subset K of $\overline{\mathbb{C}}$ is C-uniformly perfect, then the Hausdorff dimension $\dim_H(K)$ of K with respect to the spherical metric satisfies $\dim_H(K) \geq C' > 0$, where C' is a positive constant that depends only on C (Theorem 7.2 in [Su]).

Notation :

- Let Z_1 and Z_2 be two topological spaces and $g : Z_1 \to Z_2$ be a map. For any subset A of Z_2, We denote by $c(A, g)$ the set of all connected components of $g^{-1}(A)$.
- For any $y \in \overline{\mathbb{C}}$ and $\delta > 0$, we set $B(y, \delta) = \{y' \in \overline{\mathbb{C}} \mid d(y, y') < \delta\}$, where d is the spherical distance. Similarly, for any $y \in \mathbb{C}$ and $\delta > 0$, we set $D(y, \delta) = \{y' \in \mathbb{C} \mid |y - y'| < \delta\}$.

- Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. For any $y \in Y$ and $r > 0$, we set $\tilde{B}(y, r) = \{y' \in Y_{\pi(y)} \mid d_{\pi(y)}(y', y) < r\}$, where, for each $x \in X$, we denote by d_x the distance on Y_x induced by the form ω_x.

We now define the semi-hyperbolicity of fibered rational maps.

Definition 1.8. (**semi-hyperbolicity of fibered rational maps**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Let $N \in \mathbb{N}$. We denote by $SH_N(f)$ the set of points $z \in Y$ such that there exists a positive number δ, a neighborhood U of $\pi(z)$, and a local parameterization $\{i_x\}$ in U such that, for any $x \in U$, any $n \in \mathbb{N}$, any $x_n \in g^{-n}(x)$ and any $V \in c(i_x(B(i_{\pi(z)}^{-1}(z),\ \delta)),\ f_{x_n}^n)$, we have that

$$\deg(f_{x_n}^n : V \to i_x(B(i_{\pi(z)}^{-1}(z),\ \delta))) \le N.$$

We set $UH(f) = Y \setminus \bigcup_{N \in \mathbb{N}} SH_N(f)$. A point $z \in SH_N(f)$ is called a **semi-hyperbolic point of degree** N. Furthermore, we say that f is **semi-hyperbolic (along fibers)** if $\tilde{J}(f) \subset \bigcup_{N \in \mathbb{N}} SH_N(f)$. This is equivalent to $\tilde{J}(f) \subset SH_N(f)$, for some $N \in \mathbb{N}$.

Definition 1.9. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Then, we set $C(f) := \bigcup_{x \in X} \{\text{critical points of } f_x\}$ $(\subset Y)$. This is called the **fiber critical set** of fibered rational map f. Furthermore, we set $P(f) = \overline{\bigcup_{n \in \mathbb{N}} \bigcup_{x \in X} f_x^n(\text{ critical points of } f_x)}$ where the closure is taken in Y. This is called the **fiber postcritical set** of the fibered rational map f. Furthermore, we say that $f : Y \to Y$ is **hyperbolic along fibers** if $P(f) \subset F(f)$. (Then f is semi-hyperbolic along fibers with the constant $N = 1$.)

We now present a result on the John condition.

Notation: Let $y \in \mathbb{C}$ and $b \in \overline{\mathbb{C}}$ be two distinct points. Furthermore, let E be a curve in $\overline{\mathbb{C}}$ joining y to b such that $E \setminus \{b\} \subset \mathbb{C}$. Then, for any $c \ge 1$, we set

$$\text{car }(E, c, y, b) = \bigcup_{z \in E \setminus \{y, b\}} D(z,\ \frac{|y - z|}{c}).$$

This is called the c-carrot with core E and vertex y joining y to b.

Definition 1.10. Let V be a subdomain of $\overline{\mathbb{C}}$ such that $\infty \in \overline{\mathbb{C}} \setminus \partial V$. Let $c \ge 1$ be a number. Then, we say that V is a **c-John domain** if there exists a point $y_0 \in \overline{V}$ such that, for any $y \in V \setminus \{y_0\}$, there exists a curve E joining y_0 to y such that $E \setminus \{y_0\} \subset \mathbb{C}$ and car $(E, c, y, y_0) \subset V$. In what precedes, the point y_0 is called the center of the John domain V.

Definition 1.11. Let $(\pi,\ Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. A fibered rational map $f : Y \to Y$ over $g : X \to X$ is called a **fibered polynomial map over** $g : X \to X$ if f is of the form: $f(x, y) = (g(x), q_x(y))$ for $(x, y) \in X \times \overline{\mathbb{C}}$, where q_x is a polynomial for each $x \in X$. For such an f, we set $A_x(f) := \{y \in Y_x \mid \pi_{\overline{\mathbb{C}}}(f_x^n(y)) \to \infty\}$.

Theorem 1.12. *(***Johnness***) Let $(\pi, Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a semi-hyperbolic fibered polynomial map over $g : X \to X$ such that $d(x) \ge 2$, for any $x \in X$. Assume that either*

1. *$J_x(f)$ is connected for each $x \in X$, or*

2. *the map $x \mapsto f_x$ from X to the space of polynomials is locally constant (here, we identify f_x with a polynomial on $\overline{\mathbb{C}}$) and* $\inf\{d(a,b) \mid a \in \pi_{\overline{\mathbb{C}}}(J_x),\ b \in \pi_{\overline{\mathbb{C}}}(A_x(f) \cap C(f)),\ x \in X\} > 0.$

Then, there exists a positive constant c such that, for any $x \in X$, the basin of infinity $A_x(f)$ is a c-John domain.

Remark 5.
- For a fibered polynomial map f such that $d(x) \geq 2$ for each $x \in X$, we have that $J_x(f)$ is connected for each $x \in X$ if and only if $\pi_{\overline{\mathbb{C}}}(P(f)) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.
- In Theorem 1.12, if X is a set consisting of one point, then f is semi-hyperbolic if and only if the basin of infinity is a John domain([CJY]). In [Se3], O. Sester made the same assertion as that presented in Theorem 1.12 for "non-recurrent quadratic fibered polynomials" with connected fiber-wise Julia sets.

Remark 6. Johnness likewise implies many useful properties ([NV], [Jone]).For example, if V is a John domain, then the following facts hold.

- If $\infty \in V$, then the center of V is ∞.
- Let $a \in \partial V$ and $b \in V$. Then there exists a curve E joining a to b and a constant c such that car $(E, c, a, b) \subset V$. In particular, a is accessible from b.
- V is finitely connected at any point in ∂V: that is, if $y \in \partial V$, then there exists an arbitrarily small open neighborhood U of y in $\overline{\mathbb{C}}$ such that $U \cap V$ has only finitely many connected components.
- If V is simply connected, then it follows that ∂V is locally connected.
- ∂V is holomorphically removable: that is, if $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism and is holomorphic on $\overline{\mathbb{C}} \setminus \partial V$, then φ is holomorphic on $\overline{\mathbb{C}}$. From this fact, we deduce that the two-dimensional Lebesgue measure of ∂V is equal to zero.

In order to present results on porosity and other properties, let us first establish some technical conditions.

Definition 1.13 (Condition(C1)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. We say that f satisfies condition (C1) if there exists a family $\{D_x\}_{x \in X}$ of topological disks with $D_x \subset Y_x,\ x \in X$ such that the following conditions are satisfied:

1. for each $x \in X$, there exists a point $z_x \in Y_x$ and a positive number r_x such that $D_x = \tilde{B}(z_x, r_x)$,
2. $\overline{\bigcup_{x \in X} \bigcup_{n \geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$,
3. for any $x \in X$, it holds that $\operatorname{diam}(f_x^n(D_x)) \to 0$, as $n \to \infty$, and
4. $\inf_{x \in X} r_x > 0$.

Definition 1.14 (Condition(C2)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. We say that f satisfies condition (C2) if, for each $x_0 \in X$, there exists an open neighborhood O of x_0 and a family $\{D_x\}_{x \in O}$ of topological disks with $D_x \subset Y_x, x \in O$ such that the following conditions are satisfied:

1. for each $x \in O$, there exists a point $z_x \in Y_x$ and a positive number r_x such that $D_x = \tilde{B}(z_x, r_x)$,
2. $\overline{\bigcup_{x \in O} \bigcup_{n \geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$,
3. for any $x \in O$, it holds that $\mathrm{diam}(f_x^n(D_x)) \to 0$, as $n \to \infty$, and
4. $x \mapsto D_x$ is continuous in O.

Remark 7.

1. Let $(\pi,\ Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered polynomial map over $g : X \to X$ such that $d(x) \geq 2$ for each $x \in X$. Then, setting $D_x = D$, where D is a small neighborhood of infinity, for each $x \in X$, the fibered rational map f satisfies condition (C2) with the family of disks $(D_x)_{x \in X}$.
2. Let $\{h_1, \ldots h_m\}$ be non-constant rational functions on $\overline{\mathbb{C}}$ with $\deg(h_1) \geq 2$. Let $f : Y \to Y$ be the fibered rational map associated with the generator system $\{h_1, \ldots, h_m\}$ of rational semigroup $G = \langle h_1, \ldots, h_m \rangle$ (See Section 1.2). Suppose that f is semi-hyperbolic along fibers and that $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$ (See Section 1.2 and [S5]) is not equal to the Riemann sphere. Moreover, suppose that each element of Aut $(\overline{\mathbb{C}}) \cap G$ (if this is not empty) is loxodromic. Then, it follows that f satisfies condition (C2). Actually, there exists an attracting fixed point a of some element of G in $F(G)$. Since G is semi-hyperbolic, if $D_x = D(a, \epsilon)$, for each $x \in \Sigma_m$ where ϵ is a positive number, then f satisfies condition (C2) with the family of disks $(D_x)_{x \in \Sigma_m}$. For more details, see Theorem 1.35 and Remark 5 in [S4].

We now present a result on porosity.

Definition 1.15. Let (Y, d) be a metric space. Let k be a constant such that $0 < k < 1$. Let J be a subset of Y. We say that J is k**-porous** if, for any $x \in J$ and any positive number r, there exists a ball in $\{y \in Y \mid d(y, x) < r\} \setminus J$ with radius at least kr.

Remark 8. If Y is the Euclidean space $\mathbb{R}^n$ and d is the Euclidean metric, the upper box dimension of any k-porous bounded set J in $\mathbb{R}^n$ is less than $n - c(k, n)$, where $c(k, n)$ is a positive constant that depends only on k and n (Proposition 4.2 and the proof in [PR]).

Theorem 1.16. *(***Porosity***) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that f satisfies condition (C1) and that f is semi-hyperbolic. Then there exists a constant $0 < k < 1$ such that, for each $x \in X$, J_x is k-porous in Y_x. In particular, there exists a constant $0 \leq c < 2$ such that, for each $x \in X$, $\dim_H(J_x) \leq \overline{\dim}_B(J_x) \leq c$, where $\dim_H$ denotes the Hausdorff dimension and $\overline{\dim}_B$ denotes the upper box dimension with respect to the metric on Y_x induced by ω_x (ω_x is of the form in Remark 1).*

We now present a sufficient condition for $\tilde{J} = \bigcup_{x \in X} J_x$ and the complement of the backward images of non-semi-hyperbolic points in fiberwise Julia sets to be of measure zero.

Definition 1.17. (**Good points for fibered rational maps**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. We set $\tilde{J}_{good}(f) = \{z \in \tilde{J}(f) \mid \limsup_{n\to\infty} d(f^n(z), UH(f)) > 0\}$.

Theorem 1.18. *(***Measure zero***) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that all of the following conditions hold:*

1. *f satisfies condition (C1);*
2. *for each $x \in X$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x (i.e. $Y_x \setminus \partial(\hat{J}_x(f) \cap UH(f))$ is connected);*
3. *$\tilde{J}(f) \setminus \bigcup_{n\in N} f^{-n}(UH(f)) \subset \tilde{J}_{good}(f)$; and*
4. *for each $z \in \tilde{J}(f) \cap UH(f)$ and each open neighborhood V of z in $Y_{\pi(z)}$, the diameter of $f^n_{\pi(z)}(V)$ does not tend to zero as $n \to \infty$.*

Then, $\tilde{J}(f) = \bigcup_{x\in X} J_x$ and, for each $x \in X$, the two-dimensional Lebesgue measure of $J_x \setminus \bigcup_{n\in N} f^{-n}(UH(f))$ is equal to zero.

