# A proof of the Anderson localization induced by the 2-dimensional white noise

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Abstract. –

Spectral properties of a self-adjoint operator corresponding to a limit of $-\Delta + \xi_\varepsilon + c_\varepsilon$ as $\varepsilon \to 0$ are invetigated, where $\Delta$ is the Laplacian on $\mathbb{R}^2$, $\xi_\varepsilon$ is a smooth approximation of the white noise $\xi$ defined by $\exp(\varepsilon^2\Delta)\xi$, and $c_\varepsilon$ is a positive function of $\varepsilon$ diveging as $\varepsilon \to 0$. For sufficiently low energies, it is proven that phenomena of the Anderson localization occur: the spectrum is pure point and the corresponding eigenfunctions decay exponentially at infinity. For the proof, the Wegner estimate and the multi scale analysis are modified appropriately.

## 1. Introduction

Our object is a self-adjoint operator corresponding to a limit of $-\Delta + \xi_\varepsilon + c_\varepsilon$ as $\varepsilon \to 0$, which is defined in [21] as follows: for any element $u$ in

$$\mathrm{Dom}_{+0}(\widetilde{H^\xi}) := \Big\{ u \in \bigcap_{\epsilon > 0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0$$

$$\text{for any } \epsilon > 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\},$$

we set

$$
\begin{aligned}
&\widetilde{H^\xi} u \\
&= -\Delta\Phi_\xi(u) + P_\xi \Phi_\xi(u) + \Pi(\Phi_\xi(u), \xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi)) \\
&\quad + e^\Delta P_u \xi + e^\Delta{}_u P_\xi(\Delta^{-loc}\xi) + e^\Delta P_u Y_\xi \\
&\quad + C(u, \xi, \xi) + S(u, \xi, \xi) \\
&\quad + P_{Y_\xi} u + \Pi(u, Y_\xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_\xi)) \\
&\quad + P_\xi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi)) + \Pi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi), \xi) \\
&\quad + P_\xi(\Delta^{-loc} P_u Y_\xi) + \Pi(\Delta^{-loc} P_u Y_\xi, \xi),
\end{aligned} \tag{1.1}
$$

where

$$\Phi_\xi(u) := u - \Delta^{-loc} P_u \xi - \Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi) - \Delta^{-loc} P_u Y_\xi.$$

Then the operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_{+0}(\widetilde{H^\xi})$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$. We denote the corresponding self-adjoint extension by the same symbol.

For this definition, we should introduce several notations: we take a $[0,1]$-valued smooth function $\chi_0$ on $\mathbb{R}^2$ such that

$$\sum_{a\in\mathbb{Z}^2} \chi_a^2 \equiv 1 \text{ on } \mathbb{R}^2$$

and the support of $\chi_0$ is included in $\Lambda_2$, where $\chi_a(x) = \chi_0(x-a)$ for any $a \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$, and $\Lambda_r = (-r/2, r/2)^2$ for any $r > 0$. We take this function so that $\chi_0 > 0$ on $\Lambda_2$ and $\chi_0 = 0$ on $\mathbb{R}^2 \setminus \Lambda_2$. For each $\boldsymbol{a} \in \mathbb{Z}^2$, let $\Lambda_r(\boldsymbol{a}) = \boldsymbol{a} + \Lambda_r$. Referring the paracontrolled calculus in Mouzard [14], we fix a large even natural number $b$ and consider operators

$$Q_t^{(c)} := \frac{(-t\Delta)^c}{(c-1)!} e^{t\Delta} \text{ and } P_t^{(c)} := I - \int_0^t \frac{ds}{s} Q_s^{(c)} = \sum_{j=0}^{c-1} \frac{(-t\Delta)^j}{j!} e^{t\Delta}$$

for $c \in [1,b] \cap \mathbb{N}$, where $\Delta$ is the Laplacian on $\mathbb{R}^2$, and $I$ is the identity operator. For $k \in [0, 2b] \cap \mathbb{Z}$, let $StGC^k$ be the set of families of operators of the form

$$((\sqrt{t}\partial_1)^{\alpha_1} (\sqrt{t}\partial_2)^{\alpha_2} P_t^{(c)})_{t\in(0,1]}$$

with $c \in [1,b] \cap \mathbb{N}$ and $\alpha_1, \alpha_2 \in \mathbb{Z}$ satisfying $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = k$. These families of operators are colled as the standard families of Gaussian operators with cancellation of order $k$. We also set

$$StGC^I = \bigcup_{k\in I\cap\mathbb{Z}} StGC^k$$

for any interval $I$ in $[0,\infty)$. Referring [2] and [14], we decompose the product as follows:

$$fg = P_f g + \Pi(f,g) + P_g f + P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g)),$$

for appropriate distributions $f, g$ on $\mathbb{R}^2$, where

$$P_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)) \tag{1.2}$$

with a finite subset $\{(c_\nu, Q^{1,\nu}, Q^{2,\nu}, P^\nu)\}_\nu$

of $\mathbb{R} \times StGC^{[b/2,2b]} \times StGC^{[b/2,2b]} \times StGC^{[0,b/2)}$,

and

$$\Pi(f,g) := \sum_\mu c_\mu \int_0^1 \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g)) \tag{1.3}$$

with a finite subset $\{(c_\mu, Q^{1,\mu}, Q^{2,\mu}, P^\mu)\}_\mu$

of $\mathbb{R} \times StGC^{[b/2,2b]} \times StGC^{[b/2,2b]} \times StGC^{[0,b/2)}$. $P_f g$ is called as a paraproduct and $\Pi(f,g)$ is called as a resonating term. We use also

$${}_hP_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)h),$$

where $h$ is another appropriate distribution. We use

$$\Delta^{-loc} := -\int_0^1 dt e^{t\Delta}, \tag{1.4}$$

which is an approximation of the inverse of the Laplacian satisfying

$$\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^\Delta$$

and the integral kernel has a Gaussian bound:

$$\sup_{|x-y|\geq 1} \frac{\log|\Delta^{-loc}(x,y)|}{|x-y|^2} < 0.$$

We use the commutators:

$$C(f,g,h) := \Pi(\Delta^{-loc} P_f g, h) - f\Pi(\Delta^{-loc} g, h) \tag{1.5}$$

and

$$S(f,g,h) := P_h(\Delta^{-loc} P_f g) - {}_fP_h(\Delta^{-loc} g). \tag{1.6}$$

We use the Besov space $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)$ with parameters $p, q \in [1, \infty]$, $\alpha \in (-2b, 2b)$, defined by the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$(1.7)\quad \begin{aligned} &\|f\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)} \\ :=&\|e^{\Delta} f\|_{L^p(\mathbb{R}^2:dx)} \\ &+ \sup\{\|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)} : Q \in StGC^{(|\alpha|,2b]}\}. \end{aligned}$$

for any $f \in C_0^\infty(\mathbb{R}^2)$, where $C_0^\infty(\mathbb{R}^2)$ is the smooth functions with compact supports. $\mathcal{C}^{\alpha}(\mathbb{R}^2) := \mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)$ is called as the Besov $\alpha$-Hölder space, and $\mathcal{H}^{\alpha}(\mathbb{R}^2) = \mathcal{B}^{\alpha}_{2,2}(\mathbb{R}^2)$ is the Sobolev space with the index $\alpha$. It is known that $\chi_a \xi$ is $C^{-1-\epsilon}(\mathbb{R}^2)$-valued for any $\epsilon > 0$ and $a \in \mathbb{Z}^2$. As in Theorem 2.1 in [14], there exists a random field $Y_\xi$ such that

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_\xi)\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$$

for any $p \in [1, \infty)$, $\epsilon > 0$ and $a \in \mathbb{Z}^2$, where $\xi_\varepsilon := e^{\varepsilon^2 \Delta}\xi$ is a smooth approximation of $\xi$ and

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon) - \mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)].$$

The main theorem in this paper is the following:

**Theorem 1.** *There exists $E_0 \in (-\infty, 0)$ such that for almost all $\xi$, $(-\infty, E_0]$ is included in the pure point spectrum* $\mathrm{spec}_{pp}(\widetilde{H^\xi})$ *of $\widetilde{H^\xi}$ and any corresponding eigenfunction $\phi_\xi$ satisfies*

$$\overline{\lim_{|a| \to \infty}} |a|^{-1} \log \|\chi_a \phi_\xi\|_{L^2(\mathbb{R}^2)} < 0.$$

The most well developped method to prove the results on the localization is the method based on the multiscale analysis initiated by Fröhlich and Spencer [6] and developped by many works [3], [7], [17], [19]. To extend the methods to our case, we should modify the geometric resolvent inequality, since the nonlocality of our operator break the classical for of this inequality. Then we should give a Wegner type inequality applicable for the multiscale analysis based on the extended geometric resolvent inequality. On the other hand, to deduce the spectral property from the multiscale analysis, we should prepare generalized eigenfunction expansions.

Another definition of a self-adjoint operator corresponding to a limit of $-\Delta + \xi_\varepsilon + c_\varepsilon$ as $\varepsilon \to 0$ is given by Hsu and Labbé [10]. They obtain such an operator from the works by Hairer and Labbé [8], [9] on the corresponding parabolic Anderson models. Their idea is simple and they obtain the same result also on $\mathbb{R}^3$. One vantage point of our definition is that the traditional proof of the Anderson localization can be extended as in this paper.

The organization of this paper is as follows. In Section 2 we generalize the geometric resolvent inequality for our operator. In Section 3 we give an estimate of negative eigenvalues. In Section 4 we give a Wegner type estimate. In Section 5 we apply a multiscale analysis to our operator. In Section 6 we give generalized eigenfunctin expansions.for our operator. In Section 7 we complete our proof of Theorem 1.

## 2. A Geometric resolvent inequality

To obtain a Wegner type estimate in Section 4 below, we introduce a system $\overline{\xi} = (\overline{\xi_a})_{a\in\mathbb{Z}^2}$ of independently and identically distributed bounded real random variables with a smooth density independent of the white noise $\xi$, when we restrict the noise $\xi$ to bounded squares: for the system of random variable $\widetilde{\xi} = (\xi, \overline{\xi})$, any $L \in 2\mathbb{N}$, $\boldsymbol{a} \in \mathbb{Z}^2$ and $\varepsilon > 0$, we consider the random field

$$\widetilde{\xi}_{L,\boldsymbol{a}} := \overline{\xi}_{L,\boldsymbol{a}} + \xi_{L-2,\boldsymbol{a}} := \sum_{a\in\mathbb{Z}^2\cap(\Lambda_L(\boldsymbol{a})\setminus\Lambda_{L-2}(\boldsymbol{a}))} \chi_a^2\overline{\xi_a} + \sum_{a\in\mathbb{Z}^2\cap\Lambda_{L-2}(\boldsymbol{a})} \chi_a^2\xi$$

so that the random field is bounded on a neighborhood of the boundary of the square $\Lambda_L(\boldsymbol{a})$. As a corresponding operator, we consider

$$\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}} := \widetilde{H^{\xi}_{L-2,\boldsymbol{a}}} + \overline{\xi}_{L,\boldsymbol{a}}, \tag{2.1}$$

where $\widetilde{H^{\xi}_{L-2,\boldsymbol{a}}}$ is the operator treated in Section 4 in [21]: on

$$\mathrm{Dom}(\widetilde{H^{\xi}_{L-2,\boldsymbol{a}}}) := \Big\{u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,L-2,\boldsymbol{a}}(u) \in \mathcal{H}^2(\mathbb{R}^2)\Big\}$$

we define

$$\begin{aligned}
&\widetilde{H^{\xi}_{L-2,\boldsymbol{a}}}u \\
&:= -\Delta\Phi_{\xi,L-2,\boldsymbol{a}}(u) + P_{\xi_{L-2,\boldsymbol{a}}}(\Phi_{\xi,L-2,\boldsymbol{a}}(u)) + \Pi(\Phi_{\xi,L-2,\boldsymbol{a}}(u), \xi_{L-2,\boldsymbol{a}}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_{L-2,\boldsymbol{a}})) \\
&\quad + e^{\Delta}P_u\xi_{L-2,\boldsymbol{a}} + e^{\Delta}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}}) + e^{\Delta}P_uY_{\xi,L-2,\boldsymbol{a}} \\
&\quad + C(u, \xi_{L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}}) + S(u, \xi_{L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}}) \\
&\quad + P_{Y_{\xi,L-2,\boldsymbol{a}}}u + \Pi(u, Y_{\xi,L,\boldsymbol{a}}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_{\xi,L-2,\boldsymbol{a}})) \\
&\quad + P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}})) + \Pi(\Delta^{-loc}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}}), \xi_{L-2,\boldsymbol{a}}) \\
&\quad + P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}P_uY_{\xi,L-2,\boldsymbol{a}}) + \Pi(\Delta^{-loc}P_uY_{\widetilde{\xi},L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}}),
\end{aligned}$$

where $Y_{\xi,L-2,\boldsymbol{a}}$ is a random field such that

$$\lim_{\varepsilon\to 0}\mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon,L-2,\boldsymbol{a}}-Y_{\xi,L-2,\boldsymbol{a}})\|^p_{C^{-\epsilon}(\mathbb{R}^2)}]=0$$

for any $p\in[1,\infty)$ and $a\in\mathbb{Z}^2$, and

$$Y_{\xi_\varepsilon,L-2,\boldsymbol{a}}:=\Pi(\Delta^{-loc}\sum_{a\in\mathbb{Z}^2\cap\Lambda_{L-2}(\boldsymbol{a})}\chi_a^2\xi_\varepsilon,\sum_{a\in\mathbb{Z}^2\cap\Lambda_{L-2}(\boldsymbol{a})}\chi_a^2\xi_\varepsilon)$$
$$-\mathbb{E}[\Pi(\Delta^{-loc}\sum_{a\in\mathbb{Z}^2\cap\Lambda_{L-2}(\boldsymbol{a})}\chi_a^2\xi_\varepsilon,\sum_{a\in\mathbb{Z}^2\cap\Lambda_{L-2}(\boldsymbol{a})}\chi_a^2\xi_\varepsilon)],$$
$$\xi_\varepsilon:=e^{\varepsilon^2\Delta}\xi,$$
$$\Phi_{\xi,L-2,\boldsymbol{a}}(u):=u-\Delta^{-loc}P_u\xi_{L-2,\boldsymbol{a}}-\Delta^{-loc}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}})-\Delta^{-loc}P_uY_{\xi,L-2,\boldsymbol{a}}$$

In [21], the operator $\widetilde{H^{\xi}_{R,\mathbf{0}}}=:\widetilde{H^{\xi}_{R}}$ is treated. For this operator, we have the following:

**Lemma 2.1.** (i) *The operator $\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}$ is self-adjoint on $L^2(\mathbb{R}^2)$.*

(ii) *$\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}$ is the norm resolvent limit of $\widetilde{H^{\xi_\varepsilon}_{L-2,\boldsymbol{a}}}+\overline{\xi}_{L,\boldsymbol{a}}$ as $\varepsilon\to 0$.*

(iii) *The negative spectra of the operator $\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}$ are discrete.*

**Proof**.

(i) is proven by Lemma 4.10 in [21]. (ii) is proven as in Lemma 4.9 and Proposition 4.1 in [21]. (iii) is proven by the fact that $\widetilde{H^{\xi_\varepsilon}_{L-2,\boldsymbol{a}}}+\overline{\xi}_{L,\boldsymbol{a}}$ is a relatively compact perturbation of $-\Delta$. □

In this section, we give the following geometric resolvent inequality, which is also called as Simon-Lieb inequality:

**Proposition 2.1.** *For any $\epsilon>0$, there exist finite positive constants $c_*,c_1,\dots,c_6$ such that, for any $\boldsymbol{a}$, $\boldsymbol{a}'\in\mathbb{Z}^2$ and $\ell$, $\ell'\in 2\mathbb{N}$ satisfying $\Lambda_\ell(\boldsymbol{a})\subset\Lambda_{\ell'}(\boldsymbol{a}')$, and for any $a\in\Lambda_{\ell-8}(\boldsymbol{a})$, $a_*\notin\Lambda_\ell(\boldsymbol{a})$, and*

*$z \in \mathbb{C} \setminus \mathrm{spec}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}}) \setminus \mathrm{spec}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}})$, we have*

$$
\begin{aligned}
&\|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&\le \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{((\Lambda_{\ell-4}(\boldsymbol{a}) \setminus \Lambda_{\ell-6}(\boldsymbol{a}))}} \|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\chi_{a_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} c_1 \exp(-c_*|a_1 - a|) \\
&\quad + \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{((\Lambda_{\ell-4}(\boldsymbol{a}) \setminus \Lambda_{\ell-6}(\boldsymbol{a}))}, a_2 \in \mathbb{Z}^2} \|\chi_{a_2}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&\qquad \times c_2 \exp(-c_*(|a_1 - a_*| + |a_1 - a_2|^2))\Xi(\boldsymbol{a}, \ell - 2, \xi)^{1/2} \\
&\quad + \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{((\Lambda_{\ell-4}(\boldsymbol{a}) \setminus \Lambda_{\ell-6}(\boldsymbol{a}))}, a_2 \in \mathbb{Z}^2} \|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\chi_{a_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&\qquad \times \|\chi_{a_2}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} c_3 \exp(-c_*|a_1 - a_2|) \\
&\qquad \times \{|z| + (1 + \Xi(\boldsymbol{a}, \ell - 2, \xi))^{c_4}\}^{1/2}(1 + \Xi(\boldsymbol{a}, \ell - 2, \xi)))^{3/4} \\
&\quad + \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{((\Lambda_{\ell-4}(\boldsymbol{a}) \setminus \Lambda_{\ell-6}(\boldsymbol{a}))}, a_2, a_3 \in \mathbb{Z}^2} \|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\chi_{a_2}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&\qquad \times \|\chi_{a_3}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} c_5 \exp(-c_*(|a_1 - a_2| + |a_1 - a_3|^2)) \\
&\qquad \times \{|z| + (1 + \Xi(\boldsymbol{a}', \ell' - 2, \xi))^{c_6}\}^{1/2}(1 + \Xi(\boldsymbol{a}, \ell - 2, \xi)))^{1/2},
\end{aligned} \tag{2.2}
$$

*where $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^2))}$ is the operator norm of the continuous operators on $L^2(\mathbb{R}^2)$,*

$$\Xi(\boldsymbol{a}, \ell - 2, \xi) := \sup_{\widehat{a} \in \mathbb{Z}^2} \Xi(\widehat{a}, \boldsymbol{a}, \ell - 2, \xi),$$

*and*

$$
\begin{aligned}
\Xi(\widehat{a}, \boldsymbol{a}, \ell - 2, \xi) := &\sum_{j=1}^{2} \Big( \sum_{\widehat{a_1} \in \mathbb{Z}^2 \cap \Lambda_{\ell-2}(\boldsymbol{a})} \|\chi^2_{\widehat{a_1}}\xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_*|\widehat{a} - \widehat{a}_1|^2) \Big)^j \\
&+ \sum_{\widehat{a_1} \in \mathbb{Z}^2} \|\chi^2_{\widehat{a_1}} Y_{\xi,\ell-2,\boldsymbol{a}}\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_*|\widehat{a} - \widehat{a}_1|^2).
\end{aligned}
$$

*for any $\widehat{a}, \boldsymbol{a} \in \mathbb{Z}^2$ and $\ell > 2$.*

**Proof**. We take a $[0,1]$-valued smooth function $\phi$ on $\mathbb{R}^2$ so that $\phi \equiv 1$ on $\Lambda_{\ell-6}(\boldsymbol{a})$ and $\phi \equiv 0$ on $\mathbb{R}^2 \setminus \Lambda_{\ell-4}(\boldsymbol{a})$. Then we have

$$
\begin{aligned}
&(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\phi - \phi(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1} \\
=&(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\{\phi(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z) - (\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)\phi\}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1} \\
=&(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\{2(\nabla\phi)\cdot\nabla + (\Delta\phi) + \mathbb{E}[\phi\Pi(\Delta^{-loc}(\xi_{\ell'-2,\boldsymbol{a}'} - \xi_{\ell-2,\boldsymbol{a}}), \xi_{\ell'-2,\boldsymbol{a}'})] \\
&+ \mathbb{E}[\phi\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell'-2,\boldsymbol{a}'} - \xi_{\ell-2,\boldsymbol{a}})]\})(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}.
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
\le&2\|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}(\nabla\phi)\cdot\nabla(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&+ \|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}(\Delta\phi)(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&+ \|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}(\mathbb{E}[\phi\Pi(\Delta^{-loc}(\xi_{\ell'-2,\boldsymbol{a}'} - \xi_{\ell-2,\boldsymbol{a}}), \xi_{\ell'-2,\boldsymbol{a}'})] \\
&+ \mathbb{E}[\phi\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell'-2,\boldsymbol{a}'} - \xi_{\ell-2,\boldsymbol{a}})]\})(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))}.
\end{aligned}
\tag{2.3}
$$

Since

$$
\begin{aligned}
&|\mathbb{E}[\phi\Pi(\Delta^{-loc}(\xi_{\ell'-2,\boldsymbol{a}'} - \xi_{\ell-2,\boldsymbol{a}}), \xi_{\ell'-2,\boldsymbol{a}'})]| \\
&\vee |\mathbb{E}[\phi\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell'-2,\boldsymbol{a}'} - \xi_{\ell-2,\boldsymbol{a}})] \\
\le& c_2\exp(-c_1 d(\cdot, \mathbb{R}^2 \setminus \Lambda_\ell(\boldsymbol{a}))^2),
\end{aligned}
$$

the second, third and fourth terms of the right hand side of (2.3) are less than or equal to

$$
\begin{aligned}
\le& c_3 \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\Lambda_\ell(\boldsymbol{a})}} \|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}\chi_{a_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&\times \|\chi_{a_1}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))}\exp(-c_1 d(a_1, \Lambda_\ell(\boldsymbol{a})^c)^2).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}(\nabla\phi)\cdot\nabla(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
=&\sup\{|(\psi, \chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell',\boldsymbol{a}'}} - z)^{-1}(\nabla\phi)\cdot\nabla(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\varphi)_{L^2(\mathbb{R}^2)}| \\
&: \varphi, \psi \in C_0^\infty(\mathbb{R}^2), \|\varphi\|_{L^2(\mathbb{R}^2)} = \|\psi\|_{L^2(\mathbb{R}^2)} = 1\},
\end{aligned}
$$

we should estimate

$$
I_\iota = (\phi_{2,\iota}\Psi, \phi_{1,\iota}\partial_{x^\iota}\Phi)_{L^2(\mathbb{R}^2)}
$$

for $\iota \in \{1,2\}$, where

$$\Phi = (\widetilde{H^{\widetilde{\xi}}_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\varphi,$$
$$\Psi = (\widetilde{H^{\widetilde{\xi}}_{\ell',\boldsymbol{a}'}} - \overline{z})^{-1}\chi_{a_*}\psi,$$

and $\phi_{j,\iota}, \varphi, \psi \in C_0^\infty(\mathbb{R}^2)$ such that

$$\operatorname{supp}\phi_{1,\iota}, \operatorname{supp}\phi_{2,\iota} \subset \Lambda_{\ell-4}(\boldsymbol{a}) \setminus \Lambda_{\ell-6}(\boldsymbol{a})$$

and $\|\varphi\|_{L^2(\mathbb{R}^2)} = \|\psi\|_{L^2(\mathbb{R}^2)} = 1$.

We devide as $I_\iota = \sum_{j=1}^4 I_{\iota,j}$, where

$$I_{\iota,1} =(\phi_{2,\iota}\Psi, \phi_{1,\iota}\partial_{x^\iota}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi))_{L^2(\mathbb{R}^2)}$$
$$I_{\iota,2} =(\phi_{2,\iota}\Psi, \phi_{1,\iota}\partial_{x^\iota}\Delta^{-loc}P_\Phi\xi_{\ell-2,\boldsymbol{a}})_{L^2(\mathbb{R}^2)}$$
$$I_{\iota,3} =(\phi_{2,\iota}\Psi, \phi_{1,\iota}\partial_{x^\iota}\Delta^{-loc}{}_\Phi P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}})_{L^2(\mathbb{R}^2)}$$

and

$$I_{\iota,4} = (\phi_{2,\iota}\Psi, \phi_{1,\iota}\partial_{x^\iota}\Delta^{-loc}P_\Phi Y_{\xi,\ell-2,\boldsymbol{a}}))_{L^2(\mathbb{R}^2)}.$$

$I_{\iota,1}$ is dominated by

$$\sum_{a_1 \in \mathbb{Z}^2 \cap \overline{(\Lambda_{\ell-4}(\boldsymbol{a})\setminus\Lambda_{\ell-6}(\boldsymbol{a}))}} \|\chi_{a_1}\partial_{x^\iota}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)}.$$

For $a_1 \in \mathbb{Z}^2 \cap \overline{(\Lambda_{\ell-4}(\boldsymbol{a}) \setminus \Lambda_{\ell-6}(\boldsymbol{a}))}$, $\|\chi_{a_1}\nabla\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|^2_{L^2(\mathbb{R}^2)}$ is dominated by $\sum_{j=1}^8 I_j(a_1)$, where

$$I_1(a_1) =(\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi), \chi_{a_1}(z + \overline{\xi}_{\ell,\boldsymbol{a}})\Phi)_{L^2(\mathbb{R}^2)}$$
$$I_2(a_1) =(\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi), \chi_{a_1}(P_{\xi_{\ell-2,\boldsymbol{a}}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi) + \Pi(\xi_{\ell-2,\boldsymbol{a}}, \Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)))_{L^2(\mathbb{R}^2)}$$
$$\begin{aligned}I_3(a_1) =&(\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi), \chi_{a_1}(P_1^{(b)}((P_1^{(b)}\Phi)(P_1^{(b)}\xi_{\ell-2,\boldsymbol{a}})) + e^\Delta P_\Phi\xi_{\ell-2,\boldsymbol{a}}\\ &+ e^\Delta_\Phi P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}} + e^\Delta P_\Phi Y_{\xi,\ell-2,\boldsymbol{a}}))_{L^2(\mathbb{R}^2)}\end{aligned}$$
$$I_4(a_1) =(\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi), \chi_{a_1}(C(\Phi, \xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell-2,\boldsymbol{a}}) + S(\Phi, \xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell-2,\boldsymbol{a}})))_{L^2(\mathbb{R}^2)}$$
$$I_5(a_1) =(\chi_{a_1}\Phi_{\widetilde{\xi},\ell,\boldsymbol{a}}(\Phi), \chi_{a_1}(\Phi_{Y_{\xi,\ell-2,\boldsymbol{a}}}\Phi + \Pi(Y_{\xi,\ell-2,\boldsymbol{a}}, \Phi) + P_1^{(b)}((P_1^{(b)}Y_{\xi,\ell-2,\boldsymbol{a}})(P_1^{(b)}\Phi))))_{L^2(\mathbb{R}^2)}$$
$$\begin{aligned}I_6(a_1) =&(\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi), \chi_{a_1}(P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta_\Phi^{-loc}P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}\\ &+ \Pi(\xi_{\ell-2,\boldsymbol{a}}, \Delta_\Phi^{-loc}P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}})))_{L^2(\mathbb{R}^2)}\end{aligned}$$
$$\begin{aligned}I_7(a_1) =&(\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi), \chi_{a_1}(P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta^{-loc}P_\Phi Y_{\xi,\ell-2,\boldsymbol{a}}\\ &+ \Pi(\xi_{\ell-2,\boldsymbol{a}}, \Delta_\Phi^{-loc}P_{\xi_{\ell-2,\boldsymbol{a}}}\Delta^{-loc}P_\Phi Y_{\xi,\ell-2,\boldsymbol{a}})))_{L^2(\mathbb{R}^2)}\end{aligned}$$

and

$$I_8(a_1) = \|(\nabla\chi_{a_1})\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|^2_{L^2(\mathbb{R}^2)}.$$

By Lemma 3.2 (ii) (iii) in [21], $I_2(a_1)$ is dominated by

$$\sum_{a_2\in\mathbb{Z}^2\cap\Lambda_{\ell-2}(\boldsymbol{a}),a_3\in\mathbb{Z}^2} \|\chi_{a_1}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}\xi\|_{C^{-1-\varepsilon/2}(\mathbb{R}^2)}\|\chi_{a_3}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}$$
$$\times\exp(-c_4(|a_1-a_2|^2+|a_1-a_3|^2)).$$

As in the proof of Theorem 1 in [21], we have

$$\begin{aligned}&\|\chi_{a_3}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\\ \le& c_5\Big(\frac{1}{t^{(1+\epsilon)/2}}\sum_{a_4\in\mathbb{Z}^2\cap\overline{\Lambda_2(a_3)}}\|\chi_{a_4}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\\ &+t^{(1-\epsilon)/2}\sum_{a_4\in\mathbb{Z}^2\cap\overline{\Lambda_2(a_3)}}\|\chi_{a_4}\Delta\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\Big)\end{aligned}$$

for any $t\in(0,\infty)$. Since

$$(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}}-z)\Phi=\chi_a\varphi,$$

we have

$$\begin{aligned}&\|\chi_{a_4}\Delta\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\\ \le&\|\chi_{a_4}\chi_a\varphi\|_{L^2(\mathbb{R}^2)}+(|z|+\max|\overline{\xi}_0|)\|\chi_{a_4}\Phi\|_{L^2(\mathbb{R}^2)}\\ &+\sum_{a_5\in\mathbb{Z}^2}\|\chi^2_{a_5}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\exp(-c_6|a_4-a_5|^2)\\ &\times\sum_{a_6\in\mathbb{Z}^2\cap\Lambda_{\ell-2}(\boldsymbol{a})}c_7\|\chi^2_{a_6}\xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)}\exp(-c_6|a_4-a_6|^2)\\ &+\sum_{a_5\in\mathbb{Z}^2}\|\chi^2_{a_5}\Phi\|_{\mathcal{H}^{2\epsilon}(\mathbb{R}^2)}\exp(-c_6|a_4-a_5|^2)\\ &\times c_8\Big\{1+\Big(\sum_{a_6\in\mathbb{Z}^2\cap\Lambda_{\ell-2}(\boldsymbol{a})}\|\chi^2_{a_6}\xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)}\exp(-c_6|a_4-a_6|^2)\Big)^2\\ &+\sum_{a_6\in\mathbb{Z}^2}\|\chi^2_{a_6}Y_{\xi,\ell-2,\boldsymbol{a}}\|_{C^{-\varepsilon}(\mathbb{R}^2)}\exp(-c_6|a_4-a_6|^2)\Big\}^{3/2}.\end{aligned}$$

By the definition of $\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)$, we have

$$\begin{aligned}&\|\chi_{a_5}(\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)-\Phi)\|_{\mathcal{H}^{2\epsilon}(\mathbb{R}^2)}\\ \le& c_{10}\Xi(a_5,\boldsymbol{a},\ell-2,\xi)\sum_{a_6\in\mathbb{Z}^2}\|\chi^2_{a_6}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_9|a_5-a_6|^2).\end{aligned}\tag{2.4}$$

Thus we have

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\\
\le& c_{12}t^{(1-\epsilon)/2}\sum_{\widehat{a_1}\in\mathbb{Z}^2}\|\chi_{\widehat{a_1}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\exp(-c_{11}|\widehat{a_0}-\widehat{a_1}|^2)(1+\Xi(\widehat{a_0},\boldsymbol{a},\ell-2,\xi))^{3/2}\\
&+c_{13}t^{(1-\epsilon)/2}\sum_{\widehat{a_1}\in\mathbb{Z}^2\cap\overline{\Lambda_2(a_{\max})}}(\|\chi_{a_4}\chi_a\varphi\|_{L^2(\mathbb{R}^2)}+(|z|+\max|\overline{\xi}_0|)\|\chi_{a_4}\Phi\|_{L^2(\mathbb{R}^2)})\\
&+c_{14}t^{(1-\epsilon)/2}\sum_{\widehat{a_1}\in\mathbb{Z}^2}\|\chi^2_{\widehat{a_1}}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{11}|\widehat{a_0}-\widehat{a_1}|^2)(1+\Xi(\widehat{a_0},\boldsymbol{a},\ell-2,\xi))^{5/2}\\
&+\frac{c_{15}}{t^{(1+\epsilon)/2}}\sum_{\widehat{a_1}\in\mathbb{Z}^2}\|\chi_{\widehat{a_1}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

for any $\widehat{a_0}\in\mathbb{Z}^2$. By the iteration, we have

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\\
\le&(c_{17}t^{(1-\epsilon)/2})^n\sum_{\widehat{a_1},\dots,\widehat{a_n}\in\mathbb{Z}^2}\|\chi_{\widehat{a_n}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}\exp\Big(-c_{16}\sum_{j=1}^n|\widehat{a_{j-1}}-\widehat{a_j}|^2\Big)\prod_{j=0}^{n-1}(1+\Xi(\widehat{a_j},\boldsymbol{a},\ell-2,\xi))^{3/2}\\
&+\sum_{k=0}^n(c_{17}t^{(1-\epsilon)/2})^k\sum_{\widehat{a_1},\dots,\widehat{a_k}\in\mathbb{Z}^2}\exp\Big(-c_{16}\sum_{j=1}^k|\widehat{a_{j-1}}-\widehat{a_j}|^2\Big)\prod_{j=0}^{k-1}(1+\Xi(\widehat{a_j},\boldsymbol{a},\ell-2,\xi))^{3/2}\\
&\times\Big\{c_{18}t^{(1-\epsilon)/2}\sum_{\widehat{a_{k+1}}\in\mathbb{Z}^2\cap\overline{\Lambda_2(\widehat{a_k})}}(\|\chi_{\widehat{a_{k+1}}}\chi_a\varphi\|_{L^2(\mathbb{R}^2)}+(|z|+\max|\overline{\xi}_0|)\|\chi_{\widehat{a_{k+1}}}\Phi\|_{L^2(\mathbb{R}^2)})\\
&+c_{19}t^{(1-\epsilon)/2}\sum_{\widehat{a_{k+1}}\in\mathbb{Z}^2}\|\chi_{\widehat{a_{k+1}}}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{16}|\widehat{a_k}-\widehat{a_{k+1}}|^2)(1+\Xi(\widehat{a_k},\boldsymbol{a},\ell-2,\xi))^{5/2}\\
&+\frac{c_{20}}{t^{(1+\epsilon)/2}}\sum_{\widehat{a_{k+1}}\in\mathbb{Z}^2\cap\overline{\Lambda_2(\widehat{a_k})}}\|\chi_{\widehat{a_{k+1}}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\Big\}
\end{aligned}
$$

for any $n\in\mathbb{N}$. Then we have

$$
\begin{aligned}
&(c_{17}t^{(1-\epsilon)/2})^{k-1}\sum_{\widehat{a_1},\dots,\widehat{a_{k-1}}\in\mathbb{Z}^2}\exp\Big(-c_{16}\sum_{j=1}^k|\widehat{a_{j-1}}-\widehat{a_j}|^2\Big)\prod_{j=0}^{k-1}(1+\Xi(\widehat{a_j},\boldsymbol{a},\ell-2,\xi))^{3/2}\\
\le&\{c_{22}(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/2}t^{(1-\epsilon)/2}\}^{k-1}\exp\Big(-\frac{c_{21}}{k}|\widehat{a_0}-\widehat{a_k}|^2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&(c_{17}t^{(1-\epsilon)/2})^n\sum_{\widehat{a_1},\dots,\widehat{a_n}\in\mathbb{Z}^2}\exp\Big(-c_{16}\sum_{j=1}^n|\widehat{a_{j-1}}-\widehat{a_j}|^2\Big)\prod_{j=0}^{n}(1+\Xi(\widehat{a_j},\boldsymbol{a},\ell-2,\xi))^{3/2}\\
\le&\{c_{22}(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/2}t^{(1-\epsilon)/2}\}^nc_{23}n.
\end{aligned}
$$

