The space of 2-generator postcritically bounded polynomial semigroups and random complex dynamics *

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Abstract

We investigate the dynamics of 2-generator semigroups of polynomials with bounded planar postcritical set and associated random dynamics on the Riemann sphere. Also, we investigate the space $\mathcal{B}$ of such semigroups. We show that for a parameter h in the intersection of $\mathcal{B}$, the hyperbolicity locus $\mathcal{H}$ and the closure of the disconnectedness locus (the space of parameters for which the Julia set is disconnected), the corresponding semigroup satisfies either the open set condition (and Bowen's formula) or that the Julia sets of the two generators coincide. Also, we show that for such a parameter h, if the Julia sets of the two generators do not coincide, then there exists a neighborhood U of h such that for each parameter in U, the Hausdorff dimension of the Julia set of the corresponding semigroup is strictly less than 2. Moreover, we show that the intersection of the connectedness locus and $\mathcal{B} \cap \mathcal{H}$ has dense interior. By using the results on the semigroups corresponding to these parameters, we investigate the associated functions which give the probability of tending to ∞ (complex analogues of the devil's staircase or Lebesgue's singular functions) and complex analogues of the Takagi function.

1 Introduction

Some partial results of this paper have been announced in [25] without any proofs. We investigate the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps where the semigroup operation is functional composition) on the Riemann sphere $\hat{\mathbb{C}}$ and random dynamics of polynomials. We focus on the **randomness-induced phenomena** (i.e., phenomena which can hold in random dynamical systems but cannot hold in deterministic iteration dynamical systems) and the **gradation between chaos and order** in random complex dynamical systems and we study complex analogues of the devil's staircase, Lebesgue's singular functions and the Takagi function which are Hölder continuous on $\hat{\mathbb{C}}$ and vary only on **connected** thin fractal Julia sets of the associated polynomial semigroups.

The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([5]). For research on random dynamics of quadratic polynomials, see [2, 3, 6]. For recent research and motivations on random complex dynamics, see the author's works [21]–[26], [9]. In order to investigate random complex dynamical systems, it is very natural to study the dynamics of associated polynomial semigroups. In fact, it is a very powerful tool to investigate random complex dynamics, since random complex dynamics and the dynamics of polynomial semigroups are related to each other very deeply. The first study of dynamics of polynomial semigroups was conducted by A. Hinkkanen and G. J. Martin ([8]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([7]), who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ of a finitely generated polynomial semigroup G generated by $\{h_1, \ldots, h_m\}$ has "backward self-similarity," i.e., $J(G) = \cup_{j=1}^m h_j^{-1}(J(G))$ (see [14, Lemma 1.1.4]),

the study of the dynamics of polynomial semigroups can be regarded as the study of "backward iterated function systems," and also as a generalization of the study of self-similar sets in fractal geometry. For recent work on the dynamics of polynomial semigroups, see the author's papers [14]–[27], and [12, 13, 28, 29, 30, 10].

To introduce the main idea of this paper, we let G be a polynomial semigroup and denote by $F(G)$ the Fatou set of G, which is defined to be the maximal open subset of $\hat{\mathbb{C}}$ where G is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$. We call $J(G) := \hat{\mathbb{C}} \setminus F(G)$ the Julia set of G. For a polynomial map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we denote by $\mathrm{CV}(g)$ the set of critical values of $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We set $\mathrm{CV}^*(g) := \mathrm{CV}(g) \setminus \{\infty\}$. For a polynomial semigroup G, we set $P(G) := \overline{\cup_{g \in G} \mathrm{CV}(g)}$ (the closure is taken in $\hat{\mathbb{C}}$) and $P^*(G) := P(G) \setminus \{\infty\}$. The set $P(G)$ is called the postcritical set of G and $P^*(G)$ is called the planar postcritical set of G. Note that if G is generated by a family Λ of polynomials, i.e., $G = \{g_1 \circ \cdots \circ g_n \mid n \in \mathbb{N}, \forall g_j \in \Lambda\}$, then $P(G) = \overline{\cup_{h \in G \cup \{Id\}} h(\cup_{g \in \Lambda} \mathrm{CV}(g))}$. In particular, $h(P(G)) \subset P(G)$ for each $h \in G$. For a polynomial semigroup G, we set $\hat{K}(G) := \{z \in \mathbb{C} \mid \cup_{g \in G} \{g(z)\} \text{ is bounded in } \mathbb{C}\}$.

Let $\mathcal{P}$ be the space of all polynomial maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\deg(g) \geq 2$ endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Let $\mathcal{P}_n = \{f \in \mathcal{P} \mid \deg(f) = n\}$. We remark that $\deg : \mathcal{P} \to \mathbb{N}$ is continuous and for each $n \geq 2$, $\mathcal{P}_n$ is a connected component of $\mathcal{P}$, $\mathcal{P}_n$ is an open and closed subset of $\mathcal{P}$, and $\mathcal{P}_n \cong (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n$.

For each $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$, we denote by $\langle h_1, \ldots, h_m \rangle$ the polynomial semigroup generated by $\{h_1, \ldots, h_m\}$, i.e., $\langle h_1, \ldots, h_m \rangle = \{h_{i_1} \circ \cdots \circ h_{i_n} \in \mathcal{P} \mid n \in \mathbb{N}, \forall i_j \in \{1, \ldots, m\}\}$. Moreover, we set $F(h_1, \ldots h_m) := F(\langle h_1, \ldots, h_m \rangle)$, $J(h_1, \ldots, h_m) := J(\langle h_1, \ldots, h_m \rangle)$, $P(h_1, \ldots, h_m) := P(\langle h_1, \ldots, h_m \rangle)$, $P^*(h_1, \ldots, h_m) := P^*(\langle h_1, \ldots, h_m \rangle)$, and $\hat{K}(h_1, \ldots, h_m) := \hat{K}(\langle h_1 \ldots, h_m \rangle)$.

We say that a polynomial semigroup G is **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$. We say that a polynomial semigroup G is **hyperbolic** if $P(G) \subset F(G)$. We are interested in the parameter space of 2-generator postcritically bounded polynomial semigroups.

Definition 1.1. We use the following notations.

- We set $\mathcal{B} := \{(h_1, h_2) \in \mathcal{P}^2 \mid P^*(h_1, h_2) \text{ is bounded in } \mathbb{C}\}$.
- We set $\mathcal{C} := \{(h_1, h_2) \in \mathcal{P}^2 \mid J(h_1, h_2) \text{ is connected}\}$.
- We set $\mathcal{D} := \{(h_1, h_2) \in \mathcal{P}^2 \mid J(h_1, h_2) \text{ is disconnected}\}$.
- We set $\mathcal{H} := \{(h_1, h_2) \in \mathcal{P}^2 \mid \langle h_1, h_2 \rangle \text{ is hyperbolic}\}$.
- We set $\mathcal{I} := \{(h_1, h_2) \in \mathcal{P}^2 \mid J(h_1) \cap J(h_2) \neq \emptyset\}$.
- We set $\mathcal{Q} := \{(h_1, h_2) \in \mathcal{P}^2 \mid J(h_1) = J(h_2), \text{ and } J(h_1) \text{ and } J(h_2) \text{ are quasicircles}\}$.

It is well-known that for an element $f \in \mathcal{P}$, $J(f)$ is connected if and only if $P^*(f)$ is bounded in $\mathbb{C}$. However, we have $\mathcal{B} \cap \mathcal{D} \neq \emptyset$ (e.g. $(z^3, 2z^3) \in \mathcal{B} \cap \mathcal{D}$). Moreover, we have the following.

Lemma 1.2 (Lemmas 5.4, 5.1 in [23])**.** *The sets $\mathcal{H}, \mathcal{B} \cap \mathcal{H}, \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$ are non-empty and open in the product space $\mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$.*

To present the first main result, we need some notations. For each $z \in \hat{\mathbb{C}}$, we denote by $T\hat{\mathbb{C}}_z$ the complex tangent space of $\hat{\mathbb{C}}$ at z. For a holomorphic map $\varphi : V \to \hat{\mathbb{C}}$ defined on a domain V, we denote by $D\varphi_z : T\hat{\mathbb{C}}_z \to T\hat{\mathbb{C}}_{g(z)}$ the differential map of φ at z. We denote by $\|D\varphi_z\|_s$ the norm of the derivative of φ at z with respect to the spherical metric on $\hat{\mathbb{C}}$.

Definition 1.3. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ be an element. Let $\Sigma_m := \{1, \ldots, m\}^{\mathbb{N}}$ endowed with product topology. This is a compact metric space. Let $\sigma : \Sigma_m \to \Sigma_m$ be the shift map, i.e., $\sigma(w_1, w_2, \ldots) = (w_2, w_3, \ldots)$. We define the map $\tilde{h} : \Sigma_m \times \hat{\mathbb{C}} \to \Sigma_m \times \hat{\mathbb{C}}$ by $\tilde{h}(w, y) = (\sigma(w), h_{w_1}(y))$, where $w = (w_1, w_2, \ldots) \in \Sigma_m$ and $y \in \hat{\mathbb{C}}$. The map $\tilde{h} : \Sigma_m \times \hat{\mathbb{C}} \to \Sigma_m \times \hat{\mathbb{C}}$ is called the **skew product map associated with** $h = (h_1, \ldots, h_m)$. For each $w \in \Sigma_m$, we denote by $F_w(\tilde{h})$ the maximal open subset of $\hat{\mathbb{C}}$ where the family $\{h_{w_n} \circ \cdots \circ h_{w_1}\}_{n \in \mathbb{N}}$ of polynomial maps is normal. We set $J_w(\tilde{h}) := \hat{\mathbb{C}} \setminus F_w(\tilde{h})$. We set $J(\tilde{h}) := \overline{\bigcup_{w \in \Sigma_m} \{w\} \times J_w(\tilde{h})}$, where the closure is taken

in the product space $\Sigma_m \times \hat{\mathbb{C}}$. We set $F(\tilde{h}) := (\Sigma_m \times \hat{\mathbb{C}}) \setminus J(\tilde{h})$. Let $\pi : \Sigma_m \times \hat{\mathbb{C}} \to \Sigma_m$ and $\pi_{\hat{\mathbb{C}}} : \Sigma_m \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the canonical projections. For each $n \in \mathbb{N}$ and each $(w,y) \in \Sigma_m \times \hat{\mathbb{C}}$, we set $D\tilde{h}^n_{w,y} := D(h_{w_n} \circ \cdots \circ h_{w_1})_y$. For all $w = (w_1, w_2, \ldots) \in \Sigma_m$ and $n \in \mathbb{N}$, we denote by w_n the n-th coordinate of w.

Definition 1.4. Let $h = (h_1, h_2) \in \mathcal{P}^2$. For each $z \in \hat{\mathbb{C}}$ and each $t \geq 0$, we set

$$Z_{(h_1,h_2)}(z,t) := \sum_{n=1}^{\infty} \sum_{(\omega_1,\ldots,\omega_n)\in\{1,2\}^n} \sum_{y\in(h_{\omega_n}\circ\cdots\circ h_{\omega_1})^{-1}(z)} \|D(h_{\omega_n} \circ \cdots \circ h_{\omega_1})_y\|_s^{-t},$$

counting multiplicities. Here, we set $0^{-t} := \infty$. We set $Z_{(h_1,h_2)}(z) := \inf\{t \geq 0 \mid Z_{(h_1,h_2)}(z,t) < \infty\}$, where we set $\inf \emptyset := \infty$. For each $h = (h_1, h_2) \in \mathcal{H}$, we denote by $\delta_{(h_1,h_2)}$ the unique zero of the pressure function $P(t) = P(\tilde{h}|_{J(\tilde{h})}, -t \log \varphi), t \geq 0$, where $P(\cdot, \cdot)$ denotes the topological pressure, and $\varphi(\omega, y) := -\log \|D(h_{\omega_1})_y\|_s$ (for the existence and uniqueness of the zero, see [18]).

Note that $(h_1, h_2) \mapsto \delta_{(h_1,h_2)}$ is real-analytic and plurisubharmonic on $\mathcal{H}$ (see [28]). If an element $h = (h_1, h_2) \in \mathcal{H}$ satisfies the **open set condition**, i.e., there exists a non-empty open subset U of $\hat{\mathbb{C}}$ such that $\cup_{j=1}^2 h_j^{-1}(U) \subset U$ and $\cap_{j=1}^2 h_j^{-1}(U) = \emptyset$, then the dynamics of $\langle h_1, h_2 \rangle$ has many interesting properties, e.g., $\dim_H(J(h_1, h_2)) = \delta_{(h_1,h_2)}$ (this is called Bowen's formula) where $\dim_H$ denotes the Hausdorff dimension with respect to the Euclidean distance (see [18]). Thus it is very interesting to consider when $(h_1, h_2) \in \mathcal{H}$ satisfies the open set condition.

We are interested in the semigroup and random dynamical system generated by an element (h_1, h_2) of a small neighborhood of $(\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}$. Note that by Lemma 1.2, we have $(\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H} = (\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H} \subset \mathcal{C} \cap \mathcal{B} \cap \mathcal{H}$. Under the above notations, we have the following result.

Theorem 1.5. *All of the following statements 1–5 hold.*

1. *Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. Then (h_1, h_2) satisfies the open set condition.*

2. *Let $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B}$ and let $G = \langle h_1, h_2 \rangle$. Then, $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, $J(G) = \amalg_{w\in\{1,2\}^{\mathbb{N}}} J_w(\tilde{h})$ ($\amalg$ denotes the disjoint union), each $J_w(\tilde{h})$ is connected, the map $w \mapsto J_w(\tilde{h})$ is continuous on $\{1,2\}^{\mathbb{N}}$ with respect to the Hausdorff metric, and (h_1, h_2) satisfies the open set condition.*

3. *Let $(h_1, h_2) \in \overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}$ and let $G = \langle h_1, h_2 \rangle$. Then, $J(G)$ is porous and*

$$\dim_H(J(G)) \leq \overline{\dim}_B(J(G)) < 2.$$

   Here, a compact subset X of $\hat{\mathbb{C}}$ is said to be porous if there exists a constant $0 < k < 1$ such that for each $x \in X$ and for each $0 < r < \operatorname{diam} \hat{\mathbb{C}}$, there exists a ball in $\{y \in \hat{\mathbb{C}} \mid d(y,x) < r\} \setminus X$ with radius at least kr. Moreover, $\overline{\dim}_B$ denotes the upper Box dimension with respect to the Euclidean distance.

4. *Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. Let $d_j := \deg(h_j)$ for each j. Then, for each $z \in \hat{\mathbb{C}} \setminus P(h_1, h_2)$,*

$$1 < \frac{\log(d_1 + d_2)}{\sum_{j=1}^2 \frac{d_j}{d_1+d_2} \log d_j} \leq \dim_H(J(h_1, h_2)) = \overline{\dim}_B(J(h_1, h_2)) = \delta_{(h_1,h_2)} = Z_{(h_1,h_2)}(z).$$

   Moreover, in addition to the assumption, if $\frac{\log(d_1+d_2)}{\sum_{j=1}^2 \frac{d_j}{d_1+d_2} \log d_j} = \dim_H(J(h_1, h_2))$, then $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$ and there exist a transformation $\varphi(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}, \alpha \neq 0$, two complex numbers a_1, a_2 and an integer $d \geq 3$ such that $d = d_1 = d_2$ and such that for each $j = 1, 2, \varphi \circ h_j \circ \varphi^{-1}(z) = a_j z^d$.

5. *Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. Then there exist an $\epsilon > 0$ and a neighborhood V of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$ such that for each $(g_1, g_2) \in V$ and for each $z \in \hat{\mathbb{C}} \setminus P(g_1, g_2)$,*

$$\dim_H(J(g_1, g_2)) \leq \overline{\dim}_B(J(g_1, g_2)) \leq \delta_{(g_1,g_2)} = Z_{(g_1,g_2)}(z) \leq 2 - \epsilon < 2.$$

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We remark that $(z^2, \frac{1}{2}z^2) \in \mathcal{B} \cap \mathcal{H}$ and $J(z^2, \frac{1}{2}z^2) = \{z \mid 1 \leq |z| \leq 2\}$, whose interior is not empty, and the author showed that there exists an open neighborhood U of $(z^2, \frac{1}{2}z^2)$ in $\mathcal{B} \cap \mathcal{H}$ such that for a.e. $(h_1, h_2) \in U$ with respect to the Lebesgue measure on $(\mathcal{P}_2)^2$, 2-dimensional Lebesgue measure of $J(h_1, h_2)$ is positive ([30, Theorem 1.6]). We also remark that for each $(d_1, d_2) \in \mathbb{N} \times \mathbb{N}$ with $d_1 \geq 2$, $d_2 \geq 2$, $(d_1, d_2) \neq (2, 2)$, we have $(z^{d_1}, z^{d_2}) \in (\partial C) \cap \mathcal{B} \cap \mathcal{H} \cap \mathcal{Q}$, since $(z^{d_1}, (1 + \frac{1}{n})z^{d_2}) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$.

We now present results on the topology of connectedness locus in the parameter space.

Theorem 1.6. *All of the following statements 1–4 hold.*

1. $\overline{\text{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H} = \mathcal{C} \cap \mathcal{B} \cap \mathcal{H}$. *Here, we denote by* $\text{int}(\mathcal{C})$ *the set of interior points of* $\mathcal{C}$ *with respect to the topology in* $\mathcal{P}^2$. *In particular, any element in* $(\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}$ *can be approximated by a sequence in* $(\text{int}(\mathcal{C})) \cap \mathcal{B} \cap \mathcal{H}$.

2. $\mathcal{D} \cap \mathcal{Q} = \emptyset$ *and* $\mathcal{D} \cap \mathcal{B} \cap \mathcal{I} = \emptyset$. *For each connected component* $\mathcal{V}$ *of* $\mathcal{P}^2$, $\mathcal{Q} \cap \mathcal{V}$ *is included in a proper holomorphic subvariety of* $\mathcal{V}$.

3. $((\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ *is an open and dense subset of* $(\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$ *which is endowed with the relative topology from* $\mathcal{P}^2$.

4. *Suppose that* $h_1 \in \mathcal{P}$, $\langle h_1 \rangle$ *is postcritically bounded, and* h_1 *is hyperbolic. Moreover, let* $d \in \mathbb{N}, d \geq 2$ *with* $(\deg(h_1), d) \neq (2, 2)$. *Then, there exists an element* $h_2 \in \mathcal{P}$ *such that*

$$(h_1, h_2) \in ((\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I} \subset ((\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q} \text{ and } \deg(h_2) = d.$$

We next present the main results on random complex dynamical systems associated with elements $(h_1, h_2) \in \mathcal{B}$. Let τ be a Borel probability measure on $\mathcal{P}$ with compact support. We consider the independent and identically distributed (i.i.d.) random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \mathcal{P}$ according to τ. Thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the state space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ from x to A is defined as $p(x, A) = \tau(\{g \in \mathcal{P} \mid g(x) \in A\})$. For a metric space X, let $\mathfrak{M}_1(X)$ be the space of all Borel probability measures on X endowed with the topology induced by weak convergence (thus $\mu_n \to \mu$ in $\mathfrak{M}_1(X)$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{R}$). Note that if X is a compact metric space, then $\mathfrak{M}_1(X)$ is compact and metrizable. For each $\tau \in \mathfrak{M}_1(X)$, we denote by $\text{supp}\,\tau$ the topological support of τ. Let $\mathfrak{M}_{1,c}(X)$ be the space of all Borel probability measures τ on X such that $\text{supp}\,\tau$ is compact. It is very interesting and important to consider the following function of probability of tending to ∞.

Definition 1.7. For each $h = (h_1, h_2) \in \mathcal{P}^2$, each $z \in \hat{\mathbb{C}}$, and each $p \in (0, 1)$, we use the following notations. We denote by $T(h_1, h_2, p, z)$ the probability of tending to $\infty \in \hat{\mathbb{C}}$ regarding the random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose h_1 with probability p and we choose h_2 with probability $1 - p$. More precisely, setting $\tau_{h_1, h_2, p} := p\delta_{h_1} + (1 - p)\delta_{h_2} \in \mathfrak{M}_1(\{h_1, h_2\})$ (δ_{h_i} denotes the Dirac measure concentrated at h_i), we set

$$T(h_1, h_2, p, z) := \tilde{\tau}_{h_1, h_2, p}(\{\gamma \in \{h_1, h_2\}^{\mathbb{N}} \mid \gamma_n \circ \cdots \circ \gamma_1(z) \to \infty \text{ as } n \to \infty\}),$$

where $\tilde{\tau}_{h_1, h_2, p} := \otimes_{n=1}^{\infty} \tau_{h_1, h_2, p} \in \mathfrak{M}_1(\{h_1, h_2\}^{\mathbb{N}})$. (Note: $z \mapsto T(h_1, h_2, p, z)$ is locally constant on $F(h_1, h_2)$. See [24, Lemma 3.24].)

We now present results on the functions of probability of tending to ∞.

Theorem 1.8. *Statements 1 and 2 hold.*

1. *For each* $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B}) \cup ((\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H})$ *and for each* $0 < p < 1$,
   $J(h_1, h_2) = \{z_0 \in \hat{\mathbb{C}} \mid \text{for each neighborhood } U \text{ of } z_0, z \mapsto T(h_1, h_2, p, z) \text{ is not constant on } U\}$.

2. Let $(h_1, h_2) \in (\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$ and let $0 < p < 1$. Then, the function $z \mapsto T(h_1, h_2, p, z)$ is continuous on $\hat{\mathbb{C}}$ if and only if $J(h_1) \cap J(h_2) = \emptyset$.

We next give the definition of mean stable systems, minimal sets and transition operator in order to present the results on the associated random dynamical systems and further results on the functions of probability of tending to ∞.

Definition 1.9. Let Γ be a non-empty compact subset of $\mathcal{P}$. Let G be the polynomial semigroup generated by Γ, i.e., $G = \{h_1 \circ \cdots \circ h_n \in \mathcal{P} \mid n \in \mathbb{N}, \forall h_j \in \Gamma\}$. We say that Γ is **mean stable** if there exist non-empty open subsets U, V of $F(G)$ and a number $n \in \mathbb{N}$ such that all of the following (1)–(3) hold: (1) $\overline{V} \subset U$ and $\overline{U} \subset F(G)$; (2) For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$, $\gamma_n \circ \cdots \circ \gamma_1(\overline{U}) \subset V$; (3) For each point $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ such that $g(z) \in U$.

Furthermore, for a $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, we say that τ is mean stable if $\mathrm{supp}\,\tau$ is mean stable.

The author showed that if $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ is mean stable, then the associated random dynamical system has many interesting properties, e.g. the chaos of the averaged system disappears at every point of $\hat{\mathbb{C}}$ due to the cooperation of many kinds of maps in the system, even though the iteration of each map of the system has a chaotic part ([24, 26]). Also, the author showed that the set of mean stable compact subsets Γ of $\mathcal{P}$ is open and dense in the space of all non-empty compact subsets of $\mathcal{P}$ with respect to the Hausdorff metric (see [26]). Those results are called the **cooperation principle**. Note that for every $f \in \mathcal{P}$, $\{f\}$ is not mean stable and $\langle f \rangle$ is chaotic in the Julia set $J(f) \neq \emptyset$. Thus the cooperation principle is a randomness-induced phenomenon which cannot hold in the deterministic iteration dynamics of an $f \in \mathcal{P}$. Note that randomness-induced phenomena have been a central interest in the study of random dynamics. Many physicists have observed various kinds of randomness-induced phenomena (sometimes physicists call them "noise-induced phenomena") by numerical experiments (e.g., [11]). In this paper, we consider how large the set $\mathcal{MS} := \{(h_1, h_2) \in \mathcal{P}^2 \mid \{h_1, h_2\} \text{ is mean stable}\}$ (resp. $\mathcal{MS} \cap \mathcal{B}$) is in $\mathcal{P}^2$ (resp. $\mathcal{B}$).

Definition 1.10. For a metric space X, we denote by $\mathrm{Cpt}(X)$ the space of all non-empty compact subsets of X endowed with the Hausdorff metric. For a polynomial semigroup G, we say that a non-empty compact subset L of $\hat{\mathbb{C}}$ is a minimal set for $(G, \hat{\mathbb{C}})$ if L is minimal in $\{C \in \mathrm{Cpt}(\hat{\mathbb{C}}) \mid \forall g \in G, g(C) \subset C\}$ with respect to inclusion. Moreover, we set $\mathrm{Min}(G, \hat{\mathbb{C}}) := \{L \in \mathrm{Cpt}(\hat{\mathbb{C}}) \mid L \text{ is a minimal set for } (G, \hat{\mathbb{C}})\}$.

Definition 1.11. For a compact metric space X, we denote by $C(X)$ the Banach space of all complex-valued continuous functions on X endowed with supremum norm $\|\cdot\|$. Let $(h_1, h_2) \in \mathcal{P}^2$ and let $p \in (0, 1)$. We denote by $M_{h_1, h_2, p} : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ the operator defined by $M_{h_1, h_2, p}(\varphi)(z) = p\varphi(h_1(z)) + (1-p)\varphi(h_2(z))$ for each $\varphi \in C(\hat{\mathbb{C}}), z \in \hat{\mathbb{C}}$. This $M_{h_1, h_2, p}$ is called the transition operator with respect to the random dynamical system associated with $\tau_{h_1, h_2, p} = p\delta_{h_1} + (1-p)\delta_{h_2}$. We denote by $M_{h_1, h_2, p}^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ the dual map of $M_{h_1, h_2, p}$, i.e., $\int \varphi(z)d(M_{h_1, h_2, p}^*(\mu))(z) = \int M_{h_1, h_2, p}(\varphi)(z)d\mu(z)$ for each $\varphi \in C(\hat{\mathbb{C}})$ and $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$.

Definition 1.12. For each $\alpha \in (0, 1)$, we set $C^{\alpha}(\hat{\mathbb{C}}) := \{\varphi \in C(\hat{\mathbb{C}}) \mid \sup_{x,y \in \hat{\mathbb{C}}, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^{\alpha}} < \infty\}$. Moreover, for each $\varphi \in C^{\alpha}(\hat{\mathbb{C}})$, we set $\|\varphi\|_{\alpha} := \sup_{z \in \hat{\mathbb{C}}} |\varphi(z)| + \sup_{x,y \in \hat{\mathbb{C}}, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^{\alpha}}$. Note that $C^{\alpha}(\hat{\mathbb{C}})$ is a Banach space with this norm $\|\cdot\|_{\alpha}$.

Definition 1.13. For a polynomial semigroup G with $\infty \in F(G)$, we denote by $F_{\infty}(G)$ the connected component of $F(G)$ containing ∞. Also, for an element $g \in \mathcal{P}$, we set $F_{\infty}(g) := F_{\infty}(\langle g \rangle)$.

Remark 1.14. It is easy to see that if G is generated by a compact subset of $\mathcal{P}$, then $\infty \in F(G)$.

We now present the results on the random dynamical systems generated by elements in $\mathcal{B}$ and further results on the functions of probability of tending to ∞.

Theorem 1.15. *Statements 1 and 2 hold.*

1. *Let $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B} \cap \mathcal{H}) \cup (((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I})$. Let $p \in (0,1)$. Then, there exist an open neighborhood V of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$, an open neighborhood W of p in $(0,1)$ and a constant $\alpha \in (0,1)$ such that all of the following (i)–(viii) hold.*

   (i) *For each $(g_1, g_2, q) \in V \times (0,1)$, $\{g_1, g_2\}$ is mean stable and $\tau_{g_1,g_2,q}$ is mean stable.*

   (ii) *For each $(g_1, g_2, q) \in V \times W$, $z \mapsto T(g_1, g_2, q, z)$ is α-Hölder continuous on $\hat{\mathbb{C}}$.*

   (iii) *Let $(g_1, g_2) \in V$. Then there exists a unique minimal set L_{g_1,g_2} for $(\langle g_1, g_2 \rangle, \hat{K}(\langle g_1, g_2 \rangle))$ and the set of minimal sets for $(\langle g_1, g_2 \rangle, \hat{\mathbb{C}})$ is $\{\{\infty\}, L_{g_1,g_2}\}$. Moreover, $\{\infty\} \cup L_{g_1,g_2} \subset F(g_1, g_2)$ and $L_{g_1,g_2} \subset int(\hat{K}(g_1, g_2))$.*

   (iv) *For each $(g_1, g_2, q, z) \in V \times W \times \hat{\mathbb{C}}$ there exists a Borel subset $\mathcal{B}_{g_1,g_2,q,z}$ of $\{g_1, g_2\}^{\mathbb{N}}$ with $\tilde{\tau}_{g_1,g_2,p}(\mathcal{B}_{g_1,g_2,q,z}) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{B}_{g_1,g_2,q,z}$, we have $d(\gamma_n \cdots \gamma_1(z), \{\infty\} \cup L_{g_1,g_2}) \to 0$ as $n \to \infty$.*

   (v) *For each $(g_1, g_2, q) \in V \times W$, there exists a unique $M^*_{g_1,g_2,q}$-invariant Borel probability measure $\nu = \nu_{g_1,g_2,q}$ on $\hat{K}(g_1, g_2)$ such that for each $\varphi \in C(\hat{\mathbb{C}})$,*

   $$M^n_{g_1,g_2,q}(\varphi)(z) \to T(g_1, g_2, q, z) \cdot \varphi(\infty) + (1 - T(g_1, g_2, q, z)) \cdot \int \varphi d\nu$$

   uniformly on $\hat{\mathbb{C}}$ as $n \to \infty$.