The statement $\tilde{J}(f) = \bigcup_{x\in X} J_x$ in Theorem 1.18 is used to show Theorem 1.25.

We now present a result on rigidity.

Theorem 1.19. *(***A rigidity***) Let (π, Y, X) and $(\tilde{\pi}, \tilde{Y}, \tilde{X})$ be two $\overline{\mathbb{C}}$-bundles. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$ and $\tilde{f} : \tilde{Y} \to \tilde{Y}$ be a fibered rational map over $\tilde{g} : \tilde{X} \to \tilde{X}$. Let $u : Y \to \tilde{Y}$ be a homeomorphism that is a bundle conjugacy between f and $\tilde{f}$; i.e., u satisfies the condition that $\tilde{\pi} \circ u = v \circ \pi$ for some homeomorphism $v : X \to X$ and $\tilde{f} \circ u = u \circ f$. Suppose that f is semi-hyperbolic along fibers and satisfies condition (C1). For each $w \in X$, let $u_w : Y_w \to \tilde{Y}_{v(w)}$ be the restriction of u. Let $x \in X$ be a point and assume that u_x is K-quasiconformal on F_x. Then, for each $a \in \overline{\cup_{n\in\mathbb{Z}} g^n(\{x\})}$, it follows that $u_a : Y_a \to \tilde{Y}_{v(a)}$ is K-quasiconformal on the whole Y_a, with the same dilatation constant K.*

1.2 Results on rational semigroups

In this section, we present several results on rational semigroups. The proofs are given in Section 2.6. Before stating results, we will first establish some notation and definitions regarding the dynamics of rational semigroups.

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup whose semigroup operation is the functional composition. A **rational semigroup** is a subsemigroup of $\mathrm{End}(\overline{\mathbb{C}})$ without any constant elements. We say that a rational semigroup G is a **polynomial semigroup** if each element of G is a polynomial. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([HM1]), who were interested in the role that the dynamics of polynomial semigroups plays in research on various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group([GR]).

In this paper, a rational semigroup G may not be finitely generated, in general.

Definition 1.20. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \ \ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** of G, and $J(G)$ is called the **Julia set** of G. The backward orbit $G^{-1}(z)$ of z and the **set of exceptional points** $E(G)$ are defined by: $G^{-1}(z) = \cup_{g\in G} g^{-1}\{z\}$ and $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) \leq 2\}$. Furthermore, we set $G(z) := \cup_{g\in G}\{g(z)\}$. For any subset A of $\overline{\mathbb{C}}$, set $G^{-1}(A) = \cup_{g\in G} g^{-1}(A)$. We denote by $\langle h_1, h_2, \dots \rangle$ the rational semigroup generated by the family $\{h_i\}$. For a rational map g, we denote by $J(g)$ the Julia set of the dynamics of g.

We now present a result on uniform perfectness. In the following theorem, we do NOT assume semi-hyperbolicity.

Theorem 1.21. *(***Uniform perfectness***) Let Λ be a compact set in the space $\{h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid h : \text{holomorphic}, \ \deg(h) \geq 2\}$ endowed with topology induced by uniform convergence on $\overline{\mathbb{C}}$. Let G be a rational semigroup generated by the set Λ. Then there exists a positive constant C such that each subsemigroup H of G satisfies the condition that $J(H)$ is C-uniformly perfect. Furthermore, for any subsemigroup H of G, if a point $z_0 \in J(H)$ satisfies the condition that there exists an element $h \in H$ such that $h(z_0) = z_0$ and $h'(z_0) = 0$, then it follows that $z_0 \in$int $J(H)$.*

Example 1.22. Let $G = \langle h_1, h_2 \rangle$ where $h_1(z) = 2z + z^2$ and $h_2(z) = \frac{z^3}{z-a}$, $a \in \mathbb{C}$ with $a \neq 0$. Then, $0 \in J(G)$ and 0 is a superattracting fixed point of h_2. Hence, $0 \in$int $J(G)$, by Theorem 1.21. Since ∞ is a common attracting fixed point of h_1 and h_2, we have $\infty \in F(G)$. Furthermore, let H be a subsemigroup of G such that $0 \in J(H)$ and $h_2 \in H$. Then, by Theorem 1.21 again, we have $0 \in$int $J(H)$. Moreover, we have $\infty \in F(H)$.

In particular, let H_0 be a subsemigroup of G that is generated by $G \setminus \langle h_1 \rangle$. Then we have all of the following:

1. $0 \in \text{int } J(H_0)$.
2. $\infty \in F(H_0)$.
3. For any finitely generated subsemigroup H_1 of H_0, we have $0 \in F(H_1)$.

For, since $0 \in$int $J(G)$, $J(G) = \overline{\cup_{g\in G} J(g)}$ (Corollary 3.1 in [HM1]), and $J(h_1)$ is nowhere dense, we obtain that there exists a suquence (g_n) in H_0 such that $d(0, J(g_n)) \to 0$ as $n \to \infty$. Hence, $0 \in J(H_0)$. Since 0 is a superattracting fixed point of h_2 and $h_2 \in H_0$, by Theorem 1.21 we obtain $0 \in \text{int } J(H_0)$. Since $H_0 \subset G$ and $\infty \in F(G)$, we obtain $\infty \in F(H_0)$. For any finitely generated subsemigroup H_1 of H_0, since 0 is a common attracting fixed point of any element of H_0, it follows that $0 \in F(H_1)$.

Remark 9. In [HM2], by A. Hinkkanen and G. Martin, it was shown that a finitely generated rational semigroup G such that each $g \in G$ is of degree two or greater satisfies the condition that $J(G)$ is uniformly perfect. In [St3], by R. Stankewitz, it was shown that, if Λ (this is allowed to have an element of degree one) is a family of rational maps on $\overline{\mathbb{C}}$ such that the Lipschitz constant of each element of Λ with respect to the spherical metric on $\overline{\mathbb{C}}$ is uniformly bounded, then the Julia set of semigroup G generated by Λ is uniformly perfect. However, there has been no research done on the uniform perfectness of the Julia sets of subsemigroups of such a semigroup. In [HM2] and [St3], the proofs were based on the density of repelling fixed points in the Julia sets (which was shown by an application of Ahlfors's five-island theorem), whereas, in this paper, the proof of Theorem 1.21 is based on the combination of the density of repelling fixed points in the Julia set with Proposition 2.2 (potential theory).

We now define the semi-hyperbolicity of rational semigroups.

Definition 1.23. (**Semi-hyperbolicity of rational semigroups**) Let G be a rational semigroup and N a positive integer. We denote by $SH_N(G)$ the set of points $z \in \overline{\mathbb{C}}$ such that there exists a positive number δ such that, for any $g \in G$ and any $V \in c(B(z,\delta),\ g)$, it holds that $\deg(g : V \to B(z,\delta)) \leq N$. Furthermore, we set $UH(G) = \overline{\mathbb{C}} \setminus (\cup_{N\in\mathbb{N}} SH_N(G))$. A point $z \in SH_N(G)$ is called a **semi-hyperbolic point of degree** N (for G). We say that G is **semi-hyperbolic** if $J(G) \subset \bigcup_{N\in\mathbb{N}} SH_N(G)$. This is equivalent to $J(G) \subset SH_N(G)$, for some $N \in \mathbb{N}$.

Furthermore, for a rational semigroup G, we set $P(G) := \overline{\bigcup_{g\in G}\{\text{critical values of } g\}}$ $(\subset \overline{\mathbb{C}})$. This is called the **postcritical set** of G. We say that G is **hyperbolic** if $P(G) \subset F(G)$. Note that if G is hyperbolic, then G is semi-hyperbolic. Note also that if $G = \langle h_1, \dots, h_m\rangle$, then $P(G) = \overline{\bigcup_{g\in G} g\left(\bigcup_{j=1}^m\{\text{ critical values of } h_j\}\right)}$.

Definition 1.24. Let $G = \langle h_1, h_2, \dots, h_m\rangle$ be a finitely generated rational semigroup. Let U be a non-empty open set in $\overline{\mathbb{C}}$. We say that G satisfies the **open set condition** with U with respect to the generator systems $\{h_1, h_2, \dots, h_m\}$, if for each $j = 1, \dots, m$, $h_j^{-1}(U) \subset U$ and $\{h_j^{-1}(U)\}_{j=1,\dots,m}$ are mutually disjoint.

We now present a result on porosity.

Theorem 1.25. *(***Porosity***) Let $G = \langle h_1, \dots, h_m\rangle$ be a rational semigroup with an element of degree two or greater. Suppose that all of the following conditions hold:*

1. *G satisfies the open set condition with an open set U with respect to the generator system $\{f_1, \dots, f_m\}$,*
2. *$\sharp(UH(G) \cap J(G)) < \infty$ and*
3. *$UH(G) \cap J(G) \subset U$.*

Then it follows that $J(G) = \overline{U}$, or that $J(G)$ is porous (and so the upper box dimension of $J(G)$ is strictly less than 2). Moreover, the fibered rational map $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ associated with the generator system $\{h_1, \dots, h_m\}$ satisfies $\tilde{J}(f) = \bigcup_{x\in\Sigma_m} J_x$.

We now present results regarding the Hausdorff dimension of Julia sets.

Definition 1.26. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is **δ-subconformal** (for G) if, for each $g \in G$ and for each Borel-measurable set A, it holds that $\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu$, where we denote by $\| \cdot \|$ the norm of the derivative with respect to the spherical metric. We set $s(G) = \inf\{\delta \mid \exists \mu : \delta\text{-subconformal measure}\}$.

Let G be a countable rational semigroup. For each $x \in \overline{\mathbb{C}}$ and each real number s, we set $S(s,x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$, counting multiplicities and $S(x) = \inf\{s \mid S(s,x) < \infty\}$. If there is no s such that $S(s,x) < \infty$, then we set $S(x) = \infty$. Also, we set $s_0(G) = \inf\{S(x) \mid x \in \overline{\mathbb{C}}\}$.

An estimate on $s_0(G)$ exists when G satisfies the open set condition.

Proposition 1.27. *Let $G = \langle h_1, \ldots, h_m \rangle$ be a rational semigroup. When $m = 1$, assume that h_1 is neither the identity nor an elliptic Möbius transformation. Suppose that G satisfies the open set condition with an open set U with respect to the generator system $\{h_1, \ldots, h_m\}$. Suppose also that $J(G) \neq \overline{U}$. Then there exists an open set V included in $U \cap F(G)$ such that, for almost every $x \in V$ with respect to the two-dimensional Lebesgue measure, $S(2,x) < \infty$. In particular, $s_0(G) \leq 2$.*

Theorem 1.28. *(**Hausdorff dimension**) Let $G = \langle h_1, \ldots, h_m \rangle$ be a rational semigroup. When $m = 1$, we assume $J(G) \neq \overline{\mathbb{C}}$. Under the same assumptions as those of Theorem 1.25, it follows that $\dim_H(J(G)) \leq s(G) \leq s_0(G)$, where $\dim_H$ denotes the Hausdorff dimension with respect to the spherical metric in $\overline{\mathbb{C}}$.*

Example 1.29. Let $h_1(z) = z^2 + 2$, $h_2(z) = z^2 - 2$ and $U = \{|z| < 2.\}$. Then, $h_1^{-1}(U) \cup h_2^{-1}(U) \subset U$ and $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Let h_3 be a polynomial that is conjugate to h_4^n by an affine map α, where $h_4(z) = z^2 + \frac{1}{4}$ and $n \in \mathbb{N}$ is a sufficiently large number. For an appropriate α, it holds that $J(h_3) \subset U \setminus (h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}))$. For sufficiently large n, $h_3^{-1}(\overline{U}) \subset U \setminus (h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}))$. Then $G = \langle h_1, h_2, h_3 \rangle$ satisfies the conditions specified as assumptions in Theorem 1.25. In this case, $UH(G) \cap J(G)$ is one point that is a parabolic fixed point a of h_3 (For, $UH(G) \cap J(G)$ is included in $P(G) \cap J(G) = \{-2, 2, a\}$, and, by Lemma 2.14, it follows that $\{-2, 2\} \subset \overline{\mathbb{C}} \setminus (UH(G) \cap J(G))$) and $UH(G) \cap J(G) \subset U$. By Theorem 1.25, it can be seen that $J(G)$ is porous and that, in particular, the upper box dimension is strictly less than two. Furthermore, by Theorem 1.28 we obtain that $\dim_H(J(G)) \leq s(G) \leq s_0(G)$.