For any $\delta \in (0,1)$, we take $t$ as $(\delta/(1 \vee (c_{22}(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/2})))^{2/(1-\epsilon)}$. Then, by taking the limit as $n \to \infty$, we have

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \\
\leq &\sum_{k=0}^{\infty} \sum_{\widehat{a_1}\in\mathbb{Z}^2} \delta^k \exp\Big( -\frac{c_{21}}{k}|\widehat{a_0}-\widehat{a_1}|^2\Big) \\
&\times \Big\{ c_{24}\delta \sum_{\widehat{a_2}\in\mathbb{Z}^2\cap\overline{\Lambda_2(\widehat{a_1})}} (\|\chi_{\widehat{a_2}}\chi_a\varphi\|_{L^2(\mathbb{R}^2)} + (|z| + \max|\overline{\xi}_0|)\|\chi_{\widehat{a_2}}\Phi\|_{L^2(\mathbb{R}^2)}) \\
&+ c_{25}\delta \sum_{\widehat{a_2}\in\mathbb{Z}^2} \|\chi_{\widehat{a_2}}\Phi\|_{L^2(\mathbb{R}^2)} \exp(-c|\widehat{a_1}-\widehat{a_2}|^2)(1+\Xi(\widehat{a_1},\boldsymbol{a},\ell-2,\xi))^{5/2} \\
&+ c_{26}\Big(\frac{(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/2}}{\delta}\Big)^{(1+\epsilon)/(1-\epsilon)} \sum_{\widehat{a_2}\in\mathbb{Z}^2\cap\overline{\Lambda_2(\widehat{a_1})}} \|\chi_{\widehat{a_2}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\Big\}.
\end{aligned}
$$

By (2.4) and

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \delta^k \exp\Big( -\frac{c_{21}}{k}|\widehat{a_0}-\widehat{a_1}|^2\Big) \\
\leq &\sum_{k=0}^{\infty} \delta^{k/2} \exp\Big( -\inf_k \Big(\frac{k}{2}\log\frac{1}{\delta} + \frac{c_{21}}{k}|\widehat{a_0}-\widehat{a_1}|^2\Big)\Big) \\
= &\frac{1}{1-\delta^{1/2}} \exp\Big( -\sqrt{2c_{21}\log\frac{1}{\delta}}|\widehat{a_0}-\widehat{a_1}|\Big).
\end{aligned}
$$

we have

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \\
\leq &\sum_{\widehat{a_1}\in\mathbb{Z}^2} \frac{1}{1-\delta^{1/2}} \exp\Big( -\sqrt{2c_{21}\log\frac{1}{\delta}}|\widehat{a_0}-\widehat{a_1}|\Big) \\
&\times c_{27}\Big\{\delta \sum_{\widehat{a_2}\in\mathbb{Z}^2\cap\overline{\Lambda_2(\widehat{a_1})}} (\|\chi_{\widehat{a_2}}\chi_a\varphi\|_{L^2(\mathbb{R}^2)} + (|z| + \max|\overline{\xi}_0|)\|\chi_{\widehat{a_2}}\Phi\|_{L^2(\mathbb{R}^2)}) \\
&+ (1+\Xi(\boldsymbol{a},\ell-2,\xi))^{1+3(1+\epsilon)/(2(1-\epsilon))}\delta^{-(1+\epsilon)/(1-\epsilon)} \\
&\times \sum_{\widehat{a_2}\in\mathbb{Z}^2} \|\chi_{\widehat{a_2}}\Phi\|_{L^2(\mathbb{R}^2)} \exp(-c_{28}|\widehat{a_1}-\widehat{a_2}|^2)\Big\}.
\end{aligned}
$$

Thus there exist $c_{29},\dots,c_{32}$ such that

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)} \\
\leq &c_{30}\exp(-c_{29}|\widehat{a_0}-a|) \\
&+ \sum_{\widehat{a_1}\in\mathbb{Z}^2} \|\chi_{\widehat{a_1}}\Phi\|_{L^2(\mathbb{R}^2)} \exp(-c_{29}|\widehat{a_0}-\widehat{a_1}|) \\
&\times c_{31}\{|z| + (1+\Xi(\boldsymbol{a},\ell-2,\xi))^{c_{32}}\}
\end{aligned}
$$

and

$$
\begin{aligned}
&I_2(a_1)\\
\leq& c_{34}\exp(-c_{33}|a_1-a|)(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/2}\\
&\times\sum_{a_2\in\mathbb{Z}^2}\|\chi_{a_2}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{33}|a_1-a_2|)\\
&+c_{35}\{|z|+(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{c_{36}}\}(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/2}\\
&\times\Big(\sum_{a_2\in\mathbb{Z}^2}\|\chi_{a_2}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{33}|a_1-a_2|)\Big)^2.
\end{aligned}
$$

$I_j(a_1)$, $j\in\{1,3,4,\dots,8\}$, are simirarly estimated and we have

$$
\begin{aligned}
&\|\chi_{a_1}\nabla\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{L^2(\mathbb{R}^2)}\\
\leq& c_{38}\exp(-c_{37}|a_1-a|)\\
&+c_{39}\{|z|+(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{c_{40}}\}^{1/2}(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/4}\\
&\times\sum_{a_2\in\mathbb{Z}^2}\|\chi_{a_2}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{37}|a_1-a_2|)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{\iota,1}\leq&\sum_{a_1\in\mathbb{Z}^2\cap\overline{\Lambda_{\ell-4}(\boldsymbol{a})\setminus\Lambda_{\ell-6}(\boldsymbol{a})}}\|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)}\Big\{c_{38}\exp(-c_{37}|a_1-a|)\\
&+c_{39}\{|z|+(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{c_{40}}\}^{1/2}(1+\Xi(\boldsymbol{a},\ell-2,\xi))^{3/4}\\
&\times\sum_{a_2\in\mathbb{Z}^2}\|\chi_{a_2}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{37}|a_1-a_2|)\Big\}
\end{aligned}
$$

The estimate of $I_{\iota,2}$ we obtain from Lemma 3.2 in [21] is

$$
\begin{aligned}
&I_{\iota,2}\\
\leq&\sum_{a_1\in\mathbb{Z}^2\cap\overline{\Lambda_{\ell-4}(\boldsymbol{a})\setminus\Lambda_{\ell-6}(\boldsymbol{a})},a_2\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a}),a_3\in\mathbb{Z}^2}\|\chi_{a_1}^2\Psi\|_{\mathcal{H}^{2\epsilon}(\mathbb{R}^2)}\|\chi_{a_1}^2\xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}\|\chi_{a_3}^2\Phi\|_{L^2(\mathbb{R}^2)}\\
&\times c_{42}\exp(-c_{41}(|a_1-a_2|^2+|a_1-a_3|^2)).
\end{aligned}
$$

As in the estimate of $\|\chi_{\widehat{a_0}}\Phi_{\xi,\ell-2,\boldsymbol{a}}(\Phi)\|_{\mathcal{H}^{1+\epsilon}(\mathbb{R}^2)}$, we have

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Phi_{\widetilde{\xi},\ell',\boldsymbol{a}'}(\Psi)\|_{\mathcal{H}^{2\epsilon}(\mathbb{R}^2)}\\
\le& c_{44}\exp(-c_{43}|\widehat{a_0}-a_*|)\\
&+\sum_{\widehat{a_1}\in\mathbb{Z}^2}\|\chi_{\widehat{a_1}}\Psi\|_{L^2(\mathbb{R}^2)}\exp(-c_{43}|\widehat{a_0}-\widehat{a_1}|)\\
&\times c_{45}\{|z|+(1+\Xi(\boldsymbol{a}',\ell'-2,\xi))^{c_{46}}\}
\end{aligned}
$$

By using also (2.4), we have

$$
\begin{aligned}
&\|\chi_{\widehat{a_0}}\Psi\|_{\mathcal{H}^{2\epsilon}(\mathbb{R}^2)}\\
\le& c_{48}\exp(-c_{47}|\widehat{a_0}-a_*|)\\
&+\sum_{\widehat{a_1}\in\mathbb{Z}^2}\|\chi_{\widehat{a_1}}\Psi\|_{L^2(\mathbb{R}^2)}\exp(-c_{47}|\widehat{a_0}-\widehat{a_1}|)\\
&\times c_{49}\{|z|+(1+\Xi(\boldsymbol{a}',\ell'-2,\xi))^{c_{50}}\}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{\iota,2}\le&\sum_{a_1\in\mathbb{Z}^2\cap\Lambda_{\ell-4}(\boldsymbol{a})\setminus\Lambda_{\ell-6}(\boldsymbol{a})}\Big\{c_{51}\exp(-c_{47}|\widehat{a_0}-a_*|)\\
&+\sum_{a_2\in\mathbb{Z}^2}\|\chi_{a_2}\Psi\|_{L^2(\mathbb{R}^2)}\exp(-c_{47}|a_1-a_2|)\\
&\times c_{52}\{|z|+(1+\Xi(\boldsymbol{a}',\ell'-2,\xi))^{c_{50}}\}\Xi(\boldsymbol{a},\ell-2,\xi))^{1/2}\\
&\times\sum_{a_3\in\mathbb{Z}^2}\|\chi_{a_3}\Phi\|_{L^2(\mathbb{R}^2)}\exp(-c_{53}|a_1-a_3|^2)\Big\}.
\end{aligned}
$$

$I_{\iota,3}$ and $I_{\iota,4}$ are simirarly estimated and we obtain (2.2). □

## 3. An estimate of numbers of negative eigenvalues

Let $\widetilde{H_L^{\widetilde{\xi}}}=\widetilde{H_{L-2}^{\xi}}+\overline{\xi}_L$ be the operator $\widetilde{H_{L,\boldsymbol{a}}^{\widetilde{\xi}}}=\widetilde{H_{L-2,\boldsymbol{a}}^{\xi}}+\overline{\xi}_{L,\boldsymbol{a}}$ in (2.1) with $\boldsymbol{a}=\boldsymbol{0}$: we omit to write $\boldsymbol{a}$ when $\boldsymbol{a}=\boldsymbol{0}$. In this section, we denote $\ell:=L-2$.

In this section, we prove the following:

**Proposition 3.1.** *For any $\lambda>0$ and $p\ge1$, there exist finite positive constants $c_{\lambda,p,1},c_{\lambda,p,2}$ such that*

$$
\mathbb{E}[\mathrm{Tr}[\widetilde{1_{(-\infty,-\lambda]}}(\widetilde{H_L^{\widetilde{\xi}}}-t\widetilde{\chi}_L)]^p]^{1/p}\le c_{\lambda,p,1}(1+t^2)L^{c_{\lambda,p,2}}\tag{3.1}
$$

*for any $L\in2\mathbb{N}$ and $t\ge0$, where $\widetilde{\chi}_L=\sum_{a\in\mathbb{Z}^2\cap\Lambda_L}\chi_a^2$, and $\widetilde{1_{(-\infty,-\lambda]}}$ is a $[0,1]$-valued continuous function on $\mathbb{R}$ such that $\widetilde{1_{(-\infty,-\lambda]}}=1$ on $(-\infty,-\lambda]$ and $\widetilde{1_{(-\infty,-\lambda]}}=0$ on $[-\lambda/2,\infty)$.*

**Proof**

We first consider the case of $t = 0$. Since $\widetilde{H_L^{\xi}}$ is bounded below, by Lemma 2.1 and the Fatou lemma, we can show that the left hand side of (3.1) is less than or equal to

$$\varliminf_{\varepsilon \to 0} \mathbb{E}[\mathrm{Tr}[1_{\widetilde{(-\infty,-\lambda]}}(\widetilde{H_\ell^{\xi_\varepsilon}} + \overline{\xi}_L)]^p].$$

For each $\xi$ and $\varepsilon > 0$, $\mathrm{Tr}[1_{\widetilde{(-\infty,-\lambda]}}(\widetilde{H_\ell^{\xi_\varepsilon}} + \overline{\xi}_L)]$ is finite since $\widetilde{H_\ell^{\xi_\varepsilon}} + \overline{\xi}_L$ is relatively compact perturbation of $-\Delta$.

As in Section 4 in [21], we use the following products obtained by replacing the upper end 1 of the integral interval by $s \in (0,1)$ in (1.2) and (1.3):

$$P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)) \tag{3.2}$$

and

$$\Pi^s(f,g) := \sum_\mu c_\mu \int_0^s \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g)). \tag{3.3}$$

Then we have

$$fg = P_f^s g + \Pi^s(f,g) + P_g^s f + P_s^{(b)}((P_s^{(b)} f)(P_s^{(b)} g)).$$

We define a random field $Y_{\xi,\ell}^s$ by

$$\lim_{\varepsilon \to 0} \mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon,\ell}^s - Y_{\xi,\ell}^s)\|_{C^{-\epsilon}(\mathbb{R}^2)}^p] = 0$$

for any $p \in [1,\infty)$, $\epsilon > 0$ and $a \in \mathbb{Z}^2$, where $\ell = L - 2$,

$$Y_{\xi_\varepsilon,\ell}^s := \Pi^s(\Delta^{-loc}\xi_{\varepsilon,\ell}, \xi_{\varepsilon,\ell}) - \mathbb{E}[\Pi^s(\Delta^{-loc}\xi_{\varepsilon,\ell}, \xi_{\varepsilon,\ell})], \tag{3.4}$$

and $\xi_{\varepsilon,\ell} := \sum_{a \in \mathbb{Z}^2 \cap \Lambda_\ell} \chi_a^2 \xi_\varepsilon$. We use also

$$_hP_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)h),$$

$$C^s(f,g,h) := \Pi^s(\Delta^{-loc} P_f^s g, h) - f\Pi^s(\Delta^{-loc} g, h), \tag{3.5}$$

$$S^s(f,g,h) := P_h^s(\Delta^{-loc} P_f^s g) - {}_fP_h^s(\Delta^{-loc} g),. \tag{3.6}$$

and

$$\Phi_{\xi,\ell}^s(u) := u - \Delta^{-loc} P_u^s \xi_\ell - \Delta^{-loc}{}_uP_{\xi_\ell}^s(\Delta^{-loc}\xi_\ell) - \Delta^{-loc} P_u^s Y_{\xi,\ell}^s.$$

We set

$$\mathrm{Dom}(\widehat{H_\ell^{\xi,s}}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi^s_{\xi,\ell}(u) \in \mathcal{H}^2(\mathbb{R}^2) \Big\}$$

and

$$\begin{aligned}
&\widehat{H_\ell^{\xi,s}} u \\
=& -\Delta\Phi^s_{\xi,\ell}(u) + P^s_{\xi_\ell}(\Phi^s_{\xi,\ell}(u)) + \Pi^s(\Phi^s_{\xi,\ell}(u), \xi_\ell) \\
&+ e^{s\Delta} P^s_u \xi_\ell + e^{s\Delta}{}_u P^s_{\xi_\ell}(\Delta^{-loc}\xi_\ell) + e^{s\Delta} P^s_u Y^s_{\xi,\ell} \\
&+ C^s(u, \xi_\ell, \xi_\ell) + S^s(u, \xi_\ell, \xi_\ell) \\
&+ P^s_{Y^s_{\xi,\ell}} u + \Pi^s(u, Y^s_{\xi,\ell}) + P^{(b)}_s((P^{(b)}_s u)(P^{(b)}_s Y^s_{\xi,\ell})) \\
&+ P^s_{\xi_\ell}(\Delta^{-loc}{}_u P^s_{\xi_\ell}(\Delta^{-loc}\xi_\ell)) + \Pi^s(\Delta^{-loc}{}_u P^s_{\xi_\ell}(\Delta^{-loc}\xi_\ell), \xi_\ell) \\
&+ P^s_{\xi_\ell}(\Delta^{-loc} P^s_u Y^s_{\xi,\ell}) + \Pi^s(\Delta^{-loc} P^s_u Y_{\xi,\ell}, \xi_\ell)
\end{aligned} \tag{3.7}$$

for $u \in \mathrm{Dom}(\widehat{H_\ell^{\xi,s}})$. By Lemma 3.1 below, we see that

$$(\Pi - \Pi^s)(\Delta^{-loc}\xi_\ell, \xi_\ell) := \lim_{\varepsilon\to 0} \{\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell}, \xi_{\varepsilon,\ell}) - \Pi^s(\Delta^{-loc}\xi_{\varepsilon,\ell}, \xi_{\varepsilon,\ell})\}$$

is smooth as a function on $\mathbb{R}^2$ such that

$$\|\chi_a(\Pi - \Pi^s)(\Delta^{-loc}\xi_\ell, \xi_\ell)\|_{C^\beta(\mathbb{R}^2)}$$

is dominated by $\exp(-cd(a, \Lambda_L)^2)$ for any $a \in \mathbb{Z}^2$ and $\beta \in \mathbb{R}$.

$$\overline{Y^s_\ell} := \mathbb{E}[(\Pi - \Pi^s)(\Delta^{-loc}\xi_\ell, \xi_\ell)]$$

and

$$Y_{\xi,\ell} - Y^s_{\xi,\ell}$$

have the same property. Then we have

$$\mathrm{Dom}(\widehat{H_\ell^{\xi,s}}) = \mathrm{Dom}(\widetilde{H_\ell^{\xi}}).$$

and

$$\widehat{H_\ell^{\xi,s}} u + P^{(b)}_s((P^{(b)}_s u)(P^{(b)}_s \xi_\ell)) - \overline{Y^s_\ell} u = \widetilde{H_\ell^{\xi}} u$$

for $u \in \mathrm{Dom}(\widehat{H_\ell^\xi})$. Since $P_s^{(b)}((P_s^{(b)}\cdot)(P_s^{(b)}\xi_\ell))$ is a bounded symmetric operator, $\widehat{H_\ell^{\xi,s}}$ is a self-adjoint operator on $L^2(\mathbb{R}^2)$. By using Lemma 3.2, Lemma 3.3 and Lemma 3.4 below as in the proof of Lemma 4.9 in [21], we see the existence of positive constants $c_1$, $c_2$, $c_3$ such that

$$\begin{aligned}&(u, \widehat{H_\ell^{\xi,s}}u)_{L^2(\mathbb{R}^2)}\\ \geq&\frac{1}{2}\|\nabla\Phi^s_{\xi,\ell}(u)\|^2_{L^2(\mathbb{R}^2)} - c_1 s^{c_2}(1+\|\xi_\ell\|^2_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{s\in(0,1]}\|Y^s_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})^{c_3}\|u\|^2_{L^2(\mathbb{R}^2)}\end{aligned}$$

for any $u \in \mathrm{Dom}(\widehat{H_\ell^{\xi,s}})$. Thus if we set

$$s(\xi,\lambda,\ell) = (\lambda/(4c_1(1+\|\xi_\ell\|^2_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{s\in(0,1]}\|Y^s_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})^{c_3}))^{1/c_2},$$

then we have

$$\widehat{H_\ell^{\xi,s(\xi,\lambda,\ell)}} \geq -\lambda/4.$$

We take $\varepsilon > 0$ so small that

$$\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}} \geq -\lambda/2.$$

By the Birman-Schwinger principle. we have

$$\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H_\ell^{\xi_\varepsilon}} + \overline{\xi}_L)] \leq \mathrm{Tr}[1_{[1,\infty)}(\Gamma^{(\widetilde{\xi},\varepsilon)})] \leq \mathrm{Tr}[(\Gamma^{(\widetilde{\xi},\varepsilon)})^2] = \|\Gamma^{(\widetilde{\xi},\varepsilon)}\|^2_{\mathcal{I}_2},$$

where $\Gamma^{(\widetilde{\xi},\varepsilon)} = -\Gamma_0^{(\xi,\varepsilon)} + \Gamma_1^{(\xi,\varepsilon)} - \Gamma_2^{(\widetilde{\xi},\varepsilon)}$,

$$\begin{aligned}\Gamma_0^{(\xi,\varepsilon)} &= (\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}}+\lambda)^{-1/2}(P^{(b)}_{s(\xi,\lambda,\ell)}((P^{(b)}_{s(\xi,\lambda,\ell)}\xi_{\varepsilon,\ell})(P^{(b)}_{s(\xi,\lambda,\ell)}\cdot)))(\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}}+\lambda)^{-1/2},\\ \Gamma_1^{(\xi,\varepsilon)} &= (\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}}+\lambda)^{-1/2}\overline{Y_\ell^{s(\xi,\lambda,\ell)}}(\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}}+\lambda)^{-1/2},\end{aligned}$$

and

$$\Gamma_2^{(\widetilde{\xi},\varepsilon)} = (\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}}+\lambda)^{-1/2}\overline{\xi}_L(\widehat{H_\ell^{\xi_\varepsilon,s(\xi,\lambda,\ell)}}+\lambda)^{-1/2}.$$

Since

$$|(P^{(b)}_{s(\xi,\lambda,\ell)}\xi_{\varepsilon,\ell})(x)| \leq \frac{c_4\|\xi_{\varepsilon,\ell}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}}{s(\xi,\lambda,\ell)^{(1+3\epsilon)/2}}\exp\Big(\frac{-c_5 d(x,\Lambda_L)^2}{s(\xi,\lambda,\ell)}\Big),$$

the integral kernel of the operator $P^{(b)}_{s(\xi,\lambda,\ell)}((P^{(b)}_{s(\xi,\lambda,\ell)}\xi_{\varepsilon,\ell})(P^{(b)}_{s(\xi,\lambda,\ell)}\cdot))$ is estimated as

$$|(P^{(b)}_{s(\xi,\lambda,\ell)}((P^{(b)}_{s(\xi,\lambda,\ell)}\xi_{\varepsilon,\ell})(P^{(b)}_{s(\xi,\lambda,\ell)}\delta_y)))(x)| \leq \frac{c_6\|\xi_{\varepsilon,\ell}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}}{s(\xi,\lambda,\ell)^{(3+3\epsilon)/2}}\exp\Big(-\frac{c_6 d(x,\Lambda_L)^2}{s(\xi,\lambda,\ell)} - \frac{c_7|x-y|^2}{s(\xi,\lambda,\ell)}\Big).$$

Thus we have

$$\|(P^{(b)}_{s(\xi,\lambda,\ell)}((P^{(b)}_{s(\xi,\lambda,\ell)}\xi_{\varepsilon,\ell})(P^{(b)}_{s(\xi,\lambda,\ell)}\cdot))\|_{\mathcal{I}_2} \leq \frac{c_8\|\xi_{\varepsilon,\ell}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}(L+\sqrt{s(\xi,\lambda,\ell)})}{s(\xi,\lambda,\ell)^{(3+3\epsilon)/2}}.$$

By using also Lemma 3.5 below, we have

$$\mathbb{E}[\|\Gamma_0^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^p] \le c_{1,\lambda,p} L^{c_{2,\lambda,p}}$$

for any $p \ge 1$.

For $\Gamma_1^{(\xi,\varepsilon)}$, we use

$$|\overline{Y_\ell^{s(\xi,\lambda,\ell)}}| \le c_9 \exp(-c_{10} d(x,\Lambda_L)^2 / s(\xi,\lambda,\ell)) \log(1/s(\xi,\lambda,\ell)).$$

To estimate $\|\Gamma_1^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$, it is enough to estimate the $\mathcal{I}_2$-norm of

$$\Gamma_{1,0}^{(\xi,\varepsilon)} = \overline{Y_L^{s(\xi,\lambda,\ell)}} (\widetilde{H_\ell^{\xi_\varepsilon}} + k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)}))^{-1},$$

where $k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})$ is a positive polynomial of $\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}$ and $\|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)}$ such that

$$\|u\|_{L^2(\mathbb{R}^2)}^2 \le (u, (\widetilde{H_\ell^{\xi_\varepsilon}} + k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)}))u)_{L^2(\mathbb{R}^2)}$$

for any $\varepsilon \in 0,1)$ (cf. Lemma 4.9 in [21]). For this, since

$$\begin{aligned}
&(\widetilde{H_\ell^{\xi_\varepsilon}} + k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)}))^{-1} \\
=& \int_0^T \frac{dt}{2} \exp\Big(-\frac{t}{2}\widetilde{H_\ell^{\xi_\varepsilon}}\Big) \exp\Big(-\frac{t}{2} k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)}))\Big) \\
&+ \exp\Big(-\frac{T}{2}\widetilde{H_\ell^{\xi_\varepsilon}}\Big) \exp\Big(-\frac{t}{2} k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})\Big) (\widetilde{H_\ell^{\xi_\varepsilon}} + k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)}))^{-1},
\end{aligned}$$

it is enough to estimate the $\mathcal{I}_2$-norms of

$$\Gamma_{1,1}^{(\xi,\varepsilon)} = \overline{Y_\ell^{s(\xi,\lambda,\ell)}} \int_0^T \frac{dt}{2} \exp\Big(-\frac{t}{2}\widetilde{H_\ell^{\xi_\varepsilon}}\Big) \exp\Big(-\frac{t}{2} k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})\Big)$$

and

$$\Gamma_{1,2}^{(\xi,\varepsilon)} = \overline{Y_\ell^{s(\xi,\lambda,\ell)}} \exp\Big(-\frac{T}{2}\widetilde{H_\ell^{\xi_\varepsilon}}\Big) \exp\Big(-\frac{T}{2} k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})\Big)$$

for some $T \in (0,\infty)$.

For any $\varepsilon > 0$ and $p \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}^\xi[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}] \\
&= \int_{(\mathbb{R}^2)^{2p}} \Big(\prod_{j=1}^p dx_j\, \overline{Y_\ell^{s(\xi,\lambda,\ell)}}(x_j)^2\Big) \int_{[0,T]^{2p}} \Big(\prod_{j=1}^p \frac{dt_j d\underline{t_j}}{8\pi(t_j + \underline{t_j})}\Big) \\
&\quad \times \mathbb{E}^{\xi,w}\Big[\exp\Big(-\sum_{j=1}^p \int_0^{t_j+\underline{t_j}} \frac{dt}{2} (\xi_{\varepsilon,\ell} - \mathbb{E}^\xi[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell}, \xi_{\varepsilon,\ell})])(x_j + w_j(t))\Big) \\
&\quad \Big| w_j(t_j + \underline{t_j}) = 0 \text{ for any } j\Big] \exp\Big(-\sum_{j=1}^p \frac{t_j + \underline{t_j}}{2} k(\|\xi_\ell\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)})\Big),
\end{aligned}$$

where $w = (w_j)_{1\le j\le p}$ is an independent system of 2-dimensional Brownian motion starting at 0. Since $\xi$ is a Gaussian random field, we have

$$\begin{aligned}
&\mathbb{E}^\xi[\|\Gamma^{(\xi,\varepsilon)}_{1,1}\|^{2p}_{\mathcal{I}_2}] \\
&\le \int_{(\mathbb{R}^2)^{2p}} \Big(\prod_{j=1}^p dx_j\ c_{11}\exp(-c_{12}d(x_j,\Lambda_L)^2)\Big) \int_{[0,T]^{2p}} \Big(\prod_{j=1}^p \frac{dt_j d\underline{t_j}}{8\pi(t_j+\underline{t_j})}\Big) \\
&\times \mathbb{E}^w\Big[\exp\Big(\frac{1}{2}\mathbb{E}^\xi\Big[\Big(\sum_{j=1}^p \int_0^{t_j+\underline{t_j}} \frac{dt}{2}\xi_{\varepsilon,\ell}(x_j+w_j(t))\Big)^2\Big] \\
&+\sum_{j=1}^p \int_0^{t_j+\underline{t_j}} \frac{dt}{2}\mathbb{E}^\xi[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell})])(x_j+w_j(t))\Big) \\
&\Big| w_j(t_j+\underline{t_j}) = 0 \text{ for any } j\Big].
\end{aligned}$$

Since

$$\begin{aligned}
&\mathbb{E}^\xi\Big[\Big(\sum_{j=1}^p \int_0^{t_j+\underline{t_j}} \frac{dt}{2}\xi_{\varepsilon,\ell}(x_j+w_j(t))\Big)^2\Big] \\
&= \sum_{j=1}^p \int_0^{t_j+\underline{t_j}} \frac{dt}{2}\int_0^{t_j+\underline{t_j}} \frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}^2_\ell e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t')) \\
&\quad+\sum_{j\ne j'} \int_0^{t_j+\underline{t_j}} \frac{dt}{2}\int_0^{t_{j'}+\underline{t_{j'}}} \frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}^2_\ell e^{\varepsilon^2\Delta})(x_j+w_j(t),x_{j'}+w_{j'}(t')),
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{E}^\xi[\|\Gamma^{(\xi,\varepsilon)}_{1,1}\|^{2p}_{\mathcal{I}_2}] \\
&\le \int_{(\mathbb{R}^2)^{2p}} \Big(\prod_{j=1}^p dx_j\ c_{11}\exp(-c_{12}d(x_j,\Lambda_L)^2)\Big) \int_{[0,T]^{2p}} \Big(\prod_{j=1}^p \frac{dt_j d\underline{t_j}}{8\pi(t_j+\underline{t_j})}\Big) \\
&\times \Big(\prod_{j=1}^p \mathbb{E}^w\Big[\exp\Big(\int_0^{t_j+\underline{t_j}} \frac{dt}{2}\int_0^{t_j+\underline{t_j}} \frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}^2_\ell e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t')) \\
&+\int_0^{t_j+\underline{t_j}} dt\mathbb{E}^\xi[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell})])(x_j+w_j(t))\Big)\Big| w_j(t_j+\underline{t_j})=0\Big]\Big)^{1/2} \\
&\times \Big(\prod_{j\ne j'} \mathbb{E}^w\Big[\exp\Big(p(p-1)\int_0^{t_j+\underline{t_j}} \frac{dt}{2}\int_0^{t_{j'}+\underline{t_{j'}}} \frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}^2_\ell e^{\varepsilon^2\Delta})(x_j+w_j(t),x_{j'}+w_{j'}(t')\Big) \\
&\Big| w_j(t_j+\underline{t_j}) = w_{j'}(t_{j'}+\underline{t_{j'}})=0\Big]\Big)^{1/(2p(p-1))}.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^{t_j+\underline{t_j}}\frac{dt}{2}\int_0^{t_j+\underline{t_j}}\frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t'))\\
=&\int_0^{(t_j+\underline{t_j})/2}\frac{dt}{2}\int_0^{(t_j+\underline{t_j})/2}\frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t'))\\
&+\int_{(t_j+\underline{t_j})/2}^{t_j+\underline{t_j}}\frac{dt}{2}\int_{(t_j+\underline{t_j})/2}^{t_j+\underline{t_j}}\frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t'))\\
&+\int_0^{(t_j+\underline{t_j})/2}\frac{dt}{2}\int_{(t_j+\underline{t_j})/2}^{t_j+\underline{t_j}}\frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t'))\\
&+\int_{(t_j+\underline{t_j})/2}^{t_j+\underline{t_j}}\frac{dt}{2}\int_0^{(t_j+\underline{t_j})/2}\frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t')),
\end{aligned}$$

we use Lemma 3.1 in Nakao [15] to have

$$\begin{aligned}
&\mathbb{E}^\xi[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}]\\
\le&\int_{(\mathbb{R}^2)^{2p}}\Big(\prod_{j=1}^p dx_j\ c_{11}\exp(-c_{12}d(x_j,\Lambda_L)^2)\Big)\int_{[0,T]^{2p}}\Big(\prod_{j=1}^p\frac{dt_j d\underline{t_j}}{8\pi(t_j+\underline{t_j})}\Big)\\
&\times\Big(\prod_{j=1}^p 2\mathbb{E}^w\Big[\exp\Big(\int_0^{(t_j+\underline{t_j})/2}dt\int_0^{(t_j+\underline{t_j})/2}dt'(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t'))\\
&+4\int_0^{t_j+\underline{t_j}}dt\mathbb{E}^\xi[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell})])(x_j+w_j(t))\Big)\Big]\Big)^{1/4}\\
&\times\Big(\prod_{j=1}^p\mathbb{E}^w\Big[\exp\Big(\int_0^{(t_j+\underline{t_j})/2}dt\int_{(t_j+\underline{t_j})/2}^{t_j+\underline{t_j}}dt'(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_j+w_j(t'))\Big)\\
&\Big|w_j(t_j+\underline{t_j})=0\Big]\Big)^{1/4}\\
&\times\Big(\prod_{j\neq j'}\mathbb{E}^w\Big[\exp\Big(p(p-1)\int_0^{t_j+\underline{t_j}}\frac{dt}{2}\int_0^{t_{j'}+\underline{t_{j'}}}\frac{dt'}{2}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2 e^{\varepsilon^2\Delta})(x_j+w_j(t),x_{j'}+w_{j'}(t')\Big)\\
&\Big|w_j(t_j+\underline{t_j})=w_{j'}(t_{j'}+\underline{t_{j'}})=0\Big]\Big)^{1/(2p(p-1))}.
\end{aligned}$$

Thus we have

$$\overline{\lim_{\varepsilon\to0}}\,\mathbb{E}^\xi[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}]\le c_{13}L^{2p}T^p$$

if $p^2T\le c_{14}$ for some positive constant $c_{14}$ by Proposition 3.2 below. Similarly we have

$$\overline{\lim_{\varepsilon\to0}}\,\mathbb{E}^\xi[\|\Gamma_{1,2}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}]\le c_{15}L^{2p}/T^p$$

if $p^2T\le c_{16}$. The estimate of $\mathbb{E}^{\widetilde{\xi}}[\|\Gamma_2^{(\widetilde{\xi},\varepsilon)}\|_{\mathcal{I}_2}^{2p}]$ is simpler than that of $\mathbb{E}^\xi[\|\Gamma_1^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}]$.

