LIFSCHITZ TAIL OF A RANDOM PAULI HAMILTONIAN WITH AN ANOMALOUS MAGNETIC MOMENT

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1. Introduction

Let $A = (A^j_\omega(x))^2_{j=1}$ $(\omega \in \Omega, x \in \mathbb{R}^2)$ be an $\mathbb{R}^2$-valued random field such that

(i) the sample path $\mathbb{R}^2 \ni x \mapsto A_\omega(x) = (A^j_\omega(x))^2_{j=1}$ is differentiable;

(ii) $B_\omega(x) := \partial_1 A^2_\omega(x) - \partial_2 A^1_\omega(x)$ is a Gaussian random field;

(iii) $B_\omega(x)$ is stationary and ergodic with respect to the shift in the variable x,

where $\partial_j = \partial/(\partial x^j)$. For each ω, regarding $x \mapsto A_\omega(x)$ and $x \mapsto B_\omega(x)$ as a vector potential and the magnetic field, respectively, we consider a Pauli Hamiltonian defined by

$$H^{g,(\pm)}_\omega = \sum_{j=1}^{2} (i\partial_j + A^j_\omega(x))^2 \pm \frac{g}{2} B_\omega(x) \qquad (i = \sqrt{-1}), \tag{1.1}$$

where $g > 2$ is a fixed constant called the anomalous magnetic moment (cf. [2]).

In this paper we show the following:

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Theorem 1.1. *The integrated density of states $N^{g,(\pm)}(\lambda)$ of the random operator $H_\omega^{g,(\pm)}$ satisfies*

$$\lim_{\lambda\downarrow-\infty}\frac{1}{\lambda^2}\log N^{g,(\pm)}(\lambda)=\frac{-2}{(g-2)^2\beta(0)}, \tag{1.2}$$

where $\beta(x)=E[B(x)B(0)]-E[B(0)]^2$ is the covariance function.

Moreover we generalize this theorem to the case where the dimension is arbitrary and the conditions are assumed only on the magnetic field. The main theorem is Theorem 2.1 below.

The first related result was obtained by Pastur [22], Nakao [21] and Kirsch and Martinelli [15]: they showed that the integrated density of states $N_0(\lambda)$ of the random Schrödinger operator $-\sum_{j=1}^d\partial_j^2+V_\omega(x)$ satisfies

$$\lim_{\lambda\downarrow-\infty}\frac{1}{\lambda^2}\log N_0(\lambda)=\frac{-1}{8\gamma(0)}, \tag{1.3}$$

where $V_\omega(x)$ is the Gaussian random field which is stationary and ergodic in $x\in\mathbb{R}^d$, and $\gamma(x)$ is its covariance function. This result was extended by Broderix, Heldt and Leschke [3] (see also Broderix, Hundertmark and Leschke [4]) to the operator $\sum_{j=1}^d(i\partial_j+A_0^j(x))^2+V_\omega(x)$, where $A_0=(A_0^j(x))_{1\le j\le d}$ is a vector potential corresponding to a deterministic constant magnetic field. Moreover this result was extended by Matsumoto [18] and the author [26] to the operator $\sum_{j=1}^d(i\partial_j+A_\omega^j(x))^2+V_\omega(x)$ with a random magnetic field. In [18] and [26], $A=(A_\omega^j(x))_{1\le j\le d}$ and $V_\omega(x)$ were assumed to be statistically independent. Instead of this condition, the following condition on the uniform boundedness of the spectrum was also considered in [26]: there exists λ_δ for sufficiently small $\delta>0$ such that

$$P\Big(\inf\operatorname{spec}\sum_{j=1}^d(i\partial_j+A_\omega^j(x))^2_{\{|x|<\delta\}}<\lambda_\delta\Big)=1, \tag{1.4}$$

where $\operatorname{spec}\sum_{j=1}^d(i\partial_j+A_\omega^j(x))^2_{\{|x|<\delta\}}$ is the spectral set of the operator $\sum_{j=1}^d(i\partial_j+A_\omega^j(x))^2$ restricted to $\{|x|<\delta\}$ by the Dirichlet boundary condition. This condition is satisfied,

for example, when the magnetic field is bounded in ω and x. Under these conditions, the result (1.3) always holds and the leading term is determined only by the scalar potential $V_\omega(x)$. Such an asymptotics is called the classical asymptotics and is distinguished from the quantum asymptotics where the kinetic part also affects the asymptotics (cf. [23]).

Now the Pauli Hamiltonian $H_\omega^{g,(\pm)}$ is a typical operator for which the kinetic part and the scalar potential are closely related. Theorem 1.1 shows that the leading term is determined only by the difference $\pm\{(g-2)/2\}B_\omega(x)$ between $H_\omega^{g,(\pm)}$ and the Pauli Hamiltonian

$$H_\omega^{2,(\pm)} = \sum_{j=1}^{2}(i\partial_j + A_\omega^j(x))^2 \pm B_\omega(x)$$

with the nonanomalous magnetic moment $g = 2$. We prove this asymptotics by modifying the methods for the case (1.4) in [26] so that the Hamiltonian $H_\omega^{2,(\pm)}$ takes the place of the operator $\sum_{j=1}^{d}(i\partial_j + A_\omega^j(x))^2$ in [26]. The key point is that the infimum of the spectrum of the Hamiltonian $H_\omega^{2,(\pm)}$ is nonnegative and small (cf. Lemma 4.1 below). Therefore the negative spectrum of $H_\omega^{g,(\pm)}$ appears only when $g > 2$, and its distribution is determined mainly by the effect from the part $\pm\{(g-2)/2\}B_\omega(x)$ in the Hamiltonian. In the proof we use only the techniques for the classical asymptotics.

For the problem on the Lifschitz tail, the case of $0 < g \leq 2$ seems to be more complicated. For the nonanomalous magnetic moment, $g = 2$, Casher and Neuberger [8] showed

$$N^{2,(\pm)}(\lambda) \sim c\sqrt{\lambda} \qquad \text{as } \lambda \downarrow 0$$

in the case where B_ω is a white noise, and Comtet, Georges and Le Doussal [9] represented $N^{2,(\pm)}(\lambda)$ in terms of Bessel functions in the case where $A_\omega^1(x)$ is a white noise depending only on x^1 and $A_\omega^2 = 0$, by physical arguments. In these cases, $N^{2,(\pm)}(\lambda)$ decays slower than that for the free Hamiltonian $-\sum_{j=1}^{2}\partial_j^2$. This behavior is contrary to those of other random Schrödinger operators (cf.[7], [23]). To the knowledge of the author, there are no other results on the Lifschitz tail for both the cases $g > 2$, $g = 2$ and $0 < g < 2$.

Our result is equivalent to a result on the large time asymptotics of a stochastic oscillatory integral. In fact, the Tauberian argument shows that the equation (1.2) is equivalent to

$$\lim_{t\uparrow\infty}\frac{1}{t^2}\log\widetilde{N^{g,(\pm)}}(t)=\frac{(g-2)^2}{8}\beta(0), \tag{1.5}$$

where

$$\widetilde{N^{g,(\pm)}}(t):=\int_{-\infty}^{\infty}e^{-t\lambda}dN^{g,(\pm)}(\lambda)$$

is the Laplace-Stieltjes transform of $N^{g,(\pm)}(\lambda)$. This function $\widetilde{N^{g,(\pm)}}(t)$ is represented by the following oscillatory integral:

$$\widetilde{N^{g,(\pm)}}(t)=\frac{1}{4\pi t}E^{(\omega,w)}\Bigg[\exp\Big(-i\sum_{j=1}^{2}\int_0^{2t}A^j_\omega(w(s))\circ dw^j(s)$$

$$\mp\frac{g}{4}\int_0^{2t}B_\omega(w(s))ds\Big)\Bigg|w(2t)=0\Bigg], \tag{1.6}$$

where $E^{(\omega,w)}$ is the expectation with respect to ω and a two-dimensional Wiener process $w=(w^1(t),w^2(t))$, independent of ω, starting at 0, and $\circ dw^j(s)$ is the Stratonovich stochastic differential (cf. [7], [14], [23]). Now the formula (1.5) indicates that the divergence as $t\to\infty$ by the term

$$\exp\Big(\mp\frac{g}{4}\int_0^{2t}B_\omega(w(s))ds\Big)$$

under the expectation on the right hand side of (1.6) is partially canceled by the effect of the oscillation of the term

$$\exp\Big(-i\sum_{j=1}^{2}\int_0^{2t}A^j_\omega(w(s))\circ dw^j(s)\Big)$$

and that the leading term is same with that of the following integral of a nonnegative functional:

$$\frac{1}{4\pi t}E^{(\omega,w)}\Bigg[\exp\Big(\mp\frac{g-2}{4}\int_0^{2t}B_\omega(w(s))ds\Big)\Bigg|w(2t)=0\Bigg]. \tag{1.7}$$

(1.7) is the Laplace-Stieltjes transform of the integrated density of states of the operator $-\sum_{j=1}^{2}\partial_j^2 \pm \{(g-2)/2\}B_\omega$, which is the operator considered in [22], [21] and [15].

The organization of this paper is as follows: in Section 2 we state the main theorem, Theorem 2.1, in the general framework. The proof of this theorem is reduced to those of two propositions, Propositions 2.1 and 2.2. In Sections 3 and 4 we prove Proposition 2.1 and in Section 5 we prove Proposition 2.2. In Section 3 we also prove the existence of the integrated density of states (Lemma 3.1) and in Section 5 we give several examples where the operator norm of the covariance appearing in Theorem 2.1 and Proposition 2.2 can be written explicitly.

2. **Main results**

In this section we state our results in the general framework.

