Simple examples of Lifschitz tails in Gaussian random magnetic fields

by

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Abstract. – This is a first attempt to investigate the asymptotic behavior of the integrated density of states at the infimum of the spectrum for Schrödinger operators with magnetic fields which are Gaussian random fields. In simple examples it is shown that the integrate density of states decays exponentially. These examples shall give a hint to consider in more general framework.

Key words: Lifschitz tails, random Schrödinger operators, magnetic fields, Gaussian random fields

Résumé. – C'est un premier essai pour rechercher le comportement asymptotique, á la borne inférieure du spectre, de la densité d'état intégrée pour l'opérateur de Schrödinger avec champ magnétique qui est un champ aléatoire gaussiens. Nous allons montrer que la densité d'état intégrée décroissons exponentiellement en exemples simples. Ces exemples donneront une indication pour considérer en cadres plus généralaux.

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1. INTRODUCTION

This is a first attempt to consider the asymptotic behavior of the integrated density of states $N(\lambda)$ of the Schrödinger operator $L(b_\omega)$ with the magnetic field db_ω at the infimum

of the spectrum, where

$$L(b_\omega) = \frac{1}{2}\sum_{j=1}^{2}\left(i\frac{\partial}{\partial x^j} - b_\omega^j(x)\right)^2 \qquad (i = \sqrt{-1}) \tag{1.1}$$

and $b_\omega = \sum_{j=1}^{2} b_\omega^j(x)dx^j$ ($\omega \in \Omega$, $x \in \mathbb{R}^2$) is a 1-form valued Gaussian random field on a probability space Ω that is stationary and ergodic with respect to the shift in the variable $x \in \mathbb{R}^2$. We show that $N(\lambda)$ decays exponentially in the following two cases:

Case I. $b_\omega^1(x) = 0$ and $b_\omega^2(x) = g_\omega(x^1)$ depends only on the first coordinate.

Case II. $b_\omega(x) = d^* f_\omega(x)$, where $f_\omega(x)$ is a scalar valued Gaussian random field.

The main results are stated in Theorems 3.1 and 4.1 below. These simple examples shall give a hint to consider in more general framework.

The asymptotics of the integrated density of states has been studied mainly for the operator

$$H(V_\omega) = -\frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial}{\partial x^j}\right)^2 + V_\omega(x) \tag{1.2}$$

on $\mathbb{R}^d$, with a real valued random field V_ω, as other spectral properties. For this aspect, we refer Carmona-Lacroix [4] and Pastur-Figotin [21]. In particular, the exponential decay of the integrated density of states conjectured by Lifschitz [16], which is governed by both the kinetic energy and the potential energy, was first proved by the large deviation theory of Donsker and Varadhan in the case that

$$V_\omega(x) = \int u(x-y)\mu_\omega(dy), \tag{1.3}$$

where $\mu_\omega(dy)$ is a Poisson random measure and u is a nonnegative integrable function on $\mathbb{R}^d$ such that $u(x) = o(|x|^{-d-2})$ as $|x| \to \infty$ [6], [20]. For the operator

$$\frac{1}{2}\sum_{j=1}^{d}\left(i\frac{\partial}{\partial x^j} - b_\omega^j(x)\right)^2 + V_\omega(x),$$

Matsumoto [18] and the author [28] studied simple cases where the result is not affected by the magnetic field. The most studied case is that db_ω is a deterministic constant and V_ω is as in (1.3) [2], [7], [10]. This case is related with the quantum Hall effect. Broderix, Hundertmark, Kirsch and Leschke [2] showed that the integrated density of states decays exponentially and determined the leading term which are governed solely by V_ω, when u decays polynomially. Erdös [7] showed that the integrated density of states decays polynomially and determined the leading order which is governed also by the magnetic field when u is of compact support. Hupfer, Leschke and Warzel [10] studied the case that u decays exponentially and showed that the situation on the dependence of the magnetic field changes when u decays as $\exp(-C|x|^2)$. Form these results the effect of the vector potential is weaker than that of the scalar potential. Therefore our problem in this paper is important mathematically to make clear the effect of the vector potential. On the other hand, the operator with random vector potential having the same form with (1.1) is studied also in the nonlinear electromagnetic field theory (see e.g. [1], [19] and the references therein). In that case, the field b_ω is a distribution valued random field and the analysis is much harder than ours.

Our problem is equaivalent, by the Tauberian theory, to investigate the asymptotic behavior as $t \uparrow \infty$ of the Laplace-Stieltjes transform

$$\begin{aligned}\widetilde{N}(t) :=& \int_0^\infty e^{-t\lambda} dN(\lambda) \\ =& \frac{1}{2\pi t} E^{w\times b}\left[\exp\left(i \sum_{j=1}^{2} \int_0^t b_\omega^j(w(s)) \circ dw^j(s) \right) \middle| w(t) = 0 \right],\end{aligned} \tag{1.4}$$

where $E^{w\times b}$ is the expectation with respect to the 2-dimensional Wiener process $w(s) = (w^1(s), w^2(s))$, $s \geq 0$, starting at 0 and the random field b_ω. The difficult points not appearing for the operator (1.2) are that the right hand side of (1.4) is an oscillatory integral and that the phase functional is given by a stochastic integral. Therefore our problem is one of the problems on the asymptotics of the stochastic oscillatory integrals

(cf.[11],[12],[17]). However in Case I these difficulties disappear. In fact in this case the expectation with respect to $w^2(s)$ in the right hand side of (1.4) can be computed easily and our problem is reduced to the same problem for the operator (1.2) on $\mathbb{R}$ with

$$V_\omega(x) = \frac{1}{2} g_\omega(x)^2 \tag{1.5}$$

We here recall a work by Figotin [8]: he determined the leading term of the asymptotics of the integrated density of states for the operator (1.2) with

$$V_\omega(x) = \frac{1}{2} \int_{\mathbb{R}^d} u(x-y) g_\omega(x)^2 dx, \tag{1.6}$$

where u is as in (1.3), by modifying the method by Donsker and Varadhan [6], [20]. By his method, we can obtain a lower estimate of the integrated density of states. However, to obtain an upper estimate, the techniques in [8] are not enough: in [8], the function $u(x)$ plays an important role to make the potential $V_\omega(x)$ good enough to apply the large deviation theory by Donsker-Varadhan for the occupation measure by the Wiener process in the weak topology (see also [5], [6], [20], [21], [29]). On the other hand, Kirsch and Martinelli give a general upper estimate of the integrated density of states for the operator (1.2) with

$$V_\omega(x) = v(g_\omega(x)), \tag{1.7}$$

where $v(x)$ is a locally bounded positive real function on $\mathbb{R}$, polynomially bounded at infinity, by a functional analytic approach [15]. From their estimate, we can obtain an upper estimate of the integrated density of states. For Case II, we can give some upper and lower bounds which are treated samely as in Case I.

The organization of this paper is as follows. In Section 2, we follow Figotin [8] and Kirsch and Martinelli [15] to show that the integrated density of states for the operator $H(V_\omega)$ in (1.2) with the potential V_ω as that in (1.5) decays exponentially at the infimum of the spectrum. In Section 3, we consider Case I and in Section 4, we consdier Case II.

2. SQUARE OF GAUSSIAN RANDOM FIELDS

In this section we follow Figotin [8] (see also Pastur-Figotin [21]) and Kirsch and Martinelli [15] to show that the integrated density of states $N(\lambda)$ for the operator $H(V_\omega)$ in (1.2) with the potential

$$V_\omega(x) = \frac{1}{2}\sum_{j=1}^{q} g_\omega^j(x)^2 \tag{2.1}$$

decays exponentially at the infimum of the spectrum, where $\{(g_\omega^j(x))_{j=1,2,\cdots,q},\ x \in \mathbb{R}^d\}$ is the $\mathbb{R}^q$-valued stationary ergodic Gaussian random field with mean zero and the covariance $\beta_{jk}(x) = E[g_\omega^j(x) g_\omega^k(0)]$, $j,k = 1,2,\cdots,q$. In the following, we set

$$\widehat{\beta_{jk}}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\beta_{jk}(x)dx,$$

we use the polar coordinate $\xi = (\zeta,\theta)$, $\zeta > 0$, $\theta \in S^{d-1}$, and we denote the eigenvalues of any nonnegative $q \times q$ matrix A by $\mu_1(A) \le \mu_2(A) \le \cdots \le \mu_q(A)$.

