DYNAMICAL PROPERTIES AND STRUCTURE OF JULIA SETS OF POSTCRITICALLY BOUNDED POLYNOMIAL SEMIGROUPS

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ABSTRACT. We discuss the dynamic and structural properties of polynomial semigroups, a natural extension of iteration theory to random (walk) dynamics, where the semigroup G of complex polynomials (under the operation of composition of functions) is such that there exists a bounded set in the plane which contains any finite critical value of any map $g \in G$. In general, the Julia set of such a semigroup G may be disconnected, and each Fatou component of such G is either simply connected or doubly connected ([29, 33]). In this paper, we show that for any two distinct Fatou components of certain types (e.g., two doubly connected components of the Fatou set), the boundaries are separated by a Cantor set of quasicircles (with uniform dilatation) inside the Julia set of G. Important in this theory is the understanding of various situations which can and cannot occur with respect to how the Julia sets of the maps $g \in G$ are distributed within the Julia set of the entire semigroup G. We give several results in this direction and show how such results are used to generate (semi) hyperbolic semigroups possessing this postcritically boundedness condition.

1. INTRODUCTION

The dynamics of iteration of a complex analytic map has been studied quite deeply and in various contexts, e.g., rational, entire, and meromorphic maps. It is natural then to consider the generalization of this theory to the setting where the map may be changed at each point of the orbit, exactly as in a random walk. Instead of repeatedly applying the same map over and over again, one may start with a family of maps $\{h_\lambda : \lambda \in \Lambda\}$, and consider the dynamics of any iteratively defined *composition sequence* of maps, that is, any sequence $h_{\lambda_n} \circ \cdots \circ h_{\lambda_1}$ where each $\lambda_k \in \Lambda$. Assigning probabilities to the choice of map at each stage is the setting for research of random dynamics (see [9, 4, 6, 7, 5, 28, 30, 31, 32, 33] for previous work related to such dynamics). In this paper, however, we will concern ourselves with questions of dynamic stability, not just along such composition sequences one at a time, but rather we will study when such stability exists no matter which composition sequence is chosen. Restricting one's attention to the case where all h_λ are rational, one is lead to study the dynamics of rational semigroups.

A **rational semigroup** is a semigroup generated by non-constant rational maps on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation being the composition of maps. We denote by $\langle h_\lambda : \lambda \in \Lambda \rangle$ the rational semigroup generated by the family

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of maps $\{h_\lambda : \lambda \in \Lambda\}$. A **polynomial semigroup** is a semigroup generated by non-constant polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G.J. Martin in [11], who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups. Also, F. Ren, Z. Gong, and W. Zhou studied such semigroups from the perspective of random dynamical systems (see [38, 10]). Note that there is a strong connection between the study of dynamics of rational semigroups and that of random complex dynamics (see [29, 30, 31, 32, 33]). For example, for a random dynamical system generated by a family of polynomials, we consider the function $T_\infty : \overline{\mathbb{C}} \to [0,1]$ of the probability of tending to ∞. It turns out that under certain conditions, the function T_∞ is continuous on $\overline{\mathbb{C}}$ and varies only on the Julia set J of the associated polynomial semigroup G, and J is a thin fractal set. Moreover, T_∞ respects the surrounding order (see Definition 1.13). The function T_∞ is a complex analogue of the devil's staircase or Lebesgue's singular functions. For the detail of these results, see the second author's works [32, 28, 39].

As is well known, the iteration of polynomial maps $f_c(z) = z^2 + c$ for c in the Mandelbrot set (where the orbit of the sole critical point $\{f_c^n(0)\}$ is bounded in $\mathbb{C}$), provides a rich class of maps with many interesting properties. Many of these dynamic and structural properties are direct consequences the boundedness of the critical orbit. It is then natural to look at the more general situation of polynomial semigroups with bounded postcritical set. We discuss the dynamics of such polynomial semigroups as well the structure of their Julia sets. For some properties of polynomial semigroups with bounded finite postcritical set see also [26, 27, 29, 30, 31, 25, 33]. This paper is a continuation of the program initiated in [29, 30, 31, 33]. One may also see the paper [19], which is based on conference talks by the authors, where preliminary (weaker) versions of a portion of the results contained herein were announced and proven.

Definition 1.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\} \text{ and } J(G) = \overline{\mathbb{C}} \setminus F(G).$$

We call $F(G)$ the **Fatou set** of G and $J(G)$ the **Julia set** of G. The Fatou set and Julia set of the semigroup generated by a single map g is denoted by $F(g)$ and $J(g)$, respectively.

We quote the following results from [11]. The Fatou set $F(G)$ is **forward invariant** under each element of G, i.e., $g(F(G)) \subset F(G)$ for all $g \in G$, and thus $J(G)$ is **backward invariant** under each element of G, i.e., $g^{-1}(J(G)) \subset J(G)$ for all $g \in G$. Furthermore, when the cardinality $\#J(G)$ is three or more, $J(G)$ is the smallest closed subset of $\overline{\mathbb{C}}$ which contains three or more points and is backward invariant. Letting the **backward orbit** of z be denoted by $G^{-1}(z) = \cup_{g \in G} g^{-1}(\{z\})$, we have that $J(G) = \overline{G^{-1}(z)}$ for any $z \in J(G)$ whose backward orbit contains three or more points.

We should take a moment to note that the sets $F(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of *iteration* dynamics, i.e., the dynamics of semigroups generated by a single rational function. For a treatment of alternatively defined *completely* invariant Julia sets of rational semigroups the reader is referred to [13, 14, 15, 18].

Although the Julia set of a rational semigroup G may not be completely invariant, $J(G)$ has an interesting property. Namely, if G is generated by a compact family $\{h_\lambda : \lambda \in \Lambda\}$ of rational maps, then $J(G) = \bigcup_{\lambda\in\Lambda} h_\lambda^{-1}(J(G))$. This property is called the **backward self-similarity**. In particular, if $G = \langle h_1, \dots, h_m\rangle$, then $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$. From this property, the dynamics of rational semigroups can be regarded as "backward iterated function systems," and in the study of rational semigroups, we sometimes borrow and further develop techniques from iterated function systems and fractal geometry. For these things, see the second author's works [20] – [33] and [35, 36].

Note that $J(G)$ contains the Julia set of each element of G. Moreover, the following critically important result first due to Hinkkanen and Martin holds (see also [16]).

Theorem 1.2 ([11], Corollary 3.1)**.** *For rational semigroups G with $\sharp(J(G)) \geq 3$, we have*

$$J(G) = \overline{\bigcup_{f\in G} J(f)}.$$

Remark 1.3. Theorem 1.2 can be used to easily show that $F(\langle h_\lambda : \lambda \in \Lambda\rangle)$ is *precisely* the set of $z \in \overline{\mathbb{C}}$ which has a neighborhood on which every composition sequence generated by $\{h_\lambda : \lambda \in \Lambda\}$ is normal, i.e., has stable dynamics (see [38]).

In what follows we employ the following notation. The **forward orbit** of z is given by $G(z) = \cup_{g\in G} g(\{z\})$. For any subset A of $\overline{\mathbb{C}}$, we set $G^{-1}(A) = \cup_{g\in G} g^{-1}(A)$. For any polynomial g, we denote the **filled-in Julia set** of g by $K(g) := \{z \in \mathbb{C} \mid \cup_{n\in\mathbb{N}} g^n(\{z\}) \text{ is bounded in } \mathbb{C}\}$. We note that for a polynomial g with $\deg(g) \geq 2$, $J(g) = \partial K(g)$ and $K(g)$ is the polynomial hull of $J(g)$. The appropriate extension (to our situation with polynomial semigroups) of the concept of the filled-in Julia set is as follows. (See [11, 3] for other kinds of filled-in Julia sets.)

Definition 1.4. For a polynomial semigroup G, we set

$$\hat{K}(G) := \{z \in \mathbb{C} \mid G(z) \text{ is bounded in } \mathbb{C}\},$$

and call $\hat{K}(G)$ the **smallest filled-in Julia set**.

Remark 1.5. We note that for all $g \in G$, we have $\hat{K}(G) \subset K(g)$ and $g(\hat{K}(G)) \subset \hat{K}(G)$.

Definition 1.6. The **postcritical set** of a rational semigroup G is defined by

$$P(G) = \overline{\bigcup_{g\in G} \{\text{all critical values of } g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\}} \ (\subset \overline{\mathbb{C}}).$$

We say that G is **hyperbolic** if $P(G) \subset F(G)$ and we say that G is **subhyperbolic** if both $\#\{P(G) \cap J(G)\} < +\infty$ and $P(G) \cap F(G)$ is a compact subset of $F(G)$. For research on (semi-)hyperbolicity and Hausdorff dimension of Julia sets of rational semigroups see [20, 21, 22, 23, 24, 29, 30, 31, 35, 33].

Remark 1.7. It is clear that if rational semigroup G is hyperbolic, then each $g \in G$ is hyperbolic. However, the converse is not true. See Remark 5.2.

Definition 1.8. The **planar postcritical set** (or, the finite postcritical set) of a polynomial semigroup G is defined by

$$P^*(G) = P(G) \setminus \{\infty\}.$$

We say that a polynomial semigroup G is **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$.

Definition 1.9. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- each element of G is of degree at least two, and
- $P^*(G)$ is bounded in $\mathbb{C}$, i.e., G is postcritically bounded.

Furthermore, we set $\mathcal{G}_{con} = \{G \in \mathcal{G} \mid J(G) \text{ is connected}\}$ and $\mathcal{G}_{dis} = \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$.

Remark 1.10. If $G = \langle h_\lambda : \lambda \in \Lambda \rangle$, then $P(G) = \overline{\cup_{\lambda \in \Lambda} \cup_{z \in CV(h_\lambda)} (G(z) \cup \{z\})}$ where $CV(h)$ denotes the critical values of h. From this one may, in the finitely generated case, use a computer to see if $G \in \mathcal{G}$ much in the same way as one verifies the boundedness of the critical orbit for the maps $f_c(z) = z^2 + c$. The freely available software [17] can be used for this purpose.

Remark 1.11. Since $P(G)$ is forward invariant under G, we see that $G \in \mathcal{G}$ implies $P^*(G) \subset \hat{K}(G)$, and thus $P^*(G) \subset K(g)$ for all $g \in G$.

Remark 1.12. For a polynomial g of degree two or more, it is well known that $\langle g \rangle \in \mathcal{G}$ if and only if $J(g)$ is connected (see [2], Theorem 9.5.1). Hence, for any $g \in G \in \mathcal{G}$, we have that $J(g)$ is connected. We note, however, that the analogous result for polynomial semigroups does not hold as there are many examples where $G \in \mathcal{G}$, but $J(G)$ is not connected (see [37, 29, 30, 31, 33]).

See also [25] for an analysis of the number of connected components of $J(G)$ involving the inverse limit of the spaces of connected components of the realizations of the nerves of finite coverings $\mathcal{U}$ of $J(G)$, where $\mathcal{U}$ consists of backward images of $J(G)$ under finite word maps in G. In fact, the number of connected components of the Julia set of a finitely generated rational semigroup is deeply related to a new kind of cohomology (so called the "interaction cohomology"), which has been introduced by the second author of this paper. Using this cohomology, one can also investigate the number of connected components of the Fatou set of a finitely generated rational semigroup.

The aim of this paper is to investigate what can be said about the structure of the Julia sets and the dynamics of semigroups $G \in \mathcal{G}$? We begin by examining the structure of the Julia set and note that a natural order (that is respected by the backward action of the maps in G) can be placed on the components of $J(G)$, which then leads to implications on the connectedness of Fatou components.

Notation: For a polynomial semigroup $G \in \mathcal{G}$, we denote by $\mathcal{J} = \mathcal{J}_G$ the set of all connected components of $J(G)$ which do not include ∞.

Definition 1.13. We place a partial order on the space of all non-empty connected sets in $\mathbb{C}$ as follows. For connected sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" indicates that $K_1 = K_2$ or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Also, "$K_1 < K_2$" indicates $K_1 \leq K_2$ and $K_1 \neq K_2$. We call $\leq$ the *surrounding* order and read $K_1 < K_2$ as "K_1 is surrounded by K_2".

Convention: When a set K_1 is contained in the unbounded component of $\overline{\mathbb{C}} \setminus K_2$ we say that K_1 is "outside" K_2.

Theorem 1.14 ([29, 33]). *Let $G \in \mathcal{G}$ (possibly infinitely generated). Then*

(1) *$(\mathcal{J}, \leq)$ is totally ordered.*
(2) *Each connected component of $F(G)$ is either simply or doubly connected.*
(3) *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then both $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

Remark 1.15. We note that under the hypothesis of the above theorem $J_1 < J_2$ for $J_1, J_2 \in \mathcal{J}$ does not necessarily imply $g^*(J_1) < g^*(J_2)$, but only that $g^*(J_1) \leq g^*(J_2)$ as can be seen in Example 4.7. The Julia set of a $G \in \mathcal{G}_{dis}$ is shown in Figure 1.

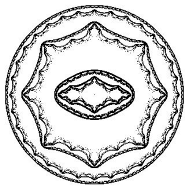

FIGURE 1. The Julia set of $G = \langle h_1, h_2 \rangle$, where we set $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$. $G \in \mathcal{G}_{dis}$ and G is hyperbolic.

We now present the main results of this paper, first giving some notation that will be needed to state our result on the existence of quasicircles in $J(G)$.

Notation: Given polynomials α_1 and α_2, we set $\Sigma_2 = \{x = (\gamma_1, \gamma_2, \dots) : \gamma_k \in \{\alpha_1, \alpha_2\}\}$. Then, for any $x = (\gamma_1, \gamma_2, \dots) \in \Sigma_2$, we set J_x equal to the set of points $z \in \overline{\mathbb{C}}$ where the sequence of functions $\{\gamma_n \circ \cdots \circ \gamma_1\}_{n \in \mathbb{N}}$ is not normal. This is sometimes called the *Julia set along the trajectory* (sequence) $x \in \Sigma_2$. See [21, 22, 24, 30, 31, 33] for much more on such *fiberwise dynamics*.