Let $d \geq 2$ be an integer and set $D = d(d-1)/2$. Let $\mathbb{B} = (B_\omega^{(j,k)}(x))_{1\leq j<k\leq d}$ ($\omega \in \Omega, x \in \mathbb{R}^d$) be an $\mathbb{R}^D$-valued Gaussian random field such that

(i) $\mathbb{B}$ is stationary and ergodic with respect to the shift in the variable x;

(ii) the sample path $x \mapsto \mathbb{B}_\omega(x)$ is continuous;

(iii) the corresponding form is closed, that is, $d\{\sum_{j<k} B_\omega^{(j,k)}(x)dx^j \wedge dx^k\} = 0$ in the sense of distributions.

We denote the mean by $M^{(j,k)} = E[B_\omega^{(j,k)}(0)]$ and the covariance function by

$$\beta_{(\ell,m)}^{(j,k)}(x) = E[(B_\omega^{(j,k)}(x) - M^{(j,k)})(B_\omega^{(\ell,m)}(0) - M^{(\ell,m)})]. \tag{2.1}$$

We can give a vector potential $A_\omega = (A_\omega^j(x))_{1\leq j\leq d}$ by

$$A_\omega^j(x) := \int_0^1 \sum_{k=1}^d B_{\omega,j}^k(tx)tx^k dt \tag{2.2}$$

(cf. (2.13.2) in [6]). In fact $A_\omega(x)$ is continuous in x and satisfies

$$d\Big\{\sum_{j=1}^d A_\omega^j(x)dx^j\Big\} = \sum_{j<k} B_\omega^{(j,k)}(x)dx^j \wedge dx^k$$

in the sense of distributions, where, for each vector $\mathbb{X} = (X^{(j,k)})_{1\leq j<k\leq d}$ in $\mathbb{R}^D$, $X = (X_k^j)_{1\leq j,k\leq d}$ is the corresponding real skew-symmetric matrix given by the relation

$$X_k^j = \begin{cases} X^{(j,k)} & \text{if } j < k, \\ -X^{(k,j)} & \text{if } j > k, \\ 0 & \text{if } j = k. \end{cases} \tag{2.3}$$

We always denote the index for the row by the superscript and the index for the column by the subscript.

Let $\gamma_1, \gamma_2, \dots, \gamma_d$ be Hermitian matrices acting on a finite dimensional complex Hilbert space V and satisfying the commutation relation

$$\gamma_j\gamma_k + \gamma_k\gamma_j = \begin{cases} 2I & \text{if } j = k, \\ O & \text{if } j \neq k, \end{cases}$$

where I and O are the identity and the zero matrices, respectively (see e.g. [10] §12.2). Then the random Pauli Hamiltonian with the magnetic moment $g > 2$ and the magnetic field B_ω is a formal operator

$$H_\omega = I\sum_{j=1}^{d}(i\partial_j + A_\omega^j(x))^2 + \frac{g}{2}\sum_{j<k} i\gamma_j\gamma_k B_\omega^{(j,k)}(x) \tag{2.4}$$

acting on V-valued functions.

REMARK 2.1. The operator H_ω in (2.4) is defined as a self-adjoint operator on $L^2(\mathbb{R}^d \to V)$ if we can take a vector potential $\widetilde{A}_\omega = (\widetilde{A}_\omega^j(x))_{1\le j\le d}$ so that $\widetilde{A}_\omega^j \in L^4_{loc}(\mathbb{R}^d)$ and $d^*(\sum_{j=1}^d \widetilde{A}_\omega^j dx^j) \in L^2_{loc}(\mathbb{R}^d)$, where d^* is the formal adjoint of d. In this case, the operator $\widetilde{H}_\omega$ obtained by replacing A_ω^j by $\widetilde{A}_\omega^j$ in (2.4) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d \to V)$. In fact, by the conditions (i), (ii) of the random field $\mathbb{B}$, Fernique's theorem [12] and Borel-Cantelli's lemma, we have

$$|B_\omega^{(j,k)}(x)| \le c_{\varepsilon,\omega}(1 + |x|^\varepsilon)$$

for any $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Then the self-adjointness is proven by the theory of Leinfelder and Simader [17]. To the knowledge of the author, the operator $\widetilde{H}_\omega$ without the condition $d^*(\sum_{j=1}^d \widetilde{A}_\omega^j dx^j) \in L^2_{loc}(\mathbb{R}^d)$ is not defined as a self-adjoint operator. In (3.2) below, we take vector potentials satisfying this condition only on bounded regions. It is enough for our purpose.

We will consider its integrated density of states $N(\lambda)$, $\lambda \in \mathbb{R}$, defined in (2.6) below: for each $R > 0$, let $\Lambda(R) := (-R/2, R/2)^d$ and $H_{\omega,R}$ be the self-adjoint operator on the space

$L^2(\Lambda(R) \to V)$ with the Dirichlet boundary condition: $H_{\omega,R}$ is the self-adjoint operator corresponding to the closure of the quadratic form

$$q_\omega(\Phi,\Psi) := \sum_{j=1}^{d} \langle (i\partial_j + A_\omega^j(x))\Phi(x), (i\partial_j + A_\omega^j(x))\Psi(x)\rangle_V + \langle \Phi(x), \frac{g}{2}\sum_{j<k} i\gamma_j\gamma_k B_\omega^{(j,k)}(x)\Psi(x)\rangle_V \tag{2.5}$$

with the domain $C_0^\infty(\Lambda(R) \to V)$, where $\langle\cdot,\cdot\rangle_V$ is the Hermitian inner product of the space $L^2(\Lambda(R) \to V)$. Then the spectra of $H_{\omega,R}$ are purely discrete. Let $N_{\omega,R}(\lambda)$ be its counting function, that is the number of eigenvalues of $H_{\omega,R}$ not exceeding λ. Then, by the usual method by Kirsch-Martinelli for the usual random Schrödinger operator (cf. [7], [15]), we can show the existence of a deterministic increasing function $N(\lambda)$ such that

$$R^{-d}N_{\omega,R}(\lambda) \longrightarrow N(\lambda) \qquad \text{as } R \to \infty \tag{2.6}$$

for any point of continuity of $N(\lambda)$ and almost all ω. Moreover we can show the existence of the Laplace-Stieltjes transform of $N(\lambda)$. For the details, see Lemma 3.1 below.

Remark 2.2. Recently the definition of the integrated density of states of the random Schrödinger operator with magnetic field was proven to be independent of the boundary condition by Doi, Iwatsuka and Mine [11], Nakamura [20] and Hupfer, Leschke, Müller and Warzel [13] under several conditions on the scalar potential and the magnetic field. This independence also holds in our case: if we define an operator $\widehat{H}_{\omega,R}$ with a Neumann boundary condition by the corresponding self-adjoint operator to the quadratic form $q_\omega(\Phi,\Psi)$ with the domain

$$W^{1,2}(\Lambda(R) \to V) := \{\Phi \in L^2(\Lambda(R) \to V) : \partial_j\Phi \in L^2(\Lambda(R) \to V) \text{ for } j = 1, 2, \dots, d\}$$

and denote the corresponding counting function of the spectra by $\widehat{N}_{\omega,R}(\lambda)$, then we have

$$R^{-d}\widehat{N}_{\omega,R}(\lambda) \longrightarrow N(\lambda) \qquad \text{as } R \to \infty \tag{2.7}$$

for any point of continuity of $N(\lambda)$ and almost all ω. In fact the only restriction in [11] is that the scalar potential is bounded below and that in [13] is that the magnetic field is constant. By combining the methods in [11] and [13], we can show (2.7).

To describe the asymptotics of $N(\lambda)$, we introduce a norm for any $D \times D$ matrix $\alpha = (\alpha^{(j,k)}_{(\ell,m)})_{1\le j<k\le d, 1\le \ell<m\le d}$, by

$$\|\alpha\|_{(1\leftarrow 2)} := \max\{\|\alpha(\mathbb{X})\|_{(1)} : \mathbb{X} \in \mathbb{R}^D, \|\mathbb{X}\|_{(2)} = 1\}, \tag{2.8}$$

where, for any $\mathbb{X} = (X^{(j,k)})_{1\le j<k\le d} \in \mathbb{R}^D$, $\alpha(\mathbb{X}) := (\sum_{\ell<m} \alpha^{(j,k)}_{(\ell,m)} X^{(\ell,m)})_{1\le j<k\le d} \in \mathbb{R}^D$, and $\|\mathbb{X}\|_{(1)}$ and $\|\mathbb{X}\|_{(2)}$ are the trace norm $\|X\|_1 := \mathrm{Tr}[\sqrt{X^*X}]$ and the Hilbert-Schmidt norm $\|X\|_2 := \sqrt{\mathrm{Tr}[X^*X]}$, respectively, of the corresponding skew-symmetric matrix $X = (X^j_k)_{1\le j,k\le d}$ (cf. [24]). The Hilbert-Schmidt norm $\|\mathbb{X}\|_{(2)} = \|X\|_2$ in (2.8) can also be replaced by $\sqrt{2}|\mathbb{X}|$ where $|\mathbb{X}| = \sqrt{\sum_{j<k}(X^{(j,k)})^2}$ is the Euclidean norm.

Then the main theorem is the following:

Theorem 2.1. *Let $N(\lambda)$ be the integrated density of states defined by (2.6) of the random Pauli Hamiltonian H_ω formally defined by (2.4). Then it holds that*

$$\lim_{\lambda\downarrow-\infty} \frac{1}{\lambda^2} \log N(\lambda) = -\left(\frac{2}{(g-2)\|\sqrt{\beta(0)}\|_{(1\leftarrow 2)}}\right)^2, \tag{2.9}$$

where $\sqrt{\beta(0)}$ is the non-negative square root of the non-negative symmetric matrix $\beta(0) = (\beta^{(j,k)}_{(\ell,m)}(0))_{1\le j<k\le d, 1\le \ell<m\le d}$ determined by the covariance functions (2.1).