We introduce the following conditions: for $\alpha > d$ and $\tau > 0$,

(P1, α, d) (i) $\sup_\xi |\zeta^\alpha \widehat{\beta_{jk}}(\xi)| < \infty$;
(ii) $\zeta^\alpha \widehat{\beta_{jk}}(\xi) \to \overset{\circ}{\beta}_{jk}(\theta)$ as $\zeta \to \infty$ for any $\theta \in S^{d-1}$ and some $\overset{\circ}{\beta}_{jk}(\theta) \in \mathbb{R}$;
(iii) $\sup_x |x|^m |\beta_{jk}(x)| < \infty$ for some $m > d + 2 + 2d/(\alpha - d)$;
(iv) vol $\{\theta \in S^{d-1}; \mu_q(\overset{\circ}{\beta}(\theta)) \neq 0\}$ is positive, where vol is the volume measure on S^{d-1}.

(E1, τ, d) (i) There exist $\sigma > 0$, $1 \le q_* \le q$ and functions $R_j(\xi)$, $1 \le j \le q_*$ such that $\sup_\theta |\mu_j(\widehat{\beta}(\xi)) \exp(\sigma\zeta^\tau + R_j(\xi)) - 1| \to 0$ as $\zeta \to \infty$, where $\sup_\theta |R_j(\xi)|\zeta^{-\tau} \to 0$ as $\zeta \to \infty$ for any $1 \le j \le q_*$ and the support of $\mu_j(\widehat{\beta}(\xi))$ is compact for any $j > q_*$;
(ii) $\sup_x |x|^m |\beta_{jk}(x)| < \infty$ for some $m > d + 2$.

(C1, d) (i) The support of $\widehat{\beta_{jk}}(\xi)$ is compact;
(ii) $\sup_x |x|^m |\beta_{jk}(x)| < \infty$ for some $m > d + 2$;
(iii) $\beta_{jj}(0) > 0$ for some j.

Then the result is the following:

THEOREM 2.1. – (i) *We assume the condition* $(\mathrm{P}1, \alpha, d)$ *for some* $\alpha > d$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-(\alpha-d+2)/\{2(\alpha d^{-1}-1)\}} \leq \log N(\lambda) \leq -c'\lambda^{-d/2}\log\lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

(ii) *We assume the condition* $(\mathrm{E}1, \tau, d)$ *for some* $\tau > 0$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-d/2}\log^{(d+\tau)/\tau}\lambda^{-1} \leq \log N(\lambda) \leq -c'\lambda^{-d/2}\log\lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

(iii) *We assume the condition* $(\mathrm{C}1, d)$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-d/2}\log\lambda^{-1} \leq \log N(\lambda) \leq -c'\lambda^{-d/2}\log\lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

Remark 2.1. – In our methods, the leading term in Theorem 2.1 (iii) is not determined. In fact, if we assume $d = q = 1$, $\beta(x) \geq 0$ and the condition $(\mathrm{C}1, 1)$, then our method gives

$$\varliminf_{\lambda\downarrow 0}\frac{\sqrt{\lambda}\log N(\lambda)}{\log\lambda^{-1}} \geq -\frac{3^2}{2^{5/2}\pi}|\operatorname{supp}\widehat{\beta}| =: -c_1(\widehat{\beta})$$

and

$$\varlimsup_{\lambda\downarrow 0}\frac{\sqrt{\lambda}\log N(\lambda)}{\log\lambda^{-1}} \geq -\frac{\pi}{3\sqrt{6}}\beta(0)\Big/\int_{\mathbb{R}}|\beta(x)|dx =: -c_2(\widehat{\beta}).$$

For these bounds it holds that

$$\sup\{c_2(\widehat{\beta})/c_1(\widehat{\beta}) : \beta \text{ satisfies } (\mathrm{C}1,1) \text{ and } \beta(x) \geq 0\} = \frac{2\pi}{27\sqrt{3}} < 1.$$

This theorem is obtained from the corollary of Theorem 2.2 and Theorem 2.3 below. For Theorem 2.2 and its corollary, we follow Figotin [8] (see also Pastur-Figotin [21]). To state the theorem, we introduce the following condition:

(H1, d) (i) $\sup_\xi \mathrm{Tr}[\widehat{\beta}(\xi)] \prod_{j=1}^d (1+|\xi^j|^{1+\gamma}) < \infty$ for some $\gamma > 0$;

(ii) $\sup_x |x|^m |\beta_{jk}(x)| < \infty$ for some $m > d + 2 + 2\gamma^{-1}$;

(iii) $\lim_{t\to\infty} \Phi(ct)/\Phi(t) = c^\kappa$ for any $c > 0$ and some $\kappa \in [0, 1)$,

where

$$\Phi(t) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \, \mathrm{Tr}[\log(I + t\widehat{\beta}(\xi)]. \tag{2.2}$$

Let $\eta(t)$, $t > 0$, be the solution of

$$\eta^{d/2}(t) \Phi\left(\frac{t}{\eta^{d/2}(t)}\right) = \frac{t}{\eta(t)}. \tag{2.3}$$

Then we have the following:

THEOREM 2.2. – *Under the condition* (H1, d), *we have*

$$\varliminf_{t\to\infty} \frac{\eta(t)}{t} \log \widetilde{N}(t) > -\infty. \tag{2.4}$$

From this theorem, we have the following:

COROLLARY. – (i) *Under the condition* (P1, α, d) *for some* $\alpha > d$, *we have*

$$\varliminf_{t\to\infty} \frac{\log \widetilde{N}(t)}{t^{(\alpha-d+2)/\{\alpha(d+2)/d-d\}}} > -\infty. \tag{2.5}$$

(ii) *Under the condition* (E1, τ, d) *for some* $\tau > 0$, *we have*

$$\varliminf_{t\to\infty} \frac{\log \widetilde{N}(t)}{t^{d/(d+2)} \log^{\{2(d+\tau)\}/\{\tau(d+2)\}} t} > -\infty. \tag{2.6}$$

(iii) *Under the condition* (C1, d), *we have*

$$\varliminf_{t\to\infty} \frac{\log \widetilde{N}(t)}{t^{d/(d+2)} \log^{2/(d+2)} t} > -\infty. \tag{2.7}$$

Theorem 2.3 is a special case of a theorem obtained by Kirsch and Martinelli [15]. To state the theorem, we introduce the following condition:

(H2, d) (i) $\beta_{jj}(x) \in L_1(\mathbb{R}^d)$;

(ii) $\beta_{jj}(x)$ is Riemannian approximable as $\lim_{\varepsilon \downarrow 0} \sum_{y \in \mathbb{Z}^d} \varepsilon^d \beta_{jj}(\varepsilon y) = \int_{\mathbb{R}^d} \beta_{jj}(x) dx$.

Then we have the following:

THEOREM 2.3. – *We assume the condition* (H2, d). *Then for any* $\varepsilon > 0$ *and* $0 < s < 1$, *there exist positive constants* c *and* λ_0 *independent of the random field* g_ω *such that*

$$N(\lambda) \leq cq \exp\Big\{ - \Big(\frac{2^{1-d/2} \pi^d d^{d/2}}{B(d+2)^{1+d/2}} - \varepsilon \Big) \frac{1}{\lambda^{d/2}} \log \sqrt{\frac{q \min_{1 \leq j \leq q} \beta_{jj}(0)}{\lambda^s}} \Big\} \tag{2.8}$$

for any $0 < \lambda < \lambda_0$, *where*

$$B = \max_{1 \leq j \leq q} \int_{\mathbb{R}^d} |\beta_{jj}(x)| dx \Big/ \beta_{jj}(0).$$

In particular we have

$$\overline{\lim_{\lambda \downarrow 0}} \frac{\lambda^{d/2} \log N(\lambda)}{\log \lambda^{-1}} \leq - \frac{\pi^d d^{d/2}}{B(d+2)^{1+d/2} 2^{d/2}}. \tag{2.9}$$

Proof of Theorem 2.3. – As in (21) in [15], we have

$$N(\lambda) \leq k(1+\alpha)^{d/2} \lambda^{d/2} P(\lambda_1(-\Delta_\Lambda^N + V_\omega) < 2\lambda),$$

where k is a positive depending only on the dimension d, $\Lambda = [-L/2, L/2]^d$, $L = \pi(1+\alpha)^{-1/2}(1+\eta)^{-1/2}(2\lambda)^{-1/2}$, α is a positive constant specified later, η is an arbitrarily small positive constant and $\lambda_1(-\Delta_\Lambda^N + V_\omega)$ is the least eigenvalue of the operator $-\Delta_\Lambda^N + V_\omega$ with the Neumann condition on Λ. We dominate the probability as

$$P(\lambda_1(-\Delta_\Lambda^N + V_\omega) < 2\lambda) \leq \sum_{j=1}^q P(\lambda_1(-\Delta_\Lambda^N + q g_\omega^j(x)^2) < 2\lambda).$$