Theorem 1.16. *Let $G \in \mathcal{G}_{dis}$ and let A and B be disjoint subsets of $\overline{\mathbb{C}}$. Suppose that A is a doubly connected component of $F(G)$ and B satisfies one of the following conditions:*

- *B is a doubly connected component of $F(G)$,*
- *B is the connected component of $F(G)$ with $\infty \in B$,*
- *$B = \hat{K}(G)$.*

Then $\partial A \cap \partial B = \emptyset$. Furthermore, $\overline{A}$ and $\overline{B}$ are separated by a Cantor family of quasicircles with uniform dilatation which all lie in $J(G)$. More precisely, there exist two elements $\alpha_1, \alpha_2 \in G$ satisfying all of the following.

(1) *There exists a non-empty open set U in $\overline{\mathbb{C}}$ with $\alpha_1^{-1}(\overline{U}) \cap \alpha_2^{-1}(\overline{U}) = \emptyset$ and $\alpha_1^{-1}(\overline{U}) \cup \alpha_2^{-1}(\overline{U}) \subset U$.*
(2) *$H = \langle \alpha_1, \alpha_2 \rangle$ is hyperbolic.*
(3) *Letting Σ_2 denote the sequence space associated with $\{\alpha_1, \alpha_2\}$, we have*

(a) $J(H) = \bigcup_{x\in\Sigma_2} J_x$ *(disjoint union),*
(b) *for any component J of $J(H)$ there exists a unique element $x \in \Sigma_2$ with $J = J_x$, and*
(c) *there exists a constant $K \geq 1$ such that any component J of $J(H)$ is a K-quasicircle.*

(4) *$\{J_x\}_{x\in\Sigma_2}$ is totally ordered with $\leq$, consisting of mutually disjoint subsets of $J(H)$.*
(5) *For each $x \in \Sigma_2$, the set J_x separates $\overline{A}$ from $\overline{B}$.*

Remark 1.17. It should be noted that in the above theorem, the quasicircles J_x are all disjoint components of $J(H)$, but may all lie in the same component of $J(G)$. See the proof of Theorem 1.23, where a semigroup is constructed such that there exist only a finite number of components of the Julia set.

Remark 1.18. There are many hyperbolic polynomial semigroups $G = \langle \alpha_1, \alpha_2 \rangle \in \mathcal{G}_{dis}$ such that for a generic $x \in \Sigma_2$, the fiberwise Julia set J_x is a Jordan curve but not quasicircle, the unbounded component of $\overline{\mathbb{C}} \setminus J_x$ is a John domain, and the bounded component of $\overline{\mathbb{C}} \setminus J_x$ is not a John domain (see [31, 33, 30]). See Figure 1. This phenomenon does not occur in the usual iteration dynamics of a single polynomial.

Example 1.19. We give an example of a semigroup $G \in \mathcal{G}$ such that $J(G)$ is a "Cantor set of round circles". Let $f_1(z) = az^k$ and $f_2(z) = bz^j$ for some positive integers $k, j \geq 2$. Then, for $|a|^{k-1} \neq |b|^{j-1}$, the sets $J(f_1)$ and $J(f_2)$ are disjoint circles centered at the origin. Let A denote the closed annulus between $J(f_1)$ and $J(f_2)$. For positive integers m_1 and m_2 each greater than or equal to 2 (if k and j are not both equal to 2 then $m_1 = m_2 = 1$ will also suffice), we see that the iterates $g_1 = f_1^{m_1}$ and $g_2 = f_2^{m_2}$ will yield $A_1 = g_1^{-1}(A) \cup g_2^{-1}(A) \subset A$ where $g_1^{-1}(A) \cap g_2^{-1}(A) = \emptyset$. Now iteratively define $A_{n+1} = g_1^{-1}(A_n) \cup g_2^{-1}(A_n)$ and note that for $G = \langle g_1, g_2 \rangle$ we have $J(G) = \cap_{n=1}^{\infty} A_n$, since $J(G)$ is the smallest closed backward invariant (under each element of G) set which contains three or more points.

For our remaining results we need to note the existence of both a minimal element and a maximal element in $\mathcal{J}$ and state a few of their properties.

Theorem 1.20 ([29, 33])**.** *Let G be a polynomial semigroup in $\mathcal{G}_{dis}$. Then there is a unique element $J_{\min}(G)$ (abbreviated by $J_{\min}$) $\in \mathcal{J}$ such that $J_{\min}$ meets (and therefore contains) $\partial\hat{K}(G)$. Also, $\infty \in F(G)$ and there exists a unique element $J_{\max}(G)$ (abbreviated by $J_{\max}$) $\in \mathcal{J}$ such that $J_{\max}$ meets (and therefore contains) ∂U_∞, where U_∞ is the simply connected component of $F(G)$ which contains ∞. Moreover, int$\hat{K}(G) \neq \emptyset$. Furthermore, we have the following*

- *$J_{\min} \leq J$ for all $J \in \mathcal{J}$,*
- *$J_{\max} \geq J$ for all $J \in \mathcal{J}$,*
- *$\hat{K}(G)$, and therefore $P^*(G)$, is contained in the polynomial hull of each $J \in \mathcal{J}$.*

Remark 1.21. We see that $\partial\hat{K}(G) \subset J(G)$ when $G \in \mathcal{G}$, but, in general, we do not have $\partial\hat{K}(G) = J(G)$, unlike in iteration theory where $\partial K(g) = J(g)$ for polynomials g of degree two or more. In fact, $\partial\hat{K}(G)$ might not even equal $J_{\min}(G)$ either (see Example 4.24).

Remark 1.22. When $G \in \mathcal{G}_{con}$ we will use the convention that $J_{\min} = J_{\max} = J(G)$ and note that it is still the case that $J_{\min}$ meets $\partial \hat{K}$ and $P^*(G)$ is contained in the polynomial hull of each $J \in \mathcal{J}$. However, it is not necessarily the case that $\infty \in F(G)$, as exhibited by the example $\langle z^2/n : n \in \mathbb{N} \rangle$.

In the proofs of many results concerning postcritically bounded polynomial semigroups, it is critical to understand the distribution of the sets $J(g)$ where $g \in G$, especially when g is a generator of G. In particular, it is important to understand the relationship between such $J(g)$ and the special components $J_{\min}$ and $J_{\max}$ of $J(G)$. In Section 4 we investigate such matters carefully providing several results including Theorem 1.23 below.

In [29], it was shown that, for each positive integer k, there exists a semigroup $G \in \mathcal{G}_{dis}$ with $2k$ generators such that $J(G)$ has exactly k components. Furthermore, in [25] it was shown that any semigroup in $\mathcal{G}$ generated by exactly three elements will have a Julia set with either one component or infinitely many components (examples where the number of components is one, $\aleph_0$ or uncountable were given). Hence we have the following question: For fixed integer $k \geq 3$, what is the fewest number of generators that can produce a semigroup $G \in \mathcal{G}_{dis}$ with $\sharp \mathcal{J} = k$? The answer to this question is four as stated in Theorem 1.23 below.

Theorem 1.23. *For any $k \in \mathbb{N}$, there exists a 4-generator polynomial semigroup $H \in \mathcal{G}$ such that $\# \mathcal{J}_H = k$. Furthermore, H can be chosen so that no $J \in \mathcal{J}_H \setminus \{J_{\min}, J_{\max}\}$ meets the Julia set of any generator of H.*

The next two results, whose proofs depend on understanding the distribution of the $J(g)$ within $J(G)$, concern the (semi-)hyperbolicity of polynomial semigroups in $\mathcal{G}$. In particular, they show how one can build larger (semi-)hyperbolic polynomial semigroups in $\mathcal{G}$ from smaller ones by including maps with certain properties. We first state two definitions.

Definition 1.24. We define $\text{Poly} = \{h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid h \text{ is a non-constant polynomial}\}$, endowed with the topology of uniform convergence on $\overline{\mathbb{C}}$ with respect to the spherical metric.

Remark 1.25. For use later we note that given integer $d \geq 1$, a sequence p_n of polynomials of degree d converges to a polynomial p in Poly if and only if the coefficients converge appropriately and p is of degree d.

Definition 1.26. A rational semigroup H is **semi-hyperbolic** if for each $z \in J(H)$ there exists a neighborhood U of z and a number $N \in \mathbb{N}$ such that for each $g \in H$ we have $\deg(g : V \to U) \leq N$ for each connected component V of $g^{-1}(U)$.

Theorem 1.27. *Let $H \in \mathcal{G}$, Γ be a compact family in* Poly*, and let $G = \langle H, \Gamma \rangle$ be the polynomial semigroup generated by H and Γ. Suppose*
(1) $G \in \mathcal{G}_{dis}$,
(2) $J(\gamma) \cap J_{\min}(G) = \emptyset$ for each $\gamma \in \Gamma$, and
(3) H is semi-hyperbolic.
Then, G is semi-hyperbolic.

Remark 1.28. Theorem 1.27 would not hold if we were to replace both instances of the word *semi-hyperbolic* with the word *hyperbolic* (see Example 5.1). However, with an additional hypothesis we do get the following result.

Theorem 1.29. *Let $H \in \mathcal{G}$, Γ be a compact family in Poly, and let $G = \langle H, \Gamma \rangle$ be the polynomial semigroup generated by H and Γ. Suppose*

(1) $G \in \mathcal{G}_{dis}$,
(2) $J(\gamma) \cap J_{\min}(G) = \emptyset$ for each $\gamma \in \Gamma$,
(3) H is hyperbolic, and
(4) for each $\gamma \in \Gamma$, the critical values of γ do not meet $J_{\min}(G)$.

Then, G is hyperbolic.

Remark 1.30. We note that hypothesis (3) can be replaced by the slightly weaker hypothesis that $P^*(H) \cap J(H) = \emptyset$ since if $\infty \in J(H) \subset J(G)$, then $J(G)$ is connected by Theorem 1.20 and so hypothesis (1) fails to hold. A similar remark can be made about hypothesis (3) in Theorem 1.27.

Remark 1.31. Theorems 1.27 and 1.29 do not require that H or G be generated by a finite, or even compact, subset of Poly.

The rest of this paper is organized as follows. In Section 2 we give the necessary background and tools required. In Section 3 we give the proof of Theorem 1.16. In Section 4 we provide a more detailed look at the distribution of $J(g)$ within $J(G)$, in particular, proving Theorem 1.23. In Section 5 we give the proofs of Theorems 1.27 and 1.29 along with Example 5.1.

2. Background and Tools

We first state some notation to be used later.

Notation: Given any set $A \subset \overline{\mathbb{C}}$ we denote by $\overline{A}$ the closure of A in $\overline{\mathbb{C}}$. For $z_0 \in \mathbb{C}$ and $r, R > 0$ we set $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}, C(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$, and $Ann(z_0; r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$. Furthermore, given any set $C \subset \mathbb{C}$ we denote the $\epsilon-$neighborhood of C by $B(C, \epsilon) = \cup_{z \in C} B(z, \epsilon)$.

Most often our understanding of the surrounding order $\leq$ given in Definition 1.13 will be applied to *compact* connected sets in $\mathbb{C}$ and so in this section we state many results which we will need later. Although not all connected compact sets in $\mathbb{C}$ are comparable in the surrounding order, we do have the following two lemmas whose proofs we leave to the reader.

Lemma 2.1. *Given two connected compact sets A and B in $\mathbb{C}$ we must have exactly one of the following:*

(1) $A < B$
(2) $B < A$
(3) $A \cap B \neq \emptyset$
(4) *A and B are* ***outside of each other****, i.e., A is a subset of the unbounded component of $\mathbb{C} \setminus B$ and B is a subset of the unbounded component of $\mathbb{C} \setminus A$.*

Definition 2.2. For a compact set $A \subset \mathbb{C}$ we define the **polynomial hull** $PH(A)$ of A to be the union of A and all bounded components of $\mathbb{C} \setminus A$.

Lemma 2.3. *Let A and B be compact connected subsets of $\mathbb{C}$ such that $PH(A) \cap PH(B) \neq \emptyset$ Then exactly one of the following holds:*

(1) $A < B$
(2) $B < A$

(3) $A \cap B \neq \emptyset$.

Remark 2.4. We note that for compact connected sets A and B in $\mathbb{C}$, it follows that $A < B$ if and only if $PH(A) < B$ since the set $PH(A)$ is also compact and connected.

