Estimates on the heat kernel of the Pauli Hamiltonian and its application to problems on hypoellipticity of the $\overline{\partial_b}$-Laplacian

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1. Introduction

In [20], we construct the smooth integral kernel of the heat semigroup generated by the $\overline{\partial_b}$-Laplacian and the $\overline{\partial}$-Laplacian with the $\overline{\partial}$-Neumann boundary condition in full generality on certain domains with infinite degeneracy and, under a restricted condition, we estimate the integral kernel to prove the C^∞-local hypoellipticity of the $\overline{\partial}$-Laplacian. In this paper we give an effective upper estimate of the integral kernels under another condition and use this estimate to prove the hypoellipticity of the $\overline{\partial_b}$-Laplacian and the $\overline{\partial}$-Laplacian. The condition for the hypoellipticity is improved and the proof is simplified. For the general fact about the $\overline{\partial_b}$-problem and the $\overline{\partial}$-Neumann problem, see Folland-Kohn [7], Kohn [11] and Sibony [15]. In the course of our proof, we use an upper estimate of the integral kerenl of the heat semigroup generated by the Pauli Hamiltonian. We will also discuss a generalization of this estimate.

Our results for the $\overline{\partial_b}$-Laplacian is stated in a simple form as follows: let $L = L_1 + L_2$ be the differential operator on $\mathbb{R}^5$ defined by

$$L_j = -\partial_{x^j}^2 - (\partial_{y^j} - g_j(x^j)\partial_u)^2 + (-1)^j h_j(x^j) i\partial_u,$$

where $i = \sqrt{-1}$, $X := (x^1, y^1, x^2, y^2, u)$ is the variable of $\mathbb{R}^5$, $g_j(x)$, $j = 1, 2$, are smooth functions on $\mathbb{R}$ and $h_j(x) = g_j'(x)$. This operator L is essentially self-adjoint (cf. [4]) and we denote the self-adjoint extension by the same symbol. For this we assume the following: for some $0 < \rho < 2$, (A1, ρ) (i) h_j, $j = 1, 2$, are nonnegative;

(ii) $h_j^{-1}(0) \subset \{0\}$;

(iii) $h_j(x) \geq \exp\left(-\dfrac{1}{|x|^\rho}\right)$ on $[-\delta, \delta]$ for some $\delta > 0$.

(B1) (i) $g_j(x)$, $j = 1, 2$, and their derivatives are all bounded;

(ii) $h_j(x)$, $j = 1, 2$, decay polynomially as $x \to \pm\infty$: $\underline{\lim}_{|x|\to\infty} \log h_j(x)/\log|x| > -\infty$.

Then the result is the following:

Theorem 1. *Under the assumptions* (A1, ρ) *for some* $0 < \rho < 2$ *and* (B1), *the heat semigroup* $\exp(-tL/2)$ *generated by the operator* $L/2$ *has the integral kernel* $k(t, X, X')$ *with respect to the Lebesgue measure, which has the representation* (1.3) *below. This function* $k(t, X, X')$ *is smooth in* $(t, X, X') \in (0, \infty) \times \mathbb{R}^5 \times \mathbb{R}^5$ *and satisfies the following estimate: for any* α, $\alpha' \in \mathbb{Z}_+^5$, $\beta \in \mathbb{Z}_+$, $\gamma \in \mathbb{R}$, N, $\varepsilon, \delta > 0$ *and* $\rho < \rho' < 2$, *there exist constants* C_1, C_2, C_3, $C_4 > 0$ *such that*

$$\begin{aligned} &\left|\partial_t^\beta \partial_X^\alpha \partial_{X'}^{\alpha'} k(t, X, X')\right| \\ &\leq C_1 \exp\left(C_2 t + C_3 \left(\frac{1}{t}\right)^{\rho'/(2-\rho')} + C_4 \left(\frac{1}{t}\right)^{2\gamma - 1} - \frac{|z - z'|^2}{2(1+\varepsilon)t} - \frac{\delta|u - u'|}{t^\gamma}\right) \end{aligned} \tag{1.1}$$

for any $t > 0$ *and* $X = (x^1, y^1, x^2, y^2, u)$, $X' = (x^{1,\prime}, y^{1,\prime}, x^{2,\prime}, y^{2,\prime}, u') \in \mathbb{R}^5$ *such that* $|x^1| + |x^2| \leq N$ *or* $|x^{1,\prime}| + |x^{2,\prime}| \leq N$, *where* $z = (x^1, y^1, x^2, y^2)$ *and* $z' = (x^{1,\prime}, y^{1,\prime}, x^{2,\prime}, y^{2,\prime})$.

The point of this theorem is that the integral kernel $k(t, X, X')$ is dominated by an exponential function of $|z - z'|^2/t$ and $|u - u'|/t^\gamma$. In [20], the integral kernel is dominated by the polynomial of the components of $X - X'$. However the polynomial estimates are not sufficient for our proof of the local hypoellipticity. In fact, to prove the C^∞-local hypoellipticity referring Kusuoka-Stroock [12], the following estimate is crucial:

Lemma 1.1. *Under the assumptions* $(\mathrm{A1}, \rho)$ *for some* $0 < \rho < 1$ *and* (B1), *it holds that*

$$\sup\left\{\left|t^{-\gamma}\partial_t^\beta \partial_X^\alpha \partial_{X'}^{\alpha'} k(t, X, X')\right| : 0 < t \leq 1, |X - X'| > \varepsilon\right\} < \infty \tag{1.2}$$

for any α, $\alpha' \in \mathbb{Z}_+^5$, β, $\gamma \in \mathbb{Z}_+$ *and* $\varepsilon > 0$.

This is proven by the estimate (1.1). By using also Lemmas 3.1, 3.2 and 3.3 below, we obtain the following:

Corollary . *Under the assumptions* $(\mathrm{A1}, \rho)$ *for some* $0 < \rho < 1$, *the operators* L *and* $\partial_t + L$ *are* C^∞*-locally hypoelliptic on* $\mathbb{R}^5$ *and* $\mathbb{R}^6$, *respectively, where* t *is an additional real variable.*

Our proof is based on the idea of the Malliavin calculus [1], [2], [12], [13]. This method is based on a probabilistic representation of the integral kernel $k(t, X, X')$. However, in the probabilistic representation, the term $(-1)^j h_j(x^j) i\partial_u$ in the operator L_j is difficult to treat. If this part does not appear, then the operator is called the Kohn Laplacian and the probabilistic representation is a standard object of the Malliavin calculus. The condition for the hypoellipticity of the Kohn Laplacian is same with that for our operator: $0 < \rho < 1$ (cf.[1], [2], [12], [13]).

To treat the term $(-1)^j h_j(x^j) i\partial_u$ in our case, we use the partial Fourier transformation

$$\widehat{\varphi}(z, \xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi u} \varphi(z, u) du$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^5)$ to transform as

$$(\widehat{L\varphi})(z, \xi) = L(\xi)\widehat{\varphi}(z, \xi),$$

where $L(\xi) = L_1(\xi) + L_2(\xi)$ and

$$L_j(\xi) = -\partial_{x^j}^2 - (\partial_{y^j} + i\xi g_j(x^j))^2 + (-1)^j \xi h_j(x^j).$$

Then, as in Theorem 1 in [20],

$$k(t,(z^1,z^2,u),(z^{1,\prime},z^{2,\prime},u')) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi(u-u')} q_1(t,z^1,z^{1,\prime};\xi) q_2(t,z^2,z^{2,\prime};\xi)d\xi \qquad (1.3)$$

is our representation of the integral kernel, where $z^j = (x^j, y^j)$, $z^{j,\prime} = (x^{j,\prime}, y^{j,\prime})$ and $q_j(t,z^j,z^{j,\prime};\xi)$ is the integral kernel of the heat semigroup $\exp(-tL_j(\xi)/2)$ generated by the self adjoint extension of $L_j(\xi)/2$ with respect to the Lebesgue measure. Now the operator $L_j(\xi)$ is the Schrödinger operator with a magnetic field for a spin 1/2 particle (the Pauli Hamiltonian) and the kernel $q_j(t,z^j,z^{j,\prime};\xi)$ is represented by the Feynman-Kac-Itô formula and the generalized Wiener functional theory by S.Watanabe as follows:

$$\begin{aligned} &q_j(t,z^j,z^{j,\prime};\xi) \\ &= E\Big[\exp\Big(i\xi\int_0^t g_j(x^j+x(s))dy(s) \qquad (1.4) \\ &\qquad -(-1)^j\frac{\xi}{2}\int_0^t h_j(x^j+x(s))ds\Big)\delta_{x^{j,\prime}}(x^j+x(t))\delta_{y^{j,\prime}}(y^j+y(t))\Big], \end{aligned}$$

where $(x(t),y(t))$, $t>0$, is the 2-dimensional standard Brownian motion and $\delta_{x^{j,\prime}}$, $\delta_{y^{j,\prime}}$ are the Dirac delta functions at $x^{j,\prime}$, $y^{j,\prime}$, respectively, on $\mathbb{R}$ (cf. [10], [22]). To prove the well definedness of the representation in (1.3), we prove the following:

Lemma 1.2. *There exist constants $C, C' > 0$ such that*

$$|q_1(t,z^1,z^{1,\prime};\xi)| \le C\frac{\xi}{t}\exp\Big(C't - \frac{|z^1-z^{1,\prime}|^2}{2t}\Big) \qquad (1.5)$$

for all $t>0$ and $\xi \ge 3$.

Lemma 1.3. *For any $\rho' \in (\rho, 2)$ and $N>0$, there exist constants C_1, C_2, $C_3 > 0$ such that*

$$\sup_{|x|\le N} E\Big[\exp\Big(-\xi\int_0^t h_j(x+x(s))ds\Big)\Big] \le C_1\exp\Big(-C_2(t\wedge C_3)(\log\xi)^{2/\rho'}\Big)$$

for all $t > 0$ *and* $j = 1, 2$.