1.3 Remark

We now make some remarks about fibered rational maps and rational semigroups.

Remark 10.

1. Dabija [D] showed that (almost) every holomorphic selfmap f of a ruled surface Y is a fibered rational map over a holomorphic map $g : X \to X$, where X is a Riemann surface.

2. Let $p(x) \in \mathbb{C}[x]$ be a polynomial of degree two or greater, and $q(x,y) \in \mathbb{C}[x,y]$ be a polynomial of the form: $q(x,y) = y^n + a_1(x)y^{n-1} + \cdots$. Let

$f : \mathbb{C}^2 \to \mathbb{C}^2$ be a map defined by $f((x, y)) = (p(x), q(x, y))$. This is called a polynomial skew product in $\mathbb{C}^2$. Dynamics of maps of this form were investigated by S.-M. Heinemann in [H1] and [H2], and by M. Jonsson in [J1]. Let X be a compact subset of $\mathbb{C}$ such that $p(X) \subset X$. (e.g., the Julia set of p.) Let $(\pi, Y = X \times \overline{\mathbb{C}}, X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Then the map $\tilde{f} : Y \to Y$ defined by $\tilde{f}((x, y)) = (p(x), q(x, y))$ is a fibered rational map over $p : X \to X$.

3. In Corollary 6.7 of [Se3], O. Sester showed that any "non-recurrent quadratic fibered polynomials" with connected fiberwise Julia sets are semi-hyperbolic.

4. Let $\{h_1, \dots , h_m\}$ be non-constant rational functions on $\overline{\mathbb{C}}$. Let $f : Y \to Y$ be the skew product map associated with the generator system $\{h_1, \dots , h_m\}$. It can be shown by means of simple arguments that $f : Y \to Y$ is semi-hyperbolic along fibers if and only if $G = \langle h_1, \dots , h_m\rangle$ is semi-hyperbolic.

5. In [S4], with G a finitely generated rational semigroup, a sufficient condition for the semi-hyperbolicity of a point $z \in J(G)$ was given, which gives a generalization of R. Mañé's work([Ma]). Furthermore, in [S4], a necessary and sufficient condition for the semi-hyperbolicity of a finitely generated rational semigroup was given. From this, it can be seen that $G = \langle z^2 + 2, z^2 - 2\rangle$ is semi-hyperbolic, and is not hyperbolic.

2 Tools and Proofs

We now present the proofs of the main results, meanwhile providing further notation and tools.

2.1 Fundamental properties of fibered rational maps

By means of definitions, the following lemma can be easily shown.

Lemma 2.1. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Then,*

1. *For each $x \in X$, $f_x^{-1}(F_{g(x)}) = F_x$, $f_x^{-1}(J_{g(x)}) = J_x$. Furthermore, $f(\tilde{J}(f)) \subset \tilde{J}(f)$.*
2. *If $g : X \to X$ is an open map, then $f^{-1}(\tilde{J}(f)) = \tilde{J}(f)$ and $f(\tilde{F}(f)) \subset \tilde{F}(f)$.*
3. *If $g : X \to X$ is a surjective and open map, then $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$ and $f^{-1}(\tilde{F}(f)) = \tilde{F}(f) = f(\tilde{F}(f))$.*

Proof. This proof is the same as that for Lemma 2.4 in [S4]. $\square$

2.2 Fundamental properties of rational semigroups

For a rational semigroup G, for each $f \in G$, it holds that $f(F(G)) \subset F(G)$ and $f^{-1}(J(G)) \subset J(G)$. Note that equality $f^{-1}(J(G)) = J(G)$ does not hold in general. If $\sharp J(G) \geq 3$, then $J(G)$ is a perfect set, $\sharp E(G) \leq 2$, $J(G)$ is the smallest closed backward invariant set containing at least three points, and $J(G)$

is the closure of the union of all repelling fixed points of the elements of G, which implies that $J(G) = \overline{\bigcup_{g\in G} J(g)}$. If a point z is not in $E(G)$, then, for every $x \in J(G)$, $x \in \overline{G^{-1}(z)}$. In particular, if $z \in J(G) \setminus E(G)$ then $\overline{G^{-1}(z)} = J(G)$. For more precise statements, see Lemma 2.3 in [S5], for which the proof is based on [HM1] and [GR]. Furthermore, if G is generated by a precompact subset Λ of End($\overline{\mathbb{C}}$), then $J(G) = \overline{\bigcup_{f\in\Lambda} f^{-1}(J(G))} = \bigcup_{h\in\overline{\Lambda}} h^{-1}(J(G))$. In particular, if Λ is compact, then $J(G) = \bigcup_{f\in\Lambda} f^{-1}(J(G))$([S4]). We call this property of the Julia set **backward self-similarity**.

Remark 11. In the context of backward self-similarity, the existing research on the Julia sets of rational semigroups may be considered as a kind of generalization of the research on self-similar sets constructed by some similarity transformations from $\mathbb{C}$ to itself, which can be regarded as the Julia sets of some rational semigroups. It can be easily seen that the Sierpiński gasket is the Julia set of a rational semigroup $G = \langle h_1, h_2, h_3\rangle$, where $h_i(z) = 2(z - p_i) + p_i, i = 1, 2, 3,$ with $p_1p_2p_3$ a regular triangle.

2.3 Potential theory and measure theory

For the proof of results on uniform perfectness, Johnness, and so on, let us borrow some notation from [J2] and [S4], concerning potential theoretic aspects. By the arguments in [J2] and [S4], for a fibered rational map $f : Y \to Y$ over $g : X \to X$ with $d(x) \geq 2$ for each $x \in X$, one can show a result corresponding to Proposition 2.5 in [S4], using the arguments in §3 in [J2] and from pp. 580-581 in [S4] . In this paper, the following statements, and especially the lower semicontinuity of $x \mapsto J_x(f)$, are necessary. (Proposition 2.2.3).

Proposition 2.2. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that $d(x) \geq 2$, for each $x \in X$. Then, for each $x \in X$, there exists a Borel probability measure μ_x on Y satisfying all of the following.*

1. *$x \mapsto \mu_x$ is continuous with respect to the weak topology of probability measures in Y.*
2. *supp$(\mu_x) = J_x$, for each $x \in X$.*
3. *$x \mapsto J_x$ is lower semicontinuous with respect to the Hausdorff metric in the space of the non-empty compact subsets of Y. That is, if $x, x_n \in X, x_n \to x$ as $n \to \infty$ and $y \in J_x$, then there exists a sequence (y_n) of points in Y with $y_n \in J_{x_n}$, for each $n \in \mathbb{N}$, such that $y_n \to y$ as $n \to \infty$.*

Furthermore , $J_x(f)$ is a non-empty perfect set, for each $x \in X$.

Proof. Since $d(x) \geq 2$ for each $x \in X$, and $x \mapsto d(x)$ is continuous, we can demonstrate these statements in the same way as in §3 in [J2], using the argument from pp. 580-581 in [S4]. The statement 3 follows easily from 1 and 2.

□

2.4 Distortion lemma and limit functions

For an exposition of the results deduced from semi-hyperbolicity, the following statements are necessary. For the research on the semi-hyperbolicity of the usual dynamics of rational functions, see [CJY] and [Ma].

Notation:

1. Let X be a compact set in $\overline{\mathbb{C}}$ and z be a point in $\overline{\mathbb{C}} \setminus X$. Then we set $\mathrm{Dist}(X, z) = \max_{y \in X} d(y, z) / \min_{y \in X} d(y, z)$.

2. For two positive numbers A and B, $A \asymp B$ means $K^{-1} \le A/B \le K$, for some constant K independent of A and B.

Lemma 2.3 ([CJY]). *(***Distortion lemma for proper maps***) For any positive integer N and real number r with $0 < r < 1$, there exists a constant $C = C(N, r)$ such that, if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with $\deg(f) = N$, then $H(f(z_0), C) \subset f(H(z_0, r)) \subset H(f(z_0), r)$ for any $z_0 \in D(0,1)$, where $H(x, s)$ denotes the hyperbolic disk in $D(0,1)$ around x with radius s. Here, the constant $C = C(N, r)$ depends only on N and r.*

The following is a generalized distortion lemma for proper maps.

Lemma 2.4 ([S4],[S6]). *Let V be a domain in $\overline{\mathbb{C}}$, K, a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N, a)$ depending only on N and a such that, for each r with $0 < r \le r(N, a)$, there exists a constant $C = C(N, r)$ depending only on N and r such that, for each connected component U of $f^{-1}(D(0, r))$, it follows that $diam_S\, U \le C$, where we denote by $diam_S$ the spherical diameter. Furthermore, it follows that $C(N, r) \to 0$ as $r \to 0$.*

As the author mentioned in Remark 6 in [S4], the results Lemma 2.13, Theorem 2.14, Lemma 2.15 and Theorem 2.17 in [S4] are generalized to the version of the dynamics of fibered rational maps on $\overline{\mathbb{C}}$-bundles, which we need to show results in this paper. The following lemma is a slightly modified version of Lemma 2.15 in [S4].

Lemma 2.5 ([S4]). *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Assume that f satisfies condition (C1). Assume $z_0 \in SH_N(f)$, for some $N \in \mathbb{N}$. Then there exists a positive number δ_0 such that, for each δ with $0 < \delta < \delta_0$, there exists a neighborhood U of $x_0 := \pi(z_0)$ in X such that, for each $n \in \mathbb{N}$, each $x \in U$ and each $x_n \in p^{-n}(x)$, it follows that each element of $c(i_x i_{x_0}^{-1} \tilde{B}(z_0, \delta),\ f_{x_n}^n)$ is simply connected.*

The following theorem relates to what happens if there exists a non-constant limit function on a component of a fiber-Fatou set. This is the key to state other results. The proof is the same as that for Lemma 2.13 in [S4].

Theorem 2.6 ([S4]). *(***Key theorem I***) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Assume that f satisfies condition (C1). Let $z \in Y$ be a point with $z \in F_{\pi(z)}$. Let (i_x) be a local parameterization. Let U be a connected open neighborhood of $i_{\pi(z)}^{-1}(z)$ in $\overline{\mathbb{C}}$. Suppose that there exists a strictly increasing sequence (n_j) of $\mathbb{N}$ such that $R_j := i_{\pi f^{n_j}(z)}^{-1} \circ f_{\pi(z)}^{n_j} \circ i_{\pi(z)}$ uniformly converges to a non-constant map ϕ on U*

as $j \to \infty$. Furthermore, suppose that $f^{n_j}_{\pi(z)}(z)$ converges to a point $z_0 \in Y$. Let $S_{i,j} = f^{n_j - n_i}_{g^{n_i}\pi(z)}$ for $1 \le i \le j$. We set

$$V = \{a \in Y_{\pi(z_0)} \mid \exists \epsilon > 0,\ \lim_{i\to\infty} \sup_{j>i} \sup_{\xi \in \tilde{B}(a,\epsilon)} d(S_{i,j} \circ \varphi(\xi),\ \xi) = 0\},$$

where φ is a map from $Y_{\pi(z_0)}$ onto $Y_{g^{n_i}\pi(z)}$ defined by the local parameterization around $\pi(z_0)$. Then V is a non-empty open proper subset of $Y_{\pi(z_0)}$, and it follows that

$$\partial V \subset \hat{J}_{\pi(z_0)}(f) \cap UH(f).$$

Corollary 2.7. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Assume that f satisfies condition (C1). Assume also that, for each $x \in X$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x. Then, for each $z \in Y$ with $z \in F_{\pi(z)}$, it follows that diam$f^n_{\pi(z)}(W) \to 0$ as $n \to \infty$, for each small connected neighborhood W of z in $Y_{\pi(z)}$, and that $d(f^n_{\pi(z)}(z), UH(f)) \to 0$ as $n \to \infty$.*

Proof. Let $z \in Y$ be a point with $z \in F_{\pi(z)}$. Suppose that the assumptions of Theorem 2.6 are satisfied, and continue with the same notation as that in Theorem 2.6. Let $x_i = g^{n_i}\pi(z)$. Let D^i be the topological disk in Y_{x_i} satisfying condition(C1). We may assume that there exists a non-empty open set A in $\overline{\mathbb{C}}$ such that $A \subset i^{-1}_{x_i}(D_i)$ for each $i \in \mathbb{N}$. Then, by the definition of V, it must be that $i_{\pi(z_0)}(A) \subset Y_{\pi(z_0)} \setminus V$. This contradicts the assumption of the corollary. Hence, it must be that diam$f^n_{\pi(z)}(W) \to 0$ as $n \to \infty$, for each small connected neighborhood W of z in $Y_{\pi(z)}$. Then, by Lemma 2.4, it follows that that $d(f^n_{\pi(z)}(z), UH(f)) \to 0$ as $n \to \infty$. □

2.5 Proofs of results on fibered rational maps

In this section, we present proofs of the results from Section 1.1. We begin with the following.

