A self-adjoint definition of a Dirac operator with the white noise magnetic field on the plane

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Abstract. –

A 2-dimensional Dirac operator in a magnetic field given by the white noise is constructed as a self-adjoint operator. The construction is based on the paracontrolled calculus and does not require renormalization. The operator is showen to be the strong resolvent limit of Dirac operators associated with smooth approximations of the noise. Moreover, its spectrum is identified with the whole real line.

1. Introduction

We study the Dirac operator

$$D^{\xi} := \sum_{\iota=1}^{2} \gamma_{\iota}(i\partial_{\iota} + A_{\iota}^{\xi}(x) + A_{\iota}^{B}(x)) \tag{1.1}$$

acting on $\mathbb{C}^2$-valued functions on $\mathbb{R}^2$, where

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

are the Pauri matrices,

$$A^{B}(x) = \frac{B}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \tag{1.2}$$

is a vector potential of a constant magnetic field $B \in \mathbb{R}$ and $A^{\xi}(x)$ is a vector potential of a magnetic field

$$\nabla \times A^{\xi}(x) = \partial_1 A_2^{\xi}(x) - \partial_2 A_1^{\xi}(x) = \xi$$

given by the white noise on the plane $\mathbb{R}^2$: $\xi = (\xi(x))_{x\in\mathbb{R}^2}$ is a Gaussian random field on $\mathbb{R}^2$ such that $\mathbb{E}[\xi(x)] = 0$ and $\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y)$ for any $x, y \in \mathbb{R}^2$, where δ is the Dirac delta distribution.

The purpose of this paper is threefold: (i) to construct the operator D^{ξ}, as a self-adjoint operator, (ii) to approximate it by the Dirac operators D^{ξ_ε} corresponding to smooth approximations ξ_ε of the noise ξ in the strong resolvent sense, and (iii) to show that its spectrum coincides with the whole real line $\mathbb{R}$.

The Dirac operator plays a central role in the description of spin-1/2 particles in both relativistic and nonrelativistic quantum mechanics (see, e.g., Thaller [17]), and it also arises as an effective Hamiltonian for electrons in graphene (see, e.g., Castro Neto, Guinea, Peres, Novoselov and Geim [4]).

For a spin-0 particle in nonrelativistic quantum mechanics, Schrödinger operators in 2 and 3 dimensions perturbed by the white noise ξ are recently investigated via renormalization techniques. For the case that the scalar potential is the white noise, the operators of the form

$$\lim_{\varepsilon\to 0}(-\Delta+\xi_\varepsilon+c_\varepsilon)$$

are constructed, where c_ε is a diverging renormalization constant. These developments rely on the recent theories on singular stochastic partial differential equations. Allez and Chouk [1] construct such an operator as a self-adjoint operator on the 2-dimensional flat torus $\mathbb{R}^2/\mathbb{Z}^2$ and show that this operator is the norm resolvent limit of the smooth approximation using the paracontrolled calculus developed by Gubinelli, Imkeller and Perkowski [6]. Gubinelli, Ugurcan and Zachhuber [7] extend these results to the 3-dimensional flat torus $\mathbb{R}^3/\mathbb{Z}^3$ and apply them to study nonlinear Schrödinger and wave equations. Ugurcan [19] extends the results to $\mathbb{R}^2$ where the diverging constant c_ε is replaced by a spatially decaying function. Labbé [13] generalizes the results to bounded domains on $(-1,1)^2$ and $(-1,1)^3$ with periodic and Dirichlet boundary conditions, applying Hairer's theory on regularity structures [8]. Hsu and Labbé [11] define the corresponding Schrödinger operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ as the self-adjoint generators of the heat semigroups obtained from the results by Hairer and Labbé [9], [10]. Mouzard [15] extend these results to compact 2-dimensional manifolds using the heat semigroup approach to paracontrolled calculus developed by Bailleul, Bernicot and Frey [2], [3]. Referring these works, the author [18] define such a self-adjoint operator on $\mathbb{R}^2$. For magnetic fields, Morin and Mouzard [14] construct the Schrödinger operator

$$\lim_{\varepsilon\to 0}\Big(\sum_{\iota=1}^{2}(i\partial_\iota+A_\iota^{\xi_\varepsilon}(x))^2-c_\varepsilon\Big)$$

and show that this operator is the norm resolvent limit of the smooth approximation on the 2-dimensional flat torus $\mathbb{R}^2/\mathbb{Z}^2$ by the heat semigroup approach to paracontrolled calculus. In this case, the renormalization constant c_ε is necessary to compensate for the divergence of the product $A_\iota^{\xi_\varepsilon}(x)^2$ as $\varepsilon\to 0$.

In contrast, the Dirac operator D^ξ is intrinsically well-defined within the paracontrolled framework without any renormalization, since it depends linearly on A^ξ and its action involves only the paraproduct between A^ξ and the spinor field. By means of a suitable gauge transformation, the singular magnetic field can be absorbed into exponential weights involving a localized inverse of the Laplacian. We introduce a natural symmetric realization of the Dirac operator on a dense core and prove that it is essentially self-adjoint almost surely. The analysis relies on paracontrolled calculus and a control of the exponential weights induced by the gauge transformation, as in [1], [7], [14], [15] and [18]. In particular, the localization of the inverse Laplacian ensures sufficient off-diagonal decay, which is crucial to consider the original operator for the magnetic field defined on the whole space. We further study approximation procedures by smooth and spatially localized magnetic fields. Our smooth approximation is given by a general mollifier and is not restricted to the one given by the heat semigroup as in [18]. We show that the corresponding Dirac operators converge in the strong resolvent sense to the singular operator constructed in this paper. This result provides robustness of the construction and justifies the interpretation of the operator as a physically meaningful limit. Finally, we establish that the spectrum of the resulting self-adjoint Dirac operator coincides almost surely with the whole real line. The proof uses a probabilistic construction of arbitrarily large regions where the operator behaves as a small perturbation of the free Dirac operator, combined with Weyls criterion. The argument is inspired by methods from random Schrödinger operators.

By our construction of the Dirac operator, its square

$$(D^\xi)^2 = \sum_{\iota=1}^{2}(i\partial_\iota + A_\iota^\xi(x) + A_\iota^B(x))^2 + (\xi(x) + B)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

reproduces the Pauli operator, again without requiring renormalization.

The organization of this paper is as follows. In Section 2 we present the definition of our operator and state the results together with a discussion of the motivation behind our formulation. In Section 3 we prove Theore 1 below, which establishes the self-adjointness of the Dirac operator. In Section 4, we prove Proposition 2.1 below, which approximates the operator. In Section 5 we prove Proposition 2.2 below, which identifies its spectral set.

2. The framework and the results

We use the Sobolev space $\mathcal{H}^\alpha(\mathbb{R}^2)$ with the index α defined, for example, by

$$\mathcal{H}^\alpha(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2) : (1+|\zeta|^2)^{\alpha/2}\widehat{f}(\zeta) \in L^2(\mathbb{R}^2)\},$$

where

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^2} dx\ \exp(-2\pi i)\zeta \cdot x) f(x)$$

denotes the Fourier transform of f. We also use the Besov α-Hölder space $\mathcal{C}^\alpha(\mathbb{R}^2)$. These spaces are particular cases of the Besov space $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)$, since $\mathcal{H}^\alpha(\mathbb{R}^2) = \mathcal{B}^\alpha_{2,2}(\mathbb{R}^2)$ and $\mathcal{C}^\alpha(\mathbb{R}^2) := \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{R}^2)$. The Besov space $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)$ with parameters $p, q \in [1,\infty], \alpha \in \mathbb{R}$ is defined, for example, via the Littlewood-Paley decomposition: we take nonnegative smooth radial functions ρ and χ on $\mathbb{R}^2$ so that $\operatorname{supp}\rho \subset \{\zeta \in \mathbb{R}^2 : 3/4 \le |\zeta| \le 8/3\}$ and $\sum_{j=-1}^{\infty} \rho_j^2 \equiv 1$, where

$$\rho_j(\zeta) = \begin{cases} \rho(\zeta/2^j) \quad (j \ge 0), \\ \chi(\zeta) \quad (j = -1). \end{cases}$$

For any $f \in \mathcal{S}'(\mathbb{R}^2)$ and $j \ge -1$, we define

$$\rho_j(i\nabla)f = \int_{\mathbb{R}^2} d\zeta\ \exp(2\pi i)\zeta \cdot x)\rho_j(2\pi\zeta)\widehat{f}(\zeta)$$

Then the Besov space is defined by

$$\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)} < \infty\},$$

where

$$\|f\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)} := \Big\{ \sum_{j=-1}^{\infty} (2^{\alpha j}\|\rho_j(i\nabla)f\|_{L^p(\mathbb{R}^2)})^q \Big\}^{1/q}$$

is a norm.