These results are generalized as we will discuss later. However we need extra techniques to prove the estimate (1.1). Now the crucial assumption is that the functions g_j and h_j depend only on one variable. By this assumption, we can easily calculate the expectation with respect to $y(\cdot)$ in (1.5) and we obtain

$$\begin{aligned} &q_j(t, z^j, z^{j,\prime}; \xi) \\ &= E\Big[\exp\Big\{i\xi(y^{j,\prime} - y^j)\frac{1}{t}\int_0^t g_j(x^j + x(s))ds \\ &\qquad -\frac{\xi^2}{2}\int_0^t \Big(g_j(x^j + x(s)) - \frac{1}{t}\int_0^t g_j(x^j + x(r))dr\Big)^2 ds \\ &\qquad -(-1)^j\frac{\xi}{2}\int_0^t h_j(x^j + x(s))ds\Big\}\delta_{x^{j,\prime}}(x^j + x(t))\delta_{y^{j,\prime}}(y^j + y(t))\Big]. \end{aligned} \tag{1.6}$$

Since the integrand becomes a positive functional except for a simple oscillation factor, the estimate of $q_j(t, z^j, z^{j,\prime}; \xi)$ becomes easier. Then we can show (1.1) by shifting the integral contour in the integral (1.3) in the complex plane. As for simple cases as above, Florchinger and Léandre [6] estimate the integral kernel of the heat semigroup generated by the operator

$$\partial_x^2 + g(x)^2\partial_y^2 + f(x)\partial_y \quad \text{ on } \mathbb{R}^2$$

in detail, where $g(x)$ satisfies the condition as $(\mathrm{A1}, \rho)$ (iii).

As for Lemma 1.2, we will discuss a generalization and give an improvement of an estimate of a stochastic oscillatory integral by Erdös [5], where an intrinsic meaning of the asymptotics is explained by a method reflecting the Brownian path. For the proof we use only the positivity of the Pauli Hamiltonian $L_j(\xi)$. For the generalization of Lemma 1.3, we can refer [20] and [21]. The only problem in our present setting is that the condition (B1) breaks the conditions in [20] and [21] since the functions $h_j(x)$ decay as $x \to \pm\infty$.

However we can overcome this problem easily by a simple localization argument as in the proof of Lemma 1.3 below.

The organization of this paper is as follows. In Section 2, we give fundamental estimates of the integral kernel of the heat semigroup generated by the Pauli Hamiltonians and prove a generalization of Lemma 1.2. In Section 3, we prove Theorem 1 and its corollary stated above. In Section 4, we state our results in a general form by usual terms in the theory of several complex variables.

2. **Estimates for the heat kernel of the Pauli Hamiltonian**

Lemma 1.2 treated the Schrödinger operator with the vector potential depending only on 1 coordinate. However this lemma is extended to the general vector potential. In this section we discuss the generalization.

We first consider on $\mathbb{R}^2$. Let $b = (b^1(x), b^2(x))$ be an $\mathbb{R}^2$-valued smooth function on $\mathbb{R}^2$ such that $B = \partial_{x^2} b^1 - \partial_{x^1} b^2$ is bounded and all derivatives of b are polynomially bounded. For $\xi \in \mathbb{R}$, we consider the self adjoint operator

$$H(\xi) = \sum_{j=1}^{2} (i\partial_{x^j} - \xi b^j)^2 - \xi B(x)$$

on $L^2(\mathbb{R}^2)$. Let $q(t, x, x'; \xi)$ be the integral kernel of the heat semigroup $\exp(-tH(\xi)/2)$ generated by $H(\xi)/2$ with respect to the Lebesgue measure. For this we prove the following:

Proposition 2.1. *For all $t > 0$ and $\xi \geq 3$, it holds that*

$$|q(t, x, x'; \xi)| \leq \frac{\xi}{4\pi t} \exp\Big(t\|B\|_\infty - \frac{|x - x'|^2}{2t}\Big), \tag{2.1}$$

where $\|B\|_\infty = \sup |B|$.

Proof. By the gauge invariance, we may assume $\nabla \cdot b = 0$. As in (1.5), we have the representation as follows:

$$\begin{aligned} &q(t, x_a, x_b; \xi) \\ &= E\Big[\exp\Big(i\xi \sum_{j=1}^{2} \int_0^t b^j(x_a + x(s)) dx^j(s) \\ &\qquad + \frac{\xi}{2}\int_0^t B(x_a + x(s)) ds\Big) \delta_{x_b}(x_a + x(t))\Big], \end{aligned} \tag{2.2}$$

where $x(t)$, $t > 0$, is the 2-dimensional standard Brownian motion and δ_{x_b} is the Dirac delta function at x_b on $\mathbb{R}^2$ (cf. [10], [22]). By fundamental properties of the pinned

Brownian motion (see e.g. [10]), we have

$$
\begin{aligned}
&q(t,x_a,x_b;\xi)\exp\Big(\frac{|x_a-x_b|^2}{2t}\Big)\\
&=E\Big[\exp\Big(i\xi\sum_{j=1}^{2}\int_0^t b^j\big(x_a+\frac{x_b-x_a}{t}s+x(s)\big)\frac{x_b^j-x_a^j}{t}ds\\
&\qquad+i\xi\sum_{j=1}^{2}\int_0^t b^j\big(x_a+\frac{x_b-x_a}{t}s+x(s)\big)dx^j(s)\\
&\qquad+\frac{\xi}{2}\int_0^t B\big(x_a+\frac{x_b-x_a}{t}s+x(s)\big)ds\Big)\delta_0(x(t))\Big].
\end{aligned}
$$

By the Lebesgue convergence theorem, the right hand side equals

$$
\lim_{n\to\infty} I_{N(n)},
$$

where

$$
\begin{aligned}
I_n=E\Big[&\exp\Big(i\xi\sum_{k=0}^{n-1}\sum_{j=1}^{2}\frac{x_b^j-x_a^j}{n}b^j\Big(x_a+\frac{x_b-x_a}{n}k+x(\frac{kt}{n})\Big)\\
&+i\xi\sum_{k=0}^{n-1}\sum_{j=1}^{2}\int_{kt/n}^{(k+1)t/n} b^j\Big(x_a+\frac{x_b-x_a}{n}k+x(s)\Big)dx^j(s)\\
&+\frac{\xi}{2}\sum_{k=0}^{n-1}\int_{kt/n}^{(k+1)t/n} B\Big(x_a+\frac{x_b-x_a}{n}k+x(s)\Big)ds\Big)\delta_0(x(t))\Big]
\end{aligned}
$$

and $\{N(n):n\in\mathbb{N}\}$ is an increasing sequence of natural numbers. For each $0\le k\le n-1$ and $\xi\ge 0$, let

$$
H_k(\xi)=\sum_{j=1}^{2}(i\partial_{x^j}-\xi b^j\big(x_a+\frac{x_b-x_a}{n}k+x\big))^2-\xi B\big(x_a+\frac{x_b-x_a}{n}k+x\big).
$$

We denote the integral kernel of the heat semigroup generated by the self adjoint extension of $H_k(\xi)/2$ by $q_k(t,x,x';\xi)$. Then I_n is written as

$$
\begin{aligned}
I_n=&\int dx_1dx_2\cdots dx_{n-1}\varphi_0(0)q_0(t/n,0,x_1;\xi)\varphi_1(x_1)q_1(t/n,x_1,x_2;\xi)\cdots\\
&\times\varphi_{n-1}(x_{n-1})q_{n-1}(t/n,x_{n-1},0;\xi),
\end{aligned}
$$

where

$$\varphi_k(x) = \exp\Big(i\xi\sum_{j=1}^{2}\frac{x_b^j - x_a^j}{n}b^j(x_a + \frac{x_b - x_a}{n}k + x)\Big).$$

Taking $0 < \delta \leq t/3$, we estimate this as

$$|I_n| \leq I_{n,1}I_{n,2}I_{n,3},$$

where

$$\begin{aligned}
I_{n,1} :=& \Big\|\int dx_1 dx_2\cdots dx_{[n\delta/t]}\varphi_0(0)q_0\Big(\frac{t}{n},0,x_1;\xi\Big)\varphi_1(x_1)q_1\Big(\frac{t}{n},x_1,x_2;\xi\Big)\cdots \\
&\times\varphi_{[n\delta/t]}(x_{[n\delta/t]})q_{[n\delta/t]}\Big(\delta - \Big[\frac{n\delta}{t}\Big]\frac{t}{n}, x_{[n\delta/t]},\cdot;\xi\Big)\Big\|, \\
I_{n,2} :=& \Big|\Big|\Big|\exp\Big(-\Big(\Big(\Big[\frac{n\delta}{t}\Big]+1\Big)\frac{t}{n}-\delta\Big)\frac{H_{[n\delta/t]}(\xi)}{2}\Big)\varphi_{[n\delta/t]+1}\exp\Big(-\frac{t}{n}\frac{H_{[n\delta/t]+1}(\xi)}{2}\Big) \\
&\times\varphi_{[n\delta/t]+2}\exp\Big(-\frac{t}{n}\frac{H_{[n\delta/t]+2}(\xi)}{2}\Big)\cdots\varphi_{[n(t-\delta)/t]-1}\exp\Big(-\frac{t}{n}\frac{H_{[n(t-\delta)/t]-1}(\xi)}{2}\Big) \\
&\times\varphi_{[n(t-\delta)/t]}\exp\Big(-\Big(t-\delta-\Big[\frac{n(t-\delta)}{t}\Big]\frac{t}{n}\Big)\frac{H_{[n(t-\delta)/t]}(\xi)}{2}\Big)\Big|\Big|\Big|, \\
I_{n,3} :=& \Big\|\int dx_{[n(t-\delta)/t]+1}dx_{[n(t-\delta)/t]+2}\cdots dx_{n-1} \\
&\times q_{[n(t-\delta)/t]}\Big(\Big(\Big[\frac{n(t-\delta)}{t}\Big]+1\Big)\frac{t}{n}-t+\delta,\cdot,x_{[n(t-\delta)/t]+1};\xi\Big) \\
&\times\varphi_{[n(t-\delta)/t]+1}(x_{[n(t-\delta)/t]+1})q_{[n(t-\delta)/t]+1}\Big(\frac{t}{n},x_{[n(t-\delta)/t]+1},x_{[n(t-\delta)/t]+2};\xi\Big) \\
&\times\varphi_{[n(t-\delta)/t]+2}(x_{[n(t-\delta)/t]+2})q_{[n(t-\delta)/t]+2}\Big(\frac{t}{n},x_{[n(t-\delta)/t]+2},x_{[n(t-\delta)/t]+3};\xi\Big)\cdots \\
&\times\varphi_{n-1}(x_{n-1})q_{n-1}\Big(\frac{t}{n},x_{n-1},0;\xi\Big)\Big\|,
\end{aligned}$$

$[\cdot]$ is the integral part, $\|\cdot\|$ is the L^2-norm on $\mathbb{R}^2$ and $|||\cdot|||$ is the operator norm. By the stochastic representation as (2.2), we have

$$|q_k(t,x,x';\xi)| \leq \frac{1}{2\pi t}\exp\Big(-\frac{|x-x'|^2}{2t}+\frac{\xi t\|B\|_\infty}{2}\Big)$$

for some $C > 0$. By this, we have

$$I_{n,1}, I_{n,3} \leq \Big\|\frac{1}{2\pi\delta}\exp\Big(-\frac{|\cdot|^2}{2\delta}+\xi\delta C\Big)\Big\| = \frac{1}{4\pi\delta}\exp\Big(\frac{\xi t\|B\|_\infty}{2}\Big).$$