This theorem is deduced from the following two propositions:

Proposition 2.1. *Let $\widetilde{N}(t)$ be the Laplace-Stieltjes transform of the integrated density of states $N(\lambda)$ defined by (2.6) of the random Pauli Hamiltonian H_ω formally defined by*

(2.4). *Then it holds that*

$$\lim_{t\uparrow\infty}\frac{1}{t^2}\log\widetilde{N}(t)=\lim_{t\uparrow\infty}\frac{1}{t^2}\log E\left[\exp\left(t\frac{g-2}{4}\|B(0)-M\|_1\right)\right], \tag{2.10}$$

where $M=E[B(0)]$.

Proposition 2.2. *For a* D*-dimensional Gaussian random variable* $\mathbb{B}=(B_\omega^{(j,k)})_{1\le j<k\le d}$ *with mean* 0 *and covariance* $\beta=(\beta_{(\ell,m)}^{(j,k)})_{1\le j<k\le d,1\le\ell<m\le d}$, *we have*

$$\lim_{t\uparrow\infty}\frac{1}{t^2}\log E[\exp(t\|B\|_1)]=\|\sqrt{\beta}\|_{(1\leftarrow 2)}^2, \tag{2.11}$$

where $B=(B_{\omega,k}^j)_{1\le j<k\le d}$ *is the skew-symmetric* $d\times d$ *matrix valued random variable identified with* $\mathbb{B}$ *by* (2.3).

We prove Proposition 2.1 in Sections 3 and 4, and prove Proposition 2.2 in Section 5.

3. **Proof of Proposition 2.1 (I): Upper estimate**

In this section we prove

$$\overline{\lim_{t\uparrow\infty}}\,\frac{1}{t^2}\log\widetilde{N}(t) \le \overline{\lim_{t\uparrow\infty}}\,\frac{1}{t^2}\log E\left[\exp\left(t\frac{g-2}{4}\|B(0)-M\|_1\right)\right]. \tag{3.1}$$

We first show the existence of $N(\lambda)$ and $\widetilde{N}(t)$, and fundamental estimates for them. For this purpose, we will use stochastic oscillatory integrals as in (1.6). Then we should change the gauge, since $d^*(\sum_j A^j_\omega(x)dx^j)$ may not be a usual function of x, where d^* is the formal adjoint of the exterior derivative. Now we define a vector potential $A^R = (A^{R,j}_\omega(x))_{1\le j\le d}$ by

$$\sum_{j=1}^d A^{R,j}_\omega(x)dx^j = d^*\Big\{\sum_{j<k} C^{R,(j,k)}_\omega(x)dx^j\wedge dx^k\Big\} \tag{3.2}$$

in the sense of distributions, where $C^{R,(j,k)}_\omega(x)$ is a C^1 function defined by

$$C^{R,(j,k)}_\omega(x) := \int k_d(x-y)\{d\rho(y/R)A_\omega(y)\}^{(j,k)}dy,$$

$$\{d\rho(y/R)A_\omega(y)\}^{(j,k)} = B^{(j,k)}_\omega(y)\rho\Big(\frac{y}{R}\Big) + \frac{1}{R}\Big((\partial_j\rho)\Big(\frac{y}{R}\Big)A^k_\omega(y) - (\partial_k\rho)\Big(\frac{y}{R}\Big)A^j_\omega(y)\Big),$$

$\rho\in C^\infty_0(\Lambda(2)\to[0,1])$ such that $\rho=1$ on $\Lambda(1)$, and $k_d(x)$ is the fundamental solution of (-1) times the Laplacian $-\Delta = -\sum_{j=1}^d\partial_j^2$. The explicit representation of $k_d(x)$ is

$$k_d(x) = \begin{cases} \dfrac{\Gamma(d/2)}{(d-2)2\pi^{d/2}|x|^{d-2}} & \text{if } d>2,\\[2ex] \dfrac{1}{2\pi}\log\dfrac{1}{|x|} & \text{if } d=2,\end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function. Thus the vector potential A^R is written more explicitly as

$$A^{R,j}_\omega(x) = \frac{-\Gamma(d/2)}{2\pi^{d/2}}\int\sum_{k\ne j}\frac{x^k-y^k}{|x-y|^d}\{d\rho(y/R)A_\omega(y)\}^{(j,k)}dy.$$

Since the de Rham-Hodge-Kodaira Laplacian $\Box := d^*d+dd^*$ acts as

$$\Box\{\psi(x)dx^j\wedge dx^k\} = -\Delta\psi(x)dx^j\wedge dx^k$$

for any $\psi \in C_0^\infty(\mathbb{R}^d)$ and

$$d\Big\{ \sum_{j<k} C_\omega^{R,(j,k)}(x) dx^j \wedge dx^k \Big\} = 0,$$

we have

$$d\Big\{ \sum_{j=1}^d A_\omega^{R,j}(x) dx^j \Big\} = d\Big\{ \rho\Big(\frac{x}{R}\Big) \sum_{j=1}^d A_\omega^j(x) dx^j \Big\}$$

in the sense of distributions. Then there exists a C^1 function $g_\omega(x)$ on $\Lambda(R)$ such that

$$dg_\omega(x) = \sum_{j=1}^d A_\omega^j(x) dx^j - \sum_{j=1}^d A_\omega^{R,j}(x) dx^j.$$

Since $\rho(x/R) = 1$ on $\Lambda(R)$, the operator $H_{\omega,R}$ is unitarily equivalent with the operator $\widetilde{H_{\omega,R}}$ by the unitary operator defined by $\exp(ig_\omega(x))$, where $\widetilde{H_{\omega,R}}$ is the operator obtained by replacing A_ω^j by $A_\omega^{R,j}$ in the definition of $H_{\omega,R}$. For the operator $\widetilde{H_{\omega,R}}$, we can use stochastic oscillatory integrals since $d^*(\sum_j A_\omega^{R,j} dx^j) = 0$.

Let $w = (w^1(t), w^2(t), \dots, w^d(t))$ be a d-dimensional Wiener process starting at 0 and $M_x(t)$ be the solution of the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt} M_x(t) = M_x(t) \Xi_x(t), \\ M_x(0) = I \end{cases} \tag{3.3}$$

on the space $\mathrm{End}(V)$ of the endomorphisms on V, where

$$\Xi_x(t) := -\frac{g}{4} \sum_{j<k} i\gamma_j\gamma_k B_\omega^{(j,k)}(x + w(t)).$$

Then the integral kernel $\exp(-t\widetilde{H_{\omega,R}}/2)(x,y)$, $(t,x,y) \in (0,\infty) \times \Lambda(R) \times \Lambda(R)$, of the heat semigroup $\exp(-t\widetilde{H_{\omega,R}}/2)$:

$$
\begin{aligned}
&\exp(-t\widetilde{H_{\omega,R}}/2)(x,y) \\
&= E^w\Bigg[\exp\Big(-i\sum_{j=1}^{d}\int_0^t A_\omega^{R,j}(x+w(s))dw^j(s)\Big)M_x(t) \\
&\qquad \times\chi\{x+w(s)\in\Lambda(R) \text{ for any } 0\le s\le t\}\Big| x+w(t)=y\Bigg] \\
&\qquad \times\frac{1}{(2\pi t)^{d/2}}\exp\left(-\frac{|x-y|^2}{2t}\right),
\end{aligned}
\tag{3.4}
$$

where E^w is the expectation with respect to the d-dimensional Wiener process w, $dw^j(s)$ is the Itô stochastic differential, and, for each Borel set A in the Wiener space, $\chi(A)$ is its indicator function (cf. [5], [14], [25]). From (3.3), we have

$$
\|M_x(t)^*\|_2 \le \|I\|_2 + \int_0^t \|\Xi_x(s)^*\|_{op}\|M_x(s)^*\|_2 ds,
$$

where $\|\cdot\|_{op}$ is the operator norm. By Gronwall's lemma, we have

$$
\|M_x(t)^*\|_2 \le \|I\|_2 \exp\left(\int_0^t \|\Xi_x(s)^*\|_{op} ds\right).
$$

Thus we have

$$
\|M_x(t)\|_2 \le c_1' \exp\Big(c_2' \int_0^t g\sum_{j<k}|B_\omega^{(j,k)}(x+w(s))|ds\Big)
$$

and

$$
E^\omega[\|M_x(t)\|_2^p] \le c_1^p \exp\Big(c_2(pgt)^2\sum_{j<k}\beta_{(j,k)}^{(j,k)}(0)\Big) \tag{3.5}
$$

for any $p \ge 1$, where $c_j', c_j > 0$ depend only on the dimension d and E^ω is the expectation with respect to ω. Since the equality $d(\sum_j A_\omega^{R,j}dx^j) = \sum_{j<k} B_\omega^{(j,k)}dx^j \wedge dx^k$ holds only on $\Lambda(R)$, we do not give an expression such as (1.6). Instead of this, we give the estimate (3.6) below:

Lemma 3.1. *There exists the integrated density of states $N(\lambda)$ defined in (2.6) of the random Pauli Hamiltonian H_ω formally defined in (2.4). Its Laplace-Stieltjes transform $\widetilde{N}(t)$ exists for any $t > 0$. This is estimated as*

$$\Big|\widetilde{N}\Big(\frac{t}{2}\Big) - \widetilde{N}_R\Big(\frac{t}{2}\Big)\Big| \le \frac{c_1}{t^{d/2}} \exp\Big(c_2 t^2 - \frac{R^2}{2t}\Big), \tag{3.6}$$

for some $c_1, c_2 > 0$ independent of t and R, where

$$\widetilde{N}_R\Big(\frac{t}{2}\Big) := \frac{1}{R^d} \int_{\Lambda(R)} E\Big[\,\mathrm{Tr}\,\Big[\exp\Big(-\frac{t}{2}\widetilde{H_{\omega,2R}}\Big)(x,x)\Big]\Big]dx.$$