We estimate the each probability in the right hand side as in the proof of Theorem 6 in [15]. Then we have

$$P(\lambda_1(-\Delta_\Lambda^N + qg_\omega^j(x)^2) < 2\lambda) \le \exp\Big(- \frac{|\Lambda|}{Q_j} G_j(\alpha,\lambda)\Big),$$

where $|\Lambda| = L^d$ is the volume,

$$\begin{aligned} Q_j =& \int_{\mathbb{R}^d} |\beta_{jj}(x)|dx \Big/ \beta_{jj}(0), \\ G_j(\alpha,\lambda) =& \sup_{t\ge 0}\left\{ \frac{t}{1+\alpha} - \log\left(\exp\left(\frac{t}{\alpha}\right) P(qg_\omega^j(0)^2 \le (2\lambda)^s) \right.\right. \\ & \left.\left. + \exp\left(\frac{2t\lambda}{(2\lambda)^s + 2\alpha\lambda}\right) P(qg_\omega^j(0)^2 > (2\lambda)^s)\right)\right\} \\ =& \frac{(2\lambda)^s + 2\alpha\lambda}{(1+\alpha)(2\lambda)^s} \log P(qg_\omega^j(0)^2 \le (2\lambda)^s)^{-1} \\ & + \left(\frac{\alpha((2\lambda)^s + 2\alpha\lambda)}{(1+\alpha)(2\lambda)^s} - \alpha(2\lambda)^{1-s}\right) \log P(qg_\omega^j(0)^2 > (2\lambda)^s)^{-1} \\ & + G_0(\alpha,\lambda) \end{aligned}$$

and $G_0(\alpha,\lambda)$ is a function depending only on s such that $\overline{\lim}_{\lambda\downarrow 0} G_0(\alpha,\lambda) > -\infty$. We estimate as $P(qg_\omega^j(0)^2 > (2\lambda)^s) \le 1$ and

$$P(qg_\omega^j(0)^2 \le (2\lambda)^s) \le \sqrt{\frac{2(2\lambda)^s}{q\pi\beta_{jj}(0)}}.$$

Then for any $\varepsilon > 0$, there exist $\lambda_1 > 0$ independent of g_ω such that

$$G_j(\alpha,\lambda) \ge \left(\frac{\alpha}{1+\alpha} - \varepsilon\right) \log \sqrt{\frac{q\beta_{jj}(0)}{\lambda^s}}$$

for any $0 < \lambda < \lambda_1$. Thus for any $\varepsilon > 0$, there exist $\lambda_2 > 0$ independent of g_ω such that

$$N(\lambda) \le k(1+\alpha)^{d/2} q \exp\left(- \left(\frac{\pi^d \alpha}{B(1+\alpha)^{1+d/2}(1+\eta)^{d/2} 2^{d/2}} - \varepsilon\right) \frac{1}{\lambda^{d/2}} \log \sqrt{\frac{q \min_j \beta_{jj}(0)}{\lambda^s}}\right)$$

for any $0 < \lambda < \lambda_2$. By taking the minimum with respect to α, we obtain (2.8). □

To prove Theorem 2.2, we prepare several lemmas:

LEMMA 2.1. – *The function $\Phi(t)$ defined in (2.2) satisfies the following:*

(i) *$\Phi(t)$ is well defined as a positive function of $t > 0$.*

(ii) Φ *is concave and monotone increasing.*

(iii) $\Phi(t)/t$ *is monotone decreasing.*

(iv) *for any* $\varepsilon > 0$, *there exists* $C(\varepsilon) > 0$ *such that* $\Phi(t) \le C(\varepsilon)t^{1/(1+\gamma)+\varepsilon}$ *for any* $t \ge 1$, *where* γ *is the constant in the condition* (H1, d) (i).

Proof. – (iv) is proven by (H1, d) (i) and the induction on the dimension d. □

LEMMA 2.2. – *There exists a positive function* $\eta(t)$, $t > 0$, *such that the equation* (2.3) *is satisfied identically. This function satisfies the following:*

(i) *the functions* $\eta(t)$ *and* $t/\eta^{d/2+1}(t)$ *are monotone increasing and tend to* ∞ *as* $t \to \infty$.

(ii) $\lim_{t\to\infty} t/\eta^{\delta}(t) = 0$ *for any* $\delta > d/2 + (1+\gamma)/\gamma$.

(iii) $\lim_{t\to\infty}(\eta(t)/t)\log\eta(t) = 0$.

Proof. – There exists a positive function $T(t)$, $t > 0$, such that $\Phi(T(t))/T(t) = (T(t)/t)^{2/d}$ by Lemma 2.1 (iii). Then the function $\eta(t) := (t/T(t))^{2/d}$ satisfies (2.3). The rest of this proposition is also proven by using Lemma 2.1. □

We next consider the functional

$$\Phi(\phi,t) = -\frac{\eta(t)}{t}\log E^g\left[\exp\Big(-\frac{t}{2}\int_{\mathbb{R}^d}\sum_{j=1}^{q}(g_\omega^j(\sqrt{\eta(t)}x))^2\phi(x)dx\Big)\right] \tag{2.10}$$

in $0 \le \phi \in L_1(\mathbb{R}^d)$ and $t > 0$, where E^g is the expectation with respect to g_ω:

LEMMA 2.3. – (i) *For any* $t > 0$, *let* $B(t)$ *be an operator on* $L_2(\mathbb{R}^d \to \mathbb{R}^q)$ *defined by*

$$\{B(t)\varphi\}^j(x) = \int_{\mathbb{R}^d}\sum_{k=1}^{q}\beta_{jk}(\sqrt{\eta(t)}(x-y))\varphi^k(y)dy.$$

Then, for any $0 \le \phi \in L_1(\mathbb{R}^d)$ *and* $t > 0$, $\Phi(\phi,t)$ *is represented as*

$$\Phi(\phi,t) = \frac{\eta(t)}{2t}\operatorname{Tr}[\log(I + t\sqrt{\phi}B(t)\sqrt{\phi})], \tag{2.11}$$

where $\mathrm{Tr}[\cdot]$ *is the trace on the space* $L_2(\mathbb{R}^d \to \mathbb{R}^q)$.

(ii) $\Phi(\phi, t)$ *is continuous in* ϕ *in the norm of* $L_1(\mathbb{R}^d)$.

(iii) $\Phi(\phi_1, t) \le \Phi(\phi_2, t)$ *if* $\phi_1 \le \phi_2$.

(iv) $\Phi(\phi(\cdot + a), t) = \Phi(\phi, t)$ *for any* $a \in \mathbb{R}^d$.

Proof. – For (i), referring that $\sqrt{\phi}B(t)\sqrt{\phi}$ is a Hilbert-Schmidt operator, we take a complete orthonormal system $\{\varphi_n\}_{n\ge1}$ of $L_2(\mathbb{R}^d \to \mathbb{R}^q)$ consisting of the eigenfunctions of $\sqrt{\phi}B(t)\sqrt{\phi}$. Let $\{\beta_n\}_{n\ge1}$ be the corresponding eigenvalues. Then we can write as

$$\Phi(\phi, t) = -\lim_{N\to\infty} \frac{\eta(t)}{t} \log E^{g\times X}\left[\exp\left(i\sum_{n=1}^{N} X_n\sqrt{t}(g_\omega(\sqrt{\eta(t)}x)\sqrt{\phi(x)}, \varphi_n(x))\right)\right],$$

where $(\cdot,\cdot)$ is the inner product of the space $L_2(\mathbb{R}^d \to \mathbb{R}^q)$ and $E^{g\times X}$ is the expectation with respect to g_ω and a system $\{X_n\}_{n\ge1}$ of independent copies of the Gaussian random variable with mean zero and covariance 1. We first calculate the expectation with respect to g_ω and then calculate the expectation with respect to $\{X_n\}$. Then we have

$$\Phi(\phi, t) = \frac{\eta(t)}{2t}\sum_{n=1}^{\infty}\log(1+t\beta_n) = \frac{\eta(t)}{2t}\mathrm{Tr}[\log(I + t\sqrt{\phi}B(t)\sqrt{\phi})].$$

(ii) – (iv) are easily obtained from the expression (2.10) . □

To treat the expression as (2.11), we prepare the following:

Lemma 2.4. – *For any nonnegative operators* B_1 *and* B_2 *on a Hilbert space of trace class, we have the following:*

(i) $\mathrm{Tr}[\log(I + B_1 + B_2)] \le \mathrm{Tr}[\log(I + B_1)] + \mathrm{Tr}[\log(I + B_2)]$.