Remark 2.1. As in [18] we can define a norm of the Besov space also by the heat semigroup approach as in [14], [15] and [18]: we fix a large even natural number b and consider operators

$$Q^{(c)}_t := \frac{(-t\Delta)^c}{(c-1)!}e^{t\Delta} \text{ and } P^{(c)}_t = Id - \int_0^t \frac{ds}{s} Q^{(c)}_s = \sum_{b=0}^{c-1} \frac{(-t\Delta)^b}{b!}e^{t\Delta}$$

for $c \in [1,b] \cap \mathbb{N}$. For $k \in [0,2b] \cap \mathbb{Z}$, let $StGC^k$ be the set of all families of operators of the form

$$((\sqrt{t}\partial_1)^{\alpha_1}(\sqrt{t}\partial_2)^{\alpha_2} P^{(c)}_t)_{t\in(0,1]}$$

with $c \in [1, b] \cap \mathbb{N}$ and $\alpha_1, \alpha_2 \in [0, \infty) \cap \mathbb{Z}$ satisfying $\alpha_1 + \alpha_2 = k$. These families of operators are colled as the standard families of Gaussian operators with cancellation of order k. We also set

$$StGC^I = \bigcup_{k \in I \cap \mathbb{Z}} StGC^k$$

for any interval I in $\mathbb{R}$. Then a norm of the Besov space $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)$ with parameters $p, q \in [1, \infty], \alpha \in \mathbb{R}$ and $\alpha \in (-2b, 2b)$ is defined by

$$\|e^{\Delta} f\|_{L^p(\mathbb{R}^2 : dx)} + \sup\{\|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)} : Q \in StGC^{(|\alpha|, 2b]}\}$$

for any $f \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)$. Thus we can use the estimates given in [14], [15] and [18].

We introduce the localized inverse of the Laplacian

$$\Delta^{-loc} := -\int_0^1 dt e^{t\Delta}, \tag{2.1}$$

which satisfies

$$\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^{\Delta}$$

and the integral kernel has a Gaussian bound:

$$\sup_{|x-y| \geq 1} \frac{\log |\Delta^{-loc}(x, y)|}{|x - y|^2} \leq \frac{-1}{4}.$$

We take a $[0, 1]$-valued smooth function χ_0 on $\mathbb{R}^2$ such that

$$\sum_{a \in \mathbb{Z}^2} \chi_a^2 \equiv 1 \text{ on } \mathbb{R}^2$$

and the support of χ_0 is included in Λ_2, where $\chi_a(x) = \chi_0(x - a)$ for any $a \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$, and $\Lambda_r = (-r/2, r/2)^2$ for any $r > 0$. Thus the support of χ_a is included in $\Lambda_2(a)$, where $\Lambda_r(a) = (a_1 - r/2, a_1 + r/2) \times (a_2 - r/2, a_2 + r/2)$ for any $a \in \mathbb{Z}^2$ and $r > 0$.

Then our definition of the Dirac operator is the following:

Definition 2.1. *Let*

$$\begin{aligned} \mathrm{Dom}_0(D^{\xi}) := \Big\{ u = \begin{pmatrix} u_- \\ u_+ \end{pmatrix} \in \bigcap_{\epsilon > 0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2 \to \mathbb{C}^2) : \\ \exp(\pm\Delta^{-loc}\xi) u_{\pm} \in \mathcal{H}^1(\mathbb{R}^2) \text{ and} \\ \limsup_{|a| \to \infty} \frac{1}{|a|} \log \|\chi_a \exp(\pm\Delta^{-loc}\xi) u_{\pm}\|_{\mathcal{H}^1\mathbb{R}^2)} < 0 \Big\}. \end{aligned} \tag{2.2}$$

For $u \in \mathrm{Dom}_0(D^\xi)$*, we define*

$$D^\xi u = \begin{pmatrix} (D^\xi u)_- \\ (D^\xi u)_+ \end{pmatrix}, \tag{2.3}$$

where

$$(D^\xi u)_\mp = \exp(\mp\Delta^{-loc}\xi)(i\partial_1 \mp \partial_2 + \widetilde{A_1^\xi} \pm i\widetilde{A_2^\xi} + A_1^B \pm iA_2^B)\exp(\pm\Delta^{-loc}\xi)u_\pm,$$

A^B *is the vector potential defined for the constant magnetic field* B *in* (1.2)*, and*

$$\widetilde{A^\xi}(x) = \int_0^1 dt\ t \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} (e^\Delta \xi)(tx) \tag{2.4}$$

is a smooth vector potential associated with the regularized field $e^\Delta\xi$*.*

The space $\mathrm{Dom}_0(D^\xi)$ is a convenient core of a self-adjoint operator by Theorem 2.1 below. The exponential decay condition on $\|\chi_a \exp(\pm\Delta^{-loc}\xi)u_\pm\|_{\mathcal{H}^1\mathbb{R}^2)}$ is imposed only for technical convenience in our discussion, as in the definition of the Schrödinger operator, Definition 2.1 in [18].

Now our main result in this paper is stated as follows:

Theorem 1. *The operator* D^ξ *in* (2.3) *is essentially self-adjoint on* $L^2(\mathbb{R}^2 \to \mathbb{C}^2)$*.*

We take a smooth approximation and a spatial restriction of ξ by

$$\xi_\varepsilon(x) := \int \frac{dy}{\varepsilon^2}\rho\Big(\frac{x-y}{\varepsilon}\Big)\xi(y) \tag{2.5}$$

with $\rho \in C^\infty(\mathbb{R}^2)$ such thst $\int_{\mathbb{R}^2}\rho(y)dy = 1$ for any $\varepsilon > 0$ and

$$\xi_R := \sum_{a \in \mathbb{Z}^2 \cap \Lambda_R} \chi_a^2 \xi \tag{2.6}$$

for any $R \in \mathbb{N}$, respectively. Then we can prove the strong convergence as in Proposition 4.1 in [18]:

Proposition 2.1. (i) *For any* $p \in [1,\infty)$*,* $\epsilon \in (0,1)$ *and* $a \in \mathbb{Z}^2$*, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|\chi_a(\xi_\varepsilon - \xi)\|^p_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}] = 0.$$

Thus for any $\epsilon \in (0,1)$*, there exists a decreasing sequence* $\{\varepsilon(m)\}_{m\in\mathbb{N}} \subset (0,1)$ *such that* $\varepsilon(m) \to 0$ *and* $\|\chi_a(\xi_\varepsilon - \xi)\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \to 0$ *as* $m \to \infty$ *for all* $a \in \mathbb{Z}^2$ *and almost all* ξ*.*

(ii) *For the sequence* $\{\varepsilon(m)\}_{m\in\mathbb{N}} \subset (0,1)$ *as in* (i), *the closure* $\overline{D^{\xi_{\varepsilon(m)}}}$ *of the operator* $D^{\xi_{\varepsilon(m)}}$ *with the domain* $C_0^\infty(\mathbb{R}^2 \to \mathbb{C}^2)$ *converges to the closure* $\overline{D^\xi}$ *of the operator* D^ξ *with the domain* $\mathrm{Dom}_0(D^\xi)$ *in the strong resolvent sense as* $m \to \infty$ *for almost all* ξ.

(ii) *The self-adjoint oeprator* D^{ξ_R} *in Lemma 3.5 below converges to the closure* $\overline{D^\xi}$ *of the operator* D^ξ *with the domain* $\mathrm{Dom}_0(D^\xi)$ *in the strong resolvent sense as* $R \to \infty$ *for almost all* ξ.

For the spectrum, we show the following:

Proposition 2.2. *The spectral set of the closure* $\overline{D^\xi}$ *of the operator* D^ξ *in* (2.3) *is* $\mathbb{R}$.

In the rest of this section, we will discuss the motivation of the definition in Definition 2.1.

Motivation of our definition of the operator

The operator is constructed by decomposing the white-noise magnetic field into a singular part $(I-e^\Delta)\xi$ and a smooth part $e^\Delta\xi$. The former is treated by the vector potential

$$\overset{\circ}{A^\xi} = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta^{-loc}\xi \tag{2.7}$$

in a Coulomb-type gauge, while the latter is treated by the potential $\widetilde{A^\xi}$ defined in (2.4) in a Poincaré gauge. Since

$$\overset{\circ}{A_1^\xi} \pm i\overset{\circ}{A_2^\xi} = \pm(i\partial_1 \mp \partial_2)\Delta^{-loc}\xi,$$

the most singular terms in

$$(D^\xi u)_\mp = (i\partial_1 \mp \partial_2)u_\pm + (\overset{\circ}{A_1^\xi} \pm i\overset{\circ}{A_2^\xi})u_\pm + (\widetilde{A_1^\xi} \pm i\widetilde{A_2^\xi} + A_1^B \pm iA_2^B)u_\pm.$$

are absorbed into exponential weigts as

$$(D^\xi u)_\mp = \exp(\mp\Delta^{-loc}\xi)\{(i\partial_1 \mp \partial_2) + (\widetilde{A_1^\xi} \pm i\widetilde{A_2^\xi} + A_1^B \pm iA_2^B)\}\exp(\pm\Delta^{-loc}\xi)u_\pm.$$

To make sense of the differential terms, we assume $\exp(\pm\Delta^{-loc}\xi)u_\pm \in \mathcal{H}^1(\mathbb{R}^2)$. To accommodate the non-differentiability of $\Delta^{-loc}\xi \in \bigcap_{\epsilon>0} C_{loc}^{1-\epsilon}(\mathbb{R}^2)$ as is showen in Lemma 3.1 below, we impose the requirement that $u_\pm \in \bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$, where $(\mathcal{B}_{p,q}^\alpha)_{loc}(\mathbb{R}^2) := \{f$: a distribution on $\mathbb{R}^2$ s.t. $\chi_a f \in \mathcal{B}_{p,q}^\alpha(\mathbb{R}^2)$ for any $a \in \mathbb{Z}^2\}$ for each Besov space $\mathcal{B}_{p,q}^\alpha(\mathbb{R}^2)$.

3. Proof of Theorem 1

Outline of the proof. We first establish regularity and localization properties of the random gauge factor $\exp(\pm\Delta^{-loc}\xi)$. Next, we introduce an operator D_R^ξ with a restricted magnetic field, prove its self-adjointness, and derive a uniform Combes–Thomas estimate. Finally, we pass to the limit $R \to \infty$ to conclude the self-adjointness of D^ξ.