On the other hand, by Lemma 2.1 below, we have

$$I_{n,2} \leq 1.$$

Therefore we have

$$I_n \leq \frac{1}{4\pi\delta} \exp(\xi\delta \|B\|_\infty)$$

and

$$|q(t, x_a, x_b; \xi)| \exp\left(\frac{|x_a - x_b|^2}{2t}\right) \leq \frac{1}{4\pi\delta} \exp(\xi\delta \|B\|_\infty).$$

By taking $\delta = t/\xi$, we obtain (2.1). □

Lemma 2.1. *$H_k(\xi)$ is a positive operator.*

Proof. For each fixed $x_0 \in \mathbb{R}^d$, if we set

$$A := (i\partial_{x^1} - \xi b^1(x_0 + x)) - i(i\partial_{x^2} - \xi b^2(x_0 + x)),$$

then we have

$$A^*A = \sum_{j=1}^{2} (i\partial_{x^j} - \xi b^j(x_0 + x))^2 - \xi B(x_0 + x),$$

which is a positive operator. □

Moreover we have the following:

Proposition 2.2. *For any α, $\alpha' \in \mathbb{Z}_+^2$ and $\varepsilon > 0$, there exist a constant $C > 0$ and a polynomial $\mathcal{P}(\xi, 1/t)$ such that*

$$|\partial_x^\alpha \partial_{x'}^{\alpha'} q(t, x, x'; \xi)| \leq \mathcal{P}\left(\xi, \frac{1}{t}\right) \exp\left(Ct - \frac{|x - x'|^2}{2(1+\varepsilon)t}\right) \tag{2.3}$$

for all $t > 0$ and $\xi \geq 3$.

Proof. By using the semigroup property, we have

$$
\begin{aligned}
&|\partial_x^\alpha \partial_{x'}^{\alpha'} q(t,x,x';\xi)| \\
&\le \int dx_1 dx_2 |\partial_x^\alpha q(\delta t, x, x_1;\xi)||q((1-2\delta)t, x_1, x_2;\xi)||\partial_{x'}^{\alpha'} q(\delta t, x_2, x';\xi)|
\end{aligned}
$$

for any $0 < \delta \le 1/3$. By the expression (2.2), we have

$$
\begin{aligned}
&|\partial_x^\alpha q(\delta t, x, x_1;\xi)| \\
&\le \mathcal{P}_\alpha\Big(\xi, \delta t, \frac{1}{\delta t}\Big) \exp\Big(-\frac{|x-x_1|^2}{2\delta t(1+\varepsilon)} + C\xi\delta t\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\partial_{x_2}^{\alpha'} q(\delta t, x_2, x';\xi)| \\
&\le \mathcal{P}_{\alpha'}\Big(\xi, \delta t, \frac{1}{\delta t}\Big) \exp\Big(-\frac{|x_2-x'|^2}{2\delta t(1+\varepsilon)} + C\xi\delta t\Big)
\end{aligned}
$$

By using also Proposition 2.1, we can complete the proof. □

We next consider on $\mathbb{R}^d$, $d \ge 2$. Let $b = (b^1(x), \dots, b^d(x))$ be an $\mathbb{R}^d$-valued smooth function on $\mathbb{R}^d$ such that $B(x) = (B_{jk})_{1\le j,k\le d}$, $B_{jk}(x) = \partial_{x^k} b^j - \partial_{x^j} b^k$ are bounded and all derivatives of b are polynomially bounded. For $\xi \in \mathbb{R}$, we consider the self adjoint operator

$$
\begin{aligned}
H(\xi) &= \Big(\sum_{j=1}^d \gamma_j (i\partial_{x^j} - \xi b^j)\Big)^2 \\
&= \sum_{j=1}^d (i\partial_{x^j} - \xi b^j)^2 - \xi \sum_{j<k} i\gamma_j\gamma_k B_{jk}
\end{aligned}
$$

on $L^2(\mathbb{R}^d \to \mathbb{C}^D)$, where D is sufficiently large integer such that a family of Hermitian matrices $\{\gamma_j\}_{1\le j\le d}$ such that $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}$ is defined on $\mathbb{C}^D$ (cf. [3] §12.2). This operator $H(\xi)$ is the Pauli Hamiltonian on $\mathbb{R}^d$. Let $q(t,x,x';\xi)$ be the integral kernel of the heat semigroup $\exp(-tH(\xi)/2)$ generated by $H(\xi)/2$ with respect to the Lebesgue measure. By the same method as above, we have the following:

Proposition 2.3. *There exist constants $C, C' > 0$ such that*

$$|q(t,x,x';\xi)| \le C \left(\frac{\xi}{t}\right)^{d/2} \exp\left(C't - \frac{|x-x'|^2}{2t}\right) \tag{2.4}$$

for all $t > 0$ and $\xi \ge 3$.

Proposition 2.4. *For any α, $\alpha' \in \mathbb{Z}_+^d$ and $\varepsilon > 0$, there exist a constant $C > 0$ and a polynomial $\mathcal{P}(\xi, 1/t)$ such that*

$$|\partial_x^\alpha \partial_{x'}^{\alpha'} q(t,x,x';\xi)| \le \mathcal{P}\Big(\xi, \frac{1}{t}\Big) \exp\left(Ct - \frac{|x-x'|^2}{2(1+\varepsilon)t}\right) \tag{2.5}$$

for all $t > 0$ and $\xi \ge 3$.

We next consider an operator acting on scalar valued functions on $\mathbb{R}^d$, $d \ge 2$: we consider the self adjoint operator

$$\mathcal{H}(\xi) = \sum_{j=1}^d (i\partial_{x^j} - \xi b^j)^2 - \xi \|B(x)\|_1/2$$

on $L^2(\mathbb{R}^d)$, where, for any matrix A, $\|A\|_1$ is its trace norm: $\|A\|_1 = \mathrm{Tr}\sqrt{{}^tA \cdot A}$. Let $\pi(t,x,x';\xi)$ be the integral kernel of the heat semigroup $\exp(-t\mathcal{H}(\xi)/2)$ generated by $\mathcal{H}(\xi)/2$ with respect to the Lebesgue measure. This is the quantity estimated by Erdös [5]. For this we prove the following:

Proposition 2.5. *If $\nabla B(x)$ is bounded, then there exist constants $C, C' > 0$ such that*

$$|\pi(t,x,x';\xi)| \le C \Big(\frac{\xi}{t}\Big)^{d/2} \exp\left(C't\xi^{2/3} - \frac{|x-x'|^2}{2t}\right) \tag{2.6}$$

for all $t > 0$ and $\xi \ge 3$.

Proof. By Example 2 in [19],

$$\sum_{j=1}^{d}(i\partial_{x^j}-\xi b^j)^2-\xi\sum_{\ell=1}^{[d/2]}B_{2\ell,2\ell-1}(x)$$

is a positive operator. For one point $x_0\in\mathbb{R}^d$ we can take the coordinate so that

$$\sum_{\ell=1}^{[d/2]}B_{2\ell,2\ell-1}(x_0)=\|B(x_0)\|_1/2.$$

Since $\nabla B(x)$ is bounded, we can take $C>0$ independently of x_0 so that

$$\sum_{\ell=1}^{[d/2]}B_{2\ell,2\ell-1}(x)\geq\|B(x)\|_1/2-C\delta\quad\text{ on }\{x\in\mathbb{R}^d:|x-x_0|<\delta\}$$

for any $\delta>0$. We now take a family of smooth functions $\{h_\iota\}_{\iota=1}^\infty$ such that, for each ι, the diameter of the support of h_ι is less than 1,

$$\sum_\iota(h_\iota)^2\equiv 1\text{ and }C':=\sup_{x\in\mathbb{R}^d}\sum_\iota\|\nabla h_\iota(x)\|^2<\infty,$$

and we set $h_\iota^\delta(x):=h_\iota(x/\delta)$. We use the IMS localization formula:

$$\sum_{j=1}^{d}(i\partial_{x^j}-\xi b^j)^2=\sum_\iota h_\iota^\delta\sum_{j=1}^{d}(i\partial_{x^j}-\xi b^j)^2h_\iota^\delta-\sum_\iota\|\nabla h_\iota^\delta\|^2$$

(see e.g. [3] Theorem 3.2). By changing the coordinate on the support of each h_ι^δ, we see that

$$h_\iota^\delta\Big\{\sum_{j=1}^{d}(i\partial_{x^j}-\xi b^j)^2-\xi\|B(x)\|_1/2+C\xi\delta\Big\}h_\iota^\delta$$

is a positive operator. Therefore

$$\mathcal{H}(\xi)+C\xi\delta+C'/\delta^2$$

is a positive operator. This is minimized when $\delta=((2C')/(C\xi))^{1/3}$ and we see that

$$\mathcal{H}(\xi)+C''\xi^{2/3}$$

is a positive operator for some $C''>0$. Then the rest of the proof is same as in the last proposition. □

Remark 2.1. This proposition is worse than Propositions 2.1 and 2.3 because of the exponential term $\exp(C' t\xi^{2/3})$. We can remove this term if $B(x)$ is written as

$$B(x) = e(x) \begin{pmatrix} & -\beta_1(x) & & & \\ \beta_1(x) & & & & \\ & & \ddots & & \\ & & & & -\beta_{[d/2]}(x) \\ & & & \beta_{[d/2]}(x) & \\ & & & & \end{pmatrix} {}^t e(x)$$

with real nonnegative smooth functions $\beta_1(x), \dots, \beta_{[d/2]}(x)$ and an orthonormal matrix valued smooth function $e(x) = (e_{jk}(x))_{1 \le j,k \le d}$ such that $\nabla e(x)$ is bounded. In this case we can use the method in [19] to show that

$$\mathcal{H}(\xi) + C$$

is a positive operator for some $C > 0$ (cf. Example 2 in [19]). Therefore we obtain

$$|\pi(t, x, x'; \xi)| \le C \left(\frac{\xi}{t}\right)^{d/2} \exp\left(C' t - \frac{|x - x'|^2}{2t}\right)$$

for all $t > 0$ and $\xi \ge 3$.

3. **Proof of Theorem 1 and its corollary**

To prove Theorem 1, we first prepare Lemmas 1.2 and 1.3. Lemma 1.2 is a special case of Proposition 2.1 proved in the last section.