Proof of Theorem 1.6. First, we prove that there exists a constant $C > 0$ such that for each $x \in X$, J_x is C-uniformly perfect. Since, for each $x \in X$, J_x is a non-empty perfect set (Proposition 2.2), it has uncountably many points. Combined with the lower semicontinuity of the map $x \mapsto J_x$ (statement 3 in Proposition 2.2) and the compactness of X, this suggests that, for any $x \in X$, one can take four points $z_{x,1}, z_{x,2}, z_{x,3}$ and $z_{x,4}$ in J_x so that $d(z_{x,i}, z_{x,j}) > C_1$, whenever $i \ne j$ and $x \in X$, for some constant C_1 independent of (i,j) and $x \in X$.

Suppose that there exists a sequence of annuli $\{D_j\}$ with $D_j \subset Y_{x_j}$, $x_j \in X$ such that D_j separates J_{x_j}, for each j and mod $D_j \to \infty$ as $j \to \infty$. Let D'_j and D''_j be the two components of $Y_{x_j} \setminus D_j$. We may assume that

$$\text{diam } D''_j \to 0 \text{ as } j \to \infty. \tag{2}$$

For, by the existence of $\{z_{x,i}\}_{x,i}$, it may be assumed that $\inf_{j\in\mathbb{N}} \text{diam } D'_j > 0$. Then, since mod $D_j \to \infty$ as $j \to \infty$, by Lemma 6.1 on p. 34 in [LV] it follows that diam $D''_j \to 0$ as $j \to \infty$.

It may also be assumed that $(\text{int}D''_j) \cap J_{x_j} \ne \emptyset$, for each $j \in \mathbb{N}$. Hence, there exists a smallest positive integer n_j such that diam $f^{n_j+1}(D''_j) \ge C_1$. Then, there exists a constant l_0 such that $l_0C_1 < \text{diam } f^{n_j}(D''_j)$, for each j. Since $\text{diam } f^{n_j}(D''_j) < C_1$, there exist three distinct points in $\{z_{x'_j,i}\}_{i=1,\dots,4}$ none of which belongs to $f^{n_j}(D''_j)$, where $x'_j = g^{n_j}(x_j)$. Since $D_j \subset F_{x_j}$, it follows that none of these three points belongs to $f^{n_j}(D_j)$, or to $f^{n_j}(D_j \cup D''_j)$. Let $\varphi_j : \{|z| < 1\} \to D_j \cup D''_j$ be a Riemann map such that $\varphi_j(0) = y_j \in D''_j$ (note that we may assume that int $D'_j \neq \emptyset$ for each j). Then, from the above, it follows that, if we set $\alpha_j = i_{x'_j}^{-1} f^{n_j} \varphi_j : \{|z| < 1\} \to \overline{\mathbb{C}}$, then $\{\alpha_j\}_{j \in \mathbb{N}}$ is normal in $\{|z| < 1\}$. But this causes a contradiction, because $\text{diam } \varphi_j^{-1}(D''_j) \to 0$ as $j \to \infty$, which follows from $\text{mod } \varphi_j^{-1}(D_j) = \text{mod } D_j \to \infty$ as $j \to \infty$, and $l_0C_1 < \text{ diam } f^{n_j}(D''_j)$, for each j. Hence, we have shown that there exists a constant $C > 0$ such that for each $x \in X$, J_x is C-uniformly perfect.

Next, we prove that there exists a constant $C > 0$ such that for each $x \in X$, $\hat{J}_x$ is C-uniformly perfect. Suppose that there exists a sequence of annuli $\{D_j\}$ with $D_j \subset Y_{x_j}$, $x_j \in X$ such that D_j separates $\hat{J}_{x_j}$, for each j, and mod $D_j \to \infty$ as $j \to \infty$. Let D'_j and D''_j be the two components of $Y_{x_j} \setminus D_j$. As in the previous paragraph, it may be assumed that diam $D''_j \to 0$ as $j \to \infty$.

Fix any $j \in \mathbb{N}$. Let $y \in D''_j \cap \hat{J}_{x_j}$ be a point. There exists a sequence $((x_{j,n}, y_{j,n}))_n$ in $X \times Y$ with $y_{j,n} \in J_{x_{j,n}}$, for each $n \in \mathbb{N}$, such that $(x_{j,n}, y_{j,n}) \to (x, y)$ as $n \to \infty$. Then we may assume that there exists a number $n(j) \in \mathbb{N}$ such that $i_{x_j,n(j)}^{-1}(J_{x_{j,n(j)}}) \subset i_{x_j}^{-1}(D'_j \cup D''_j)$. Since $i_{x_j,n(j)}^{-1}(J_{x_{j,n(j)}}) \cap i_{x_j}^{-1}(D''_j) \neq \emptyset$ (take $n(j)$ sufficiently large), by the previous paragraph it must be that $i_{x_j,n(j)}^{-1}(J_{x_{j,n(j)}}) \subset i_{x_j}^{-1}(D''_j)$, for large j. However, this contradicts the existence of $\{z_{x,i}\}_{x,i}$, because it is also the case that diam $D''_j \to 0$ as $j \to \infty$. Hence, we have proved the first and the second statements in 1. The third statement in 1 follows from the uniform perfectness of J_x , the continuity of ω_x, and Theorem 7.2 in [Su].

Next, we prove statement 2. Suppose the point z belongs to the boundary of $\hat{J}_x$ with respect to the topology of Y_x, where $x = \pi(z)$. Under a coordinate exchange, the map f_x^n around z is conjugate to $\alpha(z) = z^l$ for some $l \in \mathbb{N}$. Since $f_x^n(Y_x \setminus \hat{J}_x) \subset Y_x \setminus \hat{J}_x$, there exists an annulus A around z in Y_x that separates $\hat{J}_x$ and is isomorphic to a round annulus $A' = \{r < |z| < R\}$ in the above coordinate. Then, mod $(f_x^{ns}(A)) = $ mod $(\alpha^s(A')) \to \infty$ as $s \to \infty$. In addition, $f_x^{ns}(A)$ separates $\hat{J}_x$, for each $s \in \mathbb{N}$. This contradicts the fact that $\hat{J}_x$ is uniformly perfect. $\square$

Before proving Theorem 1.12, we need the following.

Corollary 2.8. *(Corollary of Theorem 1.6) Let $(\pi, Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered polynomial map over $g : X \to X$ such that $d(x) \geq 2$, for each $x \in X$. Let $R > 0$ be a number such that for each $x \in X$, $\pi_{\overline{\mathbb{C}}}(J_x(f)) \subset \{z \mid d(z, \infty) > R\}$, where $\pi_{\overline{\mathbb{C}}} : Y \to \overline{\mathbb{C}}$ denotes the projection and d denotes the spherical distance. (Note that such an R exists, since $d(x) \geq 2$ for each $x \in X$.) Let $\rho_x(z)|dz|$ be the hyperbolic metric on $\pi_{\overline{\mathbb{C}}}(A_x(f))$. Let $\delta_x(z) = \inf_{w \in \partial(\pi_{\overline{\mathbb{C}}}(A_x(f)))} |z - w|$ for each $z \in \pi_{\overline{\mathbb{C}}}(A_x(f)) \cap \mathbb{C}$. Then, there exists a positive constant C depending only on f and R such that for each $x \in X$ and each $z \in \pi_{\overline{\mathbb{C}}}(A_x(f)) \cap \{z \mid d(z, \infty) > R\}$, we have $C \leq \rho_x(z)\delta_x(z) \leq 1$.*

Proof. $\rho_x(z)\delta_x(z) \leq 1$ follows easily from the Schwarz lemma.

Next, as in the proof of Theorem 1.6, for any $x \in X$, we can take two points $z_{x,1}$ and $z_{x,2}$ in $J_x(f)$ so that $d(z_{x,1}, z_{x,2}) > c_0$, whenever $x \in X$, for some positive constant c_0 independent of $x \in X$. Let $\psi_x(z)$ be a Möbius transformation such that $\psi_x(z_{x,1}) = \infty$ and ψ_x preserves the spherical metric. Let $B_x = \psi_x(\pi_{\overline{\mathbb{C}}}(A_x(f)))$ $(\subset \mathbb{C})$. By Theorem 1.6 and Theorem 2.16 in [Su], there exists a positive constant c_1 such that for each $x \in X$ and each $z \in B_x$, $\rho_{x,1}(z) \inf_{w \in \partial B_x} |z - w| \geq c_1$, where $\rho_{x,1}(z)|dz|$ denotes the hyperbolic metric on B_x.

Let $z \in \pi_{\overline{\mathbb{C}}}(A_x(f)) \cap \{y \mid d(y, z_{x,1}) > c_0/2,\ d(y, \infty) > R\}$. Then, we have $\rho_x(z) = \rho_{x,1}(\psi_x(z))|\psi_x'(z)|$. Since $\psi_x(z) \in \{y \mid d(y, \infty) > c_0/2,\ d(y, \psi_x(\infty)) > R\}$ and $\psi_x^{-1}(\{y \mid d(y, \psi_x(\infty)) > R\}) \subset \{y \mid d(y, \infty) > R\}$, by the Cauchy formula, we have $|\psi_x'(z)| = |(\psi_x^{-1})'(\psi_x(z))|^{-1} \geq c_2$, where c_2 is a positive constant independent of z and x.

Next, let $w_0 \in \partial(\pi_{\overline{\mathbb{C}}}(A_x(f))) = \pi_{\overline{\mathbb{C}}}(J_x(f))$ be a point such that $\delta_x(z) = |z - w_0|$.

Suppose case (1): $w_0 \in \{y \mid d(y, z_{x,1}) \leq c_0/4\}$. Then, $|z - w_0| \geq c_3$, where c_3 is a positive constant depending only on c_0. Further, $\inf_{w \in \partial B_x} |\psi_x(z) - w| \leq |\psi_x(z) - \psi_x(z_{x,2})| \leq c_4$, where c_4 is a positive constant depending only on c_0. Hence, $\delta_x(z) \geq \frac{c_3}{c_4} \inf_{w \in \partial B_x} |\psi_x(z) - w|$.

Suppose case (2): $w_0 \in \{y \mid d(y, z_{x,1}) > c_0/4\}$. Let γ be the Euclidean segment connecting z and w_0. Then, since $|w_0 - z| = \delta_x(z)$, we obtain $d(z_{x,1}, \gamma) \geq c_5$, where c_5 is a positive constant depending only on c_0, which implies $d(\infty, \psi_x(\gamma)) \geq c_5$. Hence, by the Cauchy formula, we have $\inf_{w \in \partial B_x} |\psi_x(z) - w| \leq |\psi_x(z) - \psi_x(w_0)| \leq \sup_{w \in \gamma} |\psi_x'(w)| \cdot |z - w_0| \leq c_6 \delta_x(z)$, where c_6 is a positive constant independent of z and x_0.

From these arguments, we find that there exists a positive constant c_7 depending only on f and R, such that for each $z \in \pi_{\overline{\mathbb{C}}}(A_x(f)) \cap \{y \mid d(y, z_{x,1}) > c_0/2,\ d(y, \infty) > R\}$, we have $\rho_x(z)\delta_x(z) \geq c_7$. Similarly, we find that there exists a positive constant c_8 depending only on f and R, such that for each $z \in \pi_{\overline{\mathbb{C}}}(A_x(f)) \cap \{y \mid d(y, z_{x,2}) > c_0/2,\ d(y, \infty) > R\}$, we have $\rho_x(z)\delta_x(z) \geq c_8$. Since $d(z_{x,1}, z_{x,2}) > c_0$, we obtain the statement of the corollary.