(ii) $|\,\mathrm{Tr}[\log(I + B_1)] - \mathrm{Tr}[\log(I + B_2)]| \le \|B_1 - B_2\|(\mathrm{Tr}[\log(I + B_1)] + \mathrm{Tr}[\log(I + B_2)])$ *if* $\mathrm{Tr}[B_1] = \mathrm{Tr}[B_2]$, *where* $\|\cdot\|$ *is the operator norm.*

Proof. – (i) Since the operators B_1 and B_2 are of trace class, it is enough to show this lemma on a finite dimensional Hilbert space. If we set

$$\varphi(t) := \mathrm{Tr}[\log(I + B_1)] + \mathrm{Tr}[\log(I + tB_2)] - \mathrm{Tr}[\log(I + B_1 + tB_2)],$$

then we have

$$\varphi'(t) = \mathrm{Tr}[\{(I + tB_2)^{-1} - (I + B_1 + tB_2)^{-1}\}B_2].$$

Since $(I + tB_2)^{-1} \geq (I + B_1 + tB_2)^{-1}$, we have $\varphi'(t) \geq 0$ and $\varphi(t) \geq 0$.

(ii) By the same method in the proof of (i), we have

$$\frac{d}{dt}\,\mathrm{Tr}[\log(I + B(t)) - B(t)] = \mathrm{Tr}[(B_1 - B_2)B(t)(I + B(t))^{-1}],$$

where $B(t) := tB_1 + (1 - t)B_2$ for $0 \leq t \leq 1$. Then, by $\lambda(1 + \lambda)^{-1} \leq \log(1 + \lambda)$ for any $\lambda \geq 0$ and (i), we have

$$\begin{aligned} &|\,\mathrm{Tr}[\log(I + B_1) - B_1] - \mathrm{Tr}[\log(I + B_2) - B_2]| \\ \leq &\|B_1 - B_2\|(\mathrm{Tr}[\log(I + B_1)] + \mathrm{Tr}[\log(I + B_2)]). \end{aligned} \tag{2.12}$$

□

To consider the asymptotics as $t \to \infty$, we prepare the following:

Lemma 2.5. – *Let*

$$\Phi_R(t) := R^{-d}\,\mathrm{Tr}[\log(I + t\chi_{\Lambda(R)}B\chi_{\Lambda(R)})] \tag{2.13}$$

for $R > 0$, where $\Lambda(R) = [-R/2, R/2]^d$ and B is an operator on $L_2(\mathbb{R}^d \to \mathbb{R}^q)$ defined by

$$\{B\varphi\}^j(x) = \int_{\mathbb{R}^d} \sum_{k=1}^{q} \beta_{jk}(x - y)\varphi^k(y)dy.$$

Then we have

$$\lim_{R\to\infty} \Phi_R(t) = \Phi(t)$$

for any $t \geq 0$, where $\Phi(t)$ is the function defined in (2.2).

Proof. – For any $\ell \in \mathbb{N}$, we have

$$\lim_{R\to\infty} R^{-d} \operatorname{Tr}[(\chi_{\Lambda(R)} B \chi_{\Lambda(R)})^\ell] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \operatorname{Tr}[\widehat{\beta}(\xi)^\ell]$$

by the Lebesgue convergence theorem, the condition $(\mathrm{H1}, d)$ (ii) and the Fourier transform. Then by using also the condition $(\mathrm{H1}, d)$ (i), (ii), we see that the measure

$$m_R(d\lambda) = R^{-d} \sum_j \delta_{\mu(j;R)}(\lambda) \lambda d\lambda$$

on $[0, \infty)$ converges weakly to

$$m_\infty(d\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \sum_k \delta_{\mu(k;\xi)}(\lambda) \lambda d\lambda$$

as $R \to \infty$, where, for each $R > 0$ and $\xi \in \mathbb{R}^d$, $\{\mu(j;R)\}_j$ and $\{\mu(k;\xi)\}_k$ are the eigenvalues of the operator $\chi_{\Lambda(R)} B \chi_{\Lambda(R)}$ and the matrix $\widehat{\beta}(\xi)$, respectively. By taking a test function as $\lambda^{-1} \log(1 + t\lambda)$, we can complete the proof. □

We now consider the asymptotics of $\Phi(\phi, t)$ as $t \to \infty$:

Lemma 2.6. – *For any $c > 0$ and $\rho > 0$, we have*

$$\overline{\lim_{t\to\infty}} \Phi(c\chi_{\Lambda(\rho)}, t) \le \frac{1}{2} c^\kappa \rho^d. \tag{2.14}$$

Proof. – By Lemma 2.3 (i) and (2.3), we have

$$\Phi(c\chi_{\Lambda(\rho)}, t) = \frac{1}{2} \frac{\operatorname{Tr}[\log(I + ct\chi_{\Lambda(\rho)} B(t) \chi_{\Lambda(\rho)})]}{\eta^{d/2}(t) \Phi(ct/\eta^{d/2}(t))} \frac{\Phi(ct/\eta^{d/2}(t))}{\Phi(t/\eta^{d/2}(t))}. \tag{2.15}$$

By the condition $(\mathrm{H1}, d)$ (iii) and Lemma 2.2 (i), we have

$$\lim_{t\to\infty} \frac{\Phi(ct/\eta^{d/2}(t))}{\Phi(t/\eta^{d/2}(t))} = c^\kappa.$$

Then it is enough to show that

$$\overline{\lim_{t\to\infty}} \frac{\operatorname{Tr}[\log(I + ct\chi_{\Lambda(\rho)} B(t) \chi_{\Lambda(\rho)})]}{\eta^{d/2}(t) \Phi(ct/\eta^{d/2}(t))} \le \rho^d. \tag{2.16}$$

For any small $\varepsilon > 0$, we use Lemma 2.3 (iii) to estimate as

$$\mathrm{Tr}[\log(I + ct\chi_{\Lambda(n(\rho+\varepsilon))}B(t)\chi_{\Lambda(n(\rho+\varepsilon))})] \geq \mathrm{Tr}[\log(I + ct\Sigma + ct\Sigma')], \tag{2.17}$$

where $\Sigma = \sum_\ell \chi_{\rho,\ell}B(t)\chi_{\rho,\ell}$, $\Sigma' = \sum_{\ell\neq k}\chi_{\rho,\ell}B(t)\chi_{\rho,k}$ and $\chi_{\rho,\ell}(x) = \chi_{\Lambda(\rho)}(x - (\rho+\varepsilon)\ell)$. By Lemma 2.4 (ii) and $\mathrm{Tr}[\Sigma'] = 0$, the right hand side of (2.17) is dominated from below by

$$\mathrm{Tr}[\log(I + ct\Sigma)] - ct\|\Sigma'\|\{\mathrm{Tr}[\log(I + ct\Sigma + ct\Sigma')] + \mathrm{Tr}[\log(I + ct\Sigma)]\}.$$

By the condition (H1, d) (ii), we have

$$|\Sigma'(x,y)| \leq c_1\chi_{\{|x-y|>\varepsilon\}}(\sqrt{\eta(t)}|x-y|)^{-m}.$$

Thus we have

$$\begin{aligned}&\mathrm{Tr}[\log(I + ct\chi_{\Lambda(n(\rho+\varepsilon))}B(t)\chi_{\Lambda(n(\rho+\varepsilon))})]\\ \geq&\frac{1 - cc_2t/\eta^{m/2}(t)}{1 + cc_2t/\eta^{m/2}(t)}\,\mathrm{Tr}[\log(I + ct\sum_\ell \chi_{\rho,\ell}B(t)\chi_{\rho,\ell})].\end{aligned} \tag{2.18}$$

By the unitary transform $\phi(x) \to \eta^{d/4}(t)\phi(\sqrt{\eta(t)}x)$ and Lemma 2.3 (i),(iv), we rewrite as

$$\begin{aligned}&\mathrm{Tr}\left[\log\left(I + \frac{ct}{\eta^{d/2}(t)}\chi_{\Lambda(n(\rho+\varepsilon)\sqrt{\eta(t)})}B\chi_{\Lambda(n(\rho+\varepsilon)\sqrt{\eta(t)})}\right)\right]\\ \geq&\frac{1 - cc_2t/\eta^{m/2}(t)}{1 + cc_2t/\eta^{m/2}(t)}\,\mathrm{Tr}[\log(I + ct\chi_\rho B(t)\chi_\rho)]n^d.\end{aligned} \tag{2.19}$$