Proof. To prove not only the existence of $N(\lambda)$ but also that of $\widetilde{N}(t)$, we apply the method by Kirsch-Martinelli [15] to the Laplace-Stieltjes transform $\widetilde{N}(t)$. Let $\mathcal{I}$ be the class of sets $[\boldsymbol{a}, \boldsymbol{b})$ of the form $[\boldsymbol{a}, \boldsymbol{b}) := \{x \in \mathbb{R}^d : a^j \le x^j < b^j, j = 1, 2, \dots, d\}$, where $\boldsymbol{a} = (a^j)_{1\le j\le d}$, $\boldsymbol{b} = (b^j)_{1\le j\le d}$. For each $[\boldsymbol{a}, \boldsymbol{b}) \subset \mathcal{I}$, let $H_{\omega,[\boldsymbol{a},\boldsymbol{b})}$ be the self-adjoint operator on $L^2([\boldsymbol{a}, \boldsymbol{b}) \to V)$, corresponding to the closure of the quadratic form q_ω in (2.5) with the domain $C_0^\infty((\boldsymbol{a}, \boldsymbol{b}) \to V)$, $N_{\omega,[\boldsymbol{a},\boldsymbol{b})}(\lambda)$ be the number of eigenvalues of $H_{\omega,[\boldsymbol{a},\boldsymbol{b})}$ not exceeding λ and $\widetilde{N}_{\omega,[\boldsymbol{a},\boldsymbol{b})}(t)$ be its Laplace-Stieltjes transform, where $(\boldsymbol{a}, \boldsymbol{b}) := \{x \in \mathbb{R}^d : a^j < x^j < b^j, j = 1, 2, \dots, d\}$ for $\boldsymbol{a} = (a^j)_{1\le j\le d}$ and $\boldsymbol{b} = (b^j)_{1\le j\le d}$. By the gauge invariance, we can show that the eigenvalues of the operator $H_{\omega,[\boldsymbol{a},\boldsymbol{b})}$ are determined only by the magnetic field B_ω. Then we can show that, for each $t > 0$, $\mathcal{I} \ni [\boldsymbol{a}, \boldsymbol{b}) \mapsto \widetilde{N}_{\omega,[\boldsymbol{a},\boldsymbol{b})}(t)$ is a superadditive process:

(i) $\widetilde{N}_{T_{\boldsymbol{k}}\omega,I}(t) = \widetilde{N}_{\omega,I+\boldsymbol{k}}(t)$ for any $I \in \mathcal{I}$ and $\boldsymbol{k} \in \mathbb{R}^d$;

(ii) if $I_1, \dots, I_n$ are disjoint sets in $\mathcal{I}$ and if $I = \bigcup_{j=1}^n I_j$ is also in $\mathcal{I}$ then $\widetilde{N}_{\omega,I}(t) \ge \sum_{j=1}^n \widetilde{N}_{\omega,I_j}(t)$;

(iii) $\sup\{|I|^{-1} E[\widetilde{N}_{\omega,I}(t)] : I \in \mathcal{I}, |I| > 0\} < \infty$,

where $T_{\boldsymbol{k}} : \Omega \to \Omega$ such that $\mathbb{B}_{T_{\boldsymbol{k}}\omega}(x) = \mathbb{B}_\omega(x + \boldsymbol{k})$, and $|I| = \prod_{j=1}^d (b_j - a_j)$ for $I = [\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{a} = (a^j)_{1\le j\le d}$, $\boldsymbol{b} = (b^j)_{1\le j\le d}$. Thus by the Akcoglu-Krengel superadditive ergodic theorem, $R^{-d}\widetilde{N}_{\omega,R}(t)$ converges to a number $\widetilde{N}(t)$ depending only on t almost surely and

in L^1 (cf. [1], [16]). Since $\widetilde{N}(t)$ is monotone decreasing in t, we can modify $\widetilde{N}(t)$ so that $R^{-d}\widetilde{N}_{\omega,R}(t)$ converges to $\widetilde{N}(t)$ for any point of continuity of $\widetilde{N}(t)$ and almost all ω as in [7], [15].

We next prove (3.6). By the L^1 convergence to $\widetilde{N}(t)$, we have

$$\widetilde{N}\Big(\frac{t}{2}\Big) = \lim_{Q\to\infty} \frac{1}{Q^d} E\Big[\widetilde{N}_{\omega,Q}\Big(\frac{t}{2}\Big)\Big].$$

We use Mercer's expansion theorem, the formula (3.4), the stationarity of the random magnetic field and the gauge invariance to rewrite this as

$$\begin{aligned}
\widetilde{N}\Big(\frac{t}{2}\Big) &= \lim_{Q\to\infty} \frac{1}{Q^d} \int_{\Lambda(Q)} E\Big[\operatorname{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H_{\omega,Q}}\Big)(x,x)\Big]\Big]dx \\
&= \lim_{Q\to\infty} \int_{\Lambda(1)} E\Big[\operatorname{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H_{\omega,Q}}\Big)(Qx,Qx)\Big]\Big]dx \\
&= \lim_{Q\to\infty} \int_{\Lambda(1)} E^{(\omega,w)}\Big[\exp\Big(-i\sum_{j=1}^{d}\int_0^t A_\omega^{2Q,j}(w(s))dw^j(s)\Big)\operatorname{Tr}[M_0(t)] \\
&\qquad \times\chi\{Qx+w(s)\in\Lambda(Q) \text{ for any } 0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^{d/2}}dx.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\widetilde{N}_R\Big(\frac{t}{2}\Big) = \int_{\Lambda(1)} E^{(\omega,w)}\Big[&\exp\Big(-i\sum_{j=1}^{d}\int_0^t A_\omega^{4R,j}(w(s))dw^j(s)\Big)\operatorname{Tr}[M_0(t)] \\
&\times\chi\{Rx+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^{d/2}}dx.
\end{aligned}$$

In this expression we can replace $A_\omega^{4R,j}$ by $A_\omega^{2Q,j}$ since $dA_\omega^{4R,j} = dA_\omega^{2Q,j}$ on $\Lambda(4R)$ for large enough Q. Thus we have

$$\begin{aligned}
&\Big|\widetilde{N}\Big(\frac{t}{2}\Big) - \widetilde{N}_R\Big(\frac{t}{2}\Big)\Big| \\
&= \lim_{Q\to\infty}\Big|\int_{\Lambda(1)} E^{(\omega,w)}\Big[\exp\Big(-i\sum_{j=1}^{d}\int_0^t A_\omega^{2Q,j}(w(s))dw^j(s)\Big)\operatorname{Tr}[M_0(t)] \\
&\qquad\times\Big\{\chi\{Qx+w(s)\in\Lambda(Q) \text{ for any } 0\le s\le t\} \\
&\qquad\quad -\chi\{Rx+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^{d/2}}dx\Big|.
\end{aligned}$$

Using (3.5), we see that the right hand side is dominated by

$$\frac{c_1}{t^{d/2}}\exp(c_2t^2)\,\overline{\lim_{Q\to\infty}}\int_{\Lambda(1)}E[|\chi\{Qx+w(s)\in\Lambda(Q)\text{ for any }0\le s\le t\}$$
$$-\chi\{Rx+w(s)\in\Lambda(2R)\text{ for any }0\le s\le t\}||w(t)=0]dx.$$

Since

$$\lim_{Q\to\infty}\chi\{Qx+w(s)\in\Lambda(Q)\text{ for any }0\le s\le t\}=1$$

for each $x\in\Lambda(1-\varepsilon)$, we have

$$\begin{aligned}&\Big|\widetilde{N}\Big(\frac{t}{2}\Big)-\widetilde{N}_R\Big(\frac{t}{2}\Big)\Big|\\&\le\frac{c_1}{t^{d/2}}\exp(c_2t^2)\Big\{\int_{\Lambda(1)-\Lambda(1-\varepsilon)}dx\\&\qquad+\int_{\Lambda(1-\varepsilon)}P(Rx+w(s)\notin\Lambda(2R)\text{ for some }0\le s\le t|w(t)=0)dx\Big\}\end{aligned}$$

and

$$\begin{aligned}&\Big|\widetilde{N}\Big(\frac{t}{2}\Big)-\widetilde{N}_R\Big(\frac{t}{2}\Big)\Big|\\&\le\frac{c_1}{t^{d/2}}\exp(c_2t^2)\int_{\Lambda(1)}P(Rx+w(s)\notin\Lambda(2R)\text{ for some }0\le s\le t|w(t)=0)dx.\end{aligned}$$

From this we can obtain (3.6) by a standard argument on the Wiener process. □

We now proceed to prove (3.1). We take small $\delta>0$ and use the semigroup property as follows:

$$\begin{aligned}&\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H_{\omega,2R}}\Big)(x,x)\Big]\\&\le\int_{\Lambda(2R)}dy\Big\|\exp\Big(-\frac{\delta}{2}\widetilde{H_{\omega,2R}}\Big)(y,x)\Big\|_2^2\exp\Big(-\Big(\frac{t}{2}-\delta\Big)\inf\mathrm{spec}\,H_{\omega,2R}\Big),\end{aligned}$$

where, for any self-adjoint operator A, $\operatorname{spec} A$ is its spectral set. We take another small $\varepsilon > 0$ and use Hölder's inequality as follows:

$$\begin{aligned}
&E^\omega\Big[\operatorname{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H_{\omega,2R}}\Big)(x,x)\Big]\Big] \\
&\le E^\omega\Big[\Big(\int_{\Lambda(2R)} dy\Big\|\exp\Big(-\frac{\delta}{2}\widetilde{H_{\omega,2R}}\Big)(y,x)\Big\|_2^2\Big)^{(1+\varepsilon)/\varepsilon}\Big]^{\varepsilon/(1+\varepsilon)} \\
&\qquad\times E^\omega\Big[\exp\Big(-(1+\varepsilon)\Big(\frac{t}{2}-\delta\Big)\inf\operatorname{spec} H_{\omega,2R}\Big)\Big]^{1/(1+\varepsilon)}.
\end{aligned}$$