Proof of Lemma 1.3. By a simple localization of the proof of Proposition 3.3 in [20], we have the following: there exist $c_1, c_2, \dots, c_5, \delta > 0$, such that

$$\sup_{|x|\le N} P\left(\int_0^t h_j(x + x(s))ds < \frac{1}{R} \right) \le c_1 \exp(-c_2\sqrt{R}) + c_3 \exp(-c_4 t(\log R)^{2/\rho'})$$

for any $R \ge (2/t)^\delta \vee c_5$. By setting $\lambda = 1/R$ and taking the constants appropriately, we can rewrite this as follows: there exist $c_1, c_2, c_3, c_4 > 0$, such that

$$\sup_{|x|\le N} P\left(\int_0^t h_j(x + x(s))ds < \lambda \right) \le c_1 \exp(-c_2(t \wedge c_3)(-\log \lambda)^{2/\rho'})$$

for any $\lambda \le (t/2)^\delta \vee c_4$. By estimating the Laplace-Stieltjes transform of this, we can complete the proof $\square$

Proof of Theorem 1. We formally exchange the order of differentiations and integrations as follows:

$$\begin{aligned} &\partial_X^\alpha \partial_{X'}^{\alpha'} k(t, X, X') \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\xi)^{\alpha^u} (i\xi)^{\alpha^{u,\prime}} e^{-i\xi(u-u')} \prod_{j=1}^{2} \partial_{z^j}^{\alpha^j} \partial_{z^{j,\prime}}^{\alpha^{j,\prime}} q_j(t, z^j, z^{j,\prime}; \xi) d\xi, \end{aligned} \tag{3.1}$$

where $\alpha = (\alpha^1, \alpha^2, \alpha^u)$, $\alpha' = (\alpha^{1,\prime}, \alpha^{2,\prime}, \alpha^{u,\prime}) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+^2 \times \mathbb{Z}_+$. As in the proof of Lemma 3.1 in [20], we use the formula of the integration by parts on the Wiener space (see e.g.Section

V-9 of [10], (2.21) of [22]) in (1.7) as follows: for $j = 1, 2$,

$$\begin{aligned}
&\partial_{z^j}^{\alpha}\partial_{z^{j,\prime}}^{\alpha'} q_j(t, z^j, z^{j,\prime}; \xi) \\
&= E\Big[\Pi^j_{\alpha,\alpha'}(\xi, t, z^j - z^{j,\prime}, z(\cdot)) \exp\Big(i\xi(y^j - y^{j,\prime})\frac{1}{t}\int_0^t g_j(x^j + x(s))ds \\
&\quad -\frac{\xi^2}{2}\int_0^t \Big(g_j(x^j + x(s)) - \frac{1}{t}\int_0^t g_j(x^j + x(r))dr\Big)^2 ds \\
&\quad -(-1)^j\frac{\xi}{2}\int_0^t h_j(x^j + x(s))ds\Big)\delta_{x^{j,\prime}}(x^j + x(t))\Big] \exp\Big(-\frac{(y^j - y^{j,\prime})^2}{2t}\Big)\Big/\sqrt{2\pi t},
\end{aligned}$$

where $\Pi^j_{\alpha,\alpha'}(\xi, t, z^j - z^{j,\prime}, z(\cdot))$ is a polynomial of ξ, t, $1/t$, $x^j - x^{j,\prime}$, $y^j - y^{j,\prime}$ and finite number of elements of

$$\Big\{\int_0^t \partial^k g_j(x^j + x(s))s^\ell ds, \int_0^t \partial^k g_j^2(x^j + x(s))s^\ell ds : k, \ell \in \mathbb{Z}_+\Big\}.$$

We now introduce a holomorphic function $\varphi(\zeta)$, $\zeta \in \mathbb{C}$, by

$$\begin{aligned}
\varphi(\zeta) &:= \frac{1}{2\pi}e^{-i\zeta(u-u')}(-i\zeta)^{\alpha^u}(i\zeta)^{\alpha^{u,\prime}} \\
&\quad \times \prod_{j=1}^2 E\Big[\Pi^j_{\alpha,\alpha'}(\zeta, t, z^j - z^{j,\prime}, z(\cdot)) \\
&\quad \times \exp\Big(i\zeta(y^j - y^{j,\prime})\frac{1}{t}\int_0^t g_j(x^j + x(s))ds \\
&\quad -\frac{\zeta^2}{2}\int_0^t \Big(g_j(x^j + x(s)) - \frac{1}{t}\int_0^t g_j(x^j + x(r))dr\Big)^2 ds \\
&\quad -(-1)^j\frac{\zeta}{2}\int_0^t h_j(x^j + x(s))ds\Big)\delta_{x^{j,\prime}}(x^j + x(t))\Big] \exp\Big(-\frac{(y^j - y^{j,\prime})^2}{2t}\Big)\Big/\sqrt{2\pi t}.
\end{aligned}$$

Then we have

$$\partial_X^{\alpha}\partial_{X'}^{\alpha'} k(t, X, X') = \int_{-\infty}^{\infty} \varphi(\xi)d\xi.$$

By the condition (B1), we have the following estimate:

$$
\begin{aligned}
&|\varphi(\xi+i\eta)| \\
&\leq \exp\Big(\eta(u-u')+c_1|\eta(y-y')|+c_2\eta^2 t-\frac{|y-y'|^2}{2t}\Big) \\
&\quad\times\mathcal{P}_{\alpha,\alpha'}\Big(\xi,\eta,t,\frac{1}{t},x-x',y-y'\Big)\prod_{j=1}^{2}|q_j(t,(x^j,0),(x^{j,\prime},0);\xi)|,
\end{aligned}
$$

where c_1, $c_2>0$ and $\mathcal{P}_{\alpha,\alpha'}(\xi,\eta,t,1/t,x-x',y-y')$ is a polynomial of ξ, η, t, $1/t$, and the components of $x-x'$, $y-y'$ depending only on α and α'.

When $\xi\geq 0$, we have

$$
\begin{aligned}
&|q_2(t,(x^2,0),(x^{2,\prime},0);\xi)| \\
&\leq E\Big[\exp\Big(-\frac{\xi}{2}\int_0^t h_2(x^2+x(s))ds\Big)\Big|x^2+x(t)=x^{2,\prime}\Big]\frac{1}{2\pi t}\exp\Big(-\frac{(x^2-x^{2,\prime})^2}{2t}\Big).
\end{aligned}
$$

By the Hölder inequality, this is dominated by

$$
\begin{aligned}
&\Big\{E\Big[\exp\Big(-\frac{2+\varepsilon}{\varepsilon}\frac{\xi}{2}\int_0^t h_2(x^2+x(s))ds\Big)\Big|x^2+x(t)=x^{2,\prime}\Big]\frac{1}{2\pi t}\exp\Big(-\frac{(x^2-x^{2,\prime})^2}{2t}\Big)\Big\}^{\varepsilon/(2+\varepsilon)} \\
&\quad\times\Big(\frac{1}{2\pi t}\exp\Big(-\frac{(x^2-x^{2,\prime})^2}{2t}\Big)\Big)^{2/(2+\varepsilon)}.
\end{aligned}
$$

By the positivity of h_2 and the Markov property of the Brownian motion, the first factor is estimated as

$$
\begin{aligned}
&E\Big[\exp\Big(-\frac{2+\varepsilon}{\varepsilon}\frac{\xi}{2}\int_0^t h_2(x^2+x(s))ds\Big)\Big|x^2+x(t)=x^{2,\prime}\Big]\frac{1}{2\pi t}\exp\Big(-\frac{(x^2-x^{2,\prime})^2}{2t}\Big) \\
&\leq E\Big[\exp\Big(-\frac{2+\varepsilon}{\varepsilon}\frac{\xi}{2}\int_0^{t/2} h_2(x^2+x(s))ds\Big)\Big]\frac{1}{\sqrt{\pi t}}.
\end{aligned}
$$

By these, we obtain

$$
\begin{aligned}
&|q_2(t,(x^2,0),(x^{2,\prime},0);\xi)| \\
&\leq E\Big[\exp\Big(-\frac{2+\varepsilon}{\varepsilon}\frac{\xi}{2}\int_0^{t/2} h_2(x^2+x(s))ds\Big)\Big]^{\varepsilon/(2+\varepsilon)} \\
&\qquad\times\frac{1}{\pi t}\exp\Big(-\frac{(x^2-x^{2,\prime})^2}{2t}\Big).
\end{aligned}
$$

By using also Lemmas 1.2 and 1.3, we obtain the following: for any $p, \varepsilon > 0$ and $\rho < \rho' < 2$, these are constants $c_j > 0$, $j = 1, 2, 3, 4$, and a polynomial $\mathcal{P}_{\alpha,\alpha',p}(\eta, t, 1/t, x - x', y - y')$ of η, t, $1/t$ and the components of $x - x'$, $y - y'$, depending only on α, α', p, such that

$$
\begin{aligned}
&|\varphi(\xi + i\eta)| \\
&\leq \exp\biggl(\eta(u - u') + c_1|\eta(y - y')| - \frac{|y - y'|^2}{2t} - \frac{|x - x'|^2}{(2 + \varepsilon)t} \\
&\quad + c_2\eta^2 t + c_3 t + c_4\Bigl(\frac{1}{t}\Bigr)^{\rho'/(2-\rho')}\biggr) \\
&\quad \times \frac{1}{(1 + |\xi|)^p}\mathcal{P}_{\alpha,\alpha'}\Bigl(\eta, t, \frac{1}{t}, x - x', y - y'\Bigr),
\end{aligned} \tag{3.2}
$$

for any $\xi \geq 0$ and $t > 0$ under the condition $u \geq u'$. This estimate holds also for any $\xi \leq 0$ by changing the role of $q_1(t, (x^1, 0), (x^{1,\prime}, 0); \xi)$ and $q_2(t, (x^2, 0), (x^{2,\prime}, 0); \xi)$.