We begin with a regularity estimate for $\exp(\pm\Delta^{-loc}\xi)$:

Lemma 3.1. *For any $\epsilon \in (0,1)$,*

$$\exp(\pm\Delta^{-loc}\xi) \in \mathcal{C}_{loc}^{1-\epsilon}(\mathbb{R}^2)$$

and

$$\|\chi_a \exp(\pm\Delta^{-loc}\xi)\|_{\mathcal{C}^{1-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi} \exp\Big(C'_{\epsilon,\xi}(\log(2+|a|))^{1/2}\Big),$$

where $C_{\epsilon,\xi}$ and $C'_{\epsilon,\xi}$ depend on ϵ and on the realization of ξ.

Proof. We use the following norm of the Hölder space:

$$\|v\|_{\Lambda^{1-\epsilon}} := \sup_{x\in\mathbb{R}^2} |v(x)| + \sup_{x,y\in\mathbb{R}^2:0<|x-y|\leq 1} \frac{|v(x)-v(y)|}{|x-y|^{1-\epsilon}}.$$

A direct proof of the equivalence of this norm with $\|\cdot\|_{\mathcal{C}^{1-\epsilon}(\mathbb{R}^2)}$ is given in a general setting in Proposition 2.5 in Bailleul-Bernicot [2]. Set

$$\Xi_a := \sum_{a'\in\mathbb{Z}^2\cap\Lambda_2(a)} \chi_{a'}\Delta^{-loc}\xi.$$

A direct estimate yields

$$\begin{aligned}
&\|\chi_a \exp(\pm\Delta^{-loc}\xi)\|_{\Lambda^{1-\epsilon}} \\
\leq &\exp\Big(\|\Xi_a\|_{L^\infty}\Big) + \sup_{x,y\in\mathbb{R}^2:0<|x-y|\leq 1} \frac{|\chi_a(x)-\chi_a(y)|}{|x-y|^{1-\epsilon}} \exp(\Xi_a(x)) \\
&+ \sup_{x,y\in\mathbb{R}^2:0<|x-y|\leq 1} \frac{\chi_a(y)}{|x-y|^{1-\epsilon}} |\exp(\Xi_a(x)) - \exp(\Xi_a(y))| \\
\leq &c\exp\Big(\|\Xi_a\|_{L^\infty}\Big)\Big(1 + \sup_{x,y\in\mathbb{R}^2:0<|x-y|\leq 1} \chi_a(y)\frac{|\Xi_a(x)-\Xi_a(y)|}{|x-y|^{1-\epsilon}}\Big)
\end{aligned}$$

and

$$\sup_{x,y\in\mathbb{R}^2:0<|x-y|\leq 1} \chi_a(y)\frac{|\Xi_a(x)-\Xi_a(y)|}{|x-y|^{1-\epsilon}} \leq \|\Xi_a\|_{\Lambda^{1-\epsilon}}.$$

By Lemma 3.1 and Lemma 3.3 in [18], there exist constants $c_{\epsilon,\xi}$ and $c'_{\epsilon,\xi}$ depending on ϵ and ξ, such that

$$\|\Xi_a\|_{\Lambda^{1-\epsilon}} \leq c_{\epsilon,\xi} \sum_{a_1\in\mathbb{Z}^2} \exp(-|a-a_1|^2/5)\log(2+|a_1|)^{1/2} \leq c'_{\epsilon,\xi}\log(2+|a|)^{1/2}.$$

Thus we can complete the proof. □

Lemma 3.2. (i) *If a measurable function* $u_\pm$ *on* $\mathbb{R}^2$ *satisfies* $\exp(\pm\Delta^{-loc}\xi)u_\pm \in \mathcal{H}^1(\mathbb{R}^2)$ *and*

$$\limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \exp(\pm\Delta^{-loc}\xi)u_\pm\|_{\mathcal{H}^1(\mathbb{R}^2)} < 0, \tag{3.1}$$

then this satisfies

$$\limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u_\pm\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0. \tag{3.2}$$

and $u_\pm \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$ *for any* $\varepsilon \in (0,1)$,

(ii) $\mathrm{Dom}_0(D^\xi)$ *is dense in* $L^2(\mathbb{R}^2 \to \mathbb{C}^2)$.

Proof. (i) We use the Sobolev-Hölder product estimate (Lemma 3.2 in [18]): for any $\alpha > 0$ there exists C_α such that

$$\|fg\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \le C_\alpha \|f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}. \tag{3.3}$$

Then for any function $u_\pm$ satisfying (3.1) and any $\epsilon \in (0,1)$, we can show (3.2) by

$$\|\chi_a u_\pm\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le c_\epsilon \|\chi_a \exp(\pm\Delta^{-loc}\xi)u_\pm\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \sum_{a' \in \mathbb{Z}^2 \cap \Lambda_2(a)} \|\chi_{a'}^2 \exp(\mp\Delta^{-loc}\xi)\|_{\mathcal{C}^{1-\epsilon}(\mathbb{R}^2)}$$

and Lemma 3.1.

(ii) For any $u \in C_0^\infty(\mathbb{R}^2 \to \mathbb{C}^2)$ and $\varepsilon \in (0,1)$, we set

$$u_{\varepsilon,\pm} := \exp(\mp\Delta^{-loc}\xi)\exp(\pm\Delta^{-loc}\xi_\varepsilon)u_\pm.$$

Then $\exp(\pm\Delta^{-loc}\xi)u_\varepsilon = \exp(\pm\Delta^{-loc}\xi_\varepsilon)u_\pm$ belongs to $C_0^\infty(\mathbb{R}^2)$ and $u_{\varepsilon,\pm}$ belongs to $\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$ by (i) of this lemma. Hence we have $u_\varepsilon \in \mathrm{Dom}_0(D^\xi)$. In the estimate

$$\begin{aligned}
&\|u_{\varepsilon,\pm} - u_\pm\|_{L^2(\mathbb{R}^2)} = \|\{\exp(\pm\Delta^{-loc}(\xi_\varepsilon - \xi)) - 1\}u_\pm\|_{L^2(\mathbb{R}^2)} \\
&\le \int_0^1 dt \|\exp(\pm t\Delta^{-loc}(\xi_\varepsilon - \xi))\Delta^{-loc}(\xi_\varepsilon - \xi)u_\pm\|_{L^2(\mathbb{R}^2)} \\
&\le \int_0^1 dt \|\exp(\pm t\Delta^{-loc}\xi_\varepsilon)\|_{L^\infty(\mathrm{supp}\, u)} \|\exp(\mp t\Delta^{-loc}\xi)\|_{L^\infty(\mathrm{supp}\, u)} \|u_\pm\|_{L^2(\mathbb{R}^2)} \\
&\quad \times \|\Delta^{-loc}(\xi_\varepsilon - \xi)\|_{L^\infty(\mathrm{supp}\, u)},
\end{aligned}$$

the first and the second terms of the right hand side is finite by the same proof of Lemma 3.1, and the second term is estimated as

$$\begin{aligned}&\|\Delta^{-loc}(\xi_\varepsilon-\xi)\|_{L^\infty(\operatorname{supp} u)}\\ \le& c\sum_{a_1,a_2\in\mathbb{Z}^2,a_1\in\operatorname{supp} u}\exp(-|a_1-a_2|^2/5)\|\chi_{a_2}(\xi_\varepsilon-\xi)\|_{\mathcal{C}^{-3/2}(\mathbb{R}^2)},\end{aligned}$$

by Lemma 3.1 in [18].

We take a sequence $\{\varepsilon(m)\}_{m\in\mathbb{N}}$ in Proposition 2.1 (i). Then we have $\|u_{\varepsilon(m)}-u\|_{L^2(\mathbb{R}^2)}\to 0$ as $m\to\infty$ for almost all ξ. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, we can complete the proof. □

For any $R\in\mathbb{N}$, we set

$$\operatorname{Dom}(D_R^\xi):=\Big\{u=\begin{pmatrix}u_-\\ u_+\end{pmatrix}\in\bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2\to\mathbb{C}^2):\exp(\pm\Delta^{-loc}\xi_R)u_\pm\in\mathcal{H}^1(\mathbb{R}^2)\Big\},$$

where ξ_R is the restriction of the noise ξ defined in (2.6). For $u\in\operatorname{Dom}(D_R^\xi)$, we set

$$D_R^\xi u=\begin{pmatrix}(D_R^\xi u)_-\\ (D_R^\xi u)_+\end{pmatrix}, \tag{3.4}$$

where

$$(D_R^\xi u)_\mp=\exp(\mp\Delta^{-loc}\xi_R)(i\partial_1\mp\partial_2+\widetilde{A_{R,1}^\xi}\pm i\widetilde{A_{R,2}^\xi}+A_{R,1}^B\pm iA_{R,2}^B)\exp(\pm\Delta^{-loc}\xi_R)u_\pm,$$

$$\widetilde{A_R^\xi}=\int_0^1 dt\ t\begin{pmatrix}-x_2\\ x_1\end{pmatrix}(e^\Delta\xi_R)(tx)$$

and

$$A_R^B=B\int_0^1 dt\ t\begin{pmatrix}-x_2\\ x_1\end{pmatrix}\sum_{a\in\mathbb{Z}^2\cap\Lambda_R}\chi_a^2(tx).$$

Then, as in Lemma 3.1 and Lemma 3.2, we have the following:

Lemma 3.3. *For any $\epsilon\in(0,1)$,*

$$\exp(\pm\Delta^{-loc}\xi_R)\in\mathcal{C}^{1-\epsilon}(\mathbb{R}^2)$$

and

$$\|\chi_a\exp(\pm\Delta^{-loc}\xi_R)\|_{\mathcal{C}^{1-\epsilon}(\mathbb{R}^2)}\le\exp(c_{\epsilon,\xi}(\log(2+R))^{1/2}-d(a,\Lambda_R)^2/5),$$

where $c_{\epsilon,\xi}$ is a constant depending on ϵ and ξ.