By using Lemma 2.5 (ii) to take the limit of $n \to \infty$, we have

$$\Phi\Big(\frac{ct}{\eta^{d/2}(t)}\Big)(\rho+\varepsilon)^d\eta^{d/2}(t) \geq \frac{1 - cc_2t/\eta^{m/2}(t)}{1 + cc_2t/\eta^{m/2}(t)}\,\mathrm{Tr}[\log(I + ct\chi_\rho B(t)\chi_\rho)].$$

By Lemma 2.2 (ii), we obtain (2.16). □

Proof of Theorem 2.2. – It is well known that

$$\widetilde{N}(t) = \frac{1}{(2\pi t)^{d/2}}E^{w\times g}\left[\exp\Big(-\frac{1}{2}\int_0^t\sum_{j=1}^q(g_\omega^j(w(s)))^2ds\Big)\,\Big|\,w(t) = 0\right],$$

where $E^{w\times g}$ is the expectation with respect to the d-dimensional Wiener process $w(s)$ and $g^j_\omega(x)$ (cf.[4],[21]). We use the scaling property of the Wiener process to rewrite as

$$\begin{aligned}\widetilde{N}(t) &= \frac{1}{(2\pi t)^{d/2}} E^{w\times g}\Big[\exp\Big(-\frac{\eta(t)}{2}\int_0^{t/\eta(t)}\sum_{j=1}^{q}(g^j_\omega(\sqrt{\eta(t)}w(s)))^2 ds\Big)\Big| w\Big(\frac{t}{\eta(t)}\Big)=0\Big]\\ &= \frac{1}{\eta^{d/2}(t)} E^{g}\Big[\exp\Big(-\frac{t}{\eta(t)}H(\eta(t))\Big)(0,0)\Big],\end{aligned}$$

where $\exp(-tH(\eta))(x,y)$, $t>0$, x, $y\in\mathbb{R}^d$, is the integral kernel of the heat semigroup generated by the operator

$$H(\eta)=\frac{1}{2}\Big(-\Delta+\eta\sum_{j=1}^{q}(g^j_\omega(\sqrt{\eta}\cdot))^2\Big).$$

By Theorem (9.5) of [21] , we have

$$\widetilde{N}(t)\geq\frac{1}{\eta^{d/2}(t)|\operatorname{supp}\phi|}\exp\Big(-\frac{t}{\eta(t)}\Big(\frac{\|\nabla\phi\|^2}{2}+\Phi(\phi^2,t)\Big)\Big) \tag{2.20}$$

for any $\phi\in C_0^\infty(\mathbb{R}^d)$ such that $\|\phi\|_2=1$, where $\|\cdot\|_2$ and $(\cdot,\cdot)$ are the norm and the inner product of $L_2(\mathbb{R}^d)$, respectively. By Lemma 2.6 and Lemma 2.2 (iii), we have

$$\varliminf_{t\to\infty}\frac{\eta(t)}{t}\log\widetilde{N}(t)\geq\frac{-1}{2}(\|\nabla\phi\|_2^2+c^{2\kappa}R^d), \tag{2.21}$$

where c, $R>0$ are taken as $|\phi|\leq c$ and $\operatorname{supp}\phi\subset\Lambda(R)$. □

Corollary (i), (iii) of Theorem 2.2 are proven by Theorem 2.2 and the following:

LEMMA 2.7. – (i) *If* $\Phi(t)=Ct^A\log^B t(1+o(1))$ *as* $t\to\infty$ *for some* B, $C>0$ *and* $0\leq A<1$, *then*

$$\eta(t)=\frac{t^{A'}}{C'\log^{B'}t}(1+o(1)) \tag{2.22}$$

as $t\to\infty$, *where*

$$A'=\frac{2(1-A)}{d(1-A)+2},\quad B'=\frac{2B}{d(1-A)+2}\quad\textit{and}\quad C'=C^{2/(d(1-A)+2)}\Big(\frac{2}{d(1-A)+2}\Big)^{B'}.$$

(ii) *If*

$$\eta(t)=\frac{t^A}{C\log^B t}(1+o(1))$$

and

$$\frac{\eta(t)}{t}\log\widetilde{N}(t) = -D(1+o(1))$$

as $t \to \infty$ *for some* B, C, $D > 0$ *and* $0 < A < 1$, *then*

$$\log\widetilde{N}(t) = -CDt^{1-A}\log^B t(1+o(1)) \tag{2.23}$$

as $t \to \infty$.

This is easily proven by (2.3). For Corollary (ii), we use also the following:

LEMMA 2.8. – *For arbitrarily fixed* σ, τ, $R > 0$, *let*

$$\Phi_{\sigma,R}(t) := \int_R^\infty d\zeta\zeta^{d-1}\log(1+te^{-\sigma\zeta^\tau}), \qquad t > 0. \tag{2.24}$$

Then, for any fixed $K > 0$, *we have*

$$\Phi_{\sigma,R}(Kt) = c\log^{(d+\tau)/\tau} t(1+o(1)) \tag{2.25}$$

as $t \to \infty$, *where* $c = \tau/\{(d+\tau)d\sigma^{d/\tau}\}$.

Theorem 2.1 is obtained from Corollary of Theorem 2.2, Theorem 2.3 and the following:

LEMMA 2.9. – *If*

$$\varliminf_{t\to\infty}\frac{\log\widetilde{N}(t)}{t^A\log^B t} \geq -C$$

for some B, $C > 0$ *and* $0 < A < 1$, *then we have*

$$\varliminf_{\lambda\downarrow 0}\frac{\lambda^{A'}}{\log^{B'}(1/\lambda)}\log N(\lambda) \geq -C',$$

where $A' = A/(1-A)$, $B' = B/(1-A)$ *and* $C' = C^{1/(1-A)}(1-A)^{-B/(1-A)}$.

3. CASE I

Let $\{g_\omega(y), y \in \mathbb{R}\}$ be the stationary ergodic Gaussian random process with mean zero and the covariance $\beta(y) = E[g_\omega(y)g_\omega(0)]$. We consider the operator $L(b_\omega)$ in (1.1) with $b^1_\omega(x) = 0$ and $b^2_\omega(x) = g_\omega(x^1)$. For this operator, we introduce the integrated density of states $N(\lambda)$ by the same methods in [2], [7], [10], [18] and [28]. We use the condition $(\mathrm{P}1, \alpha, 1)$, $(\mathrm{E}1, \tau, 1)$, $(\mathrm{C}1, 1)$, $(\mathrm{H}1, 1)$ and $(\mathrm{H}2, 1)$ introduced in the last section.

Then the result is the following:

THEOREM 3.1. – (i) *We assume the condition* $(\mathrm{P}1, \alpha, 1)$ *for some* $\alpha > 1$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-(\alpha+1)/\{2(\alpha-1)\}} \leq \log N(\lambda) \leq -c'\lambda^{-p/2} \log \lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

(ii) *We assume the condition* $(\mathrm{E}1, \tau, 1)$ *for some* $\tau > 0$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-1/2} \log^{(1+\tau)/\tau} \lambda^{-1} \leq \log N(\lambda) \leq -c'\lambda^{-1/2} \log \lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

(iii) *We assume the condition* $(\mathrm{C}1, 1)$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-1/2} \log \lambda^{-1} \leq \log N(\lambda) \leq -c'\lambda^{-1/2} \log \lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

Remark 3.1. – The constants c and c' in Theorem 3.1 are same as in Theorem 2.1. Thus the leading term is not determined in Theorem 3.1 (iii) as stated in Remark 2.1.