For the extension to the case of $t>0$, we use

$$\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H_\ell^{\xi_\varepsilon}+\overline{\xi}_L-t\widetilde{\chi}_L})]\le\mathrm{Tr}[1_{[1,\infty)}(\Gamma^{(\widetilde{\xi},\varepsilon)}+t\Gamma_3^{\xi_\varepsilon})]\le2\|\Gamma^{(\widetilde{\xi},\varepsilon)}\|_{\mathcal{I}_2}^2+2t^2\|\Gamma_3^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^2,$$

where

$$\Gamma_3^{(\xi,\varepsilon)} = (H_\ell^{\widehat{\xi_\varepsilon, s(\xi,\lambda,\ell)}} + \lambda)^{-1/2} \widetilde{\chi}_L (H_\ell^{\widehat{\xi_\varepsilon, s(\xi,\lambda,\ell)}} + \lambda)^{-1/2}.$$

The estimate of $\mathbb{E}^\xi[\|\Gamma_3^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}]$ is same with that of $\mathbb{E}^{\widetilde{\xi}}[\|\Gamma_2^{(\widetilde{\xi},\varepsilon)}\|_{\mathcal{I}_2}^{2p}]$. □

In the proof of Proposition 3.1, we used also the following:

**Lemma 3.1.** *For any $s \in (0,1)$ and $\beta, \gamma_1, \gamma_2 \in \mathbb{R}$, there exist $C, k(s,\beta,\gamma_1+\gamma_2) \in (0,\infty)$ such that*

$$\begin{aligned}&\|\chi_{a_1}(\Pi(\chi_{a_2}f, \chi_{a_3}g) - \Pi^s(\chi_{a_2}f, \chi_{a_3}g)\|_{\mathcal{C}^\beta(\mathbb{R}^2)}\\ \leq& k(s,\beta,\gamma_1+\gamma_2)\|\chi_{a_2}f\|_{C^{\gamma_1}(\mathbb{R}^2)}\|\chi_{a_3}g\|_{C^{\gamma_2}(\mathbb{R}^2)}\\ &\times \exp(-C(|a_1-a_2|^2 + |a_1-a_3|^2))\end{aligned}$$

*for any $a_1, a_2, a_3 \in \mathbb{Z}^2$, $f \in C^{\gamma_1}(\mathbb{R}^2)$ and $g \in C^{\gamma_2}(\mathbb{R}^2)$.*

This lemma is proven as in the proof of Lemma 3.6 in [21]. The following are also modifications of Lemma 3.2, Lemma 3.4 and Lemma 4.3 (i) in [21]:

**Lemma 3.2.** (i) *For any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$, there exist $C_\alpha, C_{\alpha,\epsilon} \in (0,\infty)$ such that*

$$\|P_g^s f\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)} \leq C_{\alpha,\epsilon} s^{\epsilon/2} \|f\|_{\mathcal{H}^{\alpha+\epsilon}(\mathbb{R}^2)} \|g\|_{L^\infty(\mathbb{R}^2)}$$

*for any $s \in (0,1]$, $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$ and $g \in L^\infty(\mathbb{R}^2)$, and*

$$\|P_g^s f\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)} \leq C_{\alpha,\epsilon} s^{\epsilon/2} \|f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)}$$

*for any $f \in C^\alpha(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R}^2)$.*

(ii) *For any $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ and $\epsilon \in (0,1)$, there exists $C_{\alpha,\beta,\epsilon} \in (0,\infty)$ such that*

$$\|P_f^s g\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)} \leq C_{\alpha,\beta,\epsilon} s^{\epsilon/2} \|f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{H}^\beta(\mathbb{R}^2)}$$

*for any $s \in (0,1]$, $f \in C^\alpha(\mathbb{R}^2)$ and $g \in \mathcal{H}^\beta(\mathbb{R}^2)$, and*

$$\|P_f^s g\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)} \leq C_{\alpha,\beta,\epsilon} s^{\epsilon/2} \|f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{C}^\beta(\mathbb{R}^2)}$$

*for any $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$ and $g \in C^\beta(\mathbb{R}^2)$.*

(iii) *For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$, and any $\epsilon \in (0,1)$ there exists $C_{\alpha,\beta,\epsilon} \in (0,\infty)$ such that*

$$\|\Pi^s(f,g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)} \leq C_{\alpha,\beta,\epsilon} s^{\epsilon/2} \|f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{C}^\beta(\mathbb{R}^2)}$$

*for any $s \in (0,1]$, $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$ and $g \in \mathcal{C}^\beta(\mathbb{R}^2)$.*

(iv) *For any* $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ *and* $\epsilon \in (0,1)$, *there exists* $C_{\alpha,\beta,\epsilon} \in (0,\infty)$ *such that*

$$\|{}_h P^s_f g\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)} \le C_{\alpha,\beta,\epsilon} s^{\epsilon/2} \|f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \|h\|_{L^2(\mathbb{R}^2)}$$

*for any* $s \in (0,1]$, $f \in \mathcal{C}^\alpha(\mathbb{R}^2)$, $g \in \mathcal{C}^\beta(\mathbb{R}^2)$ *and* $h \in L^2(\mathbb{R}^2)$.

**Lemma 3.3.** (i) *For any* $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}, \gamma \in (-\infty, 0)$ *such that* $\beta + \gamma < 0$ *and* $\alpha + \beta + \gamma > 0$, *there exist* $C_{\epsilon,\alpha,\beta,\gamma} \in (0,\infty)$ *such that*

$$\begin{aligned}&\|C^s(f,g,h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}\\ \le& C_{\epsilon,\alpha,\beta,\gamma} s^{\epsilon/2} \|f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|h\|_{\mathcal{C}^\gamma(\mathbb{R}^2)},\end{aligned}$$

*for any* $s \in (0,1]$, $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$, $g \in \mathcal{C}^{\beta-2}(\mathbb{R}^2)$ *and* $h \in \mathcal{C}^\gamma(\mathbb{R}^2)$.

(ii) *For any* $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}$, $\gamma \in (-\infty, 0)$ *such that* $\beta + \gamma < 0$ *and* $\alpha + \beta + \gamma > 0$, *there exist* $C_{\epsilon,\alpha,\beta,\gamma} \in (0,\infty)$ *such that*

$$\begin{aligned}&\|S^s(f,g,h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}\\ \le& C_{\epsilon,\alpha,\beta,\gamma} s^{\epsilon/2} \|f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|h\|_{\mathcal{C}^\gamma(\mathbb{R}^2)},\end{aligned}$$

*for any* $s \in (0,1]$, $f \in \mathcal{H}^\alpha(\mathbb{R}^2)$, $g \in \mathcal{C}^{\beta-2}(\mathbb{R}^2)$ *and* $h \in \mathcal{C}^\gamma(\mathbb{R}^2)$.

**Lemma 3.4.** *For any* $\epsilon \in (0,1)$ *and almost all* $\xi$, *there exist* $C_{\epsilon,\xi}, C'_{\epsilon,\xi}, C_\epsilon, C'_\epsilon \in (0,\infty)$ *such that*

$$\begin{aligned}\|\chi_a Y^s_{\xi,\ell}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le& C_{\epsilon,\xi} s^{\epsilon/4} \log(2+|a|) \exp(-C_\epsilon d(\Lambda_2(a), \Lambda_{\ell+2})^2/s)\\ \le& C'_{\epsilon,\xi} s^{\epsilon/4} \log(2+\ell) \exp(-C_\epsilon d(\Lambda_2(a), \Lambda_{\ell+2})^2/s)\end{aligned}$$

*for any* $s \in (0,1]$, $a \in \mathbb{Z}^2$ *and* $\ell \in \mathbb{N}$.

**Lemma 3.5.** (i) *For any* $\epsilon \in (0,1)$ *and* $\gamma \in (2,3)$, *there exist* $c_1 \in (0,\infty)$ *and* $r_0 \in [1,\infty)$ *satisfying the following: for any* $r \ge r_0$, *there exists* $c_2 \in (0,\infty)$ *such that*

$$\iint_{0\le s_1\le s_2\le 1} \frac{\mathbb{E}[\|\chi_a(Y^{s_2}_{\xi,\ell} - Y^{s_1}_{\xi,\ell})\|^r_{C^{-\epsilon}(\mathbb{R}^2)}]}{|s_2 - s_1|^\gamma} \le c_2 \exp(-c_1 r d(a, \Lambda_\ell))$$

*for any* $a \in \mathbb{Z}^2$ *and* $\ell \in \mathbb{N}$.

(ii) *In the situation of* (i), *we have*

$$\mathbb{E}\Big[\sup_{0\le s_1\le s_2\le 1} \|\chi_a(Y^{s_2}_{\xi,\ell} - Y^{s_1}_{\xi,\ell})\|^r_{C^{-\epsilon}(\mathbb{R}^2)}\Big] \le 8\Big(\frac{8\gamma}{\gamma-2}\Big)^r c_2 \exp(-c_1 r d(a, \Lambda_\ell))$$

*In particular, we have*

$$\mathbb{E}\Big[\sup_{0\le s\le 1} \|\chi_a Y^s_{\xi,\ell}\|^r_{C^{-\epsilon}(\mathbb{R}^2)}\Big] \le 8\Big(\frac{8\gamma}{\gamma-2}\Big)^r c_2 \exp(-c_1 r d(a, \Lambda_\ell)).$$

**Proof** (i) Let $\{Q_t\}_{t\in[0,1]} \in StGC^{(0,2b]}$. As in the proof of Theorem 2.1 in [14], Lemma 3.3 and Lemma 4.3 (i) in [21], we have

$$\begin{aligned}
&\mathbb{E}[|(Q_t\chi_a(Y^{s_2}_{\xi,\ell} - Y^{s_1}_{\xi,\ell}))(x)|^2] \\
\le& c_1 \int_{s_1}^{s_2} \frac{ds}{s} \int_{s_1}^{s_2} \frac{d\underline{s}}{\underline{s}} \int_0^1 dv \int_0^1 d\underline{v}(t+s+v+\underline{s}+\underline{v})^{-1}(s+\underline{s})^{-1} \\
&\times \Big(\frac{s}{s+v}\frac{\underline{s}}{\underline{s}+\underline{v}}\Big)^{b/4} \exp(-c_2(|x-a|^2 + d(a,\Lambda_\ell)^2))
\end{aligned}$$

We estimate the right hand side as

$$\begin{aligned}
&\int_{s_1}^{s_2} \frac{ds}{s} \int_{s_1}^{s_2} \frac{d\underline{s}}{\underline{s}} \int_0^1 dv \int_0^1 d\underline{v}(t+s+v+\underline{s}+\underline{v})^{-1}(s+\underline{s})^{-1} \\
&\times \Big(\frac{s}{s+v}\frac{\underline{s}}{\underline{s}+\underline{v}}\Big)^{b/4} \\
\le& c_3 \Big(\int_{s_1}^{s_2} ds \int_0^1 dv (t+s+v)^{-1/2}(s+v)^{-3/2}\Big)^2 \\
\le& c_4 \Big(\int_{s_1}^{s_2} ds \int_0^1 dv (t+\sqrt{s^2+v^2})^{-1/2}(s^2+v^2)^{-3/4}\Big)^2.
\end{aligned}$$

By introducing the polar coordinate, we have

$$\begin{aligned}
&\int_{s_1}^{s_2} ds \int_0^1 dv (t+\sqrt{s^2+v^2})^{-1/2}(s^2+v^2)^{-3/4} \\
=& \int_0^{\tan^{-1}(1/s_2)} d\theta \int_{s_1/\cos\theta}^{s_2/\cos\theta} \frac{dR}{\sqrt{(t+R)R}} \\
&+ \int_{\tan^{-1}(1/s_2)}^{\tan^{-1}(1/s_1)} d\theta \int_{s_1/\cos\theta}^{\sqrt{1+\cos^2\theta}} \frac{dR}{\sqrt{(t+R)R}}.
\end{aligned}$$

By $t + R \ge t, R$, we have two estimates:

$$\begin{aligned}
&\int_{s_1}^{s_2} ds \int_0^1 dv (t+\sqrt{s^2+v^2})^{-1/2}(s^2+v^2)^{-3/4} \\
\le& \begin{cases} c_5(\sqrt{s_2}-\sqrt{s_1})/\sqrt{t}, \\ c_6 \log(s_2/(s_1(1-s_2))). \end{cases}
\end{aligned}$$

By these, we have

$$\begin{aligned}
&\int_0^1 ds_1 \int_0^1 ds_2 \frac{\mathbb{E}[\|t^{\varepsilon/2}(Q_t\chi_a(Y^{s_2}_{\xi,\ell} - Y^{s_1}_{\xi,\ell}))(x)\|^r_{L^r(\mathbb{R}^2\times[0,1],dxdt/t)}]}{|s_2-s_1|^\gamma} \exp(c_2 r d(a,\Lambda_\ell)) \\
\le& c_7 \iint_{0\le s_1\le s_2\le 1} \frac{ds_1 ds_2}{(\sqrt{s_2}-\sqrt{s_1})^{2\gamma}} \int_0^1 \frac{dt}{t} t^{\varepsilon r/2} \Big(\frac{\sqrt{s_2}-\sqrt{s_1}}{\sqrt{t}}\Big)^{2\gamma} \Big(\log\frac{s_2}{s_1(1-s_2)}\Big)^{r-2\gamma}.
\end{aligned}$$

This is finite if $r > 2\gamma/\varepsilon$. The rest of the proof is same with that of Theorem 2.1 in [14], Lemma 3.3 and Lemma 4.3 (i) in [21].

(ii) is proven by Theorem 2.1.3 in [20].

$\square$

We refer Chapter 2 in [4] to prove the following:

**Proposition 3.2.** (i) *For any $\varepsilon \in (0,1)$, $\ell \geq 1$, $t_1, t_2 \geq 0$ and $x_1, x_2 \in \mathbb{R}^2$, we set*

$$\alpha_\varepsilon(t_1, x_1, t_2, x_2, \ell) := \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 (e^{\varepsilon^2 \Delta} \widetilde{\chi}_\ell^2 e^{\varepsilon^2 \Delta})(x_1 + w_1(s_1), x_2 + w_2(s_2)),$$

*where $w_1$ and $w_2$ are 2 independent 2-dimensional Brownian motions starting at 0 and $\widetilde{\chi}_\ell := \sum_{a \in \mathbb{Z}^2 \cap \Lambda_\ell} \chi_a^2$. Then we have*

$$\sup_{\varepsilon \in (0,1), x_1, x_2 \in \mathbb{R}^2, \ell \geq 1} \mathbb{E}[\exp(\lambda \alpha_\varepsilon(t_1, x_1, t_2, x_2, \ell)) | w_1(t_1) = w_2(t_2) = 0] < \infty$$

*for any $\lambda, t_1, t_2 \geq 0$ such that $\lambda \sqrt{t_1 t_2} \leq 1$.*

(ii) *For any $\varepsilon \in (0,1)$, $\ell \geq 1$, $t_1, t_2 \geq 0$ and $x \in \mathbb{R}^2$, we set*

$$\beta_\varepsilon(t_1, t_2, x, \ell) := \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 (e^{\varepsilon^2 \Delta} \widetilde{\chi}_\ell^2 e^{\varepsilon^2 \Delta})(x + w(s_1), x + w(t_1 + s_2)),$$

*where $w$ is a 2-dimensional Brownian motion starting at 0. Then we have*

$$\sup_{\varepsilon \in (0,1), x \in \mathbb{R}^2, \ell \geq 1} \mathbb{E}[\exp(\lambda \beta_\varepsilon(t_1, t_2, x, \ell)) | w(t_1 + t_2) = 0] < \infty$$

*for any $\lambda, t_1, t_2 \geq 0$ such that $\lambda \sqrt{t_1 t_2} \leq 1$.*

(iii) *For any $\varepsilon \in (0,1)$, $\ell > 1$, $t \geq 0$ and $x \in \mathbb{R}^2$, we set*

$$\begin{aligned} \chi_\varepsilon(t, x, \ell) := & \int_0^t ds_1 \int_0^t ds_2 (e^{\varepsilon^2 \Delta} \widetilde{\chi}_\ell^2 e^{\varepsilon^2 \Delta})(x + w(s_1), x + w(s_2)) \\ & + 4 \int_0^t ds \mathbb{E}^\xi [\Pi(\Delta^{-loc} \xi_{\varepsilon,\ell}, \xi_{\varepsilon,\ell})](x + w(s)). \end{aligned}$$

*Then there exists a finite positive constant $c$ such that*

$$\sup_{\varepsilon \in (0,1), x \in \mathbb{R}^2, \ell \geq 1} \mathbb{E}[\exp(\lambda \chi_\varepsilon(t, x, \ell))] < \infty$$

*for any $\lambda, t \geq 0$ such that $\lambda t \leq c$.*

**Proof**

(i) For any $m \in \mathbb{N}$, we have

$$
\begin{aligned}
&\mathbb{E}[\alpha_\varepsilon(t_1,x_1,t_2,x_2,\ell)^m | w_1(t_1)=w_2(t_2)=0] \\
=&m! \sum_{\sigma \in \mathfrak{S}_m} \int_{0\le s_{1,1}\le\cdots\le s_{1,m}\le t_1} ds_{1,1}\dots ds_{1,m} \int_{0\le s_{2,\sigma(1)}\le\cdots\le s_{2,\sigma(m)}\le t_2} ds_{2,1}\dots ds_{2,m} \int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^2(y_1)\cdots\widetilde{\chi}_\ell^2(y_m) \\
&\times \int_{(\mathbb{R}^2)^m} dz_{1,1}\dots dz_{1,m} e^{\varepsilon^2\Delta}(z_{1,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{1,m}-y_m) e^{s_{1,1}\Delta/2}(x_1-z_{1,1}) \\
&\times e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}-z_{1,2})\cdots e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}-z_{1,m}) e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}-x_1)2\pi t_1 \\
&\times \int_{(\mathbb{R}^2)^m} dz_{2,1}\dots dz_{2,m} e^{\varepsilon^2\Delta}(z_{2,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{2,m}-y_m) e^{s_{2,\sigma(1)}\Delta/2}(x_1-z_{2,\sigma(1)}) \\
&\times e^{(s_{2,\sigma(2)}-s_{2,\sigma(1)})\Delta/2}(z_{2,\sigma(1)}-z_{2,\sigma(2)})\cdots e^{(s_{2,\sigma(m)}-s_{2,\sigma(m-1)})\Delta/2}(z_{2,\sigma(m-1)}-z_{2,\sigma(m)}) \\
&\times e^{(t_2-s_{2,\sigma(m)})\Delta/2}(z_{2,\sigma(m)}-x_2)2\pi t_2,
\end{aligned}
$$

where

$$e^{t\Delta}(x) = \frac{1}{4\pi t}\exp\Big(\frac{-|x|^2}{4t}\Big)$$

for any $t>0$ and $x \in \mathbb{R}^2$. This is less than or equal to

$$
\begin{aligned}
&m! \sum_{\sigma \in \mathfrak{S}_m} \int_{0\le s_{1,1}\le\cdots\le s_{1,m}\le t_1} ds_{1,1}\dots ds_{1,m} \int_{0\le s_{2,\sigma(1)}\le\cdots\le s_{2,\sigma(m)}\le t_2} ds_{2,1}\dots ds_{2,m} \\
&\times \Big\{ \int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^2(y_1)\cdots\widetilde{\chi}_\ell^2(y_m) \\
&\times \Big( \int_{(\mathbb{R}^2)^m} dz_{1,1}\dots dz_{1,m} e^{\varepsilon^2\Delta}(z_{1,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{1,m}-y_m) e^{s_{1,1}\Delta/2}(x_1-z_{1,1}) \\
&\times e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}-z_{1,2})\cdots e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}-z_{1,m}) e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}-x_1)2\pi t_1 \Big)^2 \Big\}^{1/2} \\
&\times \Big\{ \int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^2(y_1)\cdots\widetilde{\chi}_\ell^2(y_m) \\
&\times \Big( \int_{(\mathbb{R}^2)^m} dz_{2,1}\dots dz_{2,m} e^{\varepsilon^2\Delta}(z_{2,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{2,m}-y_m) e^{s_{2,\sigma(1)}\Delta/2}(x_1-z_{2,\sigma(1)}) \\
&\times e^{(s_{2,\sigma(2)}-s_{2,\sigma(1)})\Delta/2}(z_{2,\sigma(1)}-z_{2,\sigma(2)})\cdots e^{(s_{2,\sigma(m)}-s_{2,\sigma(m-1)})\Delta/2}(z_{2,\sigma(m-1)}-z_{2,\sigma(m)}) \\
&\times e^{(t_2-s_{2,\sigma(m)})\Delta/2}(z_{2,\sigma(m)}-x_2)2\pi t_2 \Big)^2 \Big\}^{1/2}.
\end{aligned}
$$

One part of this is estimated as

$$
\begin{aligned}
&\int_{(\mathbb{R}^2)^m} dy_1 \dots dy_m \widetilde{\chi}_\ell^2(y_1)\cdots\widetilde{\chi}_\ell^2(y_m) \\
&\times \Big(\int_{(\mathbb{R}^2)^m} dz_{1,1}\dots dz_{1,m} e^{\varepsilon^2\Delta}(z_{1,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{1,m}-y_m) e^{s_{1,1}\Delta/2}(x_1-z_{1,1}) \\
&\times e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}-z_{1,2})\cdots e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}-z_{1,m}) e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}-x_1)2\pi t_1\Big)^2 \\
=&\int_{(\mathbb{R}^2)^m} dz_{1,1}\dots dz_{1,m} e^{\varepsilon^2\Delta}(z_{1,1})\cdots e^{\varepsilon^2\Delta}(z_{1,m}) \int_{(\mathbb{R}^2)^m} d\underline{z_{1,1}}\dots d\underline{z_{1,m}} e^{\varepsilon^2\Delta}(\underline{z_{1,1}})\cdots e^{\varepsilon^2\Delta}(\underline{z_{1,m}}) \\
&\times \int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^2(y_1)\cdots\widetilde{\chi}_\ell^2(y_m) e^{s_{1,1}\Delta/2}(x_1-z_{1,1}-y_1) e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}+y_1-z_{1,2}-y_2) \\
&\cdots\times e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}+y_{m-1}-z_{1,m}-y_m) e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}+y_m-x_1)2\pi t_1 \\
&\times e^{s_{1,1}\Delta/2}(x_1-\underline{z_{1,1}}-y_1) e^{(s_{1,2}-s_{1,1})\Delta/2}(\underline{z_{1,1}}+y_1-\underline{z_{1,2}}-y_2) \\
&\cdots\times e^{(s_{1,m}-s_{1,m-1})\Delta/2}(\underline{z_{1,m-1}}+y_{m-1}-\underline{z_{1,m}}-y_m) e^{(t_1-s_{1,m})\Delta/2}(\underline{z_{1,m}}+y_m-x_1)2\pi t_1 \\
\le&\int_{(\mathbb{R}^2)^m} dz_{1,1}\dots dz_{1,m} e^{\varepsilon^2\Delta}(z_{1,1})\cdots e^{\varepsilon^2\Delta}(z_{1,m}) \int_{(\mathbb{R}^2)^m} d\underline{z_{1,1}}\dots d\underline{z_{1,m}} e^{\varepsilon^2\Delta}(\underline{z_{1,1}})\cdots e^{\varepsilon^2\Delta}(\underline{z_{1,m}}) \\
&\times \Big(\int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^4(y_1)\cdots\widetilde{\chi}_\ell^4(y_m) e^{s_{1,1}\Delta/2}(x_1-z_{1,1}-y_1)^2 e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}+y_1-z_{1,2}-y_2)^2 \\
&\cdots\times e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}+y_{m-1}-z_{1,m}-y_m)^2 e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}+y_m-x_1)^2(2\pi t_1)^2\Big)^{1/2} \\
&\times \Big(\int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^4(y_1)\cdots\widetilde{\chi}_\ell^4(y_m) \\
&\times e^{s_{1,1}\Delta/2}(x_1-\underline{z_{1,1}}-y_1)^2 e^{(s_{1,2}-s_{1,1})\Delta/2}(\underline{z_{1,1}}+y_1-\underline{z_{1,2}}-y_2)^2 \\
&\cdots\times e^{(s_{1,m}-s_{1,m-1})\Delta/2}(\underline{z_{1,m-1}}+y_{m-1}-\underline{z_{1,m}}-y_m)^2 e^{(t_1-s_{1,m})\Delta/2}(\underline{z_{1,m}}+y_m-x_1)^2(2\pi t_1)^2\Big)^{1/2}.
\end{aligned}
$$

One part of this is estimated as

$$
\begin{aligned}
&\int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \widetilde{\chi}_\ell^4(y_1)\cdots\widetilde{\chi}_\ell^4(y_m) e^{s_{1,1}\Delta/2}(x_1-z_{1,1}-y_1)^2 e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}+y_1-z_{1,2}-y_2)^2 \\
&\cdots\times e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}+y_{m-1}-z_{1,m}-y_m)^2 e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}+y_m-x_1)^2(2\pi t_1)^2 \\
\le&\int_{(\mathbb{R}^2)^m} dy_1\dots dy_m \frac{e^{s_{1,1}\Delta}(x_1-z_{1,1}-y_1)}{\pi s_{1,1}} \frac{e^{(s_{1,2}-s_{1,1})\Delta}(z_{1,1}+y_1-z_{1,2}-y_2)}{\pi(s_{1,2}-s_{1,1})} \\
&\cdots\times \frac{e^{(s_{1,m}-s_{1,m-1})\Delta}(z_{1,m-1}+y_{m-1}-z_{1,m}-y_m)}{\pi(s_{1,m}-s_{1,m-1})} \frac{e^{(t_1-s_{1,m})\Delta}(z_{1,m}+y_m-x_1)}{\pi(t_1-s_{1,m})}(2\pi t_1)^2 \\
=&t_1\pi^{1-m}\{s_{1,1}(s_{1,2}-s_{1,1})\cdots(s_{1,m}-s_{1,m-1})(t_1-s_{1,m})\}^{-1}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\mathbb{E}[\alpha_\varepsilon(t_1,x_1,t_2,x_2,\ell)^m | w_1(t_1)=w_2(t_2)=0]\\
\le&(m!)^2\frac{\sqrt{t_1t_2}}{\pi^{m-1}}\int\limits_{0\le s_{1,1}\le\cdots\le s_{1,m}\le t_1} ds_{1,1}\dots ds_{1,m}\{s_{1,1}(s_{1,2}-s_{1,1})\cdots(s_{1,m}-s_{1,m-1})(t_1-s_{1,m})\}^{-1/2}\\
&\times\int\limits_{0\le s_{2,1}\le\cdots\le s_{2,m}\le t_2} ds_{2,1}\dots ds_{2,m}\{s_{2,1}(s_{2,2}-s_{2,1})\cdots(s_{2,m}-s_{2,m-1})(t_2-s_{2,m})\}^{-1/2}\\
=&(m!)^2\frac{(t_1t_2)^{m/2}}{\pi^{m-1}}\Big\{B\Big(\frac{1}{2},\frac{1}{2}\Big)B\Big(\frac{2}{2},\frac{1}{2}\Big)\cdots B\Big(\frac{m}{2},\frac{1}{2}\Big)\Big\}^2\\
=&\frac{(m!)^2\pi^2(t_1t_2)^{m/2}}{\Gamma\Big(\dfrac{m+1}{2}\Big)},
\end{aligned}
$$

from which we obtain the result.

(ii) For any $m\in\mathbb{N}$, we have

$$
\begin{aligned}
&\mathbb{E}[\beta_\varepsilon(t_1,t_2,x,\ell)^m | w(t_1+t_2)=0]\\
=&m!\sum_{\sigma\in\mathfrak{S}_m}\int\limits_{0\le s_{1,1}\le\cdots\le s_{1,m}\le t_1} ds_{1,1}\dots ds_{1,m}\int\limits_{0\le s_{2,\sigma(1)}\le\cdots\le s_{2,\sigma(m)}\le t_2} ds_{2,1}\dots ds_{2,m}\int_{(\mathbb{R}^2)^m} dy_1\dots dy_m\widetilde{\chi}_\ell^2(y_1)\cdots\widetilde{\chi}_\ell^2(y_m)\int_{\mathbb{R}^2}dx'\\
&\times\int_{(\mathbb{R}^2)^m} dz_{1,1}\dots dz_{1,m}e^{\varepsilon^2\Delta}(z_{1,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{1,m}-y_m)e^{s_{1,1}\Delta/2}(x-z_{1,1})\\
&\times e^{(s_{1,2}-s_{1,1})\Delta/2}(z_{1,1}-z_{1,2})\cdots e^{(s_{1,m}-s_{1,m-1})\Delta/2}(z_{1,m-1}-z_{1,m})e^{(t_1-s_{1,m})\Delta/2}(z_{1,m}-x')\\
&\times\int_{(\mathbb{R}^2)^m} dz_{2,1}\dots dz_{2,m}e^{\varepsilon^2\Delta}(z_{2,1}-y_1)\cdots e^{\varepsilon^2\Delta}(z_{2,m}-y_m)e^{s_{2,\sigma(1)}\Delta/2}(x'-z_{2,\sigma(1)})\\
&\times e^{(s_{2,\sigma(2)}-s_{2,\sigma(1)})\Delta/2}(z_{2,\sigma(1)}-z_{2,\sigma(2)})\cdots e^{(s_{2,\sigma(m)}-s_{2,\sigma(m-1)})\Delta/2}(z_{2,\sigma(m-1)}-z_{2,\sigma(m)})\\
&\times e^{(t_2-s_{2,\sigma(m)})\Delta/2}(z_{2,\sigma(m)}-x)2\pi(t_1+t_2).
\end{aligned}
$$

The rest of the proof is same with that of (i).

(iii) By

$$
\begin{aligned}
&\iint_{0\le s_1\le s_2\le t} ds_1ds_2(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2e^{\varepsilon^2\Delta})(x+w(s_1),x+w(s_2))\\
=&\sum_{k=0}^\infty\sum_{h=0}^{2^k-1}\int_0^{t/2}\frac{ds_1}{2^k}\int_{t/2}^t\frac{ds_2}{2^k}(e^{\varepsilon^2\Delta}\widetilde{\chi}_\ell^2e^{\varepsilon^2\Delta})\Big(x+w\Big(\frac{th+s_1}{2^k}\Big),x+w\Big(\frac{th+s_2}{2^k}\Big)\Big)\\
=&\sum_{k=0}^\infty\sum_{h=0}^{2^k-1}\int_0^{t/2}\frac{ds_1}{2^k}\int_{t/2}^t\frac{ds_2}{2^k}\int dye^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2\Big(x+w\Big(\frac{th+s_1}{2^k}\Big)+y\Big)\\
&\times e^{\varepsilon^2\Delta}\Big(y+w\Big(\frac{th+s_1}{2^k}\Big)-w\Big(\frac{th+s_2}{2^k}\Big)\Big)
\end{aligned}
$$

and

$$\sum_{k=0}^{\infty}\sum_{h=0}^{2^k-1}\int_0^{t/2}\frac{ds_1}{2^k}\int_{t/2}^{t}\frac{ds_2}{2^k}\int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2\Big(x+w\Big(\frac{th+s_1}{2^k}\Big)+y\Big)$$
$$\times\,\mathbb{E}^w\Big[e^{\varepsilon^2\Delta}\Big(y+w\Big(\frac{th+s_1}{2^k}\Big)-w\Big(\frac{th+s_2}{2^k}\Big)\Big)\Big]$$
$$=2\int_0^t ds\int_0^{(t-s)/2} dr(e^{(r+\varepsilon^2)\Delta}\widetilde{\chi}_L^2 e^{\varepsilon^2\Delta})(x+w(s),x+w(s)),$$

we rewrite as $\chi_\varepsilon(t,x,\ell)=2\chi_\varepsilon^0(t,x,\ell)+4\chi_\varepsilon^1(t,x,\ell)+4\chi_\varepsilon^2(t,x,\ell)$, where

$$\chi_\varepsilon^0(t,x,\ell):=\sum_{k=0}^{\infty}\sum_{h=0}^{2^k-1}\int_0^{t/2}\frac{ds_1}{2^k}\int_{t/2}^{t}\frac{ds_2}{2^k}\int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2\Big(x+w\Big(\frac{th+s_1}{2^k}\Big)+y\Big)$$
$$\times\Big\{e^{\varepsilon^2\Delta}\Big(y+w\Big(\frac{th+s_1}{2^k}\Big)-w\Big(\frac{th+s_2}{2^k}\Big)\Big)-\mathbb{E}^w\Big[e^{\varepsilon^2\Delta}\Big(y+w\Big(\frac{th+s_1}{2^k}\Big)-w\Big(\frac{th+s_2}{2^k}\Big)\Big)\Big]\Big\},$$
$$\chi_\varepsilon^1(t,x,\ell):=\int_0^t ds\Big(\int_0^{(t-s)/2}dr-\int_0^1 dr\Big)\int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2(x+w(s)+y)e^{(r+\varepsilon^2)\Delta}(y)$$

and

$$\chi_\varepsilon^2(t,x,\ell):=\int_0^t ds\mathbb{E}^\xi[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell})-(\Delta^{-loc}\xi_{\varepsilon,\ell})\xi_{\varepsilon,\ell}](x+w(s)).$$

Since

$$\sup_{x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\ge1}|\mathbb{E}^\xi[(\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell}))(x)]-\mathbb{E}^\xi[((\Delta^{-loc}\xi_{\varepsilon,\ell})\xi_{\varepsilon,\ell})(x)]|$$
$$\le\sup_{x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\ge1}|\mathbb{E}^\xi[(P_{\Delta^{-loc}\xi_{\varepsilon,\ell}}\xi_{\varepsilon,\ell}))(x)]|+\sup_{x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\ge1}|\mathbb{E}^\xi[(P_{\xi_{\varepsilon,\ell}}\Delta^{-loc}\xi_{\varepsilon,\ell}))(x)]|$$
$$+\sup_{x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\ge1}\mathbb{E}^\xi[(P_1^{(b)}((P_1^{(b)}\Delta^{-loc}\xi_{\varepsilon,\ell})(P_1^{(b)}\xi_{\varepsilon,\ell})))(x)]|$$
$$<\infty,$$

we have

$$\sup_{x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\ge1}|\chi_\varepsilon^2(t,x,\ell)|<\infty.$$

It is also easy to see

$$\sup_{x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\ge1}|\chi_\varepsilon^1(t,x,\ell)|<\infty.$$

For any $0\le k\in\mathbb{Z}$ and $h\in\{0,1,\dots,2^k-1\}$, the moments of

$$\chi_\varepsilon^{k,h}(t,x,\ell):=\int_0^{t/2}\frac{ds_1}{2^k}\int_{t/2}^{t}\frac{ds_2}{2^k}\int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2\Big(x+w\Big(\frac{th+s_1}{2^k}\Big)+y\Big)$$
$$\times\,e^{\varepsilon^2\Delta}\Big(y+w\Big(\frac{th+s_1}{2^k}\Big)-w\Big(\frac{th+s_2}{2^k}\Big)\Big)$$

are estimated by the same method for (i) as

$$\mathbb{E}^w[\chi_\varepsilon^{k,h}(t,x,\ell)^m]\le(m!)^2\Big(\frac{t}{2^{k+1}}\Big)^m\frac{\sqrt{\pi}}{\Gamma\Big(\frac{m+1}{2}\Big)\Gamma\Big(\frac{m+2}{2}\Big)}.$$

Thus $\mathbb{E}^w[\exp(\lambda 2^k \chi_\varepsilon^{k,h}(t,x,\ell))]$ is an analytic function of $\lambda < 1/t$. By using the Taylor expansion as in the proof of Theorem 2.4.2 in [4], there exists a positive finite constant $c_1$ such that

$$\mathbb{E}^w[\exp(\lambda 2^k \widetilde{\chi_\varepsilon^{k,h}}(t,x,\ell))] \le \exp(c_1\lambda^2)$$

for $\lambda \le 1/(2t)$, where

$$\begin{aligned}\widetilde{\chi_\varepsilon^{k,h}}(t,x,\ell) := &\int_0^{t/2} \frac{ds_1}{2^k}\int_{t/2}^t \frac{ds_2}{2^k}\int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2\Big(x + w\Big(\frac{th+s_1}{2^k}\Big)+y\Big)\\ &\times \Big\{e^{\varepsilon^2\Delta}\Big(y + w\Big(\frac{th+s_1}{2^k}\Big) - w\Big(\frac{th+s_2}{2^k}\Big)\Big)\\ &- \mathbb{E}^w\Big[e^{\varepsilon^2\Delta}\Big(y + w\Big(\frac{th+s_1}{2^k}\Big) - w\Big(\frac{th+s_2}{2^k}\Big)\Big)\Big]\Big\}.\end{aligned}$$

By using the Hölder inequality as in the proof of Theorem 2.4.2 in [4], we have

$$\begin{aligned}&\mathbb{E}^w\Big[\exp\Big(\lambda\sum_{k=0}^N\sum_{h=0}^{2^k-1}\widetilde{\chi_\varepsilon^{k,h}}(t,x,\ell)\Big)\Big]\\ \le&\Big(\prod_{h=0}^{2^N-1}\mathbb{E}^w[\exp(\lambda 2^{\alpha N}\widetilde{\chi_\varepsilon^{N,h}}(t,x,\ell))]^{2^{-\alpha N}}\Big)\\ &\times\Big(\prod_{h=0}^{2^{N-1}-1}\mathbb{E}^w\Big[\exp\Big(\frac{\lambda 2^{\alpha(N-1)}}{1-2^{-\alpha N}}\widetilde{\chi_\varepsilon^{N-1,h}}(t,x,\ell)\Big)\Big]^{2^{-\alpha(N-1)}(1-2^{-\alpha N})}\Big)\\ &\times\Big(\prod_{h=0}^{2^{N-2}-1}\mathbb{E}^w\Big[\exp\Big(\frac{\lambda 2^{\alpha(N-2)}}{(1-2^{-\alpha N})(1-2^{-\alpha(N-1)})}\\ &\qquad\times\widetilde{\chi_\varepsilon^{N-2,h}}(t,x,\ell)\Big)\Big]^{2^{-\alpha(N-2)}(1-2^{-\alpha(N-1)})(1-2^{-\alpha N})}\Big)\cdots\\ &\times\Big(\prod_{h=0}^{2-1}\mathbb{E}^w\Big[\exp\Big(\frac{\lambda 2^{\alpha}}{(1-2^{-\alpha N})(1-2^{-\alpha(N-1)})\cdots(1-2^{-\alpha 2})}\\ &\qquad\times\widetilde{\chi_\varepsilon^{1,h}}(t,x,\ell)\Big)\Big]^{2^{-\alpha}(1-2^{-\alpha 2})(1-2^{-\alpha 3})\cdots(1-2^{-\alpha N})}\Big)\\ &\times\mathbb{E}^w\Big[\exp\Big(\frac{\lambda}{(1-2^{-\alpha N})(1-2^{-\alpha(N-1)})\cdots(1-2^{-\alpha 2})}\\ &\qquad\times\widetilde{\chi_\varepsilon^{1,h}}(t,x,\ell)\Big)\Big]^{(1-2^{-\alpha})(1-2^{-\alpha 2})(1-2^{-\alpha 3})\cdots(1-2^{-\alpha N})}\end{aligned}$$

for any $\alpha > 0$ and $N \in \mathbb{N}$. Since

$$\liminf_{N\to\infty}(1-2^{-\alpha N})(1-2^{-\alpha(N-1)})\cdots(1-2^{-\alpha}) \ge \exp\Big(\frac{-1}{2^\alpha(1-2^{-\alpha})^2}\Big),$$

we have

$$
\begin{aligned}
&\mathbb{E}^w\Big[\exp\Big(\lambda\sum_{k=0}^{N}\sum_{h=0}^{2^k-1}\widetilde{\chi_\varepsilon^{k,h}}(t,x,\ell)\Big)\Big]\\
\le&\exp\Big\{c_1\lambda^2\Big(2^{-(1-\alpha)N}+2^{-(1-\alpha)(N-1)}(1-2^{-\alpha N})^{-1}\\
&+2^{-(1-\alpha)(N-2)}(1-2^{-\alpha N})^{-1}(1-2^{-\alpha(N-1)})^{-1}\\
&+\cdots+2^{-(1-\alpha)}(1-2^{-\alpha N})^{-1}(1-2^{-\alpha(N-1)})^{-1}\cdots(1-2^{-\alpha 2})^{-1}\\
&+(1-2^{-\alpha N})^{-1}(1-2^{-\alpha(N-1)})^{-1}\cdots(1-2^{-\alpha})^{-1}\Big)\Big\}\\
\le&\exp\Big\{c_1\lambda^2\sum_{j=0}^{N}2^{-(1-\alpha)j}\Big/\liminf_{N\to\infty}(1-2^{-\alpha N})(1-2^{-\alpha(N-1)})\cdots(1-2^{-\alpha})\Big\}.
\end{aligned}
$$

for any $\alpha>0$ and $N\in\mathbb{N}$ if $\lambda\le\exp(-2^{-\alpha}(1-2^{-\alpha})^{-2})/(2t)$. By taking $\alpha<1$, we have

$$
\mathbb{E}^w\Big[\exp\Big(\lambda\chi_\varepsilon^0(t,x,\ell)\Big)\Big]\le\exp\Big\{c_1\frac{\lambda^2}{1-2^{-(1-\alpha)}}\exp(2^{-\alpha}(1-2^{-\alpha})^{-2})\Big\}.
$$

if $\lambda\le\exp(-2^{-\alpha}(1-2^{-\alpha})^{-2})/(2t)$. □

*Remark* 3.1. (i) As in Theorem 2.2.3 in [4] treating the intersection local time, we can show the existence of a functional $\alpha(t_1,x_1,t_2,x_2,\ell)$ of $(w_1,w_2)$ such that $\alpha_\varepsilon(t_1,x_1,t_2,x_2,\ell)$ converges to $\alpha(t_1,x_1,t_2,x_2,\ell)$ in $L^p(\mathbb{P}(\cdot|w_1(t_1)=w_2(t_2)=0))$ as $\varepsilon\to0$, for any $p\in[1,\infty)$, where $\mathbb{P}(\cdot|w_1(t_1)=w_2(t_2)=0)$ is the conditional probability of the Brownian motion. Formally $\alpha(t_1,x_1,t_2,x_2,\ell)$ is regarded as a restriction of the intersection local time to the square $\Lambda_\ell$:

$$
\alpha(t_1,x_1,t_2,x_2,\ell)=\int_0^{t_1}ds_1\int_0^{t_2}ds_2\widetilde{\chi}_\ell^2(x_1+w_1(s_1))\delta(x_1+w_1(s_1)-x_2-w_2(s_2)).
$$

The convergence also holds in $L^p(\mathbb{P})$ for any $p\in[1,\infty)$.