Lemma 3.4. $\operatorname{Dom}(D_R^\xi)$ *is dense in $L^2(\mathbb{R}^2\to\mathbb{C}^2)$.*

Moreover we can show the following:

Lemma 3.5. *The operator* D_R^ξ *with the domain* $\mathrm{Dom}(D_R^\xi)$ *is self-adjoint on* $L^2(\mathbb{R}^2 \to \mathbb{C}^2)$.

Proof. Let $\mathrm{Dom}((D_R^\xi)^*)$ denote the domain of the adjoint of D_R^ξ. For any $u \in \mathrm{Dom}((D_R^\xi)^*)$, we have

$$(i\partial_1 \mp \partial_2 + \widetilde{A_{R,1}^\xi} \pm i\widetilde{A_{R,2}^\xi} + A_{R,1}^B \pm iA_{R,2}^B) \exp(\pm\Delta^{-loc}\xi_R)u_\pm \in L^2(\mathbb{R}^2).$$

Since

$$(\widetilde{A_{R,1}^\xi} \pm i\widetilde{A_{R,2}^\xi} + A_{R,1}^B \pm iA_{R,2}^B) \exp(\pm\Delta^{-loc}\xi_R)u_\pm \in L^2(\mathbb{R}^2),$$

it follows that

$$(i\partial_1 \mp \partial_2) \exp(\pm\Delta^{-loc}\xi_R)u_\pm \in L^2(\mathbb{R}^2).$$

Hence we have

$$\exp(\pm\Delta^{-loc}\xi_R)u_\pm \in \mathcal{H}^1(\mathbb{R}^2).$$

By Lemma 3.3, this implies

$$u_\pm \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$$

for any $\epsilon > 0$. Therefore $u \in \mathrm{Dom}(D_R^\xi)$ and $(D_R^\xi)^*u = D_R^\xi u$. □

We next establish a Combes–Thomas type estimate for the resolvent of D_R^ξ (cf. [5]):

Lemma 3.6 (Combes–Thomas estimate)**.** *For any* $a, b \in \mathbb{R}^2$, *we have*

$$\|\chi_{\Lambda_1(a)}(D_R^\xi + i)^{-1}\chi_{\Lambda_1(b)}\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))} \le 6\exp(-|a-b|/2), \tag{3.5}$$

where $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}$ *is the operator norm on* $L^2(\mathbb{R}^2 \to \mathbb{C}^2)$, *and* $\chi_{\Lambda_1(a)}$ *is the operators of multiplying the characteristic function of the square* $\Lambda_1(a)$.

Proof. For any $w \in \mathbb{R}^2$ such that $|w| < 1$, since

$$e^{-w\cdot x}D_R^\xi e^{w\cdot x} = D_R^\xi + \sum_{\iota=1}^2 iw_\iota\gamma_\iota$$

and

$$\|(D_R^\xi + i)^{-1/2}\sum_{\iota=1}^2 iw_\iota\gamma_\iota(D_R^\xi + i)^{-1/2}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)\to L^2(\mathbb{R}^2\to\mathbb{C}^2)} \le |w|,$$

we have

$$
\begin{aligned}
&\|e^{-w\cdot x}(D_R^\xi+i)^{-1}e^{w\cdot x}\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}\\
\leq&\|(D_R^\xi+i)^{-1/2}\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}\\
&\times\|\{1+(D_R^\xi+i)^{-1/2}\sum_{\iota=1}^{2}iw_\iota\gamma_\iota(D_R^\xi+i)^{-1/2}\}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}\\
\leq&\sum_{n=0}^{\infty}\|(D_R^\xi+i)^{-1/2}\sum_{\iota=1}^{2}iw_\iota\gamma_\iota(D_R^\xi+i)^{-1/2}\|^n_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}\\
\leq&(1-|w|)^{-1}
\end{aligned}
$$

and

$$\|\chi_{\Lambda_1(a)}(D_R^\xi+i)^{-1}\chi_{\Lambda_1(b)}\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}\leq\sup_{a'\in\Lambda_1(a),b\in\Lambda_1(b)}\exp(w\cdot(a'-b'))(1-|w|)^{-1}.$$

By choosing $w=-(a-b)/(2|a-b|)$, we have

$$\|\chi_{\Lambda_1(a)}(D_R^\xi+i)^{-1}\chi_{\Lambda_1(b)}\|_{\mathcal{L}(L^2(\mathbb{R}^2\to\mathbb{C}^2))}\leq 2\exp\Big(-\frac{|a-b|}{2}\Big)\exp\Big(\frac{1}{2}\sup_{a'',b''\in\Lambda_1}|a''-b''|\Big)$$

and (3.5). □

Then we can prove the theorem:

Proof of Theorem 1. In this proof, c_j will be constants that may depend on ξ. For any $f\in\operatorname{Ran}(D^\xi+i)^\perp$, we will show $f=0$.

Let $\widetilde{\chi_R}$ be a $[0,1]$-valued smooth function on $\mathbb{R}^2$ such that $\widetilde{\chi_R}=0$ on $\mathbb{R}^2\setminus\Lambda_R$ and $\widetilde{\chi_R}=1$ on Λ_{R-1}. Then we have

$$\|f\|^2_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}=\lim_{R\to\infty}(f,\widetilde{\chi_R}f)_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}. \tag{3.6}$$

For any $L\in\mathbb{N}$, we set

$$\varphi_{R,L}:=(D_{R+L}^\xi+i)^{-1}\widetilde{\chi_R}f\in\operatorname{Dom}(D_{R+L}^\xi)$$

and

$$\widetilde{\varphi_{R,L,\pm}}:=\exp(\mp\Delta^{-loc}\xi\pm\Delta^{-loc}\xi_{R+L}))\varphi_{R,L,\pm}.$$

By Lemma 3.6, we have

$$\|\chi_a\varphi_{R,L}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\leq c_1\exp\Big(-d(a,\Lambda_R)/2\Big)\|\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}. \tag{3.7}$$

By combining this with Lemma 3.3, we have

$$\begin{aligned}
&\|\chi_a \exp(\pm\Delta^{-loc}\xi)\widetilde{\varphi_{R,L,\pm}}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \\
&=\|\chi_a \exp(\pm\Delta^{-loc}\xi_{R+L})\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \qquad (3.8)\\
&\leq \exp\Big(c_2(\log(2+R+L))^{1/2} - d(a,\Lambda_R)/2\Big)\|\widetilde{\chi_R} f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}.
\end{aligned}$$

We also have the following estimate of the derivative:

$$\begin{aligned}
&\|\nabla\chi_a \exp(\pm\Delta^{-loc}\xi)\widetilde{\varphi_{R,L,\pm}}\|_{L^2(\mathbb{R}^2)} \\
&=\|(i\partial_1 \pm \partial_2)\chi_a \exp(\pm\Delta^{-loc}\xi_{R+L})\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2)} \qquad (3.9)\\
&\leq \exp\Big(c_4(\log(2+R+L))^{1/2} - d(a,\Lambda_R)/2\Big)(|a|+1)\{(\log(2+R+L))^{1/2} + B\}\|\widetilde{\chi_R} f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}.
\end{aligned}$$

Indeed, by

$$D^{\xi}_{R+L}\varphi_{R,L} = \widetilde{\chi_R} f - i\varphi_{R,L},$$

we have

$$\begin{aligned}
&\|\nabla\chi_a \exp(\pm\Delta^{-loc}\xi)\widetilde{\varphi_{R,L,\pm}}\|_{L^2(\mathbb{R}^2)} \\
&=\|(i\partial_1 \pm \partial_2)\chi_a \exp(\pm\Delta^{-loc}\xi_{R+L})\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2)} \\
&\leq\|\chi_a(i\partial_1 \mp \partial_2 + \widetilde{A^{\xi}_{R+L,1}} \pm i\widetilde{A^{\xi}_{R+L,2}} + A^B_{R+L,1} \pm iA^B_{R+L,2}) \\
&\quad \times \exp(\pm\Delta^{-loc}\xi_{R+L})\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2)} \\
&\quad + \|\chi_a(\widetilde{A^{\xi}_{R+L,1}} \pm i\widetilde{A^{\xi}_{R+L,2}} + A^B_{R+L,1} \pm iA^B_{R+L,2}) \\
&\quad \times \exp(\pm\Delta^{-loc}\xi_{R+L})\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2)} \\
&\leq\|\chi_a \exp(\pm\Delta^{-loc}\xi_{R+L})(\widetilde{\chi_R} f_{\mp} - i\varphi_{R,L,\mp})\|_{L^2(\mathbb{R}^2)} \\
&\quad + \sup_{\Lambda_2(a)} |\widetilde{A^{\xi}_{R+L,1}} \pm i\widetilde{A^{\xi}_{R+L,2}} + A^B_{R+L,1} \pm iA^B_{R+L,2}|\|\chi_a \exp(\pm\Delta^{-loc}\xi_{R+L})\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Since

$$\sup_{\Lambda_2(a)} |\widetilde{A^{\xi}_{R+L}}| \leq c_3(|a|+1)(\log(2+R+L))^{1/2} \qquad (3.10)$$

and

$$\sup_{\Lambda_2(a)} |A^B_{R+L}| \leq B(|a|+1), \qquad (3.11)$$

we have (3.9)

By (3.8), (3.9) and Lemma 3.2 (i), we have

$$\begin{aligned}(3.12)\quad &\|\chi_a \exp(\pm\Delta^{-loc}\xi)\widetilde{\varphi_{R,L,\pm}}\|_{\mathcal{H}^1(\mathbb{R}^2)}\\ &\leq \exp\Big(c_5(\log(2+R+L))^{1/2} - d(a,\Lambda_R)/2\Big)(|a|+1)\{(\log(2+R+L))^{1/2}+B\}\|\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\end{aligned}$$

and $\widetilde{\varphi_{R,L}} \in \mathrm{Dom}_0(D^\xi)$.