By (3.5), we have

$$\begin{aligned}
&E^\omega\Big[\Big(\int_{\Lambda(2R)} dy\Big\|\exp\Big(-\frac{\delta}{2}\widetilde{H_{\omega,2R}}\Big)(y,x)\Big\|_2^2\Big)^{(1+\varepsilon)/\varepsilon}\Big]^{\varepsilon/(1+\varepsilon)} \\
&\le E^\omega\Big[\int_{\Lambda(2R)} dy\Big\|\exp\Big(-\frac{\delta}{2}\widetilde{H_{\omega,2R}}\Big)(y,x)\Big\|_2^{2(1+\varepsilon)/\varepsilon}\Big]^{\varepsilon/(1+\varepsilon)}(2R)^{d/(1+\varepsilon)} \\
&\le E^\omega\Big[\int_{\Lambda(2R)} dy E^w[\|M_y(\delta)\|_2^{2(1+\varepsilon)/\varepsilon}|y+w(\delta)=x] \\
&\qquad\times\exp\Big(-\frac{1+\varepsilon}{\varepsilon}\frac{|x-y|^2}{\delta}\Big)\Big]^{\varepsilon/(1+\varepsilon)}\frac{(2R)^{d/(1+\varepsilon)}}{(2\pi\delta)^d} \\
&\le c_{\varepsilon,\delta}\exp\Big(c\frac{1+\varepsilon}{\varepsilon}\delta^2\Big)R^{d/(1+\varepsilon)},
\end{aligned}$$

where $c_{\varepsilon,\delta} > 0$ depends only on d, ε and δ. Therefore we obtain

$$\widetilde{N}_R\Big(\frac{t}{2}\Big) \le c_{\varepsilon,\delta}\exp\Big(c\frac{1+\varepsilon}{\varepsilon}\delta^2\Big)R^{d/(1+\varepsilon)}\widetilde{N}_{R,0}\Big((1+\varepsilon)\Big(\frac{t}{2}-\delta\Big)\Big)^{1/(1+\varepsilon)}, \tag{3.7}$$

where

$$\widetilde{N}_{R,0}(t) := E[\exp(-t\inf\operatorname{spec} H_{\omega,2R})].$$

By the estimate

$$\begin{aligned}
q_\omega(\Phi,\Phi) &= \Big|\Big|\Big|\sum_{j=1}^d \gamma_j(i\partial_j + A_\omega^j(x))\Phi(x)\Big|\Big|\Big|_V^2 \\
&\qquad+\Big\langle\Phi(x),\Big(\frac{g}{2}-1\Big)\sum_{j<k} i\gamma_j\gamma_k B_\omega^{(j,k)}(x)\Phi(x)\Big\rangle_V \\
&\ge -\frac{g-2}{4}\langle\Phi(x),\|B_\omega(x)\|_1\Phi(x)\rangle_V,
\end{aligned}$$

we obtain

$$\inf \operatorname{spec} H_{\omega,2R} \geq -\frac{g-2}{4} \sup_{x\in\Lambda(2R)} \|B_\omega(x)\|_1,$$

where $|||\cdot|||_V$ is the L^2-norm on $L^2(\mathbb{R}^d \to V)$ (cf. Lemma 5.1 below). For arbitrarily taken small $\eta > 0$, since

$$\sup_{x\in\Lambda(2R)} \|B_\omega(x)\|_1 \leq \sup_{\iota\in\Lambda(2R)\cap((\eta/2)\mathbb{Z})^d} \sup_{x\in\Lambda_\iota(\eta)} \|B_\omega(x)\|_1,$$

we have

$$\begin{aligned}\widetilde{N}_{R,0}(t) &\leq \sum_{\iota\in\Lambda(2R)\cap((\eta/2)\mathbb{Z})^d} E\Big[\exp\Big(t\frac{g-2}{4}\sup_{x\in\Lambda_\iota(\eta)}\|B_\omega(x)\|_1\Big)\Big] \\ &\leq c_{\eta,d}R^d E\Big[\exp\Big(t\frac{g-2}{4}\sup_{x\in\Lambda(\eta)}\|B_\omega(x)\|_1\Big)\Big],\end{aligned}$$

where $\Lambda_\iota(\eta) = \Lambda(\eta) + \iota$ and $c_{\eta,d}$ is a constant depending only on η and d. Now, setting $\mathring{B}_\omega(x) = E[B_\omega(x)|B_\omega(0)] - M$ and $\overline{B}_\omega(x) = B_\omega(x) - \mathring{B}_\omega(x) - M$, we write

$$B_\omega(x) = M + \mathring{B}_\omega(x) + \overline{B}_\omega(x). \tag{3.8}$$

We note that $\mathring{B}_\omega(x)$ and $\overline{B}_\omega(x)$ are independent and that

$$\mathring{B}^{(j,k)}_\omega(x) = \sum_{\ell<m}\sum_{p<q} \beta^{(j,k)}_{(\ell,m)}(x)(\beta^{-1})^{(\ell,m)}_{(p,q)}(0)(B^{(p,q)}_\omega(0) - M^{(p,q)}). \tag{3.9}$$

By the independence we have

$$\widetilde{N}_{R,0}(t) \leq c_{\eta,d}R^d \exp\Big(t\frac{g-2}{4}\|M\|_1\Big)\widetilde{N}_{R,1}\Big(\frac{g-2}{4}t\Big)\widetilde{N}_{R,2}(\frac{g-2}{4}t\Big), \tag{3.10}$$

where

$$\widetilde{N}_{R,1}(t) := E\Big[\exp\Big(t\sup_{x\in\Lambda(\eta)}\|\mathring{B}_\omega(x)\|_1\Big)\Big]$$

and

$$\widetilde{N}_{R,2}(t) := E\Big[\exp\Big(t\sup_{x\in\Lambda(\eta)}\|\overline{B}_\omega(x)\|_1\Big)\Big].$$

By the expression (3.9), we can take a function $\ell(s)$ of $s > 0$, independent of ω, such that $\lim_{s\downarrow 0} \ell(s) = 0$,

$$|\overset{\circ}{B}{}_{\omega}^{(j,k)}(x) - (B_{\omega}^{(j,k)}(0) - M^{(j,k)})| \le \ell(|x|)\|B_{\omega}(0) - M\|_1 \tag{3.11}$$

and

$$\sup_{x\in\Lambda(\eta)} \|\overset{\circ}{B}_{\omega}(x)\|_1 \le \|B_{\omega}(0) - M\|_1(1 + \ell(\eta)).$$

Thus we have

$$\widetilde{N}_{R,1}(t) \le E[\exp\{(1 + \ell(\eta))t\|B_{\omega}(0) - M\|_1\}]. \tag{3.12}$$

On the other hand, by the continuity of the sample path $x \mapsto B(x)$, we have

$$\lim_{\eta\downarrow 0} P\Big(\sup_{x\in\Lambda(\eta)} \|\overline{B}_{\omega}(x)\|_1 \le 1\Big) = 1$$

Thus, by the theory of Fernique [12], we can take a function $k(\eta)$ such that

$$P\Big(\sup_{x\in\Lambda(\eta)} \|\overline{B}_{\omega}(x)\|_1 \ge \mu\Big) \le \exp(-\mu^2 k(\eta))$$

for any $\mu \ge 1$ and $\lim_{\eta\downarrow 0} k(\eta) = \infty$. Thus we have

$$\begin{aligned} \widetilde{N}_{R,2}(t) &= 1 + \int_0^{\infty} te^{t\mu} P\Big(\sup_{x\in B(\eta)} \|\overline{B}_{\omega}(x)\|_1 \ge \mu\Big) d\mu \\ &\le e^t + \sqrt{\frac{\pi}{k(\eta)}} t \exp\Big(\frac{t^2}{4k(\eta)}\Big). \end{aligned} \tag{3.13}$$

We now combine (3.6), (3.7), (3.10), (3.12) and (3.13), and put $R = t^2$. Then we obtain

$$\begin{aligned} &\overline{\lim_{t\uparrow\infty}} \frac{1}{(t/2)^2} \log \widetilde{N}\Big(\frac{t}{2}\Big) \\ &\le \overline{\lim_{t\uparrow\infty}} \frac{1}{t^2} \log E\Big[\exp\Big(t\frac{g-2}{4}\|B_{\omega}(0) - M\|_1\Big)\Big](1 + \varepsilon)(1 + r(\eta))^2 \\ &\quad + \frac{1 + \varepsilon}{4k(\eta)}\Big(\frac{g-2}{4}\Big)^2. \end{aligned}$$

Since η and ε are arbitrary, we obtain (3.1).