(ii) Simirarly there exists of a functional $\beta(t_1,t_2,x,\ell)$ of $w$ such that $\beta_\varepsilon(t_1,t_2,x,\ell)$ converges to $\beta(t_1,t_2,x,\ell)$ in $L^p(\mathbb{P}(\cdot|w(t_1+t_2)=0))$ as $\varepsilon\to0$, for any $p\in[1,\infty)$. Formally this is regarded as

$$
\beta(t_1,t_2,x,\ell)=\int_0^{t_1}ds_1\int_0^{t_2}ds_2\widetilde{\chi}_\ell^2(x+w(s_1))\delta(w(s_1)-w(t_1+s_2)).
$$

The convergence also holds in $L^p(\mathbb{P})$ for any $p\in[1,\infty)$.

(iii) Simirarly there exists a functional $\chi^0(t,x,\ell)$ of $w$ such that $\chi_\varepsilon^0(t,x,\ell)$ converges to $\chi^0(t,x,\ell)$ in $L^p(\mathbb{P})$ as $\varepsilon\to0$, for any $p\in[1,\infty)$. Since $\chi_\varepsilon^0(t,x,\ell)$ is written as

$$
\begin{aligned}
&\iint_{0\le s_1\le s_2\le t}ds_1ds_2\int dye^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2(x+w(s_1)+y)\\
&\times\{e^{\varepsilon^2\Delta}(y+w(s_1)-w(s_2))-\mathbb{E}^w[e^{\varepsilon^2\Delta}(y+w(s_1)-w(s_2))]\},
\end{aligned}
$$

$\chi^0(t,x,\ell)$ is formally regarded as a restriction of the renormalized intersection local time to the square $\Lambda_\ell$:

$$\iint_{0\le s_1\le s_2\le t} ds_1 ds_2 \widetilde{\chi}_\ell^2(x+w(s_1))\{\delta(w(s_1)-w(s_2))-\mathbb{E}^w[\delta(w(s_1)-w(s_2))]\}.$$

The convergence also holds in $L^p(\mathbb{P}(\cdot|w(t)=0))$ for any $p\in[1,\infty)$, since

$$\begin{aligned}&\chi_\varepsilon^0(t,x,\ell)\\ =&\chi_\varepsilon^0\Big(\frac{t}{2},x,\ell\Big)+\chi_\varepsilon^0\Big(\frac{t}{2},x,\ell;\widehat{w}\Big)-\overline{\chi_\varepsilon^0}\Big(\frac{t}{2},x,\ell;\widehat{w}\Big)+\beta_\varepsilon\Big(\frac{t}{2},\frac{t}{2},x,\ell\Big)-\overline{\overline{\chi_\varepsilon^0}}\Big(\frac{t}{2},\frac{t}{2},x,\ell\Big),\end{aligned}$$

where $\chi_\varepsilon^0(\cdot,\cdot,\cdot;\widehat{w})$ is the function defined by replacing the path $w$ by the path $\widehat{w}$ defined by $\widehat{w}(s):=w(t-s)$ for $s\in[0,t]$ in the definition of the function $\chi_\varepsilon^0(\cdot,\cdot,\cdot)$,

$$\begin{aligned}&\overline{\chi_\varepsilon^0}\Big(\frac{t}{2},x,\ell;\widehat{w}\Big)\\ =&\iint_{0\le s_1\le s_2\le t/2} ds_1ds_2\int dy e^{\varepsilon^2\Delta}(y)e^{(\varepsilon^2+(s_2-s_1)/2)\Delta}(y)(\widetilde{\chi}_\ell^2(x+\widehat{w}(s_1)+y)-\widetilde{\chi}_L^2(x+\widehat{w}(s_2)+y)),\end{aligned}$$

and

$$\overline{\overline{\chi_\varepsilon^0}}\Big(\frac{t}{2},\frac{t}{2},x,\ell\Big)=\int_0^{t/2}ds_1\int_{t/2}^t ds_2\int dy e^{\varepsilon^2\Delta}(y)e^{(\varepsilon^2+(s_2-s_1)/2)\Delta}(y)(\widetilde{\chi}_\ell^2(x+\widehat{w}(s_1)+y).$$

By Lemma 3.1 in Nakao [15], $\chi_\varepsilon^0(t/2,x,\ell)$ and $\chi_\varepsilon^0(t/2,x,\ell;\widehat{w})$ converges to $\chi^0(t/2,x,\ell)$ and $\chi^0(t/2,x,\ell;\widehat{w})$ in $L^p(\mathbb{P}(\cdot|w(t)=0))$, respecively, for any $p\in[1,\infty)$, By Ito formula,

$$\overline{\chi_\varepsilon^0}\Big(\frac{t}{2},x,\ell\Big)=-\overline{\chi_\varepsilon^{0,1}}\Big(\frac{t}{2},x,\ell\Big)-\frac{1}{2}\overline{\chi_\varepsilon^{0,2}}\Big(\frac{t}{2},x,\ell\Big),$$

where

$$\begin{aligned}&\overline{\chi_\varepsilon^{0,1}}\Big(\frac{t}{2},x,\ell\Big)\\ =&\iint_{0\le s_1\le s_2\le t/2} ds_1ds_2\int dy e^{\varepsilon^2\Delta}(y)e^{(\varepsilon^2+(s_2-s_1)/2)\Delta}(y)\int_{s_1}^{s_2}(\nabla\widetilde{\chi}_\ell^2)(x+w(s)+y)\cdot dw(s),\end{aligned}$$

and

$$\begin{aligned}&\overline{\chi_\varepsilon^{0,2}}\Big(\frac{t}{2},x,\ell\Big)\\ =&\iint_{0\le s_1\le s_2\le t/2} ds_1ds_2\int dy e^{\varepsilon^2\Delta}(y)e^{(\varepsilon^2+(s_2-s_1)/2)\Delta}(y)\int_{s_1}^{s_2}(\Delta\widetilde{\chi}_\ell^2)(x+w(s)+y)ds.\end{aligned}$$

By Lemma 3.1 in Nakao [15] and the moment inequalities for martingales, Theorem III-3.1 in [11], we have

$$\begin{aligned}
&\mathbb{E}^w\Big[\Big|\overline{\chi_\varepsilon^{0,1}}\Big(\frac{t}{2},x,\ell\Big)-\overline{\chi^{0,1}}\Big(\frac{t}{2},x,\ell\Big)\Big|^p\Big|w(t)=0\Big]^{1/p}\\
\le&\Big(\frac{1}{\pi t}\mathbb{E}^w\Big[\Big|\overline{\chi_\varepsilon^{0,1}}\Big(\frac{t}{2},x,\ell\Big)-\overline{\chi^{0,1}}\Big(\frac{t}{2},x,\ell\Big)\Big|^p\Big]\Big)^{1/p}\\
\le&\iint_{0\le s_1\le s_2\le t/2}ds_1ds_2\Big(\frac{1}{\pi t}\mathbb{E}^w\Big[\Big|\int_{s_1}^{s_2}\Big(\int dy e^{\varepsilon^2\Delta}(y)e^{(\varepsilon^2+(s_2-s_1)/2)\Delta}(y)(\nabla\widetilde{\chi}_\ell^2)(x+w(s)+y)\\
&-e^{(s_2-s_1)\Delta/2}(0)(\nabla\widetilde{\chi}_\ell^2)(x+w(s))\Big)\cdot dw(s)\Big|^p\Big]\Big)^{1/p}\\
\le&c_p\iint_{0\le s_1\le s_2\le t/2}ds_1ds_2\Big(\frac{1}{\pi t}\mathbb{E}^w\Big[\Big(\int_{s_1}^{s_2}\Big|\int dy e^{\varepsilon^2\Delta}(y)e^{(\varepsilon^2+(s_2-s_1)/2)\Delta}(y)(\nabla\widetilde{\chi}_\ell^2)(x+w(s)+y)\\
&-e^{(s_2-s_1)\Delta/2}(0)(\nabla\widetilde{\chi}_\ell^2)(x+w(s))\Big|^2ds\Big)^{p/2}\Big]\Big)^{1/p},
\end{aligned}$$

where

$$\overline{\chi^{0,1}}\Big(\frac{t}{2},x,\ell\Big)=\iint_{0\le s_1\le s_2\le t/2}ds_1ds_2e^{(s_2-s_1)\Delta/2}(0)\int_{s_1}^{s_2}(\nabla\widetilde{\chi}_\ell^2)(x+w(s))\cdot dw(s).$$

The right hand side of the above inequality converges to 0 as $\varepsilon\to 0$ by the Lebesgue convergence theorem. $\overline{\chi_\varepsilon^{0,2}}(t/2,x,\ell)$ and $\overline{\overline{\chi_\varepsilon^0}}(t/2,t/2,x,\ell)$ converge to

$$\overline{\chi^{0,2}}\Big(\frac{t}{2},x,\ell\Big)=\iint_{0\le s_1\le s_2\le t/2}ds_1ds_2e^{(s_2-s_1)\Delta/2}(0)\int_{s_1}^{s_2}(\Delta\widetilde{\chi}_\ell^2)(x+w(s))ds$$

and

$$\overline{\overline{\chi^0}}\Big(\frac{t}{2},\frac{t}{2},x,\ell\Big)=\int_0^{t/2}ds_1\int_{t/2}^t ds_2e^{(s_2-s_1)\Delta/2}(0)\widetilde{\chi}_\ell^2(x+\widehat{w}(s_1)),$$

respectively, uniformly in $w$.

Since

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathbb{E}^\xi[(P_{\Delta^{-loc}\xi_{\varepsilon,\ell}}\xi_{\varepsilon,\ell})(x)]&=\sum_\nu\int_0^1\frac{dt}{t}\int_{\mathbb{R}^2}dx_1Q_t^{1,\nu}(x,x_1)(P_t^\nu\Delta^{-loc}\widetilde{\chi}_\ell^2Q_t^{2,\nu})(x_1,x_1),\\
\lim_{\varepsilon\to 0}\mathbb{E}^\xi[(P_{\xi_{\varepsilon,\ell}}\Delta^{-loc}\xi_{\varepsilon,\ell}))(x)]&=\sum_\nu\int_0^1\frac{dt}{t}\int_{\mathbb{R}^2}dx_1Q_t^{1,\nu}(x,x_1)(P_t^\nu\widetilde{\chi}_\ell^2\Delta^{-loc}Q_t^{2,\nu})(x_1,x_1)
\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}^\xi[(P_1^{(b)}((P_1^{(b)}\Delta^{-loc}\xi_{\varepsilon,\ell})(P_1^{(b)}\xi_{\varepsilon,\ell})))(x)]=\int_{\mathbb{R}^2}dx_1P_1^{(b)}(x,x_1)(P_1^{(b)}\Delta^{-loc}P_1^{(b)})(x_1,x_1),$$

exist and are dominated by $\exp(-c_1d(x,\Lambda_\ell)^2)$ with some positive constant $c_1$,

$$\lim_{\varepsilon\to 0}\chi_\varepsilon^2(t,x,\ell)$$

exists and are bounded in $w$. It is also easy to see the existence and the boundedness of

$$\lim_{\varepsilon\to 0}\chi_\varepsilon^1(t,x,\ell).=\int_0^t ds\Big(\int_0^{(t-s)/2}dr-\int_0^1 dr\Big)(e^{r\Delta}\widetilde{\chi}_\ell^2)(x+w(s)).$$

Therefore Proposition 3.2 (iii) holds in the pinned case:

$$\sup_{\varepsilon\in(0,1),x\in\mathbb{R}^2,\ell\geq 1}\mathbb{E}[\exp(\lambda\chi_\varepsilon(t,x,\ell))|w(t)=0]<\infty$$

for any $\lambda, t \geq 0$ such that $\lambda t \leq c_2$ with some positive constant $c_2$. Therefore we can give another proof of Proposition 3.1 by proving and applying this estimate.

## 4. A Wegner type estimate

As in the last section, we consider the operator $\widetilde{H_L^{\tilde{\xi}}} = \widetilde{H_\ell^{\xi}} + \overline{\xi}_L$, where $\ell = L-2$. In this section we will show the following Wegner type estimate:

**Proposition 4.1.** *There exist finite positive constants $c_0, c_1, c_2, c_3$ such that*

$$\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\tilde{\xi}}})]]\leq c_1\eta L^{c_2}$$

*for any $E \leq -c_3$, $0<\eta\leq 1\wedge(-E/2)$ and $L\in 2\mathbb{N}$.*

**Proof**

Let $\varphi_0$ be a normalized eigenfunction of $\widetilde{H_L^{\tilde{\xi}}}$ with a negative eigenvalue $\lambda_0$. Then by the IMS localization, we have

$$\begin{aligned}\lambda_0 &= (\varphi_0, \widetilde{H_L^{\tilde{\xi}}}\varphi_0)_{L^2(\mathbb{R}^2)}\\ &=(\chi_{out}\varphi_0, \widetilde{H_L^{\tilde{\xi}}}\chi_{out}\varphi_0)_{L^2(\mathbb{R}^2)} + (\chi_{in}\varphi_0, \widetilde{H_L^{\tilde{\xi}}}\chi_{in}\varphi_0)_{L^2(\mathbb{R}^2)}\\ &\quad - (\varphi_0, (|\nabla\chi_{out}|^2+|\nabla\chi_{in}|^2)\varphi_0)_{L^2(\mathbb{R}^2)},\end{aligned}$$

where $\chi_{out}$ and $\chi_{in}$ are $[0,1]$-valued smooh functions on $\mathbb{R}^2$ so that $\chi_{out}=1$ and $\chi_{in}=0$ on $\mathbb{R}^2\setminus\Lambda_{L-1/2}$, $\chi_{out}=0$ and $\chi_{in}=1$ on $\Lambda_{L-1}$ and $\chi_{out}^2+\chi_{in}^2\equiv 1$. By Lemma 4.4 below, we have

$$(\chi_{out}\varphi_0, \widetilde{H_L^{\tilde{\xi}}}\chi_{out}\varphi_0)_{L^2(\mathbb{R}^2)} - (\varphi_0, (|\nabla\chi_{out}|^2+|\nabla\chi_{in}|^2)\varphi_0)_{L^2(\mathbb{R}^2)} \geq -c_1.$$

By Lemma 4.9 in [21], we have

$$\begin{aligned}&(\chi_{in}\varphi_0, \widetilde{H_L^{\tilde{\xi}}}\chi_{in}\varphi_0)_{L^2(\mathbb{R}^2)}\\ \geq& - c_2\|\chi_{in}\varphi_0\|^2_{L^2(\mathbb{R}^2)}(1+\|\xi_\ell\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}+\|\xi_\ell\|^2_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}+\|Y_{\xi,\ell}\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)})^{c_3}.\end{aligned}$$

As in Propositon A.1 in [1], we replace the norms by those defined by integrations:

$$\begin{aligned}&(\chi_{in}\varphi_0, \widetilde{H_L^{\tilde{\xi}}}\chi_{in}\varphi_0)_{L^2(\mathbb{R}^2)}\\ \geq& - c_4\|\chi_{in}\varphi_0\|^2_{L^2(\mathbb{R}^2)}(1+|||\xi_\ell|||_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)}+|||\xi_\ell|||^2_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)}+|||Y_{\xi,\ell}|||_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)})^{c_3},\end{aligned}$$

where we use the following Besov norm defined by the summation:

$$\begin{aligned}&|||f|||_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)}\\ :=&\|e^{\Delta}f\|_{L^p(\mathbb{R}^2:dx)}\\ &+\sum_{Q\in StGC^{(|\alpha|,2b]}}\|t^{-\alpha/2}\|Q_tf\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)}.\end{aligned}$$

Thus we have

$$\|\chi_{in}\varphi_0\|^2_{L^2(\mathbb{R}^2)}\geq(-\lambda_0-c_1)/B(\xi),$$

where

$$B(\xi):=c_4(1+|||\xi_\ell|||_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)}+|||\xi_\ell|||^2_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)}+|||Y_{\xi,\ell}|||_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)})^{c_3}.$$

If $E\leq -c_1-2$ and $\lambda_0\leq E+1$, then

$$\|\chi_{in}\varphi_0\|^2_{L^2(\mathbb{R}^2)}\geq 1/B(\xi).$$

We next vary the random variables by the constant $t\in\mathbb{R}$: for any functional $F(\widetilde{\xi})$ of the sample path $\widetilde{\xi}=((\xi(x))_{x\in\mathbb{R}^2},(\overline{\xi}_a)_{a\in\mathbb{Z}^2})$,

$$(\tau_tF)(\widetilde{\xi}):=F(\widetilde{\xi}+t),$$

where

$$\widetilde{\xi}+t=((\xi(x)+t)_{x\in\mathbb{R}^2},(\overline{\xi}_a+t)_{a\in\mathbb{Z}^2}).$$

Then we have

$$\tau_t\widetilde{H^{\widetilde{\xi}}_L}=\widetilde{H^{\widetilde{\xi}}_L}+t\widetilde{\chi}_L,$$

where $\widetilde{\chi}_L=\sum_{a\in\mathbb{Z}^2\cap\Lambda_L}\chi_a^2$ as in the last section.

Since there exists $c_5\in(0,1]$ such that

$$(\varphi_0,\widetilde{\chi}_L\varphi_0)_{L^2(\mathbb{R}^2)}\geq c_5\|\chi_{in}\varphi_0\|^2_{L^2(\mathbb{R}^2)},$$

if $\lambda_0\in[E-\eta,E+\eta]$ and $t\geq 2\eta B(\xi)/c_5$, then we have

$$(\varphi_0,(\widetilde{H^{\widetilde{\xi}}_L}+t\widetilde{\chi}_L)\varphi_0)_{L^2(\mathbb{R}^2)}\geq E+\eta$$

and

$$(\varphi_0,(\widetilde{H^{\widetilde{\xi}}_L}-t\widetilde{\chi}_L)\varphi_0)_{L^2(\mathbb{R}^2)}\leq E-\eta.$$

We take a $[0,1]$-valued smooth function $\widetilde{1}$ so that $\widetilde{1}=1$ on $(-\infty,-1]$ and $\widetilde{1}=0$ on $[1,\infty)$. For any $\eta>0$, we modify this function by $\widetilde{1_\eta}(\cdot)=\widetilde{1}(\cdot/\eta)$. Then we have

$$\begin{aligned}
&\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\widetilde{\xi}}})]] \\
=&\sum_{n=1}^{\infty}\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\widetilde{\xi}}})]:B(\xi)\in[n-1,n]] \\
\leq&\sum_{n=1}^{\infty}\mathbb{E}\Big[\ \mathrm{Tr}\ \Big[\widetilde{1_\eta}\Big(\widetilde{H_L^{\widetilde{\xi}}}-\frac{2\eta n}{c_5}\widetilde{\chi}_L-E\Big)-\widetilde{1_\eta}\Big(\widetilde{H_L^{\widetilde{\xi}}}+\frac{2\eta n}{c_5}\widetilde{\chi}_L-E\Big)\Big]\widetilde{\chi_{[n-1,n]}}(B(\xi))\Big],
\end{aligned}$$

where $\widetilde{\chi_{[n-1,n]}}$ is a $[0,1]$-valued smooth function on $\mathbb{R}$ such that $\widetilde{\chi_{[n-1,n]}}=1$ on $[n-1,n]$, $\widetilde{\chi_{[n-1,n]}}=0$ on $\mathbb{R}\setminus[n-2,n+1]$ and the derivatives of $\widetilde{\chi_{[n-1,n]}}$ are dominated by a constant independent of $n$. By the Cameron-Martin theorem and the permutation integral, the right hand side is rewritten as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{E}\Big[\Big(\prod_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)}\int_{\mathbb{R}}d\overline{\xi}_a g\Big(\overline{\xi}_a+\frac{2\eta n}{c_5}\Big)\Big)\ \mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\widetilde{\xi}}}-E)] \\
&\times\exp\Big(-\frac{2\eta n}{c_5}\int_{\Lambda_\ell}\xi(x)dx-2\Big(\frac{\eta n\ell}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}\Big(B\Big(\xi+\frac{2\eta n}{c_5}\Big)\Big) \\
&-\Big(\prod_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)}\int_{\mathbb{R}}d\overline{\xi}_a g\Big(\overline{\xi}_a-\frac{2\eta n}{c_5}\Big)\Big)\ \mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\widetilde{\xi}}}-E)] \\
&\times\exp\Big(\frac{2\eta n}{c_5}\int_{\Lambda_\ell}\xi(x)dx-2\Big(\frac{\eta n\ell}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}\Big(B\Big(\xi-\frac{2\eta n}{c_5}\Big)\Big)\Big],
\end{aligned}$$

where $g$ is the probability density of the random variable $\overline{\xi}$ (cf. Lemma 2.1.3 in [16], Theorem 1.3 in [18]). This is devided as $I_1+I_2+I_3$, where

$$\begin{aligned}
I_1=&-\sum_{n=1}^{\infty}\int_{-2\eta n/c_5}^{2\eta n/c_5}dt\mathbb{E}\Big[\Big(\prod_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)}\int_{\mathbb{R}}d\overline{\xi}_a g(\overline{\xi}_a+t)\Big)\ \mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\widetilde{\xi}}}-E)]\int_{\Lambda_\ell}\xi(x)dx \\
&\times\exp\Big(-t\int_{\Lambda_\ell}\xi(x)dx-2\Big(\frac{\eta n\ell}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}(B(\xi+t))\Big] \\
I_2=&\sum_{n=1}^{\infty}\int_{-2\eta n/c_5}^{2\eta n/c_5}dt\mathbb{E}\Big[\sum_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)}\int_{\mathbb{R}}d\overline{\xi}_a g'(\overline{\xi}_a+t)\Big(\prod_{a\neq a'\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)}\int_{\mathbb{R}}d\overline{\xi}_{a'}g(\overline{\xi}_{a'}+t)\Big) \\
&\times\mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\widetilde{\xi}}}-E)]\exp\Big(-t\int_{\Lambda_\ell}\xi(x)dx-2\Big(\frac{\eta n\ell}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}(B(\xi+t))\Big],
\end{aligned}$$

and

$$\begin{aligned}
I_3=&\sum_{n=1}^{\infty}\int_{-2\eta n/c_5}^{2\eta n/c_5}dt\mathbb{E}\Big[\Big(\prod_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)}\int_{\mathbb{R}}d\overline{\xi}_a g(\overline{\xi}_a+t)\Big)\ \mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\widetilde{\xi}}}-E)] \\
&\times\exp\Big(-t\int_{\Lambda_\ell}\xi(x)dx-2\Big(\frac{\eta n\ell}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}'(B(\xi+t))\partial_t B(\xi+t)\Big].
\end{aligned}$$

By the Cameron-Martin theorem and the permutation integral, we have

$$I_1 = \sum_{n=1}^{\infty} \int_{-2\eta n/c_5}^{2\eta n/c_5} dt \exp\Big( -\frac{L^2}{2}\Big(\Big(\frac{2\eta n}{c_5}\Big)^2 - t^2\Big)\Big)$$
$$\times \mathbb{E}[\mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\tilde{\xi}}} - t\widetilde{\chi_L} - E)]\Big(t\ell^2 - \int_{\Lambda_\ell} \xi(x)dx\Big)\widetilde{\chi_{[n-1,n]}}(B(\xi))].$$

By Proposition 3.1, we have

$$I_1 \le c_6 L^{c_7} \int_{\mathbb{R}} dt(1+|t|^3)\mathbb{P}\Big(B(\xi)+2 \ge \frac{c_5|t|}{2\eta}\Big)^{1/3}.$$

As in the proof of Lemma 3.3 (ii) in [21], there exists $c_8 \in (0,\infty)$ such that

$$\sup_{L\ge 2} \mathbb{E}\Big[\exp\Big(c_8|||\xi_\ell|||^2_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)}/\ell^{2\epsilon}\Big)\Big] < \infty$$

and

$$\sup_{L\ge 2} \mathbb{E}\Big[\exp\Big(c_8|||Y_{\xi,\ell}|||_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)}/\ell^{\epsilon}\Big)\Big] < \infty.$$

Thus we have

$$\begin{aligned}
&\mathbb{P}\Big(B(\xi)+2 \ge \frac{c_5|t|}{2\eta}\Big)\\
\le&\mathbb{P}\Big((1+|||\xi_\ell|||_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,1/\epsilon}(\mathbb{R}^2)})^2 + |||Y_{\xi,\ell}|||_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} \ge \Big(\frac{c_5|t|}{4c_4\eta}\Big)^{1/c_3}\Big) + 1_{\eta\ge c_5|t|/8}\\
\le&\mathbb{P}\Big(|||\xi_\ell|||_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,1/\epsilon}(\mathbb{R}^2)} \ge \frac{1}{2^{3/2}}\Big(\frac{c_5|t|}{4c_4\eta}\Big)^{1/(2c_3)}\Big)\\
&+\mathbb{P}\Big(|||Y_{\xi,\ell}|||_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} \ge \frac{1}{2}\Big(\frac{c_5|t|}{4c_4\eta}\Big)^{1/c_3}\Big) + 2\times 1_{\eta\ge c_9|t|}\\
\le&\mathbb{E}\Big[\exp\Big(\frac{c_8}{\ell^{2\epsilon}}\Big(|||\xi_\ell|||^2_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,1/\epsilon}(\mathbb{R}^2)} - \frac{1}{8}\Big(\frac{c_5|t|}{4c_4\eta}\Big)^{1/c_3}\Big)\Big)\Big]\\
&+\mathbb{E}\Big[\exp\Big(\frac{c_8}{\ell^{\epsilon}}\Big(|||Y_{\xi,\ell}|||_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} - \frac{1}{2}\Big(\frac{c_5|t|}{4c_4\eta}\Big)^{1/c_3}\Big)\Big)\Big] + 2\times 1_{\eta\ge c_9|t|}\\
\le&c_{10}\exp\Big(-\frac{c_{11}}{\ell^{2\epsilon}}\Big(\frac{|t|}{\eta}\Big)^{1/c_3}\Big) + 2\times 1_{\eta\ge c_9|t|}
\end{aligned}$$

and

$$\begin{aligned}
I_1 \le& c_{12}L^{c_7}\int_{\mathbb{R}} dt(1+|t|^3)\Big(\exp\Big(-\frac{c_{11}}{3\ell^{2\epsilon}}\Big(\frac{|t|}{\eta}\Big)^{1/c_3}\Big) + 1_{\eta\ge c_9|t|}\Big)\\
\le& c_{13}\eta L^{c_{14}}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
I_2 \le& c_{15}\sum_{n=1}^{\infty}\int_{-2\eta n/c_5}^{2\eta n/c_5} dt \sum_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_\ell)} \mathbb{E}\Big[\int_{\mathrm{supp}\, g} d\bar{\xi}_a\, \mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\tilde{\xi}}} - t\widetilde{\chi_L} - E)]\widetilde{\chi_{[n-1,n]}}(B(\xi))\Big]\\
\le& c_{16}\eta L^{c_{17}}
\end{aligned}$$

and

$$I_3 \leq \sum_{n=1}^{\infty} \int_{-2\eta n/c_5}^{2\eta n/c_5} dt \mathbb{E}[\mathrm{Tr}[\widetilde{1_\eta}(\widetilde{H_L^{\tilde{\xi}}} - t\widetilde{\chi_L} - E)] |\widetilde{\chi_{[n-1,n]}}'(B(\xi+t))| |(\partial_t B(\xi+t))_{t=0}|]$$

$$\leq c_{18} \eta L^{c_{19}},$$

since

$$\mathbb{E}[|(\partial_t B(\xi+t))_{t=0}|^3] \leq c_{20} L^{c_{21}}.$$

□

We next show the lemma used in the above proof.

**Lemma 4.1.** *There exists $C \in (0,\infty)$ satisfying the following:*

(i) *For any $\alpha, \beta, \gamma \in \mathbb{R}$, there exist $C_{\alpha,\beta,\gamma} \in (0,\infty)$ satisfying the following: if $a_1, a_2, a_3 \in \mathbb{Z}^2$ satisfy*

$$\max_{i,j \in \{1,2,3\}} |a_i - a_j|_\infty \geq 3, \tag{4.1}$$

*then*

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}$$

$$\leq C_{\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))$$

*for any $f \in \mathcal{C}^\beta(\mathbb{R}^2)$ and $g \in \mathcal{C}^\gamma(\mathbb{R}^2)$,*

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}$$

$$\leq C_{\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))$$

*for any $f \in \mathcal{H}^\beta(\mathbb{R}^2)$ and $g \in \mathcal{C}^\gamma(\mathbb{R}^2)$, and*

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}$$

$$\leq C_{\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))$$

*for any $f \in \mathcal{C}^\beta(\mathbb{R}^2)$ and $g \in \mathcal{H}^\gamma(\mathbb{R}^2)$. In (4.1), $|\cdot|_\infty$ is the maximum norm defined by $|x| = \max_i |x_i|$ for $x = (x_1, x_2) \in \mathbb{R}^2$.*

(ii) *For any $\alpha, \beta, \gamma \in \mathbb{R}$, there exist $C_{\alpha,\beta,\gamma} \in (0,\infty)$ satisfying the following: if $a_1, a_2, a_3 \in \mathbb{Z}^2$ satisfy (4.1), then*

$$\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}$$

$$\leq C_{\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))$$

*for any* $f \in \mathcal{C}^\beta(\mathbb{R}^2)$ *and* $g \in \mathcal{C}^\gamma(\mathbb{R}^2)$, *and*

$$
\begin{aligned}
&\|\chi_{a_1}\Pi(\chi_{a_2}f, \chi_{a_3}g)\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}\\
\le& C_{\alpha,\beta,\gamma}\|\chi_{a_2}f\|_{\mathcal{H}^\beta(\mathbb{R}^2)}\|\chi_{a_3}g\|_{\mathcal{C}^\gamma(\mathbb{R}^2)}\exp(-C(|a_1-a_2|^2+|a_1-a_3|^2))
\end{aligned}
$$

*for any* $f \in \mathcal{H}^\beta(\mathbb{R}^2)$ *and* $g \in \mathcal{C}^\gamma(\mathbb{R}^2)$.

(iii) *For any* $\alpha, \beta, \gamma \in \mathbb{R}$, *there exist* $C_{\alpha,\beta,\gamma} \in (0,\infty)$ *satisfying the following: if* $a_1, a_2, a_3, a_4 \in \mathbb{Z}^2$ *satisfy*

$$
\max_{i,j\in\{1,2,3,4\}} |a_i - a_j|_\infty \ge 3,
$$

*then*

$$
\begin{aligned}
&\|\chi_{a_1\chi_{a_2}h}P_{\chi_{a_3}f}(\chi_{a_4}g)\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}\\
\le& C_{\alpha,\beta,\gamma}\|\chi_{a_3}f\|_{\mathcal{C}^\beta(\mathbb{R}^2)}\|\chi_{a_4}g\|_{\mathcal{C}^\gamma(\mathbb{R}^2)}\|\chi_{a_2}h\|_{L^2(\mathbb{R}^2)}\\
&\times\exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2))
\end{aligned}
$$

*for any* $f \in \mathcal{C}^\beta(\mathbb{R}^2)$, $g \in \mathcal{C}^\gamma(\mathbb{R}^2)$ *and* $h \in L^2(\mathbb{R}^2)$.