   (vi) *The map $(g_1, g_2, q) \in V \times W \mapsto T(g_1, g_2, q, \cdot) \in C(\hat{\mathbb{C}})$ is continuous on $V \times W$. The map $(g_1, g_2, q) \in V \times W \mapsto \nu_{g_1,g_2,q} \in \mathfrak{M}_1(\hat{\mathbb{C}})$ is continuous on $V \times W$.*

   (vii) *For each $(g_1, g_2) \in V$, there exist an open neighborhood W_{g_1,g_2} of p in W and a constant $\beta \in (0,1)$ such that for each $q \in W_{g_1,g_2}$, we have $T(g_1, g_2, q, \cdot) \in C^\beta(\hat{\mathbb{C}})$ and the map $q \mapsto T(g_1, g_2, q, \cdot) \in (C^\beta(\hat{\mathbb{C}}), \|\cdot\|_\beta)$ is real-analytic on W_{g_1,g_2}.*

   (viii) *Let $(g_1, g_2) \in V$. Let $G = \langle g_1, g_2 \rangle$. Then for each $z \in \hat{\mathbb{C}}$, the function $q \mapsto T(g_1, g_2, q, z)$ is real-analytic on $(0,1)$. Moreover, for each $n \in \mathbb{N} \cup \{0\}$, the function $(q, z) \mapsto (\partial^n T/\partial q^n)(g_1, g_2, q, z)$ is continuous on $(0,1) \times \hat{\mathbb{C}}$. Moreover, for each $q \in (0,1)$, the function $z \mapsto T(g_1, g_2, q, z)$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that $M_{g_1,g_2,q}(\varphi) = \varphi, \varphi|_{\hat{K}(G)} \equiv 0, \varphi|_{F_\infty(G)} \equiv 1$. Furthermore, inductively, for any $n \in \mathbb{N} \cup \{0\}$ and for any $q \in (0,1)$, the function $z \mapsto (\partial^{n+1} T/\partial q^{n+1})(g_1, g_2, q, z)$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that*

   $$\varphi(z) \equiv M_{g_1,g_2,q}(\varphi)(z) + (n+1)\left(\frac{\partial^n T}{\partial q^n}(g_1, g_2, q, g_1(z)) - \frac{\partial^n T}{\partial q^n}(g_1, g_2, q, g_2(z)) \right),$$
   $$\varphi|_{\hat{K}(G) \cup F_\infty(G)} \equiv 0.$$

   Moreover, for any $n \in \mathbb{N} \cup \{0\}$, the function $z \mapsto (\partial^n T/\partial q^n)(g_1, g_2, q, z)$ is Hölder continuous on $\hat{\mathbb{C}}$ and locally constant on $F(G)$. Moreover, for each $q \in (0,1)$, there exists a $\zeta \in (0,1)$ such that setting $\psi_{n,g_1,g_2,q}(z) = (\partial^n T/\partial q^n)(g_1, g_2, q, z)$ for each $z \in \hat{\mathbb{C}}$ and $n \in \mathbb{N} \cup \{0\}$, we have that $\psi_{n+1,g_1,g_2,q} = \sum_{j=0}^{\infty} M^j_{g_1,g_2,q}((n+1)(\psi_{n,g_1,g_2,q} \circ g_1 - \psi_{n,g_1,g_2,q} \circ g_2))$ in $(C^\zeta(\hat{\mathbb{C}}), \|\cdot\|_\zeta)$ for each $n \in \mathbb{N} \cup \{0\}$.

2. *Let $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B})$ and let $G = \langle h_1, h_2 \rangle$. Then, for each $p \in (0,1)$, the function $T(h_1, h_2, p, \cdot)$ is Hölder continuous on $\hat{\mathbb{C}}$ and for each $z \in \hat{\mathbb{C}}$, the function $p \mapsto T(h_1, h_2, p, z)$ is real-analytic on $(0,1)$. Moreover, for each $n \in \mathbb{N} \cup \{0\}$, the function $(p, z) \mapsto (\partial^n T/\partial p^n)(h_1, h_2, p, z)$ is continuous on $(0,1) \times \hat{\mathbb{C}}$. For each $p \in (0,1)$, the function $z \mapsto T(h_1, h_2, p, z)$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that $M_{h_1,h_2,p}(\varphi) = \varphi, \varphi|_{\hat{K}(G)} \equiv 0, \varphi|_{F_\infty(G)} \equiv 1$.*

Furthermore, inductively, for any $n \in \mathbb{N} \cup \{0\}$, the function $z \mapsto (\partial^{n+1} T / \partial p^{n+1})(h_1, h_2, p, z)$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that
$$\varphi(z) \equiv M_{h_1, h_2, p}(\varphi)(z) + (n+1)\left((\partial^n T / \partial p^n)(h_1, h_2, p, h_1(z)) - (\partial^n T / \partial p^n)(h_1, h_2, p, h_2(z))\right),$$
$$\varphi|_{\hat{K}(G) \cup F_\infty(G)} \equiv 0.$$

Moreover, setting $\psi_{n, h_1, h_2, p}(z) = (\partial^n T / \partial p^n)(h_1, h_2, p, z)$ for each $p \in (0,1), z \in \hat{\mathbb{C}}, n \in \mathbb{N} \cup \{0\}$, we have that $\psi_{n+1, h_1, h_2, p} = \sum_{j=0}^{\infty} M_{h_1, h_2, p}^j((n+1)(\psi_{n, h_1, h_2, p} \circ h_1 - \psi_{n, h_1, h_2, p} \circ h_2))$ in $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ for each $n \in \mathbb{N} \cup \{0\}$.

In order to present results on the pointwise Hölder exponents of the functions of probability of tending to ∞, we need the following definitions.

Definition 1.16. Let $h = (h_1, h_2) \in \mathcal{P}^2$. Let $p \in (0,1)$. We set $p_1 = p, p_2 = 1 - p$. Let $\eta_p = \otimes_{n=1}^{\infty}(\sum_{j=1}^{2} p_j \delta_j) \in \mathfrak{M}_1(\Sigma_2)$ be the Bernoulli measure on Σ_2 with respect to the weight (p_1, p_2). We denote by $\pi : \Sigma_2 \times \hat{\mathbb{C}} \to \Sigma_2$ the canonical projection onto Σ_2. Also, we denote by $\pi_{\hat{\mathbb{C}}} : \Sigma_2 \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the canonical projection onto $\hat{\mathbb{C}}$. It is known that there exists a unique $\tilde{h}$-invariant ergodic Borel probability measure $\tilde{\lambda}_{h_1, h_2, p}$ on $\Sigma_2 \times \hat{\mathbb{C}}$ such that $\pi_*(\tilde{\lambda}_{h_1, h_2, p}) = \eta_p$ and $h_{\tilde{\lambda}_{h_1, h_2, p}}(\tilde{h}|\sigma) = \max_{\rho \in \mathfrak{E}_1(\Sigma_m \times \hat{\mathbb{C}}): \tilde{h}_*(\rho) = \rho, \pi_*(\rho) = \tilde{\tau}} h_\rho(\tilde{h}|\sigma) = \sum_{j=1}^{2} p_j \log(\deg(h_j))$, where $h_\rho(\tilde{h}|\sigma)$ denotes the relative metric entropy of $(\tilde{h}, \rho)$ with respect to (σ, η_p), and $\mathfrak{E}_1(\cdot)$ denotes the space of ergodic measures (see [16]). This $\tilde{\lambda}_{h_1, h_2, p}$ is called the **maximal relative entropy measure** for $\tilde{h}$ with respect to (σ, η_p). Also, we set $\lambda_{h_1, h_2, p} := (\pi_{\hat{\mathbb{C}}})_*(\tilde{\lambda}_{h_1, h_2, p})$. This is a Borel probability measure on $J(h_1, h_2)$.

Definition 1.17.

- Let $\tilde{h} : \Sigma_2 \times \hat{\mathbb{C}} \to \Sigma_2 \times \hat{\mathbb{C}}$ be the skew product associated with $h = (h_1, h_2) \in \mathcal{H}$. Let $p \in (0,1)$. We set $p_1 = p, p_2 = 1 - p$. Let ρ be an $\tilde{h}$-invariant Borel probability measure on $J(\tilde{h})$ (i.e., $\rho(A) = \rho(\tilde{h}^{-1}(A))$ for each Borel measurable subset A of $J(\tilde{h})$). Moreover, we set

$$u(h_1, h_2, p, \rho) := \frac{-\int_{\Sigma_2 \times \hat{\mathbb{C}}} \log p_{w_1} \, d\rho(w, x)}{\int_{\Sigma_2 \times \hat{\mathbb{C}}} \log \|D(h_{w_1})_x\|_s \, d\rho(w, x)} \in (0, \infty).$$

- Let V be an open subset of $\mathbb{C}$. For any function $\varphi : V \to \mathbb{R}$ and any point $y \in V$, if φ is bounded around y, we set
  $\text{Höl}(\varphi, y) := \sup\{\beta \geq 0 \mid \limsup_{z \to y, z \neq y}(|\varphi(z) - \varphi(y)|/|z - y|^\beta) < \infty\} \in [0, \infty]$, and this is called the **pointwise Hölder exponent of φ at y**. (**Note:** If $\text{Höl}(\varphi, y) < 1$, then φ is not differentiable at y. If $\text{Höl}(\varphi, y) > 1$, then φ is differentiable at y and the derivative is equal to 0.)

We now present the results on the pointwise Hölder exponents of the functions of probability of tending to ∞.

Theorem 1.18. *Statements 1 and 2 hold.*

1. (**Non-differentiability**) *Let $(h_1, h_2) \in \overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}$, $G = \langle h_1, h_2 \rangle$, and $0 < p < 1$. Then, supp $\lambda_{h_1, h_2, p} = J(G)$, $\lambda_{h_1, h_2, p}$ is non-atomic, and for almost every point $z_0 \in J(G)$ with respect to $\lambda_{h_1, h_2, p}$,*

$$\text{Höl}(T(h_1, h_2, p, \cdot), z_0) \leq u(h_1, h_2, p, \tilde{\lambda}_{h_1, h_2, p}) = -\frac{p \log p + (1-p) \log(1-p)}{p \log(\deg(h_1)) + (1-p) \log(\deg(h_2))} < 1 \tag{1}$$

and $T(h_1, h_2, p, \cdot)$ is not differentiable at z_0. In particular, there exists an uncountable dense subset A of $J(G)$ such that at every point of A, the function $T(h_1, h_2, p, \cdot)$ is not differentiable. Moreover, if $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$, then $\text{Höl}(T(h_1, h_2, p, \cdot), z_0) = u(h_1, h_2, p, \tilde{\lambda}_{h_1, h_2, p})$ for almost

every point $z_0 \in J(G)$ with respect to $\lambda_{h_1,h_2,p}$. Moreover, if $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$, then for almost every point $z_0 \in J(G)$ with respect to $\lambda_{h_1,h_2,p}$, the function $T(h_1, h_2, p, \cdot)$ is continuous at z_0.

2. *Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$, $G = \langle h_1, h_2 \rangle$, and $0 < p < 1$. Let $v = \dim_H(J(G))$ and let H^v be the v-dimensional Hausdorff measure. Then we have the all of the following.*

   (i) *$0 < H^v(J(G)) < \infty$ and there exists a unique Borel probability measure ν on $J(\tilde{h})$ such that for each $\varphi \in C(J(\tilde{h}))$, $\int_{J(\tilde{h})} \sum_{(\alpha,y) \in \tilde{h}^{-1}(\gamma,x)} \frac{\varphi(\alpha,y)}{\|D(\alpha_1)_y\|_s^v} \, d\nu(\gamma, x) = \int_{J(\tilde{h})} \varphi(\gamma, x) d\nu(\gamma, x)$. Moreover, there exists a unique element $\psi \in C(J(\tilde{h}))$ with $\psi(\gamma, x) > 0 \ (\forall (\gamma, x))$ such that for each $(\gamma, x) \in J(\tilde{h})$, we have $\sum_{(\alpha,y) \in \tilde{h}^{-1}(\gamma,x)} \frac{\psi(\alpha,y)}{\|D(\alpha_1)_y\|_s^v} = \psi(\gamma, x)$. Also, $\eta := \psi \cdot \nu$ is an $\tilde{h}$-invariant ergodic Borel probability measure on $J(\tilde{h})$. Further, $(\pi_{\hat{\mathbb{C}}})_*(\nu) = \frac{H^v|_{J(G)}}{H^v(J(G))}$.*

   (ii) *For almost every $z_0 \in J(G)$ with respect to H^v, the function $T(h_1, h_2, p, \cdot) : \hat{\mathbb{C}} \to [0, 1]$ is continuous at z_0 and $\mathrm{H\ddot{o}l}(T(h_1, h_2, p, \cdot), z_0) \leq u(h_1, h_2, p, \eta)$. If $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$, then for almost every $z_0 \in J(G)$ with respect to H^v, $\mathrm{H\ddot{o}l}(T(h_1, h_2, p, \cdot), z_0) = u(h_1, h_2, p, \eta)$.*

Theorem 1.19. *Let $h = (h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$. Let $G = \langle h_1, h_2 \rangle$. Let $p \in (0, 1)$. Let $p_1 = p, p_2 = 1 - p$. Then there exists an open neighborhood V of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$ and a number $i \in \{1, 2\}$ such that for each $(g_1, g_2) \in V$, denoting by μ_j the maximal entropy measure for $g_j : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for each $j = 1, 2$, all of the following hold.*

   (i) *For each $j = 1, 2$, for μ_j-a.e. $z_0 \in J(g_j)$, $\mathrm{H\ddot{o}l}(T(g_1, g_2, p, \cdot), z_0) \leq -\frac{\log p_j}{\log \deg(g_j)}$.*

   (ii) *For μ_i-a.e. $z_0 \in J(g_i)$, $\mathrm{H\ddot{o}l}(T(g_1, g_2, p, \cdot), z_0) \leq -\frac{\log p_i}{\log \deg(g_i)} < 1$.*

   (iii) *For each $\alpha \in (-\frac{\log p_i}{\log \deg(g_i)}, 1)$ and for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$ such that $\varphi(\infty) = 1$ and $\varphi|_{\hat{K}(G)} \equiv 0$, we have $\|M_{g_1,g_2,p}^n(\varphi)\|_\alpha \to \infty$ as $n \to \infty$.*

Remark 1.20. Let $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$ be an element. By statements 1, 2 in Theorem 1.18, it follows that if p is close enough to 0 or 1, then (1) for almost every $z_0 \in J(h_1, h_2)$ with respect to $\lambda_{h_1,h_2,p}$, $z \mapsto T(h_1, h_2, p, z)$ is not differentiable at z_0, but (2) for almost every $z_0 \in J(h_1, h_2)$ with respect to H^v, $z \mapsto T(h_1, h_2, p, z)$ is differentiable at z_0 and the derivative is equal to 0.

Remark 1.21. Theorems 1.15, 1.18, 1.19 and [26, Theorem 3.30] imply that if $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$ then there exists a neighborhood $\mathcal{A}_0$ of (h_1, h_2) in $\mathcal{P}^2$ such that for each $(g_1, g_2) \in \mathcal{A}_0$ and each $p \in (0, 1)$, the associated random dynamical system does not have chaos in the C^0 sense, but still has a kind of chaos in the C^α sense for some $0 < \alpha < 1$ (see also the similar results Theorems 2.27, 2.42, 2.56 in which we do not assume hyperbolicity). More precisely, for such an element (g_1, g_2, p), there exists a number $\alpha_0 \in (0, 1)$ such that for each $\alpha \in (0, \alpha_0)$, the system behaves well on the Banach space $C^\alpha(\hat{\mathbb{C}})$ endowed with α-Hölder norm $\|\cdot\|_\alpha$ (see [26, Theorem 1.10]), but for each $\alpha \in (\alpha_0, 1)$, the system behaves chaotically on the Banach space $C^\alpha(\hat{\mathbb{C}})$ (and on the Banach space $C^1(\hat{\mathbb{C}})$ as well). In this way, regarding the random dynamical systems, we have a kind of **gradation between chaos and order**. Note that in [24, 26], this phenomenon was found for the systems with disconnected Julia sets, but in this paper, we show that this phenomenon can hold for plenty of systems with connected Julia sets. For the related results in which we do not assume hyperbolicity, see Theorems 2.27, 2.42, 2.56 and Remark 2.57. In [9], we perform a multifractal analysis of the pointwise Hölder exponents of $T(h_1, h_2, p, \cdot)$ (and more general limit state functions of random complex dynamical systems).

Remark 1.22. From the point of view of [24, Introduction] and [26, Remark 1.14], we can say that the function $z \mapsto T(h_1, h_2, p, z)$ is a complex analogue of the devil's staircase or Lebesgue's singular functions and it is called a "**devil's coliseum**" (see Figures 2 and 3, for the definition of the devil's staircase and Lebesgue's singular functions, see [32]), and the function $z \mapsto (\partial T/\partial p)(h_1, h_2, p, z)$

(see Figure 4) is a complex analogue of the Takagi function (or the Blancmange function) (for the definition of the Takagi function, see [32]). These notions have been introduced in [24, 26], though in those papers we deal with the case having disconnected Julia sets. In this paper, we also deal with the case with **connected** Julia sets which are thin fractal sets.

Figure 1: The Julia set of $G = \langle h_1, h_2 \rangle$, where $(h_1, h_2) \in ((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$ and $\deg(h_1) = \deg(h_2) = 4$. The Julia set $J(G)$ is **connected**, (h_1, h_2) satisfies the open set condition and $\frac{3}{2} < \dim_H(J(G)) < 2$.
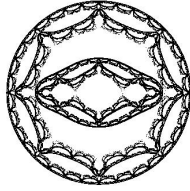
Remark 1.23. This paper is the first one in which the parameter space of polynomial semigroups is investigated. We focus on the space of parameters for which the semigroup is postcritically bounded. In particular, we study the disconnectedness locus and the connectedness locus in the above space. We combine all ideas on postcritically bounded polynomial semigroups ([21, 13]), interaction cohomology ([20]), skew products and potential theory ([16, 17, 24]), (semi-)hyperbolic semigroups and thermodynamic formalisms ([18, 31, 29]), ergodic theory, perturbation theory for linear operators, and random complex dynamics ([24, 26]). In the proofs of the results, we use the idea of the nerve of backward images of the Julia set under the elements of a polynomial semigroup and associated cohomology (interaction cohomology) from [20] and we combine this with potential theory. Also, we use the results on the dynamics of postcritically bounded polynomial semigroups from [21] and the results on hyperbolic semigroups from [18]. From these, we obtain that any element $h = (h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ satisfies the open set condition and Bowen's formula (Theorem 1.5-1, 4). This is crucial to obtain other results in this paper. Combining this with some geometric observations by using Green's functions and a result on the porosity of the Julia sets from [19], we obtain $\dim_H(J(h_1, h_2)) < 2$ (Theorem 1.5-3) and Theorem 1.5-4. Moreover, we obtain that there exists a neighborhood $\mathcal{A}_0$ of $(\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$ such that for each $(h_1, h_2) \in \mathcal{A}_0$, the set $\{h_1, h_2\}$ is mean stable (Theorem 1.15-1-(i)).

Also, it is important and interesting to study the topology of the connectedness locus and its boundary. Combining $\overline{\text{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H} = \mathcal{C} \cap \mathcal{B} \cap \mathcal{H}$ (Theorem 1.6-1), that $((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ is dense in $(\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$ (Theorem 1.6-3) and Theorem 1.5-4, we see that in any neighborhood of any point of $(\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$, there exists an open subset $\mathcal{A}_1$ of $\text{int}(\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$ such that for each $g = (g_1, g_2) \in \mathcal{A}_1$, we have that $\dim_H(J(g_1, g_2)) < 2$. Combining the results on semigroups and some results on random complex dynamics from [24, 26], and developing many new ideas, we show Theorems 1.15, 1.18, 1.19. Note that Theorem 1.15 (vii)(viii) (complex analogues of the Takagi function) are obtained by using some results from [26], which were shown by using the perturbation theory for linear operators. Combining the above results on semigroups and Theorem 1.15, we obtain that in any neighborhood of any point of $((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$, there exists an open subset $\mathcal{A}_2$ of $\text{int}(\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$ such that for each $g = (g_1, g_2) \in \mathcal{A}_2$, $\{g_1, g_2\}$ is mean stable and the function $T(g_1, g_2, p, \cdot)$ of probability of tending to ∞ is Hölder continuous on $\hat{\mathbb{C}}$ and varies only in a thin connected fractal set $J(g_1, g_2)$ whose Hausdorff dimension is strictly less than two. Moreover, if $(g_1, g_2) \in ((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$, then $T(g_1, g_2, p, \cdot)$ is Hölder continuous on $\hat{\mathbb{C}}$ and varies precisely on the connected Julia set $J(g_1, g_2)$ which is a thin fractal set (Theorems 1.15, 1.8, 1.5). Thus, we

Figure 2: The graph of $z \mapsto T(h_1, h_2, 1/2, z)$, where (h_1, h_2) is as in Figure 1. A devil's coliseum (a complex analogue of the devil's staircase or Lebesgue's singular functions). The function is Hölder continuous on $\hat{\mathbb{C}}$ and the set of varying points is equal to the **connected** Julia set in Figure 1.
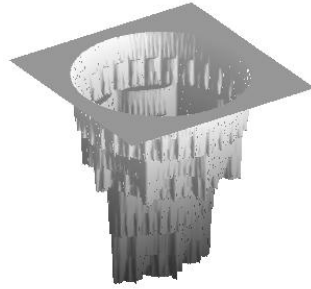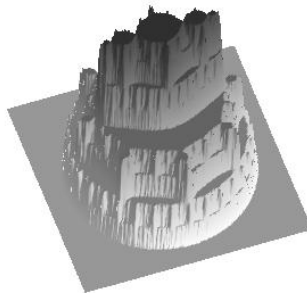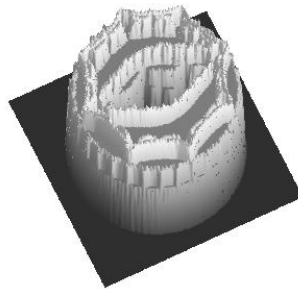

Figure 3: Figure 2 upside down.

Figure 4: The graph of $z \mapsto (\partial T/\partial p)(h_1, h_2, 1/2, z)$, where (h_1, h_2) is as in Figure 1. A complex analogue of the Takagi function (the Blancmange function). This function is Hölder continuous on $\hat{\mathbb{C}}$ and the set of varying points is included in the Julia set in Figure 1.

can say that even in the parameter region $\mathcal{C} \cap \mathcal{B} \cap \mathcal{H}$, there are plenty of examples of "**complex singular functions**" (complex analogues of the devil's staircase or Lebesgue singular functions). Note that in [24, 26], we have obtained many results on the functions of probability of tending to ∞ when the associated Julia sets are "disconnected". This paper is the first one in which we show the existence of plenty of examples such that the function of probability of tending to ∞ is continuous and "singular" when the associated Julia set is connected. (In fact, if the overlap $h_1^{-1}(J(h_1, h_2)) \cap h_2^{-1}(J(h_1, h_2))$ is not empty, then the analysis of the random dynamical systems generated by $\{h_1, h_2\}$ and $T(h_1, h_2, p, \cdot)$ is difficult.) Also, the details of such functions are given in Theorem 1.15.

In section 2, we give the proofs of the main results. Also, in section 2, we show some further results in which we do not assume hyperbolicity (Theorems 2.26, 2.27, 2.42, 2.52, 2.56). Moreover, we show a result on the Fatou components (Theorem 2.45) and some results on semi-hyperbolicity (Theorems 2.26, 2.27).

2   Proofs of the main results

In this section, we give the proofs of the main results. Also, we show some further results in which we do not assume hyperbolicity.

To recall some known facts on polynomial semigroups, let G be a polynomial semigroup with $G \subset \mathcal{P}$. Then, for each $g \in G$, $g(F(G)) \subset F(G), g^{-1}(J(G)) \subset J(G)$. Moreover, $J(G)$ is a perfect set and $J(G)$ is equal to the closure of the set of repelling cycles of elements of G. In particular, $J(G) = \overline{\cup_{g \in G} J(g)}$. We set $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp \bigcup_{g \in G} g^{-1}(\{z\}) < \infty\}$. Then $\sharp E(G) \leq 2$ and for each $z \in J(G) \setminus E(G)$, $J(G) = \overline{\bigcup_{g \in G} g^{-1}(\{z\})}$. Also, $J(G)$ is the smallest set in $\{\emptyset \neq K \subset \hat{\mathbb{C}} \mid K$ is compact, $\forall g \in G, g(K) \subset K\}$ with respect to the inclusion. If G is generated by a compact family Λ of $\mathcal{P}$, then $J(G) = \bigcup_{h \in \Lambda} h^{-1}(J(G))$ (this is called the backward self-similarity). For more details on these properties of polynomial semigroups, see [8, 12, 7, 16]. The article [12] by R. Stankewitz is a very nice introductory one for basic facts on polynomial semigroups. For the properties of the dynamics of postcritically bounded polynomial semigroups, see [21, 13].

For fundamental tools and lemmas of random complex dynamics, see [24, 26].

2.1   Proofs of Theorems 1.5 and 1.6

In this subsection, we prove Theorems 1.5 and 1.6. Also, we show some results in which we do not assume hyperbolicity (Lemma 2.24, Theorems 2.26, 2.27). We need some definitions.

Definition 2.1 ([21])**.** For any connected sets K_1 and K_2 in $\mathbb{C}$, we write $K_1 \leq_s K_2$ to indicate that $K_1 = K_2$, or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Furthermore, $K_1 <_s K_2$ indicates $K_1 \leq_s K_2$ and $K_1 \neq K_2$. Moreover, $K_2 \geq_s K_1$ indicates $K_1 \leq_s K_2$, and $K_2 >_s K_1$ indicates $K_1 <_s K_2$. Note that $\leq_s$ is a partial order in the space of all non-empty compact connected sets in $\mathbb{C}$. This $\leq_s$ is called the **surrounding order.**

Definition 2.2. For a topological space X, we denote by $\mathrm{Con}(X)$ the set of all connected components of X.

Definition 2.3. We denote by $\mathcal{G}$ the set of all postcritically bounded polynomial semigroups G with $G \subset \mathcal{P}$. We denote by $\mathcal{G}_{dis}$ the set of all $G \in \mathcal{G}$ with disconnected Julia set.

Remark 2.4. Let $G \in \mathcal{G}_{dis}$. In [21], it was shown that $J(G) \subset \mathbb{C}$, $(\mathrm{Con}(J(G)), \leq_s)$ is totally ordered, there exists a unique maximal element $J_{\max} = J_{\max}(G) \in (\mathrm{Con}(J(G)), \leq_s)$, there exists a unique minimal element $J_{\min} = J_{\min}(G) \in (\mathrm{Con}(J(G)), \leq_s)$, and each element of $\mathrm{Con}(F(G))$

is either simply connected or doubly connected. Moreover, in [21], it was shown that $\mathcal{A} \neq \emptyset$, where $\mathcal{A}$ denotes the set of all doubly connected components of $F(G)$ (more precisely, for each $J, J' \in \mathrm{Con}(J(G))$ with $J <_s J'$, there exists an $A \in \mathcal{A}$ with $J <_s A <_s J'$), $\bigcup_{A \in \mathcal{A}} A \subset \mathbb{C}$, and $(\mathcal{A}, \leq_s)$ is totally ordered. Note that each $A \in \mathcal{A}$ is bounded and multiply connected, while for a single $f \in \mathcal{P}$, we have no bounded multiply connected component of $F(f)$.