□

Proof of Theorem 1.12. This statement can be proved using an approach similar to that in [CJY], using Theorem 2.14 (3) in [S4], Corollary 2.8, Lemma 2.3 and Lemma 2.4.

The procedure is as follows: first, for any $(x, y) \in X \times \mathbb{C}$, let $G_x((x, y)) := \lim_{n \to \infty} \frac{1}{d_n(x)} \log^+ |\pi_{\overline{\mathbb{C}}}(f_x^n((x, y)))|$, where $\log^+(a) := \max\{\log a, 0\}$ for $a \in \mathbb{R}$ with $a > 0$. Then, by the arguments in [Se3] (or [Se1]), on $A_x(f)$, G_x equals the Green's function with pole (x, ∞), $G_{g(x)}(f_x(z)) = d(x)G_x(z)$ for any $x \in X$ and $z \in Y_x$, $(x, y) \mapsto G_x((x, y))$ is continuous in $X \times \mathbb{C}$, and there exist numbers $r > 0$ and $M > 0$ such that for any $(x, y) \in X \times \mathbb{C}$ with $|y| > r$, $|G_x((x, y)) - \log|y|| \leq M$. Further, for any $x \in X$, we can define a locally conformal map ψ_x that is defined on $\{(x, y) \mid |y| > R\}$, where R is a number independent of $x \in X$, such that $\psi_{g(x)} f_x = (\psi_x)^{d(x)}$ and $\log|\psi_x| = G_x$ on $\{(x, y) \mid |y| > R\}$. In particular, G_x has no critical points in $\{(x, y) \mid |y| > R\}$.

Next, for each $x \in X$, let $\rho_x(z)|dz|$ be the hyperbolic metric on $A_x(f)$. Similarly, let $\delta_x(z)|dz|$ be the quasihyperbolic metric on $A_x(f)$ ($\delta_x(z) := \inf_{w \in J_x} |\pi_{\overline{\mathbb{C}}}(z) - \pi_{\overline{\mathbb{C}}}(w)|$). By Theorem 2.14(3) in [S4], we obtain the following claim:
Claim 1: There exist constants $C > 0$ and $\alpha > 0$ such that for each $x \in X$ and each $z \in A_x(f) \cap (X \times \mathbb{C})$ with $\delta_x(z) \leq 1$, we have $\delta_x(z) \leq C G_x(z)^\alpha$.

Next, for each $x \in X$ and each $z \in A_x \cap (X \times \mathbb{C})$, we take a Green's line (a curve of steepest descent for G_x) γ_z running from z to "$J_x(f)$", such that $f_x(\gamma_z) = \gamma_{f_x(z)}$. (We choose a direction (left or right) consistently at every critical point encountered.) Then, we show the following:

Claim 2: Let $0 < C_1 < C_2 < \infty$, $0 < C_3 < C_4 < \infty$ be given numbers. Let $\tau : [a,b] \to A_x(f)$ $(0 \leq a < b < \infty)$ be a curve that parameterizes part of a Green's line, such that $\tau([a,b]) \subset \{z \in A_x(f) \mid C_1 < \delta_x(z) < C_2\}$ and $C_3 \leq l_E\tau < C_4$, where l_E denotes the Euclidean length. Then, there exist constants $C_5 > 0$ and $0 < \lambda_1 < 1$, which depend only on f, C_1, C_2, C_3, and C_4, such that $|\tau(a) - \tau(b)| \geq C_5$ and $G_x(\tau(b)) < \lambda_1 G_x(\tau(a))$.

The proof of this claim is immediate, since we assume that either $J_x(f)$ is connected for each $x \in X$ (then f_x and G_x have no critical points in $A_x(f)$) or condition 2 in the assumption of Theorem 1.12 is satisfied. Using Claim 2, $\inf\{d(a,b) \mid a \in \pi_{\overline{\mathbb{C}}}(J_x),\ b \in \pi_{\overline{\mathbb{C}}}(A_x(f) \cap C(f)),\ x \in X\} > 0$, the Koebe distortion theorem, and Corollary 2.8, we easily obtain the following:

Claim 3: Let $0 < C_6 < C_7 < \infty$ and $0 < C_8 < \infty$ be given numbers. Let $\tau : [a,b] \to A_x(f)$ be a curve that parameterizes part of a Green's line. Suppose $\tau([a,b]) \subset \{z \in A_x(f) \mid |\pi_{\overline{\mathbb{C}}}(z)| \leq C_8\}$ and $C_6 \leq l_{\rho_x}\tau \leq C_7$ (l_{ρ_x} denotes the ρ_x-length). Then, there exist constants $C_9 > 0$ and $0 < \lambda_2 < 1$ that depend only on f, C_6, C_7, and C_8, such that $C_9 \leq d_{\rho_x}(\tau(a), \tau(b)) \leq C_7$ (d_{ρ_x} denotes the distance induced by $\rho_x(z)|dz|$) and $G_x(\tau(b)) < \lambda_2 G_x(\tau(a))$.

Next, we will show the following:

Claim 4: There exist constants $C_{10} > 0, C_{11} > 0$, and $\alpha > 0$ such that for any $x \in X$ and any $z \in A_x(f)$ with $G_x(z) \leq C_{10}$, we have $l_E\gamma_z \leq C_{11}G_x(z)^\alpha$. In particular, any Green's line γ_z lands at a point in $J_x(f)$.

To show this claim, take a number C_{10} such that for any $x \in X$, $\{z \in A_x(f) \mid G_x(z) \leq C_{10}\} \subset \{z \in A_x(f) \mid \delta_x(z) \leq 1\}$. Next, for any two points w_1 and w_2 in a Green's line γ_{z_0} $(z_0 \in A_x(f))$ with $G_x(w_2) \leq G_x(w_1)$, let $\gamma_{z_0}(w_1, w_2)$ be the arc from w_1 to w_2 along γ_{z_0}. Let $\{z_n\}$ be a sequence in γ_{z_0} such that $l_{\rho_x}\gamma_{z_0}(z_n, z_{n+1}) = c$ for each $n \in \mathbb{N}$, where c is a positive constant that is sufficiently small. Then, by Corollary 2.8, Claim 1, and Claim 3, we obtain $l_E\gamma_{z_0} = \sum_n l_E\gamma_{z_0}(z_n, z_{n+1}) \leq \text{Const.}\sum_n \int_{\gamma_{z_0}(z_n,z_{n+1})} \delta_x(z)\rho_x(z)|dz| \leq$ $\text{Const.}\sum_n \int_{\gamma_{z_0}(z_n,z_{n+1})} \delta_x(z_n)\rho_x(z)|dz| \leq \text{Const.}\sum_n \delta_x(z_n) \leq \text{Const.}\sum_n G_x(z_n)^\alpha$ $\leq \text{Const.}\sum_n \lambda^{n\alpha} G_x(z_0)^\alpha \leq \text{Const.}G_x(z_0)^\alpha$, where, $\alpha > 0$ and $0 < \lambda < 1$ are constants and Const. denotes a constant. Hence, we obtain the claim.

Next, we show the following:

Claim 5: There exist constants $\delta' > 0$ and $p \in \mathbb{N}$ such that for any $x \in X$ and any $(w,z) \in A_x(f) \times A_x(f)$ with $w \in \gamma_z$, $l_{\rho_x}\gamma_z(z,w) \geq p$ and $G_x(z) < \delta'$, we have $\delta_x(w) \leq \frac{1}{2}\delta_x(z)$.

This claim can be shown using the argument on page 11 in [CJY], using Claims 3 and 4, and the distortion lemma for proper holomorphic maps. Next, we show the following:

Claim 6: There exist constants $\delta' > 0$ and $C_{12} > 0$ such that for any $x \in X$ and any $(w,z) \in A_x(f) \times A_x(f)$ with $w \in \gamma_z$ and $G_x(z) \leq \delta'$, we have $\delta_x(z) \geq C_{12}|\pi_{\overline{\mathbb{C}}}(z) - \pi_{\overline{\mathbb{C}}}(w)|$.

This claim can be shown using Claim 5 and Corollary 2.8. Finally, we show the following:

Claim 7: There exists a constant $C > 0$ such that for any $x \in X$ and any $(w,z) \in A_x(f) \times A_x(f)$ with $w \in \gamma_z$, we have $\delta_x(z) \geq C|\pi_{\overline{\mathbb{C}}}(z) - \pi_{\overline{\mathbb{C}}}(w)|$.

This claim can easily be shown using Claim 6 and the fact that there exist numbers $r > 0$ and $M > 0$ such that for any $(x, y) \in X \times \mathbb{C}$ with $|y| > r$, $|G_x((x, y)) - \log |y|| \leq M$. Hence, we have shown Theorem 1.12. $\square$

Proof of Theorem 1.16. For any $y' \in \bigcup_{x \in X} J_x$ and $r > 0$, we set

$$h(y', r) = \sup\{s \mid \exists y'' \in Y_{\pi(y')},\ \tilde{B}(y'', s) \subset F_{\pi(y')} \cap \tilde{B}(y', r)\}$$

and $h(r) = \inf\{h(y', r) \mid y' \in \bigcup_{x \in X} J_x\}$. By Theorem 2.14 and Remark 6 in [S4], it follows that $\tilde{J}(f) = \bigcup_{x \in X} J_x$. Furthermore, by condition (C1), int $J_x = \emptyset$, for any $x \in X$. Hence, it follows that $h(r) > 0$ for any $r > 0$.

Since f is semi-hyperbolic and satisfies condition (C1), by Lemma 2.5 there exists a positive number δ_1 and a number $N \in \mathbb{N}$ such that, for any $y' \in \tilde{J}(f)$, $0 < \delta \leq \delta_1$, $n \in \mathbb{N}$, any $V \in c(\tilde{B}(y', 2\delta),\ f^n)$, we have that V is simply connected and that $\deg(f^n : V \to \tilde{B}(y', 2\delta)) \leq N$.

Let $y \in \tilde{J}(f)$ and $r > 0$. We set $B_n = f^n(\tilde{B}(y, r))$ and $y_n = f^n(y)$, for each $n \in \mathbb{N}$. Since $y \in J_{\pi(y)}$, there exists a smallest positive integer n_0 such that diam $B_{n_0+1} > \delta_1$. Then there exists a constant l_0 such that $l_0\delta_1 <$diam B_{n_0}. By Corollary 2.3 in [Y], there exists a constant K that depends only on N and a ball $\tilde{B}(y_{n_0}, r_0) \subset B_{n_0}$, with $r_0 \geq$diam $B_{n_0}/K \geq \frac{l_0\delta_1}{K}$ such that the element $W \in c((\tilde{B}(y_{n_0}, r_0),\ f^{n_0})$ containing y is a subset of $\tilde{B}(y, r)$. There exists a ball $\tilde{B}(y', \frac{2}{3}h(r_0))$ included in $\tilde{B}(y_{n_0}, r_0) \cap F_{\pi(y_{n_0})}$.

Let D_0 be an element of $c((\tilde{B}(y', \frac{1}{2}h(r_0))),\ f^{n_0})$ contained in W. It follows that $D_0 \subset F_{\pi(y)}$. Let $y'' \in D_0 \cap (f^{n_0})^{-1}\{y'\}$ be a point. Then, by Corollaries 1.8 and 1.9 in [S4], it follows that Dist $(\partial D_0, y'') \leq M$, for some M that depends only on N. Further, by the method used in Corollary 1.9 in [S4] it follows that $\min_{z \in \partial D_0} d_{\pi(y'')}(y'', z) \asymp \max_{z \in \partial E} d_{\pi(y'')}(y'', z)$, where $E \in c(\tilde{B}(y_{n_0}, \delta_1),\ f^{n_0})$ with $y'' \in E$. Since $\tilde{B}(y, r) \subset E$, it follows that diam $D_0 \asymp r$. Combining this with Dist $(\partial D_0, y'') \leq M$ leads to the conclusion that there exists a constant $0 < k < 1$ that does not depend on y and r such that $B(y'', kr) \subset D_0 \subset F_{\pi(y)}$. $\square$

To show Theorem 1.18, we present some notation and propositions.