**Proof**. We will prove only the first inequality, since the other inequalities are similarly proven., (i) Since

$$
(\Delta\Delta^{-loc})^n = (1-e^\Delta)^n = \sum_{k=0}^n (-1)^k {}_nC_k e^{k\Delta}
$$

for any $n \in \mathbb{N}$, we have

$$
\|\nabla^{n_1}\chi_{a_1}P_{\chi_{a_2}f}(\chi_{a_3}g)\|_{L^\infty(\mathbb{R}^2)} \le c\sum_\nu \sum_{i,j=1}^2 I(i,j;\nu)
$$

where

$$
\begin{aligned}
I(1,1;\nu) :=& \Big\| \int_0^1 \frac{dt}{t}(\nabla^{n_1}\chi_{a_1}Q_t^{\nu,1})(((P_t^\nu\widetilde{\chi_{a_2}}\Delta^{n_2})(\Delta^{-loc})^{n_2}\chi_{a_2}f)\\
&\times((Q_t^{\nu,2}\widetilde{\chi_{a_3}}\Delta^{n_3})(\Delta^{-loc})^{n_3}\chi_{a_3}g))\Big\|_{L^\infty(\mathbb{R}^2)},\\
I(2,1;\nu) :=& \Big\| \int_0^1 \frac{dt}{t}(\nabla^{n_1}\chi_{a_1}Q_t^{\nu,1})((P_t^\nu\widetilde{\chi_{a_2}}\sum_{k=1}^{n_2}(-1)^k {}_{n_2}C_k e^{k\Delta}\chi_{a_2}f)\\
&\times((Q_t^{\nu,2}\widetilde{\chi_{a_3}}\Delta^{n_3})(\Delta^{-loc})^{n_3}\chi_{a_3}g))\Big\|_{L^\infty(\mathbb{R}^2)},\\
I(1,2;\nu) :=& \Big\| \int_0^1 \frac{dt}{t}(\nabla^{n_1}\chi_{a_1}Q_t^{\nu,1})(((P_t^\nu\widetilde{\chi_{a_2}}\Delta^{n_2})(\Delta^{-loc})^{n_2}\chi_{a_2}f)\\
&\times(Q_t^{\nu,2}\widetilde{\chi_{a_3}}\sum_{k=1}^{n_3}(-1)^k {}_{n_3}C_k e^{k\Delta}\chi_{a_3}g))\Big\|_{L^\infty(\mathbb{R}^2)},
\end{aligned}
$$

and

$$I(2,2;\nu) := \Big\| \int_0^1 \frac{dt}{t} (\nabla^{n_1}\chi_{a_1} Q_t^{\nu,1})((P_t^{\nu}\widetilde{\chi_{a_2}} \sum_{k=1}^{n_2} (-1)^k {}_{n_2}C_k e^{k\Delta}\chi_{a_2} f)$$

$$\times (Q_t^{\nu,2}\widetilde{\chi_{a_3}} \sum_{k=1}^{n_3} (-1)^k {}_{n_3}C_k e^{k\Delta}\chi_{a_3} g))\Big\|_{L^\infty(\mathbb{R}^2)}$$

for any $n_1, n_2, n_3 \in \mathbb{N}$. In these equations, $\widetilde{\chi_a}$ is a smooth function such that $\widetilde{\chi_a}\chi_a = \chi_a$ and $\operatorname{supp}\widetilde{\chi_a} \subset \Lambda_{2+\delta}(a)$ with a small positive $\delta$ for each $a$. $I(1,1;\nu)$ is dominated by

$$\begin{aligned}
&\sup_{x\in\Lambda_2(a_1)} \int_0^1 \frac{dt}{t} \int_{\mathbb{R}^2} \frac{dx_1}{t^{1+n_1/2}} \exp\Big(\frac{-|x-x_1|^2}{ct}\Big) \\
&\times \int_{\Lambda_{2+\delta}(a_2)} \frac{dx_2}{t^{1+n_2}} \exp\Big(\frac{-|x_1-x_2|^2}{ct}\Big) |(\Delta^{-loc})^{n_2}\chi_{a_2} f)(x_2)| \\
&\times \int_{\Lambda_{2+\delta}(a_3)} \frac{dx_2}{t^{1+n_3}} \exp\Big(\frac{-|x_1-x_3|^2}{ct}\Big) |(\Delta^{-loc})^{n_3}\chi_{a_3} g)(x_3)|.
\end{aligned} \tag{4.2}$$

Since

$$d(\Lambda_2(a_1), \Lambda_2(a_2)) - \delta \le |x - x_2| \le \sqrt{2(|x-x_1|^2 + |x_1 - x_2|^2)},$$

$$d(\Lambda_2(a_1), \Lambda_2(a_3)) - \delta \le |x - x_3| \le \sqrt{2(|x-x_1|^2 + |x_1 - x_3|^2)},$$

and

$$d(\Lambda_2(a_2), \Lambda_2(a_3)) - 2\delta \le |x_2 - x_3| \le \sqrt{2(|x_1-x_2|^2 + |x_1 - x_3|^2)},$$

the quantity in (4.2) is dominated by

$$\exp(-c(|a_1-a_2|^2 + |a_1-a_3|^2)) \sup_{x_2\in\Lambda_{2+\delta}(a_2)} |(\Delta^{-loc})^{n_2}\chi_{a_2} f)(x_2)| \sup_{x_3\in\Lambda_{2+\delta}(a_3)} |(\Delta^{-loc})^{n_3}\chi_{a_3} g)(x_3)| \tag{4.3}$$

under the condition (4.1). By the proof of Lemma 3.1 in [21], the quantity in (4.3) is dominated by

$$\exp(-c(|a_1-a_2|^2 + |a_1-a_3|^2)) \|\chi_{a_2} f\|_{\mathcal{C}^{\epsilon-2n_2}(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^{\epsilon-2n_3}(\mathbb{R}^2)}.$$

$I(i,j;\nu)$ with other $i$ and $j$ are also dominated by the simirar quantities by the simirar methods. □

**Lemma 4.2.** *There exist $C \in (0,\infty)$ and $C_{\alpha,\beta} \in (0,\infty)$ for each $\alpha, \beta \in \mathbb{R}$ satisfying the following: if $a_1, a_2 \in \mathbb{Z}^2$ satsfy $|a_1 - a_2|_\infty \ge 3$, then*

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \le C_{\alpha,\beta} \|\chi_{a_2} f\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \exp(-C|a_1-a_2|^2)$$

*for any $f \in \mathcal{C}^\beta(\mathbb{R}^2)$, and*

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \le C_{\alpha,\beta} \|\chi_{a_2} f\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \exp(-C|a_1-a_2|^2)$$

*for any $f \in \mathcal{H}^\beta(\mathbb{R}^2)$.*

**Proof**. As in the proof of the last lemma, $\|\nabla^{n_1}\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{L\infty(\mathbb{R}^2)}$ is dominated by $I_1+I_2$, where

$$I_1 := \int_0^1 dt\|\nabla^{n_1}\chi_{a_1}t^{t\Delta}\widetilde{\chi_{a_2}}\Delta^{n_2}(\Delta^{-loc})^{n_2}\chi_{a_2}f\|_{L\infty(\mathbb{R}^2)}$$

and

$$I_2 := \int_0^1 dt\|\nabla^{n_1}\chi_{a_1}t^{t\Delta}\widetilde{\chi_{a_2}}\sum_{k=1}^{n_2}(-1)^k{}_{n_2}C_k e^{k\Delta}\chi_{a_2}f\|_{L\infty(\mathbb{R}^2)}.$$

$I_1$ is dominated by

$$\sup_{x\in\Lambda_2(a_1)}\int_0^1 dt\int_{\Lambda_{2+\delta}(a_2)} d\frac{x_1}{t^{1+(n_1/2)+n_2}}\exp\left(\frac{-|x-x_1|^2}{ct}\right)|(\Delta^{-loc})^{n_2}\chi_{a_2}f)(x_1)|.$$

Under the condition $|a_1-a_2|_\infty \geq 3$, this qunatity is dominated by

$$\exp(-c|a_1-a_2|^2)\sup_{x_1\in\Lambda_{2+\delta}(a_2)}|(\Delta^{-loc})^{n_2}\chi_{a_2}f)(x_1)|.$$

By the proof of Lemma 3.1 in [21], this qunatity is dominated by

$$\exp(-c|a_1-a_2|^2)\|\chi_{a_2}f\|_{\mathcal{C}^{\epsilon-2n_2}(\mathbb{R}^2)}.$$

$I_2$ is also dominated by the simirar quantities by the simirar methods. $\|\nabla^{n_1}\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{L2(\mathbb{R}^2)}$ is also dominated by the simirar quantities by the simirar method. □

For any $m\in\mathbb{N}$, let $\{\chi_{a,1/m}\}_{a\in\mathbb{Z}^2/m}$ be the smooth functions defined by $\chi_{a,1/m}(x) := \chi_{ma}(mx)$. Thus these satisfy $\operatorname{supp}\chi_{a,1/m}\subset\Lambda_{2/m}(a)$ and $\sum_{a\in\mathbb{Z}^2/m}\chi_a^2\equiv 1$ on $\mathbb{R}^2$. We modify Lemmas 3.1, and 3.2 in [21] and Lemmas 4.1 and 4.2 in this section so that the functions $\{\chi_a\}_{a\in\mathbb{Z}^2}$ are replaced by the functions $\{\chi_{a,1/m}\}_{a\in\mathbb{Z}^2/m}$.. Then we have the following:

**Lemma 4.3.** *There exist $C_m\in(0,\infty)$ for each $m\in\mathbb{N}$ and $C_{m,\alpha}\in(0,\infty)$ for each $m\in\mathbb{N}$ and $\alpha\in\mathbb{R}$ such that*

$$\begin{aligned}&\|\chi_{a_1,1/m}\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}\\ \leq& C_{m,\alpha}(\sup_{a_2\in\mathbb{Z}^2/m}\|\chi_{a_2,1/m}\widetilde{\chi_{L-2}}\xi\|_{C^{-2}(\mathbb{R}^2)}+1)^2\exp(-C_m d(a_1,\Lambda_{L-2})^2)\end{aligned} \tag{4.4}$$

*and*

$$\begin{aligned}&\|\chi_{a_1,1/m}\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi)]\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}\\ \leq& C_{m,\alpha}\exp(-C_m d(a_1,\Lambda_{L-2})^2).\end{aligned} \tag{4.5}$$

*for any $a_1\in\mathbb{Z}^2/m\setminus\Lambda_{L-2+6/m}$. For any $a_1\in\mathbb{Z}^2/m\setminus\Lambda_{L-2+6/m}$, we have*

$$\lim_{\varepsilon\to 0}\|\chi_{a_1,1/m}(\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi_\varepsilon,\widetilde{\chi_{L-2}}\xi_\varepsilon)-\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi))\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}=0 \tag{4.6}$$

*and*

$$\lim_{\varepsilon\to 0}\|\chi_{a_1,1/m}(\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi_\varepsilon,\widetilde{\chi_{L-2}}\xi_\varepsilon)]-\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi)])\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}=0. \tag{4.7}$$

**Proof**. (4.4) and (4.6) are easily proven by Lemmas 3.1, and 3.2 in [21] and Lemmas 4.1 and 4.2 in this section. Since

$$\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi)]=\sum_\mu\int_0^1\frac{dt}{t}P_t^\mu(x,x_1)(Q_t^{1,\mu}\Delta^{-loc}\widetilde{\chi_{L-2}}^2Q_t^{2,\mu})(x_1,x_1),$$

we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}^2}|\nabla^n\chi_{a_1,1/m}\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi)]|\\ \le& c\sup_{x\in\Lambda_{2/m}(a_1)}\int_0^1\frac{dt}{t}\int_{\mathbb{R}^2}\frac{dx_1}{t^{1+n/2}}\exp\Big(\frac{-|x-x_1|^2}{ct}\Big)\int_{\mathbb{R}^2}\frac{dx_2}{t}\exp\Big(\frac{-|x_1-x_2|^2}{ct}\Big)\\ &\times\int_0^1 ds\int_{\Lambda_L}\frac{dx_3}{st}\exp\Big(-\frac{|x_2-x_3|^2}{ct}-\frac{|x_3-x_1|^2}{ct}\Big)\\ \le& c\exp(-cd(a_1,\Lambda_L)^2),\end{aligned}$$

for any $n\in\{0,1,2,\dots\}$.

Simirarly we can show that

$$\lim_{\varepsilon\to0}\sup_{\mathbb{R}^2}|\nabla^n\chi_{a_1,1/m}(\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi_\varepsilon,\widetilde{\chi_{L-2}}\xi_\varepsilon)]-\mathbb{E}[\Pi(\Delta^{-loc}\widetilde{\chi_{L-2}}\xi,\widetilde{\chi_{L-2}}\xi_\varepsilon)])|=0$$

for any $n\in\{0,1,2,\dots\}$. □

**Lemma 4.4.** *For any $\varphi\in\mathrm{Dom}(\widetilde{H_L^{\tilde\xi}})$, we have $\chi_{out}\varphi\in\mathrm{Dom}(\widetilde{H_L^{\tilde\xi}})\cap\mathcal{H}^2(\mathbb{R}^2)$ and*

$$\widetilde{H_L^{\tilde\xi}}\chi_{out}\varphi=\Big(-\Delta+\sum_{a\in\mathbb{Z}^2\cap(\Lambda_L\setminus\Lambda_{L-1})}\chi_a^2\overline{\xi}_a-\mathbb{E}[\Pi(\Delta^{-loc}\xi_{L-2},\xi_{L-2})]\Big)\chi_{out}\varphi. \tag{4.8}$$

**Proof**.

Since $\varphi\in\mathrm{Dom}(\widetilde{H_L^{\tilde\xi}})$, we have $\chi_{out}\varphi\in\bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$ and $\chi_{out}\Phi_{\xi,L-2}(\varphi)\in\mathcal{H}^2(\mathbb{R}^2)$. By Lemma 4.1 and Lemma 4.3, we have $\chi_{out}\Delta^{-loc}P_\varphi\xi_{L-2}$, $\chi_{out}\Delta^{-loc}{}_\varphi P_{\xi_{L-2}}\Delta^{-loc}\xi_{L-2}$, $\chi_{out}\Delta^{-loc}P_\varphi Y_{\xi,L-2}\in\mathcal{H}^n(\mathbb{R}^2)$ for any $n\in\mathbb{N}$. Thus we have $\chi_{out}\varphi\in\mathcal{H}^2(\mathbb{R}^2)$. By Lemmas 3.1, 3.2, and 4.3 in [21] and Lemmas 4.1, 4.2 and 4.3 in this section, we have $\Delta^{-loc}P_{\chi_{out}\varphi}\xi_{L-2}$, $\Delta^{-loc}{}_{\chi_{out}\varphi}P_{\xi_{L-2}}\Delta^{-loc}\xi_{L-2}$, $\Delta^{-loc}P_{\chi_{out}\varphi}Y_{\xi,L-2}\in\mathcal{H}^n(\mathbb{R}^2)$ for any $n\in\mathbb{N}$. Thus we have $\Phi_{\xi,L-2}(\chi_{out}\varphi)\in\mathcal{H}^2(\mathbb{R}^2)$ and $\chi_{out}\varphi\in\mathrm{Dom}(\widetilde{H_L^{\tilde\xi}})$. Then many terms are canceled in the equation where the definition (2.1) of the operator is applied for $\widetilde{H_L^{\tilde\xi}}\chi_{out}\varphi$, and we obtain (4.8). □

## 5. Multiscale analysis

Multiscale analysis has been applied to prove the Anderson localization since the pioneer work by Fröhlich and Spencer [6], This method has been extended, improved and simplified by many works. For this aspect, see [19] and the references therein. In particular Germinet and Klein give an effective multiscale analysis to deduce the strong dynamical localization from a weak initial estimate for a wide class of operators with short correlated potentials [7]. However our geometric resolvent inequality obtained in Section 2 is too weak. In this section, we modify multiscale analysis for our case. We use the following definition:

**Definition 5.1.** (i) *Let $m > 0$, $E < 0$, $K > 0$, $\boldsymbol{a} \in \mathbb{Z}^2$ and $L \in 6\mathbb{N}$. A square $\Lambda_L(\boldsymbol{a})$ is called $(m, E, K)$-regular for $\widetilde{\xi} = (\xi, \bar{\xi})$ if $E \notin \mathrm{spec}(\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}})$ and*

$$\|\chi_a(\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1}\chi_b\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq K\exp(-m(|a-b|_\infty \wedge d_\infty(b, \partial\Lambda_L(\boldsymbol{a})) + d(a, \Lambda_L(\boldsymbol{a}))))$$

*for any $a \in \mathbb{Z}^2$ and $b \in \mathbb{Z}^2 \cap \Lambda_{L/3}(\boldsymbol{a})$, where $|x|_\infty = |x_1| \vee |x_2|$, $|x| = \sqrt{x_1^2 + x_2^2}$ for any $x = (x_1, x_2) \in \mathbb{R}^2$, $d_\infty(x, A) = \inf_{y\in A} |x-y|_\infty$, $d(x, A) = \inf_{y\in A} |x-y|$ for any $A \subset \mathbb{R}^2$, maximal $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^2))}$ is the operator norm as a bounded operator on $L^2(\mathbb{R}^2)$, and $\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}}$ is the operator defined in (2.1). If a square $\Lambda_L(\boldsymbol{a})$ is not $(m, E, K)$-regular, then this is called $(m, E, K)$-singular.*

(ii) *For any $m > 0$, $E < 0$, $K > 0$, $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{Z}^2$, $L \in 6\mathbb{N}$ and any interval $I$ in $(-\infty, 0)$, we set the following event:*

$$\begin{aligned} &R(m, K, I, L, \boldsymbol{a}, \boldsymbol{a}') \\ :=&\{\widetilde{\xi} : \textit{for every } E \in I,\ \textit{either } \Lambda_L(\boldsymbol{a}) \textit{ or } \Lambda_L(\boldsymbol{a}') \textit{ is } (m, E, K)\textit{-regular for } \widetilde{\xi}\}. \end{aligned}$$

Then we willl prove the following:

**Proposition 5.1** (Multiscale Analysis)**.** *For any $1 \leq p < \infty$ and $1 < \alpha < 1 + p/4$, there exists $\overline{m_0} \in (0, c_*)$, $c_1, c_2, c_3 \in (0, \infty)$ satisfying the following: for any $0 < m_0 < \overline{m_0}$, $E_1 < E_0 < 0$, and $L_0 \in 6\mathbb{N}$ satisfying*

$$L_0 \geq \frac{c_1}{m_0^{\alpha/(\alpha-1)}} \vee (\log|E_1|)^2, \tag{5.1}$$

*if*

$$\mathbb{P}(R(m_0, K_0, [E_1, E_0], L_0, \boldsymbol{a}, \boldsymbol{a}')) > 1 - L_0^{-p}, \tag{5.2}$$

*for any* $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{Z}^2$ *satisfying* $|\boldsymbol{a} - \boldsymbol{a}'|_\infty > L_0 + 2$*, then we have*

$$\mathbb{P}(R(m_0/2, K_k, [E_1, E_0], L_k, \boldsymbol{a}, \boldsymbol{a}')) > 1 - L_k^{-p}$$

*for any* $k \in \mathbb{N}$ *and any* $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{Z}^2$ *satisfying* $|\boldsymbol{a} - \boldsymbol{a}'|_\infty > L_k + 2$*, where* $c_*$ *is the constant appeared in Proposition 2.1,* $\{L_k\}_{k\in\mathbb{N}}$ *is a sequence defined by* $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}} := \max\{(-\infty, L_k^\alpha] \cap (6\mathbb{N})\}$*, and* $K_k = c_2 \exp(c_3 L_k^{1/\alpha})$.

**Proof**. Let $m_k = (1 + 2^{-k})m_0/2$ for $0 \le k \in \mathbb{Z}$. Suppose

$$\mathbb{P}(R(m_k, K_k, [E_1, E_0], L_k, \boldsymbol{a}, \boldsymbol{a}')) > 1 - L_k^{-p}, \tag{5.3}$$

for any $\boldsymbol{a}, \boldsymbol{a}' \in (L_{k-1}/6)\mathbb{Z}^2$ satisfying $|\boldsymbol{a} - \boldsymbol{a}'|_\infty > L_k + 2$, where $L_{-1}/6$ is regarded as 1. Let

$$C(L_{k+1}, L_k, \boldsymbol{a}) := \Big\{\Lambda_{L_k}(\boldsymbol{a}') : \boldsymbol{a}' \in \frac{L_k}{3}\mathbb{Z}^2, \Lambda_{L_k}(\boldsymbol{a}') \subset \Lambda_{L_{k+1}-3}(\boldsymbol{a})\Big\}.$$

For $E \in [E_1, E_0]$, $\eta_k > 0, M_k \ge 1$ and $S \in 2\mathbb{Z}_+ + 1$ specified later, we consider the following events:

$$R_1(E, L_{k+1}, \boldsymbol{a}) := \{\widetilde{\xi} : d(\widetilde{\mathrm{spec}(H^{\widetilde{\xi}}_{L_{k+1},\boldsymbol{a}})}, E) > \eta_k\}.$$

$$R_2(E, L_{k+1}, \boldsymbol{a}) := \Big\{\widetilde{\xi} : d(\widetilde{\mathrm{spec}(H^{\widetilde{\xi}}_{(7j+1)L_k/3,\boldsymbol{a}'})}, E) > \eta_k \text{ for any } j \in \{1, 2, \dots, S\}$$
$$\text{and } \boldsymbol{a}' \in \Lambda_{L_{k+1}}(\boldsymbol{a}) \cap \Big(\frac{L_k}{6}\mathbb{Z}^2\Big) \text{ such that } \Lambda_{(7j+1)L_k/3}(\boldsymbol{a}') \subset \Lambda_{L_{k+1}-3}(\boldsymbol{a})\Big\}.$$

$$R_3(E, L_{k+1}, \boldsymbol{a}) := \{\widetilde{\xi} : \Lambda_{L_k}(\boldsymbol{a}_1), \dots, \Lambda_{L_k}(\boldsymbol{a}_j) \in C(L_{k+1}, L_k, \boldsymbol{a}) \text{ are } (m_k, E, K_k)\text{-singular}$$
$$\text{and } d_\infty(\Lambda_{L_k}(\boldsymbol{a}_a), \Lambda_{L_k}(\boldsymbol{a}_b)) > 2 \text{ for } a \ne b, \text{ then } j \le S\}.$$

$$R_4(L_{k+1}, \boldsymbol{a}) := \{\widetilde{\xi} : \|\chi_{\boldsymbol{a}+a}\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \le M_k(\log(2 + |a|))^{1/2} \text{ for any } a \in \mathbb{Z}^2 \cap \Lambda_{L_{k+1}}\},$$

$$R_5(L_{k+1}, \boldsymbol{a}) := \Big\{\widetilde{\xi} : \|\chi_{\boldsymbol{a}+a}Y_{\xi,L_k-2,\boldsymbol{a}+\boldsymbol{a}'}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le M_k(\log(2 + |a|)) \exp(-\widetilde{c_*}d(a, \Lambda_{L_k}(\boldsymbol{a}'))^2)$$
$$\text{for any } a \in \mathbb{Z}^2 \text{ and } \boldsymbol{a}' \in \Lambda_{L_{k+1}} \cap \Big(\frac{L_k}{3}\mathbb{Z}^2\Big)\Big\},$$

where $\widetilde{c_*}$ is the constant $C_\epsilon$ appeared in Lemma 4.3 (i) in [21].

$$R_6(L_{k+1}, \boldsymbol{a}) := \Big\{\widetilde{\xi} : \|\chi_{\boldsymbol{a}+a}Y_{\xi,((7j+1)L_k/3)-2,\boldsymbol{a}+\boldsymbol{a}'}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le M_k(\log(2 + |a|)) \exp(-\widetilde{c_*}d(a, \Lambda_{(7j+1)L_k/3}(\boldsymbol{a}'))^2)$$
$$\text{for any } a \in \mathbb{Z}^2, j \in \{1, 2, \dots, S\} \text{ and } \boldsymbol{a}' \in \frac{L_k}{6}\mathbb{Z}^2$$
$$\text{such that } \Lambda_{(7j+1)L_k/3}(\boldsymbol{a}') \subset \Lambda_{L_{k+1}-3}\Big\}.$$

$$R_7(L_{k+1}, \boldsymbol{a}) := \{\widetilde{\xi} : \|\chi_{\boldsymbol{a}+a}Y_{\xi,L_{k+1}-2,\boldsymbol{a}}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le M_k(\log(2 + |a|)) \exp(-\widetilde{c_*}d(a, \Lambda_{L_{k+1}})^2) \text{ for any } a \in \mathbb{Z}^2\Big\}.$$

Under the event $R_3(E, L_{k+1}, \boldsymbol{a})$, the following occurs:

There exist $r \in \{1, 2, \dots, S\}, y_1, \dots, y_r \in \Lambda_{L_{k+1}}(\boldsymbol{a}) \cap \Big(\frac{L_k}{3}\mathbb{Z}^2\Big), j_1, \dots, j_r \in \{1, 2, \dots, S\}$ satsfying the following:

$d_\infty(\Lambda_{7j_a L_k/3}(y_a), \Lambda_{7j_b L_k/3}(y_b)) > L_k/3$ for $a \neq b$,

$j_1 + \cdots + j_r \leq S$,

$\boldsymbol{a}' \in \Big(\frac{L_k}{3}\mathbb{Z}^2\Big) \setminus \bigcup_{h=1}^{r} \Lambda_{7j_h L_k/3}(y_h)$ implies that $\Lambda_{L_k}(\boldsymbol{a}')$ is $(m_k, E, K_k)$-regular.

Under the event $\bigcap_{i=1}^{3} R_i(E, L_{k+1}, \boldsymbol{a}) \cap \bigcap_{i=4}^{7} R_i(L_{k+1}, \boldsymbol{a})$, we estimate $\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_b\|_{\mathcal{L}(L^2(\mathbb{R}^2))}$ for any $a \in \mathbb{Z}^2$ and $b \in \mathbb{Z}^2 \cap \Lambda_{L_{k+1}/3}(\boldsymbol{a})$. If $b = b_1 \in S(r)$, then by applying Proposition 2.1 with $\Lambda_{\ell'}(\boldsymbol{a}') = \Lambda_{L_{k+1}}(\boldsymbol{a})$, $\Lambda_\ell(\boldsymbol{a}) = \Lambda_{L_k}(\boldsymbol{b}_1)$ and $(a_*, a) = (a, b)$, we have

$$\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq e_r(b_1, a) + \sum_{b_2 \in \mathbb{Z}^2} w_r(b_1, b_2)\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_2}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}, \tag{5.4}$$

where

$$S(r) := \{b_i \in \mathbb{Z}^2 : \Lambda_{L_k}(\boldsymbol{b}_i) \text{ is } (m_k, E, K_k)\text{-regular and } a \notin \Lambda_{L_k}(\boldsymbol{b}_i) \subset \Lambda_{L_{k+1}/3}(\boldsymbol{a}),$$
$$\text{where } \boldsymbol{b}_i \in (L_k/3)\mathbb{Z}^2 \text{ such that } b_i \in \Lambda_{L_k/3}(\boldsymbol{b}_1)|\},$$

$$\Lambda_{L_k/3}(\boldsymbol{b}_1)| := (\boldsymbol{b}_{1,1} - L_k/6, \boldsymbol{b}_{1,1} + L_k/6] \times (\boldsymbol{b}_{1,2} - L_k/6, \boldsymbol{b}_{1,2} + L_k/6],$$
$$e_r(b_1, a) := c_1 M_k\sqrt{\log L_k}K_k \exp\Big( - m_k\frac{L_k}{3} - c_* d(a, \Lambda_{L_k-4}(\boldsymbol{b}_1) \setminus \Lambda_{L_k-6}(\boldsymbol{b}_1))\Big),$$

and

$$w_r(b_1, b_2) := c_2\sqrt{|E|}(M_k\sqrt{\log L_k})^{c_3}K_k L_k^2 \exp\Big( - m_k\frac{L_k}{3} - c_* d(b_2, \Lambda_{L_k-4}(\boldsymbol{b}_1) \setminus \Lambda_{L_k-6}(\boldsymbol{b}_1))\Big).$$

If $b = b_1 \in S(s)$, then by applying Proposition 2.1 with $\Lambda_{\ell'}(\boldsymbol{a}') = \Lambda_{L_{k+1}}(\boldsymbol{a})$, $\Lambda_\ell(\boldsymbol{a}) = \Lambda_{(7j_{h_1}+1)L_k/3}(y_{h_1})$, $(a_*, a) = (a, b)$ and the condition in $R_2(E, L_{k+1}, \boldsymbol{a})$, we have

$$\begin{aligned}
&\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_b\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
\leq &\frac{c_4 M_k\sqrt{\log L_k}}{\eta_k}\exp\Big( - \frac{c_*}{2}|a - b| + c_*\sqrt{2}\frac{7j_{h_1}+1}{6}L_k\Big) \\
&+ \frac{c_5}{\eta_k}\sqrt{|E|}(M_k\sqrt{\log L_k})^{c_6} \\
&\times \sum_{b_1' \in \mathbb{Z}^2} \|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_1'}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
&\quad \times \exp\Big( - \frac{c_*}{2}d(b_1', \Lambda_{(7j_{h_1}+1)(L_k/3)-4}(y_{h_1}) \setminus \Lambda_{(7j_{h_1}+1)(L_k/3)-6}(y_{h_1}))\Big),
\end{aligned}$$

where

$$S(s) :=\{b_i \in \mathbb{Z}^2 : \Lambda_{L_k}(\boldsymbol{b}_i) \text{ is } (m_k, E, K_k)\text{-singular and } a \notin (\Lambda_{7j_{h_i}+3)L_k/3}(y_{h_i}) \subset \Lambda_{L_{k+1}/3}(\boldsymbol{a}),$$

$$\text{where } h_i \in \{1, 2, \dots, r\} \text{ such that } \boldsymbol{b}_i \in \Lambda_{7j_{h_i}L_k/3}(y_{h_i})\}.$$

For $b_1' \in (\Lambda_{(7j_{h_1}+2)(L_k/3)-4}(y_{h_1}) \setminus \Lambda_{7j_{h_1}L_k/3-6}(y_{h_1})) \cap \mathbb{Z}^2$, $\Lambda_{L_k}(\boldsymbol{b}_1')$ is $(m_k, E, K_k)$-regular where $\boldsymbol{b}_1' \in (L_k/3)\mathbb{Z}^2$ such that $b_1' \in \Lambda_{L_k/3}(\boldsymbol{b}_1')$ if $L_k \geq 9$. $a \notin \Lambda_{L_k}(\boldsymbol{b}_1')$ since $a \notin (\Lambda_{7j_{h_1}+3)L_k/3}(y_{h_1})$. Thus by apply Proposition 2.1 with $\Lambda_{\ell'}(\boldsymbol{a}') = \Lambda_{L_{k+1}}(\boldsymbol{a})$, $\Lambda_\ell(\boldsymbol{a}) = \Lambda_{L_k}(\boldsymbol{b}_1')$ and $(a_*, a) = (a, b_1')$, we have

$$\begin{aligned}
&\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_1'}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\
\leq & c_7 M_k \sqrt{\log(L_k)} K_k \exp(-m_k|a-b|) \\
&+ c_8 \sqrt{|E|}(M_k \sqrt{\log L_k})^{c_9} K_k \\
&\times \sum_{b_2 \in \mathbb{Z}^2} \|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_2}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \exp\Big(-m_k \frac{L_k}{3} \\
&\qquad - c_* d(b_2, \Lambda_{L_k-4}(\boldsymbol{b}_1') \setminus \Lambda_{L_k-6}(\boldsymbol{b}_1'))\Big).
\end{aligned}$$

Thus under the condition

$$c_* \geq 6m_0,$$

we obtain

$$\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq e_s(b_1, a) + \sum_{b_2 \in \mathbb{Z}^2} w_s(b_1, b_2)\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_2}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}, \tag{5.5}$$

where

$$\begin{aligned}
e_s(b_1, a) :=& \frac{c_{10}}{\eta_k} \sqrt{|E|}(M_k \sqrt{\log L_k})^{c_{11}} K_k L_k^4 S \\
&\times \exp\Big(-m_k \frac{L_k}{3} - c_* d(a, \Lambda_{(7j_{h_1}+5)L_k/3-8}(y_{h_1}) \setminus \Lambda_{(7j_{h_1}-3)L_k/3-6}(y_{h_1}))\Big)
\end{aligned}$$

and

$$\begin{aligned}
w_s(b_1, b_2) :=& \frac{c_{12}}{\eta_k} |E|(M_k \sqrt{\log L_k})^{c_{13}} K_k L_k^4 S \exp\Big(-m_k \frac{L_k}{3} \\
&- \frac{c_*}{2} d(b_2, \Lambda_{(7j_{h_1}+1)(L_k/3)-4}(y_{h_1}) \setminus \Lambda_{(7j_{h_1}+1)(L_k/3)-6}(y_{h_1}))\Big).
\end{aligned}$$

If $b = b_1 \in S(f) := \mathbb{Z}^2 \setminus S(r) \setminus S(s)$, then we use the condition $R_1(E, L_{k+1}, \boldsymbol{a})$ and Lemma 5.1 below:

$$\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq e_f(b_1, a),$$

where

$$e_f(b_1, a) := \frac{c_{14}}{\eta_k} \exp(-\overline{c_*}(|a-b_1| \wedge d(a, \Lambda_{L_{k+1}}(\boldsymbol{a}) \wedge d(b_1, \Lambda_{L_{k+1}}(\boldsymbol{a})))$$

and $\overline{c_*}$ is the constant appeared in Lemma 5.1 below.

By the iteration, we have

$$
\begin{aligned}
&\|\chi_a(\widetilde{H^{\tilde{\xi}}_{L_{k+1},\boldsymbol{a}}} - E)^{-1}\chi_{b_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}\\
\le& e_{p_1}(b_1,a)\\
&+1_{\{r,s\}}(p_1)\sum_{n=2}^{N}\sum_{p_2,\dots,p_{n-1}\in\{r,s\},p_n\in\{r,s,f\}}\sum_{b_2\in S(p_2),\dots,b_n\in S(p_n)} w_{p_1}(b_1,b_2)\\
&\quad\times w_{p_2}(b_2,b_3)\cdots w_{p_{n-1}}(b_{n-1},b_n)e_{p_n}(b_n,a)\\
&+1_{\{r,s\}}(p_1)\sum_{p_2,\dots,p_N\in\{r,s\}}\sum_{b_2\in S(p_2),\dots,b_N\in S(p_N),b_{N+1}\in\mathbb{Z}^2} w_{p_1}(b_1,b_2)\\
&\quad\times w_{p_2}(b_2,b_3)\cdots w_{p_N}(b_N,b_{N+1})\frac{1}{\eta_N}
\end{aligned}
$$

for any $N\in\mathbb{N}$, when $b_1\in S(p_1)$.