Since $(D^\xi + i)\widetilde{\varphi_{R,L}} \in \mathrm{Ran}(D^\xi + i)$, we have

$$(3.13)\qquad \|f\|^2_{L^2(\mathbb{R}^2)\to\mathbb{C}^2)} = \lim_{R\to\infty}(f, (D^\xi_{R+L}+i)\varphi_{R,L} - (D^\xi+i)\widetilde{\varphi_{R,L}})_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}.$$

Since

$$\varphi_{R,L,\pm} - \widetilde{\varphi_{R,L,\pm}} = \{1-\exp(\mp\Delta^{-loc}(\xi-\xi_{R+L}))\}\varphi_{R,L,\pm},$$

we have

$$\|\chi_a(\varphi_{R,L,\pm} - \widetilde{\varphi_{R,L,\pm}})\|_{L^2(\mathbb{R}^2)} \leq \sup_{\Lambda_2(a)}|1-\exp(\mp\Delta^{-loc}(\xi-\xi_{R+L}))|\|\chi_a\varphi_{R,L,\pm}\|_{L^2(\mathbb{R}^2)},$$

Since

$$(3.14)\qquad \sup_{\Lambda_2(a)}|1-\exp(\mp\Delta^{-loc}(\xi-\xi_{R+L}))| \leq \sup_{\Lambda_2(a)}|\Delta^{-loc}(\xi-\xi_{R+L})|\exp\Big(\sup_{\Lambda_2(a)}|\Delta^{-loc}(\xi-\xi_{R+L})|\Big)$$

and

$$\begin{aligned}&\sup_{\Lambda_2(a)}|\Delta^{-loc}(\xi-\xi_{R+L})| \leq \Big\|\sum_{a_0\in\mathbb{Z}^2\cap\Lambda_2(a)}\chi^2_{a_0}\Delta^{-loc}(\xi-\xi_{R+L})\Big\|_{\mathcal{C}^{1-2\epsilon}(\mathbb{R}^2)}\\ (3.15)\quad &\leq c_6\sum_{a_1\in\mathbb{Z}^2\setminus\Lambda_{R+L}}\exp(-|a-a_1|^2/5)\|\chi^2_{a_1}\xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}\\ &\leq c_7\sum_{a_1\in\mathbb{Z}^2\setminus\Lambda_{R+L}}\exp(-|a-a_1|^2/5)(\log(2+|a_1|))^{1/2}\\ &\leq c_8(\log(2+|a|))^{1/2}\exp(-d(a,\mathbb{R}^2\setminus\Lambda_{R+L})^2/5)\end{aligned}$$

by Lemma 3.1 and Lemma 3.3 in [18], we have

$$\begin{aligned}(3.16)\quad &\|\chi_a(\varphi_{R,L,\pm} - \widetilde{\varphi_{R,L,\pm}})\|_{L^2(\mathbb{R}^2)}\\ &\leq \exp(c_9(\log(2+R))^{1/2} - d(a,\mathbb{R}^2\setminus\Lambda_{R+L})^2/5 - d(a,\Lambda_R)/2)\|\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}.\end{aligned}$$

We also have

$$
\begin{aligned}
&(D^{\xi}_{R+L}\varphi_{R,L})_{\mp} - (D^{\xi}_{R+L}\widetilde{\varphi_{R,L}})_{\mp} \\
=&(1-\exp(\mp\Delta^{-loc}(\xi-\xi_{R+L}))(D^{\xi}_{R+L}(\widetilde{\chi_R}f_{\pm} - i\varphi_{R,L,\pm}) \\
&+\{(\widetilde{A^{\xi}_{R+L,1}} - \widetilde{A^{\xi}_1}) \pm i(\widetilde{A^{\xi}_{R+L,2}} - \widetilde{A^{\xi}_2}) + (A^B_{R+L,1} - A^B_1) \pm i(A^B_{R+L,2} - A^B_2)\} \\
&\times \exp(\mp\Delta^{-loc}(\xi-\xi_{R+L}))\varphi_{R,L,\pm}.
\end{aligned}
$$

As in (3.10) and (3.11), we have

$$
\sup_{\Lambda_2(a)} |\widetilde{A^{\xi}_{R+L}} - \widetilde{A^{\xi}}| \le c_{10}(|a|+1)(\log(2+|a|))^{1/2}\exp(-d(a,\mathbb{R}^2\setminus\Lambda_{R+L})^2/5)
$$

and

$$
\sup_{\Lambda_2(a)} |A^B_{R+L} - A^B| \le B(|a|+1)\chi_{\mathbb{R}^2\setminus\Lambda_{R+L-4}}(a).
$$

By using also (3.14) and (3.15), we have

$$
\begin{aligned}
&\|\chi_a((D^{\xi}_{R+L}\varphi_{R,L})_{\mp} - (D^{\xi}_{R+L}\widetilde{\varphi_{R,L}})_{\mp})\|_{L^2(\mathbb{R}^2)} \\
\le&\exp(c_{11}(\log(2+R))^{1/2} - cd(a,\mathbb{R}^2\setminus\Lambda_{R+L})^2 - d(a,\Lambda_R)/2)\|\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \qquad (3.17)\\
&+c_{12}(|a|+1)((\log(2+R))^{1/2}+B)\exp(-d(a,\mathbb{R}^2\setminus\Lambda_{R+L})^2/5 - d(a,\Lambda_R)/2)\|\widetilde{\chi_R}f\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}.
\end{aligned}
$$

Thus the right hand side of (3.6) is zero, and we obtain $\mathrm{Ran}(D^{\xi}+i)^{\perp} = \{0\}$ (cf.[16]). □

4. Proof of Proposition 2.1

In this section we prove Proposition 2.1, which establishes stability properties of the operator $\overline{D^{\xi}}$ under mollification of the noise and finite-volume truncations.

Proof of (i).

We present here a simple proof of a fundamental that a general mollified noise converges in negative Hölder spaces. The argument relies on the heat semigroup method. Let $Q \in StGC^r$ with $1 < r \in \mathbb{N}$. Then we have

$$
\begin{aligned}
&\mathbb{E}[\|t^{(1+\epsilon)/2}Q_t\chi_a(\xi_\varepsilon - \xi)\|^2_{L^2(\mathbb{R}^2\times[0,1]:dxdt/t)}] \\
=&\int_0^1 dt\ t^{\epsilon}\int_{\mathbb{R}^2}dx\int_{\mathbb{R}^2}dx_1\Big\{\int_{\mathbb{R}^2}dx_2\rho(x_2)(Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) - Q_t(x,x_1)\chi_a(x_1))\Big\}^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{|x|\ge R}dx\int_0^1 dt\ t^{\epsilon}\int_{\mathbb{R}^2}dx_1\Big\{\int_{\mathbb{R}^2}dx_2\rho(x_2)(Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) - Q_t(x,x_1)\chi_a(x_1))\Big\}^2 \qquad (4.1)\\
\le& c\exp(-c(R-|a|)^2_+),
\end{aligned}
$$

for any $a \in \mathbb{Z}^2$ and $\eta > 0$, there exists $R(a,\eta) \in (0,\infty)$ such that the qunatity in (4.1) is smaller than η for any $R \geq R(a,\eta)$. The remaining part can be rewritten as

$$\begin{aligned}
&\int_{|x|\leq R(a,\eta)} dx \int_0^1 dt\ t^{\epsilon} \int_{\mathbb{R}^2} dx_1 \Big\{ \int_{\mathbb{R}^2} dx_2 \rho(x_2)(Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) - Q_t(x,x_1)\chi_a(x_1)) \Big\}^2 \\
&= \int_{|x|\leq R(a,\eta)} dx \int_0^1 dt\ t^{\epsilon} \int_{\mathbb{R}^2} dx_2 \rho(x_2) \int_{\mathbb{R}^2} d\underline{x_2}\rho(\underline{x_2}) \\
&\quad \times \Big\{ \int_{\mathbb{R}^2} dx_1 Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) Q_t(x,x_1+\varepsilon \underline{x_2})\chi_a(x_1+\varepsilon \underline{x_2}) \\
&\quad - 2\int_{\mathbb{R}^2} dx_1 Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) Q_t(x,x_1)\chi_a(x_1) + \int_{\mathbb{R}^2} dx_1 Q_t(x,x_1)^2 \chi_a(x_1)^2 \Big\}.
\end{aligned} \tag{4.2}$$

Since $|Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2)|$ is dominated by an x_1-integrable function $\exp(-c|x-x_1|+c|x_2|)\chi_{\Lambda_{2+|x_2|}(a)}(x_1)$, the dominated convergence theorem yields

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} dx_1 Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) Q_t(x,x_1+\varepsilon \underline{x_2})\chi_a(x_1+\varepsilon \underline{x_2}) = \int_{\mathbb{R}^2} dx_1 Q_t(x,x_1)^2\chi_a(x_1)^2$$

and

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} dx_1 Q_t(x,x_1+\varepsilon x_2)\chi_a(x_1+\varepsilon x_2) Q_t(x,x_1)\chi_a(x_1) = \int_{\mathbb{R}^2} dx_1 Q_t(x,x_1)^2\chi_a(x_1)^2.$$

Consequently, the quantity in (4.2) converges to 0 as $\varepsilon \to 0$ by the bounded convergence theorem. Therefore we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|t^{(1+\epsilon)/2} Q_t \chi_a(\xi_\varepsilon - \xi)\|^2_{L^2(\mathbb{R}^2\times[0,1]:dxdt/t)}] = 0.$$

Since $Q_t\chi_a(\xi_\varepsilon - \xi)$ is Gaussian, it follows that

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|t^{(1+\epsilon)/2} Q_t \chi_a(\xi_\varepsilon - \xi)\|^p_{L^p(\mathbb{R}^2\times[0,1]:dxdt/t)}] = 0$$

for any $p \in [1,\infty)$. Similarly we have

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|e^{\Delta}\chi_a(\xi_\varepsilon - \xi)\|^p_{L^p(\mathbb{R}^2:dx)}] = 0.$$

Hence we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|\chi_a(\xi_\varepsilon - \xi)\|^p_{B^{-1-\epsilon}_{p,p}(\mathbb{R}^2)}] = 0.$$

By Proposition A.1 in Allez and Chouk [1], we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|\chi_a(\xi_\varepsilon - \xi)\|^p_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}] = 0. \quad \square$$

Proof of (ii).