Lemma 2.5. *Let g be a polynomial of degree at least one and suppose $B \subset PH(A)$ where $g(B) \subset B$ and $A \subset \mathbb{C}$ is compact. Then $B \subset PH(g^{-1}(A))$. In particular, if $g \in G \in \mathcal{G}$ and $P^*(G) \subset PH(A)$ where $A \subset \mathbb{C}$ is compact, then $P^*(G) \subset PH(g^{-1}(A))$.*

Proof. Suppose $B \not\subset PH(g^{-1}(A))$. Thus there exists $z_0 \in B$ in the unbounded component U of $\overline{\mathbb{C}} \setminus g^{-1}(A)$. Let γ be a curve in U connecting z_0 to ∞. Then $\Gamma = g \circ \gamma$ is a curve in $\overline{\mathbb{C}} \setminus A$ which connects $g(z_0)$ to ∞ which shows that $g(z_0) \notin PH(A)$. Since B is forward invariant we have that $g(z_0) \in B \setminus PH(A)$ which contradicts our hypothesis. $\square$

Corollary 2.6. *Let $f, g \in G \in \mathcal{G}$. If A is of the form $J \in \mathcal{J}, J(f), g^{-1}(J)$, or $g^{-1}(J(f))$, then $P^*(G) \subset PH(A)$.*

Proof. By Theorem 1.20 we have $P^*(G) \subset PH(J)$ for all $J \in \mathcal{J}$. By Remark 1.11 $P^*(G) \subset K(f) = PH(J(f))$. The other cases then follow from Lemma 2.5. $\square$

Lemma 2.7 ([12])**.** *Let X be a compact metric space and let $f : X \to X$ be a continuous open map. Let K be a compact connected subset of X. Then for each connected component B of $f^{-1}(K)$, we have $f(B) = K$.*

Lemma 2.8. *Let g be a polynomial with $d = \deg(g) \geq 1$ and let $K \subset \mathbb{C}$ be a connected compact set such that the unbounded component U of $\overline{\mathbb{C}} \setminus K$ contains no critical values of g other than ∞, i.e., the finite critical values of g lie in $PH(K)$. Then $g^{-1}(K)$ is connected. Further, if K_1 is a connected compact set such that $K < K_1$, then $g^{-1}(K) < g^{-1}(K_1)$.*

Proof. Set $V = g^{-1}(U)$ and note that V contains no finite critical points of g. Thus by the Riemann-Hurwitz relation we have $\chi(V) + \delta_g(V) = d\chi(U)$, where $\chi(\cdot)$ denotes the Euler characteristic and $\delta_g(\cdot)$ is the deficiency. Since the hypotheses on U imply $\delta_g(V) = d - 1$ and $\chi(U) = 1$, we see that $\chi(V) = 1$. Hence the open and connected set V is simply connected.

Suppose that $g^{-1}(K)$ is not connected. Then there exists a bounded component V_0 of $\mathbb{C} \setminus g^{-1}(K)$ which is not simply connected (see [2], Proposition 5.1.5). Thus there exists a Jordan curve $\gamma \subset V_0$ such that the bounded component B of $\mathbb{C} \setminus \gamma$ contains some component E of $g^{-1}(K)$. Hence $V_0 \cup B$ is open and does not meet V. By Lemma 2.7 we have $g(E) = K$. Hence $g(V_0 \cup B) \supset K \supset \partial U$, which, by the Open Mapping Theorem, implies $g(V_0 \cup B)$ meets U, and thus $V_0 \cup B$ meets V. This contradiction implies that V_0 is simply connected and hence $g^{-1}(K)$ is connected.

Now suppose $K < K_1$. Let U_0 be the bounded component of $\mathbb{C} \setminus K_1$ such that $U_0 \supset K$ and let U_1 be the unbounded component of $\mathbb{C} \setminus K_1$. Hence $g^{-1}(U_0)$ does not meet $V_1 = g^{-1}(U_1)$. Hence $PH(g^{-1}(K_1)) = \mathbb{C} \setminus V_1 \supset g^{-1}(U_0) \supset g^{-1}(K)$. Since $g^{-1}(K_1) \cap g^{-1}(K) = \emptyset$ (which follows from the fact that $K_1 \cap K = \emptyset$), we conclude that $g^{-1}(K_1) > g^{-1}(K)$. $\square$

Corollary 2.9. *Let $g, h \in G \in \mathcal{G}$ and $J \in \mathcal{J}$. Then $g^{-1}(J)$ and $g^{-1}(J(h))$ are connected. Furthermore, $J_1 < J_2$ for $J_1, J_2 \in \mathcal{J}$ implies $g^{-1}(J_1) < g^{-1}(J_2)$, and $J(h_1) < J(h_2)$ for $h_1, h_2 \in G$ implies $g^{-1}(J(h_1)) < g^{-1}(J(h_2))$.*

Proof. Any finite critical value of g must lie in $P^*(G) \subset PH(J) \cap PH(J(h))$ by Corollary 2.6. The result then follows immediately from Lemma 2.8. $\square$

Corollary 2.10. *Let $f, g \in G \in \mathcal{G}$. For any two sets A and B of the form $J \in \mathcal{J}, J(f), g^{-1}(J)$, or $g^{-1}(J(f))$, exactly one of the following must hold:*

(1) $A < B$
(2) $B < A$
(3) $A \cap B \neq \emptyset$.

Proof. This is immediate from Corollary 2.9, Corollary 2.6 and Lemma 2.3. $\square$

The following lemma will allow one to understand the surrounding order through an imbedding, of sorts, into the real numbers.

Lemma 2.11. *Suppose $z_0 \in PH(C_1) \cap PH(C_2)$ where C_1 and C_2 are disjoint compact connected sets in $\mathbb{C}$. Then $dist(z_0, C_1) < dist(z_0, C_2)$ if and only if $C_1 < C_2$ in the surrounding order.*

Proof. First suppose that $d_2 = dist(z_0, C_2) > dist(z_0, C_1)$. Then we have $B(z_0, d_2) \subset PH(C_2)$. Since $B(z_0, d_2) \cap C_1 \neq 0$ and $C_1 \cap C_2 = \emptyset$, we must have that the bounded component of $\overline{\mathbb{C}} \setminus C_2$ which contains the connected set $B(z_0, d_2)$ also contains the connected set C_1. Thus $C_2 > C_1$.

Suppose $C_1 < C_2$. Letting $d_1 = dist(z_0, C_1)$ we see that $\overline{B(z_0, d_1)} \subset PH(C_1) < C_2$ implies $\overline{B(z_0, d_1)}$ must not meet C_2, i.e., $d_1 < dist(z_0, C_2)$. $\square$

Lemma 2.12. *Let $\{C_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of non-empty compact connected sets in $\mathbb{C}$ that are linearly ordered by the surrounding order $\leq$. Suppose $\{C_\beta\}_{\beta \in \mathcal{B}}$ is a sub-collection of $\{C_\alpha\}_{\alpha \in \mathcal{A}}$ such that $\overline{\cup_{\beta \in \mathcal{B}} C_\beta} \subset \cup_{\alpha \in \mathcal{A}} C_\alpha$. Then both $\inf_{\beta \in \mathcal{B}} C_\beta$ and $\sup_{\beta \in \mathcal{B}} C_\beta$ exist and are in $\{C_\alpha\}_{\alpha \in \mathcal{A}}$.*

Proof. By compactness and the linear ordering on $\{C_\alpha\}_{\alpha \in \mathcal{A}}$, one can quickly show that the collection $\{PH(C_\alpha)\}_{\alpha \in \mathcal{A}}$ satisfies the finite intersection property. Thus there exists some $z_0 \in \cap_{\alpha \in \mathcal{A}} PH(C_\alpha)$. For each $\beta \in \mathcal{B}$, let $r_\beta = dist(z_0, C_\beta)$ and consider $r_0 = \inf r_\beta$.

We only need to consider the case where $r_0 < r_\beta$ for all $\beta \in \mathcal{B}$, since if $r_{\beta_0} = r_0$, then clearly, by Lemma 2.11, $C_{\beta_0} = \inf_{\beta \in \mathcal{B}} C_\beta$. Select a strictly decreasing sequence $r_{\beta_n} \to r_0$. By Lemma 2.11, we have that $C_{\beta_1} > C_{\beta_2} > \dots$. Let $z_{\beta_n} \in C_{\beta_n}$ be arbitrary. Without loss of generality we may assume that $z_{\beta_n} \to a_0 \in \mathbb{C}$. By hypothesis there exists C_{α_0} which contains a_0. We will now show that $C_{\alpha_0} = \inf_{\beta \in \mathcal{B}} C_\beta$.

Fixing $\beta \in \mathcal{B}$ and applying Lemma 2.11, we see that $\{a_0\} < C_\beta$, since the sequence $(z_{\beta_k})_{k \geq n}$ must lie in $PH(C_{\beta_n}) < C_\beta$ for large n (whenever $r_{\beta_n} < r_\beta$). Thus we must have $C_{\alpha_0} < C_\beta$ for all $\beta \in \mathcal{B}$. Hence C_{α_0} is a lower bound for $\{C_\beta\}_{\beta \in \mathcal{B}}$. Suppose that $C_{\alpha_1} > C_{\alpha_0}$. It must then be the case that $\{a_0\} < C_{\alpha_1}$ and so it follows that $\{z_{\beta_n}\} < C_{\alpha_1}$ for large n. Thus $C_{\beta_n} < C_{\alpha_1}$ for large n, implying that C_{α_1} is not a lower bound for $\{C_\beta\}_{\beta \in \mathcal{B}}$. We conclude that $C_{\alpha_0} = \inf_{\beta \in \mathcal{B}} C_\beta$.

The proof that $\sup_{\beta \in \mathcal{B}} C_\beta$ exist in $\{C_\alpha\}_{\alpha \in \mathcal{A}}$ follows a similar argument using $\sup r_\beta$ and Lemma 2.11. We omit the details. $\square$

By the proof of the above lemma we see that if $\overline{\cup_{\beta\in\mathcal{B}}C_\beta} = \cup_{\beta\in\mathcal{B}}C_\beta$, then both $\inf_{\beta\in\mathcal{B}} C_\beta$ and $\sup_{\beta\in\mathcal{B}} C_\beta$ are in $\{C_\beta\}_{\beta\in\mathcal{B}}$. Thus we have the following.

Lemma 2.13. *Let $\{C_\beta\}_{\beta\in\mathcal{B}}$ be a collection of compact connected sets in $\mathbb{C}$ that are linearly ordered by the surrounding order $\leq$. If $\overline{\cup_{\beta\in\mathcal{B}}C_\beta} = \cup_{\beta\in\mathcal{B}}C_\beta$, then we can conclude that both $\min_{\beta\in\mathcal{B}} C_\beta$ and $\max_{\beta\in\mathcal{B}} C_\beta$ exist.*

Lemma 2.14. *Let $f \in G \in \mathcal{G}$. Let K be a connected compact set in $\mathbb{C}$ such that $PH(K) \supset P^*(f)$.*
(a) Let $J(f) > K$. Then $J(f) > f^{-1}(K)$. Also, $f^{-1}(K) > K$ or $f^{-1}(K) \cap K \neq \emptyset$.
(b) Let $J(f) < K$. Then $J(f) < f^{-1}(K)$. Also, $f^{-1}(K) < K$ or $f^{-1}(K) \cap K \neq \emptyset$.

Proof. We now prove (a). We first note that $J(f) = f^{-1}(J(f)) > f^{-1}(K)$ follows immediately from Lemma 2.8. Since $P^*(f) \subset PH(K) < J(f)$ we see that f cannot have a Siegel disk or parabolic fixed point. Hence, f must have a finite attracting fixed point z_0. Furthermore, since $PH(K)$ is connected and $P^*(f) \subset PH(K) < J(f)$, it is clear that there can be only one attracting fixed point for f and K must lie in the immediate attracting basin $A_f(z_0)$. Since $PH(f^{-1}(K)) \supset P^*(f)$ by Lemma 2.5, we see that $f^{-1}(K)$ also lies in $A_f(z_0)$. Hence $A_f(z_0)$ must be completely invariant under f. This implies $F(f)$ has only two components $A_f(\infty)$ and $A_f(z_0)$, each which are simply connected (see [2], Theorem 5.6.1).

Letting $\varphi : A_f(z_0) \to B(0,1)$ be the Riemann map such that $z_0 \mapsto 0$, then one may apply Schwarz's Lemma to the degree greater than or equal to two (finite Blaschke product) map $B = \varphi \circ f \circ \varphi^{-1}$ to show that any point mapped to a point of maximum modulus of $\varphi(K)$ must lie outside of $\varphi(K)$. Thus either $f^{-1}(K) > K$ or $f^{-1}(K) \cap K \neq \emptyset$.

Part (b) is proved more easily than (a) since it is already known that $A_f(\infty)$ is simply connected. Then one can similarly examine the Riemann map from $A_f(\infty)$ to $B(0,1)$ such that $\infty \mapsto 0$. $\square$

We note that Theorem 1.20 along with the proof of part (a) above, with $K = J_{\min}$, proves the following (which has been already shown in [29, 33]).

Lemma 2.15. *Let $f \in G \in \mathcal{G}$ be such that $J(f) > J_{\min}$, i.e., $J(f) \cap J_{\min} = \emptyset$. Then f has an attracting fixed point $z_0 \in \mathbb{C}$ and $F(f)$ consists of just two simply connected immediate attracting basins $A_f(\infty)$ and $A_f(z_0)$.*

We note that the maps $f = h_1^2$ and $g = h_2^2$ where $h_1(z) = z^2-1$ and $h_2(z) = z^2/4$ generate $G = \langle f, g\rangle \in \mathcal{G}_{dis}$ where f has two finite attracting fixed points (see [31, 33]). Thus we see that the condition $J(f) \cap J_{\min} = \emptyset$ in the above lemma is indeed necessary. We also note that for this G, the phenomena in Remark 1.18 holds. See Figure 1.

Lemma 2.16. *Let $f \in G \in \mathcal{G}$. Let K be a connected set in $J(G)$ containing three or more points such that $f^{-n}(K)$ is also connected for each $n \in \mathbb{N}$. If $f^{-1}(K) \cap K \neq \emptyset$, then $J(f)$ and K are contained in the same component $J \in \mathcal{J}$.*

Proof. The lemma follows from the fact that the connected set $\overline{\cup_{n=0}^{\infty} f^{-n}(K)}$ in $J(G)$ must meet $J(f)$. $\square$

We now present a general topological lemma that will be used to justify a corollary which will be needed later.

Lemma 2.17. *Let $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of compact connected sets in $\overline{\mathbb{C}}$. Let $\epsilon>0$ and let C be any connected component of $\overline{\cup_{\alpha\in\mathcal{A}} C_\alpha}$. Then there exists $\alpha\in\mathcal{A}$ such that $C_\alpha\subset B(C,\epsilon)$.*

Proof. Choose any $z\in C$ and let $\alpha_n\in\mathcal{A}$ be such that $dist(z,C_{\alpha_n})\to 0$. By compactness in the topology generated by the Hausdorff metric on the space of non-empty compact subsets of $\overline{\mathbb{C}}$, we then may conclude (by passing to subsequence if necessary) that $C_{\alpha_n}\to K$ for some non-empty connected compact set K, which therefore must contain z and hence be contained in C. Thus for large n we have $B(C,\epsilon)\supset B(K,\epsilon)\supset C_{\alpha_n}$. □

Using the fact that $J(g)$ is connected whenever $g\in G\in\mathcal{G}$ we clearly obtain the following slight generalization of Lemma 4.2 in [29].