Definition 2.5. Let $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathcal{P}^{\mathbb{N}}$ be a sequence of polynomials. For each $m, n \in \mathbb{N}$ with $n \leq m$, we set $\gamma_{m,n} := \gamma_m \circ \cdots \circ \gamma_n$. We denote by F_γ the set of points $z \in \hat{\mathbb{C}}$ for which there exists a neighborhood U of z such that $\{\gamma_{n,1}\}_{n \in \mathbb{N}}$ is normal. The set F_γ is called the Fatou set of the sequence γ of polynomials. Moreover, we set $J_\gamma := \hat{\mathbb{C}} \setminus F_\gamma$. The set J_γ is called the Julia set of the sequence γ of polynomials.

Lemma 2.6. Let $(h_1, h_2) \in \mathcal{B} \cap \mathcal{D}$. Let $\Gamma = \{h_1, h_2\}$ and $G = \langle h_1, h_2 \rangle$. Then, $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, $J(G) = \amalg_{\gamma \in \Gamma^{\mathbb{N}}} J_\gamma$ (disjoint union), and the map $\gamma \mapsto J_\gamma$ is continuous on $\Gamma^{\mathbb{N}}$ with respect to the Hausdorff metric. Moreover, for each $\gamma \in \Gamma^{\mathbb{N}}$, J_γ is connected.

Proof. By [21, Proposition 2.24], we may assume that $J(h_1) \subset J_{\min}(G)$ and $J(h_2) \subset J_{\max}(G)$. Then by [21, Theorem 2.20-5], $\emptyset \neq \mathrm{int}(\hat{K}(G)) \subset \mathrm{int}(K(h_1))$. Here, for any subset A of $\hat{\mathbb{C}}$, $\mathrm{int}(A)$ denotes the set of interior points of A with respect to the topology in $\hat{\mathbb{C}}$. By [21, Theorem 2.20-5] again, $h_2(J(h_1)) \subset h_2(J_{\min}(G)) \subset \mathrm{int}(\hat{K}(G)) \subset \mathrm{int}(K(h_1))$. Therefore $h_2(\mathrm{int}(K(h_1))) \subset \mathrm{int}(K(h_1))$. Thus $\mathrm{int}(K(h_1)) \subset F(G)$.

By [20, Theorems 1.7, 1.5], $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$. Since $J(G) = h_1^{-1}(J(G)) \cup h_2^{-1}(J(G))$ ([14, Lemma 1.1.4]), it follows that $J(G) = \amalg_{\gamma \in \Gamma^{\mathbb{N}}} J(G)_\gamma$, where $J(G)_\gamma := \cap_{n=1}^{\infty} (\gamma_1^{-1} \cdots \gamma_n^{-1}(J(G)))$. Since $J_\gamma = \gamma_1^{-1} \cdots \gamma_n^{-1}(J_{(\gamma_{n+1}, \gamma_{n+2}, \dots)}) \subset \gamma_1^{-1} \cdots \gamma_n^{-1}(J(G))$ for each γ and each $n \in \mathbb{N}$, it follows that for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma \subset J(G)_\gamma$.

Let $\gamma \in \Gamma^{\mathbb{N}}$. In order to prove $J_\gamma = J(G)_\gamma$, suppose that there exists a point $y_0 \in J(G)_\gamma \setminus J_\gamma$. We now consider the following two cases. Case 1: $\sharp\{n \in \mathbb{N} \mid \gamma_n \neq h_1\} = \infty$. Case 2: $\sharp\{n \in \mathbb{N} \mid \gamma_n \neq h_1\} < \infty$.

Suppose that we have Case 1. Then there exist an open neighborhood U of y_0 in $\hat{\mathbb{C}}$, a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers, and a map $\varphi : U \to \hat{\mathbb{C}}$, such that $\gamma_{n_j+1} = h_2$ for each $j \in \mathbb{N}$, and such that $\gamma_{n_j,1} \to \varphi$ uniformly on U as $j \to \infty$. Since $\gamma_{n_j,1}(y_0) \in J(G)$ for each j, and since $\mathrm{int}(\hat{K}(G)) \subset F(G)$, [22, Lemma 5.6] implies that φ is constant. By [23, Lemma 3.13], it follows that $d(\gamma_{n_j,1}(y_0), P^*(G)) \to 0$ as $j \to \infty$. Moreover, since $\gamma_{n_j+1} = h_2$, we obtain $\gamma_{n_j,1}(y_0) \in h_2^{-1}(J(G))$ for each j. Furthermore, by [21, Theorem 2.20-2,5], $h_2^{-1}(J(G)) \subset \hat{\mathbb{C}} \setminus P^*(G)$. This is a contradiction. Hence, we cannot have Case 1.

Suppose we have Case 2. Let $r \in \mathbb{N}$ be a number such that for each $s \in \mathbb{N}$ with $s \geq r$, $\gamma_s = h_1$. Then $h_1^n(\gamma_{r,1}(y_0)) \in J(G)$ for each $n \geq 0$. Since $y_0 \notin J_\gamma$, we have $\gamma_{r,1}(y_0) \notin J(h_1)$. Moreover, since $\gamma_{r,1}(y_0) \in J(G)$ and $\mathrm{int}(\hat{K}(h_1)) \subset F(G)$, it follows that $\gamma_{r,1}(y_0)$ belongs to $F_\infty(h_1)$. It implies that $h_1^n(\gamma_{r,1}(y_0)) \to \infty$ as $n \to \infty$. However, this contradicts that $h_1^n(\gamma_{r,1}(y_0)) \in J(G)$ for each $n \geq 0$. Therefore, we cannot have Case 2.

Thus, for each $\gamma \in \Gamma$, $J_\gamma = J(G)_\gamma$. Combining this with [23, Lemma 3.5] and [19, Proposition 2.2(3)], we obtain that the map $\gamma \mapsto J_\gamma$ is continuous on $\Gamma^{\mathbb{N}}$ with respect to the Hausdorff metric.

Finally, by [23, Lemma 3.8], J_γ is connected for each $\gamma \in \Gamma^{\mathbb{N}}$. Thus we have proved our lemma. $\square$

Proposition 2.7. Let $(h_1, h_2) \in \mathcal{B} \cap \mathcal{D}$. Then either (1) $K(h_1) \subset \mathrm{int}(K(h_2))$ or (2) $K(h_2) \subset \mathrm{int}(K(h_1))$ holds. If (1) holds, then setting $U := (\mathrm{int}(K(h_2))) \setminus K(h_1)$, we have that (h_1, h_2) satisfies the open set condition with U, that $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$, that h_2 is hyperbolic, and that $J(h_2)$ is a quasicircle.

Proof. Let $\Gamma := \{h_1, h_2\}$. Let $G = \langle h_1, h_2 \rangle$. By Lemma 2.6, we have $J(G) = \amalg_{\gamma \in \Gamma^{\mathbb{N}}} J_\gamma$. Combining this with [21, Theorem 2.7], we obtain that either $J(h_1) <_s J(h_2)$ or $J(h_2) <_s J(h_1)$ holds. We now assume that $J(h_1) <_s J(h_2)$ (which is equivalent to $K(h_1) \subset \mathrm{int}(K(h_2))$). Then, by [21,

Proposition 2.24], $J(h_1) \subset J_{\min}(G)$ and $J(h_2) \subset J_{\max}(G)$. By Lemma 2.6 again, it follows that $J(h_1) = J_{\min}(G)$ and $J(h_2) = J_{\max}(G)$. Let $A = K(h_2) \setminus \operatorname{int}(K(h_1))$. We now prove the following claim.

Claim 1. $h_1^{-1}(A) \cup h_2^{-1}(A) \subset A$.

To prove this claim, let $\alpha = (h_2, h_1, h_1, \ldots) \in \Gamma^{\mathbb{N}}$. Then $J_\alpha = h_2^{-1}(J(h_1))$. Since $J(h_1) = J_{\min}(G)$, we obtain that $J(h_1) <_s J_\alpha = h_2^{-1}(J(h_1))$. Therefore $h_2^{-1}(A) \subset A$. Similarly, letting $\beta = (h_1, h_2, h_2, \ldots) \in \Gamma^{\mathbb{N}}$, we have $J_\beta = h_1^{-1}(J(h_2)) <_s J(h_2)$ and $h_1^{-1}(A) \subset A$. Thus we have proved Claim 1.

We have that $h_1^{-1}(A)$ and $h_2^{-1}(A)$ are connected compact sets. We prove the following claim.

Claim 2. $J_\beta = h_1^{-1}(J(h_2)) <_s J_\alpha = h_2^{-1}(J(h_1))$. In particular, $h_1^{-1}(A) <_s h_2^{-1}(A)$.

To prove this claim, suppose that $J_\beta <_s J_\alpha$ does not hold. Then by [21, Theorem 2.7], $J_\alpha <_s J_\beta$. This implies that $A = h_1^{-1}(A) \cup h_2^{-1}(A)$. By [8, Corollary 3.2], we have $J(G) \subset A$. Since $J(G)$ is disconnected (assumption) and since A is connected, $F(G) \cap A \neq \emptyset$. Let $y \in F(G) \cap A$. Since $A = h_1^{-1}(A) \cup h_2^{-1}(A)$, there exists an element $\gamma \in \Gamma^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\gamma_{n,1}(y) \in A$. Since $y \in A \cap F(G)$ and $\bigcup_{g \in G} g(F(G)) \subset F(G)$, $\gamma_{n,1}(y) \in F_\infty(h_1) \cap A$ for each $n \in \mathbb{N}$. Therefore there exists a strictly increasing sequence $\{n_j\}_{j=1}^\infty$ in $\mathbb{N}$ such that for each j, $\gamma_{n_j+1} = h_2$. Since $y \in F_\gamma$, we may assume that there exists an open neighborhood U of y in $\hat{\mathbb{C}}$ and a holomorphic map $\varphi : U \to \hat{\mathbb{C}}$ such that $\gamma_{n_j,1} \to \varphi$ uniformly on U as $j \to \infty$. Since $\gamma_{n_j,1}(y) \in F_\infty(h_1) \cap A \subset (\hat{\mathbb{C}} \setminus \hat{K}(G)) \cap A$ for each j, [22, Lemma 5.6] implies that there exists a constant $c \in \mathbb{C}$ such that $\varphi = c$ on U. By [23, Lemma 3.13], it follows that $c \in P^*(G)$. Moreover, $\gamma_{n_j+1,1} \to h_2(c) \in P^*(G)$ as $j \to \infty$ uniformly on U. Since $P^*(G) \subset K(h_1)$ and since $\gamma_{n_j+1,1}(y) \in F_\infty(h_1)$ for each j, it follows that $d(\gamma_{n_j+1,1}(y), J(h_1)) \to 0$ as $j \to \infty$. Combining this with that $\gamma_{n_j+1} = h_2$ for each j, we obtain that $d(\gamma_{n_j,1}(y), h_2^{-1}(J(h_1))) \to \infty$. Since $J(h_1) <_s h_2^{-1}(J(h_1))$, it follows that $c \in F_\infty(h_1)$. However, this is a contradiction, Hence, $J_\beta <_s J_\alpha$. Thus we have proved Claim 2.

From Claims 1 and 2, we obtain that (h_1, h_2) satisfies the open set condition with U and $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$.

By [21, Theorem 2.20-4], h_2 is hyperbolic and $J(h_2)$ is a quasicircle.

Thus we have proved our proposition. $\qquad\square$

Lemma 2.8. *Statement 2 in Theorem 1.5 holds.*

Proof. Statement 2 in Theorem 1.5 follows from Lemma 2.6 and Proposition 2.7. $\qquad\square$

The following proposition is the key to proving many results in this paper.

Proposition 2.9. *Let $(h_1, h_2) \in \overline{\mathcal{D}} \cap (\mathcal{B} \cap \mathcal{H})$. Then, either (1) $K(h_1) \subset K(h_2)$ or (2) $K(h_2) \subset K(h_1)$ holds. If (1) holds then, setting $U_{h_1,h_2} := (\operatorname{int}(K(h_2))) \setminus K(h_1)$, we have all of the following.*

(a) $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$.

(b) $h_1^{-1}(U_{h_1,h_2}) \amalg h_2^{-1}(U_{h_1,h_2}) \subset U_{h_1,h_2}$.

(c) $J(h_2)$ *is a quasicircle.*

Proof. Let $(h_1, h_2) \in \overline{\mathcal{D}} \cap (\mathcal{B} \cap \mathcal{H})$. Then there exists a sequence $\{(h_{1,n}, h_{2,n})\}_{n \in \mathbb{N}}$ in $\mathcal{D} \cap \mathcal{H} \cap \mathcal{B}$ such that $(h_{1,n}, h_{2,n}) \to (h_1, h_2)$ as $n \to \infty$. By Proposition 2.7, we may assume that for each $n \in \mathbb{N}$, $K(h_{1,n}) \subset \operatorname{int}(K(h_{2,n}))$,

$$K(h_{1,n}) \subset h_{1,n}^{-1}(K(h_{2,n})) \subset h_{2,n}^{-1}(K(h_{1,n})) \subset K(h_{2,n}), \tag{2}$$

and $h_{1,n}^{-1}(U_n) \amalg h_{2,n}^{-1}(U_n) \subset U_n$, where $U_n := (\operatorname{int}(K(h_{2,n}))) \setminus K(h_{1,n})$. Suppose that $h_1^{-1}(K(h_2)) \cap (\hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1))) \neq \emptyset$. Then,

$$\partial(h_1^{-1}(K(h_2))) \setminus h_2^{-1}(K(h_1)) \neq \emptyset. \tag{3}$$

For, let $z \in h_1^{-1}(K(h_2)) \cap (\hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1))) \neq \emptyset$ be a point. If $z \in \text{int}(h_1^{-1}(K(h_2)))$, then since $\hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1))$ is connected (this is because $(h_1, h_2) \in \mathcal{B}$), there exists a curve $\gamma : [0,1] \to \hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1))$ such that $\gamma(0) = z \in \text{int}(h_1^{-1}(K(h_2)))$ and $\gamma(1) = \infty \in \hat{\mathbb{C}} \setminus h_1^{-1}(K(h_2))$. Therefore there exists a $t \in [0,1]$ with $\gamma(t) \in \partial(h_1^{-1}(K(h_2)))$. Thus (3) holds. Let V be an open disc neighborhood of ∞ such that for each $n \in \mathbb{N}$, $V \subset F(h_{1,n}, h_{2,n}) \cap F(h_1, h_2)$. By (3), there exist a point $w \in h_1^{-1}(J(h_2))$ and a positive integer l such that $h_1^l h_2(w) \in V$. Since $J(h_{2,n}) \to J(h_2)$ as $n \to \infty$ with respect to the Hausdorff metric, for a sufficiently large $n \in \mathbb{N}$, there exists a point $w_0 \in h_{1,n}^{-1}(J(h_{2,n}))$ such that $h_{1,n}^l h_{2,n}(w_0) \in V$. Therefore $w_0 \in \mathbb{C} \setminus h_{2,n}^{-1}(K(h_{1,n}))$. Hence, $h_{1,n}^{-1}(J(h_{2,n})) \cap (\mathbb{C} \setminus h_{2,n}^{-1}(K(h_{1,n}))) \neq \emptyset$. However, this contradicts (2). Thus, we should have that $h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1))$. Similarly, by using the fact $K(h_{1,n}) \subset \text{int}(K(h_{2,n}))$ for each n, we obtain $K(h_1) \subset K(h_2)$. Therefore, $K(h_1) \subset h_1^{-1}(K(h_1)) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset h_2^{-1}(K(h_2)) = K(h_2)$. Thus statement (a) holds.

We next prove statement (b). Let $U = (\text{int}(K(h_2))) \setminus K(h_1)$. By statement (a), we have

$$h_1^{-1}(U) = h_1^{-1}((\text{int}(K(h_2))) \setminus K(h_1)) = (\text{int}(h_1^{-1}(K(h_2)))) \setminus h_1^{-1}(K(h_1)) = \text{int}(h_1^{-1}(K(h_2))) \setminus K(h_1) \tag{4}$$

$$\subset (\text{int}(K(h_2))) \setminus K(h_1) = U. \tag{5}$$

Similarly, by statement (a), we have

$$h_2^{-1}(U) = h_2^{-1}((\text{int}(K(h_2))) \setminus K(h_1)) = h_2^{-1}(\text{int}(K(h_2))) \setminus h_2^{-1}(K(h_1)) = (\text{int}(K(h_2))) \setminus h_2^{-1}(K(h_1)) \tag{6}$$

$$\subset (\text{int}(K(h_2))) \setminus K(h_1) = U. \tag{7}$$

Moreover, since we have $h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1))$ (statement (a)),

$$(\text{int}(h_1^{-1}(K(h_2)))) \cap ((\text{int}(K(h_2))) \setminus h_2^{-1}(K(h_1))) = \emptyset. \tag{8}$$

Combining (4), (6), and (8), we obtain $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Therefore statement (b) holds. Since $J(h_{2,n})$ is a quasicircle for each n, since h_2 is hyperbolic, and since $h_{2,n} \to h_2$, we obtain that $J(h_2)$ is a quasicircle. Hence statement (c) holds.

Thus we have proved Proposition 2.9. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.10. *Statement 1 in Theorem 1.5 holds.*

Proof. Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. Then by Proposition 2.9, either $K(h_1) \subset K(h_2)$ or $K(h_2) \subset K(h_1)$. We may assume $K(h_1) \subset K(h_2)$. By Proposition 2.9 again, we obtain that setting $U := (\text{int}(K(h_2))) \setminus K(h_1)$, $h_1^{-1}(U) \amalg h_2^{-1}(U) \subset U$. Moreover, $J(h_2)$ is a quasicircle. Since $(h_1, h_2) \notin \mathcal{Q}$, it follows that $K(h_1) \subsetneq K(h_2)$. Therefore, $U \neq \emptyset$. Thus, (h_1, h_2) satisfies the open set condition with U. Hence statement 1 in Theorem 1.5 holds. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.11. *Let $(h_1, h_2) \in (\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}$. Then $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) \neq \emptyset$.*

Proof. Suppose $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) = \emptyset$. By Proposition 2.9, we may assume that $K(h_1) \subset K(h_2)$. Combining Proposition 2.9 and $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) = \emptyset$, we obtain that

$$h_1^{-1}(K(h_2)) \subset \text{int}(h_2^{-1}(K(h_1))). \tag{9}$$

Let $U = (\text{int}(K(h_2))) \setminus K(h_1)$. By Proposition 2.9-(b) and [8, Corollary 3.2], we have $J(G) \subset \overline{U}$. By (4) and (6), we have that

$$h_1^{-1}(\overline{U}) \subset h_1^{-1}(K(h_2)) \text{ and } h_2^{-1}(\overline{U}) \subset K(h_2) \setminus (\text{int}(h_2^{-1}(K(h_1)))). \tag{10}$$

Since we are assuming $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) = \emptyset$, from (9) and (10) it follows that $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) \subset h_1^{-1}(\overline{U}) \cap h_2^{-1}(\overline{U}) = \emptyset$. Since $J(G) = h_1^{-1}(J(G)) \cup h_2^{-1}(J(G))$ ([14, Lemma 1.1.4]), we

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obtain that $J(G)$ is disconnected. It implies $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$. However, since $\mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$ is an open subset of $\mathcal{P}^2$ (Lemma 1.2), it contradicts $(h_1, h_2) \in (\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}$. Therefore, we must have that $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) \neq \emptyset$.

Thus we have proved our lemma. $\qquad \square$

Lemma 2.12. *Let* $(h_1, h_2) \in \mathcal{B} \cap \mathcal{H}$. *Suppose* $h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1)) \neq \emptyset$. *Let* $z_0 \in h_1^{-1}(J(h_2)) \cap h_2^{-1}(J(h_1))$. *For each* $b \in \mathbb{C}$, *let* $S_b(z) = z + b$ *and let* $h_{3,b} = S_b \circ h_2 \circ S_b^{-1}$. *Then, for each* $\epsilon > 0$, *there exists a number* $c \in \{b \in \mathbb{C} \mid |b| < \epsilon\}$ *such that* $h_1^{-1}(J(h_{3,c})) \setminus h_{3,c}^{-1}(K(h_1)) \neq \emptyset$ *and* $h_1^{-1}(K(h_{3,c})) \neq h_{3,c}^{-1}(K(h_1))$.

Proof. Let $w_0 := h_1(z_0) \in J(h_2)$. Then,

$$S_0(w_0) = w_0, \ S_b(w_0) \in J(h_{3,b}) \text{ for each } b \in \mathbb{C}, \text{ and } b \mapsto S_b(w_0) \text{ is holomorphic on } \mathbb{C}. \quad (11)$$

Let V be an unbounded simply connected subdomain of $\mathbb{C}$ with $0 \in V$ such that the set $V_0 := \{S_b(w_0) \mid b \in V\}$ does not contain any critical value of h_1. Let $\zeta : V_0 \to \mathbb{C}$ be a well-defined inverse branch of h_1 such that $\zeta(w_0) = z_0$. Let $\alpha : V \to \mathbb{C}$ be the map defined by $\alpha(b) = (h_{3,b} \circ \zeta \circ S_b)(w_0), b \in V$. Let $l := \deg(h_1), n := \deg(h_2)$ and let $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ be numbers such that $h_1(z) = a_1 z^l + \cdots$ and $h_2(z) = a_2 z^n + \cdots$. Then,

$$|\zeta(S_b(w_0))| = |\zeta(w_0 + b)| \sim \left| \frac{w_0 + b}{a_1} \right|^{\frac{1}{l}} = \left(\frac{1}{|a_1|} \right)^{\frac{1}{l}} \cdot |b|^{\frac{1}{l}}(1 + o(1)), \text{ as } b \to \infty \text{ in } V.$$

Hence $|\zeta(S_b(w_0)) - b| = |b|(1 + o(1)|$ as $b \to \infty$ in V. Therefore $|h_2(\zeta(S_b(w_0)) - b)| = |a_2||b|^n(1 + o(1))$ as $b \to \infty$ in V. Thus

$$|\alpha(b)| = |b + h_2(\zeta(S_b(w_0)) - b)| = |a_2||b|^n(1 + o(1)) \text{ as } b \to \infty \text{ in } V.$$

Hence $\alpha : V \to \mathbb{C}$ is not constant on V. Since $\alpha(0) = h_2(z_0) \in J(h_1)$, it follows that for each $\epsilon > 0$ there exists a number $c \in \{b \in \mathbb{C} \mid |b| < \epsilon\}$ such that $\alpha(c) \in \mathbb{C} \setminus K(h_1)$. Combining this with $S_c(w_0) \in J(h_{3,c})$, we obtain that $(h_{3,c}^{-1}(\mathbb{C} \setminus K(h_1))) \cap (h_1^{-1}(J(h_{3,c}))) \neq \emptyset$. Therefore $(h_1^{-1}(J(h_{3,c}))) \setminus (h_{3,c}^{-1}(K(h_1))) \neq \emptyset$.

Thus we have proved our lemma. $\qquad \square$

Lemma 2.13. *Statement 1 in Theorem 1.6 holds.*

Proof. It suffices to prove that $(\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H} \subset \overline{\text{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H}$. In order to prove it, let $(h_1, h_2) \in (\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$. Then $(h_1, h_2) \in (\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}$. By Proposition 2.9, we may assume $K(h_1) \subset K(h_2)$. Let W be a small neighborhood of h_2 in $\mathcal{P}$ with $\{h_1\} \times W \subset \mathcal{B} \cap \mathcal{H}$. By Lemmas 2.11 and 2.12, there exists an element $h_3 \in W$ such that $(h_1^{-1}(J(h_3))) \setminus (h_3^{-1}(K(h_1))) \neq \emptyset$. Then there exists an open neighborhood B of (h_1, h_3) in $\mathcal{B} \cap \mathcal{H}$ such that

$$(g_1^{-1}(J(g_2))) \setminus (g_2^{-1}(K(g_1))) \neq \emptyset \text{ for each } (g_1, g_2) \in B. \quad (12)$$

We consider the following two cases. Case 1. $K(h_1) \subsetneq K(h_2)$. Case 2. $K(h_1) = K(h_2)$.

We now suppose that we have case 1. Then, letting W and B so small, we obtain that

$$K(g_2) \setminus K(g_1) \neq \emptyset \text{ for each } (g_1, g_2) \in B. \quad (13)$$

Then, for each $(g_1, g_2) \in B$, $J(g_1, g_2)$ is connected. Indeed, suppose that there exists an element $(\beta_1, \beta_2) \in B$ such that $J(\beta_1, \beta_2)$ is disconnected. Then $(\beta_1, \beta_2) \in \mathcal{D} \cap \mathcal{B}$. By Proposition 2.7 and (13), we obtain $K(\beta_1) \subset \text{int}(K(\beta_2))$. By Proposition 2.7 again, we get that $\beta_1^{-1}(K(\beta_2)) \subset \beta_2^{-1}(K(\beta_1))$. However, this contradicts (12). Therefore, for each $(g_1, g_2) \in B$, $J(g_1, g_2)$ is connected. Thus, $B \subset \overline{\text{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H}$. Since W was an arbitrary small neighborhood of h_2, it follows that $(h_1, h_2) \in \overline{\text{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H}$.

We now suppose that we have case 2. For each $b \in \mathbb{C}$, let $S_b(z) = z+b$ and $h_{3,b}(z) = S_b \circ h_2 \circ S_b^{-1}$. Let $A := K(h_1) = K(h_2)$. Let $x_1, x_2 \in A$ be two points such that $|x_1 - x_2| = \mathrm{diam}_E(A)$, where $\mathrm{diam}_E(A_0) := \sup_{\alpha, \beta \in A_0} |\alpha - \beta|$ for a subset A_0 of $\mathbb{C}$. Then $x_1, x_2 \in J(h_1) = J(h_2)$. Moreover, we have

$$|S_b(x_1) - S_b(x_2)| = \mathrm{diam}_E(K(h_{3,b})) = \mathrm{diam}_E(K(h_1)). \tag{14}$$

Let $\epsilon > 0$ be any small number. Since $\mathrm{int}(K(h_1)) = \mathrm{int}(K(h_2)) \neq \emptyset$ (see Proposition 2.9-(c)), there exists an element $c \in \{b \in \mathbb{C} \mid |b| < \epsilon\}$ such that

$$S_c(x_1) \in \mathrm{int}(K(h_1)). \tag{15}$$

By (14) and (15), we obtain

$$S_c(x_2) \in \mathbb{C} \setminus K(h_1). \tag{16}$$

By (15) and (16), it follows that

$$J(h_{3,c}) \cap \mathrm{int}(K(h_1)) \neq \emptyset \text{ and } J(h_{3,c}) \cap (\mathbb{C} \setminus K(h_1)) \neq \emptyset.$$

Therefore there exists a neighborhood B of $(h_1, h_{3,c})$ in $\mathcal{B} \cap \mathcal{H}$ such that for each $(g_1, g_2) \in B$,

$$J(g_2) \cap \mathrm{int}(K(g_1)) \neq \emptyset \text{ and } J(g_2) \cap (\mathbb{C} \setminus K(g_1)) \neq \emptyset.$$

Since $J(g_1)$ is connected, we obtain that for each $(g_1, g_2) \in B$, $J(g_1) \cap J(g_2) \neq \emptyset$. Thus $J(g_1) \cup J(g_2)$ is included in a connected component of $J(g_1, g_2)$. By [21, Theorem 2.1], it follows that for each $(g_1, g_2) \in B$, $J(g_1, g_2)$ is connected. Hence $B \subset \mathrm{int}(\mathcal{C})$. From these arguments, we obtain that $(h_1, h_2) \in \overline{\mathrm{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H}$.

Therefore $(\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H} \subset \overline{\mathrm{int}(\mathcal{C})} \cap \mathcal{B} \cap \mathcal{H}$. Thus statement 1 in Theorem 1.6 holds. $\square$

Lemma 2.14. *There exists a neighborhood U of $\overline{\mathcal{B} \cap \mathcal{D}}$ in $\mathcal{P}^2$ such that for each $(g_1, g_2) \in U$, $(\deg(g_1), \deg(g_2)) \neq (2, 2)$. Moreover, for each connected component A of $\mathcal{P}^2$ with $A \cap \overline{\mathcal{B} \cap \mathcal{D}} \neq \emptyset$, $(\deg(g_1), \deg(g_2)) \neq (2, 2))$ for each $(g_1, g_2) \in A$.*

Proof. By [21, Theorem 2.15], for each $(g_1, g_2) \in \mathcal{B} \cap \mathcal{D}$ we have $(\deg(g_1), \deg(g_2)) \neq (2, 2)$. Since the function $\deg : \mathcal{P} \to \mathbb{N} \subset \mathbb{R}$ is continuous, the statement of our lemma holds. $\square$

Definition 2.15. Let D be a domain in $\hat{\mathbb{C}}$ with $\infty \in D$. We denote by $\varphi(D, z)$ Green's function on D with pole at ∞.