We rewrite the each factor as

$$w_r(b_i,b_{i+1}) = \widetilde{w_r}(b_i,b_{i+1})\widehat{w_r}(b_i,b_{i+1}),$$

$$w_s(b_i,b_{i+1}) = \widetilde{w_s}(b_i,b_{i+1})\widehat{w_s}(b_i,b_{i+1}),$$

$$e_r(b_i,a)\le \overline{w_r}\widehat{w_r}(b_i,a),$$

and

$$e_s(b_i,a)\le \overline{w_s}\widehat{w_s}(b_i,a),$$

where

$$
\begin{aligned}
\widetilde{w_r}(b_i,b_{i+1}) :=& c_2\sqrt{|E|}(M_k\sqrt{\log L_k})^{c_3}K_kL_k^2\exp\Big(-(m_k-m_{k+1})\frac{L_k}{3}\\
&-\frac{c_*}{2}d(b_{i+1},\Lambda_{L_k-4}(\boldsymbol{b}_i)\setminus\Lambda_{L_k-6}(\boldsymbol{b}_i))\Big),\\
\widehat{w_r}(b_i,b_{i+1}) :=& \exp\Big(-m_{k+1}\frac{L_k}{3}-\frac{c_*}{2}d(b_{i+1},\Lambda_{L_k-4}(\boldsymbol{b}_i)\setminus\Lambda_{L_k-6}(\boldsymbol{b}_i))\Big),\\
\widetilde{w_s}(b_i,b_{i+1}) :=& \frac{c_{12}}{\eta_k}|E|(M_k\sqrt{\log L_k})^{c_{13}}K_kL_k^4S\exp\Big(-(m_k-m_{k+1})\frac{L_k}{3}\\
&-\frac{c_*}{4}d(b_{i+1},\Lambda_{(7j_{h_i}+1)(L_k/3)-4}(y_{h_i})\setminus\Lambda_{(7j_{h_i}+1)(L_k/3)-6}(y_{h_i}))\Big),\\
\widehat{w_s}(b_i,b_{i+1}) :=& \exp\Big(-m_{k+1}\frac{L_k}{3}-\frac{c_*}{4}d(b_{i+1},\Lambda_{(7j_{h_i}+1)(L_k/3)-4}(y_{h_i})\setminus\Lambda_{(7j_{h_i}+1)(L_k/3)-6}(y_{h_i}))\Big),\\
\overline{w_r} :=& c_1M_k\sqrt{\log L_k}K_k\exp\Big(-(m_k-m_{k+1})\frac{L_k}{3}\Big),
\end{aligned}
$$

and

$$\overline{w_s} := \frac{c_{10}}{\eta_k}\sqrt{|E|}(M_k\sqrt{\log L_k})^{c_{11}}K_kL_k^4S\exp\Big(-(m_k-m_{k+1}\frac{L_k}{3}\Big).$$

We now introduce the condition

$$\frac{c_{15}R}{\eta_k}|E|(M_k\sqrt{\log L_k})^{c_{16}}K_kL_k^4S^2 \le \exp\Big(\frac{m_0}{2^{k+2}}\frac{L_k}{3}\Big), \tag{5.6}$$

where $c_{15}$ and $c_{16}$ are sufficiently large positive finite constants, and $R$ is a positive integer specified later.

Then we have

$$\sum_{b_{i+1}\in\mathbb{Z}^2} \widetilde{w_r}(b_i,b_{i+1}) \le \frac{1}{R}$$

and

$$\sum_{b_{i+1}\in\mathbb{Z}^2} \widetilde{w_s}(b_i,b_{i+1}) \le \frac{1}{R}.$$

Thus, if $b_1 \in S(p_1)$ for $p_1 \in \{r,s\}$, then we have

$$\begin{aligned}
&\sum_{p_2,\dots,p_n\in\{r,s\}}\ \sum_{b_2\in S(p_2),\dots,b_n\in S(p_n)} w_{p_1}(b_1,b_2)\\
&\qquad \times w_{p_2}(b_2,b_3)\cdots w_{p_{n-1}}(b_{n-1},b_n)e_{p_n}(b_n,a)\\
\le&\Big(\frac{2}{R}\Big)^{n-1}\overline{w_{p_n}}\sup\{\widetilde{w_{p_1}}(b_1,b_2)\widetilde{w_{p_2}}(b_2,b_3)\cdots\widetilde{w_{p_n}}(b_n,a)\\
&\qquad : p_1,\dots,p_n\in\{r,s\}, b_1\in S(p_1),\dots,b_n\in S(p_n)\}\\
\le&\Big(\frac{2}{R}\Big)^{n-1}\overline{w_{p_n}}\exp\Big(-m_{k+1}(|a-b|_\infty\wedge d_\infty(b.\partial\Lambda_{L_{k+1}}(\boldsymbol{a}))-\frac{c_*}{4}d(a,\Lambda_{L_{k+1}}(\boldsymbol{a}))+c_{17}SL_k\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{p_2,\dots,p_{n-1}\in\{r,s\}}\ \sum_{b_2\in S(p_2),\dots,b_{n-1}\in S(p_{n-1}),b_n\in S(f)} w_{p_1}(b_1,b_2)\\
&\qquad \times w_{p_2}(b_2,b_3)\cdots w_{p_{n-1}}(b_{n-1},b_n)e_f(b_n,a)\\
\le&\Big(\frac{2}{R}\Big)^{n-1}\sup\{\widetilde{w_{p_1}}(b_1,b_2)\widetilde{w_{p_2}}(b_2,b_3)\cdots\widetilde{w_{p_{n-1}}}(b_{n-1},b_n)e_f(b_n,a)\\
&\qquad : p_1,\dots,p_n\in\{r,s\}, b_1\in S(p_1),\dots,b_{n-1}\in S(p_{n-1}),b_n\in S(f)\}\\
\le&\Big(\frac{2}{R}\Big)^{n-1}\frac{c_{18}}{\eta_k}\exp\Big(-m_{k+1}(|a-b|_\infty\wedge d_\infty(b.\partial\Lambda_{L_{k+1}}(\boldsymbol{a}))-\frac{c_*}{4}d(a,\Lambda_{L_{k+1}}(\boldsymbol{a}))+c_{19}SL_k\Big)
\end{aligned}$$

for any $n\ge 2$. We also have

$$e_{p_1}(b_1,a)\le\overline{w_{p_1}}\exp\Big(-m_{k+1}(|a-b|_\infty\wedge d_\infty(b.\partial\Lambda_{L_{k+1}}(\boldsymbol{a}))-\frac{c_*}{4}d(a,\Lambda_{L_{k+1}}(\boldsymbol{a}))+c_{20}SL_k\Big)$$

for $p_1\in\{r,s\}$, and

$$e_f(b_1,a)\le\frac{c_{21}}{\eta_k}\exp\Big(-m_{k+1}(|a-b|_\infty\wedge d_\infty(b.\partial\Lambda_{L_{k+1}}(\boldsymbol{a}))-\frac{c_*}{4}d(a,\Lambda_{L_{k+1}}(\boldsymbol{a}))+c_{22}SL_k\Big).$$

By these, we have

$$
\begin{aligned}
&\|\chi_a(\widetilde{H^{\widetilde{\xi}}_{L_{k+1},\boldsymbol{a}}}-E)^{-1}\chi_{b_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}\\
\le &K_{k+1}\exp\Big(-m_{k+1}(|a-b|_\infty\wedge d_\infty(b.\partial\Lambda_{L_{k+1}}(\boldsymbol{a}))-\frac{c_*}{4}d(a,\Lambda_{L_{k+1}}(\boldsymbol{a}))\Big)
\end{aligned}
$$

for $R>2$, where

$$
K_{k+1}:=\Big(\frac{c_{23}}{\eta_k}+\frac{2}{1-2/R}\Big)\exp(c_{24}SL_k).
$$

In this setting, we have

$$
\begin{aligned}
&R(m_{k+1},L_{k+1},[E_1,E_0],\boldsymbol{a},\boldsymbol{a}')\\
\supset&\bigcap_{E\in[E_1,E_0]}\Big(\Big(\bigcap_{j_{\boldsymbol{a}}=1}^{3}R_{j_{\boldsymbol{a}}}(E,L_{k+1},\boldsymbol{a})\cap\bigcap_{j_{\boldsymbol{a}}=4}^{7}R_{j_{\boldsymbol{a}}}(L_{k+1},\boldsymbol{a})\Big)\cup\Big(\bigcap_{j_{\boldsymbol{a}'}=1}^{3}R_{j_{\boldsymbol{a}'}}(E,L_{k+1},\boldsymbol{a}')\cap\bigcap_{j_{\boldsymbol{a}'}=4}^{7}R_{j_{\boldsymbol{a}'}}(L_{k+1},\boldsymbol{a}')\Big)\Big)
\end{aligned}
$$

for $\boldsymbol{a},\boldsymbol{a}'\in\mathbb{Z}^2$.

We next consider the probability:

$$
\begin{aligned}
&\mathbb{P}(R(m_{k+1},L_{k+1},[E_1,E_0],\boldsymbol{a},\boldsymbol{a}'))\\
\ge&1-\mathbb{P}\Big(\bigcup_{E\in[E_1,E_0]}\Big(\Big(\bigcup_{j_{\boldsymbol{a}}=1}^{2}R_{j_{\boldsymbol{a}}}(E,L_{k+1},\boldsymbol{a})^c\Big)\cap\Big(\bigcup_{j_{\boldsymbol{a}'}=1}^{2}R_{j_{\boldsymbol{a}'}}(E,L_{k+1},\boldsymbol{a}')^c\Big)\Big)\\
&-\sum_{\boldsymbol{a}''\in\{\boldsymbol{a},\boldsymbol{a}'\}}\mathbb{P}\Big(\bigcup_{E\in[E_1,E_0]}R_3(E,L_{k+1},\boldsymbol{a}'')^c\Big)-\sum_{j=4}^{7}\sum_{\boldsymbol{a}''\in\{\boldsymbol{a},\boldsymbol{a}'\}}\mathbb{P}\Big(R_j(L_{k+1},\boldsymbol{a}'')^c\Big)
\end{aligned}
\tag{5.7}
$$

We assume $|\boldsymbol{a}-\boldsymbol{a}'|_\infty>L_{k+1}+2$. The second term of (5.7) is estimated as

$$
\begin{aligned}
&\mathbb{P}\Big(\bigcup_{E\in[E_1,E_0]}\Big(\Big(\bigcup_{j_{\boldsymbol{a}}=1}^{2}R_{j_{\boldsymbol{a}}}(E,L_{k+1},\boldsymbol{a})^c\Big)\cap\Big(\bigcup_{j_{\boldsymbol{a}'}=1}^{2}R_{j_{\boldsymbol{a}'}}(E,L_{k+1},\boldsymbol{a}')^c\Big)\Big)\\
\le&\sum_{(\ell_1,\boldsymbol{a}_1)\in S(L_{k+1},\boldsymbol{a})}\ \sum_{(\ell_2,\boldsymbol{a}_2)\in S(L_{k+1},\boldsymbol{a}')}\mathbb{E}[\mathbb{P}(\{\exists E\in[E_1,E_0]\text{ s.t. }:d(\mathrm{spec}\,\widetilde{H^{\xi.\bar{\xi}}_{\ell_1,\boldsymbol{a}_1}},E)\le\eta_k\\
&\text{and }d(\mathrm{spec}\,\widetilde{H^{\xi.\bar{\xi}}_{\ell_2,\boldsymbol{a}_2}},E)\le\eta_k|\mathcal{F}(\Lambda_{\ell_1+1}(\boldsymbol{a}_1)))],
\end{aligned}
$$

where $\mathcal{F}(\Lambda_{\ell_1+1}(\boldsymbol{a}_1)$ is the $\sigma$-field generated by $(\xi.\bar{\xi})$ on $\Lambda_{\ell_1+1}(\boldsymbol{a}_1)$, and

$$
\begin{aligned}
S(L_{k+1},\boldsymbol{a}):=&\{(L_{k+1},\boldsymbol{a})\}\\
&\cup\{((7j+1)L_k/3,\boldsymbol{a}''):j\in\{1,2,\dots,S\}\boldsymbol{a}''\in\Lambda_{L_{k+1}}(\boldsymbol{a})\cap((L_k/6)\mathbb{Z}^2),\\
&\Lambda_{(7j+4)L_k/3}(\boldsymbol{a}'')\subset\Lambda_{L_{k+1}-3}(\boldsymbol{a})\}.
\end{aligned}
$$

By Proposition 4.1, we have

$$
\begin{aligned}
&\mathbb{P}(\{\exists E \in [E_1, E_0] \text{ s.t. } : d(\operatorname{spec} \widetilde{H^{\xi\cdot\bar{\xi}}_{\ell_1,\boldsymbol{a}_1}}, E) \leq \eta_k \\
&\quad \text{and } d(\operatorname{spec} \widetilde{H^{\xi\cdot\bar{\xi}}_{\ell_2,\boldsymbol{a}_2}}, E) \leq \eta_k | \mathcal{F}(\Lambda_{\ell_1+1}(\boldsymbol{a}_1)))] \\
&\leq \sum_{\mu \in \operatorname{spec} \widetilde{H^{\xi\cdot\bar{\xi}}_{\ell_1,\boldsymbol{a}_1}} \text{ s.t. } d(\mu,[E_1,E_0]) \leq \eta_k} \mathbb{P}(d(\operatorname{spec} \widetilde{H^{\xi\cdot\bar{\xi}}_{\ell_2}}, E) \leq 2\eta_k) \\
&\leq c_{25}\eta_k \ell_2^{c_{26}} \operatorname{Tr}[E([E_1 - \eta_k, E_0 + \eta_k] : \widetilde{H^{\xi\cdot\bar{\xi}}_{\ell_1,\boldsymbol{a}_1}})].
\end{aligned}
$$

By using also Proposition 3.1, we have

$$
\begin{aligned}
&\mathbb{P}\Big(\bigcup_{E\in[E_1,E_0]} \Big(\Big(\bigcup_{j_{\boldsymbol{a}}=1}^{2} R_{j_{\boldsymbol{a}}}(E, L_{k+1}, \boldsymbol{a})^c\Big) \cap \Big(\bigcup_{j_y=1}^{2} R_{j_y}(E, L_{k+1}, y)^c\Big)\Big) \\
&\leq c_{27} S^2 (L_{k+1}/L_k)^4 \eta_k L_{k+1}^{c_{28}},
\end{aligned}
$$

The third term of (5.7) is estimated as

$$
\begin{aligned}
&\mathbb{P}\Big(\bigcup_{E\in[E_1,E_0]} R_3(E, L_{k+1}, \mathbf{0})^c\Big) \\
&\leq \mathbb{P}\Big(\text{There exist } E \in [E_1, E_0] \text{ and } \Lambda_{L_k}(\boldsymbol{a}_1), \ldots, \Lambda_{L_k}(\boldsymbol{a}_{S+1}) \in C(L_{k+1}, L_k, \mathbf{0}) \text{ such that} \\
&\quad \Lambda_{L_k}(\boldsymbol{a}_1), \ldots, \Lambda_{L_k}(\boldsymbol{a}_{S+1}) \text{ are } (m_k, E, K_k)\text{-singular} \\
&\quad \text{and } d_\infty(\Lambda_{L_k}(\boldsymbol{a}_a), \Lambda_{L_k}(\boldsymbol{a}_b)) > 2 \text{ for } a \neq b\Big) \\
&\leq \mathbb{P}\Big(\text{There exist } E \in [E_1, E_0] \text{ and } \Lambda_{L_k}(\boldsymbol{a}_1), \Lambda_{L_k}(\boldsymbol{a}_2) \in C(L_{k+1}, L_k, \mathbf{0}) \text{ such that} \\
&\quad \Lambda_{L_k}(\boldsymbol{a}_1), \Lambda_{L_k}(\boldsymbol{a}_2) \text{ are } (m_k, E, K_k)\text{-singular and } d_\infty(\Lambda_{L_k}(\boldsymbol{a}_1), \Lambda_{L_k}(\boldsymbol{a}_2)) > 2\Big)^{(S+1)/2} \\
&\leq \Big(\frac{3L_{k+1}}{L_k}\Big)^{2(S+1)} \frac{1}{L_k^{p(S+1)/2}}.
\end{aligned}
$$

The fourth term of (5.7) is estimated as follows:

$$
\begin{aligned}
&\mathbb{P}\Big(R_4(L_{k+1}, \mathbf{0})^c\Big) \\
&\leq \sum_{a\in\mathbb{Z}^2\cap\Lambda_{L_{k+1}}} \mathbb{P}\Big(\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \geq M_k (\log(2 + |a|))^{1/2}\Big) \\
&\leq \sum_{a\in\mathbb{Z}^2\cap\Lambda_{L_{k+1}}} \frac{\mathbb{E}\Big[\exp\Big(c_{29}\|\chi_a\xi\|^2_{C^{-1-\epsilon}(\mathbb{R}^2)}\Big)\Big]}{\exp(c_{29} M_k^2 \log(2 + |a|))} \\
&\leq \frac{c_{31}}{2^{M_k}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\Big(R_5(L_{k+1},\mathbf{0})^c\Big)\\
\leq&\sum_{a\in\mathbb{Z}^2}\sum_{\boldsymbol{a}'\in\Lambda_{L_{k+1}}\cap\left(\frac{L_k}{3}\mathbb{Z}^2\right)}\mathbb{P}\Big(\|\chi_a Y^{\xi.\bar{\xi}}_{L_k,\boldsymbol{a}'}\|_{C^{-\epsilon}(\mathbb{R}^2)}\geq M_k(\log(2+|a|))\exp(-\widetilde{c_*}d(a,\Lambda_{L_k}(\boldsymbol{a}')))\Big)\\
\leq&\sum_{a\in\mathbb{Z}^2}\sum_{\boldsymbol{a}'\in\Lambda_{L_{k+1}}\cap\left(\frac{L_k}{3}\mathbb{Z}^2\right)}\frac{\mathbb{E}\Big[\exp\Big(c_{30}\|\chi_a Y^{\xi.\bar{\xi}}_{L_k,\boldsymbol{a}'}\|_{C^{-\epsilon}(\mathbb{R}^2)}\exp(\widetilde{c_*}d(a,\Lambda_{L_k}(\boldsymbol{a}')))\Big)\Big]}{\exp(c_{30}M_k\log(2+|a|)\Big)}\\
\leq&\frac{c_{31}}{2^{M_k}}\Big(\frac{L_{k+1}}{L_k}\Big)^2.
\end{aligned}
$$

Simirarly we have

$$
\mathbb{P}\Big(R_6(L_{k+1},\mathbf{0})^c\Big)\leq S\frac{c_{31}}{2^{M_k}}\Big(\frac{L_{k+1}}{L_k}\Big)^2
$$

and

$$
\mathbb{P}\Big(R_7(L_{k+1},\mathbf{0})^c\Big)\leq S\frac{c_{31}}{2^{M_k}}\Big(\frac{L_{k+1}}{L_k}\Big)^2.
$$

Thus, a sufficient condition for

$$
\mathbb{P}(R(m_{k+1},K_{k+1},[E_1,E_0],L_{k+1},\boldsymbol{a},\boldsymbol{a}'))\geq 1-L_{k+1}^{-p}
$$

is

$$
\begin{aligned}
c_{27}S^2(L_{k+1}/L_k)^4\eta_k L_{k+1}^{c_{28}}\leq&\frac{1}{3L_{k+1}^p}\\
\Big(\frac{3L_{k+1}}{L_k}\Big)^{2(S+1)}\frac{1}{L_k^{p(S+1)/2}}\leq&\frac{1}{6L_{k+1}^p}
\end{aligned}
$$

and

$$
S\frac{c_{31}}{2^{c_{32}M_k}}\Big(\frac{L_{k+1}}{L_k}\Big)^2\leq\frac{1}{6L_{k+1}^p}.
$$

These conditions and the condition in (5.6) are satisfied if

$$
\begin{aligned}
S=&\min\Big\{\Big[\frac{2p\alpha}{p-4(\alpha-1)},\infty\Big)\cap(2\mathbb{Z}_++1)\Big\},\\
\eta_k=&\frac{c_{33}}{L_{k+1}^{p+c_{34}-1/\alpha}S^4},\\
M_k=&(p+2)\alpha\log L_k,
\end{aligned}
$$

and $L_0$ is sufficiently large so that (5.1) is satisfied. □

The following lemma used in the above proof is based on Lemma 4.4:

**Lemma 5.1.** *There exists* $\overline{c_*}, c_1, c_2 \in (0,\infty)$ *such that*

$$\|\chi_{a_*}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - E)^{-1}\chi_a\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}$$

$$\leq c_1 \exp(-\overline{c_*}(|a_* - a| \wedge d(a_*, \Lambda_\ell(\boldsymbol{a}))$$

*for any* $E < -c_2$*,*

**Proof**. We may assume $|a_* - a| \wedge d(a_*, \Lambda_\ell(\boldsymbol{a}) \geq 3$. We take a $[0,1]$-valued smooth function $\phi$ on $\mathbb{R}^2$ so that

$$\phi = \begin{cases} 1 & \text{on } B(a_*, d(a_*, \Lambda_\ell(\boldsymbol{a})) - 2), \\ 0 & \text{on } \mathbb{R}^2 \setminus B(a_*, d(a_*, \Lambda_\ell(\boldsymbol{a})) - 1) \end{cases}$$

if $a \in \Lambda_\ell(\boldsymbol{a})$, and

$$\phi = \begin{cases} 1 & \text{on } B(a_*, |a_* - a| - 2), \\ 0 & \text{on } \mathbb{R}^2 \setminus B(a_*, |a_* - a| - 1) \end{cases}$$

if $a \notin \Lambda_\ell(\boldsymbol{a})$, where $B(a,r) := \{x \in \mathbb{R}^2 : |x-a| < r\}$ for any $a \in \mathbb{R}^2$ and $r > 0$. Then , since

$$\begin{aligned}&(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - E)^{-1}\phi - \phi(-\Delta - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell-2,\boldsymbol{a}} - E)^{-1} \\ =&(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - E)^{-1}\{2(\nabla\phi)\cdot\nabla + (\Delta\phi))\}(-\Delta - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell-2,\boldsymbol{a}} - E)^{-1},\end{aligned}$$

we have

$$\begin{aligned}&\|\chi_a(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}} - E)^{-1}\chi_{a_*}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\ \leq& \frac{c}{d(E, \operatorname{spec}(\widetilde{H^{\tilde{\xi}}_{\ell,\boldsymbol{a}}}))}\Big(\sum_{a_1\in\mathbb{Z}^2\cap\operatorname{supp}(\nabla\phi)} \|\chi_{a_1}\nabla(-\Delta - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell-2,\boldsymbol{a}})] - E)^{-1}\chi_{a_*}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\ &+ \sum_{a_1\in\mathbb{Z}^2\cap\operatorname{supp}(\nabla\phi)} \|\chi_{a_1}(-\Delta - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{\ell-2,\boldsymbol{a}})] - E)^{-1}\chi_{a_*}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}\Big).\end{aligned}$$

Then the rest of the proof is simirar with that of the Combes-Thomas type estimate below. □

To obtain the initial estimate (5.2), we use the following estimate:

**Lemma 5.2** (Combes-Thomas type estimate (cf. [5]))**.** *There exists* $c \in (0,\infty)$ *such that*

$$\begin{aligned}&\|\chi_a(\widetilde{H^{\tilde{\xi}}_L} - E)^{-1}\chi_b\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\ \leq& \frac{3}{d(E, \operatorname{spec}\widetilde{H^{\tilde{\xi}}_L})} \\ &\times \exp\Big(\frac{-(|a-b| - 2\sqrt{2})_+ d(E, \operatorname{spec}\widetilde{H^{\tilde{\xi}}_L})}{2\sqrt{d(E, \operatorname{spec}\widetilde{H^{\tilde{\xi}}_L}) + c(1 + \sup_{a\in\Lambda_{L-2}\cap\mathbb{Z}^2}\|\chi_a\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{a\in\mathbb{Z}^2}\|\chi_a Y_{\xi,L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)})}}\Big),\end{aligned} \tag{5.8}$$

*for any* $a, b \in \mathbb{Z}^2$*,* $L \in \mathbb{N}$ *and* $E < \inf \operatorname{spec} \widetilde{H^{\widetilde{\xi}}_L}$*,*

To prove this lemma, we modify the proof of Lemma 4.11 in [21] by the following

**Lemma 5.3.** *There exists* $c \in (0, \infty)$ *such that*

$$\begin{aligned}&\|(\widetilde{H^{\widetilde{\xi}}_L} - |v|^2 + i)^{-1/2} v \cdot \nabla (\widetilde{H^{\widetilde{\xi}}_L} - |v|^2 + i)^{-1/2}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \\ \le & c|v| \frac{1 + \sup_{a \in \Lambda_{L-2} \cap \mathbb{Z}^2} \|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^1)} + \sup_{a \in \mathbb{Z}^2} \|\chi_a Y_{\xi, L-2}\|_{C^{-\epsilon}(\mathbb{R}^1)}}{d(E, \operatorname{spec} \widetilde{H^{\widetilde{\xi}}_L}) - |v|^2}\end{aligned}$$

*for any* $L \in \mathbb{N}$*,* $E < \inf \operatorname{spec} \widetilde{H^{\widetilde{\xi}}_L}$ *and* $v \in \mathbb{R}^2$ *such that* $|v|^2 < d(E, \operatorname{spec} \widetilde{H^{\widetilde{\xi}}_L})$*.*

This lemma is proven by modifying the proof of Lemma 4.12 in [21] under the restriction tha $E < \inf \operatorname{spec} \widetilde{H^{\xi}_L}$.

We finally consider the initial estimate:

**Lemma 5.4.** *For any* $1 \le p < \infty$*,* $1 < \alpha < 1 + p/4$ *and* $m_0 > 0$*, there exists* $\overline{E_0} \in (-\infty, 0)$ *satisfying the following: for any* $E_0 \le \overline{E_0}$*, there exist* $E_1 \in (-\infty, E_0)$ *and* $L_0 \in 6\mathbb{N}$ *such that* (5.1) *and* (5.2) *are satisfied.*

**Proof**. As in the proof of Lemma 3.3 (ii) and Lemma 4,3 (i), we have

$$\sup_{a \in \mathbb{Z}^2} \mathbb{E}[\exp(h_1 \|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)})] < \infty$$

and

$$\sup_{a \in \mathbb{Z}^2} \mathbb{E}[\exp(h_2 \|\chi_a Y_{\xi, L_0 - 2, \mathbf{0}}\|_{C^{-\epsilon}(\mathbb{R}^2)} \exp(\widetilde{c_*} d(a, \Lambda_{L_0}))] < \infty$$

for some $h_1, h_2, \widetilde{c_*} \in (0, \infty)$. From these, we have

$$\mathbb{P}(\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \ge \Xi_1 \sqrt{\log(2 + |a|)} \text{ for some } a \in \mathbb{Z}^2) \le 1/(2L_0^{p/2}),$$

$$\mathbb{P}(\|\chi_a Y_{\xi, L_0 - 2, \mathbf{0}}\|_{C^{-\epsilon}(\mathbb{R}^2)} \ge \Xi_2 \log(2 + |a|) \exp(-\widetilde{c_*} d(a, \Lambda_{L_0})) \text{ for some } a \in \mathbb{Z}^2) \le 1/(2L_0^{p/2})$$

and

$$\begin{aligned}\mathbb{P}(&\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \le \Xi_1 \sqrt{\log(2 + |a|)} \text{ and} \\ &\|\chi_a Y_{\xi, L_0 - 2, \mathbf{0}}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le \Xi_2 \log(2 + |a|) \exp(-\widetilde{c_*} d(a, \Lambda_{L_0})) \text{ for any } a \in \mathbb{Z}^2) \ge 1 - 1/L_0^{p/2}\end{aligned}$$

for some $\Xi_1, \Xi_2 \in (0, \infty)$. Under the event in this probability, we have

$$\inf \operatorname{spec} \widetilde{H^{\widetilde{\xi}}_{L_0, \mathbf{0}}} \ge -\Xi_3 (\log L_0)^{\Xi_4}$$

for some $\Xi_3, \Xi_4 \in (0,\infty)$, by the proof of Lemma 4.9 in [21]. We now take $E_0$ as

$$E_0 < -\Xi_3(\log L_0)^{\Xi_4} - 2m_0^2 - m_0\sqrt{m_0^2 + c(\Xi_1\sqrt{\log(2+L_0)} + \Xi_2\log(2+L_0))}. \tag{5.9}$$

and

$$K_0 \geq \frac{3}{-E_0}\exp\Big(\frac{\sqrt{2}(-E_1)}{\sqrt{c(\Xi_1\sqrt{\log(2+L_0)} + \Xi_2\log(2+L_0)) - E_1 - \Xi_3(\log L_0)^{\Xi_4}}}\Big),$$

where $c$ is the constant appeared in Lemma 5.2. Then, by Lemma 5.2, we see that $\Lambda_{L_0}$ is $(m_0, E, K_0)$-regular for any $E \in [E_1, E_0]$. Thus we have (5.2) by the stationarity and the spatially independence of the white noise. For the function

$$f(\ell) := \ell - \log\{\Xi_3(\ell)^{\Xi_4} + 2m_0^2 + m_0\sqrt{m_0^2 + c(\Xi_1\sqrt{\log(2+\ell)} + \Xi_2\log(2+\ell))}$$

of $\ell > 0$, there exists $\ell_0 \in (0,\infty)$ such that $f(\ell) > 0$ for any $\ell > \ell_0$, since $f(\ell) \to \infty$ as $\ell \to \infty$. Now for any

$$\begin{aligned} E_0 <& -\Xi_3\Big(\log\Big(\ell_0 \vee \frac{c_1}{m_0^{\alpha/(\alpha-1)}}\Big)\Big)^{\Xi_4} - 2m_0^2 \\ & - m_0\Big\{m_0^2 + c\Big(\Xi_1\Big(\log\Big(2+\ell_0 \vee \frac{c_1}{m_0^{\alpha/(\alpha-1)}}\Big)\Big)^{1/2} + \Xi_2\log\Big(2+\ell_0\vee\frac{c_1}{m_0^{\alpha/(\alpha-1)}}\Big)\Big)\Big\}^{1/2}, \end{aligned}$$

and some $E_1 \leq E_0$, (5.9) and (5.1) are satisfied by

$$L_0 \geq \frac{c_1}{m_0^{\alpha/(\alpha-1)}} \vee (\log|E_1|)^2 \vee \ell_0.$$

□

## 6. Generalized eigenfunction expansions

Let $T$ be the operator of multiplication with the function $(1+|x|^2)^\nu$, where $\nu$ is a fixed number greater than 1/2. Let $\mathcal{H}_\pm$ be the weighted spaces defined by

$$\mathcal{H}_\pm := L^2(\mathbb{R}^2, (1+|x|^2)^{\pm 2\nu}dx).$$

$\psi \in \mathcal{H}_-$ is called a generalized eigenfunction of $\widetilde{H^\xi}$ with generalized eigenvalue $E$ if

$$\int (\widetilde{H^\xi\varphi})\psi dx = E\int \varphi\psi dx$$

for any $\varphi \in \mathrm{Dom}_{+0}(\widetilde{H^\xi})$. This condition is meaningful since $\mathrm{Dom}_{+0}(\widetilde{H^\xi}) \subset \mathcal{H}_+$ and $\widetilde{H^\xi}(\mathrm{Dom}_{+0}(\widetilde{H^\xi})) \subset \mathcal{H}_+$. For the generalized eigenfunctions, we show the following, which is called an *eigenfunction decay inequality* in [7]:

**Proposition 6.1.** *There exist finite positive constants $c_1, c_2, c_3$ such that, for any $\boldsymbol{a} \in \mathbb{Z}^2$, $L \in 2\mathbb{N}$ and any generalized eigenfunction $\psi$ of $\widetilde{H^\xi}$ with generalized eigenvalue $E \in \mathbb{R}$, it holds that*

$$
\begin{aligned}
&\|\chi_{\boldsymbol{a}}\psi\|_{L^2(\mathbb{R}^2)} \\
\le& c_3 \sum_{a_1 \in \mathbb{Z}^2} \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)} (1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^{3/2} \Xi_c(a_1, \boldsymbol{a}, \ell, \xi)^{1/2} (1 \vee \Xi_c(a_1, \boldsymbol{a}, \ell, \xi))^{1/2} \exp(-c_1|a_1 - \boldsymbol{a}|) \\
&+ c_3 \sum_{a_1 \in \mathbb{Z}^2} \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)} (1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^2 \exp(-c_1|a_1 - a_2| - c_1 d(a_1, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
&+ c_3 \sum_{a_1, a_2 \in \mathbb{Z}^2} \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)} \sum_{a_3 \in \mathbb{Z}^2 \cap \overline{\Lambda_2(\boldsymbol{a})}} \|\chi_{a_2} (\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1} \chi_{a_3}^2\|_{\mathcal{L}(L^2(\mathbb{R}^2))} (\log L)^{c_2} \\
&\quad \times (1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^2 (1 \vee \Xi_c(a_2, \boldsymbol{a}, \ell, \xi))^{3/2} \Xi_c(a_2, \boldsymbol{a}, \ell, \xi)^{1/2} (1 \vee \Xi_c(a_2, \xi))^{1/2} \\
&\quad \times (\max|\overline{\xi_0}| + |E| + \Xi(\boldsymbol{a}, \ell, \xi)) \exp(-c_1|a_1 - a_2|) \\
&+ c_3 \sum_{a_1, a_2 \in \mathbb{Z}^2} \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)} \sum_{a_3 \in \mathbb{Z}^2 \cap \overline{\Lambda_2(\boldsymbol{a})}} \|\chi_{a_2} (\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1} \chi_{a_3}^2\|_{\mathcal{L}(L^2(\mathbb{R}^2))} (\log L)^{c_2} \\
&\quad \times (1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^{5/2} (1 \vee \Xi_c(a_2, \boldsymbol{a}, \ell, \xi)) (\max|\overline{\xi_0}| + |E| + \Xi(\boldsymbol{a}, \ell, \xi)) \\
&\quad \times \exp(-c_1|a_1 - a_2| - c_1 d(a_1, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))),
\end{aligned}
\tag{6.1}
$$

*where $\ell = L - 2$, $\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}$ is the operator defined in* (2.1)*, $\Xi(\boldsymbol{a}, \ell, \xi)$ is the quantity defined in Proposition 2.1,*

$$
\begin{aligned}
\Xi_c(a_1, \boldsymbol{a}, \ell, \xi) :=& \sum_{a_2 \in \mathbb{Z}^2 \setminus \Lambda_\ell(\boldsymbol{a})} \|\chi_{a_2}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|) \\
&+ \sum_{(a_2, a_3) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \setminus \Lambda_\ell(\boldsymbol{a})^2} \prod_{j=2}^{3} \|\chi_{a_j}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_j|) \\
&+ \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 (Y_\xi - Y_{\xi,\ell,\boldsymbol{a}})\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|),
\end{aligned}
$$

*and*

$$
\begin{aligned}
\Xi_c(a_1, \xi) :=& \sum_{j=1}^{2} \Big( \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|) \Big)^j \\
&+ \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 Y_\xi\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|)
\end{aligned}
$$

*for any $a_1, \boldsymbol{a} \in \mathbb{Z}^2$ and $\ell > 0$.*

**Proof**. We set $v := \chi_{\Lambda_L(\boldsymbol{a})}\psi$ and $u := (\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1} \chi_{\boldsymbol{a}}^2 \psi$. We take a $[0,1]$-valued smooth function $\phi$ on $\mathbb{R}^2$ so that $\phi \equiv 1$ on $\Lambda_{L/3}(\boldsymbol{a})$ and $\phi \equiv 0$ on $\mathbb{R}^2 \setminus \Lambda_{2L/3}(\boldsymbol{a})$. Since $u \in \mathrm{Dom}(\widetilde{H^{\xi}_{\ell,\boldsymbol{a}}})$ for $\ell = L - 2$, we have