Corollary 2.18. *Let $\{g_\lambda\}_{\lambda\in\Lambda}\subset G\in\mathcal{G}$. Let $\epsilon>0$ and let C be any connected component of $\overline{\cup_{\lambda\in\Lambda} J(g_\lambda)}$. Then there exists $\lambda\in\Lambda$ such that $J(g_\lambda)\subset B(C,\epsilon)$.*

In particular, we apply Theorem 1.2 to obtain that if $\{g_\lambda\}_{\lambda\in\Lambda}=G\in\mathcal{G}$ and $J\in\mathcal{J}$, then for every $\epsilon>0$ there exists $g\in G$ such that $J(g)\subset B(J,\epsilon)$.

3. Proof of Theorem 1.16

We first present a definition and a lemma that will assist in the proof of Theorem 1.16.

Definition 3.1. For compact connected sets K_1 and K_2 in $\mathbb{C}$ such that $K_1<K_2$ we define $Ann(K_1,K_2)=U\setminus PH(K_1)$ where U is the bounded component of $\mathbb{C}\setminus K_2$ which contains K_1. Thus $Ann(K_1,K_2)$ is the open doubly connected region "between" K_1 and K_2.

Remark 3.2. For any compact connected set $A\subset Ann(K_1,K_2)$ we immediately see that $A<K_2$ and, by Lemma 2.1, either K_1 and A are outside of each other or $K_1<A$.

Lemma 3.3. *Let $f,g\in G\in\mathcal{G}$ be such that $J(f)$ and $J(g)$ lie in different components of $J(G)$ with $J(f)<J(g)$. Then for any fixed $n,m\in\mathbb{N}$ there exists $h,k\in G$ such that $f^{-(n+1)}(J(g))<J(h)<f^{-n}(J(g))$ and $g^{-m}(J(f))<J(k)<g^{-(m+1)}(J(f))$.*

Proof. Corollary 2.6, Lemma 2.14(a), and Lemma 2.16 show that $g^{-1}(J(f))>J(f)$. Set $X=g^{-1}(J(f)), A=g^{-m}(J(f))$ and $B=g^{-(m+1)}(J(f))$ and note that $J(f)<A<B$ by Lemma 2.8. Keeping Lemma 2.15 in mind, we may choose $\ell\in\mathbb{N}$ large enough so that $f^{-\ell}(B)\subset Ann(J(f),X)$. Then $g^{-m}(f^{-\ell}(\overline{Ann(A,B)}))\subset g^{-m}(Ann(J(f),X))\subset Ann(A,B)\subset\overline{Ann(A,B)}$ which implies that $k=f^\ell\circ g^m\in G$ is such that $J(k)\subset Ann(A,B)$. Since by Corollary 2.10 we must have either $J(k)<A$ or $A<J(k)$, we see by construction that $A<J(k)$ must hold.

The other result is proved similarly. □

We will require the following result which was proved via *fiberwise* quasiconformal surgery by the second author.

Proposition 3.4 ([31], Proposition 2.25)**.** *Let $G=\langle\alpha_1,\alpha_2\rangle\in\mathcal{G}$ be hyperbolic such that $P^*(G)$ is contained in a single component of $int(\hat{K}(G))$. Then there exists $K\geq 1$ such that for all sequences $x\in\Sigma_2$, the set J_x is a K-quasicircle.*

Remark 3.5. Under the hypotheses above we know that for each $g \in G$, the set $J(g)$ is a quasicircle (see [8], p. 102). But the above result shows much more as it shows that the Julia sets along *sequences* are also all quasicircles, and that all such quasicircles have uniform dilations.

We now can present the proof of Theorem 1.16.

Proof of Theorem 1.16. We first give a proof in the case that A and B are doubly connected components of $F(G)$. Since the doubly connected components of $F(G)$ are linearly ordered by $\leq$, we may assume without loss of generality that $A < B$.

Let γ_A be a non-trivial curve in A (i.e., γ_A separates the components of $\mathbb{C} \setminus A$) and let γ_B be a non-trivial curve in B. Since $J(G) = \overline{\cup_{g \in G} J(g)}$, the bounded component of $\mathbb{C} \setminus \gamma_A$ and $Ann(\gamma_A, \gamma_B)$ both meet $J(G)$, and both A and B do not meet $J(G)$, there must exists maps $f, g \in G$ such that $J(f) < \gamma_A$ and $\gamma_A < J(g) < \gamma_B$. Note then that $J(g) < B$ since $J(g) \cap B = \emptyset$. Since $J(f)$ and $J(g)$ lie indifferent components of $J(G)$ (separated by A), $J(g) \subset \overline{\cup_{n \in \mathbb{N}} g^{-n}(J(f))}$, and each $g^{-n}(J(f)) \cap A = \emptyset$, there exists $n_0 \in \mathbb{N}$ such that $g^{-n_0}(J(f)) > \gamma_A$ and thus $g^{-n_0}(J(f)) > A$. By Lemma 3.3 there exists $k \in G$ such that $A < g^{-n_0}(J(f)) < J(k) < g^{-(n_0+1)}(J(f)) < J(g)$.

We now find a sub-semigroup H' that satisfies conclusions (1) - (4) of the theorem. Keeping Lemma 2.15 in mind, we see that we may choose $m_1, m_2 \in \mathbb{N}$ large (as in Example 1.19), such that $\beta_1 = k^{m_1}$ and $\beta_2 = g^{m_2}$ generate a sub-semigroup H' of G where $J(H')$ is disconnected and contained in $\overline{Ann(J(k), J(g))}$. Further, H' is hyperbolic since $P^*(H') \subset P^*(G) \subset K(f) < J(k)$. By choosing U to be a suitable open set containing $\overline{Ann(J(k), J(g))}$ we see that H' satisfies parts (1) and (2) of the theorem.

By Theorem 2.14(2) in [22], the hyperbolicity of H' implies $J(H') = \cup_{x \in \Sigma_2'} J_x$, where Σ_2' is the sequence space corresponding to the maps β_1 and β_2. The fact that $J_{x_1} \neq J_{x_2}$ when $x_1 \neq x_2$ follows in much the same way as the proof that the standard middle-third Cantor set is totally disconnected. We present the details now. First we define σ to be the shift map on Σ_2' given by $\sigma(\gamma_1, \gamma_2, \dots) = (\gamma_2, \gamma_3, \dots)$. Then, for $x = (\gamma_1, \gamma_2, \dots)$, one can show by using the definition of normality $J_x = \gamma_1^{-1}(J_{\sigma(x)})$ and thus by induction $J_x = \gamma_1^{-1}(J_{\sigma(x)}) = \cdots = (\gamma_n \circ \cdots \circ \gamma_1)^{-1}(J_{\sigma^n(x)}) \subset (\gamma_n \circ \cdots \circ \gamma_1)^{-1}(J(H'))$. Thus $J_x \subset \cap_{n=1}^{\infty} (\gamma_n \circ \cdots \circ \gamma_1)^{-1}(J(H'))$. But by (induction on) condition (1) we can see that this intersection will produce distinct sets for distinct sequences in Σ'. Thus we have shown that $J_{x_1} \neq J_{x_2}$ when $x_1 \neq x_2$

Each J_x is connected by Lemma 3.6 in [31]. Hence we have shown parts 3(a) and 3(b). Now part (4) is then clear by 3(a), 3(b), and Theorem 1.14(1). Part 3(c) now follows directly from Proposition 3.4.

We have thus shown that $H' = \langle \beta_1, \beta_2 \rangle$ satisfies items (1) - (4) of the theorem, but it is not certain that $J(g)$ does not meet $\overline{B}$, and so (5) remains in question. However, letting $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_1 \circ \beta_2$ (note $J(\alpha_2) < J(\beta_2) < B$), we see that $H = \langle \alpha_1, \alpha_2 \rangle$ will satisfy (1) - (5). We have thus proved the result in the case that A and B are both doubly connected Fatou components.

Consider the case where B is the unbounded component of $F(G)$ containing ∞. As above we obtain $f, g \in G$ such that $J(f) < \gamma_A < J(g)$ where γ_A is a non-trivial curve in A. We then follow the above method to complete the proof.

Finally, we consider the case where $B = \hat{K}$. As above we obtain $f, g \in G$ such that $J(g) < \gamma_A < J(f)$ where γ_A is a non-trivial curve in A. We then follow the above method, noting that the surrounding order inequalities are now reversed from above, to complete the proof. □

4. Structural properties of $\mathcal{J}$

In this section we discuss issues related to the topological nature of $\mathcal{J}$ as well as discuss issues related to the question of where the "small" Julia sets $J(g)$ for $g \in G$ reside inside of the larger Julia set $J(G)$. In particular, we investigate the question of when it is the case that a given $J \in \mathcal{J}$ must contain $J(g)$ for some $g \in G$. Since $J_{\min}$ and $J_{\max}$ play special roles we will be particularly interested in when these components of $J(G)$ have this property. When $G = \langle g_\lambda : \lambda \in \Lambda \rangle$, it is of particular interest to know which $J \in \mathcal{J}$ meet $J(g_\lambda)$ for some $\lambda \in \Lambda$. The first result in this direction is the following, which appears as Proposition 2.24 in [29].

Proposition 4.1. *If $G \in \mathcal{G}$ is generated by a compact family in Poly, then both $J_{\min}$ and $J_{\max}$ must contain the Julia set of one of the generating maps of G.*

In order to succinctly discuss such issues we make the following definitions.

Definition 4.2. Let $G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}$. We say that $J \in \mathcal{J}$ has property $(\star)$ if J contains $J(g)$ for some $g \in G$. We say that $J \in \mathcal{J}$ has property $(\star\lambda)$ if J contains $J(h_\lambda)$ for some generator $h_\lambda \in G$.

Remark 4.3. A given rational semigroup G may have multiple generating sets. For example, the whole semigroup itself can always be taken as a generating set. However, in this paper when it is written that $G = \langle h_\lambda : \lambda \in \Lambda \rangle$, it is assumed that this generating set is fixed and thus the property $(\star\lambda)$ is always in relation to this given generating set.

Lemma 4.4. *Let $G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}$.*
a) If $J_{\min}$ has property $(\star)$, then $J_{\min}$ has property $(\star\lambda)$.
b) If $J_{\max}$ has property $(\star)$, then $J_{\max}$ has property $(\star\lambda)$.

Remark 4.5. Lemma 4.4 does not apply to a general $J \in \mathcal{J}$. Indeed, in Example 1.19 we see that only $J_{\min}$ and $J_{\max}$ have property $(\star\lambda)$, although infinitely many other $J \in \mathcal{J}$ have property $(\star)$.

Proof. Suppose $J(g) \subset J_{\min}$ for $g = h_{\lambda_1} \circ \cdots \circ h_{\lambda_k}$ and $J_{\min} \cap J(h_\lambda) = \emptyset$ for all $\lambda \in \Lambda$. By Corollary 2.10 we have $J_{\min} < J(h_\lambda)$ for all $\lambda \in \Lambda$, and thus by Lemma 2.14, Lemma 2.16 and Theorem 1.20 we have $J_{\min} < h_\lambda^{-1}(J_{\min})$. So it also follows from Corollary 2.10 that $J(g) < h_\lambda^{-1}(J(g))$ for all $\lambda \in \Lambda$. Thus $J(g) < h_{\lambda_1}^{-1}(J(g))$ and by Lemma 2.8 $J(g) < h_{\lambda_2}^{-1}(J(g)) < h_{\lambda_2}^{-1} h_{\lambda_1}^{-1}(J(g))$. By repeated application of this argument we then get that $J(g) < h_{\lambda_k}^{-1}(J(g)) < \cdots < h_{\lambda_k}^{-1} \dots h_{\lambda_1}^{-1}(J(g)) = g^{-1}(J(g)) = J(g)$, which is a contradiction. From this part (a) follows.

Part (b) follows in a similar manner. □

Corollary 4.6. *If $J_{\min}$ (respectively, $J_{\max}$) has non-empty interior, then $J_{\min}$ (respectively, $J_{\max}$) has property $(\star\lambda)$.*

Proof. Suppose $int(J_{\min}) \neq \emptyset$. Since $J(G) = \overline{\cup_{g \in G} J(g)}$ some $J(g)$ must meet $int(J_{\min})$. Thus it follows from Lemma 4.4 that $J_{\min}$ has property $(\star\lambda)$. □

It is not always the case, however, that $J_{\min}$ and $J_{\max}$ have property $(\star\lambda)$.

Example 4.7. We will give an example of an infinitely generated $G \in \mathcal{G}_{dis}$ such that

(1) $J_{\max}$ does not have property $(\star\lambda)$,
(2) $\#\mathcal{J} = \aleph_0$, and
(3) there exists $J', J'' \in \mathcal{J}$ and $g \in G$ such that $J' < J''$ and $g^*(J') = g^*(J'')$.