For any $v \in L^2(\mathbb{R}^2 \to \mathbb{C}^2)$, we set

$$u := (\overline{D^{\xi}} + i)^{-1} v.$$

For any $\eta > 0$, there exists $u_0 \in \mathrm{Dom}_0(D^\xi)$ such that

$$\|u - u_0\|_{L^2(\mathbb{R}^2 \to \mathbb{C}^2)}, \quad \|\overline{D^\xi} u - D^\xi u_0\|_{L^2(\mathbb{R}^2 \to \mathbb{C}^2)} < \eta.$$

For any $\varepsilon > 0$, we set

$$u_{\varepsilon,\pm} := \exp(\mp \Delta^{-loc}\xi_\varepsilon) \exp(\mp \Delta^{-loc}\xi) u_{0,\pm}.$$

Then we have

$$\|\chi_a(u_{0,\pm} - u_{\varepsilon,\pm})\|_{L^2(\mathbb{R}^2)} \le \sup_{\Lambda_2(a)} |1 - \exp(\mp \Delta^{-loc}(\xi_\varepsilon - \xi))| \|\chi_a u_{0,\pm}\|_{L^2(\mathbb{R}^2)}.$$

Moreover we have

$$\sup_{\Lambda_2(a)} |1 - \exp(\mp \Delta^{-loc}(\xi - \xi_\varepsilon))| \le \sup_{\Lambda_2(a)} |\Delta^{-loc}(\xi - \xi_\varepsilon)| \exp\Big(\sup_{\Lambda_2(a)} |\Delta^{-loc}(\xi - \xi_\varepsilon)|\Big) \tag{4.3}$$

and

$$\sup_{\Lambda_2(a)} |\Delta^{-loc}(\xi - \xi_\varepsilon)| \le c \sum_{a_1 \in \mathbb{Z}^2} \exp(-c|a - a_1|^2) \|\chi_{a_1}^2 \xi - \xi_\varepsilon\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \tag{4.4}$$

as in (3.14) and (3.15). Thus, we have

$$\lim_{m \to \infty} \|u_{0,\pm} - u_{\varepsilon(m),\pm}\|_{L^2(\mathbb{R}^2)} = 0 \tag{4.5}$$

by using also Lemma 3.2. We next consider

$$\begin{aligned}
&(D^\xi u_0)_\mp - (D^{\xi_\varepsilon} u_\varepsilon)_\mp \\
=&(1 - \exp(\mp \Delta^{-loc}(\xi_\varepsilon - \xi))(D^\xi u_0)_\mp \\
&- \exp(\mp \Delta^{-loc}\xi_\varepsilon)((\widetilde{A_1^\xi} \pm i\widetilde{A_2^\xi}) - (\widetilde{A_1^{\xi_\varepsilon}} \pm i\widetilde{A_2^{\xi_\varepsilon}})) \exp(\pm \Delta^{-loc}\xi)) u_{0,\pm}.
\end{aligned}$$

The first term is estimated similarly as

$$\|\chi_a(1 - \exp(\mp \Delta^{-loc}(\xi_\varepsilon - \xi))(D^\xi u_0)_\mp\|_{L^2(\mathbb{R}^2)} \le \sup_{\Lambda_2(a)} |1 - \exp(\mp \Delta^{-loc}(\xi_\varepsilon - \xi))| \|\chi_a(D^\xi u_0)_\mp\|_{L^2(\mathbb{R}^2)},$$

By (1.2), we have

$$\sup_{\Lambda_2(a)} |A^B| \le \frac{B}{2}(1 + |a|).$$

By (2.4) and Lemma 3.3 (ii) in [18], we have

$$\sup_{\Lambda_2(a)} |\widetilde{A^\xi}| \le c(1 + |a|)^3 (\log(2 + |a|))^{1/2}.$$

By using also Lemma 3.1 and Lemma 3.2, we obtain

$$\|\chi_a(D^\xi u_0)_\mp\|_{L^2(\mathbb{R}^2)} \le c \exp(-c|a|),$$

Hence, by (4.3) and (4.4), we have

$$\lim_{m\to\infty} \|(1-\exp(\mp\Delta^{-loc}(\xi_{\varepsilon(m)}-\xi))(D^{\xi}u_0)_{\mp}\|_{L^2(\mathbb{R}^2)} = 0$$

Furthermore, by (2.4), we have

$$\sup_{\Lambda_2(a)} |\widetilde{A^{\xi}} - \widetilde{A^{\xi_\varepsilon}}| \le c(1+|a|)\sum_{a_1\in\mathbb{Z}^2} \exp\Big(-cd\Big(a_1, \bigcup_{t\in[0,1]} t\Lambda_2(a)\Big)^2\Big)\|\xi-\xi_\varepsilon\|_{C^{-1-\epsilon}(\mathbb{R}^2)}.$$

Combining this with (4.4), we have

$$\lim_{m\to\infty} \|\exp(\mp\Delta^{-loc}\xi_\varepsilon)((\widetilde{A_1^{\xi}} \pm i\widetilde{A_2^{\xi}}) - (\widetilde{A_1^{\xi_\varepsilon}} \pm i\widetilde{A_2^{\xi_\varepsilon}}))\exp(\pm\Delta^{-loc}\xi))u_{0,\pm}\|_{L^2(\mathbb{R}^2)} = 0.$$

Therefore we conclude that

$$\lim_{m\to\infty} \|(D^{\xi}u_0)_{\mp} - (D^{\xi_{\varepsilon(m)}}u_{\varepsilon(m)})_{\mp}\|_{L^2(\mathbb{R}^2)} = 0. \tag{4.6}$$

Finally we note that

$$\begin{aligned}
&\|(D^{\xi_\varepsilon}+i)^{-1}v - (D^{\xi}+i)^{-1}v\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
\le&\|(D^{\xi_\varepsilon}+i)^{-1}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)\to L^2(\mathbb{R}^2\to\mathbb{C}^2)}\|(D^{\xi}+i)u-(D^{\xi}+i)u_0\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
&+\|(D^{\xi_\varepsilon}+i)^{-1}(D^{\xi}+i)u_0 - u_\varepsilon\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
&+\|u_\varepsilon-u_0\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} + \|u_0-u\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
<&\|D^{\xi}u_0 - D^{\xi_\varepsilon}u_\varepsilon\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} + \|u_0-u_\varepsilon\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} + 3\eta.
\end{aligned}$$

Therefore we obtain

$$\lim_{m\to\infty} \|(D^{\xi_{\varepsilon(m)}}+i)^{-1}v - (D^{\xi}+i)^{-1}v\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} = 0. \tag{4.7}$$

□

Proof of (iii).

As in the proof of (ii), we take $v\in L^2(\mathbb{R}^2\to\mathbb{C}^2)$ arbitrarily, and we set

$$u := (\overline{D^{\xi}}+i)^{-1}v.$$

We take $\eta>0$ arbitrarily and choose $u_0\in \mathrm{Dom}_0(D^{\xi})$ so that

$$\|u-u_0\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)},\quad \|\overline{D^{\xi}}u - D^{\xi}u_0\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} < \eta,$$

For any $R>0$, we set

$$u_{R,\pm} := \exp(\mp\Delta^{-loc}\xi_R)\exp(\mp\Delta^{-loc}\xi)u_{0,\pm}.$$

Now we have $\|\chi_1(\xi_r - \xi)\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \to 0$ as $R \to \infty$ for any $\epsilon \in (0,1)$, $a \in \mathbb{Z}^2$ and almost all ξ by Lemma 3.3 in [18]. Thus as in the proof of (4.5) and (4.6), we have

$$\lim_{R\to\infty} \|u_{0,\pm} - u_{R,\pm}\|_{L^2(\mathbb{R}^2)} = 0 \tag{4.8}$$

and

$$\lim_{R\to\infty} \|(D^{\xi} u_0)_{\mp} - (D^{\xi_R} u_{\varepsilon(m)})_{\mp}\|_{L^2(\mathbb{R}^2)} = 0. \tag{4.9}$$

Therefore, as in (4.7), we obtain

$$\lim_{m\to\infty} \|(D^{\xi_{\varepsilon(m)}} + i)^{-1} v - (D^{\xi} + i)^{-1} v\|_{L^2(\mathbb{R}^2 \to \mathbb{C}^2)} = 0. \tag{4.10}$$

□

5. Proof of Proposition 2.2

In this section we prove Proposition 2.2, namely that the spectrum of D^{ξ} coincides almost surely withthe whole real line. The argument follows the strategy of Section 5 in [18], based on the construction of compactly supported Weyl sequences.