THEOREM 3.2. – *Under the condition* $(\mathrm{H}1,1)$, *we have*

$$\varliminf_{t\to\infty}\frac{\eta(t)}{t}\log\widetilde{N}(t)>-\infty. \tag{3.1}$$

COROLLARY. – (i) *Under the condition* $(\mathrm{P}1,\alpha,1)$ *for some* $\alpha>1$, *we have*

$$\varliminf_{t\to\infty}\frac{\log\widetilde{N}(t)}{t^{(\alpha-1)/(3\alpha-1)}}>-\infty. \tag{3.2}$$

(ii) *Under the condition* $(\mathrm{E}1,\tau,p)$ *for some* $\tau>0$, *we have*

$$\varliminf_{t\to\infty}\frac{\log\widetilde{N}(t)}{t^{1/3}\log^{2(\tau+1)/(3\tau)}t}>-\infty. \tag{3.3}$$

(iii) *Under the condition* $(\mathrm{C}1,p)$, *we have*

$$\varliminf_{t\to\infty}\frac{\log\widetilde{N}(t)}{t^{1/3}\log^{2/3}t}>-\infty. \tag{3.4}$$

THEOREM 3.3. – *Under the condition* $(\mathrm{H}2,1)$, *we have*

$$\varlimsup_{\lambda\downarrow 0}\frac{\lambda^{1/2}\log N(\lambda)}{\log\lambda^{-1}}\le-\frac{\pi}{B3^{3/2}2^{1/2}}, \tag{3.5}$$

where

$$B=\int_{\mathbb{R}}|\beta(x)|dx\Big/\beta(0).$$

Theorem 3.1 is obtained from the corollary of Theorem 3.2 and Theorem 3.3. For the proof of Theorems 3.2 and 3.3, we prepare the following:

LEMMA 3.1. – (i) *It holds that*

$$\widetilde{N}(t)=\frac{1}{(2\pi)^{3/2}t^{1/2}}\int_{\mathbb{R}}d\xi E^{w\times g}\left[\exp\Big(-\frac{1}{2}\int_0^t(g_\omega(w(s))+\xi)^2ds\Big)\Big|w(t)=0\right], \tag{3.6}$$

where $E^{w\times g}$ *is the expectation with respect to the* 1*-dimensional Wiener process* w *and* g_ω.

(ii) *It holds that*

$$\widetilde{N}(t)=\frac{1}{2\pi t}E^{w\times g}\left[\exp\Big(-\frac{t}{2}V(g_\omega,w,t)\Big)\Big|w(t)=0\right], \tag{3.7}$$

where

$$V(g_\omega, w, t) = \int_0^t \frac{ds}{t} \left(g_\omega(w(s)) - \int_0^t g_\omega(w(r)) \frac{dr}{t} \right)^2$$

is the sample variance.

(iii) *It holds that*

$$\widetilde{N}(t) \geq \frac{1}{2\pi t} E^{w \times g} \left[\exp\left(-\frac{1}{2} \int_0^t g_\omega(w(s))^2 ds \right) \middle| w(t) = 0 \right]. \tag{3.8}$$

(iv) *It holds that*

$$\widetilde{N}(t) \leq \frac{(1 + t\beta(0))^{1/2}}{2\pi t} E^{w \times g} \left[\exp\left(-\frac{1}{2} \int_0^t g_\omega(w(s))^2 ds \right) \middle| w(t) = 0 \right]. \tag{3.9}$$

Proof. – (i) By Theorem 6.1 in Broderix, Hundertmark and Leschke [3], the integral kernel of the heat semigroup generated by the operator $L(b_\omega)$ is represented as

$$\begin{aligned} e^{-tL(b_\omega)}(x, y) =& E^w \left[\exp\left(i \int_0^t g_\omega(x^1 + w^1(s)) dw^2(s) \right) \middle| x + w(t) = y \right] \\ & \times \frac{1}{2\pi t} \exp\left(-\frac{|x - y|^2}{2t} \right), \end{aligned}$$

where E^w is the expectation with respect to the 2-dimensional Wiener process $w(s) = (w^1(s), w^2(s))$ (cf.[11], [25], [26], [27]). Then by using the Fourier transform we have

$$\begin{aligned} e^{-tL(b_\omega)}(0, 0) =& \frac{1}{(2\pi)^2 t} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} dy^2 \exp(i\xi y^2) E^w \left[\exp\left(i \int_0^t g_\omega(w^1(s)) dw^2(s) \right) \right. \\ & \left. \middle| w^1(t) = 0, w^2(t) = y^2 \right] \exp\left(-\frac{(y^2)^2}{2t} \right), \\ =& \frac{1}{(2\pi)^{3/2} t^{1/2}} \int_{\mathbb{R}} d\xi E^w \left[\exp\left(i \int_0^t (g_\omega^j(w^1(s)) + \xi) dw^2(s) \right) \middle| w^1(t) = 0 \right]. \end{aligned}$$

By calculating the expectation with respect to $w^2(s)$ and taking the expectation with respect to b_ω, we obtain (3.6).

(ii) and (iii) are easily obtained from (i) and (ii), respectively.

(iv) We rewrite (3.6) as

$$\widetilde{N}(t) = \frac{1}{(2\pi)^{3/2} t^{1/2}} \int_{\mathbb{R}} d\xi E^w[\exp(-t\Phi(\mu_w^t, t, \xi)) | w(t) = 0], \tag{3.10}$$

where

$$\Phi(\mu,t,\xi)=-\frac{1}{t}\log E\left[\exp\left(-\frac{t}{2}\int_{\mathbb{R}}(g_\omega(x)+\xi)^2\mu(dx)\right)\right] \tag{3.11}$$

for any probability measure μ on $\mathbb{R}$ and μ_w^t is the empirical distribution of the Wiener process: $\{w(s):0\le s\le t\}\mapsto\mu_w^t(B):=t^{-1}(\text{ the measure of }\{s:0\le s\le t, w(s)\in B\})$ for any Borel set B in $\mathbb{R}$. If we define an operator $B(\mu)$ on $L_2(\mathbb{R},d\mu)$ by

$$\{B(\mu)\varphi\}(x)=\int_{\mathbb{R}}\beta(x-y)\varphi(y)\mu(dy)$$

for each probability measure μ on $\mathbb{R}$, then we easily see that $B(\mu)$ is a Hilbert-Schmidt operator. Accordingly we take a complete orthonormal system $\{\varphi_n\}_{n\ge1}$ of $L_2(\mathbb{R},d\mu)$ consisting of the eigenfunctions of $B(\mu)$. Let $\{\beta_n\}_{n\ge1}$ be the corresponding eigenvalues. Then by the same method as in the proof of Lemma 2.3 (i), we can write as

$$\Phi(\mu,t,\xi)=\frac{1}{2t}\sum_{n=1}^{\infty}\log(1+t\beta_n)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\alpha_n^2}{1+t\beta_n}, \tag{3.12}$$

where $\alpha_n=(\xi,\varphi_n)_\mu$ and $(\cdot,\cdot)_\mu$ is the inner product of the space $L_2(\mathbb{R},d\mu)$. The first term is equal to $\Phi(\mu,t,0)$ and the second term is estimated as

$$\Phi(\mu,t,\xi)-\Phi(\mu,t,0)\ge\frac{1}{2(1+t\beta(0))}\sum_{n=1}^{\infty}\alpha_n^2=\frac{|\xi|^2}{2(1+t\beta(0))}$$

since the operator norm of $B(\mu)$ is estimated as $\|B(\mu)\|\le\beta(0)$. By applying this to (3.10), we obtain (3.9). □

Now Theorem 3.2 and its corollary are easily obtained from Lemma 3.1 (iii), Theorem 2.2 and its corollary. To prove Theorem 3.3, we further use the following:

LEMMA 3.2. – (i) *If*

$$\overline{\lim_{t\to\infty}}\frac{\log\widetilde{N}(t)}{t^A\log^B t}\le -C \tag{3.13}$$

for some B, $C > 0$ and $0 < A < 1$, then we have

$$\overline{\lim_{\lambda \downarrow 0}} \frac{\lambda^{A'}}{\log^{B'}(1/\lambda)} \log N(\lambda) \le -C', \tag{3.14}$$

where $A' = A/(1-A)$, $B' = B/(1-A)$ and $C' = C^{1/(1-A)} A^{A/(1-A)} (1-A)^{1-B/(1-A)}$.