4. **Proof of Proposition 2.1 (II): Lower estimate**

In this section we prove

$$\varliminf_{t\uparrow\infty} \frac{1}{t^2}\log \widetilde{N}(t) \geq \varliminf_{t\uparrow\infty} \frac{1}{t^2}\log E\left[\exp\left(t\frac{g-2}{4}\|B(0)-M\|_1\right)\right]. \tag{4.1}$$

By the Akcoglu-Krengel superadditive ergodic theorem (cf. [1], [7], [15], [16]), we have

$$\widetilde{N}(t) \geq E[\widetilde{N}_{\omega,1}(t)] = E[\mathrm{Tr}[\exp(-tH_{\omega,1})]] \geq E[\exp(-t\inf\mathrm{spec}\, H_{\omega,1})]$$

$$\geq E[\exp(-t\inf\mathrm{spec}\, H_{\omega,\eta})]$$

for any $0<\eta<1$. According to the expression (3.8), we write the vector potential as $A_\omega = \mathring{A}_\omega(x) + \overline{A}_\omega(x)$, where $\mathring{A}_\omega(x)$ and $\overline{A}_\omega(x)$ are vector potentials obtained by replacing B_ω by $\mathring{B}_\omega$ and $M+\overline{B}_\omega$, respectively, in (2.2). We denote the Dirac operator with the magnetic field $\mathring{B}_\omega$ by $\mathring{D}_\omega$:

$$\mathring{D}_\omega = \sum_{j=1}^{d}\gamma_j(i\partial_j + \mathring{A}^j_\omega).$$

Let $(\mathring{D}^2_\omega)_\eta$ be the self-adjoint operator corresponding to the closure of the quadratic form

$$r_\omega(\Phi,\Psi) := \langle \mathring{D}_\omega\Phi(x), \mathring{D}_\omega\Psi(x)\rangle_V$$

with the domain $C_0^\infty(\Lambda(\eta)\to V)$. For arbitrarily fixed $\varepsilon>0$, we take $\mathring{\Phi}_\omega \in C_0^\infty(\Lambda(\eta)\to V)$ so that $|||\mathring{\Phi}_\omega|||_V = 1$ and $r_\omega(\mathring{\Phi}_\omega,\mathring{\Phi}_\omega) \leq \inf\mathrm{spec}((\mathring{D}^2_\omega)_\eta)+\varepsilon$. Then, for arbitrarily fixed

$\delta > 0$, we have

$$
\begin{aligned}
&\inf \operatorname{spec} H_{\omega,\eta} \\
&\le \sum_{j=1}^{d} |||(i\partial_j + \mathring{A}^j_\omega + \overline{A}^j_\omega)\mathring{\Phi}_\omega|||^2_V + \langle \mathring{\Phi}_\omega, \frac{g}{2}\sum_{j<k} i\gamma_j\gamma_k B^{(j,k)}_\omega \mathring{\Phi}_\omega\rangle_V \\
&\le (1+\delta) r_\omega(\mathring{\Phi}_\omega, \mathring{\Phi}_\omega) + \Big(1+\frac{1}{\delta}\Big)\Big|\Big|\Big|\sum_{j=1}^{d}\gamma_j \overline{A}^j_\omega \mathring{\Phi}_\omega\Big|\Big|\Big|^2_V + \frac{g-2}{2}\Big\langle \mathring{\Phi}_\omega, \sum_{j<k} i\gamma_j\gamma_k \mathring{B}^{(j,k)} \mathring{\Phi}_\omega\Big\rangle_V \\
&\quad + \frac{g-2}{2}\Big\langle \mathring{\Phi}_\omega, \sum_{j<k} i\gamma_j\gamma_k (M^{(j,k)} + \overline{B}^{(j,k)}_\omega(x))\mathring{\Phi}_\omega\Big\rangle_V \\
&\le (1+\delta)\inf \operatorname{spec}((\mathring{D}^2_\omega)_\eta) + (1+\delta)\varepsilon + c\Big(1+\frac{1}{\delta}\Big)\sup_{x\in\Lambda(\eta)} |\overline{A}_\omega(x)|^2 \\
&\quad + \frac{g-2}{2}\Big\langle \mathring{\Phi}_\omega, \sum_{j<k} i\gamma_j\gamma_k \mathring{B}^{(j,k)} \mathring{\Phi}_\omega\Big\rangle_V + c\frac{g-2}{2}\sup_{x\in\Lambda(\eta)} |\mathbb{M} + \overline{\mathbb{B}}_\omega(x)|
\end{aligned}
$$

for some $c > 0$. By the independence, we have

$$\widetilde{N}(t) \ge \widetilde{N}^0(t)\widetilde{N}^1(t),$$

where

$$
\begin{aligned}
\widetilde{N}^0(t) := E\Big[\exp\Big\{ -t\Big((1+\delta)\inf \operatorname{spec}((\mathring{D}^2_\omega)_\eta) \\
+ \frac{g-2}{2}\Big\langle \mathring{\Phi}_\omega, \sum_{j<k} i\gamma_j\gamma_k \mathring{B}^{(j,k)}_\omega \mathring{\Phi}_\omega\Big\rangle_V\Big)\Big\}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{N}^1(t) := \exp(-t(1+\delta)\varepsilon) E\Big[\exp\Big\{ -tc\Big(\Big(1+\frac{1}{\delta}\Big)\sup_{x\in\Lambda(\eta)} |\overline{A}_\omega(x)|^2 \\
+ \frac{g-2}{2}\sup_{x\in\Lambda(\eta)} |\mathbb{M} + \overline{\mathbb{B}}_\omega(x)|\Big)\Big\}\Big].
\end{aligned}
$$

By Jensen's inequality, we have

$$
\begin{aligned}
&\varliminf_{t\uparrow\infty} \frac{1}{t^2}\log \widetilde{N}^1(t) \\
&\ge -\varlimsup_{t\uparrow\infty} \frac{c}{t} E\Big[\Big(1+\frac{1}{\delta}\Big)\sup_{x\in\Lambda(\eta)} |\overline{A}_\omega(x)|^2 + \frac{g-2}{2}\sup_{x\in\Lambda(\eta)} |\mathbb{M} + \overline{\mathbb{B}}_\omega(x)|\Big].
\end{aligned}
$$

Since

$$E\Big[\sup_{x\in\Lambda(\eta)}|\overline{A}_\omega(x)|^2\Big] \ \text{ and } \ E\Big[\sup_{x\in\Lambda(\eta)}|\mathbb{M}+\overline{\mathbb{B}}_\omega(x)|\Big]$$

are finite, we have

$$\varliminf_{t\uparrow\infty}\frac{1}{t^2}\log\widetilde{N}^1(t)\geq 0.$$

By the continuity (3.11), we can choose $\overset{\circ}{\Phi}_\omega$ so that

$$\Big\langle\overset{\circ}{\Phi}_\omega,\sum_{j<k}i\gamma_j\gamma_k\overset{\circ}{B}{}^{(j,k)}\overset{\circ}{\Phi}_\omega\Big\rangle_V\leq-\frac{1}{2}(1-2D\ell(\eta))\|B_\omega(0)-M\|_1.$$

By using also Lemma 4.1 below, we have

$$\widetilde{N}^0(t)\geq\exp(-t(1+\delta)c\eta^{-2})E\Big[\exp\Big\{t\|B_\omega(0)-M\|_1\Big(\frac{g-2}{4}(1-2D\ell(\eta))-cr(\eta)\Big)\Big\}\Big].$$

Since η is arbitrary, we obtain (4.1).

Lemma 4.1. *We assume $M=0$. Then there exist a constant $c>0$ such that*

$$\inf\operatorname{spec}((\overset{\circ}{D}{}^2_\omega)_\eta)\leq c(\eta^{-2}+\|B_\omega(0)\|_1\ell(\eta)) \tag{4.2}$$

for small enough $\eta>0$, where $\ell(\eta)$ is the function taken in (3.11).

Proof. By (3.11) we have

$$|\mathring{A}_\omega(x)-\widehat{A}_\omega(x)|\leq c\ \ell(\eta)\|B_\omega(0)\|_1|x|$$

on $x\in\Lambda(\eta)$, where $\widehat{A}_\omega=(\widehat{A}^j_\omega)_{1\leq j\leq d}$ is

$$\widehat{A}^j_\omega=\frac{1}{2}\sum_{k=1}^d B^k_{\omega,j}(0)x^k.$$

Then the proof is reduced to an estimate in the case of the uniform magnetic field as follows:

$$\begin{aligned}
&\inf \operatorname{spec}((\mathring{D}_\omega^2)_\eta) \\
&\le c \inf \left\{ \left|\!\left|\!\left| \sum_{j=1}^d \gamma_j \Big(i\partial_j + \frac{1}{2} \sum_{k=1}^d B_{\omega,j}^k(0) x^k \Big) \Phi(x) \right|\!\right|\!\right|_V^2 + \ell(\eta)^2 \|B_\omega(0)\|_1^2 \left|\!\left|\!\left| |x|\Phi \right|\!\right|\!\right|_V^2 \right. \\
&\qquad \left. : \Phi \in C_0^\infty(\Lambda(\eta) \to V), |||\Phi|||_V = 1 \right\}.
\end{aligned}$$

Changing the variable $x \to x/\sqrt{\|B_\omega(0)\|_1}$, we rewrite

$$\begin{aligned}
&\inf \operatorname{spec}((\mathring{D}_\omega^2)_\eta) \\
&\le c\|B_\omega(0)\|_1 \inf \left\{ \left|\!\left|\!\left| \sum_{j=1}^d \gamma_j \Big(i\partial_j + \frac{1}{2} \sum_{k=1}^d \frac{B_{\omega,j}^k(0)}{\sqrt{\|B_\omega(0)\|_1}} x^k \Big) \Phi(x) \right|\!\right|\!\right|_V^2 + \ell(\eta)^2 \left|\!\left|\!\left| |x|\Phi \right|\!\right|\!\right|_V^2 \right. \\
&\qquad \left. : \Phi \in C_0^\infty(\Lambda(\eta\sqrt{\|B_\omega(0)\|_1}) \to V), |||\Phi|||_V = 1 \right\}.
\end{aligned}$$