On the 2-dimensional flat torus $\mathbb{T}_L^2 := \mathbb{R}^2/(L\mathbb{Z})^2$ with any $L \in \mathbb{N}$, we take an orthonormal basis $\{\varphi_{\boldsymbol{n}}^L\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ of $L^2(\mathbb{T}_L^2)$ defined by

$$\varphi^L_{(n_1,n_2)}(x_1, x_2) = \phi^L_{n_1}(x_1)\phi^L_{n_2}(x_2)$$

and

$$\phi^L_{n_1}(x_1) = \begin{cases} \sqrt{2/L}\cos(2\pi n_1 x_1/L) & \text{for } 0 < n_1 \in \mathbb{Z}, \\ \sqrt{1/L} & \text{for } n_1 = 0, \\ \sqrt{2/L}\sin(2\pi n_1 x_1/L) & \text{for } 0 > n_1 \in \mathbb{Z}. \end{cases}$$

Then the white noise ξ^L on $\mathbb{T}_L^2$ is represented as

$$\xi^L(x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} X_{\boldsymbol{n}}(\xi^L)\varphi^L_{\boldsymbol{n}}(x)$$

in the Besov Hölder space $\mathcal{C}^{-1-\epsilon}(\mathbb{T}_L^2)$ on $\mathbb{T}_L^2$ for any $\epsilon > 0$, where $\{X_{\boldsymbol{n}}(\xi^L)\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ is a system of independently identically distributed random variables having the standard normal distribution. Let $\widetilde{\chi_L}$ and $\widetilde{\chi_L}^c$ be $[0,1]$-valued smooth function on $\mathbb{R}^2$ such that $\widetilde{\chi_L} = 0$ on $\mathbb{R}^2\backslash\Lambda_L$, $\widetilde{\chi_L} = 1$ on Λ_{L-1} and $\widetilde{\chi_L}^2 + (\widetilde{\chi_L}^c)^2 = 1$ on $\mathbb{R}^2$.

We represent the white noise as

$$\xi = \widetilde{\chi_L}\xi^L + \widetilde{\chi_L}^c \xi^{L,c}, \tag{5.1}$$

where ξ^L and $\xi^{L,c}$ are white noises on $\mathbb{T}_L^2$ and $\mathbb{R}^2$, respectively, such that ξ^L and $\xi^{L,c}$ are independent as random fields. (5.1) is justified by showing that the probability distribution of the pairing of the right hand side with any $\phi \in L^2(\mathbb{R}^2)$ is the normal distribution with the mean 0 and the variance $\|\phi\|^2_{L^2(\mathbb{R}^2)}$.

For any $N \in \mathbb{N}$, we decompose ξ^L into low- and high-frequency components:

$$\xi^L_{N\geq}(x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2\cap\Lambda_N} X_{\boldsymbol{n}}(\xi^L)\varphi^L_{\boldsymbol{n}}(x) \quad \text{and} \quad \xi^L_{N<}(x) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\Lambda_N} X_{\boldsymbol{n}}(\xi^L)\varphi^L_{\boldsymbol{n}}(x) \tag{5.2}$$

For any $L \in \mathbb{N}$, we define the event $\Omega(L)$ by

$$\Big\{\xi : \text{In the representation of (5.1) and (5.2) with } N = L^{10}, \text{ it holds that}$$

$$|X_{\boldsymbol{0}}(\xi^L) + LB|, |X_{\boldsymbol{n}}(\xi^L)| \leq N^{-2} \text{ for any } \boldsymbol{n} \in \mathbb{Z}^2 \cap \Lambda_N \setminus \{\boldsymbol{0}\}, \text{ and}$$

$$\|\chi_a \xi^L_{N<}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \leq \begin{cases} L^{-\epsilon} \ (a \in \mathbb{Z}^2 \cap \Lambda_{L/2}), \\ |a|^\epsilon \ (a \in \mathbb{Z}^2 \cap (\Lambda_L \setminus \Lambda_{L/2})) \end{cases} \Big\},$$

where $\epsilon \in (0,1)$ and $B \in \mathbb{R}$ are arbitrarily fixed. The positivity of this event is proven as in the proof of Lemma 5.3 in [18]:

Lemma 5.1. *There exists $L_0 \in \mathbb{N}$ such that $\mathbb{P}(\Omega(L)) > 0$ for any $L_0 \leq L \in \mathbb{N}$.*

Under the event $\Omega(L)$, we set

$$\xi^{L,0}_{N\geq}(x) = (X_{\boldsymbol{0}}(\xi^L) + LB)\varphi^L_{\boldsymbol{0}}(x) + \sum_{\boldsymbol{n}\in\mathbb{Z}^2\cap\Lambda_N\setminus\{\boldsymbol{0}\}} X_{\boldsymbol{n}}(\xi^L)\varphi^L_{\boldsymbol{n}}(x)$$

and

$$\xi^{L,0} = \widetilde{\chi_L}(\xi^{L,0}_{N\geq} + \xi^L_{N<}) + \widetilde{\chi_L}^c \xi^{L,c}.$$

Then $\xi = \xi^{L,0} - \widetilde{\chi_L}B$ and $\xi^{L,0}(x)$ is small for $x \in \Lambda_L$. Now let $D_0^{\xi^{L,0}}$ be the Dirac operator obtained by replacing (ξ, B) by $(\xi^{L,0}, 0)$ in the definition (2.3): for u in

$$\mathrm{Dom}_0(D_0^{\xi^{L,0}}) := \Big\{ u = \begin{pmatrix} u_- \\ u_+ \end{pmatrix} \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2 \to \mathbb{C}^2) : \exp(\pm\Delta^{-loc}\xi^{L,0})u_\pm \in \mathcal{H}^1(\mathbb{R}^2),$$

$$\limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \exp(\pm\Delta^{-loc}\xi^{L,0})u_\pm\|_{\mathcal{H}^1(\mathbb{R}^2)} < 0 \Big\},$$

we set

$$D_0^{\xi^{L,0}} u = \begin{pmatrix} (D_0^{\xi^{L,0}} u)_- \\ (D_0^{\xi^{L,0}} u)_+ \end{pmatrix},$$

where

$$(D_0^{\xi^{L,0}} u)_\mp = \exp(\mp\Delta^{-loc}\xi^{L,0})(i\partial_1 \mp \partial_2 + \widetilde{A_1^{\xi^{L,0}}} \pm i\widetilde{A_2^{\xi^{L,0}}}) \exp(\pm\Delta^{-loc}\xi^{L,0}) u_\pm,$$

and

$$\widetilde{A^{\xi^{L,0}}}(x) = \int_0^1 dt\ t \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} (e^{\Delta}\xi^{L,0})(tx).$$

Then we have

$$D^\xi = D_0^{\xi^{L,0}} + B\gamma \cdot \overset{\circ}{A},$$

where

$$\overset{\circ}{A}(x) = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix} \Delta^{-loc}\widetilde{\chi_L} + \int_0^1 dt\ t \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} (e^{\Delta}\widetilde{\chi_L})(tx) + \frac{1}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Since

$$\nabla \times \overset{\circ}{A} = 1 - \widetilde{\chi_L},$$

there exists $\overset{\circ}{\phi} \in C^\infty(\mathbb{R}^2 \to \mathbb{R})$ such that

$$\nabla\overset{\circ}{\phi} = \overset{\circ}{A} - \widehat{A},$$

where

$$\widehat{A}(x) := \int_0^1 dt\ t \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} (1 - \widetilde{\chi_L})(tx).$$

Thus we have

$$D^\xi = \exp(iB\overset{\circ}{\phi})(D_0^{\xi^{L,0}} + B\gamma \cdot \widehat{A}) \exp(-iB\overset{\circ}{\phi}).$$

We fix $B, \mu \in \mathbb{R}$ and $\epsilon \in (0,1)$ arbitrarily. For any $\varepsilon \in (0,1)$, we can take $\varphi_\varepsilon =^t (\varphi_{\varepsilon,-}, \varphi_{\varepsilon,+}) \in C_0^\infty(\mathbb{R}^2 \to \mathbb{C}^2)$ and $R_\varepsilon \in (0,\infty)$ such that $\|(i\partial_1 \mp \partial_2)\varphi_{\varepsilon,\pm} - \mu\varphi_{\varepsilon,\mp}\|_{L^2(\mathbb{R}^2)} < \varepsilon$, $\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} = 1$ and $\operatorname{supp}\varphi_\varepsilon \subset \Lambda_{R_\varepsilon}$.

Under the event $\Omega(L)$, we set $\widetilde{\varphi_{\varepsilon,\pm}} := \exp(\mp\Delta^{-loc}\xi^{L,0})\varphi_{\varepsilon,\pm}$. Then we have $\widetilde{\varphi_\varepsilon} \in \operatorname{Dom}(D_0^{\xi^{L,0}})$. As in the proof of Theorem 1, we have the following:

Lemma 5.2. *For any $\varepsilon \in (0,1)$, there exists $L_\varepsilon \in \mathbb{N}$ satisfying the following: under the event $\Omega(L_\varepsilon)$,*

$$\|\widetilde{\varphi_\varepsilon} - \varphi_\varepsilon\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \leq c_\xi \varepsilon$$

and

$$\|(D^\xi - \mu)\exp(iB\overset{\circ}{\phi})\widetilde{\varphi_\varepsilon}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \leq c_\xi \varepsilon,$$

where c_ε is a finite constant depending on ξ.