Set $b_n = 2 - 1/n$ for $n \in \mathbb{N}$ and $\epsilon_n = \min\{\frac{b_{n+1}-b_n}{10}, \frac{b_n - b_{n-1}}{10}\}$. Set $A = \overline{Ann(0;1/2,2)}, A_n = \overline{Ann(0; b_n - \epsilon_n, b_n + \epsilon_n)}$ and $A'_n = \overline{Ann(\epsilon_n; b_n - \epsilon_n, b_n + \epsilon_n)}$. Note that the set $\tilde{A_n} := \overline{A_n \cup A'_n} \subset \overline{Ann(0; b_n - 2\epsilon_n, b_n + 2\epsilon_n)}$ and so, by the choice of the ϵ_n, the $\tilde{A_n}$ are disjoint. Choose polynomials f_n and g_n such that $J(f_n) = C(0, b_n)$ and $J(g_n) = C(\epsilon_n, b_n)$. Choose $m_n \in \mathbb{N}$ large enough so that $h_n = f_n^{m_n}$ yields $h_n^{-1}(A) \subset A_n, h_n(\overline{B(0,1/2)}) \subset B(0,1/2)$ and $h_n(\{|z| \geq 2\}) \subset \{|z| > 5\}$. Choose $j_n \in \mathbb{N}$ large enough so that $k_n = g_n^{j_n}$ yields $k_n^{-1}(A) \subset A'_n, k_n(\overline{B(0,1/2)}) \subset B(0,1/2)$ and $k_n(\{|z| \geq 2\}) \subset \{|z| > 5\}$. Note that $\tilde{A} := \overline{\cup_{n=1}^{\infty} \tilde{A_n}} \subset A$. Let $G = \langle h_n, k_n : n \in \mathbb{N}\rangle$ and note that $P^*(G) \subset B(0,1/2)$ and $G^{-1}(\tilde{A}) \subset G^{-1}(A) \subset \tilde{A}$ which implies $J(G) \subset \overline{G^{-1}(\tilde{A})} \subset \tilde{A}$.

We see by forward invariance that $\{|z| > 2\} \subset F(G)$, but note that $C(0,2) \in \mathcal{J}$ (since the open sets $Ann(\tilde{A}_n, \tilde{A}_{n+1})$ are all in $F(G)$). Also, for no $g \in G$ does $J(g)$ meet $C(0,2)$ else there would exist $z_0 \in C(0,2) \cap J(g)$ such that $|g(z_0)| \leq 2$, contradicting the fact that each $g \in G$ maps $C(0,2)$ into $\{|z| > 5\}$. Thus $J_{\max} = C(0,2)$ fails to have property $(\star)$.

We now show that $\#\mathcal{J} = \aleph_0$. Letting $J_n \in \mathcal{J}$ be such that J_n contains the overlapping sets $J(h_n)$ and $J(k_n)$ we note that, since $J_n \subset \tilde{A_n}$ and the $\tilde{A_n}$ are separated from each other, each J_n is isolated from the other J_m, i.e., for each n there exists $\epsilon_n > 0$ such that the ϵ_n-neighborhood $B(J_n, \epsilon_n)$ does not meet any other $J_m \in \mathcal{J}$.

Let $C = C(0,2)$. Since $C \subset J(G)$ we see that $J(G) = \overline{G^{-1}(C)}$. We now show that for each $g \in G$, the set $g^{-1}(C) \subset J_n$ for some n. Write $g = g_{i_1} \circ \ldots g_{i_j}$ where each g_{i_ℓ} is a generator for G. Suppose that $g_{i_j} = h_n$ for some n. Then, by the backward invariance of A under each map in G, we have that $g^{-1}(C) \subset g^{-1}(A) \subset h_n^{-1}(A) \subset A_n$. Since $g^{-1}(C)$ is connected, $g^{-1}(C) > B(0,1/2)$, and $J(k_n) \cup J(h_n)$ meets both the inner boundary and outer boundary of A_n, we must have that $g^{-1}(C)$ meets $J(k_n) \cup J(h_n)$ and thus $g^{-1}(C) \subset J_n$. Note that the same argument (using A'_n) holds if $g_{i_j} = k_n$. Thus we have shown that $G^{-1}(C) \subset \cup_{n\in\mathbb{N}} J_n$. Since the J_n are isolated from each other and accumulate only to C, it follows that $J(G) = \overline{G^{-1}(C)} \subset \overline{\cup_{n\in\mathbb{N}} J_n} \subset C \cup \bigcup_{n\in\mathbb{N}} J_n$ and so $\#\mathcal{J} = \aleph_0$. Note also then that we must have $J_{\min} = J_1$ and so $J_{\min}$ does have property $(\star\lambda)$.

We now show that $J' < J''$ for $J', J'' \in \mathcal{J}$ does not necessarily imply $g^*(J') < g^*(J'')$. Indeed, we see that $h_n^*(J) = J_n$ for all $J \in \mathcal{J}$.

We note that we could easily adapt this example (by letting $b_n = 0.5 + 1/n$) to produce $G_1 \in \mathcal{G}$ such that $J_{\min}(G_1)$ does not have property $(\star\lambda)$, but $J_{\max}$ does. Or we could produce $G_2 = \langle G, G_1\rangle \in \mathcal{G}$ such that neither $J_{\min}$ nor $J_{\max}$ has property $(\star\lambda)$.

Note that in the above example(s) where $J_{\min}$ (respectively $J_{\max}$) did not meet $\cup_{\lambda\in\Lambda} J(g_\lambda)$, it was true that $J_{\min}$ (respectively $J_{\max}$) was contained in $\overline{\cup_{\lambda\in\Lambda} J(g_\lambda)}$. We will prove in Theorem 4.9 that this is indeed always the case. First we need to prove the following lemma.

Lemma 4.8. *Let $\{g_\lambda\}_{\lambda\in\Lambda} \subset G \in \mathcal{G}_{dis}$. Let $\mathcal{C}$ denote the connected components of $\overline{\cup_{\lambda\in\Lambda} J(g_\lambda)}$. Then both $M' = \min_{C\in\mathcal{C}} C$ and $M'' = \max_{C\in\mathcal{C}} C$ exist (with respect to the surrounding order $\leq$). Also, $PH(C) \supset \hat{K}(G) \supset P^*(G)$ for each $C \in \mathcal{C}$.*

Proof. First we note that $\infty \in F(G)$ by Theorem 1.20 and so all sets in $\mathcal{C}$ are contained in $\mathbb{C}$. Let $C \in \mathcal{C}$. Suppose that $z_0 \in \hat{K}(G) \setminus PH(C)$. Let γ be a curve in $\overline{\mathbb{C}} \setminus PH(C)$ connecting z_0 to ∞ and set $\epsilon = dist(\gamma, PH(C))$. By Corollary 2.18, there exists $\lambda \in \Lambda$ such that $J(g_\lambda) \subset B(C, \epsilon)$. Hence, we see that γ is outside $J(g_\lambda)$ implying that $g_\lambda^n(z_0) \to \infty$ and thus contradicting the fact that $z_0 \in \hat{K}(G)$.

Lemma 2.3 shows that the compact connected sets in $\mathcal{C}$ are linearly ordered with respect to the surrounding order. The existence of M' and M'' then follows directly from Lemma 2.13. □

Theorem 4.9. *Consider $G = \langle g_\lambda : \lambda \in \Lambda\rangle \in \mathcal{G}_{dis}$. Let $A = \cup_{\lambda\in\Lambda} J(g_\lambda)$ and denote by M' and M'' the minimal and maximal connected components of $\overline{A}$, respectively. Then both $J_{\min} \supset M'$ and $J_{\max} \supset M''$ and, in particular, both $J_{\min} \cap \overline{A} \neq \emptyset$ and $J_{\max} \cap \overline{A} \neq \emptyset$. Furthermore, we have the following.*

1. *If $J_{\min} \cap A = \emptyset$ (i.e., $J_{\min}$ does not have property $(\star\lambda)$), then $J_{\min} = M'$ and $J_{\min}$ is the boundary of the unbounded component of $\mathbb{C} \setminus J_{\min}$.*
2. *If $J_{\max} \cap A = \emptyset$ (i.e., $J_{\max}$ does not have property $(\star\lambda)$), then $J_{\max} = M''$ and $J_{\max}$ is the boundary of the bounded component of $\mathbb{C} \setminus J_{\max}$ which contains $J_{\min}$.*

Remark 4.10. In the above theorem, if $J(G)$ is connected , then $J_{\min} = J(G) = J_{\max}$ meets all $J(g)$ such $g \in G$ and thus meets A.

Open Question: We notice in Example 4.7 that $J_{\max} \cap A = \emptyset$ and $J_{\max}$ is a simple closed curve. However, it is not clear, in general, whether the hypothesis $J_{\max} \cap A = \emptyset$ for $G = \langle g_\lambda : \lambda \in \Lambda\rangle \in \mathcal{G}_{dis}$, must *necessarily* lead to the conclusion that $J_{\max}$ is a simple closed curve. It is also not clear under this hypothesis whether $J_{\max}$ must be the common boundary of exactly two complementary domains. So we state these as open questions (noting the corresponding questions regarding $J_{\min}$ are also open).

Remark 4.11. We note that we could also use Theorem 4.9 to see that $J_{\min}$ in Example 4.7 has property $(\star\lambda)$, since $J_{\min}$ must meet $\overline{\cup_{j=1}^\infty J(k_j) \cup J(h_j)} = C(0,2) \cup \cup_{j=1}^\infty (J(k_j) \cup J(h_j))$, but does not meet $C(0,2)$.

Proof. Let $J', J'' \in \mathcal{J}$ be such that $M' \subset J'$ and $M'' \subset J''$. Fix $\lambda \in \Lambda$. By the minimality of M', we have either $J(g_\lambda) \subset J'$ or $J(g_\lambda) > J'$. Then $g_\lambda^*(J') \geq J'$ by Lemmas 2.14 and 2.16 and Theorem 1.20. Since $g_\lambda^*(J') \geq J'$ for all $\lambda \in \Lambda$ we must have that $J' = J_{\min}$ (because the closed set $\cup_{\{J\in\mathcal{J}: J \geq J'\}} J$ is then backward invariant under each $g \in G$). Similarly we see that $J'' = J_{\max}$. Thus both $J_{\min} \cap \overline{A} \neq \emptyset$ and $J_{\max} \cap \overline{A} \neq \emptyset$.

We now prove (2) by first showing that $J(G) \subset J_{\min} \cup M'' \cup Ann(J_{\min}, M'')$. Fix $\lambda \in \Lambda$. Since $PH(M'') \supset P^*(G)$ by Lemma 4.8, we see that $g_\lambda^{-1}(M'')$ is connected using Lemma 2.8. Thus $g_\lambda^{-1}(M'') \cap M'' = \emptyset$, else $J(g_\lambda)$ meets $J_{\max}$ by Lemma 2.16 which violates our hypothesis that $J_{\max} \cap A = \emptyset$. Hence $g_\lambda^{-1}(M'') < M''$ by Lemma 2.14(b).

From the facts that $g_\lambda^{-1}(J_{\min}) \geq J_{\min}$ and $g_\lambda^{-1}(M'') < M''$ for all $\lambda \in \Lambda$, we deduce from Lemma 2.8 that the closed annulus-type region $A_1 = J_{\min} \cup M'' \cup$

$Ann(J_{\min}, M'')$ is backward invariant under each generator (and thus under each $g \in G$). Hence $J(G) \subset J_{\min} \cup M'' \cup Ann(J_{\min}, M'')$ as desired.

We now suppose there exists $w \in J_{\max} \setminus M''$. Such a point w must necessarily then lie in $Ann(J_{\min}, M'')$ (since $w \in J_{\min}$ would imply $J_{\min} = J_{\max} = J(G)$ and thus $J_{\max}$ clearly meets A). Let U be the connected component of $\mathbb{C} \setminus M''$ which contains $J_{\min}$. Note that $w \in U$ by definition of $Ann(J_{\min}, M'')$. Recall that $\partial \hat{K}(G) \subset J_{\min}$. Let γ be a curve in U which connects w to some $z_0 \in \hat{K}(G)$ and set $\epsilon = dist(\gamma, M'') > 0$. By Corollary 2.18 there exists a generator $g_{\lambda_0} \in G$ such that $J(g_{\lambda_0}) \subset B(M'', \epsilon)$. Thus $\gamma \cap J(g_{\lambda_0}) = \emptyset$. Since $\hat{K}(G) \subset PH(J(g_{\lambda_0}))$, we see that $z_0 \in \gamma \cap PH(J(g_{\lambda_0}))$ and so $\gamma < J(g_{\lambda_0})$. Hence $\{w\} < J(g_{\lambda_0})$ which implies (by Corollary 2.10) either $J_{\max} < J(g_{\lambda_0})$ or $J_{\max} \cap J(g_{\lambda_0}) \neq \emptyset$. Since neither of these can occur we conclude that no such w exists and thus $J_{\max} = M''$.

Recall that U is the bounded component of $\mathbb{C} \setminus J_{\max}$ which contains $J_{\min}$. Since $J_{\max} \cap A = \emptyset$, we have that for every $\lambda \in \Lambda$, the set $J(g_\lambda)$ is contained in U. Hence $\overline{A} \subset \overline{U}$ and so $J_{\max} = M'' \subset \overline{A} \subset \overline{U}$, which implies $J_{\max} = \partial U$.

The proof for case (1) is similar, but simpler. In this case the point ∞ can play the role of z_0 in order to help demonstrate that any point in $J_{\min} \setminus M'$ must lie "outside" of some $J(g_\lambda)$ (which is a contradiction). We omit the details. □

Corollary 4.12. *When $G = \langle g_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}$ with $\overline{\cup_{\lambda \in \Lambda} J(g_\lambda)} = \cup_{\lambda \in \Lambda} J(g_\lambda)$, then both $J_{\min}$ and $J_{\max}$ must have property $(\star\lambda)$.*

Remark 4.13. The above corollary applies, for example, when $G = \langle g_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}_{dis}$ has $\cup_{\lambda \in \Lambda} J(g_\lambda) = \cup_{k=1\dots n} J(g_{\lambda_k})$. Such non compactly generated examples can easily be constructed. Other more "exotic" examples can also be constructed to satisfy the hypotheses of the corollary.