Lemma 2.16. *Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. For each $i = 1, 2$, let $d_i := \deg(h_i)$ and we denote by c_i the leading coefficient of h_i. Then $\frac{1}{d_1 - 1} \log |c_1| \neq \frac{1}{d_2 - 1} \log |c_2|$. Moreover, $h_1^{-1}(K(h_2)) \neq h_2^{-1}(K(h_1))$.*

Proof. Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. By Proposition 2.9, we may assume that $K(h_1) \subset K(h_2)$. Then by Proposition 2.9 again, we have $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$. For each $i = 1, 2$, let $A_i := \hat{\mathbb{C}} \setminus K(h_i)$. Since $K(h_1) \subset K(h_2)$, we have that

$$\varphi(A_2, z) - \varphi(A_1, z) \leq 0 \text{ on } \mathbb{C} \setminus K(h_2). \tag{17}$$

Therefore by letting $z \to \infty$ in (17), we obtain that

$$\frac{1}{d_2 - 1} \log |c_2| - \frac{1}{d_1 - 1} \log |c_1| \leq 0. \tag{18}$$

Since the function $\varphi(A_2, z) - \varphi(A_1, z)$ is harmonic and bounded in $\mathbb{C} \setminus K(h_2)$, the maximum principle implies that the equality holds in (18) if and only if $\varphi(A_2, z) - \varphi(A_1, z) \equiv 0$ on $\mathbb{C} \setminus K(h_2)$, which is equivalent to $J(h_1) = J(h_2)$. Since $K(h_1) \subset K(h_2)$, Proposition 2.9-(c) implies that $J(h_2)$

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is a quasicircle. Therefore, $J(h_1) = J(h_2)$ is equivalent to $(h_1, h_2) \in \mathcal{Q}$. Since we are assuming $(h_1, h_2) \notin \mathcal{Q}$, it follows that

$$\frac{1}{d_2 - 1} \log |c_2| - \frac{1}{d_1 - 1} \log |c_1| \neq 0. \tag{19}$$

It is easy to see that $\varphi(\hat{\mathbb{C}} \setminus h_1^{-1}(K(h_2)), z) = \frac{1}{d_1} \varphi(A_2, h_1(z))$ for each $z \in \hat{\mathbb{C}} \setminus h_1^{-1}(K(h_2))$ and $\varphi(\hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1)), z) = \frac{1}{d_2} \varphi(A_1, h_2(z))$ for each $z \in \hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1))$. Since $h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1))$, we obtain that $\varphi(\hat{\mathbb{C}} \setminus h_2^{-1}(K(h_1)), z) - \varphi(\hat{\mathbb{C}} \setminus h_1^{-1}(K(h_2)), z) \leq 0$ for each $z \in \mathbb{C} \setminus h_2^{-1}(K(h_1))$. Therefore

$$\frac{1}{d_2} \varphi(A_1, h_2(z)) - \frac{1}{d_1} \varphi(A_2, h_1(z)) \leq 0 \text{ for each } z \in \mathbb{C} \setminus h_2^{-1}(K(h_1)). \tag{20}$$

Since $\varphi(A_i, z) = \log |z| + \frac{1}{d_i - 1} \log |c_i| + O(\frac{1}{|z|})$ as $z \to \infty$ for each $i = 1, 2$, by letting $z \to \infty$ in (20) we obtain that

$$\frac{1}{d_2}\left(\log |c_2| + \frac{1}{d_1 - 1} \log |c_1|\right) - \frac{1}{d_1}\left(\log |c_1| + \frac{1}{d_2 - 1} \log |c_2|\right) \leq 0. \tag{21}$$

The function $\frac{1}{d_2} \varphi(A_1, h_2(z)) - \frac{1}{d_1} \varphi(A_2, h_1(z))$ is harmonic and bounded in $\mathbb{C} \setminus h_2^{-1}(K(h_1))$. Therefore, the maximum principle implies that the equality holds in (21) if and only if

$$\frac{1}{d_2} \varphi(A_1, h_2(z)) - \frac{1}{d_1} \varphi(A_2, h_1(z)) \equiv 0 \text{ on } \mathbb{C} \setminus h_2^{-1}(K(h_1)),$$

which is equivalent to that $h_1^{-1}(K(h_2)) = h_2^{-1}(K(h_1))$. Moreover, (21) is equivalent to

$$(d_2(d_1 - 1)(d_2 - 1) - d_1(d_2 - 1)) \log |c_1| \geq (d_1(d_2 - 1)(d_1 - 1) - d_2(d_1 - 1)) \log |c_2|. \tag{22}$$

It is easy to see that (22) is equivalent to

$$(d_1 d_2 - d_1 - d_2)((d_2 - 1) \log |c_1| - (d_1 - 1) \log |c_2|) \geq 0. \tag{23}$$

Since $(d_1, d_2) \neq (2, 2)$ (Lemma 2.14), (23) is equivalent to

$$(d_2 - 1) \log |c_1| \geq (d_1 - 1) \log |c_2|. \tag{24}$$

Therefore, (21) is equivalent to (24), and the equality holds in (21) if and only if the equality holds in (24). By (19), it follows that $h_1^{-1}(K(h_2)) \neq h_2^{-1}(K(h_1))$.

Thus we have proved our lemma. $\square$

Lemma 2.17. *Statement 3 in Theorem 1.5 holds.*

Proof. It suffices to prove that if $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$, then $J(h_1, h_2)$ is porous and $\dim_H(J(h_1, h_2)) \leq \dim_B(J(h_1, h_2)) < 2$. In order to prove it, let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. By Proposition 2.9, we may assume that $K(h_1) \subset K(h_2)$. By Proposition 2.9 again, we obtain that $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$ and setting $U := (\text{int}(K(h_2))) \setminus K(h_1)$, (h_1, h_2) satisfies the open set condition with U. Moreover, $h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}) \subset \overline{U}$. We now show the following claim.
Claim. $h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}) \neq \overline{U}$.

To prove this claim, suppose that $h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}) = \overline{U}$. Since $\overline{U} \subset K(h_2) \setminus \text{int}(K(h_1))$, we obtain that

$$\overline{U} = h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}) \subset \left(h_1^{-1}(K(h_2)) \setminus \text{int}(K(h_1))\right) \cup \left(K(h_2) \setminus \text{int}(h_2^{-1}(K(h_1)))\right). \tag{25}$$

Moreover, by Lemma 2.16, $h_1^{-1}(K(h_2)) \subsetneq h_2^{-1}(K(h_1))$. Since $\overline{\mathrm{int}(h_2^{-1}(K(h_1)))} = h_2^{-1}(K(h_1))$, we obtain that $(\mathrm{int}(h_2^{-1}(K(h_1)))) \setminus h_1^{-1}(K(h_2)) \neq \emptyset$. Therefore

$$\emptyset \neq (\mathrm{int}(h_2^{-1}(K(h_1)))) \setminus h_1^{-1}(K(h_2)) \subset (\mathrm{int}(K(h_2))) \setminus K(h_1) = U.$$

However, this contradicts (25). Thus we have proved our claim.

Let $G = \langle h_1, h_2 \rangle$. By [8, Corollary 3.2], $J(G) \subset \overline{U}$. Moreover, by [14, Lemma 1.1.4]), $J(G) = h_1^{-1}(J(G)) \cup h_2^{-1}(J(G))$. Therefore, Claim above implies that $J(G) = h_1^{-1}(J(G)) \cup h_2^{-1}(J(G)) \subset h_1^{-1}(\overline{U}) \subset h_2^{-1}(\overline{U}) \subsetneq \overline{U}$. Hence, $J(G) \neq \overline{U}$. Combining this with [19, Theorem 1.25], we see that $J(G)$ is porous and $\dim_H(J(G)) \leq \overline{\dim}_B(J(G)) < 2$. Thus statement 3 in Theorem 1.5 holds. $\quad\square$

Lemma 2.18. *Statement 4 in Theorem 1.5 holds.*

Proof. Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. By Lemma 2.10, (h_1, h_2) satisfies the open set condition. Thus, by [18, Theorem 1.1, 1.2], for each $z \in \hat{\mathbb{C}} \setminus P(\langle h_1, h_2 \rangle)$, $\dim_H(J(h_1, h_2)) = \overline{\dim}_B(J(h_1, h_2)) = \delta_{(h_1, h_2)} = Z_{(h_1, h_2)}(z)$. Moreover, by [31, Theorem 3.15], we have that $\frac{\log(d_1 + d_2)}{\sum_{j=1}^2 \frac{d_j}{d_1 + d_2} \log d_j} \leq \dim_H(J(h_1, h_2))$.

Suppose that $\frac{\log(d_1 + d_2)}{\sum_{j=1}^2 \frac{d_j}{d_1 + d_2} \log d_j} = \dim_H(J(h_1, h_2))$. Then [31, Theorem 3.15] implies that there exist a transformation $\varphi(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}, \alpha \neq 0$, two complex numbers a_1, a_2 and a positive integer d such that we have $d = d_1 = d_2$ and for each $j = 1, 2, \varphi \circ h_j \circ \varphi^{-1}(z) = a_j z^d$. By Lemma 2.14, we obtain that $d \geq 3$. From these it is easy to see that $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$. $\quad\square$

Lemma 2.19. *Statement 5 in Theorem 1.5 holds.*

Proof. There exists a neighborhood W of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$. By [18, Theorem 1.1], for each $(g_1, g_2) \in W$ and for each $z \in \hat{\mathbb{C}} \setminus P(\langle g_1, g_2 \rangle)$, we have $\dim_H(J(g_1, g_2)) \leq \overline{\dim}_B(J(g_1, g_2)) \leq \delta_{(g_1, g_2)} = Z_{(g_1, g_2)}(z)$. Furthermore, by [28], the map $(g_1, g_2) \mapsto \delta_{(g_1, g_2)}$ is continuous in W. Combining these arguments with Lemmas 2.17, 2.18, we see that Statement 5 in Theorem 1.5 holds. $\quad\square$

Lemma 2.20. *Let $L_1 := \{(h_1, h_2) \in \mathcal{P}^2 \mid J(h_1) = J(h_2)\}$. Then, for each connected component $\mathcal{V}$ of $\mathcal{P}^2$, $L_1 \cap \mathcal{V}$ is included in a proper holomorphic subvariety of $\mathcal{V}$.*

Proof. Let $(h_1, h_2) \in \mathcal{P}^2$. Let $n := \deg(h_1), m := \deg(h_2)$. We write $h_1(z) = a_n(h_1) z^n + a_{n-1}(h_1) z^{n-1} + \cdots + a_0(h_1)$ and $h_2(z) = b_m(h_2) z^m + b_{m-1}(h_2) z^{m-1} + \cdots + b_0(h_2)$. Let $\zeta(h_1) := -\frac{a_{n-1}(h_1)}{n \cdot a_n(h_1)}$. Let

$$\Sigma(h_1) := \{\alpha(z) = a(z - \zeta(h_1)) + \zeta(h_1) \mid a \in \mathbb{C}, |a| = 1, \alpha(J(h_1)) = J(h_1)\}.$$

By [1, Theorems 1,5], $J(h_1) = J(h_2)$ if and only if there exists an element $\eta \in \Sigma(h_1)$ such that $h_1 \circ h_2 = \eta \circ h_2 \circ h_1$. Moreover, either (1) $\sharp \Sigma(h_1) \leq n$ or (2) $\sharp \Sigma(h_1) = \infty$. Moreover, (1) holds if and only if setting $\rho(z) = (\frac{1}{a_n(h_1)})^{\frac{1}{n-1}} \cdot z$, we have $\rho^{-1} \circ T_{\zeta(h_1)}^{-1} \circ h_1 \circ T_{\zeta(h_1)} \circ \rho(z) = z^c h_{1,0}(z^d)$, where $c, d \geq 0$ are maximal for this form, and $h_{1,0}$ is a non-constant monic polynomial. Note that in this case, $\Sigma(h_1) = \{\alpha(z) = a(z - \zeta(h_1)) + \zeta(h_1) \mid \alpha(J(h_1)) = J(h_1), \alpha^d = 1\}$. Furthermore, "$J(h_1) = J(h_2)$ and (2)" holds if and only if $h_1(z) = a_n(h_1)(z - \zeta(h_1))^n + \zeta(h_1)$ and $h_2(z) = b_m(h_2)(z - \zeta(h_1))^m + \zeta(h_1)$. Thus the statement of our lemma holds. $\quad\square$

Lemma 2.21. *Statement 2 in Theorem 1.6 holds.*

Proof. By the definition of $\mathcal{Q}$, it is easy to see that $\mathcal{Q} \cap \mathcal{D} = \emptyset$. Moreover, by Lemma 2.6, for each $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B}$, $h_1^{-1}(J(h_1, h_2)) \cap h_2^{-1}(J(h_1, h_2)) = \emptyset$. Thus $\mathcal{D} \cap \mathcal{B} \cap \mathcal{I} = \emptyset$. Let $L_1 := \{(h_1, h_2) \in \mathcal{P}^2 \mid J(h_1) = J(h_2)\}$. Let $\mathcal{V}$ be any connected component of $\mathcal{P}^2$. Since $\mathcal{Q} \subset L_1$, Lemma 2.20 implies that $\mathcal{Q} \cap \mathcal{V}$ is included in a proper subvariety of $\mathcal{V}$. Thus statement 2 in Theorem 1.6 holds. $\quad\square$

Lemma 2.22. *Statement 3 in Theorem 1.6 holds.*

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Proof. It is easy to see that $((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ is an open subset of $(\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$. In order to show that $((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ is dense in $(\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$, let $(h_1, h_2) \in (\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$. Let W be any open polydisc neighborhood of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$. By Lemma 2.13 and Lemma 2.21, there exists an element $(g_1, g_2) \in ((\operatorname{int}(\mathcal{C})) \cap W) \setminus \mathcal{Q}$. By Lemma 2.21, $W \setminus \mathcal{Q}$ is connected. Therefore there exists a curve γ in $W \setminus \mathcal{Q}$ which joins (g_1, g_2) and a point in $\mathcal{D}$. Then $\gamma \cap \partial\mathcal{D} \neq \emptyset$. Therefore $\gamma \cap (((\partial\mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}) \neq \emptyset$. Thus $((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$ is dense in $(\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$. Hence statement 3 in Theorem 1.6 holds. $\qquad\square$

Lemma 2.23. *Statement 4 in Theorem 1.6 holds.*

Proof. Let $h_1 \in \mathcal{P}$ and suppose $\langle h_1 \rangle \in \mathcal{G}$ and h_1 is hyperbolic. Then $\operatorname{int}(K(h_1)) \neq \emptyset$. Let $d_1 := \deg(h_1)$ and let $d \in \mathbb{N}$. Suppose $(d_1, d) \neq (2, 2)$. Let $b \in \operatorname{int}(K(h_1))$ be a point. Here, if $h_1(z)$ is of the form $c_1(z - c_2)^{d_1} + c_2$, then we need the additional condition that $b \neq c_2$. Let $z_0 \in J(h_1)$ be a point such that $|z_0 - b| = \sup_{z_1 \in K(h_1)} |z_1 - b|$. Let $s := |z_0 - b|$. We show the following claim.
Claim 1. $\{z \in \mathbb{C} \mid |z - b| = s\} \setminus J(h_1) \neq \emptyset$.

In order to prove Claim 1, let $C := \{z \in \mathbb{C} \mid |z - b| = s\}$. By the way of the choice of b, we have $C \neq J(h_1)$. (Indeed, if $J(h_1) = C$, then $h_1(z) = c_1(z - b)^{d_1} + b$ for some $c_1 \in \mathbb{C}$.) Suppose $C \subset J(h_1)$. Then $C \subsetneq J(h_1)$. By the definition of s, we have $J(h_1) \subset \{z \in \mathbb{C} \mid |z - b| \leq s\}$. Therefore $J(h_1) \setminus C \subset \{z \in \mathbb{C} \mid |z - b| < s\}$. Let $w_1 \in J(h_1) \setminus C$ be a point. Let W be any neighborhood of w_1 in $\mathbb{C}$. Then there exists a point $w_2 \in W \cap (\hat{\mathbb{C}} \setminus K(h_1)) \cap \{z \in \mathbb{C} \mid |z - b| < s\}$. Since $\hat{\mathbb{C}} \setminus K(h_1)$ is connected, there exist a curve γ in $\hat{\mathbb{C}} \setminus K(h_1)$ which joins w_2 and ∞. Then $\emptyset \neq \gamma \cap C \subset \hat{\mathbb{C}} \setminus (K(h_1))$, which contradicts the assumption that $C \subset J(h_1)$. Thus, we must have that $C \not\subset J(h_1)$. Hence claim 1 holds.

Let $z_1 \in \{z \in \mathbb{C} \mid |z - b| = s\} \setminus J(h_1)$. Let $\theta \in \mathbb{R}$ be a number such that $e^{i\theta} \frac{1}{s^{d-1}}(z_0 - b)^d + b = z_1$. Let $r > 0$ be a number such that $D(b, r) \subset \operatorname{int}(K(h_1))$. Let $c_1 \in \mathbb{C} \setminus \{0\}$ be the leading coefficient of h_1. Then $h_1(z) = c_1 z^{d_1}(1 + O(\frac{1}{z}))$ as $z \to \infty$. Let $R \in \mathbb{R}$ be a number such that

$$R > \exp\left(\frac{1}{dd_1 - d - d_1}(-d_1 \log r + d_1 d \log 2 - d \log |c_1|)\right). \tag{26}$$

Taking R large enough, we may assume that R satisfies the following

$$D\left(b, \sqrt[d_1]{\frac{R}{|c_1|}} \cdot \frac{3}{4}\right) \subset h_1^{-1}(D(b, R)) \subset D\left(b, \sqrt[d_1]{\frac{R}{|c_1|}} \cdot \frac{3}{2}\right) \subset\subset D(b, R), \tag{27}$$

where $A \subset\subset B$ means that $\overline{A}$ is a compact subset of $\operatorname{int} B$. For each $t > 0$, let $g_t(z) = te^{i\theta}(z - b)^d + b$. Then for each $t > 0$, we have $g_t^{-1}(D(b, r)) = D\left(b, \sqrt[d]{\frac{r}{t}}\right)$. Let $t_0 = \frac{1}{R^{d-1}}$. Taking R so large, we may assume that

$$t_0 < \frac{1}{s^{d-1}}. \tag{28}$$

Since $(d_1, d) \neq (2, 2)$, it is easy to see that (26) is equivalent to

$$\sqrt[d]{\frac{r}{t_0}} > 2\sqrt[d_1]{\frac{R}{|c_1|}}. \tag{29}$$

By the definition of t_0, we have

$$J(g_{t_0}) = \left\{z \in \mathbb{C} \mid |z - b| = \sqrt[d-1]{\frac{1}{t_0}}\right\} = \{z \in \mathbb{C} \mid |z - b| = R\}. \tag{30}$$

Moreover, taking R so large, we may assume

$$D\left(b, \sqrt[d_1]{\frac{R}{|c_1|}} \cdot \frac{1}{2}\right) \supset K(h_1), \ D(0, R) \supset K(h_1). \tag{31}$$

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By (31), (27), (29), (30), we obtain

$$K(h_1) \subset D\left(b, \sqrt[d_1]{\frac{R}{|c_1|}}\frac{1}{2}\right) \subset\subset D\left(b, \sqrt[d_1]{\frac{R}{|c_1|}}\frac{3}{4}\right) \subset h_1^{-1}(K(g_{t_0})) \subset D\left(b, \sqrt[d_1]{\frac{R}{|c_1|}} \cdot \frac{3}{2}\right) \tag{32}$$

$$\subset\subset D\left(b, \sqrt[d]{\frac{r}{t_0}}\right) = g_{t_0}^{-1}(D(b,r)) \subset g_{t_0}^{-1}(K(h_1)) \subset\subset K(g_{t_0}). \tag{33}$$

Let

$$t_1 := \sup\{t \in [t_0, \frac{1}{s^{d-1}}] \mid \forall u \in [t_0, t), K(h_1) \subset\subset h_1^{-1}(K(g_u)) \subset\subset g_u^{-1}(K(h_1)) \subset\subset K(g_u)\}. \tag{34}$$

Note that by (32), (33) and (28), t_1 is well-defined. It is easy to see that

$$K(h_1) \subset h_1^{-1}(K(g_{t_1})) \subset g_{t_1}^{-1}(K(h_1)) \subset K(g_{t_1}). \tag{35}$$

Therefore, for each $t \in [t_0, t_1]$, we have $g_t(\mathrm{CV}^*(h_1) \cup \mathrm{CV}^*(g_t)) \subset g_t(K(h_1)) \subset K(h_1)$ and $h_1(\mathrm{CV}^*(h_1) \cup \mathrm{CV}^*(g_t)) \subset h_1(K(h_1)) = K(h_1)$. In particular,

$$(h_1, g_t) \in \mathcal{B} \text{ for each } t \in [t_0, t_1]. \tag{36}$$

Moreover, for each $t \in [t_0, t_1)$,

$$\left(h_1^{-1}(K(g_t) \setminus \mathrm{int}(K(h_1)))\right) \amalg \left(g_t^{-1}(K(g_t) \setminus \mathrm{int}(K(h_1)))\right) \subset K(g_t) \setminus \mathrm{int}(K(h_1)).$$

Therefore $J(h_1, g_t) \subset K(g_t) \setminus \mathrm{int}(K(h_1))$ and by [14, Lemma 1.1.4]

$$J(h_1, g_t) = h_1^{-1}(J(h_1, g_t)) \cup g_t^{-1}(J(h_1, g_t)) = h_1^{-1}(J(h_1, g_t)) \amalg g_t^{-1}(J(h_1, g_t)).$$

Thus

$$(h_1, g_t) \in \mathcal{D} \text{ for each } t \in [t_0, t_1). \tag{37}$$

In particular,

$$(h_1, g_{t_1}) \in \overline{\mathcal{D}} \cap \mathcal{B}. \tag{38}$$

We now show the following claim.

Claim 2. $t_1 < \frac{1}{s^{d-1}}$.

To prove this claim, suppose to the contrary that $t_1 = \frac{1}{s^{d-1}}$. Then $J(g_{t_1}) = \{z \in \mathbb{C} \mid |z - b| = s\}$. Since $|z_0 - b| = s$, it follows that $z_0 \in J(g_{t_1}) \cap J(h_1)$. By (35), we have $g_{t_1}(K(h_1)) \subset K(h_1)$. Therefore, $g_{t_1}(z_0) \in J(g_{t_1}) \cap K(h_1)$. Moreover, by the way of the choice of θ, we have $g_{t_1}(z_0) = z_1 \notin J(h_1)$. Hence, we obtain $g_{t_1}(z_0) \in J(g_{t_1}) \cap \mathrm{int}(K(h_1))$. In particular, $J(g_{t_1}) \cap \mathrm{int}(K(h_1)) \neq \emptyset$. However, this contradicts (35). Thus, we have proved Claim 2.

By Claim 2 and that $K(h_1) \subset \overline{D(b, s)}$, we obtain that

$$J(h_1) \cap J(g_{t_1}) = \emptyset. \tag{39}$$

Combining (39) and (35), we also obtain that

$$K(h_1) \subset \mathrm{int}(h_1^{-1}(K(g_{t_1}))) \text{ and } g_{t_1}^{-1}(K(h_1)) \subset \mathrm{int}(K(g_{t_1})). \tag{40}$$

Moreover, by the definition of t_1 and (40), we obtain $h_1^{-1}(J(g_{t_1})) \cap g_{t_1}^{-1}(J(h_1)) \neq \emptyset$. In particular, $h_1^{-1}(J(h_1, g_{t_1})) \cap g_{t_1}^{-1}(J(h_1, g_{t_1})) \neq \emptyset$. Combining this with (36) and [20, Theorems 1.5, 1.7], we obtain that $(h_1, g_{t_1}) \in \mathcal{C}$. Combining this with (38), we see that

$$(h_1, g_{t_1}) \in (\partial \mathcal{D}) \cap \mathcal{B}. \tag{41}$$

We next prove the following claim.

Claim 3. $g_{t_1}^{-1}(J(h_1)) \cap J(h_1) = \emptyset$.

To prove this claim, suppose to the contrary that there exists a point $w \in g_{t_1}^{-1}(J(h_1)) \cap J(h_1)$. Then by (35), we have $w \in h_1^{-1}(K(g_{t_1}))$. If we would have that $w \in h_1^{-1}(\text{int}(K(g_{t_1})))$, then (35) implies that $w \in g_{t_1}^{-1}(\text{int}(K(h_1)))$. However, this contradicts $w \in g_{t_1}^{-1}(J(h_1))$. Therefore, we must have that $w \in h_1^{-1}(J(g_{t_1}))$. Hence $w \in h_1^{-1}(J(g_{t_1})) \cap J(h_1)$. Therefore $h_1(w) \in J(g_{t_1}) \cap J(h_1)$. However, this contradicts (39). Thus, we have proved Claim 3.

By (35) and Claim 3, we obtain that

$$K(h_1) \subset \text{int}(g_{t_1}^{-1}(K(h_1))). \text{ In particular, } K(h_1) \cap g_{t_1}^{-1}(J(h_1)) = \emptyset. \tag{42}$$

We next prove the following claim.

Claim 4. For each $t \in [t_0, t_1]$, $P^*(h_1, g_t) \subset \text{int}(K(h_1))$.

To prove this claim, let $t \in [t_0, t_1]$. By (36), we have that $\text{CV}^*(h_1) \cup \text{CV}^*(g_t) \subset K(h_1)$. Moreover, by (35), $g_t(K(h_1)) \cup h_1(K(h_1)) \subset K(h_1)$. Therefore

$$P^*(h_1, g_t) \subset K(h_1). \tag{43}$$

Combining this with (42), we obtain that there exists a constant $\epsilon_1 > 0$ such that for each $z \in g_t^{-1}(J(h_1))$, for each $h \in \langle h_1, g_t \rangle$ and for each connected component V_1 of $h^{-1}(D(z, \epsilon_1))$,

$$h : V_1 \to D(z, \epsilon_1) \text{ is bijective.} \tag{44}$$

Since $\text{CV}^*(g_t) = \{b\} \subset \text{int}(K(h_1))$, there exists a number $\epsilon_2 > 0$ such that for each $z \in J(h_1)$, for each connected component V_2 of $g_t^{-1}(D(z, \epsilon_2))$, we have that

$$\text{diam}(V_2) < \epsilon_1 \text{ and } g_t : V_2 \to D(z, \epsilon_2) \text{ is bijective.} \tag{45}$$

Since h_1 is hyperbolic, there exists a number $\epsilon_3 > 0$ such that for each $z \in J(h_1)$, for each $n \in \mathbb{N}$ and for each connected component V_3 of $h_1^{-n}(D(z, \epsilon_3))$, we have that

$$\text{diam}(V_3) < \epsilon_2 \text{ and } h_1^n : V_3 \to D(z, \epsilon_3) \text{ is bijective.} \tag{46}$$

By (44), (45) and (46), it follows that for each $z \in J(h_1)$, for each $g \in \langle h_1, g_t \rangle$, and for each connected component W of $g^{-1}(D(z, \epsilon_3))$, we have that $g : W \to D(z, \epsilon_3)$ is bijective. Thus $J(h_1) \cap P^*(h_1, g_t) = \emptyset$. Combining this with (43), we obtain that $P^*(h_1, g_t) \subset \text{int}(K(h_1))$. Thus we have proved Claim 4.

By (34) and (35), for each $t \in [t_0, t_1]$, $g_t(K(h_1)) \subset K(h_1)$ and $h_1(K(h_1)) \subset K(h_1)$. Thus

$$\text{int}(K(h_1)) \subset F(h_1, g_t) \text{ for each } t \in [t_0, t_1]. \tag{47}$$

Combining this with Claim 4, it follows that for each $t \in [t_0, t_1]$, $P(h_1, g_t) \subset F(h_1, g_t)$. Thus

$$(h_1, g_t) \in \mathcal{H} \text{ for each } t \in [t_0, t_1]. \tag{48}$$

By (41), (39) and (48), it follows that $(h_1, g_{t_1}) \in ((\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I} \subset ((\partial \mathcal{D}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{Q}$. Moreover, $\deg(g_{t_1}) = d$. Therefore, statement 4 in Theorem 1.6 holds. Thus we have proved Lemma 2.23. $\square$

We show some results which are related to statement 4 in Theorem 1.6. In order to do so, we need the following lemma.