Definition 2.9. (**Conical set for fibered rational maps**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Let $N \in \mathbb{N}$ and $r > 0$. We denote by $\tilde{J}_{con}(f, N, r)$ the set of points $z \in \tilde{J}(f)$ such that, for any $\epsilon > 0$, there exists a positive integer n such that the element $U \in c(\tilde{B}(f^n(z), r),\ f^n|_{Y_{\pi(z)}})$ containing z satisfies the following conditions:

1. diam $U \leq \epsilon$,
2. U is simply connected, and
3. $\deg(f^n : U \to \tilde{B}(f^n(z), r)) \leq N$.

We set $\tilde{J}_{con}(f, N) = \bigcup_{r>0} \tilde{J}_{con}(f, N, r)$ and $\tilde{J}_{con}(f) = \bigcup_{N \in \mathbb{N}} \tilde{J}_{con}(f, N)$.

Proposition 2.10. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that $\hat{J}_x$ has no interior points with respect to the topology in Y_x, for each $x \in X$. Then the two-dimensional Lebesgue measure of $\tilde{J}_{con}(f) \cap J_x$ is equal to zero, for each $x \in X$.*

Proof. Fix $N \in \mathbb{N}$. Suppose that there exists a point $x \in X$ such that $\tilde{J}_{con}(f,N) \cap J_x$ has positive measure. Then there exists a Lebesgue density point $y \in \tilde{J}_{con}(f,N) \cap J_x$. Let $y_m = f_x^m(y)$ and $x_m = g^m(x)$, for any $m \in \mathbb{N}$. Let $\delta > 0$ be a number such that $y \in \tilde{J}_{con}(f,N,\delta)$. Let U_m, U'_m be the respective elements of $c(\tilde{B}(y_m,\delta/2),\ f_x^m),\ \ c(\tilde{B}(y_m,\delta),\ f_x^m)$ containing y. Since $y \in \tilde{J}_{con}(f,N,\delta)$, there exists a subsequence (n) in $\mathbb{N}$ with $n \to \infty$ such that U'_n is simply connected, $\deg(f_x^n : U'_n \to \tilde{B}(y_n,\delta)) \leq N$, for each n, and diam $U'_n \to 0$ as $n \to \infty$. By Corollary 1.9 in [S4], for any local parameterization i_x,

$$\lim_{n\to\infty} \frac{m(i_x^{-1}(U_n \cap J_x))}{m(i_x^{-1}(U_n))} = 1, \tag{3}$$

where m denotes the spherical measure of $\overline{\mathbb{C}}$. Using an argument from the proof of Theorem 4.4 in [S4], from (3) we can show that

$$\lim_{n\to\infty} \frac{m(\imath_{x_n}^{-1}(\tilde{B}(y_n,\delta/2) \cap F_{x_n}))}{m(i_{x_n}^{-1}(\tilde{B}(y_n,\delta/2)))} = 0, \tag{4}$$

where i_{x_n} denotes a local parameterization. There exists a subsequence (n_j) of (n), a point $y_\infty \in Y$, and a point $x_\infty \in X$ such that $y_{n_j} \to y_\infty$ and $x_{n_j} \to x_\infty$ as $j \to \infty$. By (4), it follows that $\tilde{B}(y_\infty,\ \delta/2) \subset \hat{J}_{x_\infty}$. On the other hand, by assumption, for any $a \in X$, $\hat{J}_a$ has no interior point. This is a contradiction. □

Proposition 2.11. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that f satisfies condition (C1). Then the following hold.*

1. $$\tilde{J}_{good}(f) \cap \bigcup_{x\in X} J_x \subset \tilde{J}_{con}(f).$$

2. *If it is assumed, furthermore, that, for each $x \in X$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x, then*
$$\tilde{J}_{good}(f) \subset \tilde{J}_{con}(f) \cap \bigcup_{x\in X} J_x.$$

Proof. We will first prove the first statement. Let $z \in \bigcup_{x\in X} J_x$ be a point such that $\limsup_{n\to\infty} d(f^n(z), UH(f)) > 0$. For each $m \in \mathbb{N}$, let $z_m = f^m(z)$ and $x_m = \pi(f^m(z))$. For each $m \in \mathbb{N}$ and each $r > 0$, let $U_m(r), U'_m(r)$ respectively, be the elements of $c(\tilde{B}(z_m,r/2),\ f_{\pi(z)}^m), c(\tilde{B}(z_m,r),\ f_{\pi(z)}^m)$ containing z. There exists a positive number δ, a positive integer N, and a sequence (n) in $\mathbb{N}$ such that $\deg(f_{\pi(z)}^n : U'_n(\delta) \to \tilde{B}(z_n,\delta)) \leq N$. By Lemma 2.5, taking δ sufficiently small it can be assumed that $U'_n(\delta)$ is simply connected.

Suppose that diam $(U_n(\delta))$ does not tend to zero as $n \to \infty$ in (n). Then, by the distortion lemma for proper maps, there exists a subsequence (n_j) of (n) with $n_j \to \infty$ and a positive number r such that $U_{n_j}(\delta) \supset \tilde{B}(z,r)$, for each j. Hence,

$$f^{n_j}(\tilde{B}(z,r)) \subset \tilde{B}(f_{n_j}(z),\delta) \tag{5}$$

for each j. By condition (C1), if δ is sufficiently small, then (5) contradicts that $z \in \cup_{x\in X} J_x$. Hence, it follows that diam $U_n(\delta) \to 0$ as $n \to \infty$ in (n). Therefore, $z \in \tilde{J}_{con}(f)$.

The second statement follows from Corollary 2.7 and the first statement. $\square$

Corollary 2.12. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that $\tilde{J}(f) = \bigcup_{x\in X} J_x$, and that f satisfies condition (C1). Then, for each $x \in X$, it follows that $d(f_x^n(y), UH(f)) \to 0$, as $n \to \infty$, for almost every $y \in J_x$, with respect to the Lebesgue measure in Y_x.*

Proof. By condition (C1), $\hat{J}_x = J_x$ has no interior points, for each $x \in X$. By Proposition 2.10 and Proposition 2.11, the following statement results. $\square$

Proof of Theorem 1.18. Suppose that there exists a point $z \in \tilde{J}(f)$ such that $z \in F_{\pi(z)}$. By Corollary 2.7, for each small connected neighborhood W of z in $F_{\pi(z)}$, it follows that diam $f^n(W) \to 0$ and $d(f^n(z), UH(f)) \to 0$ as $n \to \infty$. However, by conditions three and four in the assumptions of our theorem, this causes a contradiction. Hence, we have shown that $\tilde{J}(f) = \bigcup_{x\in X} J_x$. By Corollary 2.12, it also follows that the two-dimensional Lebesgue measure of $J_x \setminus \bigcup_{n\in\mathbb{N}} f^{-n}(UH(f))$ is equal to zero. $\square$

Proof of Theorem 1.19. Let $x' = v(x)$. We now show the following.

Claim: There exists a constant $C > 0$ such that, for each $y \in \overline{\mathbb{C}}$,

$$\limsup_{r\to 0} \sup\{\frac{d_{x'}(u_x(i_x(y)),\ u_x(i_x(y')))}{d_{x'}(u_x(i_x(y)),\ u_x(i_x(y'')))} \mid d(y, y') = d(y, y'') = r\} \leq C, \tag{6}$$

where $d_{x'}$ is the metric on $\tilde{Y}_{x'}$ induced by the form $\tilde{\omega}_{x'}$.

If $i_x(y) \in F_x(f)$, then (6) holds with a constant $C = C(K)$ that depends only on K. Let $i_x(y) \in J_x(f)$ be a point. Let δ_1 and $N \in \mathbb{N}$ be the numbers from the proof of Theorem 1.16. Let $r > 0$ be a number. We set $B_n = f^n(i_x(B(y, r)))$ and $y_n = f^n(i_x(y))$, for each $n \in \mathbb{N}$. Since $i_x(y) \in J_x(f)$, there exists a smallest positive integer n_0 such that diam $B_{n_0+1} > \delta_1$. Furthermore, there exists a constant l_0 such that $l_0\delta_1 <$diam B_{n_0}. By Corollary 2.3 in [Y], there exists a constant A that depends only on N, and a ball $\tilde{B}(y_{n_0}, r_0) \subset B_{n_0}$ with $r_0 \geq$diam $B_{n_0}/A \geq \frac{l_0\delta_1}{A}$ such that the component of $(f^{n_0} i_x)^{-1}(\tilde{B}(y_{n_0}, r_0))$ containing y is a subset of $B(y, r)$. Then there exist constants $\delta' > 0$ and $0 < c < 1$ such that

$$\tilde{B}(u(y_{n_0}),\ c\delta') \subset u(\tilde{B}(y_{n_0}, r_0)) \subset u(B_{n_0}) \subset \tilde{B}(u(y_{n_0}),\ \delta').$$

Since f is semi-hyperbolic and satisfies condition(C1), the same thing holds for $\tilde{f}$. Applying Lemma 2.5, Lemma 2.3, and the Koebe distortion theorem to the map $\tilde{f}^{n_0}$, which maps a neighborhood $u(i_x(B(y, r)))$ of $u(i_x(y))$ in $\tilde{Y}_{x'}$ to a neighborhood of $u(y_{n_0})$ in $\tilde{Y}_{\tilde{g}^{n_0}(x')}$, it follows that (6), for some constant C' that does not depend on $i_x(y) \in J_x(f)$. Hence, the claim holds.

By this claim, $u_x : Y_x \to \tilde{Y}_{x'}$ is quasiconformal. Since the two-dimensional Lebesgue measure of $J_x(f)$ is zero, which is a consequence of Theorem 1.18, it is K-quasiconformal on Y_x. Since $\tilde{f}_{v(w)} \circ u_w = u_{g(w)} \circ f_w$ on Y_w, and f_w and $\tilde{f}_{v(w)}$ are holomorphic for each $w \in X$, it follows that $u_{a'} : Y_{a'} \to \tilde{Y}_{v(a')}$ is K-quasiconformal for each $a' \in \cup_{n\in\mathbb{Z}} g^n(\{x\})$. Hence, for each $a \in \overline{\cup_{n\in\mathbb{Z}} g^n(\{x\})}$, the map $u_a : Y_a \to \tilde{Y}_{v(a)}$ is also K-quasiconformal, with the same dilatation constant K. $\square$

2.6 Proofs of results on rational semigroups

In this section, we prove the results given in Section 1.2. We now prove Theorem 1.21.

Proof of Theorem 1.21. Let $f : \Lambda^{\mathbb{N}} \times \overline{\mathbb{C}} \to \Lambda^{\mathbb{N}} \times \overline{\mathbb{C}}$ be the fibered rational map over the shift map $\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ $(\sigma(h_1, h_2, h_3, \dots) = (h_2, h_3, \dots))$ defined as $f((h_1, h_2, \dots),\ y) = ((h_2, h_3, \dots),\ h_1(y))$. Then, by Theorem 1.6, there exist positive constants C_0 and C_1 such that each $g \in G$ satisfies the conditions that $J(g)$ is C_0-uniformly perfect and that diam $J(g) > C_1$.

Let H be any subsemigroup of G. Let A be an annulus that separates $J(H)$. Let V_1 and V_2 be two connected components of $\overline{\mathbb{C}} \setminus A$. Since $J(G) = \overline{\bigcup_{g \in G} J(g)}$ (Corollary 3.1 in [HM1]) , there exist elements g_1 and g_2 in H such that $J(g_1) \cap V_1 \neq \emptyset$ and $J(g_2) \cap V_2 \neq \emptyset$.

If $J(g_1) \cap V_2 \neq \emptyset$ or $J(g_2) \cap V_1 \neq \emptyset$, then mod $A \leq C_0$. If $J(g_1) \cap V_2 = \emptyset$ and $J(g_2) \cap V_1 = \emptyset$, then, by Lemma 6.1 on p. 34 in [LV], there exists a constant $C = C(C_1)$ that depends only on C_1 such that mod $A \leq C$. The second statement follows from the uniform perfectness of $J(H)$ and Theorem 4.1 in [HM2]. □

To show Theorem 1.25, we present some notation and lemmas.

Notation: Throughout the rest of this section, for a fixed generator system $\{h_1, \dots, h_m\}$, let $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the fibered rational map over the shift map $\sigma : \Sigma_m \to \Sigma_m$, where $\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$, associated with the generator system $\{h_1, \dots, h_m\}$. We set $q_x^{(n)}(y) = \pi_{\overline{\mathbb{C}}}(f_x^n(y))$, for any $(x, y) \in \Sigma_m \times \overline{\mathbb{C}}$.