Example 4.14. We note that without the hypothesis that $J_{\min} \cap A = \emptyset$ in Theorem 4.9(1), the conclusion that $J_{\min} = M'$ might not hold. Set $f_1(z) = z^2, f_2(z) = (z-\epsilon)^2 + \epsilon$, and $f_3(z) = z^2/4$. For $\epsilon > 0$ small and m_1, m_2, m_3 all large we set $g_1 = f_1^{m_1}, g_2 = f_2^{m_2}$, and $g_3 = f_3^{m_3}$ and note that $G = \langle g_1, g_2, g_3 \rangle \in \mathcal{G}_{dis}$. Thus $M' = J(f_1) \cup J(f_2) = C(0,1) \cup C(\epsilon, 1)$. However, the real point in $g_1^{-1}(\{1+\epsilon\})$ is clearly in $g_1^{-1}(J(f_2)) \subset J_{\min}$, but not in M'.

Theorem 4.15. *Let $G = \langle g_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}_{dis}$ and suppose $J_{\min} \cap \cup_{\lambda \in \Lambda} J(g_\lambda) = \emptyset$. Then $\partial \hat{K}(G) \subset M' \subset \overline{\cup_{\lambda \in \Lambda} J(g_\lambda)}$, where M' is the minimal connected component of $\overline{\cup_{\lambda \in \Lambda} J(g_\lambda)}$*

Proof. By Theorem 4.9(1), we see that $\partial \hat{K}(G) \subset J(G) \subset M' \cup J_{\max} \cup Ann(M', J_{\max})$. Moreover, by Lemma 4.8, we have $\partial \hat{K}(G) \subset PH(M')$. Hence $\partial \hat{K}(G) \subset M'$. □

Having discussed properties $(\star)$ and $(\star\lambda)$ with respect to $J_{\min}$ and $J_{\max}$ we now turn our attention to a general $J \in \mathcal{J}$. In particular, we investigate what can be said about which $J \in \mathcal{J}$ have property $(\star)$ or $(\star\lambda)$. We also concern ourselves with the question of when does every $J \in \mathcal{J}$ have property $(\star)$ or $(\star\lambda)$. We begin with the following definition.

Definition 4.16. Let $G \in \mathcal{G}$. We say that $J \in \mathcal{J}$ is **isolated** in $\mathcal{J}$ if there exists $\epsilon > 0$ such that $B(J, \epsilon)$ does not meet any other set in $\mathcal{J}$.

Lemma 4.17. *Let $G \in \mathcal{G}$ with $J \in \mathcal{J}$ isolated in $\mathcal{J}$. Then J has property $(\star)$.*

Proof. Assume that $\epsilon > 0$ is such that $B(J, \epsilon)$ does not meet any other set in $\mathcal{J}$. Since $J(G) = \overline{\cup_{g \in G} J(g)}$ by Theorem 1.2, we see that any point in J must have, within a distance ϵ, a point in some $J(g)$, where $g \in G$. It must then be the case that $J(g)$, which lies in some set in $\mathcal{J}$, must lie entirely in J. □

Remark 4.18. If $G \in \mathcal{G}$ is such that $\#\mathcal{J} < +\infty$, then clearly each $J \in \mathcal{J}$ is isolated in $\mathcal{J}$ and so each $J \in \mathcal{J}$ has property $(\star)$. We note, however, that if each $J \in \mathcal{J}$ is isolated in $\mathcal{J}$, then it is not necessarily the case that each $J \in \mathcal{J}$ has property $(\star\lambda)$. See the proof of Theorem 1.23 where, for any positive integer k, a semigroup $G' \in \mathcal{G}$ is constructed such that $\#\mathcal{J} = k$, but only $J_{\min}$ and $J_{\max}$ have properly $(\star\lambda)$.

Remark 4.19. If $G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}$ where $\#\Lambda \leq \aleph_0$ and $\#\mathcal{J}$ is uncountable, then since $\#G = \aleph_0$ we see that some $J \in \mathcal{J}$ must fail to have property $(\star)$. An example of this is the Cantor set of circles in Example 1.19.

Example 4.20. Suppose $G \in \mathcal{G}$ and $\#\mathcal{J} = \aleph_0$. Then it is possible that not all $J \in \mathcal{J}$ have property $(\star)$ as in Example 4.7. But it is also possible that all $J \in \mathcal{J}$ do have property $(\star)$ as in [29] Theorem 2.26 where both $J_{\min}$ and $J_{\max}$ have property $(\star\lambda)$ and each other component of $J(G)$ is isolated in $\mathcal{J}$.

We saw above that isolated $J \in \mathcal{J}$ have property $(\star)$. We now show that this is also true for the components of $J(G)$ which contain the pre-image of an isolated $J \in \mathcal{J}$.

Claim 4.21. *Let $G \in \mathcal{G}$. If $J_1 \in \mathcal{J}$ is isolated in $\mathcal{J}$ and $h^{-1}(J_1) \subset J \in \mathcal{J}$ for some $h \in G$, then J has property $(\star)$.*

Proof. Since J_1 is isolated in $\mathcal{J}$, Lemma 4.17 implies there exists $g \in G$ such that $J(g) \subset J_1$. Thus, since J_1 is isolated in $\mathcal{J}$, for large $n \in \mathbb{N}$ we have $g^{-n}(J) \subset J_1$. Hence $h^{-1}(g^{-n}(J)) \subset h^{-1}(J_1) \subset J$ which implies $J(g^n \circ h) \subset J$. □

Open Question: If $\#\mathcal{J} = \aleph_0$ and $G \in \mathcal{G}$ is finitely generated, then must every $J \in \mathcal{J}$ have property $(\star)$? Note that the finitely generated condition is required by Example 4.7. Also, Example 1.19 shows that if $\#\mathcal{J}$ is uncountable, $\mathcal{J}$ can have (uncountably many) J which fail to have property $(\star)$.

We now turn our attention to considering those semigroups where $J_{\min}$ has property $(\star\lambda)$. In particular, we examine the generating maps whose Julia sets meet $J_{\min}$ as well as the sub-semigroup generated by just these special maps.

Definition 4.22. Let $G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}_{dis}$. We set

$$B_{\min} = B_{\min}(G) := \{\lambda \in \Lambda : J(h_\lambda) \subset J_{\min}(G)\}.$$

If $B_{\min} \neq \emptyset$, let $H_{\min}(G)$ be the sub-semigroup of G which is generated by $\{h_\lambda : \lambda \in B_{\min}\}$.

The following proposition has been shown in [29, 33].

Proposition 4.23. *Let $G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}_{dis}$. If $\{h_\lambda : \lambda \in \Lambda\}$ is compact in Poly, then $B_{\min}$ is a proper non-empty subset of Λ under the above notation.*

Proof. The result follows since by Proposition 4.1 both $J_{\min}$ and $J_{\max}$ have property $(\star\lambda)$. □

It is natural to investigate the relationship between $H_{\min}(G)$ and G. Specifically we ask, and answer, the following questions for a semigroup $G \in \mathcal{G}_{dis}$:

(1) Must $J(H_{\min}(G)) = J_{\min}(G)$? (2) Must $J(H_{\min}(G))$ be connected?
(3) Must $J_{\min}(H_{\min}(G)) = J_{\min}(G)$? (4) Must $H_{\min}(H_{\min}(G)) = H_{\min}(G)$?

The answer to each of these questions is NO, as we see in this next example.

Example 4.24. We will construct a single 3-generator polynomial semigroup $G \in \mathcal{G}_{dis}$ which negatively answers questions (1)-(4). Furthermore, we will show that $\#\mathcal{J} = \aleph_0$.

Let $h_1(z) = -z^2$ and $f_2(z) = z^2/\sqrt{2}$ and note that $J(h_1) = C(0,1)$ and $J(f_2) = C(0, \sqrt{2})$. We set $h_2 = f_2^{m_2}$ where conditions on the large $m_2 \in \mathbb{N}$ will be specified later. Choose point $P \in h_2^{-1}(J(h_1))$ such that $P > 0$. Note that $P = \sqrt{2} - \delta$ where δ is small for m_2 large. Hence $P^2 = 2 - 2\sqrt{2}\delta + \delta^2$ and $P^4 = 4 - 8\sqrt{2}\delta + O(\delta^2)$.

Setting $r = \frac{P^4+P^2}{2} = 3 - M_1\delta + O(\delta^2)$ and $\epsilon = \frac{P^4-P^2}{2} = 1 - M_2\delta + O(\delta^2)$ where $M_1, M_2 > 0$, we see that for $f_3(z) = \frac{(z-\epsilon)^2}{r} + \epsilon$ both $h_1(P) = -P^2 = \epsilon - r$ and $h_1^2(e^{i\pi/4}P) = P^4 = r + \epsilon$ lie in $C(\epsilon, r) = J(f_3)$.

Suppose there exists $w \in h_1^{-1}(J(f_3)) \cap J(f_3)$, i.e., $h_1(w) \in J(f_3)$ and $w \in J(f_3)$. Then $|h_1(w)| = |w|^2 \geq |-P^2|^2 = P^4$ since $-P^2$ is the point in $J(f_3)$ of smallest modulus. Since P^4 is the point in $J(f_3)$ of largest modulus, we see that $h_1(w)$ could only be in $J(f_3)$ if $w = -P^2$. But $h_1(-P^2) = -P^4 \notin J(f_3)$, and so we have $h_1^{-1}(J(f_3)) \cap J(f_3) = \emptyset$ and thus $h_1^{-1}(J(f_3)) < J(f_3)$ (see Figure 1).

Since for m_2 large we clearly have $h_2^{-1}(J(f_3)) < J(f_3)$, we are then free to choose A' to be any closed annulus such that $A' \subset B(\epsilon, r), int(A') \neq \emptyset$, and $A' > h_1^{-1}(J(f_3)) \cup h_2^{-1}(J(f_3))$.

Choose $m_3 \in \mathbb{N}$ large enough so that $h_3 = f_3^{m_3}$ maps A' into $B(0,1)$ (which can be done since the super attracting fixed point of f_3 is $\epsilon \in B(0,1)$). This then implies that $B(0,1)$ is forward invariant under each map h_1, h_2 and h_3 and $P^*(G) \subset B(0,1) = int\hat{K}(G) \subset F(G)$ where $G = \langle h_1, h_2, h_3 \rangle$.

Let $A = J(h_2) \cup h_2^{-1}(J(h_1)) \cup h_1^{-1}(J(h_3)) \subset J(G)$. Because $h_1^{-1}(J(h_3))$ is connected (Corollary 2.9) and contains both the point iP^2, which lies outside the circle $J(h_2)$, and the point $P \in h_2^{-1}(J(h_1))$, which lies inside the circle $J(h_2)$, we see that the set A is connected.

Note that $e^{i\pi/4}P \in h_2^{-1}(J(h_1)) \subset A$ and $e^{i\pi/4}P \in h_1^{-1}(h_1^{-1}(J(h_3))) \subset h_1^{-1}(A)$ since $h_1^2(e^{i\pi/4}P) = P^4 \in J(h_3)$. Thus $h_1^{-1}(A) \cap A \neq \emptyset$ and since $h_1^{-n}(A)$ is connected for each $n \in \mathbb{N}$ by Lemma 2.8, we see that Lemma 2.16 implies A and $J(h_1)$ are contained in a single $J \in \mathcal{J}$. Since $P^*(G) \subset B(0,1) \subset F(G)$, we see that $J = J_{\min}$. Thus both $J(h_1)$ and $J(h_2)$ are contained in $J_{\min}(G)$. Note that $\partial\hat{K}(G) = J(h_1) \neq J_{\min}(G)$.

Both h_1 and h_2 map A' into the unbounded component of $\overline{\mathbb{C}} \setminus J(h_3)$ (since A' is outside of both $h_1^{-1}(J(h_3))$ and $h_2^{-1}(J(h_3))$), which is forward invariant under each map h_1, h_2 and h_3. The map h_3 maps A' into $B(0,1)$, which is also forward invariant under each map h_1, h_2 and h_3. Hence for any $g \in G$ we have that $g(A') \cap A' = \emptyset$ and so $int(A') \subset F(G)$. We conclude that $J(h_3)$ is not contained in $J_{\min}$.

Thus we have that $H_{\min}(G) = \langle h_1, h_2 \rangle$. One can easily show that $J(H_{\min}(G))$ is disconnected (Cantor set of circles) and thus $J(H_{\min}(G)) \neq J_{\min}(G)$. Also $H_{\min}(H_{\min}(G)) = \langle h_1 \rangle \neq \langle h_1, h_2 \rangle = H_{\min}(G)$ and $J_{\min}(G) \neq J_{\min}(H_{\min}(G))$.

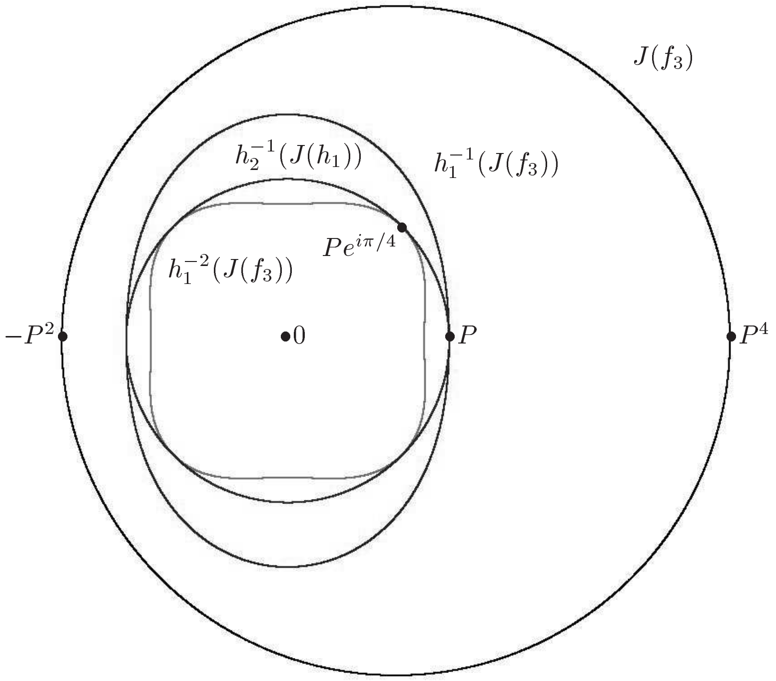

FIGURE 2. This figure shows the sets (from outside to inside) $J(f_3), h_1^{-1}(J(f_3)), h_2^{-1}(J(h_1))$, and $h_1^{-2}(J(f_3))$. For this picture we chose $m_2 = 5$ and $P = 2^{31/64} \approx 1.39898$.