Lemma 2.24. *Let* $(h_1, h_2) \in \mathcal{B}$ *with* $(\deg(h_1), \deg(h_2)) \neq (2, 2)$. *Suppose that* $K(h_1) \subset \text{int}(K(h_2))$ *and suppose that* (h_1, h_2) *satisfies the open set condition with* $U := (\text{int}(K(h_2))) \setminus K(h_1)$. *Then* $h_1^{-1}(K(h_2)) \subsetneqq h_2^{-1}(K(h_1))$ *and* $\text{int}(J(h_1, h_2)) = \emptyset$.

Proof. Since $(h_1, h_2) \in \mathcal{B}$ and (h_1, h_2) satisfies the open set condition with U, we obtain that $K(h_1) \subset h_2^{-1}(K(h_1)) \subset h_1^{-1}(K(h_2)) \subset K(h_2)$. Combining this, the assumption $(\deg(h_1), \deg(h_2)) \neq (2, 2)$, and the method in the proof of Lemma 2.16, we obtain that $h_1^{-1}(K(h_2)) \subsetneq h_2^{-1}(K(h_1))$. Hence $h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}) \subsetneq \overline{U}$. Let $G = \langle h_1, h_2 \rangle$. Since $J(G) \subset \overline{U}$ (see [8, Corollary 3.2]) and $h_1^{-1}(J(G)) \cup h_2^{-1}(J(G)) = J(G)$ (see [14, Lemma 1.1.4]), it follows that $J(G) \neq \overline{U}$. Combining this with [17, Proposition 4.3], we obtain that $\mathrm{int}(J(G)) = \emptyset$. $\square$

Definition 2.25. A polynomial semigroup G is said to be semi-hyperbolic if there exists an $N \in \mathbb{N}$ and a $\delta > 0$ such that for each $z \in J(G)$, for each $g \in G$ and for each connected component V of $g^{-1}(B(z, \delta))$, we have $\deg(g : V \to B(z, \delta)) \leq N$.

Theorem 2.26. *Under the assumptions of Lemma 2.24, suppose that h_1 is semi-hyperbolic (i.e. $\langle h_1 \rangle$ is semi-hyperbolic). Then $\langle h_1, h_2 \rangle$ is semi-hyperbolic, $J(h_1, h_2)$ is porous and $\dim_H(J(h_1, h_2)) = \overline{\dim}_B(J(h_1, h_2)) = \inf\{Z_{(h_1, h_2)}(z) \mid z \in \hat{\mathbb{C}}\} < 2$.*

Proof. Since $K(h_1) \subset \mathrm{int}(K(h_2))$, we have $K(h_1) \subset \mathrm{int}(h_1^{-1}(K(h_2)))$. By Lemma 2.24, it follows that $K(h_1) \subset \mathrm{int}(h_2^{-1}(K(h_1)))$. By using the similar method to that in the proof of Lemma 2.23, it is easy to see that $\langle h_1, h_2 \rangle$ is semi-hyperbolic. Let A be the connected component of $\mathrm{int}(K(h_2))$ with $K(h_1) \subset A$. Since $K(h_1) \subset \mathrm{int}(K(h_2))$, we have $h_2^{-1}(K(h_1)) \subset \mathrm{int}(K(h_2))$. Since (h_1, h_2) satisfies the open set condition with U, we have $h_2^{-1}(K(h_1)) \supset K(h_1)$. Moreover, since $(h_1, h_2) \in \mathcal{B}$, we have that $h_2^{-1}(K(h_1))$ is connected. It follows that $h_2^{-1}(K(h_1)) \subset A$. Thus $h_2^{-1}(A) \subset A$. It implies that $\mathrm{int}(K(h_2))$ is connected. Since $h_2(K(h_1)) \subset K(h_1) \subset \mathrm{int}(K(h_2))$, it follows that $J(h_2)$ is a quasicircle. Since $J(h_2)$ is a quasicircle and since $F_\infty(h_1)$ is a John domain (this is because h_1 is semi-hyperbolic, see [4]), it follows that there exists a constant $\alpha \in (0, 1)$ such that for each $r \in (0, 1]$ and for each $x \in \overline{U}$, we have $l_2(U \cap D(x, r)) \geq \alpha l_2(D(x, r))$, where l_2 denotes the 2-dimensional Lebesgue measure on $\mathbb{C}$. Since (h_1, h_2) satisfies the open set condition with $U := (\mathrm{int}(K(h_2))) \setminus K(h_1)$, [29, Theorem 1.11] implies that $\dim_H(J(h_1, h_2)) = \inf\{Z_{(h_1, h_2)}(z) \mid z \in \hat{\mathbb{C}}\}$.

Moreover, since $\langle h_1. h_2 \rangle$ is semi-hyperbolic, (h_1, h_2) satisfies the open set condition with U, and $J(h_1, h_2) \neq \overline{U}$ (see Lemma 2.24), [19, Theorem 1.25] implies that $J(h_1, h_2)$ is porous and $\dim_H(J(h_1, h_2)) \leq \overline{\dim}_B(J(h_1, h_2)) < 2$. Hence we have proved our theorem. $\square$

Semi-hyperbolic polynomial semigroups with open set condition have many interesting properties. For the results, see [29].

By using the method in the proof of Lemma 2.23, we can show the following theorem.

Theorem 2.27. *Suppose that $h_1 \in \mathcal{P}$, $\langle h_1 \rangle$ is postcritically bounded and $\mathrm{int}(K(h_1)) \neq \emptyset$. Moreover, let $d \in \mathbb{N}$, $d \geq 2$ and suppose that $(\deg(h_1), d) \neq (2, 2)$. Then there exists an element $h_2 \in \mathcal{P}$ such that all of the following hold.*

1. *$(h_1, h_2) \in ((\partial\mathcal{D}) \cap \mathcal{C} \cap \mathcal{B}) \setminus \mathcal{I}$ and $\deg(h_2) = d$.*

2. *We have that $K(h_1) \subset \mathrm{int}(K(h_2))$, $K(h_1) \subset \mathrm{int}(h_1^{-1}(K(h_2)))$ and $h_2^{-1}(K(h_1)) \subset \mathrm{int}(K(h_2))$. Moreover, (h_1, h_2) satisfies the open set condition with $U := (\mathrm{int}(K(h_2)) \setminus K(h_1)$. Furthermore, $\mathrm{int}(J(h_1, h_2)) = \emptyset$.*

3. *If, in addition to the assumptions of our theorem, h_1 is semi-hyperbolic, then $\langle h_1, h_2 \rangle$ is semi-hyperbolic, $J(h_1, h_2)$ is porous and and $\dim_H(J(h_1, h_2)) = \overline{\dim}_B(J(h_1, h_2)) = \inf\{Z_{(h_1, h_2)}(z) \mid z \in \hat{\mathbb{C}}\} < 2$.*

Proof. By using the method in the proof of Lemma 2.23 and Lemma 2.24, we can show that there exists an element $h_2 \in \mathcal{P}$ which satisfies properties 1,2. We now suppose h_1 is semi-hyperbolic. Then by Theorem 2.26, we obtain that $\langle h_1, h_2 \rangle$ is semi-hyperbolic, $J(h_1, h_2)$ is porous and $\dim_H(J(h_1, h_2)) = \overline{\dim}_B(J(h_1, h_2)) = \inf\{Z_{(h_1, h_2)}(z) \mid z \in \hat{\mathbb{C}}\} < 2$. $\square$

2.2 Proof of Theorem 1.8

In this subsection, we prove Theorem 1.8. We need the following.

Definition 2.28. Let Rat be the space of all non-constant rational maps on $\hat{\mathbb{C}}$, endowed with the distance κ which is defined by $\kappa(f,g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. All the notations and definitions in section 1 are generalized to the settings of rational semigroups (i.e., subsemigroups of Rat) and random dynamical systems of rational maps.

Definition 2.29. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$. Let U be a non-empty open subset of $\hat{\mathbb{C}}$. We say that $h = (h_1, \ldots, h_m)$ satisfies the open set condition with U if $\cup_{j=1}^m h_j^{-1}(U) \subset U$ and $h_i^{-1}(U) \cap h_j^{-1}(U) = \emptyset$ for each (i,j) with $i \neq j$. We say that $h = (h_1, \ldots, h_m)$ satisfies the open set condition if there exists a non-empty open set U such that $h = (h_1, \ldots, h_m)$ satisfies the open set condition with U.

Definition 2.30. For a rational semigroup G and a subset A of $\hat{\mathbb{C}}$, we set $G(A) := \cup_{g \in G} g(A)$ and $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$.

Definition 2.31. We denote by $B(\hat{\mathbb{C}})$ the set of all bounded Borel measurable complex-valued functions on $\hat{\mathbb{C}}$. For each $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, we denote by $M_\tau : B(\hat{\mathbb{C}}) \to B(\hat{\mathbb{C}})$ the operator defined by $M_\tau(\varphi)(z) = \int_{\mathrm{Rat}} \varphi(g(z)) d\tau(g)$ for each $\varphi \in B(\hat{\mathbb{C}})$ and $z \in \hat{\mathbb{C}}$. Note that $M_\tau(C(\hat{\mathbb{C}})) \subset C(\hat{\mathbb{C}})$. This M_τ is called the transition operator with respect to the random dynamical system associated with τ.

Lemma 2.32. Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h_1, \ldots, h_m \in \mathrm{Rat}$ and let $G = \langle h_1, \ldots, h_m \rangle$. Let $(p_1, \ldots, p_m) \in (0,1)^m$ with $\sum_{j=1}^m p_j = 1$. Let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Let $\varphi \in B(\hat{\mathbb{C}})$ such that $M_\tau(\varphi) = a\varphi$ for some $a \in \mathbb{C}$ with $|a| = 1$. Suppose that for each connected component Ω of $F(G)$, $\varphi|_\Omega$ is constant. Let

$$A = \{z_0 \in \hat{\mathbb{C}} \mid \text{for each neighborhood } V \text{ of } z_0 \text{ in } \hat{\mathbb{C}}, \varphi|_V \text{ is not constant on } V\}.$$

Suppose that $(h_1, \ldots, h_m)$ satisfies the open set condition with an open set U. Then either $A = J(G)$ or $A \subset \partial U$.

Proof. From the assumption of our lemma, we have $A \subset J(G)$. Moreover, by the open set condition, [16, Lemma 2.3(f)] implies that $J(G) \subset \overline{U}$. Suppose that there exists a point $z_0 \in A \cap U$. We now prove the following claim.

Claim 1. If $w_0 \in h_i^{-1}(z_0)$ for some i, then $w_0 \in A \cap U$.

To prove claim 1, let $h_i(w_0) = z_0$. Then there exists an open disk neighborhood W_0 of w_0 such that $h_i(W_0) \subset U$. By the open set condition, we have that for each j with $j \neq i$, $h_j(W_0) \subset (\hat{\mathbb{C}} \setminus \overline{U}) \subset F(G)$. Combining this with $A \subset J(G)$, we obtain that for each j with $j \neq i$ and for each $w_1, w_2 \in W_0$, $\varphi(h_j(w_1)) = \varphi(h_j(w_2))$. Hence, by $M_\tau(\varphi) = a\varphi$, we get that for each $w_1, w_2 \in W_0$, $a(\varphi(w_1) - \varphi(w_2)) = M_\tau(\varphi)(w_1) - M_\tau(\varphi)(w_2) = \sum_{j=1}^m p_j(\varphi(h_j(w_1)) - \varphi(h_j(w_2))) = p_i(\varphi(h_j(w_1)) - \varphi(h_j(w_2)))$. Since $z_0 \in A$, it follows that $w_0 \in A$. Moreover, by the open set condition, $w_0 \in U$. Therefore, claim 1 holds.

By claim 1, we obtain that $\overline{G^{-1}(z_0)} \subset A$.

We now prove the following.

Claim 2. $z_0 \notin E(G)$.

To prove claim 2, we first observe that $\{(h_{w_1} \cdots h_{w_n})^{-1}(U)\}_{(w_1, \ldots, w_n) \in \{1, \ldots, m\}^n}$ are mutually disjoint because of the open set condition. Therefore $\sharp \cup_{(w_1, \ldots, w_n) \in \{1, \ldots, m\}^n} (h_{w_1} \cdots h_{w_n})^{-1}(z_0) \geq m^n$. Thus $\sharp G^{-1}(z_0) = \infty$ and $z_0 \notin E(G)$. Hence we have proved claim 2.

By the fact $\overline{G^{-1}(z_0)} \subset A$, claim 2 and [16, Lemma 2.3(e)], it follows that $J(G) \subset \overline{G^{-1}(z_0)} \subset A$. Thus we have proved Lemma 2.32. $\qquad\square$

Definition 2.33. Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. We set $\tilde{\tau} := \otimes_{n=1}^\infty \tau \in \mathfrak{M}_1(\mathcal{P}^\mathbb{N})$. Moreover, for each $z \in \hat{\mathbb{C}}$, we set $T_{\infty,\tau}(z) = \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{P}^\mathbb{N} \mid \gamma_n \cdots \gamma_1(z) \to \infty \text{ as } n \to \infty\})$. Furthermore, we denote by G_τ the polynomial semigroup generated by $\mathrm{supp}\,\tau$. Namely, $G_\tau = \{h_n \circ \cdots \circ h_1 \mid n \in \mathbb{N}, \forall h_j \in \mathrm{supp}\,\tau\}$. Moreover, we set $\Gamma_\tau := \mathrm{supp}\,\tau$ and $X_\tau := (\mathrm{supp}\,\tau)^\mathbb{N}$.

Lemma 2.34. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $\text{int}(\hat{K}(G_\tau)) \neq \emptyset$. Then $\text{int}(T_{\infty,\tau}^{-1}(\{1\})) \subset F(G_\tau)$.*

Proof. We first prove the following claim.

Claim. For each $z_0 \in T_{\infty,\tau}^{-1}(\{1\})$, there exists no $g \in G_\tau$ with $g(z_0) \in \text{int}(\hat{K}(G_\tau))$.

To prove this claim, let $z_0 \in T_{\infty,\tau}^{-1}(\{1\})$ and suppose there exists an element $g \in G_\tau$ with $g(z_0) \in \text{int}(\hat{K}(G_\tau))$. Let $h_1, \ldots, h_m \in \Gamma_\tau$ be some elements with $g = h_m \circ \cdots \circ h_1$. Then there exists a neighborhood W of $(h_1, \ldots, h_m)$ in Γ_τ^m such that for each $\omega = (\omega_1, \ldots, \omega_m) \in W$, $\omega_m \cdots \omega_1(z_0) \in \text{int}(\hat{K}(G_\tau))$. Therefore for each $\gamma \in X_\tau$ with $(\gamma_1, \ldots, \gamma_m) \in W$, $\{\gamma_{n,1}(z_0)\}_{n \in \mathbb{N}}$ is bounded. Thus $T_{\infty,\tau}(z_0) \leq 1 - \tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_m) \in W\}) < 1$. This is a contradiction. Hence we have proved the claim.

From this claim, $G_\tau(\text{int}(T_{\infty,\tau}^{-1}(\{1\}))) \subset \hat{\mathbb{C}} \setminus \text{int}(\hat{K}(G_\tau))$. Therefore $\text{int}(T_{\infty,\tau}^{-1}(\{1\})) \subset F(G_\tau)$. Thus we have proved our lemma. $\qquad\square$

Lemma 2.35. *Let $(h_1, h_2) \in \overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}$. Suppose that $K(h_1) \subsetneqq K(h_2)$. Then $\overline{(\text{int}(K(h_2))) \setminus K(h_1)} = K(h_2) \setminus \text{int}(K(h_1))$.*

Proof. It is clear that $\overline{(\text{int}(K(h_2))) \setminus K(h_1)} \subset K(h_2) \setminus \text{int}(K(h_1))$. Let $A_1 := J(h_2) \cap (\mathbb{C} \setminus K(h_1))$, $A_2 := (\text{int}(K(h_2))) \cap (\mathbb{C} \setminus K(h_1))$, $A_3 := J(h_2) \cap J(h_1)$ and $A_4 := (\text{int}(K(h_2))) \cap J(h_1)$. Then we have

$$
\begin{aligned}
K(h_2) \setminus \text{int}(K(h_1)) &= (J(h_2) \cup \text{int}(K(h_2))) \cap (\mathbb{C} \setminus \text{int}(K(h_1))) \\
&= (J(h_2) \cup \text{int}(K(h_2))) \cap ((\mathbb{C} \setminus K(h_1)) \cup J(h_1)) \\
&= A_1 \cup A_2 \cup A_3 \cup A_4.
\end{aligned}
$$

We want to see that $A_i \subset \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$ for each $i = 1, \ldots, 4$.

It is clear that $A_2 \subset \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$.

Suppose $z_0 \in A_1$. Let $\epsilon_0 > 0$ be a small number such that $B(z_0, \epsilon_0) \subset \mathbb{C} \setminus K(h_1)$. Then for each $\epsilon \in (0, \epsilon_0)$ there exists a point $z_1 \in B(z_0, \epsilon) \cap \text{int}(K(h_2))$. Thus $z_1 \in (B(z_0, \epsilon) \cap \text{int}(K(h_2))) \setminus K(h_1)$. Hence $z_0 \in \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$. Therefore $A_1 \subset \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$.

We now let $z_0 \in A_4$. Then there exists a number $\epsilon_0 > 0$ such that $B(z_0, \epsilon_0) \subset \text{int}(K(h_2))$. For each $\epsilon \in (0, \epsilon_0)$, there exists an element $z_1 \in B(z_0, \epsilon) \cap (\mathbb{C} \setminus K(h_1))$. Hence $z_1 \in B(z_0, \epsilon) \cap (\mathbb{C} \setminus K(h_1)) \cap \text{int}(K(h_2))$. Therefore $z_0 \in \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$. Thus $A_4 \subset \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$.

We now let $z_0 \in A_3$. Since we are assuming $K(h_1) \subsetneqq K(h_2)$, there exists a point $z_1 \in J(h_2) \setminus K(h_1)$. Then for each $\epsilon > 0$ there exists a point $a_n \in h_2^{-n}(z_1)$ for some $n \in \mathbb{N}$ such that $a_n \in B(z_0, \epsilon)$. There exists a point $w_1 \in (\text{int}(K(h_2))) \setminus K(h_1)$ arbitrarily close to z_1. Hence there exists a point $b_n \in h_2^{-n}(w_1)$ arbitrarily close to a_n. Since $h_2^{-1}((\text{int}(K(h_2))) \setminus K(h_1)) \subset (\text{int}(K(h_2))) \setminus K(h_1)$ (see Proposition 2.9), it follows that $b_n \in (\text{int}(K(h_2))) \setminus K(h_1)$. Therefore we obtain that $z_0 \in \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$. Thus $A_3 \subset \overline{(\text{int}(K(h_2))) \setminus K(h_1)}$. From these arguments, it follows that $\overline{(\text{int}(K(h_2))) \setminus K(h_1)} = K(h_2) \setminus \text{int}(K(h_1))$.

Thus we have proved Lemma 2.35. $\qquad\square$

The following notion is the key to investigating random complex dynamical systems which are associated with rational semigroups.

Definition 2.36. Let G be a rational semigroup. We set $J_{\ker}(G) = \bigcap_{g \in G} g^{-1}(J(G))$. This $J_{\ker}(G)$ is called the *kernel Julia set* of G. Moreover, for a finite subset $\{h_1, \ldots, h_m\}$ of Rat, we set $J_{\ker}(h_1, \ldots, h_m) := J_{\ker}(\langle h_1, \ldots, h_m \rangle)$.

Note that $J_{\ker}(G)$ is the largest forward invariant subset of $J(G)$ under the action of G. We remark that if G is a group or if G is a commutative semigroup, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup G generated by a family of rational maps h with $\deg(h) \geq 2$, it may happen that $\emptyset = J_{\ker}(G) \neq J(G)$.

Proposition 2.37. *Let $(h_1, h_2) \in (\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$. Let $G = \langle h_1, h_2 \rangle$. Then we have all of the following.*

(1) $J_{\mathrm{ker}}(G) = J(h_1) \cap J(h_2)$.

(2) *Either $K(h_1) \subset K(h_2)$ or $K(h_2) \subset K(h_1)$.*

(3) *If $K(h_1) \subset K(h_2)$ then $F_\infty(G) = \hat{\mathbb{C}} \setminus K(h_2)$ and $\hat{K}(G) = K(h_1)$.*

Proof. If $K(h_1) = K(h_2)$, then $J(h_1) = J(h_2)$ and $J(G) = J(h_1) = J(h_2)$. Thus (1)–(3) hold.

We now suppose $K(h_1) \neq K(h_2)$. By Proposition 2.9, either $K(h_1) \subset K(h_2)$ or $K(h_2) \subset K(h_1)$. We suppose $K(h_1) \subset K(h_2)$. Since we are assuming $K(h_1) \neq K(h_2)$, we obtain $K(h_1) \subsetneqq K(h_2)$. Therefore $U := (\mathrm{int} K(h_2)) \setminus K(h_1)$ is a non-empty open set. Moreover, by Proposition 2.9, we have that (h_1, h_2) satisfies the open set condition with U. Therefore $J(G) \subset \overline{U}$. Moreover, by Lemma 2.35, we have $\overline{U} = K(h_2) \setminus \mathrm{int}(K(h_1))$. Thus we obtain $J(G) \subset K(h_2) \setminus \mathrm{int}(K(h_1))$. Therefore

$$F_\infty(G) = \hat{\mathbb{C}} \setminus K(h_2). \tag{49}$$

We now prove the following claim.

Claim. $\hat{K}(G) = K(h_1)$.

To prove this claim, it is easy to see that $\hat{K}(G) \subset K(h_1)$. By Proposition 2.9, we have that

$$h_j^{-1}((\mathrm{int}(K(h_2))) \setminus K(h_1)) \subset (\mathrm{int}(K(h_2))) \setminus K(h_1) \text{ for each } j = 1, 2. \tag{50}$$

Suppose $h_2(K(h_1)) \cap (\mathbb{C} \setminus K(h_1)) \neq \emptyset$. Then $h_2(\mathrm{int}(K(h_1))) \cap (\mathbb{C} \setminus K(h_1)) \neq \emptyset$. Since $h_2(\mathrm{int}(K(h_1))) \subset h_2(\mathrm{int}(K(h_2))) \subset \mathrm{int}(K(h_2))$, it follows that $h_2^{-1}((\mathrm{int}(K(h_2))) \setminus K(h_1)) \cap \mathrm{int}(K(h_1)) \neq \emptyset$. However, this contradicts (50). Thus we must have that $h_2(K(h_1)) \subset K(h_1)$. Hence $K(h_1) \subset \hat{K}(G)$. Therefore $K(h_1) = \hat{K}(G)$. Thus we have proved the above claim.

By Claim and (49), we obtain $J(h_1) \cap J(h_2) \subset \hat{K}(G) \cap \overline{F_\infty(G)}$. Therefore for each $j = 1, 2$, we have

$$h_j(J(h_1) \cap J(h_2)) \subset \hat{K}(G) \cap \overline{F_\infty(G)} = K(h_1) \cap (\hat{\mathbb{C}} \setminus \mathrm{int}(K(h_2))).$$

Since $K(h_1) \subset K(h_2)$, we have $\mathrm{int}(K(h_1)) \subset \mathrm{int}(K(h_2))$. Hence

$$K(h_1) \cap (\hat{\mathbb{C}} \setminus \mathrm{int}(K(h_2))) \subset J(h_1) \cap (\mathbb{C} \setminus \mathrm{int}(K(h_2))) = J(h_1) \cap ((\mathbb{C} \setminus K(h_2)) \cup J(h_2)) = J(h_1) \cap J(h_2).$$

It follows that $h_j(J(h_1) \cap J(h_2)) \subset J(h_1) \cap J(h_2)$ for each $j = 1, 2$. Therefore $J(h_1) \cap J(h_2) \subset J_{\mathrm{ker}}(G)$.

We now let $z_0 \in J_{\mathrm{ker}}(G)$. Then we have $z_0 \in K(h_1) \cap K(h_2) = K(h_1)$. By Proposition 2.9, we have $h_2(\mathrm{int}(K(h_1))) \subset \mathrm{int}(K(h_1))$. Therefore $g(\mathrm{int}(K(h_1))) \subset \mathrm{int}(K(h_1))$ for each $g \in G$. It implies that $\mathrm{int}(K(h_1)) \subset F(G)$. Since $z_0 \in J(G) \cap K(h_1)$, it follows that $z_0 \in J(h_1)$. We now want to show that $z_0 \in J(h_2)$. Suppose that $z_0 \in \mathrm{int}(K(h_2))$. By Proposition 2.9, $J(h_2)$ is a quasicircle and h_2 has a unique attracting fixed point $c \in \mathrm{int}(K(h_2))$. Since $c \in P(G) \subset F(G)$, it follows that there exists a number $n \in \mathbb{N}$ such that $h_2^n(z_0) \in F(G)$. However, this contradicts $z_0 \in J_{\mathrm{ker}}(G)$. Therefore $z_0 \in J(h_2)$. Thus $z_0 \in J(h_1) \cap J(h_2)$. Hence $J_{\mathrm{ker}}(G) \subset J(h_1) \cap J(h_2)$. Therefore $J_{\mathrm{ker}}(G) = J(h_1) \cap J(h_2)$. Thus we have proved Proposition 2.37. $\square$

Lemma 2.38. *Statement 1 in Theorem 1.8 holds.*

Proof. Let $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B}) \cup ((\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H})$ and let $0 < p < 1$. Let $\varphi(z) = T(h_1, h_2, p, z)$. Let

$$A = \{z_0 \in \hat{\mathbb{C}} \mid \text{ for each neighborhood } V \text{ of } z_0, \varphi|_V \text{ is not constant}\}.$$

Let $G = \langle h_1, h_2 \rangle$. Since $\emptyset \neq P^*(G) \subset \hat{K}(G)$, $\varphi \not\equiv 1$. Hence $\varphi : \hat{\mathbb{C}} \to [0, 1]$ is not constant. If $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B}$, then by Lemma 2.6 and [24, Lemmas 3.75, 3.73, Theorem 3.22], it follows that $J(G) = A$.

We now suppose $(h_1, h_2) \in (\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$. We consider the following two cases. Case 1. $(h_1, h_2) \in \mathcal{Q}$. Case 2. $(h_1, h_2) \notin \mathcal{Q}$. If we have case 1, then $\hat{K}(G) = K(h_1) = K(h_2)$ and $J(G) = J(h_1) = J(h_2)$. Therefore $A = J(G) = J(h_1) = J(h_2)$. Thus, the remaining case is Case 2. Let $(h_1, h_2) \in$

$((\partial\mathcal{C})\cap\mathcal{B}\cap\mathcal{H})\setminus\mathcal{Q}$. By Proposition 2.9, we may assume that $K(h_1) \subset K(h_2)$. Then by Proposition 2.9 again, we have

$$K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2) \tag{51}$$

and (h_1, h_2) satisfies the open set condition with $U := (\mathrm{int}(K(h_2)))\setminus K(h_1)$. Moreover, by Lemma 2.16, we have $h_1^{-1}(K(h_2)) \neq h_2^{-1}(K(h_1))$. Therefore $h_1^{-1}(J(h_2)) \setminus h_2^{-1}(J(h_1)) \neq \emptyset$. Moreover, by Proposition 2.37 (1), that $K(h_1) \subset K(h_2)$ and that $(h_1, h_2) \notin \mathcal{Q}$, we obtain that $J(h_2) \cap J(h_1)$ is a nowhere dense subset of $J(h_2)$. It follows that $(h_1^{-1}(J(h_2))) \setminus (h_2^{-1}(J(h_1)) \cup J(h_1)) \neq \emptyset$. Let $z_0 \in h_1^{-1}(J(h_2)) \setminus (h_2^{-1}(J(h_1)) \cup J(h_1))$ be a point. Since $h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1))$ and $z_0 \notin h_2^{-1}(J(h_1))$, we obtain $z_0 \in h_2^{-1}(\mathrm{int}(K(h_1)))$. Hence

$$h_2(z_0) \in \mathrm{int}(K(h_1)). \tag{52}$$

Since $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$, we obtain $h_i(K(h_1)) \subset K(h_1)$ for each $i = 1, 2$. Therefore $\mathrm{int}(K(h_1)) \subset F(G)$. Combining this with (52), we obtain

$$h_2(z_0) \in F(G). \tag{53}$$

We now show the following claim.

Claim 1. $z_0 \in U$.