(ii) *Conversely if* (3.14) *holds, then* (3.13) *also holds.*

Proof. – The proof of (i) is almost same with that of Fukushima [9] as in Lemma 2.9. Now if (3.14) holds, then for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that $N(\lambda) \le \exp(-(C' - \varepsilon)\lambda^{-A'} \log^{B'} \lambda^{-1})$ for any $0 < \lambda < \lambda_0$. Acoordingly we devide as

$$\widetilde{N}(t) = \int_0^{\lambda_0} e^{-t\lambda} dN(\lambda) + \int_{\lambda_0}^{\infty} e^{-t\lambda} dN(\lambda) =: \widetilde{N}_1(t) + \widetilde{N}_2(t)$$

and estimate as

$$\widetilde{N}_2(t) \le e^{-t(1-\varepsilon)\lambda_0} \widetilde{N}(\varepsilon t) \le e^{-t(1-\varepsilon)\lambda_0} \Big/ (2\pi\varepsilon t)$$

and

$$\widetilde{N}_1(t) \le t \int_0^{\lambda_0} e^{-g(\lambda)} d\lambda + e^{-g(\lambda_0)},$$

where $g(\lambda) := t\lambda + (C' - \varepsilon)\lambda^{-A'} \log^{B'} \lambda^{-1}$. For each $t > 0$, we see that $\min_{0 \le \lambda \le \lambda_0} g(\lambda) = g(\lambda(t))$ where $\lambda(t)$ is the solution of

$$\lambda(t) = (C' - \varepsilon)^{1/(A'+1)} t^{-1/(A'+1)} (B' \log^{B'-1} \lambda(t)^{-1} + A' \log^{B'} \lambda(t)^{-1})^{1/(A'+1)}.$$

By the iteration, we have

$$\min_{0 \le \lambda \le \lambda_0} g(\lambda) \sim t^A \log^B t C(\varepsilon)$$

as $t \to \infty$, where $C(\varepsilon) = (C' - \varepsilon)^{1/(A'+1)} (A'+1)^{-B} ((A')^{1/(A'+1)} + (A')^{-A})$. From this we obtain (3.13). □

4. CASE II

Let $\{f_\omega(x), x \in \mathbb{R}^2\}$ be the real valued stationary ergodic Gaussian random field with mean zero and the covariance $\beta(x) = E[f_\omega(x) f_\omega(0)]$. Under the integrability of $\beta(x)$ and $\widehat{\beta}(\xi)(1+|\xi|^2)$, $\{\nabla f_\omega(x), x \in \mathbb{R}^2\}$ is the stationary ergodic Gaussian random field with mean zero and the covariance

$$E\left[\frac{\partial f_\omega}{\partial x^j}(x)\frac{\partial f_\omega}{\partial x^k}(0)\right] = -\frac{\partial^2 \beta}{\partial x^j \partial x^k}(x) \tag{4.1}$$

for any $j, k = 1, 2$. Then, for the operator $L(b_\omega)$ in (1.1) with

$$b^1_\omega(x) = \frac{\partial f_\omega}{\partial x^2}(x) \quad \text{and} \quad b^2_\omega(x) = -\frac{\partial f_\omega}{\partial x^1}(x), \tag{4.2}$$

we have the integrated density of states $N(\lambda)$ as in the last section.

We introduce the following conditions: for $\alpha > 4$ and $\tau > 0$,

(P2, α) $\widehat{\beta} \in C^\infty$ and there exists $R > 0$ such that $\widehat{\beta}(\xi) = |\xi|^{-\alpha}$ for $|\xi| \geq R$.

(E2, τ) $\widehat{\beta} \in C^\infty$ and there exists $R > 0$ such that $\widehat{\beta}(\xi) = \exp(-|\xi|^\tau)$ for $|\xi| \geq R$.

(C2) $0 \neq \widehat{\beta} \in C_0^\infty$.

Then the result is the following:

THEOREM 4.1. – (i) *We assume the condition* (P2, α) *for some* $\alpha > 4$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-(\alpha-2)/(\alpha-4)} \leq \log N(\lambda) \leq -c'\lambda^{-1/2}\log\lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

(ii) *We assume the condition* (E2, τ) *for some* $\tau > 0$. *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-1}\log^{(\tau+2)/\tau}\lambda^{-1} \leq \log N(\lambda) \leq -c'\lambda^{-1/2}\log\lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

(iii) *We assume the condition* (C2). *Then for small enough* $\varepsilon > 0$, *there exist* c, $c' > 0$ *such that*

$$-c\lambda^{-1}\log\lambda^{-1} \le \log N(\lambda) \le -c'\lambda^{-1/2}\log\lambda^{-1}$$

for any $0 < \lambda < \varepsilon$.

As in Theorems 2.1 and 3.1, this theorem is obtained from the corollary of Theorem 4.2 and Therem 4.3 below. Therefore the condtions $(\mathrm{P2},\alpha)$, $(\mathrm{E2},\tau)$ and (C2) can be weaken. To state Theorem 4.2, we introduce the following condition:

(H3) (i) $\sup_\xi \widehat{\beta}(\xi)(1+|\xi|)^{\gamma+4} < \infty$ for some $\gamma > 0$;

(ii) β is C^2 and $\sup_x |x|^n \left|\dfrac{\partial^2\beta(x)}{\partial x^j \partial x^k}\right| < \infty$ for any $j,k = 1,2$ and some $n > 4(\gamma^{-1}+1)$;

(iii) $\lim_{t\to\infty} \Phi(ct)/\Phi(t) = c^\kappa$ for any $c > 0$ and some $\kappa \in [0,1)$,

where

$$\Phi(t) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\xi\, \mathrm{Tr}[\log(I + tM(\xi,\widehat{\beta})], \tag{4.3}$$

$M(\xi,\widehat{\beta}) = (M_{jk}(\xi,\widehat{\beta}))_{1\le j,k\le 2}$ and

$$\begin{aligned} M_{jk}(\xi,\widehat{\beta}) &= \int_{\mathbb{R}^2} e^{-i\xi\cdot x} E[b^j_\omega(x) b^k_\omega(0)]dx \\ &= \left(\sum_{\ell>j,m>k}\xi^\ell\xi^m + \sum_{\ell<j,m<k}\xi^\ell\xi^m - \sum_{\ell>j,m<k}\xi^\ell\xi^m - \sum_{\ell<j,m>k}\xi^\ell\xi^m\right)\widehat{\beta}(\xi). \end{aligned}$$

Let $\eta(t)$, $t > 0$, be the solution of (2.3) where $\Phi(t)$ is replaced by that defined in (4.3).

Then we have the following:

THEOREM 4.2. – *Under the condition* (H3), *we have*

$$\varliminf_{t\to\infty} \frac{\eta(t)}{t}\log\widetilde{N}(t) > -\infty.$$

COROLLARY. – (i) *Under the condition* (P2, α) *for some* $\alpha > 4$, *we have*

$$\varliminf_{t\to\infty} \frac{\log \widetilde{N}(t)}{t^{(\alpha-2)/(2\alpha-6)}} > -\infty.$$

(ii) *Under the condition* (E2, τ) *for some* $\tau > 0$, *we have*

$$\varliminf_{t\to\infty} \frac{\log \widetilde{N}(t)}{t^{1/2}\log^{(\tau+2)/(2\tau)} t} > -\infty.$$

(iii) *Under the condition* (C2), *we have*

$$\varliminf_{t\to\infty} \frac{\log \widetilde{N}(t)}{t^{1/2}\log^{1/2} t} > -\infty.$$

To state Theorem 4.3, we introduce the following condition:

(H4) (i) $\widehat{\beta}(\xi)$ is C^4;

(ii) $\sup_{\theta\in S^1} \int_0^\infty d\zeta(1+\zeta)^3 \left|\left(\frac{\partial}{\partial\zeta}\right)^k \widehat{\beta}(\zeta,\theta)\right| < \infty$ for $k = 0, 1, 2$;

(iii) $\inf_{\theta\in S^1} \int_0^\infty d\zeta\zeta^3\widehat{\beta}(\zeta,\theta) > 0$,

where (ζ,θ), $\zeta \geq 0$, $\theta \in S^1$, is the polar coordinate of ξ.

Then we have the following:

THEOREM 4.3. – *Under the condition* (H4), *we have*

$$\varlimsup_{\lambda\downarrow 0} \frac{\lambda^{1/2}\log N(\lambda)}{\log\lambda^{-1}} \leq -\frac{\pi}{3^{3/2}B}, \tag{4.4}$$

where

$$B = \sup_{\theta\in S^1} \int_{\mathbb{R}} dx^1 \left| \int_0^\infty d\zeta \cos(\zeta x^1)\zeta^3\widehat{\beta}(\zeta,\theta)\right| \Big/ \int_0^\infty d\zeta\zeta^3\widehat{\beta}(\zeta,\theta).$$

To prove Theorem 4.3, we prepare the following:

LEMMA 4.1. – *Under the condition* (H4), *it holds that*

$$\widetilde{N}(t) \leq \frac{1}{2\pi}\int_{S^1} \widetilde{N}_\theta(t)d\theta, \tag{4.5}$$

where

$$\widetilde{N}_\theta(t) = E^{w\times g}\left[\exp\left(i\int_0^t g_{\theta,\omega}(w^1(s))dw^2(s)\right)\middle| w(t)=0\right]\frac{1}{2\pi t} \tag{4.6}$$

is the Laplace-Stieltjes transform of the integrated density of states for the Schrödinger operator $L(b_\omega)$ in (1.1) with $b^1_\omega = 0$ and $b^2_\omega = g_{\theta,\omega}(x^1)$, $\{g_{\theta,\omega}(x^1), x^1\in\mathbb{R}\}$ is the stationary ergodic Gaussian random field with mean zero and the covariance

$$\beta_\theta(x^1) = \frac{1}{2\pi}\int_0^\infty d\zeta\zeta^3\widehat{\beta}(\zeta,\theta)\cos(\zeta x^1), \tag{4.7}$$

and $E^{w\times g}$ is the expectation with respect to $w(s) = (w^1(s), w^2(s))$ and $g_{\theta,\omega}(x^1)$.