By the estimate (3.2), we first see that the right hand side of (1.3) is a well defined smooth function in (t, X, X') and that the formal eschange of differentiations and integrations in (3.1) is justified. Then, as in [18] and [20], we can show that the right hand side of (1.3) is really the integral kernel of the heat semigroup. Moreover, by Cauchy's integral formula, we have

$$
\begin{aligned}
0 &= \int_{-R}^{R} \varphi(\xi)d\xi + \int_{0}^{\delta/t^\gamma} \varphi(R + i\eta)id\eta \\
&\quad - \int_{-R}^{R} \varphi(\xi + i\delta/t^\gamma)d\xi - \int_{0}^{\delta/t^\gamma} \varphi(-R + i\eta)id\eta \\
&=: I_1(R) + I_2(R) - I_3(R) - I_4(R)
\end{aligned}
$$

for any $R > 0$. By (3.2) we see that $\lim_{R\to\infty} I_1(R)$ and $\lim_{R\to\infty} I_3(R)$ exist and that $\lim_{R\to\infty} I_2(R) = \lim_{R\to\infty} I_4(R) = 0$. Therefore we have

$$
\partial_X^\alpha \partial_{X'}^{\alpha'} k(t, X, X') = \int_{-\infty}^{\infty} \varphi(\xi + i\delta/t^\gamma)d\xi.
$$

By (3.2), we see that this is dominated by

$$c_1 \exp\left(\frac{\delta(u-u')}{t^\gamma} - \frac{|z-z'|^2}{2(1+\varepsilon)t} + c_2\left(\frac{1}{t}\right)^{\rho'/(2-\rho')} + c_3\left(\frac{1}{t}\right)^{2\gamma-1} + c_4 t\right).$$

□

The proof of Corollary of Theorem 1 is reduced to the following, which is essentially due to Kusuoka-Stroock [12]:

Lemma 3.1. *If Lemmas 1.1, 3.2 and 3.3 below hold, then the operators L and $\partial_t + L$ are C^∞-locally hypoelliptic on $\mathbb{R}^5$ and $\mathbb{R}^6$, respectively.*

Lemma 3.2. *For any $m \in \mathbb{Z}_+$, there exists $C > 0$ such that*

$$\sup_{t\in(0,1]} \|e^{-tL}\phi\|_m \le C\|\phi\|_m \tag{3.3}$$

for any $\phi \in \mathcal{S}(\mathbb{R}^5)$, where

$$\|\phi\|_m = \sum_{\alpha\in\mathbb{Z}_+^5:|\alpha|\le m} \|\partial^\alpha\phi\|,$$

$\|\cdot\|$ is the L^2-norm.

This lemma is proven similarly as in the proof of Proposition 5.3 in [20]. The proof is easier than that of Proposition 5.3 in [20] by the condition (B1).

Lemma 3.3. *For any $\alpha \in \mathbb{Z}_+^5$ and $\psi \in \mathcal{S}(\mathbb{R}^5)$, $\partial_X^\alpha e^{-tL}\psi \to \partial_X^\alpha\psi$ in $L^2(\mathbb{R}^5)$ as $t \to 0$.*

Proof. By the Parseval equality, it is enough to show that $\|\xi^\beta\partial_z^\alpha(e^{-tL(\xi)}\phi - \phi)\| \to 0$ as $t \to 0$ for any $\alpha \in \mathbb{Z}_+^4$, $\beta \in \mathbb{Z}_+$ and $\phi = \phi(z,\xi) \in \mathcal{S}(\mathbb{R}^5)$, where $\|\cdot\|$ is the L^2-norm on $\mathbb{R}^5$ with the variable $(z,\xi)\in\mathbb{R}^4\times\mathbb{R}$. By (1.7) and making the change of variables $(v=(y'-y)/\sqrt{t})$, we have

$$
\begin{aligned}
&(e^{-tL(\xi)}\phi)(z,\xi)\\
&=\int_{\mathbb{R}^2}dvE\Big[\exp\Big\{i\xi\sum_{j=1}^{2}\frac{v^j}{\sqrt{t}}\int_0^t g_j(x^j+x^j(s))ds\\
&\qquad -\frac{\xi^2}{2}\int_0^t\sum_{j=1}^{2}\Big(g_j(x^j+x^j(s))-\frac{1}{t}\int_0^t g_j(x^j+x^j(r))dr\Big)^2ds\\
&\qquad +\frac{\xi}{2}\int_0^t(h_1(x^1+x^1(s))-h_2(x^2+x^2(s)))ds\Big\}\\
&\qquad \times\phi(x+x(t),y+\sqrt{t}v,\xi)\Big]\frac{1}{2\pi}\exp\Big(-\frac{|v|^2}{2}\Big),
\end{aligned}
$$

where $x(t)=(x^1(t),x^2(t))$ is the 2-dimensional Brownian motion. Then we have

$$
\begin{aligned}
&\|\xi^\beta\partial_z^\alpha(e^{-tL(\xi)}\phi-\phi)\|\\
&\le\|e^{-tL(\xi)}\xi^\beta\partial_z^\alpha\phi-\xi^\beta\partial_z^\alpha\phi\|\\
&\quad+\Big\|\int_{\mathbb{R}^2}dvE\Big[\sum_{|\gamma|\le|\alpha|}\Pi_\gamma(t,x,v,x(\cdot))\exp\Big\{i\xi\sum_{j=1}^{2}\frac{v^j}{\sqrt{t}}\int_0^t g_j(x^j+x^j(s))ds\\
&\qquad -\frac{\xi^2}{2}\int_0^t\sum_{j=1}^{2}\Big(g_j(x^j+x^j(s))-\frac{1}{t}\int_0^t g_j(x^j+x^j(r))dr\Big)^2ds\\
&\qquad +\frac{\xi}{2}\int_0^t(h_1(x^1+x^1(s))-h_2(x^2+x^2(s)))ds\Big\}\\
&\qquad \times\xi^\beta\partial_z^\gamma\phi(x+x(t),y+\sqrt{t}v,\xi)\Big]\frac{1}{2\pi}\exp\Big(-\frac{|v|^2}{2}\Big)\Big\|\\
&=:I_1(t)+I_2(t),
\end{aligned}
$$

where, for each $\gamma\in\mathbb{Z}_+^4$ such that $|\gamma|\le|\alpha|$, $\Pi_\gamma(t,x,v,x(\cdot))$ is a polynomial of elements of

$$
\begin{aligned}
&\Big\{\frac{v^j}{\sqrt{t}}\int_0^t g_j^{(k)}(x^j+x^j(s))ds,\frac{1}{\sqrt{t}}\int_0^t g_j^{(k)}(x^j+x^j(s))ds,\\
&\int_0^t(g_j^2)^{(k)}(x^j+x^j(s))ds,\int_0^t g_j^{(k)}(x^j+x^j(s))ds:k\in\mathbb{Z}_+,j=1,2\Big\}
\end{aligned}
$$

whose 0-th order term is 0. We know that $I_1(t) \to 0$ as $t \to 0$. By the condition (B1), we have

$$I_2(t) \le C\sqrt{t} \sum_{|\gamma| \le |\alpha|} I(t; \xi^\beta \partial_z^\gamma \phi)$$

for $0 < t \le 1$, where, for any $\phi = \phi(z, \xi) \in \mathcal{S}(\mathbb{R}^5)$,

$$\begin{aligned} I(t;\phi) = \Big\| \int_{\mathbb{R}^2} dv E\Big[\exp\Big\{ -\frac{\xi^2}{2} \int_0^t \sum_{j=1}^2 \Big(g_j(x^j + x^j(s)) - \frac{1}{t} \int_0^t g_j(x^j + x^j(r)) dr \Big)^2 ds \\ +\frac{\xi}{2} \int_0^t (h_1(x^1 + x^1(s)) - h_2(x^2 + x^2(s))) ds \Big\} \\ \times |\phi(x + x(t), y + \sqrt{t} v, \xi)| \Big] \frac{1}{2\pi} \exp\Big(-\frac{|v|^2}{2} \Big) \Big\|. \end{aligned}$$

$I(t;\phi)$ is rewritten as

$$\begin{aligned} I(t;\phi) = \Big\{ \int_{\mathbb{R}} d\xi \int_{\mathbb{R}^4} dz \Big| \int_{\mathbb{R}^4} dz' \prod_{j=1}^2 q_j(t, (x^j, 0), (x^{j,\prime}, 0); \xi) \\ \times \exp\Big(-\frac{|y - y'|^2}{2t} \Big) |\phi(z', \xi)| \Big|^2 \Big\}^{1/2}. \end{aligned}$$

By Lemma 1.2, we can show that

$$\sup_{0 < t \le 1} I(t;\phi) < \infty.$$

By these we can complete the proof. □

4. **General statement**

In this section we state our theorem in a general form by usual terms in the theory of several complex variables. We recall the notions in [20]. For a real valued smooth function f on $\mathbb{C}^n$, we consider a domain

$$D_f = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \ \mathrm{Im}\ w > f(z)\}$$

with a Hermitian metric for which $\{Z_j\}_{j=1}^{n+1}$ is a unitary frame of the holomorphic tangent bundle, where

$$Z_j = \frac{\partial}{\partial z^j} + 2i\frac{\partial f}{\partial z^j}\frac{\partial}{\partial w}, \quad j = 1, 2, \ldots, n, \quad Z_{n+1} = i\sqrt{2}\frac{\partial}{\partial w}.$$

For related examples of domains, see [8], [14], [16] and [20]. The Levi form of this domain is the matrix valued function

$$\left(\frac{\partial^2 f}{\partial z^j \partial \overline{z^k}}\right)_{1 \leq j,k \leq n}$$

on the boundary bD_f (cf. [7] Proposition (3.2.1)). We will consider the case where this degenerates infinitely.

We use as coordinates on D_f (z, u, r) where $u = \mathrm{Re}\ w$ and $r = \mathrm{Im}\ w - |z|^2$. Then we can regard D_f as the product of some domain B_f, which is identified with the boundary bD_f, and $\mathbb{R}_+$. (z, u) is a coordinate of B_f and r is a coordinate of $\mathbb{R}_+$. In terms of these coordinates,

$$Z_j = \frac{1}{2}(X_j - iY_j), \ j = 1, 2, \ldots, n, \quad Z_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r} + i\frac{\partial}{\partial u}\right),$$

where

$$X_j = \frac{\partial}{\partial x^j} + \frac{\partial f}{\partial y^j}\frac{\partial}{\partial u}, \quad Y_j = \frac{\partial}{\partial y^j} - \frac{\partial f}{\partial x^j}\frac{\partial}{\partial u}$$

and $z^j = x^j + iy^j$. The domain B_f is just the Euclidean space $\mathbb{R}^{2n+1}$ with the Cauchy-Riemann structure spanned by the complex vector fields $\{Z_j\}_{j=1}^n$ and the metric for which $\{X_1/\sqrt{2}, Y_1/\sqrt{2}, \ldots, X_n/\sqrt{2}, Y_n/\sqrt{2}, \partial/\partial u\}$ is an orthonormal frame. The metric on the domain D_f is the product of the metric on B_f and the standard metric on $\mathbb{R}_+$. The volume element of the domain B_f is $2^n dx^1 dy^1 \cdots dx^n dy^n du$ and that of the domain D_f is $2^n dx^1 dy^1 \cdots dx^n dy^n du dr$.