To prove this claim, since $z_0 \in h_2^{-1}(J(h_1))$, the fact $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$ implies that $z_0 \in K(h_2)$. If $z_0 \in J(h_2)$, then (52) implies that $h_2(z_0) \in J(h_2) \cap \mathrm{int}(K(h_1))$. However, this contradicts $K(h_1) \subset K(h_2)$. Therefore, we must have that

$$z_0 \in \mathrm{int}(K(h_2)). \tag{54}$$

If $z_0 \in K(h_1)$, then the fact $z_0 \in h_1^{-1}(J(h_2)) \setminus (h_2^{-1}(J(h_1)) \cup J(h_1))$ implies $z_0 \notin J(h_1)$. Hence $z_0 \in \mathrm{int}(K(h_1))$. Therefore $z_0 \in (\mathrm{int}(K(h_1))) \cap h_1^{-1}(J(h_2))$. However, this contradicts $K(h_1) \subset h_1^{-1}(K(h_2))$. Thus, we must have that

$$z_0 \notin K(h_1). \tag{55}$$

By (54)(55), we obtain that $z_0 \in U$. Thus claim 1 holds.

Since $z_0 \in h_1^{-1}(J(h_2))$, we have $h_1(z_0) \in J(h_2)$. Moreover, by (51), we have $h_1(F_\infty(h_2)) \subset F_\infty(h_2)$. Therefore for each $g \in G$, we have $g(F_\infty(h_2)) \subset F_\infty(h_2)$. It follows that $F_\infty(h_2) \subset F(G)$. Therefore $F_\infty(h_2) = F_\infty(G)$. Thus $h_1(z_0) \in \overline{F_\infty(G)} \cap J(G)$. We now prove the following claim.

Claim 2. For each neighborhood W of $h_1(z_0)$ in $\hat{\mathbb{C}}$, the function $\varphi|_W : W \to [0, 1]$ is not constant.

To prove claim 2, suppose that there exists a neighborhood W of $h_1(z_0)$ in $\hat{\mathbb{C}}$ and a constant $c \in [0, 1]$ such that $\varphi|_W \equiv c$. Since $h_1(z_0) \in \overline{F_\infty(G)}$, [24, Lemma 5.24] implies that $c = 1$. Thus $h_1(z_0) \in \mathrm{int}(\varphi^{-1}(1))$. Combining this with Lemma 2.34, we obtain that $h_1(z_0) \in F(G)$. However, this contradicts $h_1(z_0) \in J(G)$. Therefore, claim 2 holds.

By the equation $\varphi(z) = p\varphi(h_1(z)) + (1 - p)\varphi(h_2(z))$, the fact $\varphi : F(G) \to [0, 1]$ is locally constant (see [24, Lemma 5.27]), (53), and claim 2, we obtain that $z_0 \in A$. From this, claim 1, Lemma 2.32 and the fact that (h_1, h_2) satisfies the open set condition with U, it follows that $A = J(G)$.

Thus statement 1 in Theorem 1.8 holds. $\qquad\square$

We now prove statement 2 in Theorem 1.8.

Lemma 2.39. *Statement 2 in Theorem 1.8 holds.*

Proof. Let $(h_1, h_2) \in (\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$ and let $0 < p < 1$. Let $G = \langle h_1, h_2 \rangle$. Let $\varphi(z) = T(h_1, h_2, p, z)$. Suppose $J(h_1) \cap J(h_2) = \emptyset$. Then by Proposition 2.37, we obtain that $J_{\mathrm{ker}}(G) = \emptyset$. By [24, Theorem 3.22], it follows that $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ is continuous on $\hat{\mathbb{C}}$.

We now suppose that $J(h_1) \cap J(h_2) \neq \emptyset$. By Proposition 2.37 (3), we have $J(h_1) \cap J(h_2) \subset \overline{F_\infty(G)} \cap \hat{K}(G)$. Since $\varphi|_{F_\infty(G)} \equiv 1$ (see [24, Lemma 5.24]) and $\varphi|_{\hat{K}(G)} \equiv 0$, it follows that $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ is not continuous at each point in $J(h_1) \cap J(h_2)$. Thus statement 2 in Theorem 1.8 holds. $\qquad\square$

2.3 Proof of Theorem 1.15

In this subsection, we prove Theorem 1.15. We also show a result on the Fatou components (Theorem 2.45) and a result in which we do not assume hyperbolicity (Theorem 2.42).

Definition 2.40. For an element $\tau \in \mathfrak{M}_1(\mathrm{Rat})$, we denote by U_τ the space of all linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where a non-zero element $\varphi \in C(\hat{\mathbb{C}})$ is said to be a unitary eigenvector of M_τ if there exists an element $a \in \mathbb{C}$ with $|a| = 1$ such that $M_\tau(\varphi) = a\varphi$.

Lemma 2.41. *Statement 1 in Theorem 1.15 holds.*

Proof. Let $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B} \cap \mathcal{H}) \cup (((\partial\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I})$. Let $p \in (0, 1)$. By Lemma 2.8 and Proposition 2.37, we obtain $J_{\ker}(h_1, h_2) = \emptyset$. From this, that $\langle h_1, h_2 \rangle$ is hyperbolic, and [24, Proposition 3.63], it follows that there exists an open neighborhood V of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$ such that for each $(g_1, g_2, q) \in V \times (0, 1)$, we have that $\{g_1, g_2\}$ is mean stable and $\tau_{g_1, g_2, q}$ is mean stable. By [26, Remark 5.11], taking a small open neighborhood W of p, and shrinking V if necessary, we obtain that there exists a number $\alpha \in (0, 1)$ such that for each $(g_1, g_2, q) \in V \times W$, $T(g_1, g_2, q, \cdot) \in C^\alpha(\hat{\mathbb{C}})$.

We now let $(g_1, g_2, q) \in V \times W$. Since $\emptyset \neq \hat{K}(g_1, g_2)$ and $\langle g_1, g_2 \rangle (\hat{K}(g_1, g_2)) \subset \hat{K}(g_1, g_2)$, Zorn's lemma implies that there exists a minimal set L_0 for $(\langle g_1, g_2 \rangle, \hat{\mathbb{C}})$ with $L_0 \subset \hat{K}(g_1, g_2)$. Let L be a minimal set for $(\langle g_1, g_2 \rangle, \hat{\mathbb{C}})$ with $L \neq \{\infty\}$. Then $L \subset \hat{K}(g_1, g_2)$. By Proposition 2.9, we may assume that $K(h_1) \subset K(h_2)$. Then, by Proposition 2.9 again, we obtain that $J(h_2)$ is a quasicircle and h_2 has an attracting fixed point c in $K(h_2)$. Shrinking V if necessary, we obtain that $J(g_2)$ is a quasicircle and g_2 has an attracting fixed point c' in $K(g_2)$. Moreover, since $\{g_1, g_2\}$ is mean stable, $L \subset F(g_1, g_2)$. Therefore $L \subset \mathrm{int}(K(g_2))$. It follows that for each $z \in L$, $g_2^n(z) \to c'$ as $n \to \infty$. In particular, $c' \in L$. Thus, there exists a unique minimal set L_{g_1, g_2} for $(\langle g_1, g_2 \rangle, \hat{\mathbb{C}})$ with $L_{g_1, g_2} \subset \mathbb{C}$. Hence the set of all minimal sets for $(\langle g_1, g_2 \rangle, \hat{\mathbb{C}})$ is $\{\{\infty\}, L_{g_1, g_2}\}$. Moreover, from the above argument we have $L_{g_1, g_2} \subset \mathrm{int}(\hat{K}(g_1, g_2)) \subset F(g_1, g_2)$. By [24, Theorem 3.15-7], item (iv) of statement 1 in Theorem 1.15 follows. Since L_{g_1, g_2} contains a fixed point g_2, [24, Theorem 3.15-12] implies that the number r_L in [24, Theorem 3.15-8] for $L = L_{g_1, g_2}$ is equal to 1. Therefore, by [24, Theorem 3.15-1, 13], there exist two functions $\varphi_1, \varphi_2 \in C(\hat{\mathbb{C}})$ with $M_{g_1, g_2, q}(\varphi_i) = \varphi_i$ and a Borel probability measure $\nu = \nu_{g_1, g_2, q}$ on L_{g_1, g_2} with $M^*_{g_1, g_2, q}(\nu) = \nu$ such that for each $\varphi \in C(\hat{\mathbb{C}})$,

$$M_{g_1, g_2, q}^n(\varphi) \to \varphi(\infty) \cdot \varphi_1 + \left(\int \varphi \, d\nu\right) \cdot \varphi_2 \text{ in } C(\hat{\mathbb{C}}) \text{ as } n \to \infty,$$

$\mathrm{supp}\, \nu = L_{g_1, g_2}$, $\varphi_1(\infty) = 1, \varphi_1|_{L_{g_1, g_2}} \equiv 0$, $\varphi_2(\infty) = 0$, and $\varphi_2|_{L_{g_1, g_2}} \equiv 1$. Combining these with item (iv), we see that $\varphi_1(z) = T(g_1, g_2, q, z)$ and $\varphi_2(z) = 1 - T(g_1, g_2, q, z)$. From these arguments item (v) of statement 1 in Theorem 1.15 follows. By [26, Theorem 3.24], shrinking V and W if necessary, item (vi) of statement 1 in Theorem 1.15 holds. By [26, Theorem 3.32], item (vii) of statement 1 in Theorem 1.15 holds.

We now prove item (viii). Let $(g_1, g_2) \in V$ and let $G = \langle g_1, g_2 \rangle$. Since the statement in item (vii) holds for arbitrary $p \in (0, 1)$, we obtain that the function $q \mapsto T(h_1, h_2, q, z)$ is real-analytic on $(0, 1)$ for any $z \in \hat{\mathbb{C}}$, that the function $(q, z) \mapsto (\partial^n T/\partial q^n)(g_1, g_2, q, z)$ is continuous on $(0, 1) \times \hat{\mathbb{C}}$ for any $n \in \mathbb{N} \cup \{0\}$, and that the function $z \mapsto (\partial^n T/\partial q^n)(g_1, g_2, q, z)$ is Hölder continuous on $\hat{\mathbb{C}}$ for any $n \in \mathbb{N} \cup \{0\}$ and any $q \in (0, 1)$. Moreover, for any $n \in \mathbb{N} \cup \{0\}$ and any $q \in (0, 1)$, since $z \mapsto T(g_1, g_2, q, z)$ is locally constant on $F(G)$ (see [24, Lemma 3.24]), it follows that the function $z \mapsto (\partial^n T/\partial q^n)(g_1, g_2, q, z)$ is locally constant on $F(G)$. By [24, Proposition 3.26], for each $q \in (0, 1)$, the function $z \mapsto T(g_1, g_2, q, z)$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that $M_{g_1, g_2, q}(\varphi) = \varphi, \varphi|_{\hat{K}(G)} \equiv 0, \varphi|_{F_\infty(G)} \equiv 1$. For each $q \in (0, 1)$ and for each $n \in \mathbb{N} \cup \{0\}$, we set $\varphi_{n, q}(z) = (\partial^n T/\partial q^n)(g_1, g_2, q, z)$. Since $\varphi_{0, q}|_{F_\infty(G)} \equiv 1$ and $\varphi_{0, q}|_{\hat{K}(G)} \equiv 0$, we have $\varphi_{n, q}|_{F_\infty(G) \cup \hat{K}(G)} \equiv 0$ for each $n \geq 1$. By [26, Theorem 3.32], the function $\varphi_{1, q}$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that $\varphi_{1, q}(z) = M_{g_1, g_2, q}(\varphi_{1, q})(z) + (\varphi_{0, q}(g_1(z)) - $

$\varphi_{0,q}(g_2(z)))$, $\varphi_{1,q}|_{F_\infty(G)\cup \hat{K}(G)} \equiv 0$. Let $k \geq 0$ and suppose that $\varphi_{k+1,q}(z) = M_{g_1,g_2,q}(\varphi_{k+1,q})(z) + (k+1)(\varphi_{k,q}(g_1(z)) - \varphi_{k,q}(g_2(z)))$. By taking the partial derivatives of both hand sides of this equation with respect to the parameter q, we obtain that $\varphi_{k+2,q}(z) = M_{g_1,g_2,q}(\varphi_{k+2,q})(z) + (k+2)(\varphi_{k+1,q}(g_1(z)) - \varphi_{k+1,q}(g_2(z)))$. Therefore for each $n \in \mathbb{N} \cup \{0\}$, we have $\varphi_{n+1,q}(z) = M_{g_1,g_2,q}(\varphi_{n+1,q})(z) + (n+1)(\varphi_{n,q}(g_1(z)) - \varphi_{n,q}(g_2(z)))$. Let $n \in \mathbb{N}, q \in (0,1)$ and let $\varphi \in C(\hat{\mathbb{C}})$ be an element such that

$$\varphi(z) = M_{g_1,g_2,q}(\varphi)(z) + (n+1)(\varphi_{n,q}(g_1(z)) - \varphi_{n,q}(g_2(z))) \text{ and } \varphi|_{F_\infty(G)\cup\hat{K}(G)} \equiv 0. \qquad (56)$$

We want to show that $\varphi = \varphi_{n+1,q}$. Let $\phi_n(z) = (n+1)(\varphi_{n,q}(g_1(z)) - \varphi_{n,q}(g_2(z)))$. Then $\phi_n \in C^\gamma(\hat{\mathbb{C}})$ for some $\gamma \in (0,1)$. Since $\tau_{g_1,g_2,q}$ is mean stable, there exists a direct decomposition $C(\hat{\mathbb{C}}) = U_{\tau_{g_1,g_2,q}} \oplus \{\psi \in \hat{\mathbb{C}} \mid M^n_{g_1,g_2,q}(\psi) \to 0 \text{ as } n \to \infty\}$ (see [24, Theorem 3.15]). Let $\pi_{\tau_{g_1,g_2,q}} : C(\hat{\mathbb{C}}) \to U_{\tau_{g_1,g_2,q}}$ be the projection map regarding the direct decomposition. Moreover, by [26, Theorem 3.30] and its proof, there exist constants $\zeta \in (0,\gamma], \lambda \in (0,1), C > 0$ such that for each $\psi \in C^\zeta(\hat{\mathbb{C}})$, $\|M^k_{g_1,g_2,q}(\psi - \pi_{\tau_{g_1,g_2,q}}(\psi))\|_\zeta \leq C\lambda^k \|\psi - \pi_{\tau_{g_1,g_2,q}}(\psi)\|_\zeta$ for each $k \in \mathbb{N}$. By the definition of ϕ_n, we have $\phi_n|_{\{\infty\}\cup\hat{K}(G)} \equiv 0$. Therefore by [24, Theorem 3.15-2], we have $\pi_{\tau_{g_1,g_2,q}}(\phi_n) = 0$. It follows that $\|M^k_{g_1,g_2,q}(\phi_n)\|_\zeta \leq C\lambda^k \|\phi_n\|_\zeta$. By (56), we have

$$(I - M^k_{g_1,g_2,q})(\varphi) = \sum_{j=0}^{k-1} M^k_{g_1,g_2,q}(\phi_n) \text{ for each } k \in \mathbb{N}. \qquad (57)$$

Moreover, by $\varphi|_{F_\infty(G)\cup\hat{K}(G)} \equiv 0$ and [24, Theorem 3.15-2] we have $\pi_{\tau_{g_1,g_2,q}}(\varphi) = 0$. Hence $M^k_{g_1,g_2,q}(\varphi) \to 0$ in $C(\hat{\mathbb{C}})$ as $k \to \infty$. Therefore, letting $k \to \infty$ in (57), we obtain $\varphi = \sum_{j=0}^\infty M^j_{g_1,g_2,q}(\phi_n)$ in $C(\hat{\mathbb{C}})$. (In fact, this equation holds even in $C^\zeta(\hat{\mathbb{C}})$.) Thus, there exists a unique element $\varphi \in C(\hat{\mathbb{C}})$ which satisfies (56). Hence we have proved item (viii). Thus we have proved statement 1 in Theorem 1.15. $\qquad \square$

Theorem 2.42. *Let* $(h_1, h_2) \in \mathcal{B}$. *Let* $p \in (0,1)$. *Suppose that* $K(h_1) \subset (int(K(h_2)))$. *Let* $U := (int(K(h_2))) \setminus K(h_1)$. *Suppose that* (h_1, h_2) *satisfies the open set condition with U. Then, we have all of the following.*

(i) *$J_{\text{ker}}(h_1, h_2) = \emptyset$. Moreover, $T(h_1, h_2, p, \cdot)$ is Hölder continuous on $\hat{\mathbb{C}}$ and locally constant on $F(G)$. Moreover, $J(G) = \{z_0 \in \hat{\mathbb{C}} \mid$ for each neighborhood V of $z_0, T(h_1, h_2, p, \cdot)|_V$ is not constant$\}$.*

(ii) *There exists a unique minimal set L for $(\langle h_1, h_2 \rangle, \hat{K}(h_1, h_2))$ and the set of minimal sets for $(\langle h_1, h_2 \rangle, \hat{\mathbb{C}})$ is $\{\{\infty\}, L\}$.*

(iii) *For each $z \in \hat{\mathbb{C}}$ there exists a Borel subset $\mathcal{B}_z$ of $\{h_1, h_2\}^\mathbb{N}$ with $\tilde{\tau}_{h_1,h_2,p}(\mathcal{B}_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{B}_z$, we have $d(\gamma_n \cdots \gamma_1(z), \{\infty\} \cup L) \to 0$ as $n \to \infty$.*

(iv) *There exists a unique $M^*_{\tau_{h_1,h_2,p}}$-invariant Borel probability measure ν on $\hat{K}(h_1, h_2)$ such that for each $\varphi \in C(\hat{\mathbb{C}})$,*

$$M^n_{h_1,h_2,p}(\varphi)(z) \to T(h_1, h_2, p, z) \cdot \varphi(\infty) + (1 - T(h_1, h_2, p, z)) \cdot \int \varphi d\nu \text{ as } n \to \infty$$

uniformly on $\hat{\mathbb{C}}$.

(v) *For each $z \in \hat{\mathbb{C}}$, the function $p \mapsto T(h_1, h_2, p, z)$ is real-analytic on $(0,1)$. Moreover, for each $n \in \mathbb{N} \cup \{0\}$, the function $(p, z) \mapsto (\partial^n T / \partial p^n)(h_1, h_2, p, z)$ is continuous on $(0,1) \times \hat{\mathbb{C}}$. The function $z \mapsto T(h_1, h_2, p, z)$ is characterized by the unique element $\varphi \in C(\hat{\mathbb{C}})$ such that $M_{h_1,h_2,p}(\varphi) = \varphi, \varphi|_{\hat{K}(G)} \equiv 0, \varphi|_{F_\infty(G)} \equiv 1$. Furthermore, inductively, for any $n \in \mathbb{N} \cup \{0\}$, the*

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function $z \mapsto (\partial^{n+1}T/\partial p^{n+1})(h_1, h_2, p, z)$ *is characterized by the unique element* $\varphi \in C(\hat{\mathbb{C}})$ *such that*

$$\varphi(z) \equiv M_{h_1,h_2,p}(\varphi)(z) + (n+1)\left((\partial^n T/\partial p^n)(h_1, h_2, p, h_1(z)) - (\partial^n T/\partial p^n)(h_1, h_2, p, h_2(z))\right),$$
$$\varphi|_{\hat{K}(G) \cup F_\infty(G)} \equiv 0.$$

Moreover, the function $z \mapsto (\partial^{n+1}T/\partial p^{n+1})(h_1, h_2, p, z)$ *is locally constant on* $F(G)$. *Moreover, setting* $\psi_{n,h_1,h_2,p}(z) = (\partial^n T/\partial p^n)(h_1, h_2, p, z)$ *for each* $p \in (0,1), z \in \hat{\mathbb{C}}, n \in \mathbb{N} \cup \{0\}$, *we have that* $\psi_{n+1,h_1,h_2,p} = \sum_{j=0}^\infty M_{h_1,h_2,p}^j((n+1)(\psi_{n,h_1,h_2,p} \circ h_1 - \psi_{n,h_1,h_2,p} \circ h_2))$ *in* $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$ *for each* $n \in \mathbb{N} \cup \{0\}$.

Proof. Let A be the connected component of $\text{int}(K(h_2))$ with $K(h_1) \subset A$. Since $K(h_1) \subset \text{int}(K(h_2))$, we have $h_2^{-1}(K(h_1)) \subset \text{int}(K(h_2))$. Since (h_1, h_2) satisfies the open set condition with U, we have $h_2^{-1}(K(h_1)) \supset K(h_1)$. Moreover, since $(h_1, h_2) \in \mathcal{B}$, we have that $h_2^{-1}(K(h_1))$ is connected. It follows that $h_2^{-1}(K(h_1)) \subset A$. Thus $h_2^{-1}(A) \subset A$. It implies that $\text{int}(K(h_2))$ is connected. Since $h_2(K(h_1)) \subset K(h_1) \subset \text{int}(K(h_2))$, it follows that there exists an attracting fixed point z_0 of h_2 in $K(h_1)$. Since (h_1, h_2) satisfies the open set condition with U, we have $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Since $U = (\text{int}(K(h_2))) \setminus K(h_1)$ and $\text{int}(K(h_2)) = U \cup K(h_1)$, we obtain that $h_2^{-1}(U) \cap (h_1^{-1}(\text{int}(K(h_2)))) = h_2^{-1}(U) \cap (h_1^{-1}(U) \cup K(h_1)) \subset h_2^{-1}(U) \cap K(h_1) \subset U \cap K(h_1) = \emptyset$. Therefore $\overline{h_2^{-1}(U)} \subset \hat{\mathbb{C}} \setminus h_1^{-1}(\text{int}(K(h_2)))$. We now want to show that $z_0 \in \text{int}(K(h_1))$. Suppose to the contrary that $z_0 \in J(h_1)$. Then $z_0 \in h_2^{-1}(z_0) \subset h_2^{-1}(\overline{U}) = \overline{h_2^{-1}(U)}$. It implies that $z_0 \in \hat{\mathbb{C}} \setminus h_1^{-1}(\text{int}(K(h_2))) \subset \hat{\mathbb{C}} \setminus h_1^{-1}(K(h_1)) = \hat{\mathbb{C}} \setminus K(h_1)$, which contradicts $z_0 \in J(h_1)$. Thus we must have that $z_0 \in \text{int}(K(h_1))$.

Since $h_2(K(h_1)) \subset K(h_1)$, setting $G = \langle h_1, h_2 \rangle$, we have $g(K(h_1)) \subset K(h_1)$. Hence

$$\hat{K}(G) = K(h_1). \tag{58}$$

Therefore $G(\text{int}(K(h_1))) \subset \text{int}(K(h_1))$. Thus $\text{int}(K(h_1)) \subset F(G)$. Since z_0 is an attracting fixed point of h_2 and it belongs to $\text{int}(K(h_1)) \subset F(G)$, it follows that for each $z \in K(h_1)$, there exists a number $n \in \mathbb{N}$ such that $h_2^n(z) \in F(G)$. Moreover, if $z \in \hat{\mathbb{C}} \setminus K(h_1)$, there exists a number m such that $h_1^m(z) \in F_\infty(G) \subset F(G)$. Hence, we obtain that $J_{\ker}(G) = \emptyset$. Combining this with [26, Theorem 3.29] and [24, Theorem 3.22], we obtain that the function $\psi_p := T(h_1, h_2, p, \cdot)$ is Hölder continuous on $\hat{\mathbb{C}}$ and $M_{h_1,h_2,p}(\psi_p) = \psi_p$. By [24, Lemma 3.24], ψ_p is locally constant on $F(G)$. By Lemma 2.24 and the arguments in the proof of Lemma 2.34, it follows that $J(G) = \{z_0 \in \hat{\mathbb{C}} \mid \text{for each neighborhood } V \text{ of } z_0, \psi_p|_V \text{ is not constant}\}$. By using the arguments in the proof of Lemma 2.41, we can show that items (ii)–(iv) of our theorem hold. Moreover, by [24, Proposition 3.26], ψ_p is characterized by the unique element $\psi \in C(\hat{\mathbb{C}})$ such that $\psi = M_{h_1,h_2,p}(\psi) = \psi, \psi_{F_\infty(G)} \equiv 1, \psi_{\hat{K}(G)} \equiv 0$.

Since (h_1, h_2) satisfies the open set condition with U, we have $h_1^{-1}(\text{int}(K(h_2))) \subset h_1^{-1}(U \cup K(h_1)) \subset U \cup K(h_1) = \text{int}(K(h_2))$. Hence $h_1^{-1}(K(h_2)) \subset K(h_2)$. Therefore $h_1(F_\infty(h_2)) \subset F_\infty(h_2)$. Thus $G(F_\infty(h_2)) \subset F_\infty(h_2)$. It follows that

$$F_\infty(G) = F_\infty(h_2). \tag{59}$$

Since ψ_p is continuous on $\hat{\mathbb{C}}$ and $M_{h_1,h_2,p}(\psi_p) = \psi_p$, we obtain that

$$\psi_p(z) = p\psi_p(h_1(z)) + (1-p)\psi_p(h_2(z)) \text{ for each } z \in \hat{\mathbb{C}}, \ \psi_p|_{\hat{K}(G)} \equiv 0, \ \psi_p|_{\overline{F_\infty(G)}} \equiv 1. \tag{60}$$

Let $\varphi \in C(\hat{\mathbb{C}})$ be an element such that $\varphi_{\hat{K}(G)} \equiv 0, \varphi|_{\overline{F_\infty(G)}} \equiv 1$. By items (ii) and (iii) of our theorem, which have been already proved, we obtain that $T(h_1, h_2, p, z) = \lim_{n \to \infty} M_{h_1,h_2,p}^n(\varphi)(z)$ for each $z \in \hat{\mathbb{C}}$. Let $A := \{p \in \mathbb{C} \mid |p| < 1, |1-p| < 1\}$. For each $p \in A$ and for each $\psi \in C(\hat{\mathbb{C}})$, we set $M_p(\psi)(z) = p\psi(h_1(z)) + (1-p)\psi(h_2(z))$. For each $p \in A$, we set $p_1 = p$ and $p_2 = 1 - p$. For each $n \in \mathbb{N}$ and for each $w = (w_1, \ldots, w_n) \in \{1,2\}^n$, we set $h_w = h_{w_n} \cdots h_{w_1}$ and $p_w =$

$p_{w_n} \cdots p_{w_1}$. Moreover, we set $B_{n,z} = \{(w_1, \ldots, w_n) \in \{1,2\}^n \mid h_{w_n} \cdots h_{w_1}(z) \in \overline{F_\infty(G)}\}$ and $C_{n,z} = \{(w_1, \ldots, w_n) \in \{1,2\}^n \mid h_{w_n} \cdots h_{w_1}(z) \in U\}$. Furthermore, for each $w = (w_1, \ldots, w_n) \in \{1,2\}^n$ and for each $m \leq n$, we set $w|_m := (w_1, \ldots, w_m) \in \{1,2\}^m$. Then for each $p \in A$, for each $n \in \mathbb{N}$ and for each $z \in \hat{\mathbb{C}}$, we have

$$|M_p^{n+1}(\varphi)(z) - M_p^n(\varphi)(z)|$$

$$= \left| \sum_{\omega \in B_{n+1,z}} p_\omega + \sum_{\omega \in C_{n+1,z}} p_\omega \varphi(h_\omega(z)) - \sum_{\gamma \in B_{n,z}} p_\gamma - \sum_{\gamma \in C_{n,z}} p_\gamma \varphi(h_\gamma(z)) \right|$$

$$= \left| \sum_{\omega \in B_{n+1,z}, \omega|_n \in B_{n,z}} p_\omega + \sum_{\omega \in B_{n+1,z}, \omega|_n \in C_{n,z}} p_\omega + \sum_{\omega \in C_{n+1,z}} p_\omega \varphi(h_\omega(z)) \right.$$
$$\left. - \sum_{\gamma \in B_{n,z}} p_\gamma - \sum_{\gamma \in C_{n,z}} p_\gamma \varphi(h_\gamma(z)) \right|.$$

Since $\sum_{\omega \in B_{n+1,z}, \omega|_n \in B_{n,z}} p_\omega = \sum_{\gamma \in B_{n,z}} p_\gamma$, we obtain

$$|M_p^{n+1}(\varphi)(z) - M_p^n(\varphi)(z)| = \left| \sum_{\omega \in B_{n+1,z}, \omega|_n \in C_{n,z}} p_\omega + \sum_{\omega \in C_{n+1,z}} p_\omega \varphi(h_\omega(z)) - \sum_{\gamma \in C_{n,z}} p_\gamma \varphi(h_\gamma(z)) \right|. \tag{61}$$

Let K be a non-empty compact subset of A. Then there exists a constant $c_K \in (0,1)$ such that $K \subset \{|z| < c_K\}$. By (61) and that (h_1, h_2) satisfies the open set condition with U, it follows that $|M_p^{n+1}(\varphi)(z) - M_p^n(z)| \leq c_K^n + (c_K^{n+1} + c_K^n) \sup_{a \in \hat{\mathbb{C}}} |\varphi(a)|$. Therefore $S_{\infty,p}(z) := \lim_{n \to \infty} M_p^n(\varphi)(z)$ exists in $A \times \hat{\mathbb{C}}$, $(p,z) \to S_{\infty,p}(z)$ is continuous on $A \times \hat{\mathbb{C}}$, and

$$\text{for each } z \in \hat{\mathbb{C}}, \text{ the function } p \mapsto S_{\infty,p}(z) \text{ is holomorphic in } A. \tag{62}$$

Combining these with Cauchy's integral formula, we obtain that for each $n \in \mathbb{N}$, $\frac{\partial^n S_{\infty,p}(z)}{\partial p^n}$ is continuous on $A \times \hat{\mathbb{C}}$. From these arguments, it follows that for each $z \in \hat{\mathbb{C}}$, the function $p \mapsto T(h_1, h_2, p, z)$ is real-analytic on $(0,1)$ and for each $n \in \mathbb{N} \cup \{0\}$, the function $(p,z) \mapsto (\partial^n T/\partial p^n)(h_1, h_2, p, z)$ is continuous on $(0,1) \times \hat{\mathbb{C}}$.