Proof. – By Theorem 6.1 in Broderix, Hundertmark and Leschke [3] and [28], we have

$$\widetilde{N}(t) = E^{w\times f}[e^{i\Psi(w,\omega)}|w(t)=0]\frac{1}{(2\pi t)^{d/2}}, \tag{4.8}$$

where

$$\Psi(w,\omega) = \int_0^t\left(\frac{\partial f_\omega}{\partial x^2}(w(s))dw^1(s) - \frac{\partial f_\omega}{\partial x^1}(w(s))dw^2(s)\right)$$

and $E^{w\times f}$ is the expectation with respect to $w(s)$ and f_ω. By calculating the expectation with respect to f_ω, we have

$$\widetilde{N}(t) = E^w\left[\exp\left(-\frac{1}{2}E^f[\Psi(w,\omega)^2]\right)\middle| w(t)=0\right]\frac{1}{2\pi t}, \tag{4.9}$$

where E^f is the expectation with respect to f_ω.

In terms of the covariance β, we can write as

$$E^f[\Psi(w,\omega)^2] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\xi\widehat{\beta}(\xi)\left|\int_0^t e^{i\xi\cdot w(s)}d\xi\wedge w(s)\right|^2, \tag{4.10}$$

where $\xi\wedge w(s) = \xi^2 w^1(s) - \xi^1 w^2(s)$. In fact, if we take a division Δ of the interval $[0,t]$ given by $0 = t_0 < t_1 < \cdots < t_n = t$, then we have $I(\Delta) = J(\Delta)$, where

$$I(\Delta) := E^f\left[\left\{\sum_{a=0}^{n-1}\left(\frac{\partial f_\omega}{\partial x^2}(w(t_a))\Delta w^1(t_a) - \frac{\partial f_\omega}{\partial x^1}(w(t_a))\Delta w^2(t_a)\right)\right\}^2\right],$$

$$J(\Delta) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d\xi\widehat{\beta}(\xi)\left|\sum_{a=0}^{n-1} e^{i\xi\cdot w(t_a)}\Delta\xi\wedge w(t_a)\right|^2$$

and $\Delta w(t_a) = w(t_{a+1}) - w(t_a)$. By the definition of the Itô integral, we see that $I(\Delta)$ and $J(\Delta)$ converge the left and right hand sides of (4.10) respectively in $L_1(P^W(\cdot|w(t)=0))$ as $|\Delta| := \max_a |t_{a+1} - t_a| \to 0$ where P^W is the Wiener measure.

We now use Jensen's inequality in (4.9) as follows:

$$\widetilde{N}(t) \le \frac{1}{2\pi} \int_{S^1} \breve{N}_\theta(t) d\theta, \tag{4.11}$$

where

$$\breve{N}_\theta(t) := E\Big[\exp\Big(-\frac{1}{2} \int_0^\infty d\zeta \cdot \zeta \widehat{\beta}(\zeta, \theta) \Big| \int_0^t e^{i\xi \cdot w(s)} d\xi \wedge w(s) \Big|^2 \Big| w(t) = 0 \Big] \frac{1}{2\pi t}.$$

By the $O(2)$-invariance, the process

$$\Big(\frac{\xi \cdot w(s)}{\zeta}, \frac{\xi \wedge w(s)}{\zeta} \Big)_s$$

is a 2-dimensional Wiener process independent of ζ. Thus we have

$$\breve{N}_\theta(t) = E\Big[\exp\Big(-\frac{1}{2} I(\theta, w) \Big) \Big| w(t) = 0 \Big] \frac{1}{2\pi t},$$

where

$$I(\theta, w) = \frac{1}{2\pi} \int_0^\infty d\zeta \cdot \zeta^3 \widehat{\beta}(\zeta, \theta) \Big| \int_0^t e^{i\zeta w^1(s)} dw^2(s) \Big|^2.$$

By the same method as in (4.10), we have

$$E^g[G_\theta(w, \omega)^2] = I(\theta, w)$$

for any $\theta \in S^1$, where

$$G_\theta(w, \omega) = \int_0^t g_{\theta,\omega}(w^1(s)) dw^2(s)$$

and E^g is the expectation with respect to $g_{\theta,\omega}$. Now we see that $\breve{N}_\theta(t) = \widetilde{N}_\theta(t)$. □

Proof of Theorem 4.3. – By Lemma 4.1 and (3.9), we have

$$\widetilde{N}(t) \le \sup_{\theta \in S^1} \sqrt{\frac{1 + t\beta_\theta(0)}{2\pi t}} \widetilde{N_{0,\theta}}(t) =: \breve{N}(t)$$

for any $(j,k) \in S$, where $\widetilde{N_{0,\theta}}(t)$ is the Laplace-Stieltjes transform of the integrated density of states for the operator $H(V_\omega)$ in (1.2) on $L_2(\mathbb{R})$ with $V_\omega(x) = g_{\theta,\omega}(x)^2/2$. By (2.8) we see that, for any small $\varepsilon > 0$ and any $0 < s < 1$, there exist c, $\lambda_0 > 0$ such that

$$N_{0,\theta}(\lambda) \le c \exp\left\{ -\left(\frac{2^{1/2}\pi}{B3^{3/2}} - \varepsilon\right)\frac{1}{\lambda^{1/2}} \log\sqrt{\frac{\beta_0}{\lambda^s}}\right\}$$

for any $0 < \lambda < \lambda_0$, where $N_{0,\theta}(\lambda)$ is the integrated density of states for the operator $H(V_\omega)$ in (1.2) on $L_2(\mathbb{R})$ with $V_\omega(x) = g_{\theta,\omega}(x)^2/2$ and $\beta_0 = \sup_{\theta\in S^1}\beta_\theta(0)$. By the same method as in the proof of Lemma 3.2 (ii), we have

$$\overline{\lim_{t\to\infty}} \frac{\log \breve{N}(t)}{t^{1/3}\log^{2/3} t} \le -\left(\frac{\pi^2}{2}\right)^{1/3}\frac{1}{(3B)^{2/3}}.$$

From this and Lemma 3.2 (i), we obtain (4.4). □

Proof of Theorem 4.2. – We use the scaling property of the Wiener process to obtain

$$e^{-tL(b_\omega)}(0,0) = \exp\left(-\frac{t}{\eta(t)} L(\sqrt{\eta(t)}b_\omega(\sqrt{\eta(t)}\cdot))\right)(0,0)\frac{1}{\eta(t)^{d/2}},$$

where $\eta(t)$ is the function defined in (2.3). As in (2.20), by Theorem (9.5) of [21], we have

$$\widetilde{N}(t) \ge \frac{1}{\eta(t)|\operatorname{supp}\phi|} E^f\left[\exp\left(-\frac{t}{\eta(t)}(L(\sqrt{\eta(t)}b_\omega(\sqrt{\eta(t)}\cdot))\phi,\phi)\right)\right]$$

for any $\phi \in C_0^\infty(\mathbb{R}^2 \to \mathbb{C})$. By restricting the function ϕ to be of real valued, we have

$$\widetilde{N}(t) \ge \frac{1}{\eta(t)|\operatorname{supp}\phi|} \exp\left\{ -\frac{t}{\eta(t)}\left(\frac{\|\nabla\phi\|^2}{2} + \Phi(\phi^2,t)\right)\right\}$$

for any $\phi \in C_0^\infty(\mathbb{R}^2 \to \mathbb{R})$, where

$$\Phi(\phi^2,t) = -\frac{\eta(t)}{t}\log E^f\left[\exp\left(-\frac{t}{2}\int_{\mathbb{R}^2} |b_\omega(\sqrt{\eta(t)}x)|^2\phi^2(x)dx\right)\right].$$

Thus we can complete the proof as in the proof of Theorem 2.2. □

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