Lemma 2.13. *Let E be a finite subset of $\overline{\mathbb{C}}$. Let $\langle h_1, \dots, h_m \rangle$ be a rational semigroup. Then, for any number $M > 0$, there exists a positive integer n_0 such that, for any $(n, x, y) \in \mathbb{N} \times \Sigma_m \times E$ with $n \geq n_0$ that satisfies all of the following conditions:*

1. *$q_x^{(j)}(y) \in E$ for $j = 0, \dots, n$*
2. *$(q_x^{(n)})'(y) \neq 0$ and*
3. *for any $i \in \mathbb{N}$ and $j \in \mathbb{N}$ with $i + j \leq n$, if $q_{\sigma^i(x)}^{(j)}(q_x^{(i)}(y)) = q_x^{(i)}(y)$, then $|(q_{\sigma^i(x)}^{(j)})'(q_x^{(i)}(y))| > 1$,*

we have that $\|(q_x^{(n)})'(y)\| > M$, where $\| \cdot \|$ denotes the norm of the derivative with respect to the spherical metric.

Proof. This lemma can be shown by induction on $\sharp E$ by means of the same method that was employed in Lemma 1.32 in [S4]. □

Lemma 2.14. *Let $G = \langle h_1, \dots, h_m \rangle$ be a finitely generated rational semigroup. Suppose $\sharp(UH(G) \cap J(G)) < \infty$ and $UH(G) \cap J(G) \neq \emptyset$. Then, for each $z \in UH(G) \cap J(G)$, there exists an element $g \in G$, an element $h \in G$, and a point $w \in UH(G) \cap J(G)$ such that $h(w) = z$, $g(w) = w$ and $|g'(w)| \leq 1$.*

Proof. Suppose that there exists a point $z \in UH(G) \cap J(G)$ for which there exists no (g, h, w) in the conclusion of our lemma. Then, by Lemma 2.13 and the Koebe distortion theorem, it can easily be seen that, for an arbitrarily small $\epsilon > 0$, there exists a positive number δ and a positive constant N such that, if a point $w_0 \in UH(G) \cap J(G)$ and an element $g_0 \in G$ satisfy $g_0(w_0) = z$, then the diameter of the component V of $g_0^{-1}(B(z,\delta))$ containing w_0 is less than ϵ and $\deg(g_0 : V \to B(z,\delta)) \leq N$. Then, for ϵ sufficiently small, since G is finitely generated and $\sharp(UH(G) \cap J(G)) < \infty$, we can easily obtain that there exists a positive constant N' such that, for any element $g_1 \in G$, and any component W of $g_1^{-1}(V)$, it follows that $\deg(g_1 : W \to V) \leq N'$. This implies that $z \in SH_{N+N'}(G)$, and this contradicts that $z \in UH(G)$. $\square$

In the following Lemma 2.15, Lemma 2.16, and Lemma 2.17, we suppose the assumption of Theorem 1.25. Furthermore, we assume either (1) $m \geq 2$, or (2) $m = 1$ and $J(G) \neq \overline{\mathbb{C}}$.

Lemma 2.15. *If $m \geq 2$, then there exists a disk D in $F(G)$ such that*

1. *$\overline{\bigcup_{g \in G} g(D)} \subset F(G)$ and*
2. *diam $q_x^{(n)}(D) \to 0$ as $n \to \infty$ uniformly on $x \in \Sigma_m$.*

In particular, the fibered rational map f satisfies condition (C2).

Proof. Suppose the case $m \geq 2$. Let $h \in G$ be an element of degree two or greater. Since $\emptyset \neq \overline{\mathbb{C}} \setminus \overline{U} \subset F(G)$ and $UH(G) \cap J(G) \subset U$, there exists an attracting periodic point z_0 of h in $F(G) \setminus U$. Since $z_0 \in UH(G)$ and $UH(G) \cap J(G) \subset U$ again, it follows that there exists a disk D around z_0 such that $\overline{\bigcup_{g \in G} g(D)} \subset F(G)$. Now the second statement of the lemma results from Lemma 1.30 in [S4].

$\square$

Lemma 2.16. *If $UH(G) \cap J(G) \neq \emptyset$, then, for each point $z \in UH(G) \cap J(G)$, there exists an element $h \in G$ such that z is a parabolic fixed point of h. Furthermore, $\{\beta \in G \mid \beta(z) = z\} = \{g_z^n \mid n \in \mathbb{N}\}$ for some $g_z \in G$ such that z is a parabolic fixed point of g_z.*

Proof. Let $z \in UH(G) \cap J(G)$ be a point. Then $z \in U$. By Lemma 2.14, there exists an element $g \in G$, an element $\alpha \in G$, and a point $w \in UH(G) \cap J(G)$ such that $\alpha(w) = z$, $g(w) = w$, and $|g'(w)| \leq 1$. We write $\alpha = h_{i_k} \cdots h_{i_1}$ and $g = h_{j_l} \cdots h_{j_1}$. We may assume $l \geq k$. Since $w \in g^{-1}(U) \cap \alpha^{-1}(U)$, by the open set condition we obtain $(i_1, \dots i_k) = (j_1, \dots, j_k)$. Let $h := h_{j_k} \cdots h_{j_1} \cdot h_{j_l} \cdots h_{j_{k+1}}$. Then we easily obtain $h(z) = z$. Furthermore, $|h'(z)| = |g'(w)| \leq 1$.

If $\deg(h) = 1$, then, by Lemma 2.15, it follows that z is a repelling fixed point of h. However, this is a contradiction. If $\deg(h) \geq 2$, then, since it is assumed that $\sharp(UH(G) \cap J(G)) < \infty$, there exists no recurrent critical point c of h in $J(h)$. From [Ma], it follows that z is an attracting or parabolic fixed point of h. Suppose that z is an attracting fixed point of h. Then there exists an open neighborhood V of z in U such that $h(V) \subset V$. Let $x \in \Sigma_m$ be a point such that $h_{x_n} \cdots h_{x_1} = h$, for some n, where $x = (x_1, x_2, \dots)$. Then, by the open set condition, it follows that $h_{x'_n} \cdots h_{x'_1}(V) \subset \overline{\mathbb{C}} \setminus U$, for any $(x'_1, \dots, x'_n) \in \{1, \dots, m\}^n \setminus \{(x_1, \dots, x_n)\}$. Hence, G is normal in V, and this is also a contradiction.

Let g_z be an element in $G_z := \{\beta \in G \mid \beta(z) = z\}$ with minimal word length with respect to the generator system $\{h_1, \dots, h_m\}$. Let $\beta \in G_z$. Then $z \in \beta^{-1}(U) \cap g_z^{-1}(U)$. By the open set condition, it follows that $\beta = \gamma g_z$ for some $\gamma \in G$. Then $\gamma \in G_z$. Applying the same argument again to γ, we see that $\beta = g_z^n$ for some $n \in \mathbb{N}$. $\square$

Lemma 2.17. *We have that for each* $(x,y) \in \pi_{\overline{\mathbb{C}}}^{-1}\left(G^{-1}(J(G)\setminus UH(G))\right) \cap \tilde{J}(f)$, $\limsup_{n\to\infty} d(q_x^{(n)}(y),\ UH(G)) > 0$.

Proof. Let (x,y) be a point in $\pi_{\overline{\mathbb{C}}}^{-1}\left(G^{-1}\left(J(G)\setminus UH(G)\right)\right)\cap\tilde{J}(f)$. Then $q_x^{(n)}(y) \in J(G)\setminus UH(G)$, for each $n\in\mathbb{N}$.

Assume that $\lim_{n\to\infty} d(q_x^{(n)}(y), UH(G)) = 0$. We now derive a contradiction. Let $\epsilon > 0$ be a small number and let A_ϵ be the ϵ-neighborhood of $UH(G)\cap J(G)$ in $\overline{\mathbb{C}}$. Then there exists a number $n_0\in\mathbb{N}$ such that $q_x^{(n)}(y)\in A_\epsilon$, for each $n\geq n_0$.

For each $n\geq n_0$, let $z_n\in UH(G)\cap J(G)$ be the unique point such that $d(z_n, q_x^{(n)}(y)) < \epsilon$. Since $g(UH(G))\subset UH(G)$, for each $g\in G$, it may be assumed that $q^{(1)}_{\sigma^n(x)}(z_n) = z_{n+1}$, for each $n\geq n_0$.

Since $\sharp(UH(G)\cap J(G)) < \infty$, there exists a positive integer $n_1\geq n_0$ and $l\in\mathbb{N}$ such that $z_{n_1+l} = z_{n_1}$. Let $g_1\in G$ be an element such that $g_1(z_{n_1}) = z_{n_1}$. Let $w\in\{1,\dots,m\}^l$ be the word such that $h_{w_l}\circ\cdots\circ h_{w_1} = g_1$. Then, by the open set condition, $\sigma^{n_1}(x) = w^\infty$. Since we assume that $d(q_x^{(n)}(y),\ UH(G))\to 0$ as $n\to\infty$, by $z_{n_1+l} = z_{n_1}$ it follows that $g_1^k(q_x^{(n_1)}(y))\to z_{n_1}$ as $k\to\infty$. Hence, by Lemma 2.16 it must be that z_{n_1} is a parabolic fixed point of g_1, and $q_x^{(n_2)}(y)$ belongs to $W\cap\mathcal{P}$, where W is a small neighborhood of z_{n_1} in U, $\mathcal{P}$ is the union of the attracting petals of g_1 at z_{n_1}, and n_2 is a large positive number with $n_2\geq n_1$. Then there exists an open neighborhood V of y such that $q_x^{(n_2)}(V)\subset W\cap\mathcal{P}$. Taking W sufficiently small and n_2 sufficiently large, it may be assumed that $g_1^s(q_x^{(n_2)}(V))\subset W\cap\mathcal{P}$, for any $s\in\mathbb{N}$. It follows that $q_x^{(n)}(V)\subset U$ for each $n\in\mathbb{N}$. By the open set condition, for each $n\in\mathbb{N}$, it follows that $q_{x'}^{(n)}(V)\subset\overline{\mathbb{C}}\setminus U$, for each $x'\in\Sigma_m$ with $(x'_1,\dots,x'_n)\in\{1,\dots,m\}^n\setminus\{(x_1,\dots,x_n)\}$. Hence, G is normal in V, and this contradicts that $y\in J(G)$.

□

We now present a proof of Theorem 1.25.

Proof of Theorem 1.25. By Lemma 2.3 in [S5], we have $J(G)\subset\overline{U}$. Suppose $J(G)\neq\overline{U}$. Then, by Proposition 4.3 in [S4], $\text{int}J(G) = \emptyset$. For any $y'\in J(G)$ and $r > 0$, we set

$$h(y',r) = \sup\{s \mid \exists y''\in\overline{\mathbb{C}},\ B(y'',s)\subset F(G)\cap B(y',r)\cap U\}$$

and $h(r) = \inf\{h(y',r)\mid y'\in J(G)\}$. Then, since int $J(G)=\emptyset$ and $J(G)\subset\overline{U}$, it follows that $h(r) > 0$ for any $r > 0$.

Let $\delta_0 > 0$ be a small number. Let B be the δ_0-neighborhood of $UH(G)\cap J(G)$ in $\overline{\mathbb{C}}$. By Lemma 2.5 and Lemma 2.15, there exists a positive number δ_1 and a number $N\in\mathbb{N}$ such that, for any $y'\in J(G)\setminus B$, $0<\delta\leq\delta_1$ and any $V\in c(B(y',2\delta),\ g)$, V is simply connected and $\deg(g: V\to B(y',2\delta))\leq N$. By Lemma 2.16 and the open set condition, the condition 4 in Theorem 1.18 holds. Hence by Lemma 2.17 and Theorem 1.18, we obtain

$$\tilde{J}(f) = \bigcup_{x\in\Sigma_m} J_x. \tag{7}$$

Let $y\in J(G)$ be a point. Since $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$ (Proposition 3.2 in [S5]), by (7) there exists a point $x\in\Sigma_m$ such that $y\in J_x$.

Let $\delta_2 = \min\{\delta_0, \delta_1\}$. Let r be a positive number. We set $B_n = q_x^{(n)}(B(y,r))$ and $y_n = q_x^{(n)}(y)$, for each $n \in \mathbb{N}$. Since $y \in J_x$, there exists a smallest positive integer n_0 such that diam $B_{n_0+1} > \delta_2$. Then there exists a constant l_0 such that $l_0\delta_2 <$diam B_{n_0}.