Let $\psi_{n,p}(z) = \frac{\partial^n T}{\partial p^n}(h_1, h_2, p, z)$ for each $n \in \mathbb{N} \cup \{0\}$ and $z \in \hat{\mathbb{C}}$. By taking the n-th derivatives of equations in (60), we obtain that

$$\psi_{n+1,p}(z) = M_p(\psi_{n+1,p})(z) + (n+1)(\psi_{n,p}(h_1(z)) - \psi_{n,p}(h_2(z))), \psi_{n+1,p}|_{\overline{F_\infty(G)} \cup \hat{K}(G)} \equiv 0 \tag{63}$$

for each $n \in \mathbb{N} \cup \{0\}$, $p \in (0,1)$ and $z \in \hat{\mathbb{C}}$. Let $\phi_{n,p}(z) := (n+1)(\psi_{n,p}(h_1(z)) - \psi_{n,p}(h_2(z)))$. Then $\phi_{n,p}|_{\overline{F_\infty(G)} \cup \hat{K}(G)} \equiv 0$. Let $\psi \in C(\hat{\mathbb{C}})$ be an element such that $\psi_{\overline{F_\infty(G)} \cup \hat{K}(G)} \equiv 0$. Then $M_p^k(\psi)(z) = \sum_{\omega \in \{1,2\}^k} p_\omega \psi(h_\omega(z)) = \sum_{\omega \in C_{k,z}} p_\omega \psi(h_\omega(z))$. Since (h_1, h_2) satisfies the open set condition with U, it follows that

$$\|M_p^k(\psi)\|_\infty \leq (\max\{p, 1-p\})^k \|\psi\|_\infty. \tag{64}$$

If $\alpha \in C(\hat{\mathbb{C}})$ is an element such that

$$\alpha(z) = M_p(\alpha)(z) + (n+1)(\psi_{n,p}(h_1(z)) - \psi_{n,p}(h_2(z))), \alpha|_{\overline{F_\infty(G)} \cup \hat{K}(G)} \equiv 0 \tag{65}$$

then we have

$$(I - M_p^k)(\alpha)(z) = \sum_{j=0}^{k} M_p^j(\phi_{n,p})(z) \text{ for each } z \in \hat{\mathbb{C}}, k \in \mathbb{N}. \tag{66}$$

By (64), letting $k \to \infty$ in (66) we obtain that $\alpha = \sum_{j=0}^{\infty} M_p^j(\phi_{n,p})$ in $(C(\hat{\mathbb{C}}), \|\cdot\|_\infty)$. Therefore, for each $n \in \mathbb{N} \cup \{0\}$, the element $\psi_{n+1,p} \in C(\hat{\mathbb{C}})$ is characterized by the unique element $\alpha \in C(\hat{\mathbb{C}})$ such that $\alpha(z) = M_p(\alpha)(z) + (n+1)(\psi_{n,p}(h_1(z)) - \psi_{n,p}(h_2(z))), \alpha|_{\overline{F_\infty(G)} \cup \hat{K}(G)} \equiv 0$. Since $\psi_p = T(h_1, h_2, p, \cdot)$ is locally constant on $F(G)$, $\psi_{n+1,p}$ is locally constant on $F(G)$.

Combining all of these arguments, we see that item (v) of our theorem holds.

Thus we have proved Theorem 2.42. $\qquad \square$

Remark 2.43. Let $h_1 \in \mathcal{P}$ and $d \in \mathbb{N}$ with $d \geq 2$ and $p \in (0,1)$. Suppose that $\langle h_1 \rangle$ is postcritically bounded, $\mathrm{int}(K(h_1)) \neq \emptyset$ and $(\deg(h_1), d) \neq (2,2)$. Then, by Theorem 2.27, there exists an element $h_2 \in \mathcal{P}$ with $\deg(h_2) = d$ such that $(h_1, h_2) \in ((\partial D) \cap \mathcal{C} \cap \mathcal{B}) \setminus \mathcal{I}$ and such that (h_1, h_2, p) satisfies the assumptions of Theorem 2.42. Note that if h_1 has a parabolic cycle or a Siegel disk cycle, then the above h_2 can be taken so that $\{h_1, h_2\}$ is **not** mean stable. In fact, in order to have such an h_2, we take a point $b \in \mathrm{int}(K(h_1))$ so that b belongs to the basin of parabolic cycle or the Siegel disk cycle of h_1 and then use the method in the proof of Lemma 2.23. In [26], the author showed several results on the random dynamical systems for which the associated kernel Julia sets are empty and all minimal sets are included in the Fatou sets. However, the author did not deal with the case for which some minimal sets meet the Julia sets. We remark that if $h_1 \in \mathcal{P}$ has a parabolic cycle, $\langle h_1 \rangle$ is postcritically bounded, and $d \geq 2$ satisfies $(\deg(h_1), d) \neq (2,2)$, then we can take an $h_2 \in \mathcal{P}$ so that $\deg(h_2) = d$, $(h_1, h_2) \in ((\partial \mathcal{D}) \cap \mathcal{C} \cap \mathcal{B}) \setminus \mathcal{I}$, (h_1, h_2) satisfies the assumptions of Theorem 2.42, $J_{\mathrm{ker}}(h_1, h_2) = \emptyset$ and the bounded minimal set of $\langle h_1, h_2 \rangle$ meets $J(h_1, h_2)$ (we take a point b in the basin of the parabolic cycle of h_1 and use the method of Lemma 2.23). Thus, Theorems 2.27 and 2.42 deal with a new case.

Lemma 2.44. *Statement 2 in Theorem 1.15 holds.*

Proof. Let $(h_1, h_2) \in (\mathcal{D} \cap \mathcal{B})$. Then by Proposition 2.7, we may assume that $K(h_1) \subset \mathrm{int}(K(h_2))$. By Proposition 2.7 again, (h_1, h_2) satisfies the open set condition with $U := (\mathrm{int}(K(h_2)) \setminus K(h_1)$. Thus, Theorem 2.42 implies that statement 2 in Theorem 1.15 holds. $\qquad \square$

We now give a result on the set of connected components of the Fatou set, which is shown by applying Theorem 1.15-1-(i) and Theorem 1.5-5.

Theorem 2.45. *Let $(h_1, h_2) \in ((\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$. Then there exists a neighborhood V of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$ with $V \cap int(\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H} \neq \emptyset$ such that for each $(g_1, g_2) \in V$, the set of connected components U of $F(g_1, g_2)$ with $U \cap (F_\infty(g_1, g_2) \cup \hat{K}(g_1, g_2)) = \emptyset$ is infinite. In particular, for each $(g_1, g_2) \in V$, there are infinitely many connected components of $F(g_1, g_2)$.*

Proof. By Theorem 1.15-1-(i) and Theorem 1.5-5, there exists a neighborhood V of (h_1, h_2) in $\mathcal{B} \cap \mathcal{H}$ such that for each $(g_1, g_2) \in V$, the set $\{g_1, g_2\}$ is mean stable and $\dim_H(J(g_1, g_2)) < 2$. Thus for each $(g_1, g_2) \in V$, we have $\hat{K}(g_1, g_2) \neq \emptyset$, $J_{\mathrm{ker}}(g_1, g_2) = \emptyset$ and $\mathrm{int}(J(g_1, g_2)) = \emptyset$. Combining this with [24, Theorem 3.34], we see that for each $(g_1, g_2) \in V$, the set of connected components U of $F(g_1, g_2)$ with $U \cap (F_\infty(g_1, g_2) \cup \hat{K}(g_1, g_2)) = \emptyset$ is infinite. In particular, for each $(g_1, g_2) \in V$, there are infinitely many connected components of $F(g_1, g_2)$. Since $(g_1, g_2) \in (\partial \mathcal{C}) \cap \mathcal{B} \cap \mathcal{H}$, Theorem 1.6-1 implies that $V \cap \mathrm{int}(\mathcal{C}) \cap \mathcal{B} \cap \mathcal{H} \neq \emptyset$. Thus we have proved our theorem. $\qquad \square$

2.4 Proof of Theorem 1.18

In this subsection, we prove Theorems 1.18. Also, we show some related results in which we do not assume hyperbolicity (Theorems 2.52, 2.56). We need the following.

Definition 2.46. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$. Let $p = (p_1, p_2, \ldots, p_m) \in (0,1)^m$ with $\sum_{j=1}^{m} p_j = 1$. Let $\tilde{\mu}$ be an $\tilde{h}$-invariant Borel probability measure on $J(\tilde{h})$ (i.e., $\tilde{\mu}(A) = \tilde{\mu}(\tilde{h}^{-1}(A))$ for each Borel subset A of $J(\tilde{h})$). We set

$$v(h, p, \tilde{\mu}) := \frac{-\int_{\Sigma_m \times \hat{\mathbb{C}}} \log p_{w_1} d\tilde{\mu}(w, x)}{\int_{\Sigma_m \times \hat{\mathbb{C}}} \log \|D(h_{w_1})_x\|_s d\tilde{\mu}(w, x)} \in (0, \infty)$$

(when the denominator is positive).

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Definition 2.47. Let $(h_1, \ldots, h_m) \in (\mathrm{Rat})^m$ and let $G = \langle h_1, \ldots, h_m \rangle$. Let L be a minimal set for $(G, \hat{\mathbb{C}})$. We say that L is attracting (for $(G, \hat{\mathbb{C}})$) if there exist non-empty open subsets U, V of $F(G)$ and a number $n \in \mathbb{N}$ such that both of the following hold.

- $L \subset V \subset \overline{V} \subset U \subset \overline{U} \subset F(G), \sharp(\hat{\mathbb{C}} \setminus V) \geq 3$.
- For each $(w_1, \ldots, w_n) \in \{1, \ldots, m\}^n$, $h_{w_1} \cdots h_{w_n}(\overline{U}) \subset V$.

Definition 2.48. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ and let L be a minimal set for $(G_\tau, \hat{\mathbb{C}})$. For each $z \in \hat{\mathbb{C}}$, we set $T_{L,\tau}(z) = \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{Rat})^{\mathbb{N}} \mid d(\gamma_n \cdots \gamma_1(z), L) \to 0 \text{ as } n \to \infty\})$.

Definition 2.49. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$. We denote by $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ the dual map of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, i.e., $\int_{\hat{\mathbb{C}}} \varphi(z) d(M_\tau^*(\mu))(z) = \int_{\hat{\mathbb{C}}} M_\tau(\varphi)(z) d\mu(z)$ for each $\varphi \in C(\hat{\mathbb{C}})$ and $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$. By using the topological embedding $z \in \hat{\mathbb{C}} \mapsto \delta_z \in \mathfrak{M}_1(\hat{\mathbb{C}})$, we regard $\hat{\mathbb{C}}$ as a compact subset of the compact metric space $\mathfrak{M}_1(\hat{\mathbb{C}})$. We denote by $F_{pt}^0(\tau)$ the set of points $z_0 \in \hat{\mathbb{C}}$ for which the sequence $\{(M_\tau^*)^n|_{\hat{\mathbb{C}}} : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous at the one point z_0.

Lemma 2.50. *Let $(h_1, \ldots, h_m) \in (\mathrm{Rat})^m$ and let $G = \langle h_1, \ldots, h_m \rangle$. Let L be an attracting minimal set for $(G, \hat{\mathbb{C}})$. Let $p = (p_1, \ldots, p_m) \in (0, 1)^m$ with $\sum_{j=1}^m p_j = 1$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Then $T_{L,\tau}$ is locally constant on $F(G)$ and $M_\tau(T_{L,\tau}) = T_{L,\tau}$. Moreover, $T_{L,\tau}$ is continuous at every point of $F_{pt}^0(\tau)$.*

Proof. Let U be the open set coming from Definition 2.47 for L. By [26, Remark 3.5], for each $z \in U$ and for each $\gamma \in \Sigma_m$, we have $d(h_{\gamma_n} \circ \cdots \circ h_{\gamma_1}(z), L) \to 0$ as $n \to \infty$ and $\|D(h_{\gamma_n} \circ \cdots \circ h_{\gamma_1})_z\|_s \to 0$ as $n \to \infty$. Hence, for each $\omega \in \Sigma_m$, for each connected component W of $F(G)$, if x is a point of W and $d(h_{\omega_n} \circ \cdots \circ h_{\omega_1}(x), L) \to 0$ as $n \to \infty$, then for any $y \in W$, we have $d(h_{\omega_n} \circ \cdots \circ h_{\omega_1}(y), L) \to 0$ as $n \to \infty$. Therefore $T_{L,\tau}$ is constant on W. Thus $T_{L,\tau}$ is locally constant on $F(G)$.

Let $\epsilon > 0$ be a small number such that $B(L, \epsilon) \subset U$. Let $\varphi \in C(\hat{\mathbb{C}})$ be an element such that $\varphi|_L \equiv 1$ and $\varphi|_{\hat{\mathbb{C}} \setminus B(L, \epsilon)} \equiv 0$. Since for each $z \in U$ and for each $\gamma \in \Sigma_m$, we have $d(h_{\gamma_n} \circ \cdots \circ h_{\gamma_1}(z), L) \to 0$ as $n \to \infty$, it follows that $M_\tau^n(\varphi)(z) \to T_{L,\tau}(z)$ as $n \to \infty$ for each $z \in \hat{\mathbb{C}}$. Therefore by [24, Lemma 4.2-2], we obtain that $T_{L,\tau}$ is continuous at every point of $F_{pt}^0(\tau)$. Moreover, for each $z \in \hat{\mathbb{C}}$, $M_\tau(T_{L,\tau})(z) = M_\tau(\lim_{n \to \infty} M_\tau^n(\varphi))(z) = T_{L,\tau}(z)$. Thus we have proved our lemma. $\square$

Definition 2.51. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$. Let $w = (w_1, \ldots, w_n) \in \{1, \ldots, m\}^n$. We set $h_w = h_{w_n} \circ \cdots \circ h_{w_1}$. Moreover, for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Sigma_m$ and for each $n \in \mathbb{N}$, we set $\gamma|_n = (\gamma_1, \ldots, \gamma_n) \in \{1, \ldots, m\}^n$.

Theorem 2.52. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$. Let $G = \langle h_1, \ldots, h_m \rangle$. Suppose that $\sharp J(G) \geq 3$. Let $p = (p_1, \ldots, p_m) \in (0, 1)^m$ with $\sum_{j=1}^m p_j = 1$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Suppose that h satisfies the open set condition with an open set U. Let $\tilde{\mu}$ be an $\tilde{h}$-invariant ergodic Borel probability measure on $J(\tilde{h})$ such that $\int \log \|D(\gamma_1)_x\|_s d\tilde{\mu}(\gamma, x) > 0$. Let $\mu := (\pi_{\hat{\mathbb{C}}})_*(\tilde{\mu})$. Suppose that $\mu(U \setminus P(G)) > 0$. Let L be an attracting minimal set for $(G, \hat{\mathbb{C}})$. Suppose that there exists a point $\xi \in U$ such that $T_{L,\tau}$ is not constant on any neighborhood of ξ. Then for μ-a.e. $z_0 \in J(G)$, we have that $z_0 \in F_{pt}^0(\tau)$, $T_{L,\tau}$ is continuous at z_0 and $\mathrm{Höl}(T_{L,\tau}, z_0) \leq v(h, p, \tilde{\mu})$.*

Proof. We assume that there exists an attracting minimal set L for $(G, \hat{\mathbb{C}})$. Let $\tilde{U} := \pi_{\hat{\mathbb{C}}}^{-1}(U \setminus P(G))$. Then $\tilde{h}^{-1}(\tilde{U}) \subset \tilde{U}$ and $\tilde{\mu}(\tilde{U}) > 0$. Hence $\tilde{\mu}(\cap_{n=0}^\infty \tilde{h}^{-n}(\tilde{U})) = \lim_{n \to \infty} \tilde{\mu}(\tilde{h}^{-n}(\tilde{U})) = \tilde{\mu}(\tilde{U}) > 0$. Therefore by the ergodicity of $\tilde{\mu}$, we obtain $\tilde{\mu}(\cap_{n=0}^\infty \tilde{h}^{-n}(\tilde{U})) = 1$. Let $(\omega, a) \in \mathrm{supp}(\tilde{\mu}) \cap \cap_{n=0}^\infty \tilde{h}^{-n}(\tilde{U}) \cap J(\tilde{h})$ be a point. By Birkhoff's ergodic theorem, we have that for $\tilde{\mu}$-a.e. $(\gamma, x) \in J(\tilde{h})$, there exists a strictly increasing sequence $\{n_j\}$ in $\mathbb{N}$ such that $\tilde{h}^{n_j}(\gamma, x) \to (\omega, a)$ as $j \to \infty$. Let $\tilde{A} := \{(\gamma, x) \in J(\tilde{h}) \mid \exists \{n_j\} \to \infty \text{ s.t. } \tilde{h}^{n_j}(\gamma, x) \to (\omega, a)\}$. Then $\tilde{\mu}(\tilde{A}) = 1$. We now prove the following claim.

Claim. For each $b \in \pi_{\hat{\mathbb{C}}}(\cap_{n=0}^\infty \tilde{h}^{-n}(\tilde{U})) \cap J(G)$, there exists a unique $\alpha \in \Sigma_m$ such that $b \in \cap_{j=1}^\infty h_{w_1}^{-1} \cdots h_{w_j}^{-1}(J(G))$.

To prove this claim, since $b \in \pi_{\hat{\mathbb{C}}}(\cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U}))$, there exists an element $\alpha \in \Sigma_m$ such that $(\alpha, b) \in \cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U})$. Therefore $b \in h_{\alpha|_n}^{-1}(U)$ for each n. Moreover, since $b \in J(G) = \pi_{\hat{\mathbb{C}}}(J(\tilde{h}))$ (see [23, Lemma 3.5]), there exists an element $\alpha' \in \Sigma_m$ such that $(\alpha', b) \in J(\tilde{h})$. Then $h_{\alpha'|_n}(b) \in J(G)$ for each n. Since $J(G) \subset \overline{U}$, we obtain $b \in \overline{h_{\alpha'|_n}^{-1}(U)}$. Hence $h_{\alpha|_n}^{-1}(U) \cap \overline{h_{\alpha'|_n}^{-1}(U)} \neq \emptyset$. Therefore $h_{\alpha|_n}^{-1}(U) \cap h_{\alpha'|_n}^{-1}(U) \neq \emptyset$. Since h satisfies the open set condition with U, it follows that $\alpha|_n = \alpha'|_n$ for each $n \in \mathbb{N}$. Hence $\alpha = \alpha'$. Therefore $b \in \cap_{n=0}^{\infty} h_{\alpha|_n}^{-1}(J(G))$. Let $\beta \in \Sigma_m$ be an element such that $b \in \cap_{n=0}^{\infty} h_{\beta|_n}^{-1}(J(G))$. Then $b \in h_{\alpha|_n}^{-1}(U) \cap h_{\beta|_n}^{-1}(J(G)) = h_{\alpha|_n}^{-1}(U) \cap \overline{h_{\beta|_n}^{-1}(U)}$ for each $n \in \mathbb{N}$. Since h satisfies the open set condition with U, we obtain that $\alpha|_n = \beta|_n$ for each $n \in \mathbb{N}$. Hence $\alpha = \beta$. Therefore our claim holds.

By the above claim and [24, Lemma 4.3], we obtain that $\pi_{\hat{\mathbb{C}}}(\cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U})) \cap J(G) \subset F_{pt}^0(\tau)$. By Lemma 2.50, for each point $z_0 \in \pi_{\hat{\mathbb{C}}}(\cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U})) \cap J(G)$, the function $T_{L,\tau}$ is continuous at z_0. Since $a \in \pi_{\hat{\mathbb{C}}}(\cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U})) \cap J(G)$, it follows that $T_{L,\tau}$ is continuous at a. Moreover, let $\{K_n\}$ be a sequence of compact subsets of $\cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U})$ such that $\tilde{\mu}((\cap_{n=0}^{\infty} \tilde{h}^{-n}(\tilde{U})) \setminus \cup_{n=1}^{\infty} K_n) = 0$. Then $B := \pi_{\hat{\mathbb{C}}}(\cup_{n=1}^{\infty} K_n) \cap J(G)$ is Borel measurable and $\mu(B) = 1$. Thus we obtain that for μ-a.e. $z_0 \in J(G)$, we have $z_0 \in F_{pt}^0(\tau)$ and $T_{L,\tau}$ is continuous at z_0.

We now want to prove that for μ-a.e. $z_0 \in J(G)$, $\mathrm{Höl}(T_{L,\tau}, z_0) \leq v(h, p, \tilde{\mu})$. We have $v(h, p, \tilde{\mu}) < \infty$, i.e., $\int |\log \|D(\gamma_1)_x\|_s| d\tilde{\mu}(\gamma, x) < \infty$. For each $k \in \mathbb{N}$, let $\theta_k = v(h, p, \tilde{\mu}) + \frac{1}{k}$. Then $\int |\log(p_{\gamma_1} \|D(h_{\gamma_1})_x\|_s^{\theta_k})| d\tilde{\mu}(\gamma, x) < \infty$. Let

$$\tilde{A}_k := \{(\gamma, x) \in J(\tilde{h}) \mid \frac{1}{n} \log(p_{\gamma_n} \cdots p_{\gamma_1} \|D(h_{\gamma|_n})_x\|_s^{\theta_k}) \to \int \log(p_{\gamma_1} \|D(h_{\gamma_1})_x\|_s^{\theta_k}) d\tilde{\mu}(\gamma, x) \text{ as } n \to \infty\}$$

for each $k \in \mathbb{N}$. By Birkhoff's ergodic theorem, we have $\tilde{\mu}(\tilde{A}_k) = 1$. Moreover, since

$$\int \log(p_{\gamma_1} \|D(h_{\gamma_1})_x\|_s^{\theta_k}) d\tilde{\mu}(\gamma, x) > 0,$$

we obtain that for each $(\gamma, x) \in \tilde{A}_k$, $p_{\gamma_n} \cdots p_{\gamma_1} \|D(h_{\gamma|_n})_x\|_s^{\theta_k} \to \infty$ as $n \to \infty$. Let $(\gamma, x) \in \tilde{A} \cap \tilde{A}_k$. Then there exists a strictly increasing sequence $\{n_j\}$ in $\mathbb{N}$ such that $h_{\gamma|_{n_j}}(x) \to a$ as $j \to \infty$. Let W be a small open disk neighborhood of a with $\overline{W} \subset U \setminus P(G)$. We may assume that $h_{\gamma|_{n_j}}(x) \in W$ for each j. Let $\zeta_j : W \to \hat{\mathbb{C}}$ be the inverse branch of $h_{\gamma|_{n_j}}$ with $\zeta_j(h_{\gamma|_{n_j}}(x)) = x$. Then $\zeta_j(W) \subset h_{\gamma|_{n_j}}^{-1}(W) \subset h_{\gamma|_{n_j}}^{-1}(U)$. Since h satisfies the open set condition with U and $J(G) \subset \overline{U}$, we obtain that for each $(\rho_1, \ldots, \rho_{n_j}) \in \{1, \ldots, m\}^{n_j}$ with $(\gamma_1, \ldots, \gamma_{n_j}) \neq (\rho_1, \ldots, \rho_{n_j})$, we have $\zeta_j(W) \cap (h_{\rho_{n_j}} \cdots h_{\rho_1})^{-1}(J(G)) = \emptyset$. In particular, $h_{\rho_{n_j}} \cdots h_{\rho_1}(\zeta_j(W))$ is included in $F(G)$. Since $h_{\rho_{n_j}} \cdots h_{\rho_1}(\zeta_j(W))$ is connected, it is included in a connected component of $F(G)$. Since $T_{L,\tau}$ is locally constant on $F(G)$ and $M_\tau(T_{L,\tau}) = T_{L,\tau}$ (see Lemma 2.50), it follows that for each $y \in \zeta_j(W)$, we have

$$|T_{L,\tau}(x) - T_{L,\tau}(y)| = p_{\gamma_{n_j}} \cdots p_1 |T_{L,\tau}(h_{\gamma|_{n_j}}(x)) - T_{L,\tau}(h_{\gamma|_{n_j}}(y))|.$$

Since we are assuming that there exists a point $\xi \in U$ such that $T_{L,\tau}$ is not constant in any neighborhood of ξ, Lemma 2.32 implies that $T_{L,\tau}$ is not constant in any neighborhood of a. Therefore there exists a point $b \in W$ such that $T_{L,\tau}(a) \neq T_{L,\tau}(b)$. Let $\eta = |T_{L,\tau}(a) - T_{L,\tau}(b)| > 0$. Let $c_j := \zeta_j(b) \in \zeta_j(W)$. Since $T_{L,\tau}$ is continuous at a, for each large j, we have

$$|T_{L,\tau}(h_{\gamma|_{n_j}}(x)) - T_{L,\tau}(h_{\gamma|_{n_j}}(c_j))| = |T_{L,\tau}(h_{\gamma|_{n_j}}(x)) - T_{L,\tau}(b)| \geq \frac{\eta}{2}.$$

Therefore for each large j, we have $|T_{L,\tau}(x) - T_{L,\tau}(c_j)| \geq p_{\gamma_{n_j}} \cdots p_1 \frac{\eta}{2}$. By the Koebe distortion theorem, there exists a constant $C > 0$ such that $c_j \in B(x, C\|D(h_{\gamma|_{n_j}})_x\|_s^{-1})$ for each j. Then

$$\sup_{y \in B(x, C\|D(h_{\gamma|_{n_j}})_x\|_s^{-1})} \frac{|T_{L,\tau}(x) - T_{L,\tau}(y)|}{C^{\theta_k} \|D(h_{\gamma|_{n_j}})_x\|_s^{-\theta_k}} \geq \frac{p_{\gamma_{n_j}} \cdots p_1}{C^{\theta_k} \|D(h_{\gamma|_{n_j}})_x\|_s^{-\theta_k}} \frac{\eta}{2} \to \infty \text{ as } j \to \infty.$$

Therefore $\limsup_{y\to x, y\neq x}\frac{|T_{L,\tau}(x)-T_{L,\tau}(y)|}{d(y,x)^{\theta_k}} = \infty$. Thus $\mathrm{H\ddot{o}l}(T_{L,\tau},x)\leq\theta_k$. Hence for each $(\gamma,x)\in\tilde{A}\cap$ $\cap_{k=1}^{\infty}\tilde{A}_k$, $\mathrm{H\ddot{o}l}(T_{L,\tau},x)\leq v(h,p,\tilde{\mu})$. Let $\{E_n\}$ be a sequence of compact subsets of $\tilde{A}\cap\cap_{k=1}^{\infty}\tilde{A}_k$ such that $\tilde{\mu}((\tilde{A}\cap\cap_{k=1}^{\infty}\tilde{A}_k)\setminus\cup_{n=1}^{\infty}E_n)=0$. Then $D:=\pi_{\hat{\mathbb{C}}}(\cup_{n=1}^{\infty}E_n)$ is Borel measurable and $\mu(D)=1$. Moreover, for each $x\in D$, $\mathrm{H\ddot{o}l}(T_{L,\tau},x)\leq v(h,p,\tilde{\mu})$. Thus we have proved Theorem 2.52. $\square$

Lemma 2.53. *Let $h=(h_1,h_2)\in(\overline{\mathcal{D}}\cap\mathcal{B}\cap\mathcal{H})\setminus\mathcal{Q}$. Let $G=\langle h_1,h_2\rangle$. Then (h_1,h_2) satisfies the open set condition with an open set U for which the following (i) and (ii) holds. (i) $(U\cap J(G))\setminus P(G)\neq\emptyset$. (ii) There exists a point $\xi\in U$ such that $T(h_1,h_2,p,\cdot)$ is not constant in any neighborhood of ξ.*

Moreover, if $\tilde{\mu}$ is an $\tilde{h}$-invariant ergodic Borel probability measure on $J(\tilde{h})$ with $\mathrm{supp}\tilde{\mu}=J(\tilde{h})$, then for each $p\in(0,1)$, setting $\mu=(\pi_{\hat{\mathbb{C}}})_*(\tilde{\mu})$ and $\tau=p\delta_{h_1}+(1-p)\delta_{h_2}$, for μ-a.e. $z_0\in J(G)$, we have $z_0\in F_{pt}^0(\tau)$, $T(h_1,h_2,p,\cdot)$ is continuous at z_0, and $\mathrm{H\ddot{o}l}(T(h_1,h_2,p,\cdot),z_0)\leq u(h_1,h_2,p,\tilde{\mu})$.