Let $\Lambda^{p,q}(B_f)$ be the vector bundle on B_f spanned by $\omega^J \wedge \overline{\omega^K}$, $J \in \mathcal{I}(p)$, $K \in \mathcal{I}(q)$, where $\mathcal{I}(p) = \{J = (j_1, j_2, \dots, j_p) : 1 \leq j_1 < j_2 < \cdots < j_p \leq n\}$, $\omega^J = \omega^{j_1} \wedge \omega^{j_2} \wedge \cdots \wedge \omega^{j_p}$ for $J = (j_1, j_2, \dots j_p) \in \mathcal{I}(p)$, $\{\omega^j\}_{j=1}^{n+1}$ is the dual frame of $\{Z_j\}_{j=1}^{n+1}$ and $\overline{\omega^K}$ is the complex conjugate of ω^K. For any subbundle U of $\bigoplus_{p,q=0}^n \Lambda^{p,q}(B_f)$, let $\mathcal{S}(U)$ be the space of sections on B_f to U whose coefficients relative to $\omega^J \wedge \overline{\omega^K}$ are rapidly decreasing functions of $(x^1, y^1, \dots, x^n, y^n, u) \in \mathbb{R}^{2n+1}$, $L^2(U)$ be that of square integrable sections with respect to the Lebesgue measure $dx^1 dy^1 \cdots dx^n dy^n du$ and $\Gamma_0^\infty(U)$ be that of smooth sections with compact supports.

The $\overline{\partial_b}$-Laplacian is defined on $\Gamma_0^\infty(\Lambda^{p,q}(B_f))$ by

$$\Box_b = \overline{\partial_b}\vartheta_b + \vartheta_b\overline{\partial_b},$$

where $\overline{\partial_b}$ is a differential operator from $\Gamma_0^\infty(\Lambda^{p,q}(B_f))$ to $\Gamma_0^\infty(\Lambda^{p,q+1}(B_f))$ defined by

$$\overline{\partial_b}\varphi = \sum_{j=1}^n (\overline{Z_j}\varphi_{J\overline{K}})\overline{\omega^j} \wedge \omega^J \wedge \overline{\omega^K}$$

for any $\varphi = \varphi_{J\overline{K}}\omega^J \wedge \overline{\omega^K} \in \Gamma_0^\infty(\Lambda^{p,q}(B_f))$, and ϑ_b is the formal adjoint of $\overline{\partial_b}$. This operator $\Box_b$ is essentially self-adjoint as an operator on $L^2(\Lambda^{p,q}(B_f))$ (cf. [4]) and we denote the self-adjoint extension also by the same symbol. This operator is concretely expressed as

$$\Box_b = -\frac{1}{4}\sum_{j=1}^n (\widetilde{X_j^2} + \widetilde{Y_j^2}) - Fi\widetilde{\frac{\partial}{\partial u}}$$

on the space $\mathcal{S}(\Lambda^{p,q}(B_f))$, where, for any differential operator A on a space of scalar valued functions, $\widetilde{A}$ is the operator on the sections acting only on the coefficients of $\omega^J \wedge \overline{\omega^K}$ as

$$\widetilde{A}(\sum \varphi_{J\overline{K}}\omega^J \wedge \overline{\omega^K}) = \sum (A\varphi_{J\overline{K}})\omega^J \wedge \overline{\omega^K},$$

F is a multiplication operator defined by

$$F = \sum_{j=1}^{n} \frac{-1}{4}\Big(\frac{\partial^2 f}{\partial(x^j)^2} + \frac{\partial^2 f}{\partial(y^j)^2}\Big)[\mathrm{int}(\omega^j), \mathrm{ext}(\overline{\omega^j})] + \sum_{j\neq k} 2\frac{\partial^2 f}{\partial z^j \partial \overline{z^k}}\,\mathrm{ext}(\overline{\omega^k})\,\mathrm{int}(\omega^j),$$

for any $\omega \in \Lambda^{p,q}(B_f)$, $\mathrm{ext}(\omega)$ is the exterior multiplication and $\mathrm{int}(\overline{\omega})$ is the interior multiplication, i.e., the dual operator of $\mathrm{ext}(\omega)$. This $\mathrm{int}(\omega)$ is complex linear in ω. Therefore the operator L in Theorem 1 is just $4\Box_b$ on the vector bundle spanned by $\overline{\omega^1}$ when $n = 2$ and

$$f(z) = \sum_{j=1}^{2} \int_0^{x_j} g_j(x)dx + (\text{a constant}).$$

To analyze this operator $\Box_b$, we use the partial Fourier transformation as in [20]:

$$\widehat{\varphi}(z,\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi u}\varphi(z,u)du$$

for $\varphi \in \mathcal{S}(\Lambda^{p,q}(B_f))$. We here note that the metric, the differential structure and the Cauchy-Riemann structure of the domain B_f are all uniform in the variable u. Then we have

$$\widehat{(\Box_b\varphi)}(z,\xi) = \Box_b(\xi)\widehat{\varphi}(z,\xi),$$

where

$$\Box_b(\xi) = -\frac{1}{4}\sum_{j=1}^{n}(\widetilde{X_j(\xi)^2} + \widetilde{Y_j(\xi)^2}) - \xi F,$$

$$X_j(\xi) = \frac{\partial}{\partial x^j} - i\xi\frac{\partial f}{\partial y^j}, \quad Y_j(\xi) = \frac{\partial}{\partial y^j} + i\xi\frac{\partial f}{\partial x^j}.$$

This operator $\Box_b(\xi)$ is also essentially self-adjoint (cf. [4]) and we denote the self-adjoint extension by the same symbol. The heat semigroup $e^{-t\Box_b(\xi)}$ generated by $\Box_b(\xi)$ has the integral kernel $q(t,z,z';\xi)$, $t > 0$, z, $z' \in \mathbb{C}^n$, with respect to the Lebesgue measure $dx^1dy^1\cdots dx^ndy^n$. This $q(t,z,z';\xi)$ is represented in terms of the generalized Wiener functional theory by S.Watanabe as follows:

$$\begin{aligned}
&q(t,z,z';\xi)\\
&= E\Big[\exp\Big(-i\xi\int_0^t \nabla f(z+z(s))*dz(s)\\
&\qquad +\xi\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big],
\end{aligned}$$

where $z(t) := (z^1(t), z^2(t), \dots, z^n(t))$, $z^j(t) := (x^j(t) + iy^j(t))/\sqrt{2}$, $j = 1,2,\dots,n$, $(x^1(t), y^1(t), \dots, x^n(t), y^n(t))$ is the $2n$-dimensional standard Brownian motion,

$$\begin{aligned}
&\int_0^t \nabla f(z+z(s))*dz(s)\\
&:= \frac{1}{\sqrt{2}}\sum_{j=1}^n \int_0^t \Big(\frac{\partial f}{\partial y^j}(z+z(s))\cdot dx^j(s)\\
&\qquad -\frac{\partial f}{\partial x^j}(z+z(s))\cdot dy^j(s)\Big),
\end{aligned}$$

$\cdot dx^j(s)$, $dy^j(s)$, $j = 1,\dots,n$ are the Itô integral and $\delta_{z'}$ is the Dirac delta function at z' on $\mathbb{C}^n$ (cf. [10], [22]).

Then a candidate of the integral kernel of the heat semigroup $\exp(-t\Box_b)$ generated by the $\overline{\partial}_b$-Laplacian $\Box_b$ with respect to the Lebesgue measure $dx^1dy^1\cdots dx^ndy^ndu$ is

$$k(t,(z,u),(z',u')) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi e^{-i\xi(u-u')}q(t,z,z';\xi). \tag{4.1}$$

To justify this expression and give a good estimate, we assume the following: for some $0<\rho<2$,

(A2, ρ) (i) $f(z) = \sum_{j=1}^n f_j(x^j)$;

(ii) $f_j'' \geq 0$, $j = 1,2,\dots,n$;

(iii) The zero sets of f_j'', $j = 1,2,\dots,n$, are discrete;

(iv) For any zero point a of f_j'', $\lim_{x\to a}|x-a|^\rho \log\dfrac{1}{f_j''(x)} = 0$.

(B2) (i) ∇f and its derivatives are all bounded;

(ii) $\nabla^2 f$ decays polynomially as $|z| \to \infty$: $\underline{\lim}_{|z|\to\infty} \log|\nabla^2 f(z)|/\log|z| > -\infty$.

Under the condition $(\mathrm{A2}, \rho)$, the operator $\Box_b(\xi)$ splits as a direct sum:

$$\Box_b(\xi) = \sum_{j=1}^{n} \Box_b^j(\xi),$$

where

$$\Box_b^j(\xi) = -\frac{1}{4}(\widetilde{X_j(\xi)^2} + \widetilde{Y_j(\xi)^2}) + \frac{\xi}{4} f_j''(x^j)[\mathrm{int}(\omega^j), \mathrm{ext}(\overline{\omega^j})].$$

Accordingly the integral kernel $q(t, z, z'; \xi)$ splits as

$$q(t, z, z'; \xi) = \prod_{j=1}^{n} q_j(t, z^j, z^{j,\prime}; \xi) \tag{4.2}$$

where $q_j(t, z^j, z^{j,\prime}; \xi)$ is the integral kernel of the heat semigroup $\exp(-t\Box_b^j(\xi))$ generated by $\Box_b^j(\xi)$. This kernel splits as

$$q_j(t, z^j, z^{j,\prime}; \xi) = q_j^+(t, z^j, z^{j,\prime}; \xi) P_+^j + q_j^-(t, z^j, z^{j,\prime}; \xi) P_-^j,$$

where $q_j^\pm(t, z^j, z^{j,\prime}; \xi) P_\pm^j$ is the integral kernel of the heat semigroup $\exp(-t\Box_b^{j,\pm}(\xi))$ generated by

$$\Box_b^{j,\pm}(\xi) = -\frac{1}{4}(\widetilde{X_j(\xi)^2} + \widetilde{Y_j(\xi)^2}) \mp \frac{\xi}{4} f_j''(x^j),$$

P_+^j is the projection to the forms containing $\overline{\omega^j}$ and P_-^j is that to the forms not containing $\overline{\omega^j}$. Since the function f_j depends only on one variable x^j, the representation of