$\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u) \in \mathcal{H}^2(\mathbb{R}^2)$ and $(\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u) \in \mathrm{Dom}_{+0}(\widetilde{H^{\xi}})$ by Lemma 4.1 and Lemma 4.2 in [21], where $\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a}) = (s(a;\epsilon,\xi,\delta,\boldsymbol{a}), s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a}), s_2(a;\epsilon,\xi,\delta,\boldsymbol{a}))_{a,a'\in\mathbb{Z}^2}$ is the numbers defined by modifying the definition of the numbers $\boldsymbol{s}(\epsilon,\xi,\delta)$ in Lemma 4.1 in [21] as follows:

$$\begin{aligned} s(a;\epsilon,\xi,\delta,\boldsymbol{a}) &= s(\epsilon,\xi(\cdot+\boldsymbol{a}))\Big(\frac{\delta}{(\log(2+|a-\boldsymbol{a}|))^{1/2}}\Big)^{M(\epsilon)}, \\ s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a}) &= s_1(\epsilon,\xi(\cdot+\boldsymbol{a}))\Big(\frac{\delta}{(\log(2+|a-\boldsymbol{a}|))^{1/2}(\log(2+|a'-\boldsymbol{a}|))^{1/2}}\Big)^{M_1(\epsilon)} \end{aligned}$$

and

$$s_2(a;\epsilon,\xi,\delta,\boldsymbol{a}) = s_2(\epsilon,\xi(\cdot+\boldsymbol{a}))\Big(\frac{\delta}{\log(2+|a-\boldsymbol{a}|)}\Big)^{M_2(\epsilon)}.$$

Then we have

$$\begin{aligned} &\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)}^2 \\ =&(\psi,\phi(\widetilde{H_{L,\boldsymbol{a}}^{\tilde{\xi}}}-E)u) \\ =&I+EJ, \end{aligned}$$

where

$$I = (\psi, \phi\widetilde{H_{L,\boldsymbol{a}}^{\tilde{\xi}}}u - \widetilde{H^{\tilde{\xi}}}(\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)}$$

and

$$J = (\psi, (\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u) - \phi u)_{L^2(\mathbb{R}^2)}.$$

We devide as $I=\sum_{j=1}^{7} I_j$, where

$$I_1 =(\psi, -\phi\Delta\Phi_{\xi,\ell,\boldsymbol{a}}(u)+\Delta\Phi_\xi((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)},$$

$$I_2 =(\psi, \phi P_{\xi_{\ell,\boldsymbol{a}}}\Phi_{\xi,\ell,\boldsymbol{a}}(u)-P_\xi\Phi_\xi((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)}$$
$$+(\psi,\phi\Pi(\xi_{\ell,\boldsymbol{a}},\Phi_{\xi,\ell,\boldsymbol{a}}u)-\Pi(\xi,\Phi_\xi((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)},$$

$$I_3 =(\psi,\phi P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_{\ell,\boldsymbol{a}}))-P_1^{(b)}(P_1^{(b)}(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))(P_1^{(b)}\xi)))_{L^2(\mathbb{R}^2)}$$

$$I_4 =(\psi,\phi e^{\Delta}(P_u\xi_{\ell,\boldsymbol{a}}+_u P_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell,\boldsymbol{a}}+P_uY_{\xi.\ell,\boldsymbol{a}})$$
$$-e^{\Delta}(P_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\xi+_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P_\xi\Delta^{-loc}\xi+P_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}Y_\xi)_{L^2(\mathbb{R}^2)},$$

$$I_5 =(\psi,\phi C(u,\xi_{\ell,\boldsymbol{a}},\xi_{\ell,\boldsymbol{a}})-C((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u),\xi,\xi))_{L^2(\mathbb{R}^2)}$$
$$+(\psi,\phi S(u,\xi_{\ell,\boldsymbol{a}},\xi_{\ell,\boldsymbol{a}})-S((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u),\xi,\xi))_{L^2(\mathbb{R}^2)},$$

$$I_6 =(\psi,\phi P_{Y_{\xi,\ell,\boldsymbol{a}}}u-P_{Y_\xi}((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)}$$
$$+(\psi,\phi\Pi(Y_{\xi,\ell,\boldsymbol{a}},u)-\Pi(Y_\xi,(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)}$$
$$+(\psi,\phi(P_1^{(b)}((P_1^{(b)}Y_{\xi,\ell,\boldsymbol{a}})(P_1^{(b)}u))-(P_1^{(b)}((P_1^{(b)}Y_\xi)(P_1^{(b)}(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)},$$

and

$$I_7 =(\psi,\phi P_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}({}_uP_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell,\boldsymbol{a}}+P_uY_{\xi,\ell,\boldsymbol{a}})$$
$$-P_\xi\Delta^{-loc}({}_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P_\xi\Delta^{-loc}\xi+P_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}Y_\xi))_{L^2(\mathbb{R}^2)}$$
$$+(\psi,\phi\Pi(\xi_{\ell,\boldsymbol{a}},\Delta^{-loc}({}_uP_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell,\boldsymbol{a}}+P_uY_{\xi,\ell,\boldsymbol{a}}))$$
$$-\Pi(\xi,\Delta^{-loc}({}_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P_\xi\Delta^{-loc}\xi+P_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}Y_\xi)))_{L^2(\mathbb{R}^2)}.$$

Since

$$-\phi\Delta\Phi_{\xi,\ell,\boldsymbol{a}}(u)+\Delta\Phi_\xi((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)$$
$$=([\Delta,\phi]\Phi_{\xi,\ell,\boldsymbol{a}}(u)-\Delta\phi\Phi_{\xi,\ell,\boldsymbol{a}}(u))+(\Delta\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)+\Delta(\Phi_\xi-\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))$$
$$=[\Delta,\phi]\Phi_{\xi,\ell,\boldsymbol{a}}(u)-\Delta\phi(\Phi_{\xi,\ell,\boldsymbol{a}}-\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(u)+\Delta(\Phi_{\xi,\ell,\boldsymbol{a}}-\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))$$
$$+\Delta(\Phi_\xi-\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}-\Phi_{\xi,\ell,\boldsymbol{a}}+\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)$$

and

$$[\Delta,\phi]=2(\nabla\phi)\cdot\nabla+(\Delta\phi),$$

$I_1$ is divided as

$$2I_{1,1}+\sum_{j=2}^{5} I_{1,j},$$

where

$$I_{1,1} =(\psi,(\nabla\phi)\nabla\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)},$$

$$I_{1,2} =(\psi,(\Delta\phi)\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)},$$

$$I_{1,3} =(\psi,-\phi\Delta(\Phi_{\xi,\ell,\boldsymbol{a}}-\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(u)+\Delta(\Phi_{\xi,\ell,\boldsymbol{a}}-\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(\phi u))_{L^2(\mathbb{R}^2)},$$

$$I_{1,4} =(\psi,\Delta(\Phi_{\xi,\ell,\boldsymbol{a}}-\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(-\phi u+(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)},$$

$$I_{1,5} =(\psi,\Delta(\Phi_{\xi,\ell,\boldsymbol{a}}-\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(u))_{L^2(\mathbb{R}^2)},$$

and

$$I_{1,6} = (\psi,\Delta(\Phi_{\xi}-\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}-\Phi_{\xi,\ell,\boldsymbol{a}}+\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})(\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)}.$$

$I_{1,1}$ is dominated by

$$\sum_{a_1\in\mathbb{Z}^2\cap\mathrm{supp}\,\nabla\phi}\|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_1}\nabla\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}.$$

Now we estimate $I(a_1):=\|\chi_{a_1}\Delta\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}$. This is dominated by $\sum_{j=1}^{8} I_j(a_1)$, where

$$I_1(a_1) =\|\chi_{a_1}(\widetilde{H_{L,\boldsymbol{a}}^{\widetilde{\xi}}}-E)u\|_{L^2(\mathbb{R}^2)}=\|\chi_{a_1}\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)},$$

$$I_2(a_1) =\|\chi_{a_1}(\overline{\xi}_{L,\boldsymbol{a}}-E)u\|_{L^2(\mathbb{R}^2)}\le(\max|\overline{\xi}_0|+|E|)\|\chi_{a_1}u\|_{L^2(\mathbb{R}^2)},$$

$$I_3(a_1) =\|\chi_{a_1}P_{\xi_{\ell,\boldsymbol{a}}}\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}+\|\chi_{a_1}\Pi(\xi_{\ell,\boldsymbol{a}},\Phi_{\xi,\ell,\boldsymbol{a}}(u))\|_{L^2(\mathbb{R}^2)},$$

$$I_4(a_1) =\|\chi_{a_1}P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_{\ell,\boldsymbol{a}}))\|_{L^2(\mathbb{R}^2)},$$

$$I_5(a_1) =\|\chi_{a_1}e^{\Delta}(P_u\xi_{\ell,\boldsymbol{a}}+_u P_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell,\boldsymbol{a}}+P_uY_{\xi.\ell,\boldsymbol{a}})\|_{L^2(\mathbb{R}^2)},$$

$$I_6(a_1) =\|\chi_{a_1}C(u,\xi_{\ell,\boldsymbol{a}},\xi_{\ell,\boldsymbol{a}})\|_{L^2(\mathbb{R}^2)}+\|\chi_{a_1}S(u,\xi_{\ell,\boldsymbol{a}},\xi_{\ell,\boldsymbol{a}})\|_{L^2(\mathbb{R}^2)},$$

$$I_7(a_1) =\|\chi_{a_1}P_{Y_{\xi,\ell,\boldsymbol{a}}}u\|_{L^2(\mathbb{R}^2)}+\|\chi_{a_1}\Pi(Y_{\xi,\ell.\boldsymbol{a}},u)\|_{L^2(\mathbb{R}^2)}+\|\chi_{a_1}P_1^{(b)}((P_1^{(b)}Y_{\xi,\ell,\boldsymbol{a}})(P_1^{(b)}u))\|_{L^2(\mathbb{R}^2)},$$

and

$$I_8(a_1) =\|\chi_{a_1}P_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}({}_uP_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell,\boldsymbol{a}}+P_uY_{\xi,\ell,\boldsymbol{a}})\|_{L^2(\mathbb{R}^2)},$$

$$+\|\chi_{a_1}\Pi(\xi_{\ell,\boldsymbol{a}},\Delta^{-loc}({}_uP_{\xi_{\ell,\boldsymbol{a}}}\Delta^{-loc}\xi_{\ell,\boldsymbol{a}}+P_uY_{\xi,\ell,\boldsymbol{a}})\|_{L^2(\mathbb{R}^2)}.$$

By Lemma 3.2 (ii), (iii) in [21], $I_3(a_1)$ is dominated by

$$\Xi(a_1,\boldsymbol{a},\ell,\xi)^{1/2}\sum_{a_2\in\mathbb{Z}^2}\|\chi_{a_2}^2\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{\mathcal{H}^{1+2\epsilon}(\mathbb{R}^2)}\exp(-c|a_1-a_2|^2),$$

where $\Xi(a_1, \boldsymbol{a}, \ell, \xi)$ is the quatity appeared in Proposition 2.1. As in the proof of Theorem 1 in [21], we have

$$\|\chi_{a_2}^2 \Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{\mathcal{H}^{1+2\epsilon}(\mathbb{R}^2)} \leq \frac{c_2}{t_1^{(1+2\epsilon)/2}} \Big( \|\chi_{a_2}^2 \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} + t_1 \|\chi_{a_2}^2 \Delta \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \Big)$$

for any $t_1 \in (0, \infty)$. By Lemma 3.4 in [21], $I_6(a_1)$ is dominated by

$$\Xi(a_1, \boldsymbol{a}, \ell, \xi) \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{\mathcal{H}^{3\epsilon}(\mathbb{R}^2)} \exp(-c|a_1 - a_2|^2)$$

As in the proof of Theorem 1 in [21], we have

$$\begin{aligned}
&\|\chi_{a_2}^2 u\|_{\mathcal{H}^{3\epsilon}(\mathbb{R}^2)} \\
\leq &\|\chi_{a_2}^2 \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{\mathcal{H}^{3\epsilon}(\mathbb{R}^2)} + \|\chi_{a_2}^2 \Delta^{-loc} P_u \xi_{\ell,\boldsymbol{a}}\|_{\mathcal{H}^{3\epsilon}(\mathbb{R}^2)} \\
&+ \|\chi_{a_2}^2 \Delta^{-loc}{}_u P_{\xi_{\ell,\boldsymbol{a}}} \Delta^{-loc} \xi_{\ell,\boldsymbol{a}}\|_{\mathcal{H}^{3\epsilon}(\mathbb{R}^2)} + \|\chi_{a_2}^2 \Delta^{-loc} P_u Y_{\xi,\Lambda_\ell(\boldsymbol{a})}\|_{\mathcal{H}^{3\epsilon}(\mathbb{R}^2)} \\
\leq &\frac{c}{t_2^{3\epsilon/2}} \Big( \sum_{a_3 \in \mathbb{Z}^2 \cap \overline{\Lambda_2(a_2)}} \|\chi_{a_3}^2 \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} + t_2^{1/2} \|\chi_{a_2}^2 \Delta \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \Big) \\
&+ c\Xi(a_1, \boldsymbol{a}, \ell, \xi) \sum_{a_3 \in \mathbb{Z}^2} \|\chi_{a_3}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_2 - a_3|^2)
\end{aligned}$$

for any $t_2 \in (0, \infty)$. The other terms are similarly estimated and we obtain

$$\begin{aligned}
&\|\chi_{a_1} \Delta \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \\
\leq &\|\chi_{a_1} \chi_{\boldsymbol{a}}^2 \psi\|_{L^2(\mathbb{R}^2)} + (\max |\overline{\xi}_0| + |E|) \|\chi_{a_1} u\|_{L^2(\mathbb{R}^2)} \\
&+ c\Xi(a_1, \boldsymbol{a}, \ell, \xi)^2 \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2|^2) \\
&+ c\Xi(a_1, \boldsymbol{a}, \ell, \xi) \sum_{a_2 \in \mathbb{Z}^2} \Big( \frac{\|\chi_{a_2}^2 \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)}}{t^{(1+2\epsilon)/2}} + t^{(1-2\epsilon)/2} \|\chi_{a_2}^2 \Delta \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \Big) \exp(-c|a_1 - a_2|^2)
\end{aligned}$$

for any $t \in (0,1]$. By the iteration, we have

$$
\begin{aligned}
&\|\chi_{a_1}\Delta\Phi_{\xi,\ell,\boldsymbol{a}}u\|_{L^2(\mathbb{R}^2)}\\
&\le \sum_{j=0}^{N}(c\Xi(\boldsymbol{a},\ell,\xi)t^{(1-2\epsilon)/2})^j \sum_{a_2,a_3,\dots,a_{j+1}\in\mathbb{Z}^2}\exp\Big(-c\sum_{k=1}^{j}|a_k-a_{k+1}|^2\Big)\\
&\quad\times\{\|\chi_{a_{j+1}}\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)}+(\max|\bar{\xi}_0|+|E|)\|\chi_{a_{j+1}}u\|_{L^2(\mathbb{R}^2)}\\
&\quad+c\Xi(\boldsymbol{a},\ell,\xi)^2\sum_{a_{j+2}\in\mathbb{Z}^2}\|\chi_{a_{j+2}}^2u\|_{L^2(\mathbb{R}^2)}\exp\Big(-c|a_{j+1}-a_{j+2}|^2\Big)\\
&\quad+\frac{c\Xi(\boldsymbol{a},\ell,\xi)}{t^{(1+2\epsilon)/2}}\|\chi_{a_{j+2}}^2\Phi_{\xi,\ell,\boldsymbol{a}}u\|_{L^2(\mathbb{R}^2)}\exp(-c|a_{j+1}-a_{j+2}|^2)\Big\}\\
&\quad+(c\Xi(\boldsymbol{a},\ell,\xi)t^{(1-2\epsilon)/2})^{N+1}\sum_{a_2,a_3,\dots,a_{N+2}\in\mathbb{Z}^2}\exp\Big(-c\sum_{k=1}^{N+1}|a_k-a_{k+1}|^2\Big)\|\chi_{a_{N+2}}^2\Delta\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}\\
&\le \sum_{a_2\in\mathbb{Z}^2}\Big\{\Big\{\delta_{a_1a_2}+\sum_{j=1}^{N}(c\Xi(\boldsymbol{a},\ell,\xi)t^{(1-2\epsilon)/2})^j\exp\Big(-\frac{c}{j}|a_1-a_2|^2\Big)\Big\}\\
&\quad\times\{\|\chi_{a_2}\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)}+(\max|\bar{\xi}_0|+|E|)\|\chi_{a_2}u\|_{L^2(\mathbb{R}^2)}\}\\
&\quad+\sum_{j=0}^{N}(c\Xi(\boldsymbol{a},\ell,\xi)t^{(1-2\epsilon)/2})^j\exp\Big(-\frac{c}{j+1}|a_1-a_2|^2\Big)\\
&\quad\times\Big(\Xi(\boldsymbol{a},\ell,\xi)\|\chi_{a_2}^2u\|_{L^2(\mathbb{R}^2)}+\frac{\|\chi_{a_2}^2\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}}{t^{(1+2\epsilon)/2}}\Big)\\
&\quad+(c\Xi(\boldsymbol{a},\ell,\xi)t^{(1-2\epsilon)/2})^{N+1}\exp\Big(-\frac{c|a_1-a_2|}{N+1}\Big)\|\chi_{a_{N+2}}^2\Delta\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}\Big\}.
\end{aligned}
$$

We take $t \le 1/(4c\Xi(\boldsymbol{a},\ell,\xi))^{2/(1-2\epsilon)}$ so that

$$
c\Xi(\boldsymbol{a},\ell,\xi)t^{(1-2\epsilon)/2} \le \frac{1}{2^2}.
$$

Then, since

$$
\frac{c|a_1-a|^2}{j}+j\log\frac{1}{2}\ge 2\sqrt{c\log\frac{1}{2}}|a_1-a_2|,
$$

we have

$$
\begin{aligned}
&\|\chi_{a_1}\Delta\Phi_{\xi,\ell,\boldsymbol{a}}u\|_{L^2(\mathbb{R}^2)}\\
&\le \sum_{a_2\in\mathbb{Z}^2}c\exp(-c|a_1-a_2|)\Big\{\|\chi_{a_2}\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)}\\
&\quad+(\max|\bar{\xi}_0|+|E|+\Xi(\boldsymbol{a},\ell,\xi))\|\chi_{a_2}u\|_{L^2(\mathbb{R}^2)}\\
&\quad+(1\vee\Xi(\boldsymbol{a},\ell,\xi)^{(1+2\epsilon)/(1-2\epsilon)})\|\chi_{a_2}^2\Phi_{\xi,\ell,\boldsymbol{a}}(u)\|_{L^2(\mathbb{R}^2)}\Big\}
\end{aligned}
$$

by taking the limit $N \to \infty$. Since it is easy to obtain

$$\begin{aligned}
&\|\chi_{a_2}^2 \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \\
\leq& c(1 \vee \Xi(\boldsymbol{a}, \ell, \xi)) \sum_{a_3 \in \mathbb{Z}^2} \|\chi_{a_3}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_2 - a_3|^2),
\end{aligned} \tag{6.2}$$

we have

$$\begin{aligned}
&\|\chi_{a_1} \Delta \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \\
\leq& c \exp(-c|a_1 - \boldsymbol{a}|) \|\chi_{\boldsymbol{a}}^2 \psi\|_{L^2(\mathbb{R}^2)} \\
&+ \sum_{a_2 \in \mathbb{Z}^2} c \exp(-c|a_1 - a_2|) \|\chi_{a_2} u\|_{L^2(\mathbb{R}^2)} \\
&\times (1 \vee \Xi(\boldsymbol{a}, \ell, \xi)^{(1+2\epsilon)/(1-2\epsilon)})(\max |\overline{\xi}_0| + |E| + \Xi(\boldsymbol{a}, \ell, \xi)).
\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}
&\|\chi_{a_1} \nabla \Phi_{\xi,\ell,\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \\
\leq& c(1 \vee \Xi(\boldsymbol{a}, \ell, \xi)) \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2} \chi_{\boldsymbol{a}}^2 \psi\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2|) \\
&+ c(1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^{1/(1-2\epsilon)} (\max |\overline{\xi}_0| + |E| + \Xi(\boldsymbol{a}, \ell, \xi))^{1/2} \\
&\times \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2} u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2|).
\end{aligned}$$

Thus we have

$$\begin{aligned}
|I_{1,1}| \leq& c \sum_{a_1 \in \mathbb{Z}^2 \cap \operatorname{supp} \nabla \phi} \|\chi_{a_1} \psi\|_{L^2(\mathbb{R}^2)} \\
&\times \Big\{ (1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^2 \|\chi_{\boldsymbol{a}}^2 \psi\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - \boldsymbol{a}|) \\
&+ (1 \vee \Xi(\boldsymbol{a}, \ell, \xi))^{1+1/(1-2\epsilon)} (\max |\overline{\xi}_0| + |E| + \Xi(\boldsymbol{a}, \ell, \xi))^{1/2} \\
&\times \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2} u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2|) \Big\}.
\end{aligned}$$

We also have

$$\begin{aligned}
|I_{1,2}| \leq& c \sum_{a_1 \in \mathbb{Z}^2 \cap \operatorname{supp} \nabla \phi} \|\chi_{a_1} \psi\|_{L^2(\mathbb{R}^2)} \\
&\times (1 \vee \Xi(\boldsymbol{a}, \ell, \xi)) \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2|^2).
\end{aligned}$$

$I_{1,3}$ is decomposed as $\sum_{j=1}^{3} I_{1,3,j}$, where

$$
\begin{aligned}
I_{1,3,1} :=&(\psi, \phi\Delta\Delta^{-loc} \sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} (P_u\chi_a^2\xi - P_u^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2\xi) \\
&- \Delta\Delta^{-loc} \sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} (P_{\phi u}\chi_a^2\xi - P_{\phi u}^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2\xi))_{L^2(\mathbb{R}^2)}, \\
I_{1,3,2} :=&(\psi, \phi\Delta\Delta^{-loc} \sum_{a,a'\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} ({}_uP_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi -{}_u P_{\chi_a^2\xi}^{s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a})}\Delta^{-loc}\chi_{a'}^2\xi) \\
&- \Delta\Delta^{-loc} \sum_{a,a'\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} ({}_{\phi u}P_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi -{}_{\phi u} P_{\chi_a^2\xi}^{s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a})}\Delta^{-loc}\chi_{a'}^2\xi))_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

and

$$
\begin{aligned}
I_{1,3,3} := (\psi, \phi\Delta\Delta^{-loc} &\sum_{a\in\mathbb{Z}^2} (P_u\chi_a^2 Y_{\xi,\ell,\boldsymbol{a}} - P_u^{s_2(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2 Y_{\xi,\ell,\boldsymbol{a}}) \\
- \Delta\Delta^{-loc} &\sum_{a\in\mathbb{Z}^2} (P_{\phi u}\chi_a^2 Y_{\xi,\ell,\boldsymbol{a}} - P_{\phi u}^{s_2(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2 Y_{\xi,\ell,\boldsymbol{a}}))_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

$I_{1,3,1}$ is decomposed as $\sum_{j=1}^{2} I_{1,3,1,j}$, where

$$
I_{1,3,1,1} := (\psi, (e^{\Delta}\phi - \phi e^{\Delta}) \sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} (P_u\chi_a^2\xi - P_u^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2\xi))_{L^2(\mathbb{R}^2)}
$$

and

$$
\begin{aligned}
I_{1,3,1,2} :=&(\psi, \Delta\Delta^{-loc}\phi \sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} (P_u\chi_a^2\xi - P_u^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2\xi) \\
&- \Delta\Delta^{-loc} \sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} (P_{\phi u}\chi_a^2\xi - P_{\phi u}^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2\xi))_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

$I_{1,3,1,1}$ is estimated as

$$
\begin{aligned}
I_{1,3,1,1} = &\int dx_1\psi(x_1) \int dx_2 e^{\Delta}(x_1,x_2) \int_0^1 dr (x_2 - x_1)\cdot(\nabla\phi)(rx_2 + (1-r)x_1) \\
&\times \sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} (P_u\chi_a^2\xi - P_u^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}\chi_a^2\xi)(x_2)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{1,3,1,1}| \le &c \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_2\in\mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2|^2 - cd(a_2, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \Xi(\boldsymbol{a}, \ell, \xi)^{1/2}.
\end{aligned}
$$

By using Lemma 3.6 in [21], $I_{1,3,1,2}$ is estimated as

$$
\begin{aligned}
I_{1,3,1,2} =& \int dx_1 dx_2 dx_3 dx_4 \psi(x_1) \int_0^1 ds(-\Delta e^{s\Delta})(x_1,x_2) \\
&\times \Big( \sum_{\nu} c_\nu \sum_{a\in\mathbb{Z}^2\cap\Lambda_L(\boldsymbol{a})} \int_{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}^1 \frac{dt}{t} Q_t^{1,\nu}(x_2,x_3)(Q_t^{2,\nu}\chi_a^2\xi)(x_3)P_t^{\nu}(x_3,x_4)u(x_4) \\
&\times \int_0^1 dr(x_2-x_4)\cdot(\nabla\phi)(rx_2+(1-r)x_4)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{1,3,1,2}| \le& c \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_2\in\mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1-a_2|^2 - cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_3\in\mathbb{Z}^2\cap\Lambda_L(\boldsymbol{a})} s(a_3;\epsilon,\xi,\delta,\boldsymbol{a})^{-\epsilon-1/2} \|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \exp(-c|a_2-a_3|^2).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
|I_{1,3,1}| \le& c \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_2\in\mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1-a_2|^2 - cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_3\in\mathbb{Z}^2\cap\Lambda_L(\boldsymbol{a})} s(a_3;\epsilon,\xi,\delta,\boldsymbol{a})^{-\epsilon-1/2} \|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \exp(-c|a_2-a_3|^2).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
|I_{1,3,2}| \le& c \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_2\in\mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1-a_2|^2 - cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_3,a_4\in\mathbb{Z}^2\cap\Lambda_L(\boldsymbol{a})} s_1(a_3,a_4,;\epsilon,\xi,\delta,\boldsymbol{a})^{-\epsilon} \|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \|\chi_{a_4}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \\
&\times \exp(-c(|a_2-a_3|^2+|a_3-a_4|^2))
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{1,3,3}| \le& c \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_2\in\mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1-a_2|^2 - cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
&\times \sum_{a_3\in\mathbb{Z}^2} s_2(a_3;\epsilon,\xi,\delta,\boldsymbol{a})^{-\epsilon/2} \|\chi_{a_3}^2 Y_{\xi,\ell,\boldsymbol{a}}\|_{C^{-\epsilon}(\mathbb{R}^2)} \exp(-c|a_2-a_3|^2 - cd(a_3,\Lambda_L(\boldsymbol{a}))^2).
\end{aligned}
$$

For $I_{1,4}$, we prepare

$$
\begin{aligned}
&\phi u - (\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)) \\
=&(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(\phi u) - \phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)).
\end{aligned}
$$

We use Lemma 4.1 in [21] to obtain

$$
\begin{aligned}
&\|\chi_{a_0}(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}v\|_{L^2(\mathbb{R}^2)} \\
\leq &\sum_{a_1\in\mathbb{Z}^2} c\exp(-c|a_0-a_1|)\|\chi_{a_1}v\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Moreover we prepare

$$
\begin{aligned}
&\|\chi_{a_1}(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(\phi u) - \phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u))\|_{L^2(\mathbb{R}^2)} \\
\leq &\sum_{a_2\in\mathbb{Z}^2:d_\infty(\overline{a_1a_2},\Lambda_{2L/3}\setminus\Lambda_{L/3}))\leq 1} c\exp(-c|a_1-a_2|^2)\|\chi_{a_2}u\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\|\chi_{a_0}(\phi u - (\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u))\|_{L^2(\mathbb{R}^2)} \\
\leq &\sum_{a_1\in\mathbb{Z}^2} c\exp(-c|a_0-a_1| - cd(a_0,\Lambda_{2L/3}\setminus\Lambda_{L/3}))\|\chi_{a_1}u)\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

We apply this to each term of the right hand side of $I_{1,4} = \sum_{j=1}^3 I_{1,4,j}$, where

$$
\begin{aligned}
I_{1,4,1} :=&(\psi, \Delta\Delta^{-loc}\sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}(P_{\phi u-(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)}\chi_a^2\xi) \\
&- P^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}_{\phi u-(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)}\chi_a^2\xi))_{L^2(\mathbb{R}^2)}, \\
I_{1,4,2} :=&(\psi, \Delta\Delta^{-loc}\sum_{a,a'\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}({}_{\phi u-(\Phi^{\boldsymbol{s}(\epsilon\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)}P_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi \\
&-{}_{\phi u-(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)}\ P^{s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a})}_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
I_{1,4,3} :=&(\psi, \Delta\Delta^{-loc}\sum_{a\in\mathbb{Z}^2}(P_{\phi u-(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)}\chi_a^2 Y_{\xi,\ell,\boldsymbol{a}} \\
&- P^{s_2(a;\epsilon,\xi,\delta,\boldsymbol{a})}_{\phi u-(\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}})^{-1}\phi\Phi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}_{\xi,\ell,\boldsymbol{a}}(u)}\chi_a^2 Y_{\xi,\ell,\boldsymbol{a}}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|I_{1,4,1}| \le & c \sum_{a_1 \in \mathbb{Z}^2} \|\chi_{a_1}^2 \psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
& \times \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2| - cd(a_2, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
& \times \sum_{a_3 \in \mathbb{Z}^2 \cap \Lambda_\ell(\boldsymbol{a})} s(a_3; \epsilon, \xi, \delta, \boldsymbol{a})^{-(1+\epsilon)/2} \|\chi_{a_3}^2 \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \exp(-c|a_2 - a_3|), \\
|I_{1,4,2}| \le & c \sum_{a_1 \in \mathbb{Z}^2} \|\chi_{a_1}^2 \psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
& \times \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2| - cd(a_2, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
& \times \sum_{a_3, a_4 \in \mathbb{Z}^2 \cap \Lambda_\ell(\boldsymbol{a})} s_1(a_3, a_4; \epsilon, \xi, \delta, \boldsymbol{a})^{-\epsilon} \|\chi_{a_3}^2 \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \|\chi_{a_4}^2 \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \\
& \times \exp(-c(|a_2 - a_3| + |a_3 - a_4|))
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{1,4,3}| \le & c \sum_{a_1 \in \mathbb{Z}^2} \|\chi_{a_1}^2 \psi\|_{L^2(\mathbb{R}^2)} \exp(-cd(a_1, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
& \times \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \exp(-c|a_1 - a_2| - cd(a_2, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a}))) \\
& \times \sum_{a_3 \in \mathbb{Z}^2} s_2(a_3; \epsilon, \xi, \delta, \boldsymbol{a})^{-\epsilon/2} \|\chi_{a_3}^2 Y_{\xi,\ell,\boldsymbol{a}}\|_{C^{-\epsilon}(\mathbb{R}^2)} \exp(-c|a_2 - a_3|).
\end{aligned}
$$

For $I_{1,5}$, we also prepare

$$
\begin{aligned}
& ((\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - (\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}) \phi \Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u) \\
= & (\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} (\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})} - \Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}) (\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} \phi \Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)
\end{aligned}
$$

and

$$
\begin{aligned}
& \|\chi_{a_1} (\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})} - \Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}) v\|_{L^2(\mathbb{R}^2)} \\
\le & \sum_{a_2 \in \mathbb{Z}^2} c \exp(-c|a_1 - a_2|^2) \Xi_c(a_1, \boldsymbol{a}, \ell, \xi) \|\chi_{a_2} v\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\|\chi_{a_0}((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)\|_{L^2(\mathbb{R}^2)}\\
\le&\sum_{a_1\in\mathbb{Z}^2}c\exp(-c|a_0-a_1|)\\
&\times\sum_{a_2\in\mathbb{Z}^2}c\exp(-c|a_1-a_2|^2)\Xi_c(a_1,\boldsymbol{a},\ell,\xi)\\
&\times\sum_{a_3\in\mathbb{Z}^2\cap\Lambda_{2L/3}(\boldsymbol{a})}c\exp(-c|a_2-a_3|)\\
&\times\sum_{a_4\in\mathbb{Z}^2}c\exp(-c|a_3-a_4|^2)\|\chi_{a_4}u\|_{L^2(\mathbb{R}^2)}\\
\le&\sum_{a_1\in\mathbb{Z}^2}c\exp(-c|a_0-a_1|-cd(a_0,\Lambda_{2L/3}(\boldsymbol{a})))\Xi_c(a_0,\boldsymbol{a},\ell,\xi)\|\chi_{a_1}u\|_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

by changing the constant appearing in the exponential terms of $\Xi_c(\cdot)$ when we move to the last line. We apply this to each term of the right hand side of $I_{1,5}=\sum_{j=1}^3 I_{1,5,j}$, where

$$
\begin{aligned}
I_{1,5,1}:=&(\psi,\Delta\Delta^{-loc}\sum_{a\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}(P_{((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2\xi)\\
&-P^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}_{((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2\xi))_{L^2(\mathbb{R}^2)},\\
I_{1,5,2}:=&(\psi,\Delta\Delta^{-loc}\sum_{a,a'\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}(_{((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi\\
&-_{((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P_{\chi_a^2\xi}^{s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a})}\Delta^{-loc}\chi_{a'}^2\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
I_{1,5,3}:=&(\psi,\Delta\Delta^{-loc}\sum_{a\in\mathbb{Z}^2}(P_{((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2Y_{\xi,\ell,\boldsymbol{a}}\\
&-P^{s_2(a;\epsilon,\xi,\delta,\boldsymbol{a})}_{((\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}-(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2Y_{\xi,\ell,\boldsymbol{a}}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|I_{1,5,1}|\le& c\sum_{a_1\in\mathbb{Z}^2}\sum_{a_2\in\mathbb{Z}^2}\sum_{a_3\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2u\|_{L^2(\mathbb{R}^2)}s(a_3;\epsilon,\xi,\delta)^{-(1+\epsilon)/2}\|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)}\\
&\times\Xi_c(a_2,\boldsymbol{a},\ell,\xi)\exp(-c|a_1-a_2|-c|a_1-a_3|^2-cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})))\\
|I_{1,5,2}|\le& c\sum_{a_1\in\mathbb{Z}^2}\sum_{a_2\in\mathbb{Z}^2}\sum_{a_3\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}\sum_{a_4\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})}\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2u\|_{L^2(\mathbb{R}^2)}\\
&\times s_1(a_3,a_4;\epsilon,\xi,\delta)^{-3\epsilon/2}\|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)}\|\chi_{a_4}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)}\\
&\times\Xi_c(a_2,\boldsymbol{a},\ell,\xi)\exp(-c\max_{1\le j<k\le4}|a_j-a_k|-c\max_{1\le j\le4}d(a_j,\Lambda_{2L/3}(\boldsymbol{a}))),
\end{aligned}
$$

and

$$|I_{1,5,3}| \leq c \sum_{a_1,a_2,a_3\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} s_2(a_3;\epsilon,\xi,\delta)^{-\epsilon/2} \|\chi_{a_3}^2 Y_{\xi,\ell,\boldsymbol{a}}\|_{C^{-\epsilon}(\mathbb{R}^2)}$$
$$\times \Xi_c(a_2,\boldsymbol{a},\ell,\xi)\exp(-c|a_1-a_2|-c|a_1-a_3|^2-cd(a_2,\Lambda_{2L/3}(\boldsymbol{a}))).$$