We now show $\#\mathcal{J}_G = \aleph_0$. Consider the set $B = J_{\min}(G) \cup \bigcup_{n=1}^{\infty} h_3^{-n}(J_{\min}(G)) \cup J(h_3) \subset J(G)$, which clearly contains more than three points. We have $J(h_3) \subset J_{\max}(G)$ and $h_1^{-1}(J(h_3)) \subset J_{\min}(G)$. By Theorem 1.14 or Corollary 2.9, we obtain $h_1^{-1}(B) \subset h_1^{-1}(J(G)) \subset J_{\min}(G) \subset B$. Similarly, taking m_2 so large, we may assume $h_2^{-1}(J(h_3)) \subset J_{\min}(G)$, and it implies $h_2^{-1}(B) \subset h_2^{-1}(J(G)) \subset J_{\min}(G) \subset B$. Since B is closed and backward invariant under each generator of G (and hence under every $g \in G$), we must have that $B = J(G)$. Also, since $h_3^{-1}(J_{\min}(G))$ is connected (by Corollary 2.9) and does not meet $J_{\min}(G)$, we see that $J_{\min}(G) < h_3^{-1}(J_{\min}(G))$. Repeated application of Lemma 2.8 shows us that $J_{\min}(G) < h_3^{-1}(J_{\min}(G)) < h_3^{-2}(J_{\min}(G)) < \cdots < h_3^{-n}(J_{\min}(G)) < \ldots$. From this we may conclude that $\mathcal{J} = \{J_{\min}(G), J(h_3), h_3^{-n}(J_{\min}(G)) : n \in \mathbb{N}\}$, thus demonstrating that $\#\mathcal{J}_G = \aleph_0$.

Remark 4.25. Note that Example 4.24 does not settle (in the negative) the open question stated above since Claim 4.21 with $J_1 = J_{\min}$ shows that each $J \in \mathcal{J}_G$ contains $J(g)$ for some $g \in G$. One could also note that every $J \in \mathcal{J}_G$ other than $J(h_3)$ is isolated in $\mathcal{J}_G$ and so from Lemma 4.17 each such J has property $(\star)$.

Question: Does there exists an example of some $G \in \mathcal{G}$ which can negatively answer questions (1)-(4) addressed by Example 4.24, but where $\#\mathcal{J}_G$ is finite? The answer, as we see in the next example, is YES. We will also see that this example will settle two other questions that naturally arise when considering the two following results. In [29, 33] it was shown that, for each positive integer k, there exists a semigroup $G \in \mathcal{G}_{dis}$ with $2k$ generators such that $J(G)$ has exactly k components. Furthermore, in [25] it was shown that any semigroup in $\mathcal{G}$ generated by exactly three elements will have a Julia set with either one or infinitely many components. Hence we have the following questions.

(5) What is the fewest number of generators that can produce a semigroup $G \in \mathcal{G}_{dis}$ with $\#\mathcal{J} = 3$?

(6) For fixed integer $k > 3$, what is the fewest number of generators that can produce a semigroup $G \in \mathcal{G}_{dis}$ with $\#\mathcal{J} = k$?

The answer to both of these questions is four as stated in Theorem 1.23 whose proof is given now.

Proof of Theorem 1.23. Fix $k \geq 2$ since the $k = 1$ case is trivial. Let maps h_1, f_2, and f_3 and integer $m_2 \in \mathbb{N}$ be defined as in Example 4.24. Again, we set $h_2 = f_2^{m_2}$ and $h_3 = f_3^{m_3}$ where large $m_3 \in \mathbb{N}$ will be specified to fit the stipulations given below. Letting γ_1 denote $C(0,1)$ and γ_2 denote the boundary of the unbounded component of $\mathbb{C} \setminus (h_1^{-1}(J(f_3)) \cup h_2^{-1}(J(f_3)))$, we set $B = \overline{Ann(\gamma_1, \gamma_2)}$. Let A' be any closed annulus such that $A' \subset B(\epsilon, r), int(A') \neq \emptyset$, and $A' > B$. We choose $m_3 \in \mathbb{N}$ large enough so that $h_3^{-1}(B) > A' > B$. Set $G = \langle h_1, h_2, h_3 \rangle$ and note that for m_3 large enough, $\gamma_1 \cup \gamma_2 \subset J_{\min}(G) \subset B$, as in Example 4.24.

Set $\ell = k - 2$. Let $C = C(\epsilon_0, r_0)$ be the circle which is internally tangent to the circle $J(h_3)$ at the point $\epsilon + r = P^4$ such that C meets $h_3^{-(\ell+1)}(B)$ and $\overline{B(\epsilon_0, r_0)} \supset h_3^{-(\ell+1)}(B)$. Hence C must necessarily meet $h_3^{-(\ell+1)}(\gamma_2) \subset h_3^{-(\ell+1)}(J_{\min}(G))$ and $C > h_3^{-\ell}(B) > \cdots > h_3^{-1}(B) > B$. Note that as $m_3 \to \infty$, we have $\epsilon_0 \to \epsilon$ and $r_0 \to r$. We may assume then that m_3 has been chosen large enough so that $\epsilon_0 \in B(0,1)$.

Set $f_4(z) = \frac{(z-\epsilon_0)^2}{r_0} + \epsilon_0$ and observe that $J(f_4) = C$. Let $R > 0$ be large enough so that $f_j(\mathbb{C} \setminus \overline{B(0,R)}) \subset \mathbb{C} \setminus \overline{B(0,R)}$ for $j = 1, \ldots, 4$ (where $f_1 := h_1$). Let A'' be a closed annulus such that $int(A'') \neq \emptyset$ and $h_3^{-\ell}(B) < A'' < h_3^{-(\ell+1)}(B)$. Then $h_3^{\ell+1}(A'') \subset B(0,1)$. Let $A_0 := A' \cup \cup_{j=0}^{\ell} h_3^j(A'')$. We define $h_4 = f_4^{m_4}$ where $m_4 \in \mathbb{N}$ is large enough such that (i) $h_4(A_0) \subset B(0,1)$, (ii) $h_4^{-1}(\gamma_1)$ meets $h_3^{-(\ell+1)}(J_{\min}(G))$ (this is possible since the connected set $h_3^{-(\ell+1)}(J_{\min}(G))$ meets, but is not contained in, C), (iii) $h_4(\cup_{j=1}^2 \cup_{h \in \langle h_1, h_2, h_3 \rangle \cup \{id\}} h \circ h_j(A_0)) \subset \mathbb{C} \setminus \overline{B(0,R)}$ (note that $\overline{h_1(A_0) \cup h_2(A_0)} \subset \overline{\mathbb{C}} \setminus K(h_3)$, which is equal to the connected component of $F(\langle h_1, h_2, h_3 \rangle)$ containing ∞), and (iv) $h_4^{-1}(\gamma_1) > h_3^{-\ell}(B)$.

Set $G' = \langle h_1, h_2, h_3, h_4 \rangle$. Since $B(0,1)$ is forward invariant under each map in G', we conclude $B(0,1) \subset F(G')$ and $P^*(G') \subset B(0,1)$. Thus $G' \in \mathcal{G}$. Also, $int(A_0) \subset F(G')$ since one can show that $g(A_0) \subset A_0 \cup B(0,1) \cup \hat{\mathbb{C}} \setminus K(h_3)$ for all $g \in G'$. Hence $G' \in \mathcal{G}_{dis}$. By applying Lemma 2.16 and Lemma 2.14 (noting that $J_{\min}(G') < int(A') < J(h_3)$), we have that $h_3^{-n}(J_{\min}(G')) < h_3^{-(n+1)}(J_{\min}(G'))$ for all $n \geq 0$. Further, $J_{\max}(G')$, which must contain $J(h_3)$ and $J(h_4)$ by Proposition 4.1, must also contain $h_3^{-n}(J_{\min}(G'))$ for all $n > \ell$. By examining the dynamics one can then show that $J(G') = J_{\min}(G') \cup J_{\max}(G') \cup h_3^{-1}(J_{\min}(G')) \cup \cdots \cup h_3^{-\ell}(J_{\min}(G'))$, since this set is closed and backward invariant under each generator of G'. Moreover, since $h_3^{-\ell}(\gamma_1) < A'' < h_3^{-(\ell+1)}(\gamma_1)$, $h_3^{-\ell}(\gamma_1) \subset h_3^{-\ell}(J_{\min}(G'))$, $h_3^{-(\ell+1)}(\gamma_1) \subset J_{\max}(G')$

and $int(A'') \subset F(G')$, we have $h_3^{-\ell}(J_{\min}(G')) < int(A'') < J_{\max}(G')$. Thus we see that $J(G')$ has exactly k components. $\square$

Remark 4.26. The addition of one generating function in the proof of Theorem 1.23 to the semigroup in Example 4.24 illustrates something of a general principle (which we decline to attempt to make precise) at work when dealing with the dynamics of semigroups in $\mathcal{G}_{dis}$. Namely, if one adds a generator (or a whole family of generators) whose Julia set does not meet $J_{\min}$ of the new larger semigroup, then key properties of the dynamics can often be preserved. See for example Theorems 1.27 and 1.29.

However, as we see in this next lemma, adding "too many" new functions will necessarily destroy certain critical aspects of the dynamics. In particular, if we look to produce a new semigroup in $\mathcal{G}$ by adding "too many" generating polynomials of small degree (such that G_k defined in the lemma is not pre-compact) to a semigroup $G \in \mathcal{G}_{dis}$, then the new semigroup will necessarily have a **connected** Julia set.

Lemma 4.27. *Let $G = \langle h_\lambda : \lambda \in \Lambda \rangle \in \mathcal{G}_{dis}$. Then each $G_k = \{g \in G : \deg(g) \leq k\}$ is pre-compact in* Poly *and, in particular, each $\{h_\lambda : \lambda \in \Lambda\} \cap G_k$ is pre-compact in* Poly.

Remark 4.28. As stated earlier, a possibly generating set for G is G itself, which is necessarily not pre-compact (since it contains elements of arbitrarily high degree). Thus it is impossible to strengthen Lemma 4.27 to conclude that $\{h_\lambda : \lambda \in \Lambda\}$ is pre-compact.

Proof. Note that $J(G)$ is bounded in $\mathbb{C}$ since Theorem 1.20 yields $\infty \in F(G)$. Choose $R > 0$ such that $J(G) \subset \overline{B(0, R)}$. Then $\text{Cap}(J(g)) \leq R$ for all $g \in G$, where $\text{Cap}(E)$ denotes the logarithmic capacity of the set E (see [1] for definition and properties). Also, since $G \in \mathcal{G}_{dis}$ we have $int(\hat{K}(G)) \neq \emptyset$ (see Theorem 1.20 or [29]), and so there exists a ball of some radius $r > 0$ in $\hat{K}(G)$. Thus $\text{Cap}(J(g)) \geq r$ for all $g \in G$.

Let $H_n = \{g \in G : \deg(g) = n\}$. In order to show that $G_k = \cup_{n=1}^k H_n$ is pre-compact, it suffices to show that each H_n is pre-compact. We now fix $g(z) = a_n z^n + \cdots + a_0$ in H_n and proceed to show that $|a_n|$ is uniformly bounded below by R^{1-n} and uniformly bounded above by r^{1-n}, and that the remaining coefficients $a_{n-1}, \ldots, a_0$ of $g(z)$ are uniformly bounded (above) by positive constants which only depend on r, R and n. Recalling Remark 1.25, it follows then that H_n is pre-compact. Since $|a_n|^{-1/(n-1)} = \text{Cap}(J(g))$ (see [8], p. 35), we see that $r^{1-n} \geq |a_n| \geq R^{1-n}$. Express $g'(z) = \beta(z - \alpha_1) \ldots (z - \alpha_{n-1})$ where $\beta = n a_n$ and the α_j are the critical points of g which, since $G \in \mathcal{G}$ and $\infty \in F(G)$, must lie in $\mathbb{C} \setminus F_\infty(G) \subset \overline{B(0, R)}$, where $F_\infty(G)$ denotes the connected component of $F(G)$ containing ∞.

One can multiply out the terms in the expansion of $g'(z)$ and find an anti-derivative to see that the $a_{n-1}, \ldots, a_1$ coefficients of $g(z)$ are also bounded by constants which depend only on r, R and n. Now fix $z_0 \in \hat{K}(G)$. Since $g(\hat{K}(G)) \subset \hat{K}(G) \subset \overline{B(0, R)}$, we have $|g(z_0)| = |a_n z_0^n + \cdots + a_0| \leq R$. Thus, since $|a_n|, \ldots, |a_1|$ are bounded by constants depending only on r, R and n, the same is true for $|a_0|$. $\square$

Remark 4.29. The proof of Lemma 4.27 also holds for any $G \in \mathcal{G}$ such that there exists both lower and upper bounds on $\text{Cap}(J(g))$ for all $g \in G$ (e.g., when $\hat{K}(G)$ contains some non-degenerate continuum and $\infty \in F(G)$).