$q_j^{\pm}(t, z^j, z^{j,\prime}; \xi)$ in terms of the generalized Wiener functional theory is rewritten as follows:

$$
\begin{aligned}
& q_j^{\pm}(t, z^j, z^{j,\prime}; \xi) \\
&= E\Big[\exp\Big(i\frac{\xi}{\sqrt{2}} \int_0^t f_j'(x^j + \frac{x^j(s)}{\sqrt{2}}) dy^j(s) \\
&\qquad \pm \frac{\xi}{4} \int_0^t f_j''(x^j + \frac{x^j(s)}{\sqrt{2}}) ds \Big) \delta_{z^{j,\prime}}(z^j + z^j(t)) \Big] \\
&= E\Big[\exp\Big(i\xi(y^{j,\prime} - y^j) \frac{1}{t} \int_0^t f_j'(x^j + \frac{x^j(s)}{\sqrt{2}}) ds \\
&\qquad - \frac{\xi^2}{4} \int_0^t \Big(f_j'(x^j + \frac{x^j(s)}{\sqrt{2}}) - \frac{1}{t} \int_0^t f_j'(x^j + \frac{x^j(r)}{\sqrt{2}}) dr \Big)^2 ds \\
&\qquad \pm \frac{\xi}{4} \int_0^t f_j''(x^j + \frac{x^j(s)}{\sqrt{2}}) ds \Big) \delta_{x^{j,\prime}}(x^j + \frac{x^j(t)}{\sqrt{2}}) \Big] \frac{1}{\sqrt{\pi t}} \exp\Big(- \frac{(y^j - y^{j,\prime})^2}{t} \Big).
\end{aligned} \tag{4.3}
$$

As in the last section to show the integrability in the variable ξ in (4.1), the existence of both the factor $q_j^+(t, z^j, z^{j,\prime}; \xi)$ for some j and the factor $q_k^-(t, z^k, z^{k,\prime}; \xi)$ for some k is important. This is satisfied on $\Lambda^{p,q}(B_f)$ with $0 < q < n$ and we obtain the following:

Theorem 2. *We assume* $(\mathrm{A2}, \rho)$ *for some* $0 < \rho < 2$ *and* (B2). *Then on the bundles* $\Lambda^{p,q}(B_f)$ *with* $0 < q < n$, *the heat semigroup* $\exp(-t\Box_b)$ *generated by the* $\overline{\partial_b}$*-Laplacian* $\Box_b$ *has the integral kernel* $k(t, X, X')$, $t > 0$, X, $X' \in B_f$, *with respect to the Lebesgue measure* $dX = dx^1 dy^1 \cdots dx^n dy^n du$, *which has the representation* (4.1) *with* (4.2) *and* (4.4). *This kernel* $k(t, X, X')$ *is smooth in* $(t, X, X') \in (0, \infty) \times B_f \times B_f$ *and satisfies the following estimate: for any* α, $\alpha' \in \mathbb{Z}_+^{2n+1}$, $\beta \in \mathbb{Z}_+$, $\gamma \in \mathbb{R}$, *compact* $K \subset B_f$ *and* ε, $\delta > 0$, *there are constants* C_1, C_2, C_3, $C_4 > 0$ *such that*

$$
\begin{aligned}
& \big\| \partial_t^\beta \partial_X^\alpha \partial_{X'}^{\alpha'} k(t, X, X') \big\| \\
& \le C_1 \exp\Big(C_2 t + C_3 \Big(\frac{1}{t}\Big)^{\rho/(2-\rho)} + C_4 \Big(\frac{1}{t}\Big)^{2\gamma - 1} - \frac{|z - z'|^2}{(1+\varepsilon)t} - \frac{\delta |u - u'|}{t^\gamma} \Big)
\end{aligned} \tag{4.4}
$$

for any $t > 0$ and X, $X' \in B_f$ such that $X \in K$ or $X' \in K$, where $\|\cdot\|$ is the Hilbert-Schmidt norm. The derivatives of $k(t, X, X')$ can be calculated by differentiating under the integral sign in (4.1).

The restriction of the space to $\Lambda^{p,q}(B_f)$ with $0 < q < n$ is same for the general strictly pseudoconvex case. Theorem 1 is the special case of this theorem with $n = 2$.

From this theorem we obtain the hypoellipticity as in the last section:

Corollary . *We assume* $(\mathrm{A}2, \rho)$ *for some* $0 < \rho < 1$. *Then on the bundles* $\Lambda^{p,q}(B_f)$ *with* $0 < q < n$, *the* $\overline{\partial_b}$*-Laplacian* $\Box_b$ *and the operator* $\partial_t + \Box_b$ *are* C^∞*-locally hypoelliptic on* B_f *and* $\mathbb{R} \times B_f$, *respectively.*

We next consider the $\overline{\partial}$-Neumann problem. Let $\Lambda^{p,q}(D_f)$ be the bundle of the (p, q)-forms on D_f. For any subbundle V of $\bigoplus_{p,q=0}^{n+1} \Lambda^{p,q}(D_f)$, let $\mathcal{S}(V)$ be the space of forms on D_f to V whose coefficients relative to $\omega^J \wedge \overline{\omega^K}$, can be extended to rapidly decreasing functions of $(x^1, x^2, \ldots, x^{2n}, u, r) \in \mathbb{R}^{2n+2}$, $L^2(V)$ be that of square integrable sections with respect to the Lebesgue measure $dx^1 dx^2 \cdots dx^{2n} du dr$ and $\Gamma_0^\infty(V)$ be that of smooth sections with compact supports in D_f. $\overline{\partial}$ maps $\mathcal{S}(\Lambda^{p,q}(D_f))$ to $\mathcal{S}(\Lambda^{p,q+1}(D_f))$ and has the smallest closed extension to $L^2(\Lambda^{p,q}(D_f))$, which we also denote by $\overline{\partial}$. Let $\overline{\partial}^*$ be the adjoint of $\overline{\partial}$ on $L^2(\Lambda^{p,q}(D_f))$. We define the $\overline{\partial}$-Laplacian $\Box$ by

$$\Box = \overline{\partial}\, \overline{\partial}^* + \overline{\partial}^* \overline{\partial}$$

$$\mathrm{Dom}\ (\Box) = \{\varphi \in L^2(\Lambda^{p,q}(D_f)) : \varphi \in \mathrm{Dom}\ \overline{\partial} \cap \mathrm{Dom}\ \overline{\partial}^*,$$

$$\overline{\partial}\varphi \in \mathrm{Dom}\ \overline{\partial}^*,\ \overline{\partial}^*\varphi \in \mathrm{Dom}\ \overline{\partial}\}.$$

Then the operator $\Box$ is a positive self-adjoint operator on $L^2(\Lambda^{p,q}(D_f))$ (cf. [7] Proposition 1.3.8). For the domain we have

$$\begin{aligned}&\mathrm{Dom}(\Box)\cap\mathcal{S}(\Lambda^{0,q}(D_f))\\&=\{\varphi\in\mathcal{S}(\Lambda^{0,q}(D_f)):\ (\widetilde{\partial}_r-i\widetilde{\partial}_u)P_t\varphi=P_n\varphi=0\text{ on }bD_f\},\end{aligned}\tag{4.5}$$

where P_t is the projection to the forms not containing $\overline{\omega^{n+1}}$ and P_n is the projection to the forms containing $\overline{\omega^{n+1}}$. The boundary condition in (4.5) is the $\overline{\partial}$-Neumann boundary condition. We now consider only on the $(0,q)$-forms $\Lambda^{0,q}(D_f)$ to avoid inessential complexity. Then we have

$$\Box=\Box_b-\frac{1}{2}\left(\widetilde{\partial}_u^2+\widetilde{\partial}_r^2\right)\quad\text{on }\ \mathrm{Dom}(\Box)\cap\mathcal{S}(\Lambda^{0,q}(D_f)).$$

Then the result is the following:

Theorem 3. *We assume* $(\mathrm{A2},\rho)$ *for some* $0<\rho<2$ *and* $(\mathrm{B2})$. *Then, on the bundles* $\Lambda^{0,q}(D_f)$ *with* $0<q\le n+1$, *the heat semigroup* $e^{-t\Box}$ *generated by the* $\overline{\partial}$*-Laplacian* $\Box$ *has the integral kernel* $h(t,Z,Z')$, $t>0$, $Z,\ Z'\in D_f$, *with respect to the Lebesgue measure* $dZ=dx^1dy^1\cdots dx^ndy^ndudr$, *which has the representation*

$$\begin{aligned}&h(t,(z,u,r),(z',u',r'))\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi e^{-i\xi(u-u')}q(t,z,z';\xi)E(t,r,r',\xi),\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}E(t,r,r',\xi)=&\left[e^{-\xi^2t/2}\{e(t,r-r')+e(t,r+r')\}-\xi\sqrt{\frac{2}{\pi}}\varepsilon(t,r+r',\xi)\right]P_t\\&+e^{-\xi^2t/2}\{e(t,r-r')-e(t,r+r')\}P_n,\end{aligned}$$

$$e(t,r)=\frac{e^{-r^2/(2t)}}{\sqrt{2\pi t}},\quad \varepsilon(t,r,\xi)=e^{r\xi}\int_{r/\sqrt{t}+\xi\sqrt{t}}^{\infty}e^{-\mu^2/2}d\mu$$

and $q(t,z,z';\xi)$ is represented as (4.2) and (4.4). This kernel $h(t,Z,Z')$ is smooth in $(t,Z,Z') \in (0,\infty)\times\overline{D_f}\times\overline{D_f}$ and satisfies the following estimate: for any α, $\alpha' \in \mathbb{Z}_+^{2n+2}$, $\beta\in\mathbb{Z}_+$, $\gamma\in\mathbb{R}$, compact $K\subset\overline{D_f}$ and ε, $\delta>0$, there exist constants C_1, C_2, C_3, $C_4>0$ such that

$$
\begin{aligned}
&\left\|\partial_t^\beta\partial_Z^\alpha\partial_{Z'}^{\alpha'}h(t,Z,Z')\right\| \\
&\le C_1\exp\left(C_2t+C_3\Big(\frac{1}{t}\Big)^{\rho/(2-\rho)}+C_4\Big(\frac{1}{t}\Big)^{2\gamma-1}\right. \\
&\qquad\left.-\frac{|z-z'|^2}{(1+\varepsilon)t}-\frac{\delta|u-u'|}{t^\gamma}-\frac{(r-r')^2}{2(1+\varepsilon)t}\right)
\end{aligned}
\tag{4.7}
$$

for any $t>0$ and Z, $Z'\in\overline{D_f}$ such that $Z\in K$ or $Z'\in K$. The derivatives of $h(t,Z,Z')$ can be calculated by differentiating under the integral sign in (4.6).

The restriction to the $(0,q)$-forms with $q>0$ is same for the general strictly pseudoconvex case. The expression (4.6) was first given by Stanton on the strictly pseudoconvex Siegel domain [16], [17].