Since

$$U_\omega^* B_\omega(0) U_\omega = \begin{pmatrix} B_\omega^{\langle 1 \rangle} & & & \\ & \ddots & & \\ & & B_\omega^{\langle [d/2] \rangle} & \\ & & & (0) \end{pmatrix}, \qquad B_\omega^{\langle j \rangle} = \begin{pmatrix} 0 & B_\omega^{(j)} \\ -B_\omega^{(j)} & 0 \end{pmatrix}, \qquad B_\omega^{(j)} \ge 0,$$

for some orthogonal matrix U_ω, we have

$$\inf \operatorname{spec}((\mathring{D}_\omega^2)_\eta) \le c \sum_{h=0}^{[d/2]} I_\omega^h,$$

where

$$\begin{aligned}
I_\omega^0 := \|B_\omega(0)\|_1 \inf \Big\{ &|||\phi'(x)|||_1^2 + \ell(\eta)^2 |||x\phi(x)|||_1^2 \\
&: \phi \in C_0^\infty(\Lambda_1(\eta\sqrt{\|B_\omega(0)\|_1/d}) \to \mathbb{R}), |||\phi|||_1 = 1 \Big\}
\end{aligned}$$

and

$$I_\omega^h := \|B_\omega(0)\|_1 \inf\Big\{ \Big|\Big|\Big| \Big\{ \gamma_1 \Big(i\partial_1 - \frac{B_\omega^{(h)} x^2}{2\|B_\omega(0)\|_1} \Big) + \gamma_2 \Big(i\partial_2 + \frac{B_\omega^{(h)} x^1}{2\|B_\omega(0)\|_1} \Big) \Big\} \Phi(x) \Big|\Big|\Big|_2^2 + \ell(\eta)^2 \Big|\Big| |x| \Phi \Big|\Big|\Big|_2^2 : \Phi \in C_0^\infty(\Lambda_2(\eta\sqrt{\|B_\omega(0)\|_1/d}) \to \mathbb{C}^2), |||\Phi|||_2 = 1 \Big\}$$

for $1 \le h \le [d/2]$. In these expressions, $|||\cdot|||_j$ is the norm of $L^2(\mathbb{R}^j \to \mathbb{C}^j)$ and $\Lambda_j(R) := (-R/2, R/2)^j$ for $j = 1$, 2 and $R > 0$, and γ_1, γ_2 are the matrices defined by

$$\gamma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

By changing the variable $x \to x/\sqrt{\ell(\eta)}$, we have

$$I_\omega^0 = \|B_\omega(0)\|_1 \ell(\eta) \inf\{ |||\phi'|||_1^2 + |||x\phi|||_1^2 : \phi \in C_0^\infty(\Lambda_1(\eta\sqrt{\ell(\eta)\|B_\omega(0)\|_1/d}) \to \mathbb{R}), |||\phi|||_1 = 1 \}. \tag{4.3}$$

By choosing $\phi(x) = \zeta(x/(\eta\sqrt{\ell(\eta)\|B_\omega(0)\|_1/d}))$ with $\zeta \in C_0^\infty(\Lambda_1(1) \to [0,1])$ such that $\zeta = 1$ on $\Lambda_1(1/2)$, we have

$$I_\omega^0 \le c(\eta^{-2} + \eta^2 \ell(\eta)^2 \|B_\omega(0)\|_1^2). \tag{4.4}$$

This estimate is efficient only when $\|B_\omega(0)\|_1$ is small so that $\eta\sqrt{\ell(\eta)\|B_\omega(0)\|_1/d} \le 1$. When $\eta\sqrt{\ell(\eta)\|B_\omega(0)\|_1/d} \ge 1$, we replace $\eta\sqrt{\ell(\eta)\|B_\omega(0)\|_1/d}$ by 1 in (4.3). Then we have

$$\begin{aligned} I_\omega^0 &\le \|B_\omega(0)\|_1 \ell(\eta) \inf\{ |||\phi'|||_1^2 + |||x\phi|||_1^2 \\ &\qquad : \phi \in C_0^\infty(\Lambda_1(1) \to \mathbb{R}), |||\phi|||_1 = 1 \} \\ &\le c\|B_\omega(0)\|_1 \ell(\eta). \end{aligned}$$

Thus we always have

$$I_\omega^0 \le c(\eta^{-2} + \|B_\omega(0)\|_1 \ell(\eta)).$$

For I^h_ω, we take $\Phi(x) = (\phi(x^1)\phi(x^2), 0)^*$ with a real valued function ϕ of one variable. Then we have

$$I^h_\omega \le 2\|B_\omega(0)\|_1 \inf\Big\{\Big|\Big|\Big|\phi'(x) + \frac{B^{(h)}_\omega x}{2\|B_\omega(0)\|_1}\phi(x)\Big|\Big|\Big|_1^2 + \ell(\eta)^2|||x\phi|||_1^2$$

$$: \phi \in C^\infty_0(\Lambda_1(\eta\sqrt{\|B_\omega(0)\|_1/d}) \to \mathbb{R}), |||\phi|||_1 = 1\Big\}. \tag{4.5}$$

By

$$\Big|\Big|\Big|\phi'(x) + \frac{B^{(h)}_\omega x}{2\|B_\omega(0)\|_1}\phi(x)\Big|\Big|\Big|_1^2 \le 2\Big(|||\phi'(x)|||_1^2 + \Big|\Big|\Big|\frac{B^{(h)}_\omega x}{2\|B_\omega(0)\|_1}\phi(x)\Big|\Big|\Big|_1^2\Big),$$

we can estimate I^h_ω as in our estimate of I^0_ω:

$$\begin{aligned} I^h_\omega &\le c\|B_\omega(0)\|_1 \inf\Big\{|||\phi'|||_1^2 + \Big(\frac{(B^{(h)}_\omega)^2}{\|B_\omega(0)\|_1^2} + \ell(\eta)^2\Big)|||x\phi|||_1^2 \\ &\qquad : \phi \in C^\infty_0(\Lambda_1(\eta\sqrt{\|B_\omega(0)\|_1/d}) \to \mathbb{R}), |||\phi|||_1 = 1\Big\} \\ &= c\sqrt{(B^{(h)}_\omega)^2 + \|B_\omega(0)\|_1^2\ell(\eta)^2}\inf\Big\{|||\phi'|||_1^2 + |||x\phi|||_1^2 \\ &\qquad : \phi \in C^\infty_0(\Lambda_1(\eta((B^{(h)}_\omega)^2 + \|B_\omega(0)\|_1^2\ell(\eta)^2)^{1/4}/\sqrt{d}) \to \mathbb{R}), |||\phi|||_1 = 1\Big\}, \end{aligned}$$

from which we obtain

$$I^h_\omega \le c\Big(\eta^{-2} + \sqrt{(B^{(h)}_\omega)^2 + \|B_\omega(0)\|_1^2\ell(\eta)^2}\Big). \tag{4.6}$$

On the other hand we choose ϕ in (4.5) as

$$\phi(x) = \zeta\Big(\frac{x}{\eta}\sqrt{d/\|B_\omega(0)\|_1}\Big)\exp\Big(-\frac{B^{(h)}_\omega x^2}{4\|B_\omega(0)\|_1}\Big),$$

where ζ is the function used in (4.4). Then we have

$$\phi'(x) + \frac{B^{(h)}_\omega x}{2\|B_\omega(0)\|_1}\phi(x) = \frac{\sqrt{d/\|B_\omega(0)\|_1}}{\eta}\zeta'\Big(\frac{x}{\eta}\sqrt{d/\|B_\omega(0)\|_1}\Big)\exp\Big(-\frac{B^{(h)}_\omega x^2}{4\|B_\omega(0)\|_1}\Big).$$

By a simple estimate we have

$$\begin{aligned} &\|\phi\|_1^2 \ge c\{(\eta\sqrt{\|B_\omega(0)\|_1}) \wedge \sqrt{\|B_\omega(0)\|_1/B^{(h)}_\omega}\}, \\ &\Big|\Big|\Big|\phi'(x) + \frac{B^{(h)}_\omega x}{2\|B_\omega(0)\|_1}\phi(x)\Big|\Big|\Big|_1^2 \le \frac{c}{\eta^2\|B_\omega(0)\|_1}\exp\Big(-c'B^{(h)}_\omega\eta^2\Big)\{(\eta\sqrt{\|B_\omega(0)\|_1}) \wedge \sqrt{\|B_\omega(0)\|_1/B^{(h)}_\omega}\} \end{aligned}$$

and

$$\|x\phi\|_1^2 \le c\{(\eta\sqrt{\|B_\omega(0)\|_1}) \wedge (\sqrt{\|B_\omega(0)\|_1/B_\omega^{(h)}})\}^3,$$

from which we obtain

$$I_\omega^h \le c\Big(\frac{1}{\eta^2} + \ell(\eta)^2\|B_\omega(0)\|_1\Big(\frac{\|B_\omega(0)\|_1}{B_\omega^{(h)}}\Big)^2\Big). \tag{4.7}$$

We use (4.7) when $B_\omega^{(h)} \ge \|B_\omega(0)\|_1\ell(\eta)$, and use (4.6) when $B_\omega^{(h)} \le \|B_\omega(0)\|_1\ell(\eta)$. Then we always obtain (4.2). $\square$

5. **Skew-symmetric matrix valued Gaussian random variables**

In this section we first prove Proposition 2.2 and discuss the representation of the norm $\|\cdot\|_{(1\leftarrow 2)}$.