Case 1. $y_{n_0} \in J(G) \setminus B$.

By Corollary 2.3 in [Y], there exists a constant K that depends only on N, and a ball $B(y_{n_0}, r_0) \subset B_{n_0}$ with $r_0 \geq$diam $B_{n_0}/K \geq \frac{l_0\delta_2}{K}$ such that the element $W \in c(B(y_{n_0}, r_0),\ q_x^{(n_0)})$ containing y is a subset of $B(y,r)$. There exists a ball $B(y', \frac{2}{3}h(r_0))$ included in $B(y_{n_0}, r_0) \cap F(G) \cap U$.

Let D_0 be the element of $c(B(y', \frac{1}{2}h(r_0)),\ q_x^{(n_0)})$ contained in W. By the open set condition, we obtain that $g^{-1}(U \cap F(G)) \subset U \cap F(G)$, for each $g \in G$. Hence, it follows that $D_0 \subset F(G) \cap U$. Let $y'' \in D_0 \cap (q_x^{(n_0)})^{-1}\{y'\}$ be a point. Then, by Corollaries 1.8 and 1.9 in [S4], Dist $(\partial D_0, y'') \leq M$ for some M that depends only on N. Further, by the same argument that was used in the proof of Theorem 1.16, diam $D_0 \asymp r$. Hence, there exists a constant $0 < k < 1$ that does not depend on y and r such that $B(y'', kr) \subset D_0 \subset F(G) \cap B(y,r)$.

Case 2. $y_{n_0} \in B$.

Making use of Lemma 2.16, the fact that $UH(G) \cap J(G) \subset U$, taking a sufficiently small δ_0, and the method used on pp. 286-287 in [Y], one can show that there exists a ball $B(y'', k'r)$ in $B(y,r) \cap F(G)$, where k' is a constant with $0 < k' < 1$ that does not depend on y and r. $\square$

We now prove Proposition 1.27.

Proof of Proposition 1.27. By the open set condition and Lemma 2.3 (f) in [S5], $J(G) \subset \overline{U}$. We now show the following.

Claim 1: There exists an open set V' included in $U \cap F(G)$ such that $h^{-1}(V') \cap V' = \emptyset$, for each $h \in G$.

Before supporting this claim, we would like the reader to note that the following claim can easily be shown.

Claim 2: If there exists a point $z \in U \cap F(G)$ such that $z \in \overline{\mathbb{C}} \setminus \overline{G(z)}$, then Claim 1 holds with a small open neighborhood V' of z.

As a proof of Claim 1, by the open set condition, we obtain that

$$\bigcup_{j=1}^{m} h_j^{-1}(U \cap F(G)) \subset U \cap F(G). \tag{8}$$

Suppose the equality does not hold in (8). Then there exists a point $z \in U \cap F(G)$ such that $h_j(z) \in \overline{\mathbb{C}} \setminus U$ for each $j = 1, \dots, m$. Hence, by the open set condition, $z \in \overline{\mathbb{C}} \setminus \overline{G(z)}$. By Claim 2, Claim 1 holds.

Hence, it may be assumed that

$$\bigcup_{j=1}^{m} h_j^{-1}(U \cap F(G)) = U \cap F(G). \tag{9}$$

Let $\alpha : U \cap F(G) \to U \cap F(G)$ be the map defined as: $\alpha(z) = h_j(z)$ if $z \in h_j^{-1}(U \cap F(G))$. This is well defined by (9) and the open set condition.

Let $z \in U \cap F(G)$ be a point. If $z \in \overline{\mathbb{C}} \setminus \overline{G(z)}$, then, by Claim 2, Claim 1 follows. Hence, it may be assumed that $z \in \overline{G(z)}$; i.e.,

$$z \in \overline{\bigcup_{n=0}^{\infty} \{\alpha^n(z)\}}. \tag{10}$$

Let W be the connected component of $U \cap F(G)$ containing z. By (10), there exists a smallest positive integer n with $\alpha^n(W) \subset W$. By (10) and the open set condition, one of the following two cases holds.

Case 1: W is included in an attracting or a parabolic basin of an element $g \in G$, z is a fixed point in the basin, and $g|_W = \alpha^n|_W$.

Case 2: W is included in a Siegel disk or a Herman ring of an element $g \in G$ of degree 2 or greater, and $g|_W = \alpha^n|_W$.

If Case 1 holds, then there exists an open set V' included in W with $\alpha^{-l}(V') \cap V' = \emptyset$, for each $l \in \mathbb{N}$; i.e., $h^{-1}(V') \cap V' = \emptyset$, for each $h \in G$.

If Case 2 holds, then, for V' in a connected component A of $\alpha^{-n}(W)$ with $A \cap W = \emptyset$, it follows that $\alpha^{-l}(V') \cap V' = \emptyset$, for each $l \in \mathbb{N}$; i.e., $h^{-1}(V') \cap V' = \emptyset$, for each $h \in G$.

Hence, we have proved Claim 1. Let V' be an open set included in $U \cap F(G)$ such that $h^{-1}(V') \cap V' = \emptyset$ for each $h \in G$. Then, by the open set condition, $g^{-1}(V') \cap h^{-1}(V') = \emptyset$, if $g, h \in G$ and $g \neq h$. Furthermore, the postcritical set $P(G)$ of G, which equals $\overline{\bigcup_{g \in G} g\left(\bigcup_{j=1}^{m}\{\text{critical values of } h_j\}\right)}$, does not accumulate in V'. Let V be an open disk included in $V' \setminus P(G)$. Then, it follows that $\int_V \sum_{h \in G} \sum_\alpha \|\alpha'(z)\|^2 \, dm(z) < \infty$, where α runs over all the well-defined inverse branches of h on V. Hence, for almost every $x \in V$ with respect to the Lebesgue measure, it follows that $S(2, x) < \infty$. □

We now prove Theorem 1.28. We first present several lemmas.

Lemma 2.18. *Let G be a rational semigroup. Assume that $\infty \in F(G)$ and that, for each $x \in E(G)$, there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Let A be a subset of $J(G)$. Suppose that there exist positive constants a_1, a_2 and c with $0 < c < 1$ such that, for each $x \in A$, there exist two sequences (r_n) and (R_n) of positive real numbers and a sequence (g_n) of the elements of G satisfying all of the following conditions:*

1. *$r_n \to 0$ and, for each n, $0 < \frac{r_n}{R_n} < c$ and $g_n(x) \in J(G)$.*
2. *for each n, $g_n(D(x, R_n)) \subset D(g_n(x), a_1)$.*
3. *for each n $g_n(D(x, r_n)) \supset D(g_n(x), a_2)$.*

Then, $dim_H(A) \leq s(G)$.

Proof. We may assume that $\sharp(J(G)) \geq 3$. Let $\delta \geq s(G)$ be a number and μ a δ-subconformal measure. Using the method employed in the proof of Lemma 5.5 in [S4], one can show that there exists a constant $c' > 0$ that does not depend on $n \in \mathbb{N}$ and $x \in A$ such that $\mu(D(x, r_n)) \geq c' r_n^\delta$. From this and Theorem 7.2 in [Pe], it follows that $\dim_H A \leq \delta$. □

Definition 2.19. (**Conical set for rational semigroups**) Let G be a rational semigroup. Let $N \in \mathbb{N}$ and $r > 0$. We denote by $J_{con}(G, N, r)$ the set of points

$z \in J(G)$ such that, for any $\epsilon > 0$, there exists an element $g \in G$ such that $g(z) \in J(G)$ and the element $U \in c(B(g(z), r),\ g)$ containing z satisfies the following conditions:

1. $\operatorname{diam} U \leq \epsilon$,
2. U is simply connected, and
3. $\deg(g : U \to B(g(z), r)) \leq N$.

We set $J_{con}(G, N) = \cup_{r>0} J_{con}(G, N, r)$ and $J_{con}(G) = \cup_{N \in \mathbb{N}} J_{con}(G, N)$.

Proposition 2.20. *Let G be a rational semigroup. Assume that $F(G) \neq \emptyset$ and that, for each $x \in E(G)$, there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Then, $dim_H(J_{con}(G)) \leq s(G)$.*

Proof. We now have only to show the following:

Claim: For fixed $N \in \mathbb{N}$ and $r > 0$, $\dim_H(J_{con}(G, N, r)) \leq s(G)$.

We may assume that $\infty \in F(G)$. Let $x \in J_{con}(G, N, r)$ be a point. Then there exists a sequence (g_n) in G such that, for each $n \in \mathbb{N}$, we have that $g_n(x) \in J(G)$, $\deg(g_n : V_n(r) \to D(g_n(x), r)) \leq N$, $V_n(r)$ is simply connected and $\operatorname{diam} V_n(r) \to 0$ as $n \to \infty$, where $V_n(r)$ is the element of $c(D(g_n(x), r),\ g_n)$ containing x. Let $\varphi_n : D(0, 1) \to V_n(r)$ be the Riemann map such that $\varphi_n(0) = x$. By the Koebe distortion theorem, for each n, $V_n(r) \supset D(x, \frac{1}{4}|\varphi_n'(0)|)$. By Lemma 2.3 and the Koebe distortion theorem, there exists an $\epsilon > 0$ such that, for each $n \in \mathbb{N}$, $V_n(\epsilon r) \subset D(x, \frac{1}{8}|\varphi_n'(0)|)$. Since $\operatorname{diam} V_n(r) \to 0$ as $n \to \infty$, it follows that $|\varphi_n'(0)| \to 0$ as $n \to \infty$. The application of Lemma 2.18 proves the claim. □

Definition 2.21. (**Good points for finitely generated rational semigroups**) Let $G = \langle h_1, \ldots, h_m \rangle$ be a rational semigroup. Let $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the fibered rational map associated with the generator system $\{h_1, \ldots, h_m\}$. Then we set $J_{good}(G) = \pi_{\overline{\mathbb{C}}}(\tilde{J}_{good}(f))$. Note that this definition does not depend on the choice of any generator system of G that consists of finitely many elements.

We now prove the following theorem.

Theorem 2.22. *Let $G = \langle h_1, \ldots, h_m \rangle$ be a finitely generated rational semigroup with $F(G) \neq \emptyset$. Let $f : Y \to Y$ be the fibered rational map associated with the generator system $\{h_1, \ldots, h_m\}$, where $Y = \Sigma_m \times \overline{\mathbb{C}}$. Suppose that f satisfies condition (C1) and that, for each $x \in \Sigma_m$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x. Then, $J_{good}(G) \subset J_{con}(G)$ and $\dim_H(J_{good}(G)) \leq s(G) \leq s_0(G)$.*

Proof. We may assume that $\sharp(J(G)) \geq 3$. We now show the following:

Claim: If $E(G) \neq \emptyset$, then, for each $x \in E(G)$, there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$.

If there exists an element $h \in G$ with $\deg(h) \geq 2$, then this claim is trivial. Suppose that each element of G is of degree one. By Lemma 2.3 (d) in [S5], $\sharp(E(G)) \leq 2$. Since f satisfies condition (C1), for each i, h_i is loxodromic. Since $h_i(E(G)) = E(G)$, for each i, it must be that each $x \in E(G)$ is fixed by h_i, for each i. Let $x \in E(G)$ be a point. Suppose $|h_i'(x)| > 1$, for each i. Then,

$J(G) = \{x\}$, and this is a contradiction, since we assume that $\sharp(J(G)) \geq 3$. Hence, $|h_i'(x)| < 1$ for some i. Hence, the claim holds.

The statement of this theorem follows from the claim, the second statement in Proposition 2.11, Proposition 2.20 and Theorem 4.2 in [S2]. □

We now prove Theorem 1.28.

Proof of Theorem 1.28. This follows from Lemma 2.15, $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$ (Proposition 3.2 in [S5]), Lemma 2.17 and Theorem 2.22, when $m \geq 2$.

We now suppose $m = 1$ and $J(G) \neq \overline{\mathbb{C}}$. Then, by [Ma], $J_{good}(G) \subset J_{con}(G)$. By Lemma 2.17 and Proposition 2.20, it follows that $\dim_H(J(G)) \leq s(G)$. Moreover, by Theorem 4.2 in [S2], we have $s(G) \leq s_0(G)$.

Hence, we have proved Theorem 1.28. □

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