We next consider the hypoellipticity. As in [20], to consider the hypoellipticity of the operator $\partial_t+\Box$, we should introduce several notions. Since the $\overline{\partial}$-Laplacian acts diagonally on the $(0,q)$-forms under the assumption $(\mathrm{A2},\rho)$, we may regard the $\overline{\partial}$-Laplacian acts on scalar valued functions. For any domain Ω, let $\mathcal{D}(\Omega)$ be the space of all test functions, $\mathcal{D}'(\Omega)$ be that of all distributions and $<\cdot,\cdot>$ be the pairing that is complex linear in the first variable and conjugate linear in the second variable. For any $\Phi\in\mathcal{D}'(\mathbb{R}\times\overline{D_f})$, let L_Φ be the continuous complex linear map from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\overline{D_f})$ defined by

$$<\psi, L_\Phi\phi>=<\overline{\phi}\cdot\psi,\Phi>$$

for any $\phi\in\mathcal{D}(\mathbb{R})$ and $\psi\in\mathcal{D}(\overline{D_f})$. For any $\Phi\in\mathcal{D}'(\mathbb{R}\times\overline{D_f})$ such that $L_\Phi(\mathcal{D}(\mathbb{R}))\subset \mathrm{Dom}(\Box)$, let $\Box\Phi\in\mathcal{D}'(\mathbb{R}\times\overline{D_f})$ be the distribution corresponding to the continuous linear map $\Box L_\Phi$ from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\overline{D_f})$ defined by the composition of L_Φ and $\Box$ (cf. [9] the kernel theorem). Then the result is the following:

Corollary . *We assume $(\mathrm{A2},\rho)$ for some $0<\rho<1$. Then on the bundles $\Lambda^{0,q}(D_f)$ with $0<q\le n+1$, the $\overline{\partial}$-Laplacian $\Box$ and the operator $\partial_t+\Box$ are C^∞-locally hypoelliptic on $\overline{D_f}$ and $\mathbb{R}\times\overline{D_f}$, respectively, in the following sense:*

(i) *If $\Phi\in\mathrm{Dom}(\Box)\cap L^2(\Lambda^{0,q}(D_f))$ and $\Box\Phi$ is smooth on an open set W of $\overline{D_f}$, then Φ is smooth on W.*

(ii) *If $\Phi\in\mathcal{D}'(\mathbb{R}\times\overline{D_f})$ such that $L_\Phi(\mathcal{D}(\mathbb{R}))\subset\mathrm{Dom}(\Box)$ and $\partial_t\Phi+\Box\Phi$ is smooth on an open set W of $\mathbb{R}\times\overline{D_f}$, then Φ is smooth on W.*

In [20], the condition for the hypoellipticity was $\rho<2/3$. However the condition is weakened as in this paper.

Finally we discuss a generalization of our theorem on the $\overline{\partial}$-Neumann problem by referring [20] and [21]. We assume the following instead of $(\mathrm{A2},\rho)$: for some $0<\rho<2$ and $0<n_+\le n$,

$(\mathrm{A3},\rho,n_+)$ (i) $f(z)=f_+(z_+)+\sum_{j=n_+1}^{n}f_j(x^j)$ where $z_+=(z^1,z^2,\dots,z^{n_+})$;
(ii) $f_j''\ge 0$, $j=n_++1,n_++2,\dots,n$;
(iii) The zero sets of f_j'', $j=n_++1,n_++2,\dots,n$, are discrete;
(iv) For any zero point a of f_j'', $j=n_++1,n_++2,\dots,n$, $\lim_{x\to a}|x-a|^\rho\log(1/f_j''(x))=0$;
(v) $\partial^2 f_+/(\partial z^j\partial\overline{z^k})=0$ for $j\ne k$ and $\Delta_+f_+\ge\prod_{\mu=1}^N h\circ\psi_\mu$, where $\Delta_+=\sum_{j=1}^{n_+}\partial^2/(\partial z^j\partial\overline{z^j})$, $N\in\mathbb{N}$, and $h(\lambda):\mathbb{R}\to[0,\infty)$ and $\psi_\mu(z_+):\mathbb{C}^{n_+}\to\mathbb{R}$ are functions satisfying the following:

(v-i) h is even, non-decreasing on $[0,\infty)$ and $h^{-1}(0)=\{0\}$;

(v-ii) ψ_μ is C^∞, the derivatives of ψ_μ are all bounded and

$$\inf\{\psi_\mu(z_+)^2+\|d\psi_\mu(z_+)\|^2:|z_+|\le K\}>0$$

for any $K > 0$;

(v-iii) $\lim_{\lambda \downarrow 0} \lambda^\rho \log \dfrac{1}{h(\lambda)} = 0$.

Moreover we introduce the following conditions:

(B3) (i) ∇f and its derivatives are all bounded;

(ii) for each $j = n_+ + 1, n_+ + 2, \ldots, n$, the function $f_j(x^j)$ decays polynomially as $x^j \to \pm\infty$: $\underline{\lim}_{x^j \to \pm\infty} \log |f_j''(x^j)| / \log |x^j| > -\infty$;

(iii) for each $\mu = 1, 2, \ldots, N$, the set $\psi_\mu^{-1}(0)$ is compact.

(C) $\Delta_+ \psi_\mu = 0$ and $\nabla \Delta_+ \psi_\mu = 0$ on $\psi_\mu^{-1}(0)$ for any $\mu = 1, 2, \ldots, N$.

Let $\Lambda^{(p), <n_+>}(D_f)$ be the vector bundle on D_f spanned by $\omega^{j_1} \wedge \omega^{j_2} \wedge \cdots \wedge \omega^{j_p} \wedge \overline{\omega^1} \wedge \overline{\omega^2} \wedge \cdots \wedge \overline{\omega^{n_+}}$, $1 \leq j_1 < j_2 < \cdots < j_p \leq n$, and $\omega^{j_1} \wedge \omega^{j_2} \wedge \cdots \wedge \omega^{j_p} \wedge \overline{\omega^1} \wedge \overline{\omega^2} \wedge \cdots \wedge \overline{\omega^{n_+}} \wedge \overline{\omega^{n_+ + 1}}$. On the space $\Lambda^{(p), <n_+>}(D_f)$, the integral kernel is represented as (4.6) and

$$q(t, z, z'; \xi) = q_+(t, z_+, z'_+; \xi) \prod_{j=n_+ + 1}^{n} q_j^-(t, z^j, z^{j,\prime}; \xi), \tag{4.8}$$

where

$$\begin{aligned} &q_+(t, z_+, z'_+; \xi) \\ &= E\Big[\exp\Big(-i\xi \int_0^t \nabla f_+(z_+ + z_+(s)) * dz_+(s) \\ &\qquad + \frac{\xi}{4} \int_0^t \Delta_+ f_+(z_+ + z_+(s)) ds \Big) \delta_{z'_+}(z_+ + z_+(t)) \Big], \end{aligned} \tag{4.9}$$

$z_+(t) = (z^1(t), z^2(t), \ldots, z^{n_+}(t))$ and

$$\begin{aligned} &\int_0^t \nabla f_+(z_+ + z_+(s)) * dz_+(s) \\ &= \frac{1}{\sqrt{2}} \sum_{j=1}^{n_+} \int_0^t \Big(\frac{\partial f_+}{\partial y^j}(z_+ + z_+(s)) \cdot dx^j(s) - \frac{\partial f_+}{\partial x^j}(z_+ + z_+(s)) \cdot dy^j(s) \Big). \end{aligned}$$

Then we have the following:

Theorem 4. (i) *We assume* (B2), (C), (A3, ρ, n_+) *for some* $n_+ > 0$ *and* $0 < \rho < 2$. *Then on the bundles* $\Lambda^{(p),<n_+>}(D_f)$ *with* $0 \leq p \leq n$, *the same results of Theorem 3 hold, where the estimate* (4.7) *is replaced by the following: for any* α, $\alpha' \in \mathbb{Z}_+^{2n+2}$, $\beta \in \mathbb{Z}_+$, $\gamma \in \mathbb{R}$, *compact* $K \subset \overline{D_f}$ *and* ε, $\delta > 0$, *there are constants* C_1, C_2, C_3, $C_4 > 0$ *such that*

$$\begin{aligned}
&\left\|\partial_t^\beta \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t, Z, Z')\right\| \\
&\leq C_1 \exp\left(C_2 t + C_3\Big(\frac{1}{t}\Big)^{\rho/(2-\rho)} + C_4\Big(\frac{1}{t}\Big)^{2\gamma-1}\right. \\
&\qquad \left. - \frac{|z-z'|^2}{(1+\varepsilon)t} - \frac{\delta|u-u'|}{t^\gamma} - \frac{(r-r')^2}{2(1+\varepsilon)t}\right)
\end{aligned} \tag{4.10}$$

for any $t > 0$ *and* Z, $Z' \in \overline{D_f}$ *such that* $Z \in K$ *or* $Z' \in K$.

(ii) *We assume* (B2) *and* (A3, ρ, n_+) *for some* $n_+ > 0$ *and* $0 < \rho < 1$. *Then on the bundles* $\Lambda^{(p),<n_+>}(D_f)$ *with* $0 \leq p \leq n$, *the same results of Theorem 3 hold, where the estimate* (4.7) *is replaced by the following: for any* α, $\alpha' \in \mathbb{Z}_+^{2n+2}$, $\beta \in \mathbb{Z}_+$, $\gamma \in \mathbb{R}$, *compact* $K \subset \overline{D_f}$ *and* ε, δ, $\mu > 0$, *there are constants* C_1, C_2, C_3, $C_4 > 0$ *such that*

$$\begin{aligned}
&\left\|\partial_t^\beta \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t, Z, Z')\right\| \\
&\leq C_1 \exp\left(C_2 t + C_3\Big(\frac{1}{t}\Big)^{\mu+\rho/(1-\rho)} + C_4\Big(\frac{1}{t}\Big)^{2\gamma-1}\right. \\
&\qquad \left. - \frac{|z-z'|^2}{(1+\varepsilon)t} - \frac{\delta|u-u'|}{t^\gamma} - \frac{(r-r')^2}{2(1+\varepsilon)t}\right)
\end{aligned} \tag{4.11}$$

for any $t > 0$ *and* Z, $Z' \in \overline{D_f}$ *such that* $Z \in K$ *or* $Z' \in K$.

Thus we have the following:

Corollary . *We assume* (A3, ρ, n_+) *for some* $n_+ > 0$ *and* $0 < \rho < 1$. *Moreover we assume* (C) *or* $\rho < 1/2$. *Then on the bundles* $\Lambda^{(p),<n_+>}(D_f)$ *with* $0 \leq p \leq n$, *the same results of the corollary of Theorem 3 hold.*

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