Proof of Proposition 2.2. Since $\mathbb{B}$ can be realized as

$$B_\omega^{(j,k)} = \sum_{1\le \ell<m\le d} \sqrt{\beta}_{(\ell,m)}^{(j,k)} X_\omega^{(\ell,m)},$$

where $X_\omega^{(\ell,m)}$, $1\le \ell<m\le d$, are independent unit normal random variables, we have

$$\begin{aligned} I(t) &:= E[\exp(t\|B\|_1)] \\ &= \int_{\mathbb{R}^D} \exp\Big\{ t\Big\| \Big(\sum_{1\le \ell<m\le d} \sqrt{\beta}_{(\ell,m)}^{(\cdot,\cdot)} x^{(\ell,m)} \Big)\Big\|_{(1)} - \frac{|x^{(\cdot,\cdot)}|^2}{2} \Big\} \prod_{1\le \ell<m\le d} \frac{dx^{(\ell,m)}}{\sqrt{2\pi}}. \end{aligned}$$

By changing the variable $x^{(\cdot,\cdot)} \to t x^{(\cdot,\cdot)}$, we have

$$\begin{aligned} I(t) &:= E[\exp(t\|B\|_1)] \\ &= t^D \int_{\mathbb{R}^D} \exp\Big\{ t^2\Big(\Big\| \Big(\sum_{1\le \ell<m\le d} \sqrt{\beta}_{(\ell,m)}^{(\cdot,\cdot)} x^{(\ell,m)} \Big)\Big\|_{(1)} - \frac{|x^{(\cdot,\cdot)}|^2}{2} \Big)\Big\} \prod_{1\le \ell<m\le d} \frac{dx^{(\ell,m)}}{\sqrt{2\pi}}. \end{aligned}$$

Thus we obtain

$$\lim_{t\uparrow\infty} \frac{1}{t^2} \log I(t) = \sup\Big\{ \Big\| \Big(\sum_{1\le \ell<m\le d} \sqrt{\beta}_{(\ell,m)}^{(\cdot,\cdot)} x^{(\ell,m)} \Big)\Big\|_{(1)} - \frac{|x^{(\cdot,\cdot)}|^2}{2} : (x^{(\cdot,\cdot)}) \in \mathbb{R}^D \Big\}.$$

It is easy to see that the right hand side coincides with the right hand side of (2.11). □

In the case of $d = 2$ or 3, the trace norm $\|X\|_1$ and the Hilbert-Schmidt norm $\|X\|_2$ are same up to constants and we can represent the norm $\|\alpha\|_{(1\leftarrow 2)}$ explicitly as follows:

Example 1. In the case of $d = 2$, $\mathbb{X}$ and α are scalars. Then it is easy to see that $\|\mathbb{X}\|_{(1)} = \sqrt{2}\|\mathbb{X}\|_{(2)} = 2|\mathbb{X}|$ and $\|\alpha\|_{(1\leftarrow 2)} = \sqrt{2}|\alpha|$. Therefore we obtain Theorem 1.1 from Theorem 2.1.

EXAMPLE 2. In the case of $d = 3$, the eigenvalues of X are $i|\mathbb{X}|$, $-i|\mathbb{X}|$, 0. Thus we have $\|\mathbb{X}\|_{(1)} = \sqrt{2}\|\mathbb{X}\|_{(2)} = 2|\mathbb{X}|$ and

$$\begin{aligned}\|\alpha\|_{(1\leftarrow 2)} &= \sqrt{2}\max\{|\alpha(\mathbb{X})| : \mathbb{X} \in \mathbb{R}^3, |\mathbb{X}| = 1\Big\} \\ &= \sqrt{2}\mu(\alpha),\end{aligned}$$

where $\mu(\alpha)$ is the largest eigenvalue of α. Therefore the result of Proposition 2.2 is

$$\lim_{t\uparrow\infty} \frac{1}{t^2} \log E[\exp(t\|B\|_1)] = 2\mu(\beta)$$

and the result of Theorem 2.1 is

$$\lim_{\lambda\downarrow-\infty} \frac{1}{\lambda^2} \log N(\lambda) = \frac{-2}{(g-2)^2\mu(\beta(0))}.$$

For higher dimensional cases, we give the following:

Lemma 5.1. (i) *For any* $\mathbb{X} = (X^{(j,k)})_{1\le j<k\le d} \in \mathbb{R}^D$, *we have*

$$\|\mathbb{X}\|_{(1)} = 2\|\widehat{\mathbb{X}}\|_{op},$$

where $\|\widehat{\mathbb{X}}\|_{op}$ *is the operator norm of*

$$\widehat{\mathbb{X}} = \sum_{1\le j<k\le d} X^{(j,k)} i\gamma_j\gamma_k$$

in $\mathrm{End}(V)$.

(ii) *For any* $D \times D$ *matrix* $\alpha = (\alpha^{(j,k)}_{(\ell,m)})_{1\le j<k\le d, 1\le \ell<m\le d}$, *we have*

$$\begin{aligned}\|\alpha\|^2_{(1\leftarrow 2)} = 2\max\Big\{ &\sum_{1\le j<k\le d} \alpha^{(j,k)}(\mathbb{X})^2 + 2\sum_{1\le j<k<\ell<m\le d} (-\alpha^{(j,k)}(\mathbb{X})\alpha^{(\ell,m)}(\mathbb{X}) \\ &+\alpha^{(j,\ell)}(\mathbb{X})\alpha^{(k,m)}(\mathbb{X}) - \alpha^{(j,m)}(\mathbb{X})\alpha^{(k,\ell)}(\mathbb{X}))(v, \gamma_j\gamma_k\gamma_\ell\gamma_m v)_V \\ &: \mathbb{X} \in \mathbb{R}^D, |\mathbb{X}| = 1, v \in V, |v|_V = 1\Big\},\end{aligned}$$

where $(\cdot,\cdot)_V$ *and* $|\cdot|_V$ *are the inner product and the norm of* V, *respectively.*

We here note that $(v, \gamma_j\gamma_k\gamma_\ell\gamma_m v)_V$ is real since $\gamma_j\gamma_k\gamma_\ell\gamma_m$ is an Hermitian matrix. By this lemma we have the following:

EXAMPLE 3. In the case of $d = 4$, since

$$\{(v, \gamma_1\gamma_2\gamma_3\gamma_4 v)_V : v \in V, |v|_V = 1\} = [-1, 1],$$

we have

$$\begin{aligned}
&\|\alpha\|^2_{(1\leftarrow 2)} \\
&= 2\max\Big\{ \sum_{j<k} \alpha^{(j,k)}(\mathbb{X})^2 + 2\tau(\alpha^{(1,2)}(\mathbb{X})\alpha^{(3,4)}(\mathbb{X}) - \alpha^{(1,3)}(\mathbb{X})\alpha^{(2,4)}(\mathbb{X}) + \alpha^{(1,4)}(\mathbb{X})\alpha^{(2,3)}(\mathbb{X})) \\
&\qquad : \mathbb{X} \in \mathbb{R}^6, |\mathbb{X}| = 1, -1 \le \tau \le 1\Big\} \\
&= 2\big(\max\{(\alpha^{(1,2)}(\mathbb{X}) + \alpha^{(3,4)}(\mathbb{X}))^2 + (\alpha^{(1,3)}(\mathbb{X}) - \alpha^{(2,4)}(\mathbb{X}))^2 + (\alpha^{(1,4)}(\mathbb{X}) + \alpha^{(2,3)}(\mathbb{X}))^2 \\
&\qquad : \mathbb{X} \in \mathbb{R}^6, |\mathbb{X}| = 1\} \\
&\quad \vee \max\{(\alpha^{(1,2)}(\mathbb{X}) - \alpha^{(3,4)}(\mathbb{X}))^2 + (\alpha^{(1,3)}(\mathbb{X}) + \alpha^{(2,4)}(\mathbb{X}))^2 + (\alpha^{(1,4)}(\mathbb{X}) - \alpha^{(2,3)}(\mathbb{X}))^2 \\
&\qquad : \mathbb{X} \in \mathbb{R}^6, |\mathbb{X}| = 1\}\big).
\end{aligned}$$

In particular if $\alpha^{(j,k)}_{(\ell,m)} = \delta_{j\ell}\delta_{km}\alpha^{(j,k)}$, which corresponds to the case where the random variables $B^{(j,k)}(0)$, $1 \le j < k \le 4$, are mutually independent, then we have

$$\|\alpha\|^2_{(1\leftarrow 2)} = 2\{((\alpha^{(1,2)})^2 + (\alpha^{(3,4)})^2) \vee ((\alpha^{(1,3)})^2 + (\alpha^{(2,4)})^2) \vee ((\alpha^{(1,4)})^2 + (\alpha^{(2,3)})^2\}.$$

At present we do not have other simple representations for general higher dimensional cases, except for the following:

EXAMPLE 4. Consider the case where the components $B^{(j,k)}_\omega(0)$, $1 \le j < k \le d$, of the value of the magnetic field at a point are independently and identically distributed with

the variance $\beta(0)$. This corresponds to the case where the $D \times D$ matrix α is a constant times the identity matrix I, for which

$$\|I\|_{(1\leftarrow 2)} = \max\{\|\mathbb{X}\|_{(1)} : \mathbb{X} \in \mathbb{R}^D, \|\mathbb{X}\|_{(2)} = 1\}.$$

We note that, if the nonzero eigenvalues of X are $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_h$, $0 \leq h \leq [d/2]$, then $\|X\|_1 = 2(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_h|)$ and $\|X\|_2 = \sqrt{2(|\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_h|^2)}$, and that h varies over $\{0, 1, \dots, [d/2]\}$ and $(\lambda_1, \lambda_2, \dots, \lambda_h)$ varies over $(\mathbb{R} - 0)^h$ as $\mathbb{X}$ varies over $\mathbb{R}^D$. Thus we obtain

$$\|I\|_{(1\leftarrow 2)} = \sqrt{2\Big[\frac{d}{2}\Big]}.$$

Thus the result of Theorem 2.1 is

$$\lim_{\lambda\downarrow -\infty} \frac{1}{\lambda^2} \log N(\lambda) = \frac{-2}{(g-2)^2[d/2]\beta(0)}.$$

A statement corresponding to Proposition 2.2 is the following: if $B_\omega^{(j,k)}$, $1 \leq j < k \leq d$ are independently and identically distributed Gaussian random variable with the mean 0 and the variance β, then the corresponding random skew-symmetric matrix $B = (B_{\omega,k}^j)$ satisfies

$$\lim_{t\uparrow\infty} \frac{1}{t^2} \log E[\exp(t\|B\|_1)] = 2\Big[\frac{d}{2}\Big]\beta.$$

In this case we also know the explicit form of the joint probability distribution of the eigenvalues (cf. Mehta [19] §3.4).

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