5. Proof of Theorems 1.27 and 1.29

Example 5.1. Let $f_1(z) = z^2 + c$ where $c > 0$ is small (thus $J(f_1)$ is a quasi-circle). Let $z_0 \in \mathbb{R}$ denote the finite attracting fixed point of f_1. Note that $f_1^k(0)$ increases to z_0. Choose $f_2(z) = \frac{(z-z_0)^2}{(c-z_0)} + z_0$ and note that $J(f_2) = C(z_0, |c - z_0|)$. For $m_1, m_2 \in \mathbb{N}$ large $h_1 = f_1^{m_1}$ and $h_2 = f_2^{m_2}$ each map $B(z_0, |c - z_0|)$ into itself and $J(G)$ is disconnected for $G = \langle h_1, h_2 \rangle$. Note that $P^*(G) \subset \overline{B(z_0, |c - z_0|)}$ and so $G \in \mathcal{G}$. We have $H = \langle h_2 \rangle$ is hyperbolic, but since $f_1(0) = c \in J(h_2) \subset J(G)$, the semigroup $G = \langle H, h_1 \rangle$ is not hyperbolic even though $J(h_1) \cap J_{\min}(G) = \emptyset$.

By conjugating h_2 by a suitable rotation we may assume that $\{h_2^k(c) : k \in \mathbb{N}\}$ is dense in $J(h_2)$ and therefore we see that H can be hyperbolic and have G fail to even be sub-hyperbolic. However, Theorem 1.27 does imply that $G = \langle H, h_1 \rangle$ is semi-hyperbolic.

Remark 5.2. In contrast to the analogous behavior of Iterated Function Systems where contraction in each generating map leads to a semigroup (IFS) that is overall contracting, we see that in Example 5.1 each map of the semigroup G is hyperbolic, yet the entire semigroup G fails to be hyperbolic. To see this, note that each map h_2^n is hyperbolic and for each map $g \in G \setminus \{h_2^n\}$ we have $P^*(g) \subset P^*(G) \subset \overline{B(z_0, |c - z_0|)}$ and $J(g) > \overline{B(z_0, |c - z_0|)}$ which implies g is hyperbolic.

We now state a lemma which we will use the proof of Theorem 1.27.

Lemma 5.3. *Let H_1 be a polynomial semigroup in $\mathcal{G}$ and let Γ be a compact family in* Poly. *Let $H_2 = \langle H_1, \Gamma \rangle$ be the semigroup generated by H_1 and Γ. Suppose*
(1) $H_2 \in \mathcal{G}_{dis}$, and
(2) $J(\gamma) \cap J_{\min}(H_2) = \emptyset$ for $\gamma \in \Gamma$.

Then $\hat{K}(H_1) = \hat{K}(H_2)$, which then implies $J_{\min}(H_1) \subset J_{\min}(H_2)$ since $\partial \hat{K}(H_1) \subset J_{\min}(H_1)$ and $\partial \hat{K}(H_2) \subset J_{\min}(H_2)$.

Remark 5.4. We recall the facts given in [29, 33] that for any $G \in \mathcal{G}$ we have $int\hat{K}(G) = \hat{K}(G) \cap F(G)$. Moreover, for any $G \in \mathcal{G}_{dis}$, we have $int\hat{K}(G) \neq \emptyset$ and $g(\hat{K}(G) \cup J_{\min}(G)) \subset int\hat{K}(G)$ for any $g \in G$ such that $J(g) \cap J_{\min}(G) = \emptyset$.

Proof. We begin by first showing that $J_{\min}(H_1) \subset J_{\min}(H_2)$. Let $\mathcal{C}$ be the set of all connected components of $\overline{\bigcup_{\gamma \in \Gamma} J(\gamma)}$. By Lemma 4.8, $M_1 := \min_{C \in \mathcal{C}} C$ exists with respect to the surrounding order. Let $J_1 \in \mathcal{J}_{H_2}$ be the element containing M_1. Let $J_2 \in \mathcal{J}_{H_2}$ be the element containing $J_{\min}(H_1)$. Let $J_0 := \min\{J_1, J_2\} \in \mathcal{J}_{H_2}$. Then for each $g \in H_1 \cup \Gamma$, either $J_0 < J(g)$ or $J(g) \subset J_0$. By Lemma 2.14 and Lemma 2.16, we obtain that for each $g \in H_1 \cup \Gamma$, either $g^{-1}(J_0) > J_0$ or $g^{-1}(J_0) \subset J_0$. By Corollary 2.9 or Theorem 1.14, it follows that $A := \bigcup_{J \in \mathcal{J}_{H_2}, J \geq J_0} J$ is closed, $\sharp A \geq 3$, and $g^{-1}(A) \subset A$ for each $g \in H_2$. Therefore $J(H_2) \subset A$. Thus $J(H_2) = A$ and hence $J_0 = J_{\min}(H_2)$. From assumption (2), however, it must be the case that $J_0 = J_2$. Therefore $J_{\min}(H_1) \subset J_2 = J_{\min}(H_2)$ as desired.

By Remark 5.4 and Theorem 1.20, it follows that for each $\gamma \in \Gamma$, we have $\gamma(\partial \hat{K}(H_1)) \subset \gamma(J_{\min}(H_1)) \subset \gamma(J_{\min}(H_2)) \subset int\hat{K}(H_2)$. Let $\gamma \in \Gamma$ and let U be any connected component of $int\hat{K}(H_1)$, which we note is simply connected by the maximum principle. Then $\partial\gamma(U) \subset \gamma(\partial U) \subset int\hat{K}(H_2)$. Let V be the connected component of $int\hat{K}(H_2)$ containing the connected set $\gamma(\partial U)$. Then $\partial\gamma(U) \subset V$. By the maximum principle, $\overline{\mathbb{C}} \setminus \overline{V}$ is connected and unbounded. Hence $\gamma(U) \subset V$. From this argument, it follows that for each $\gamma \in \Gamma$, we have $\gamma(\hat{K}(H_1)) \subset \hat{K}(H_2) \subset \hat{K}(H_1)$, where the last inclusion holds since $H_1 \subset H_2$. Thus since $\hat{K}(H_1)$ is forward invariant under each $\gamma \in \Gamma$ and under each map in H_1, it is also forward invariant under each map in H_2. We then conclude that $\hat{K}(H_1) \subset \hat{K}(H_2)$, which together with the reverse inclusion already noted gives $\hat{K}(H_1) = \hat{K}(H_2)$. □

Definition 5.5. Let G be a rational semigroup and let N be a positive integer. We define $SH_N(G)$ to be the set of all $z \in \overline{\mathbb{C}}$ such that there exists a neighborhood U of z such that for all $g \in G$ we have $\deg(g : V \to U) \leq N$ for each connected component V of $g^{-1}(U)$.

Definition 5.6. Let G be a rational semigroup. We define

$$UH(G) = \overline{\mathbb{C}} \setminus \cup_{N=1}^{\infty} SH_N(G).$$

Remark 5.7. For a rational semigroup G we note that each $SH_N(G)$ is open and thus $UH(G)$ is closed.

Remark 5.8. For a rational semigroup G we see that $UH(G) \subset P(G)$. This holds since for $z \notin P(G)$ and $U = B(z, \delta)$ such that $U \cap P(G) = \emptyset$ it must be the case (by an application of the Riemann-Hurwitz relation) that $\deg(g : V \to U) = 1$ for each connected component V of $g^{-1}(U)$.

Remark 5.9. We note from Lemma 1.14 in [22] that, the attracting cycles of g, parabolic cycles of g, and the boundary of every Siegel disk of g are contained in $UH(\langle g \rangle)$, for any polynomial g with $\deg(g) \geq 2$. Hence we may conclude that such points are also in $UH(G)$ for any G containing g.

Proof of Theorem 1.27. Assume the conditions stated in the hypotheses. By the definition of semi-hyperbolic, our goal is to show $J(G) \subset SH_K(G)$ for some $K \in \mathbb{N}$. We will show the equivalent statement that $J(G) \cap UH(G) = \emptyset$. Since $UH(G) \subset P(G)$ and $P^*(G) \cap J(G) \subset J_{\min}(G)$, we have only to show $J_{\min}(G) \subset \mathbb{C} \setminus UH(G)$.

By Theorem 1.20 or [29, 33] we know that $\sharp J_{\min}(G) \geq 3$. Thus hypothesis (2) and Lemma 2.16 imply $\gamma^{-1}(J_{\min}(G)) \cap J_{\min}(G) = \emptyset$ for $\gamma \in \Gamma$, which in turn implies (by Lemma 2.14) $\gamma^{-1}(J_{\min}(G)) > J_{\min}(G)$. Thus, for all $\gamma \in \Gamma$,

$$\gamma^{-1}(J(G)) \cap A = \emptyset \tag{I}$$

where $A = PH(J_{\min})$.

Since Γ is compact in Poly, $d = \min_{\gamma \in \Gamma} dist(\gamma^{-1}(J(G)), A) > 0$. By (I) there exists $d_1 > 0$ such that for all $\gamma \in \Gamma$, for all $z \in J(G)$, and all components U of $\gamma^{-1}(B(z, d_1))$ we have

$$U \cap B(A, d/2) = \emptyset. \tag{II}$$

Now by Lemma 5.3 and by hypothesis (3) we have $UH(H) \cap \mathbb{C} \subset P^*(H) \cap F(H) \subset \hat{K}(H) \cap F(H) = int\hat{K}(H) = int\hat{K}(G) \subset F(G)$ and so, taking complements, $J_{\min}(G) \subset \mathbb{C} \setminus UH(H)$.

Claim: There exists $b \in UH(H) \cap int\hat{K}(H)$.

Proof of claim: Lemma 5.3 and Remark 5.4 show that $int\hat{K}(H) = int\hat{K}(G) \neq \emptyset$. Let $g_0 \in H$ and consider the iterates $\{g_0^n\}$ at any $w \in int\hat{K}(H) \subset F(H)$. Hypothesis (3) implies $UH(H) \cap \mathbb{C} \subset F(H)$ which implies that g_0 cannot have a cycle of Siegel disks nor a parabolic cycle (see Remark 5.9). Thus by Sullivan's No Wandering Domains Theorem the orbit $\{g_0^n(w)\}$ must be drawn toward an attracting cycle in $\mathbb{C}$. By replacing, if necessary, g_0 by an iterate we may assume that $g_0^n(w)$ approaches a finite fixed point b of g_0. Thus $b \in UH(H) \cap \mathbb{C} \subset P^*(H) \cap F(H) \subset \hat{K}(H) \cap F(H) = int\hat{K}(H)$ which completes the proof of the claim.

Now let $z \in J_{\min}(G) \subset \mathbb{C} \setminus UH(H)$. Then there exists $\delta > 0$ such that $B(z, 2\delta) \subset \mathbb{C} \setminus UH(H)$. Since $g(UH(H)) \subset UH(H)$ for each $g \in H$, we must have $g(b) \notin B(z, 2\delta)$. Since H is normal at b, there exists $\epsilon_1 > 0$ such that $g \in H$ gives $g(B(b, \epsilon_1)) \cap B(z, \delta) = \emptyset$, which implies $g^{-1}(B(z, \delta)) \cap B(b, \epsilon_1) = \emptyset$. Since $z \in \mathbb{C} \setminus UH(H)$ there exists $\delta_1 < \delta$ and $N \in \mathbb{N}$ such that for all $h \in H$ and for all components V of $h^{-1}(B(z, \delta_1))$ we have $\deg(h : V \to B(z, \delta_1)) \leq N$.

Fix $h \in H$ and consider a component V of $h^{-1}(B(z, \delta_1))$ and note that the maximum principle implies that V is simply connected. Let $\phi_{V,h} : B(0, 1) \to V$ be the Riemann map chosen such that $h \circ \phi_{V,h}(0) = z$. By applying the distortion Lemma 1.10 in [22], there exists $0 < \delta_2 < \delta_1$ such that the component W of $(h \circ \phi_{V,h})^{-1}(B(z, \delta_2))$ containing 0 is such that $diam W \leq c$ where $c > 0$ is a small number independent of h, to be specified later.

Note that, in the above, the set V does depend on $h \in H$. Yet for each $h \in H$, the set $\phi_{V,h}(B(0, 1)) = V$ does not meet $B(b, \epsilon_1)$ and so the family $\{\phi_{V,h}\}_{h \in H}$ is normal on $B(0, 1)$. Thus

$$diam \phi_{V,h}(W) < d_1/10 \tag{III}$$

when c is sufficiently small.

Let $g \in G$. If $g \in H$, then (since $\delta_2 < \delta_1$) we have $\deg(g : V \to B(z, \delta_2)) \leq N$ where V is any component of $g^{-1}(B(z, \delta_2))$. If $g \notin H$, then we write $g = h\gamma g_1$ where $g_1 \in G \cup \{id\}$, $h \in H \cup \{id\}$ and $\gamma \in \Gamma$. Let V_0 be a component of $\gamma^{-1} h^{-1}(B(z, \delta_2))$. Thus we have $\deg(h\gamma : V_0 \to B(z, \delta_2)) \leq NM$ where $M = \max_{\gamma \in \Gamma}\{\deg \gamma\}$. By (III) we have $diam \gamma(V_0) < d_1/10$. By the definition of d_1 we have $V_0 \cap B(A, d/2) = \emptyset$ and thus $V_0 \cap P(G) = \emptyset$. Using the maximum principle applied to the polynomial $h\gamma$ implies V_0 is simply connected and hence each branch of g_1^{-1} is well defined on V_0. So for all components V_1 of $g^{-1}(B(z, \delta_2))$ we have $\deg(g : V_1 \to B(z, \delta_2)) \leq NM$.

In the above, N depends on z, but what we have shown is that $z \in J_{\min}(G)$ implies $z \in J_{\min}(G) \cap SH_N(H)$ for some N, which in turn implies $z \in J_{\min}(G) \cap SH_{NM}(G)$, thus giving $z \notin UH(G)$. □

Proof of Theorem 1.29. The proof follows the same line as the proof of Theorem 1.27. We note that the usual Koebe Distortion Theorem applies (without needing to invoke the distortion Lemma 1.10 in [22]), and on the domains of interest in the proof each γ is one-to-one by hypothesis (4) and each $h \in H$ is one-to-one by hypothesis (3). We omit the details. □

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