In the following, $\{c_j\}_{j\in\mathbb{N}}$ are positive finite constants, and $\{c_{\xi,j}\}_{j\in\mathbb{N}}$ are positive finite constants depending on ξ.

Proof. Under the event $\Omega(L)$, it holds that

$$\begin{aligned}
&\|\chi_a \Delta^{-loc}\xi^{L,0}\|_{\mathcal{C}^{1-2\epsilon}(\mathbb{R}^2)} \\
&\leq \sum_{a_1\in\mathbb{Z}^2\cap\Lambda_L} c_1\exp(-c_2|a-a_1|^2)L^{-1} + \sum_{a_1\in\mathbb{Z}^2\cap\Lambda_{L/2}} c_1\exp(-c_2|a-a_1|^2)L^{-\epsilon} \\
&\quad + \sum_{a_1\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_{L/2})} c_1\exp(-c_2|a-a_1|^2)|a_1|^\epsilon + \sum_{a_1\in\mathbb{Z}^2\setminus\Lambda_{L-1}} c_{\xi,1}\exp(-c_2|a-a_1|^2)(\log(2+|a_1|))^{1/2}.
\end{aligned}$$

For $a \in \Lambda_{R_\varepsilon}$, it holds that

$$\begin{aligned}
&\|\Delta^{-loc}\xi^{L,0}\|_{L^\infty(\Lambda_2(a))} \\
\leq& c_3L^{-\epsilon} + c_4\exp(-c_2 d(a,\Lambda_L\setminus\Lambda_{L/2})^2)(1+|a|^\epsilon) + c_{\xi,2}\exp(-c_2 d(a,\mathbb{R}^2\setminus\Lambda_L)^2)(\log(1+|a|))^{1/2} \\
\leq& c_3L^{-\epsilon} + c_4(1+R_\varepsilon^\epsilon)\exp(-c_2(L-2R_\varepsilon)_+^2)) + c_{\xi,2}(\log(2+R_\varepsilon))^{1/2}\exp(-c_2(L-R_\varepsilon)_+^2)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\|\widetilde{\varphi_\varepsilon} - \varphi_\varepsilon\|^2_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \\
&\leq \sum_{a\in\mathbb{Z}^2\cap\Lambda_R} \exp(2\|\Delta^{-loc}\xi^{L,0}\|_{L^\infty(\Lambda_2(a))})\|\Delta^{-loc}\xi^{L,0}\|^2_{L^\infty(\Lambda_2(a))}\|\chi_a\varphi_{\mu,\varepsilon}\|^2_{L^2(\mathbb{R}^2\to\mathbb{C}^2)} \\
&\leq \exp\{2c_3L^{-\epsilon} + 2c_4(1+R_\varepsilon^\epsilon)\exp(-c_2(L-2R_\varepsilon)_+^2)) + 2c_{\xi,2}(\log(2+R_\varepsilon))^{1/2}\exp(-c_2(L-R_\varepsilon)_+^2))\} \\
&\quad \times \{c_3L^{-\epsilon} + c_4(1+R_\varepsilon^\epsilon)\exp(-c_2(L-2R_\varepsilon)_+^2)) + c_{\xi,2}(\log(2+R_\varepsilon))^{1/2}\exp(-c_2(L-R_\varepsilon)_+^2))\}^2.
\end{aligned} \tag{5.3}$$

By using also

$$\begin{aligned}
&((D_0^{\xi^{L,0}} - \mu)\widetilde{\varphi_\varepsilon})_\mp \\
=& \exp(\mp\Delta^{-loc}\xi^{L,0})\{((i\partial_1 \mp \partial_2)\varphi_{\varepsilon,\pm} - \mu\varphi_{\varepsilon,\mp}) + (\widetilde{A_1^{\xi^{L,0}}} \pm i\widetilde{A_2^{\xi^{L,0}}})\varphi_{\varepsilon,\pm} \\
&+ \mu(\exp(\mp\Delta^{-loc}\xi^{L,0}) - \exp(\pm\Delta^{-loc}\xi^{L,0}))\varphi_{\varepsilon,\mp}
\end{aligned}$$

and

$$
\begin{aligned}
&\|\widetilde{A^{\xi^{L,0}}}\|_{L^\infty(\Lambda_2(a))}\\
\le & c_5(1+|a|)\sum_{a_1\in\mathbb{Z}^2}\int_0^1 dt\exp(-c_2|ta-a_1|^2)\|\chi_{a_1}\xi^{L,0}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}\\
\le & c_6(1+R_\varepsilon)L^{-\epsilon}+c_7(1+R_\varepsilon^{1+\epsilon})\exp(-c_2(L-2R_\varepsilon)_+^2))\\
&+c_{\xi,3}(1+R_\varepsilon(\log(2+R_\varepsilon))^{1/2})\exp(-c_2(L-R_\varepsilon)_+^2))
\end{aligned}
$$

for $a\in\Lambda_{R_\varepsilon}$, we have

$$
\begin{aligned}
&\|(D_0^{\xi^{L,0}}-\mu)\widetilde{\varphi_\varepsilon}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
\le & \exp\{c_3L^{-\epsilon}+c_4(1+R_\varepsilon^{\epsilon})\exp(-c_2(L-2R_\varepsilon)_+^2))+c_{\xi,2}(\log(2+R_\varepsilon))^{1/2}\exp(-c_2(L-R_\varepsilon)_+^2))\}\\
&\times\{c_8\varepsilon+c_9(1+R_\varepsilon)L^{-\epsilon}+c_{10}(1+R_\varepsilon^{1+\epsilon})\exp(-c_2(L-2R_\varepsilon)_+^2))\\
&+c_{\xi,4}(1+R_\varepsilon(\log(2+R_\varepsilon))^{1/2})\exp(-c_2(L-R_\varepsilon)_+^2))\},
\end{aligned}
\tag{5.4}
$$

Thus, since $\operatorname{supp}\widehat{A}\subset\mathbb{R}^2\setminus\Lambda_{L-1}$, we have

$$
\begin{aligned}
&\|(D^{\xi}-\mu)\exp(iB\mathring{\phi})\widetilde{\varphi_\varepsilon}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
= & \|(D_0^{\xi^{L,0}}-\mu+B\gamma\cdot\widehat{A})\widetilde{\varphi_\varepsilon}\|_{L^2(\mathbb{R}^2\to\mathbb{C}^2)}\\
\le & \exp\{c_3L^{-\epsilon}+c_4(1+R_\varepsilon^{\epsilon})\exp(-c_2(L-2R_\varepsilon)_+^2))+c_{\xi,2}(\log(2+R_\varepsilon))^{1/2}\exp(-c_2(L-R_\varepsilon)_+^2))\}\\
&\times\{c_8\varepsilon+c(1+R_\varepsilon)L^{-\epsilon}+c_9(1+R_\varepsilon^{1+\epsilon})\exp(-c_2(L-2R_\varepsilon)_+^2))\\
&+c_{\xi,4}(1+R_\varepsilon(\log(2+R_\varepsilon))^{1/2})\exp(-c_2(L-R_\varepsilon)_+^2))\}
\end{aligned}
\tag{5.5}
$$

if $L\ge R_\varepsilon+1$.

We can take L so large that the right hand sides in (5.3), (5.4) and (5.5) are less than $c_\xi\varepsilon$, □

Proof of Proposition 2.2. For any $x_0\in\mathbb{Z}^2$, we set

$$\Omega(x_0,L_\varepsilon):=\{\xi:\xi(\cdot-x_0)\in\Omega(L_\varepsilon)\}.$$

Then by the spacial independence and the stationarity of the white noise and Lemma 5.1, we have

$$\mathbb{P}\Big(\bigcap_{x_0\in L_\varepsilon\mathbb{Z}^2}\Omega(x_0,L_\varepsilon)^c\Big)=\prod_{x_0\in L_\varepsilon\mathbb{Z}^2}\Big(1-\mathbb{P}(\Omega(L_\varepsilon))\Big)=0.$$

Thus we have

$$\mathbb{P}\Big(\bigcup_{x_0\in L_\varepsilon\mathbb{Z}^2}\Omega(x_0,L_\varepsilon)\Big)=1.$$

For any $x_0 \in L_\varepsilon \mathbb{Z}^2$ and $\xi \in \Omega(x_0, L_\varepsilon)$, we define

$$\widetilde{\varphi_{\varepsilon,x_0}}(x) := \widetilde{\varphi_\varepsilon}(x; \xi(\cdot - x_0))$$

for any $x \in \mathbb{R}^2$, where $\widetilde{\varphi_\varepsilon}(\cdot;\xi)$ is the function $\widetilde{\varphi_\varepsilon}(\cdot)$ used in Lemma 5.2 whose dependence on ξ is denoted. Then we have $\exp(iB\overset{\circ}{\phi}(\cdot + x_0))\widetilde{\varphi_{\varepsilon,x_0}}(\cdot + x_0) \in \mathrm{Dom}_0(D^\xi)$,

$$|\|\exp(iB\overset{\circ}{\phi}(\cdot + x_0))\widetilde{\varphi_{\varepsilon,x_0}}(\cdot + x_0)\|_{L^2(\mathbb{R}^2)} - 1| \le c_\xi \varepsilon,$$

and

$$\|(D^\xi - \mu)\exp(iB\overset{\circ}{\phi}(\cdot + x_0))\widetilde{\varphi_{\varepsilon,x_0}}(\cdot + x_0)\|_{L^2(\mathbb{R}^2)} \le c_\xi \varepsilon.$$

Since ε is taken arbitrarily small, we can prove that μ belongs to the spectral set of $\overline{D^\xi}$, by Weyl's criterion (cf. Hislop and Sigal [12], Theorem 5.10), □

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