Proof. By Proposition 2.9, we may assume that $K(h_1)\subset K(h_2)$. By Proposition 2.9 again, we obtain that (h_1,h_2) satisfies the open set condition with $U:=(\mathrm{int}(K(h_2)))\setminus K(h_1)$. Let $a\in J(h_2)\setminus J(h_1)$ and let $b\in J(h_1)\setminus J(h_2)$. Then there exists a sequence $\{a_j\}$ of points with $a_j\in h_1^{-n_j}(a)$ for some $n_j\in\mathbb{N}$, such that $a_j\to b$ as $j\to\infty$. Then, for a large j, we have $a_j\in(\mathrm{int}(K(h_2)))\setminus K(h_1)=U$. Since $a_j\in J(G)$, it follows that $U\cap J(G)\neq\emptyset$. Moreover, by Lemma 2.38, for each $z_0\in J(G)$, the function $T(h_1,h_2,p,\cdot)$ is not constant in any neighborhood of z_0. Hence there exists a point $\xi\in U$ such that $T(h_1,h_2,p,\cdot)$ is not constant in any neighborhood of ξ. Furthermore, since G is hyperbolic and $U\cap J(G)\neq\emptyset$, we obtain $(U\cap J(G))\setminus P(G)\neq\emptyset$.

Let $\tilde{\mu}$ be an $\tilde{h}$-invariant ergodic Borel probability measure on $J(\tilde{h})$ with $\mathrm{supp}\,\tilde{\mu}=J(\tilde{h})$. Let $p\in(0,1)$. We set $\mu:=(\pi_{\hat{\mathbb{C}}})_*(\tilde{\mu})$ and $\tau:=p\delta_{h_1}+(1-p)\delta_{h_2}$. Since $\mathrm{supp}\,\mu=\pi_{\hat{\mathbb{C}}}(J(\tilde{h}))=J(G)$ (see [23, Lemma 3.5]), we obtain that $\mu(U\setminus P(G))>0$. By Theorem 2.52, it follows that for μ-a.e. $z_0\in J(G)$, we have $z_0\in F_{pt}^0(\tau)$, $T(h_1,h_2,p,\cdot)$ is continuous at z_0, and $\mathrm{H\ddot{o}l}(T(h_1,h_2,p,\cdot),z_0)\leq u(h_1,h_2,p,\tilde{\mu})$. Thus we have proved our lemma. $\square$

We now prove Theorem 1.18.

Lemma 2.54. *Statement 1 in Theorem 1.18 holds.*

Proof. Let $h=(h_1,h_2)\in\overline{\mathcal{D}}\cap\mathcal{B}\cap\mathcal{H}$ and let $0<p<1$. By [16, Theorem 4.3], $\mathrm{supp}\,\lambda_{h_1,h_2,p}=J(G)$. Let $p_1=p,p_2=1-p$. Since $\pi_*(\tilde{\lambda}_{h_1,h_2,p})$ is equal to the Bernoulli measure $\otimes_{n=1}^{\infty}(\sum_{i=1}^2 p_i\delta_i)$ on Σ_2, we have $\int_{\Sigma_2\times\hat{\mathbb{C}}}-\log p_{\gamma_1}d\tilde{\lambda}_{h_1,h_2,p}(\gamma,x)=-\sum_{i=1}^2 p_i\log p_i$. Moreover, by [24, Lemma 5.52], $\int_{\Sigma_2\times\hat{\mathbb{C}}}\log\|D(h_{\gamma_1})_x\|_s d\tilde{\lambda}_{h_1,h_2,p}(\gamma,x)=\sum_{i=1}^2 p_i\log\deg(h_i)$. Therefore we get that $u(h_1,h_2,p,\tilde{\lambda}_{h_1,h_2,p})=\frac{-\sum_{i=1}^2 p_i\log p_i}{\sum_{i=1}^2 p_i\log(\deg(h_i))}$. Since $\sum_{i=1}^2-p_i\log p_i\leq\log 2$ and $(\deg(h_1),\deg(h_2))\neq(2,2)$ (see Lemma 2.14), we obtain $\frac{-\sum_{i=1}^2 p_i\log p_i}{\sum_{i=1}^2 p_i\log(\deg(h_i))}<1$.

If $(h_1,h_2)\in\mathcal{Q}$, then $J(G)=J(h_1)=J(h_2)$, $F_{\infty}(G)=F_{\infty}(h_1)=F_{\infty}(h_2)$, and $\hat{K}(G)=K(h_1)=K(h_2)$. Since $T(h_1,h_2,p,\cdot)|_{F_{\infty}(G)}\equiv 1$ (see [24, Lema 5.24]) and $T(h_1,h_2,p,\cdot)|_{\hat{K}(G)}\equiv 0$, we obtain that for each $z_0\in J(G)$, the function $T(h_1,h_2,p,\cdot)$ is not continuous at z_0. In particular, for each $z_0\in J(G)$, $0=\mathrm{H\ddot{o}l}(T(h_1,h_2,p,\cdot),z_0)\leq u(h_1,h_2,p,\tilde{\lambda}_{h_1,h_2,p})$.

We now suppose that $(h_1,h_2)\in(\overline{\mathcal{D}}\cap\mathcal{B}\cap\mathcal{H})\setminus\mathcal{Q}$. By [16, Theorem 4.3], $\mathrm{supp}\tilde{\lambda}_{h_1,h_2,p}=J(\tilde{h})$. Hence, Lemma 2.53 implies that for $\lambda_{h_1,h_2,p}$-a.e. $z_0\in J(G)$, we have that the function $T(h_1,h_2,p,\cdot):\hat{\mathbb{C}}\to\mathbb{R}$ is continuous at z_0 and $\mathrm{H\ddot{o}l}(T(h_1,h_2,p,\cdot),z_0)\leq u(h_1,h_2,p,\tilde{\lambda}_{h_1,h_2,p})$.

We now suppose that $(h_1,h_2)\in\mathcal{D}\cap\mathcal{B}\cap\mathcal{H}$. Then by Lemma 2.6, $h_1^{-1}(J(G))\cap h_2^{-1}(J(G))=\emptyset$. By [24, Theorem 3.82], it follows that for $\lambda_{h_1,h_2,p}$-a.e. $z_0\in J(G)$, $\mathrm{H\ddot{o}l}(T(h_1,h_2,p,\cdot),z_0)=u(h_1,h_2,p,\tilde{\lambda}_{h_1,h_2,p})$. Thus statement 1 in Theorem 1.18 holds. $\square$

Lemma 2.55. *Statement 2 in Theorem 1.18 holds.*

Proof. Let $h=(h_1,h_2)\in(\overline{\mathcal{D}}\cap\mathcal{B}\cap\mathcal{H})\setminus\mathcal{Q}$. Then by Lemma 2.10, (h_1,h_2) satisfies the open set condition. By [18], it follows that item (i) of statement 2 holds. Moreover, we have $\mathrm{supp}\,\nu=J(\tilde{h})$. Hence $\mathrm{supp}\,\eta=J(\tilde{h})$. By Lemma 2.53 and item (i) of statement 2, it follows that for almost every

$z_0 \in J(G)$ with respect to H^v, the function $T(h_1, h_2, p, \cdot) : \hat{\mathbb{C}} \to [0,1]$ is continuous at z_0 and $\text{Höl}(T(h_1, h_2, p, \cdot), z_0) \le u(h_1, h_2, p, \eta)$.

We now suppose that $(h_1, h_2) \in \mathcal{D} \cap \mathcal{B} \cap \mathcal{H}$. Then by Lemma 2.6, we have $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$. By [24, Theorem 3.84], it follows that for almost every $z_0 \in J(G)$ with respect to H^v, $\text{Höl}(T(h_1, h_2, p, \cdot), z_0) = u(h_1, h_2, p, \eta)$.

Thus statement 2 in Theorem 1.18 holds. □

We now show the following result which is proved by using Theorem 2.52.

Theorem 2.56. *Let $(h_1, h_2) \in \mathcal{B}$ with $(\deg(h_1), \deg(h_2)) \ne (2,2)$. Let $G = \langle h_1, h_2 \rangle$. Let $p \in (0,1)$. Suppose that $K(h_1) \subset (\text{int}(K(h_2)))$. Let $U := (\text{int}(K(h_2))) \setminus K(h_1)$. Suppose that (h_1, h_2) satisfies the open set condition with U. Then, $\text{supp}\, \lambda_{h_1, h_2, p} = J(G)$, $\lambda_{h_1, h_2, p}$ is non-atomic, and for almost every point $z_0 \in J(G)$ with respect to $\lambda_{h_1, h_2, p}$,*

$$\text{Höl}(T(h_1, h_2, p, \cdot), z_0) \le u(h_1, h_2, p, \tilde{\lambda}_{h_1, h_2, p}) = -\frac{p \log p + (1-p) \log(1-p)}{p \log(\deg(h_1)) + (1-p) \log(\deg(h_2))} < 1 \quad (67)$$

and $T(h_1, h_2, p, \cdot)$ is not differentiable at z_0. In particular, there exists an uncountable dense subset A of $J(G)$ such that at every point of A, the function $T(h_1, h_2, p, \cdot)$ is not differentiable. Moreover, for each $\alpha \in (u(h_1, h_2, p, \tilde{\lambda}_{h_1, h_2, p}), 1)$ and for each $\varphi \in C^\alpha(\hat{\mathbb{C}})$ such that $\varphi(\infty) = 1$ and $\varphi|_{\hat{K}(G)} \equiv 0$, we have $\|M_{h_1, h_2, p}^n(\varphi)\|_\alpha \to \infty$ as $n \to \infty$.

Proof. From our assumption, we have $J(h_1) \cap J(h_2) = \emptyset$. Also, by Lemma 2.24, we have $h_1^{-1}(K(h_2)) \subsetneq h_2^{-1}(K(h_1))$. By Theorem 2.42 (i), we have that for each open set W in $\hat{\mathbb{C}}$ with $J(G) \cap W \ne \emptyset$, $T(h_1, h_2, p, \cdot)$ is not constant on W. Let $a \in J(h_2)$ and let $b \in J(h_1)$. Then there exists a sequence $\{a_j\}$ of points with $a_j \in h_1^{-n_j}(a)$ for some $n_j \in \mathbb{N}$, such that $a_j \to b$ as $j \to \infty$. Then, for a large j, we have $a_j \in (\text{int}(K(h_2))) \setminus K(h_1) = U$. Since $a_j \in J(G)$ and $U \subset (\mathbb{C} \setminus K(\hat{h}_1)) \subset \mathbb{C} \setminus P^*(G)$, it follows that $U \cap J(G) \cap (\hat{\mathbb{C}} \setminus P(G)) \ne \emptyset$. Also, by [16, Theorem 4.3], $\text{supp}\tilde{\lambda}_{h_1, h_2, p} = J(G)$. Hence $\lambda_{h_1, h_2, p}(U \setminus P(G)) > 0$. Moreover, by [24, Lemma 5.52], we have $\int \log \|D(\gamma_1)_x\|_s d\tilde{\lambda}_{h_1, h_2, p}(\gamma, x) = p \log \deg(h_1) + (1-p) \log \deg(h_2) > 0$. Combining these with Theorem 2.52, we see that for $\lambda_{h_1, h_2, p}$-a.e.$z_0 \in J(G)$,

$$\text{Höl}(T(h_1, h_2, p, \cdot), z_0) \le u(h_1, h_2, p, \tilde{\lambda}_{h_1, h_2, p}) = -\frac{p \log p + (1-p) \log(1-p)}{p \log(\deg(h_1)) + (1-p) \log(\deg(h_2))}.$$

Since $-p \log p - (1-p) \log(1-p) \le \log 2$ and $(\deg(h_1), \deg(h_2)) \ne (2,2)$, we have that $-\frac{p \log p + (1-p) \log(1-p)}{p \log(\deg(h_1)) + (1-p) \log(\deg(h_2))} < 1$. In particular, for $\lambda_{h_1, h_2, p}$-a.e.$z_0 \in J(G)$, $T(h_1, h_2, p, \cdot)$ is not differentiable at z_0. Moreover, by [16, Lemma 5.1], $\lambda_{h_1, h_2, p}$ is non-atomic. Hence there exists an uncountable dense subset A of $J(G)$ such that at every point of A, the function $T(h_1, h_2, p, \cdot)$ is not differentiable.

We now let $\alpha \in (u(h_1, h_2, p, \tilde{\lambda}_{h_1, h_2, p}), 1)$ and let $\varphi \in C^\alpha(\hat{\mathbb{C}})$ such that $\varphi(\infty) = 1$ and $\varphi|_{\hat{K}(G)} \equiv 0$. By Theorem 2.42, we have $M_{h_1, h_2, p}^n(\varphi)(z) \to T(h_1, h_2, p, z)$ as $n \to \infty$ uniformly on $\hat{\mathbb{C}}$. If there exists a constant $C > 0$ and a strictly increasing sequence $\{n_j\}$ in $\mathbb{N}$ such that $\|M_{h_1, h_2, p}^{n_j}(\varphi)\|_\alpha \le C$ for each j, then we obtain $T(h_1, h_2, p, \cdot) \in C^\alpha(\hat{\mathbb{C}})$. However, this is a contradiction. Hence $\|M_{h_1, h_2, p}^n(\varphi)\|_\alpha \to \infty$ as $n \to \infty$. Thus we have proved our theorem. □

Remark 2.57. Let $h_1 \in \mathcal{P}$ and $d \in \mathbb{N}$ with $d \ge 2$ and $p \in (0,1)$. Suppose that $\langle h_1 \rangle$ is postcritically bounded, $\text{int}(K(h_1)) \ne \emptyset$ and $(\deg(h_1), d) \ne (2,2)$. Then, as in Remark 2.43, by Theorem 2.27, there exists an element $h_2 \in \mathcal{P}$ with $\deg(h_2) = d$ such that $(h_1, h_2) \in ((\partial D) \cap \mathcal{C} \cap \mathcal{B}) \setminus \mathcal{I}$ and such that (h_1, h_2, p) satisfies the assumptions of Theorems 2.42 and 2.56. These two theorems imply that the associated random dynamical system does not have chaos in the C^0 sense (note that this is a randomness-induced phenomenon which cannot hold in the deterministic iteraton dynamics of an $f \in \mathcal{P}$), but still has a kind of chaos in the C^α sense for some $0 < \alpha < 1$. More precisely, there exists a number $\alpha_0 \in (0,1)$ such that for each $\alpha \in (\alpha_0, 1)$, the system behaves chaotically on the

Banach space $C^\alpha(\hat{\mathbb{C}})$ (and on the Banach space $C^1(\hat{\mathbb{C}})$ as well). Namely, as in Remark 1.21, the above results indicate that regarding the random dynamical systems, we have a kind of gradation between chaos and order. Note that in Theorems 2.42 and 2.56 we do not assume hyperbolicity, and as in Remark 2.43, if h_1 has a parabolic cycle or Siegel disk cycle, then the above h_2 can be taken so that $\langle h_1, h_2 \rangle$ is not mean stable. Moreover, as in Remark 2.43 again, if h_1 has a parabolic cycle, then the above h_2 can be taken so that $J_{\mathrm{ker}}(h_1, h_2) = \emptyset$ and a minimal set of $\langle h_1, h_2 \rangle$ meets the Julia set of $\langle h_1, h_2 \rangle$. Thus, regarding the gradation between the chaos and order, Theorems 2.42 and 2.56 deal with a new case.

2.5   Proof of Theorem 1.19

In this subsection, we prove Theorem 1.19. We need several lemmas.

Lemma 2.58. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \ldots, h_m) \in (\mathrm{Rat})^m$. Let $G = \langle h_1, \ldots, h_m \rangle$. Let $p = (p_1, \ldots, p_m) \in (0,1)^m$ with $\sum_{j=1}^m p_j = 1$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Let L be an attracting minimal set for $(G, \hat{\mathbb{C}})$. Let $\tilde{\mu}$ be an $\tilde{h}$-invariant ergodic Borel probability measure on $J(\tilde{h})$. Let $\mu = (\pi_{\hat{\mathbb{C}}})_*(\tilde{\mu})$. Suppose that all of the following holds.*

(a) $\mathrm{supp}\,\mu \subset J(G) \setminus \bigcup_{(i,j):i \neq j}(h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)))$.

(b) *There exists a point $a \in \mathrm{supp}\,\mu \setminus P(G)$ such that $T_{L,\tau}$ is not constant in any neighborhood of a.*

(c) $\int \log \|D(\gamma_1)_x\|_s \, d\tilde{\mu}(\gamma, x) > 0.$

Then, $T_{L,\tau}$ is continuous at each point of $\mathrm{supp}\,\mu$ and for μ-a.e. $z_0 \in J(G)$, $\mathrm{H\ddot{o}l}(T_{L,\tau}, z_0) \leq v(h, p, \tilde{\mu})$.

Proof. Let $x \in \mathrm{supp}\,\mu$. Then there exists an element $\gamma \in \Sigma_m$ such that $(\gamma, x) \in \mathrm{supp}\,\tilde{\mu}$. Since $\tilde{h}^n(\mathrm{supp}\,\tilde{\mu}) \subset \mathrm{supp}\,\tilde{\mu}$ for each $n \in \mathbb{N}$, we obtain that

$$h_{\gamma|_n}(x) \in \mathrm{supp}\,\mu \subset J(G) \setminus \bigcup_{(i,j):i \neq j}(h_i^{-1}(J(G)) \cap h_j^{-1}(J(G))) \text{ for each } n \in \mathbb{N}. \qquad (68)$$

Let $\alpha \in \Sigma_m$ be an element such that $x \in \cap_{n=0}^\infty h_{\alpha|_n}^{-1}(J(G))$. Then by (68), it follows that $\alpha = \gamma$. Therefore $\{\omega \in \Sigma_m \mid x \in \cap_{n=0}^\infty h_{\omega|_n}^{-1}(J(G))\} = \{\gamma\}$. By [24, Lemma 4.3], we obtain that $\mathrm{supp}\,\mu \subset F_{pt}^0(\tau)$. By Lemma 2.50, it follows that $T_{L,\tau}$ is continuous at every point of $\mathrm{supp}\,\mu$.

Since $a \in \mathrm{supp}\,\mu$, there exists a point $\omega \in \Sigma_m$ such that $(\omega, a) \in \mathrm{supp}\,\tilde{\mu}$. Let

$$\tilde{A} := \{(\gamma, x) \in J(\tilde{h}) \mid \exists \{n_j\} \to \infty \text{ s.t. } \tilde{h}^{n_j}(\gamma, x) \to (\omega, a) \text{ as } j \to \infty\}.$$

Then by Birkhoff's ergodic theorem, we have $\tilde{\mu}(\tilde{A}) = 1$. By using the assumptions of our lemma, (68) and the method in the proof of Theorem 2.52, it is easy to see that for μ-a.e. $z_0 \in J(G)$, $\mathrm{H\ddot{o}l}(T_{L,\tau}, z_0) \leq v(h, p, \tilde{\mu})$. Thus we have proved our lemma. $\qquad \square$

Lemma 2.59. *Let $(h_1, h_2) \in (\overline{\mathcal{D}} \cap \mathcal{B} \cap \mathcal{H}) \setminus \mathcal{I}$. Then $\cup_{j=1}^2 J(h_j) \subset \hat{\mathbb{C}} \setminus (h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)))$. Moreover, if, in addition to the assumption, $K(h_1) \subset K(h_2)$, then $h_2(K(h_1)) \subset int(K(h_1))$ and $h_1(\overline{F_\infty(h_2)}) \subset F_\infty(h_2)$.*

Proof. By Proposition 2.9, either $K(h_1) \subset K(h_2)$ or $K(h_2) \subset K(h_1)$. We assume $K(h_1) \subset K(h_2)$. By Proposition 2.9 again, we obtain $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$ and (h_1, h_2) satisfies the open set condition with $U = (int(K(h_2))) \setminus K(h_1)$. Then $J(G) \subset \overline{U} \subset K(h_2) \setminus int(K(h_1))$. Suppose $J(h_1) \cap h_2^{-1}(J(G)) \neq \emptyset$. Then $J(h_1) \cap h_2^{-1}(J(G)) \subset J(h_1) \cap h_2^{-1}(K(h_2) \setminus int(K(h_1))) \subset J(h_1) \cap h_2^{-1}(J(h_1))$. Hence $J(h_1) \cap h_2^{-1}(J(h_1)) \neq \emptyset$. Since $J(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1))$, we obtain that $\emptyset \neq J(h_1) \cap h_2^{-1}(J(h_1)) \subset J(h_1) \cap h_1^{-1}(J(h_2))$. Thus $J(h_1) \cap J(h_2) \neq \emptyset$. However, this contradicts $(h_1, h_2) \notin \mathcal{I}$. Hence we must have that $J(h_1) \cap h_2^{-1}(J(G)) = \emptyset$. Similarly, we can show that $J(h_2) \cap h_1^{-1}(J(G)) = \emptyset$. Since $K(h_1) \subset h_1^{-1}(K(h_2)) \subset h_2^{-1}(K(h_1)) \subset K(h_2)$, $J(h_1) \cap h_2^{-1}(J(G)) = \emptyset$, and $J(h_2) \cap h_1^{-1}(J(G)) = \emptyset$, we obtain that $h_2(K(h_1)) \subset int(K(h_1))$ and $h_1(\overline{F_\infty(h_2)}) \subset F_\infty(h_2)$. Thus we have proved our lemma. $\qquad \square$

Lemma 2.60. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $\infty \in F(G_\tau)$. Then $\text{int}(T_{\infty,\tau}^{-1}(\{0\})) \subset F(G_\tau)$.*

Proof. Since $\infty \in F(G_\tau)$, [24, Lemma 5.24] implies that for each $\gamma \in X_\tau$, $\gamma_{n,1} \to \infty$ locally uniformly on $F_\infty(G_\tau)$. We now prove the following claim.

Claim. For each $z_0 \in T_{\infty,\tau}^{-1}(\{0\})$, there exists no $g \in G_\tau$ with $g(z_0) \in F_\infty(G_\tau)$.

To prove this claim, let $z_0 \in T_{\infty,\tau}^{-1}(\{0\})$ and suppose there exists an element $g \in G_\tau$ with $g(z_0) \in F_\infty(G_\tau)$. Let $h_1, \ldots, h_m \in \Gamma_\tau$ be some elements with $g = h_m \circ \cdots \circ h_1$. Then there exists a neighborhood W of $(h_1, \ldots, h_m)$ in Γ_τ^m such that for each $\omega = (\omega_1, \ldots, \omega_m) \in W$, $\omega_m \cdots \omega_1(z_0) \in F_\infty(G_\tau)$. Therefore for each $\gamma \in X_\tau$ with $(\gamma_1, \ldots, \gamma_m) \in W$, $\gamma_{n,1}(z_0) \to \infty$ as $n \to \infty$. Thus $T_{\infty,\tau}(z_0) \geq \tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_m) \in W\}) > 0$. This is a contradiction. Hence the claim holds.

From this claim, $G_\tau(\text{int}(T_{\infty,\tau}^{-1}(\{0\}))) \subset \hat{\mathbb{C}} \setminus F_\infty(G_\tau)$. Therefore $\text{int}(T_{\infty,\tau}^{-1}(\{0\})) \subset F(G_\tau)$. $\square$

We now prove Theorem 1.19.

Proof of Theorem 1.19. By Lemma 2.14, we have $\deg(h_j) \geq 2$ for each j and $(\deg(h_1), \deg(h_2)) \neq (2, 2)$. Therefore there exists an $i \in \{1, 2\}$ such that $-\frac{\log p_i}{\log \deg(h_i)} < 1$.

By Proposition 2.9, we may assume that $K(h_1) \subset K(h_2)$. Since $(g_1, g_2) \mapsto J(g_1, g_2)$ is continuous in a neighborhood of (h_1, h_2) with respect to the Hausdorff metric (see [14, Theorem 2.4.1]), Lemma 2.59 implies that there exists an open neighborhood V of (h_1, h_2) such that for each $(g_1, g_2) \in V$, we have $\cup_{j=1}^2 J(g_j) \subset \hat{\mathbb{C}} \setminus (g_1^{-1}(J(g_1, g_2)) \cap g_2^{-1}(J(g_1, g_2)))$, $g_2(K(g_1)) \subset \text{int}(K(g_1))$ and $g_1(\overline{F_\infty(g_2)}) \subset F_\infty(g_2)$. Then for each $(g_1, g_2) \in V$, $\hat{K}(g_1, g_2) = K(g_1)$ and $F_\infty(g_1, g_2) = F_\infty(g_2)$. If V is small enough, then Lemma 2.41 implies that for each $g = (g_1, g_2) \in V$, $T(g_1, g_2, p, \cdot)$ is continuous on $\hat{\mathbb{C}}$. Hence for each $g = (g_1, g_2) \in V$, $T(g_1, g_2, p, \cdot)|_{\overline{F_\infty(g_1, g_2)}} \equiv 1$. Let $g = (g_1, g_2) \in V$. By Lemma 2.34, it follows that for each point $a \in \partial F_\infty(g_1, g_2) = J(g_2)$, the function $T(g_1, g_2, p, \cdot)$ is not constant in any neighborhood of a. Moreover, by lemma 2.60, for each point $a \in \partial \hat{K}(g_1, g_2) = J(g_1)$, the function $T(g_1, g_2, p, \cdot)$ is not constant in any neighborhood of a. Furthermore, for each $j = 1, 2$, there exists a $\tilde{g}$-invariant Borel probability measure $\tilde{\mu}_j$ on $J(\tilde{g})$ such that $(\pi_{\hat{\mathbb{C}}})_*(\tilde{\mu}_j) = \mu_j$. Since μ_j is ergodic with respect to g_j, we obtain that $\tilde{\mu}_j$ is ergodic. Moreover, for each $j = 1, 2$, we have $\int \log \|D(\gamma_1)_x\|_s d\tilde{\mu}_j(\gamma, x) = \int \log \|D(g_j)_x\|_s d\mu_j(x) = \log \deg(g_j) > 0$. Hence, Lemma 2.58 implies that for μ_j-a.e. $z_0 \in J(g_j)$, we have $\text{Höl}(T(g_1, g_2, p, \cdot), z_0) \leq -\frac{\log p_j}{\int \log \|D(g_j)_x\|_s d\mu_j(x)} = -\frac{\log p_j}{\log \deg g_j}$. Therefore items (i) (ii) of our theorem hold. We now prove item (iii) of our theorem. Let $\alpha \in (-\frac{\log p_i}{\log \deg(g_i)}, 1)$ and let $\varphi \in C^\alpha(\hat{\mathbb{C}})$ be an element such that $\varphi(\infty) = 1$ and $\varphi|_{\hat{K}(G)} \equiv 0$. By Lemma 2.41, if V is small enough, item (v) in statement 1 in Theorem 1.15 holds. Thus $M_{g_1, g_2, p}^n(\varphi)(z) \to T(g_1, g_2, p, z)$ as $n \to \infty$ uniformly on $\hat{\mathbb{C}}$. If there exists a constant $C > 0$ and a strictly increasing sequence $\{n_j\}$ in $\mathbb{N}$ such that $\|M_{g_1, g_2, p}^{n_j}(\varphi)\|_\alpha \leq C$ for each j, then we obtain $T(g_1, g_2, p, \cdot) \in C^\alpha(\hat{\mathbb{C}})$. However, this contradicts item (ii) of Theorem 1.19, which we have already proved. Therefore, item (iii) of our theorem holds. Thus, we have proved Theorem 1.19. $\square$

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