Moreover we apply

$$\begin{aligned}
&\|\chi_{a_0}(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)\|_{L^2(\mathbb{R}^2)}\\
&\leq \sum_{a_1\in\mathbb{Z}^2\cap\Lambda_{2L/3}(\boldsymbol{a})} c\exp(-c|a_0-a_1|)\\
&\quad\times \sum_{a_2\in\mathbb{Z}^2} c\exp(-c|a_1-a_2|^2)\|\chi_{a_2}u\|_{L^2(\mathbb{R}^2)}\\
&\leq \sum_{a_1\in\mathbb{Z}^2} c\exp(-c|a_0-a_1|-cd(a_0,\Lambda_{2L/3}(\boldsymbol{a})))\|\chi_{a_1}u\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

to each term of the right hand side of $I_{1,6}=\sum_{j=1}^3 I_{1,6,j}$, where

$$\begin{aligned}
I_{1,6,1} :=&(\psi,\Delta\Delta^{-loc}\sum_{a\in\mathbb{Z}^2\setminus\Lambda_\ell(\boldsymbol{a})}(P_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2\xi)\\
&-P^{s(a;\epsilon,\xi,\delta,\boldsymbol{a})}_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2\xi))_{L^2(\mathbb{R}^2)},\\
I_{1,6,2} :=&(\psi,\Delta\Delta^{-loc}\sum_{(a,a')\in\mathbb{Z}^2\times\mathbb{Z}^2\setminus\Lambda_\ell(\boldsymbol{a})^2}(_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi\\
&-_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}P^{s_1(a,a';\epsilon,\xi,\delta,\boldsymbol{a})}_{\chi_a^2\xi}\Delta^{-loc}\chi_{a'}^2\xi),
\end{aligned}$$

and

$$\begin{aligned}
I_{1,6,3} :=&(\psi,\Delta\Delta^{-loc}\sum_{a\in\mathbb{Z}^2}(P_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2(Y_\xi-Y_{\xi,\ell,\boldsymbol{a}})\\
&-P^{s_2(a;\epsilon,\xi,\delta,\boldsymbol{a})}_{(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)}\chi_a^2(Y_\xi-Y_{\xi,\ell,\boldsymbol{a}})).
\end{aligned}$$

Then we have

$$\begin{aligned}
|I_{1,6,1}| \leq& c\sum_{a_1\in\mathbb{Z}^2}\sum_{a_2\in\mathbb{Z}^2}\sum_{a_3\in\mathbb{Z}^2\setminus\Lambda_\ell(\boldsymbol{a})}\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2u\|_{L^2(\mathbb{R}^2)}s(a_3;\epsilon,\xi,\delta,\boldsymbol{a})^{-(1+\epsilon)/2}\|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)}\\
&\times\exp(-c|a_1-a_2|-c|a_1-a_3|^2-cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})))\\
|I_{1,6,2}| \leq& c\sum_{a_1\in\mathbb{Z}^2}\sum_{a_2\in\mathbb{Z}^2}\sum_{(a_3,a_4)\in\mathbb{Z}^2\times\mathbb{Z}^2\setminus\Lambda_\ell(\boldsymbol{a})^2}\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2u\|_{L^2(\mathbb{R}^2)}\\
&\times s_1(a_3,a_4;\epsilon,\xi,\delta,\boldsymbol{a})^{-3\epsilon/2}\|\chi_{a_3}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)}\|\chi_{a_4}^2\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)}\\
&\times\exp(-c\max_{1\leq j<k\leq 4}|a_j-a_k|-c\max_{1\leq j\leq 4}d(a_j,\Lambda_{2L/3}(\boldsymbol{a}))),
\end{aligned}$$

and

$$|I_{1,6,3}| \le c \sum_{a_1,a_2,a_3\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} s_2(a_3;\epsilon,\xi,\delta,\boldsymbol{a})^{-\epsilon/2}\|\chi_{a_3}^2(Y_\xi - Y_{\xi,\ell,\boldsymbol{a}})\|_{C^{-\epsilon}(\mathbb{R}^2)}$$

$$\times \exp(-c|a_1-a_2| - c|a_1-a_3|^2 - cd(a_2,\Lambda_{2L/3}(\boldsymbol{a}))).$$

$I_2$ is decomposed as $\sum_{j=1}^4 I_{2,j}$, where

$$I_{2,1} :=(\psi, \phi P_{\xi_{\ell,\boldsymbol{a}}}\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u) - P_{\xi_{\ell,\boldsymbol{a}}}\phi\Phi_{\xi,\ell,\boldsymbol{a}}(u))_{L^2(\mathbb{R}^2)}$$

$$+ (\psi, \phi\Pi(\xi_{\ell,\boldsymbol{a}}, \Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)) - \Pi(\xi_{\ell,\boldsymbol{a}}, \phi\Phi_{\xi,\ell,\boldsymbol{a}}(u)))_{L^2(\mathbb{R}^2)},$$

$$I_{2,2} :=(\psi, P_{\xi_{\ell,\boldsymbol{a}}}(\phi\Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - \Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi)\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)}$$

$$+ (\psi, \Pi(\xi_{\ell,\boldsymbol{a}}, (\phi\Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - \Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi)\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)},$$

$$I_{2,3} :=(\psi, P_{\xi_{\ell,\boldsymbol{a}}}(\Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - \Phi_\xi(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)}$$

$$+ (\psi, \Pi(\xi_{\ell,\boldsymbol{a}}, (\Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - \Phi_\xi(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1})\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)},$$

and

$$I_{2,4} :=(\psi, P_{\xi_{\ell,\boldsymbol{a}}-\xi}\Phi_\xi(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u))_{L^2(\mathbb{R}^2)}$$

$$+ (\psi, \Pi(\xi_{\ell,\boldsymbol{a}} - \xi, \Phi_\xi(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)))_{L^2(\mathbb{R}^2)}.$$

In

$$I_{2,1} := \sum_\nu c_\nu \sum_{a_1,a_2\in\mathbb{Z}^2} \int dx_1 \int_0^1 \frac{dt}{t}(\chi_{a_1}^2\psi)(x_1)Q_t^{1,\nu}(x_1,x_2)(P_t^\nu\xi_{\ell,\boldsymbol{a}})(x_2)$$

$$\times Q_t^{2,\nu}(x_2,x_3)(\chi_{a_2}^2(\Phi_{\xi,\ell,\boldsymbol{a}}(u))(x_3)(\phi(x_1)-\phi(x_2))$$

$$+ \sum_\mu c_\mu \sum_{a\in\mathbb{Z}^2} \int dx_1 \int_0^1 \frac{dt}{t}(\chi_{a_1}^2\psi)(x_1)P_t^\mu(x_1,x_2)(Q_t^{1,\mu}\xi_{\ell,\boldsymbol{a}})(x_2)$$

$$\times Q_t^{2,\mu}(x_2,x_3)(\chi_{a_2}^2(\Phi_{\xi,\ell,\boldsymbol{a}}(u))(x_3)(\phi(x_1)-\phi(x_3)),$$

a neccesary condition for

$$\phi(x_1) - \phi(x_3) = (x_1 - x_3)\cdot \int_0^1 dr(\nabla\phi)(rx_1 + (1-r)x_4) \neq 0$$

is

$$d_\infty(\overline{a_1 a_2}, \Lambda_{2L/3}(\boldsymbol{a}) \setminus \Lambda_{L/3}(\boldsymbol{a})) \le 1.$$

By Lemma 3.2 (ii) (iii) in [21] and (6.3)-(6.2), we have

$$
\begin{aligned}
|I_{2,1}| \le & c \sum_{a_1\in\mathbb{Z}^2} \sum_{a_2\in\mathbb{Z}^2\cap\Lambda_\ell(\boldsymbol{a})} \sum_{a_3\in\mathbb{Z}^2, d_\infty(\overline{a_1 a_2}, \Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))\le 1} \\
& \times \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)} \|\chi_{a_2}^2 \xi\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)} \|\chi_{a_3}^2 \Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}(u)\|_{\mathcal{H}^{1+2\epsilon}(\mathbb{R}^2)} \\
& \times \exp(-c(|a_1-a_2|^2 + |a_1-a_3|^2)) \\
\le & c \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)} \\
& \times \exp(-c|a_1-\boldsymbol{a}|^2 - c(d(a_1, \Lambda_{2L/3}\setminus\Lambda_{L/3}) \vee L)) \\
& + c \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \Xi(\boldsymbol{a},\ell,\xi)(1\vee\Xi(\boldsymbol{a},\ell,\xi))^{(1+2\epsilon)/(1-2\epsilon)} \\
& \times (\max|\overline{\xi_0}| + |E| + \Xi(\boldsymbol{a},\ell,\xi)) \\
& \times \exp(-c|a_1-a_2| - c\max_{j\in\{1,2\}} d(a_j, \Lambda_{2L/3}\setminus\Lambda_{L/3}))).
\end{aligned}
$$

By the decomposition

$$
\begin{aligned}
& \phi\Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - \Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi \\
= & \phi(\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - (\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi \\
= & \{\phi(\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}}) - (\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})\phi\}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} \\
& + (\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})\{\phi(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - (\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\phi\},
\end{aligned}
$$

we have

$$
\begin{aligned}
|I_{2,2}| \le & c(\log L)^c \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)} \|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \\
& \times \Xi(\boldsymbol{a},\ell,\xi)^{3/2} \exp(-c|a_1-a_2| - c\max_{j\in\{1,2\}} d(a_j, \Lambda_{2L/3}\setminus\Lambda_{L/3})).
\end{aligned}
$$

By the decomposition

$$
\begin{aligned}
& \Phi_{\xi,\ell,\boldsymbol{a}}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - \Phi_\xi(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} \\
= & (\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - (\underline{\Phi_\xi} - \underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}) \\
= & \{(\underline{\Phi_{\xi,\ell,\boldsymbol{a}}} - \underline{\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}}) - (\underline{\Phi_\xi} - \underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})\}(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} \\
& + (\underline{\Phi_\xi} - \underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})\{(\Phi_{\xi,\ell,\boldsymbol{a}}^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} - (\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1}\},
\end{aligned}
$$

we have

$$
\begin{aligned}
|I_{2,3}| \leq & c(\log L)^c \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_4}^2 u\|_{L^2(\mathbb{R}^2)} \\
& \times \Xi_c(a_2,\boldsymbol{a},\ell,\xi)(1\vee\Xi_c(a_2,\boldsymbol{a},\ell,\xi))\Xi(\boldsymbol{a},\ell,\xi)^{1/2}(1\vee\Xi(\boldsymbol{a},\ell,\xi)) \\
& \times \{\exp(-c|a_1-a_2|-cd(a_2,\Lambda_{2L/3}\setminus\Lambda_{L/3}(\boldsymbol{a}))).
\end{aligned}
$$

By the decomposition

$$
\Phi_\xi(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1} = 1 + (\underline{\Phi_\xi} - \underline{\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})}})(\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta,\boldsymbol{a})})^{-1},
$$

we have

$$
\begin{aligned}
|I_{2,4}| \leq & \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)} c\exp(-c|a_1-\boldsymbol{a}|)\Xi_c(a_1,\boldsymbol{a},\ell,\xi)^{1/2} \\
& + \sum_{a_1,a_2\in\mathbb{Z}^2} c(\log(2+|a_1-\boldsymbol{a}|))^c\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)} \\
& \times \exp(-c|a_1-a_2|-cd(a_2,\Lambda_{2L/3}\setminus\Lambda_{L/3}(\boldsymbol{a})))\Xi_c(a_1,\boldsymbol{a},\ell,\xi)^{1/2} \\
& \times ((1\vee\Xi(\boldsymbol{a},\ell,\xi)^{(1+2\epsilon)/(1-2\epsilon)})(\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},\ell,\xi))+\Xi_c(a_2,\xi)).
\end{aligned}
$$

The other terms are simirarly estimated and we obtain

$$
\begin{aligned}
\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)}^2 \le \sum_{a_1\in\mathbb{Z}^2} & \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)} \\
& \times c(1\vee\Xi(\boldsymbol{a},\ell,\xi))^{3/2}\Xi_c(a_1,\boldsymbol{a},\ell,\xi)^{1/2}(1\vee\Xi_c(a_1,\boldsymbol{a},\ell,\xi))^{1/2} \\
& \times \exp(-c|a_1-\boldsymbol{a}|) \\
& + \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)} \\
& \times c(1\vee\Xi(\boldsymbol{a},\ell,\xi))^2\exp(-c|a_1-\boldsymbol{a}|-cd(a_1,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))) \\
& + \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)}c(\log L)^c \\
& \times (1\vee\Xi(\boldsymbol{a},\ell,\xi))^2\Xi_c(a_2,\boldsymbol{a},\ell,\xi)^{1/2} \\
& \times (1\vee\Xi_c(a_2,\boldsymbol{a},\ell,\xi))^{3/2}(1\vee\Xi_c(a_2,\xi))^{1/2} \\
& \times (\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},\ell,\xi))\exp(-c|a_1-a_2|) \\
& + \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\|\chi_{a_2}^2 u\|_{L^2(\mathbb{R}^2)}c(\log L)^c \\
& \times (1\vee\Xi(\boldsymbol{a},\ell,\xi))^{5/2}(1\vee\Xi_c(a_2,\boldsymbol{a},\ell,\xi))(\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},\ell,\xi)) \\
& \times \exp(-c|a_1-a_2|-cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))).
\end{aligned}
$$

Since

$$
\|\chi_{a_2}u\|_{L^2(\mathbb{R}^2)} \le \sum_{a_3\in\mathbb{Z}^2\cap\overline{\Lambda_2(\boldsymbol{a})}} \|\chi_{a_2}(\widetilde{H_{L,\boldsymbol{a}}^{\xi}}-E)^{-1}\chi_{a_3}^2\|_{\mathcal{L}(L^2(\mathbb{R}^2))}\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)},
$$

we have

$$\|\chi_{\boldsymbol{a}}^2\psi\|_{L^2(\mathbb{R}^2)} \le \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)} c(1\vee\Xi(\boldsymbol{a},\ell,\xi))^{3/2}\Xi_c(a_1,\boldsymbol{a},\ell,\xi)^{1/2}$$

$$\times(1\vee\Xi_c(a_1,\boldsymbol{a},\ell,\xi))^{1/2}\exp(-c|a_1-\boldsymbol{a}|)$$

$$+\sum_{a_1\in\mathbb{Z}^2}\|\chi_{a_1}\psi\|_{L^2(\mathbb{R}^2)}c(1\vee\Xi(\boldsymbol{a},\ell,\xi))^2$$

$$\times\exp(-c|a_1-\boldsymbol{a}|-cd(a_1,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a})))$$

$$+\sum_{a_1,a_2\in\mathbb{Z}^2}\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\sum_{a_3\in\mathbb{Z}^2\cap\overline{\Lambda_2(\boldsymbol{a})}}\|\chi_{a_2}(\widetilde{H_{L,\boldsymbol{a}}^{\xi}}-E)^{-1}\chi_{a_3}^2\|_{\mathcal{L}(L^2(\mathbb{R}^2))}$$

$$\times c(\log L)^c(1\vee\Xi(\boldsymbol{a},\ell,\xi))^2\Xi_c(a_2,\boldsymbol{a},\ell,\xi)^{1/2}$$

$$\times(1\vee\Xi_c(a_2,\boldsymbol{a},\ell,\xi))^{3/2}(1\vee\Xi_c(a_2,\xi))^{1/2}$$

$$\times(\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},\ell,\xi))\exp(-c|a_1-a_2|)$$

$$+\sum_{a_1,a_2\in\mathbb{Z}^2}\|\chi_{a_1}^2\psi\|_{L^2(\mathbb{R}^2)}\sum_{a_3\in\mathbb{Z}^2\cap\overline{\Lambda_2(\boldsymbol{a})}}\|\chi_{a_2}(\widetilde{H_{L,\boldsymbol{a}}^{\xi}}-E)^{-1}\chi_{a_3}^2\|_{\mathcal{L}(L^2(\mathbb{R}^2))}$$

$$\times c(\log L)^c(1\vee\Xi(\boldsymbol{a},\ell,\xi))^{5/2}(1\vee\Xi_c(a_2,\boldsymbol{a},\ell,\xi))(\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},\ell,\xi))$$

$$\times\exp(-c|a_1-a_2|-cd(a_2,\Lambda_{2L/3}(\boldsymbol{a})\setminus\Lambda_{L/3}(\boldsymbol{a}))).$$

□

For the existense of generalized eigenfunctions, we apply Theorem 3.1 and Corollary 3.1 in Klein, Koines and Seifert [12] to obtain Lemma 6.1 below: let $\mu^\xi$ be a measure defined by $\mu^\xi(\cdot)=\mathrm{Tr}[T^{-1}E(\cdot:\widetilde{H^\xi})T^{-1}]$. Let $\iota_+:\mathcal{H}_+\to L^2(\mathbb{R}^2,dx)$ and $\iota_-:L^2(\mathbb{R}^2,dx)\to\mathcal{H}_-$ be the natural injections. Let $\mathcal{L}(L^2(\mathbb{R}^2,dx))$ be the space of bounded operators on $L^2(\mathbb{R}^2,dx)$, and $\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)$ be that of bounded operators from $\mathcal{H}_+$ to $\mathcal{H}_-$. Let $\tau$ be a Banach space isomorphism from $\mathcal{L}(L^2(\mathbb{R}^2,dx))$ to $\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)$ defined by $\tau(C)=TCT$ for any $C\in\mathcal{L}(L^2(\mathbb{R}^2,dx))$. Let $\mathcal{I}_{1,+}(L^2(\mathbb{R}^2,dx))$ be the space of non-negative definite operators on $L^2(\mathbb{R}^2,dx)$ with finite traces. We set $\mathcal{I}_{1,+}(\mathcal{H}_+,\mathcal{H}_-)=\tau\mathcal{I}_{1,+}(L^2(\mathbb{R}^2,dx))$. Then the results we need are the following:

**Lemma 6.1.** *For almost all $\xi$, there exits a $\mu^\xi$-locally integrable $\mathcal{I}_{1,+}(\mathcal{H}_+,\mathcal{H}_-)$-valued function $P^\xi(\lambda)$ of $\lambda\in\mathbb{R}$ such that*

$$\iota_-E(I:\widetilde{H^\xi})\iota_+=\int_I P^\xi(\lambda)\mu^\xi(d\lambda)$$

*in the sense of the Bochner integral for any bounded interval $I$.*

*For $\mu^\xi$-almost every $\lambda\in\mathbb{R}$, $P^\xi(\lambda)\phi$ is a generalized eigenfunction of $\widetilde{H^\xi}$ with a generalized eigenvalue $\lambda$ for any $\phi\in\mathcal{H}_+$ if $P^\xi(\lambda)\phi\neq 0$.*

In Klein, Koines and Seifert [12], these results are proven under several hypotheses. Among them, we should show that $\mu^\xi(\cdot)$ is a locally finite measure. For this, we show the following estimate, which is called the assumption on the strong generalized eigenfunction expansion in Germinet and Klein [7]:

**Proposition 6.2.** *For any bounded interval $I$ and $p > 0$, we have*

$$\mathbb{E}[\mathrm{Tr}[T^{-1}E(I:\widetilde{H^\xi})T^{-1}]^p] < \infty. \tag{6.4}$$

**Proof**. Suppose $\mathbb{E}[\mathrm{Tr}[T^{-1}E(I:\widetilde{H^\xi})T^{-1}]^p] = \infty$. Then for any $R > 0$, there exists $N \in \mathbb{N}$ such that

$$\mathbb{E}\Big[\Big\{\sum_{n=1}^{N}\|E(I:\widetilde{H^\xi})\psi_n\|_{L^2(\mathbb{R}^2)}^2\Big\}^p\Big] \geq R, \tag{6.5}$$

where $\{T\psi_n\}_{n=1}^\infty$ is a complete orthonormal basis of $L^2(\mathbb{R}^2)$. Since $\widetilde{H^\xi}$ is the limit of $\widetilde{H^{\xi_\varepsilon}}$ in the strong resolvent sense as $\varepsilon \to 0$, we have

$$\sum_{n=1}^{N}\|E(I:\widetilde{H^\xi})\psi_n\|_{L^2(\mathbb{R}^2)}^2 \leq X_0^N(\xi) = \lim_{\varepsilon\to 0} X_\varepsilon^N(\xi),$$

where

$$X_0^N(\xi) = \sum_{n=1}^{N}\|\widetilde{\chi_I}(\widetilde{H^\xi})\psi_n\|_{L^2(\mathbb{R}^2)}^2,$$

$$X_\varepsilon^N(\xi) = \sum_{n=1}^{N}\|\widetilde{\chi_I}(\widetilde{H^\xi})\psi_n\|_{L^2(\mathbb{R}^2)}^2,$$

and $\widetilde{\chi_I}$ is a $[0,1]$-valued continuous function such that $\widetilde{\chi_I} = 1$ on $I$ and $\widetilde{\chi_I}(\mu) = 0$ if $d(\mu, I) \geq 1$. To show the contradiction of (6.5), it is enough to show that $\sup_{\varepsilon\in(0,1],N\in\mathbb{N}}\mathbb{E}[X_\varepsilon^N(\xi)^m] < \infty$ for any $m \in \mathbb{N}$.

For any $t > 0$, we have

$$\begin{aligned}
&\mathbb{E}[X_\varepsilon^N(\xi)^m] \\
\leq&\mathbb{E}\Big[\Big(\sum_{n=1}^{N}\int_{\mathbb{R}^2} dy(\exp(-t\widetilde{H^{\xi_\varepsilon}}/4)(y,\cdot),\psi_n)_{L^2(\mathbb{R}^2)}^2\Big)^m\Big]\exp(mt(\sup I + 1)/2) \\
=&\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2}\frac{dx}{(1+|x|^2)^{2\nu}}\exp(-t\widetilde{H^{\xi_\varepsilon}}/2)(x,x)\Big)^m\Big]\exp(mt(\sup I + 1)/2).
\end{aligned}$$

We introduce an $m$-independent system of 2-dimensional Brownian motions $(w_1(\cdot),\ldots,w_m(\cdot))$ and use the Feynman-Kac formula to represent

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{R}^2}\frac{dx}{(1+|x|^2)^{2\nu}}\exp\Big(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}}\Big)(x,x)\Big)^m\Big]\\
=&\Big(\prod_{j=1}^m\int_{\mathbb{R}^2}\frac{dx_j}{(1+|x_j|^2)^{2\nu}}\Big)\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^m\int_0^t\frac{ds}{2}\xi_\varepsilon(x_j+w_j(s))\Big)\\
&\Big|w_1(t)=\cdots=w_m(t)=0\Big]\exp\Big(\frac{mt}{2}\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Big)(2\pi t)^{-m}\\
=&\Big(\prod_{j=1}^m\int_{\mathbb{R}^2}\frac{dx_j}{(1+|x_j|^2)^{2\nu}}\Big)\mathbb{E}\Big[\exp\Big(\frac{1}{8}\sum_{j=1}^m\int_0^t ds_1\int_0^t ds_2 e^{2\varepsilon^2\Delta}(w_j(s_1),w_j(s_2))\\
&+\frac{1}{4}\sum_{1\le j<k\le m}\int_0^t ds_1\int_0^t ds_2 e^{2\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)\\
&\Big|w_1(t)=\cdots=w_m(t)=0\Big]\exp\Big(\frac{mt}{2}\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Big)(2\pi t)^{-m}.
\end{aligned}$$

By the Schwarz inequality, this is less than or equal to

$$\begin{aligned}
&(2\pi t)^{-m}\exp\Big(\frac{mt}{2}\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Big)\mathbb{E}\Big[\exp\Big(\frac{1}{4}\int_0^t ds_1\int_0^t ds_2 e^{2\varepsilon^2\Delta}(w_1(s_1),w_1(s_2))\Big|w_1(t)=0\Big]^{m/2}\\
&\times\Big(\prod_{j=1}^m\int_{\mathbb{R}^2}\frac{dx_j}{(1+|x_j|^2)^{2\nu}}\Big)\mathbb{E}\Big[\exp\Big(\frac{1}{2}\sum_{1\le j<k\le m}\int_0^t ds_1\int_0^t ds_2 e^{2\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)\\
&\Big|w_1(t)=\cdots=w_m(t)=0\Big]^{1/2}.
\end{aligned}$$

Theorem 3.7 (iii) in Matsuda [13] showes

$$\begin{aligned}
\sup_{\varepsilon'\in(0,\infty)}\mathbb{E}\Big[\exp\Big(t\int_0^1 ds_1\int_0^1 ds_2(e^{(\varepsilon')^2\Delta}(w_1(s_1),w_1(s_2))\\
-\mathbb{E}[e^{(\varepsilon')^2\Delta}(w_1(s_1),w_1(s_2))|w_1(1)=0]\Big)\Big|w_1(1)=0\Big]
\end{aligned}$$

is finite for sufficiently small $t>0$ and Lemma 3.4 in Matsuda [13] showes

$$\int_0^t ds_1\int_0^t ds_2\mathbb{E}[e^{\varepsilon^2\Delta}(w_1(s_1),w_1(s_2))|w_1(t)=0]=\frac{t}{\pi}\log\frac{t}{2\varepsilon^2}+o(1)$$

as $\varepsilon\to 0$. On the other hand, Remark 3.1 in [21] showes

$$\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]=\frac{-1}{4\pi}\log\frac{1}{2\varepsilon^2}+o(1)$$

as $\varepsilon\to 0$. Thus

$$\sup_{\varepsilon\in(0,1]}(2\pi t)^{-m}\exp(mt\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)])\mathbb{E}\Big[\exp\Big(\frac{1}{4}\int_0^t ds_1\int_0^t ds_2 e^{\varepsilon^2\Delta}(w_1(s_1),w_1(s_2))\Big|w_1(t)=0\Big]^{m/2}$$

is finite for sufficiently small $t > 0$. By the Hölder inequality and the time symmetry of the Brownian motion, we have

$$\mathbb{E}\Big[\exp\Big(\frac{1}{4}\sum_{1\le j<k\le m}\int_0^t ds_1\int_0^t ds_2 e^{\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)\Big|w_1(t)=\cdots=w_m(t)=0\Big]$$
$$\le \prod_{1\le j<k\le m}\mathbb{E}\Big[\exp\Big({}_mC_2\int_0^{t/2} ds_1\int_0^{t/2} ds_2 e^{\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)\Big|w_j(t)=w_k(t)=0\Big]^{1/{}_mC_2}.$$

By Lemma 3.1 in Nakao [15], we have

$$\mathbb{E}\Big[\exp\Big({}_mC_2\int_0^{t/2} ds_1\int_0^{t/2} ds_2 e^{\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)\Big|w_j(t)=w_k(t)=0\Big]$$
$$\le 4\mathbb{E}\Big[\exp\Big({}_mC_2\int_0^{t/2} ds_1\int_0^{t/2} ds_2 e^{\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)\Big].$$

As in (4.3.8) in Chen [4] and (3.5) in Matsuda [13], we have

$$\mathbb{E}\Big[\Big(\int_0^t ds_1\int_0^t ds_2 e^{\varepsilon^2\Delta}(x_j+w_j(s_1),x_k+w_k(s_2))\Big)^n\Big]$$
$$=\Big(\prod_{h=1}^n\int_0^t ds_{1,h}\int_0^t ds_{2,h}\int_{\mathbb{R}^2} d\zeta_h\Big)\mathbb{E}\Big[\exp\Big(\sum_{h=1}^n 2\pi i\zeta_h\cdot(x_j+w_j(s_{1,h})-x_k-w_k(s_{2,h}))-4\varepsilon^2\pi^2|\zeta_h|\Big)\Big]$$
$$=\Big(\prod_{h=1}^n\int_0^t ds_{1,h}\int_0^t ds_{2,h}\int_{\mathbb{R}^2} d\zeta_h\Big)\exp\Big(\sum_{h=1}^n 2\pi i\zeta_h\cdot(x_j-x_k)-\frac{1}{2}\mathbb{E}\Big[\Big(\sum_{h=1}^n 2\pi\zeta_h\cdot(w_j(s_{1,h})-w_k(s_{2,h}))\Big)^2\Big]-4\varepsilon^2\pi^2|\zeta_h|\Big)$$
$$\le\Big(\prod_{h=1}^n\int_0^t ds_{1,h}\int_0^t ds_{2,h}\int_{\mathbb{R}^2} d\zeta_h\Big)\exp\Big(-\frac{1}{2}\mathbb{E}\Big[\Big(\sum_{h=1}^n 2\pi\zeta_h\cdot(w_j(s_{1,h})-w_k(s_{2,h}))\Big)^2\Big]\Big)$$
$$=\mathbb{E}\Big[\Big(\int_0^t ds_1\int_0^t ds_2\delta_0(w_j(s_1)-w_k(s_2))\Big)^n\Big]$$

for any $n \in \mathbb{N}$, $t, \varepsilon > 0$ and $x_j, x_k \in \mathbb{R}^2$. If we estimate the right hand side as in Chapter 3 in Chen [4], we have the following bound:

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t ds_1 \int_0^t ds_2\ \delta_0(w_j(s_1) - w_k(s_2))\Big)^n\Big] \\
=&\mathbb{E}\Big[\Big(\int_0^t ds_1 \int_0^t ds_2 \int_{\mathbb{R}^2} dy\ \delta_y(w_j(s_1))\delta_y(w_k(s_2))\Big)^n\Big] \\
=& \sum_{\sigma_1,\sigma_2 \in \mathfrak{S}_n} \int_{(\mathbb{R}^2)^n} dy_1 \cdots dy_n \int_{0 \le s_{1,\sigma_1(1)} \le \cdots \le s_{1,\sigma_1(n)} \le t} ds_{1,1} \cdots ds_{1,n} \\
&\times \mathbb{E}\Big[\delta_{y_{\sigma_1(1)}}(w_j(s_{1,\sigma_1(1)})) \cdots \delta_{y_{\sigma_1(n)}}(w_j(s_{1,\sigma_1(n)}))\Big] \\
&\times \int_{0 \le s_{2,\sigma_2(1)} \le \cdots \le s_{2,\sigma_2(n)} \le t} ds_{2,1} \cdots ds_{2,n} \mathbb{E}\Big[\delta_{y_{\sigma_2(1)}}(w_k(s_{2,\sigma_2(1)})) \cdots \delta_{y_{\sigma_2(n)}}(w_k(s_{2,\sigma_2(n)}))\Big] \\
\le& \sum_{\sigma_1,\sigma_2 \in \mathfrak{S}_n} \int_{0 \le s_{1,\sigma_1(1)} \le \cdots \le s_{1,\sigma_1(n)} \le t} ds_{1,1} \cdots ds_{1,n} \int_{0 \le s_{2,\sigma_2(1)} \le \cdots \le s_{2,\sigma_2(n)} \le t} ds_{2,1} \cdots ds_{2,n} \\
&\times \Big(\int_{(\mathbb{R}^2)^n} dy_1 \cdots dy_n \mathbb{E}\Big[\delta_{y_{\sigma_1(1)}}(w_j(s_{1,\sigma_1(1)})) \cdots \delta_{y_{\sigma_1(n)}}(w_j(s_{1,\sigma_1(n)}))\Big]^2\Big)^{1/2} \\
&\times \Big(\int_{(\mathbb{R}^2)^n} dy_1 \cdots dy_n \mathbb{E}\Big[\delta_{y_{\sigma_2(1)}}(w_k(s_{2,\sigma_2(1)})) \cdots \delta_{y_{\sigma_2(n)}}(w_k(s_{2,\sigma_2(n)}))\Big]^2\Big)^{1/2}.
\end{aligned}
$$

The right hand side is less than or equal to

$$
\Big(\frac{t}{4}\Big)^n \Big(\frac{n!}{\Gamma((n+2)/2)}\Big)^2.
$$

Thus we see that

$$
\begin{aligned}
&\sup_{x_j, x_k} \mathbb{E}\Big[\exp\Big(\lambda \int_0^t ds_1 \int_0^t ds_2 e^{\varepsilon^2 \Delta}(x_j + w_j(s_1), x_k + w_k(s_2))\Big)\Big] \\
\le&\mathbb{E}\Big[\exp\Big(\lambda \int_0^t ds_1 \int_0^t ds_2 \delta_0(w_j(s_1) - w_k(s_2))\Big)\Big] \\
<&\infty
\end{aligned}
$$

if $0 \le \lambda t < 2$. Thus we can complete the proof. □

## 7. Proof of Theorem 1

In this section we prove Theorem 1 by using the results in the preceding sections.

For any $1 \le p < \infty$, $1 < \alpha < 1 + p/4$ and $0 < m_0 < \overline{m_0}$, there exists $\overline{E_0} \in (-\infty, 0)$ such that the consequence of Lemma 5.4 holds, where $\overline{m_0}$ is the number given in Proposition 5.1. Thus for any $E_0 \le \overline{E_0}$, there exist $E_1 < E_0$ and $L_0 \in 6\mathbb{N}$ such that

$$
\mathbb{P}(R(m_0/2, K_k, [E_1, E_0], L_k, \boldsymbol{a}, \boldsymbol{a}')) > 1 - L_k^{-p}
$$

for any $k \in \mathbb{N}$ and any $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{Z}^2$ satisfying $|\boldsymbol{a}-\boldsymbol{a}'|_\infty > L_k+2$ by Proposition 5.1 and Lemma 5.4. We set $A_{k+1}(\boldsymbol{a}') := \Lambda_{2b_1 L_{k+1}}(\boldsymbol{a}') \setminus \Lambda_{2L_k+2}(\boldsymbol{a}')$ and $E_k(\boldsymbol{a}') := \{\xi :$ there exists $\lambda \in [E_1, E_0]$ and $\boldsymbol{a} \in A_{k+1}(\boldsymbol{a}') \cap \mathbb{Z}^2$ such that $\Lambda_{L_k}(\boldsymbol{a}')$ and $\Lambda_{L_k}(\boldsymbol{a})$ are $(m_k, K_k, \lambda)$-singular$\}$, where $b_1$ is taken from $(1 + 2/L_0, \infty)$ so that $\bigcup_{k=0}^{\infty} A_{k+1}(\boldsymbol{a}') = \mathbb{R}^2 \setminus \Lambda_{2L_0+2}(\boldsymbol{a}')$. Since

$$E_k(\boldsymbol{a}') = \bigcup_{\boldsymbol{a} \in A_{k+1}(\boldsymbol{a}') \cap \mathbb{Z}^2} R(m_0/2, K_k, [E_1, E_0], L_k, \boldsymbol{a}, \boldsymbol{a}')^c,$$

we have

$$\sum_{k=0}^{\infty} \mathbb{P}(E_k(\boldsymbol{a}')) < \infty$$

by taking $p$ larger than $2\alpha$. By the Borel and Cantelli lemma, for almost all $\xi$ and any $\boldsymbol{a}' \in \mathbb{Z}^2$, there exists $k(\boldsymbol{a}', \xi) \in \mathbb{N}$ such that $\Lambda_{L_k}(\boldsymbol{a}')$ or $\Lambda_{L_k}(\boldsymbol{a})$ is $(m_0/2, K_k, \lambda)$-regular for any $k \geq k(\boldsymbol{a}', \xi)$, $\lambda \in [E_1, E_0]$ and $\boldsymbol{a} \in A_{k+1}(\boldsymbol{a}') \cap \mathbb{Z}^2$. Let $u$ be a generalized eigenfunction of $\widetilde{H^\xi}$ with a generalized eigenvalue $\lambda \in [E_1, E_0]$. We take $\boldsymbol{a}' \in \mathbb{Z}^2$ so that $\|\chi_{\boldsymbol{a}'} u\|_{L^2(\mathbb{R}^2)} > 0$. Then there exists $k'(\xi) \in \mathbb{N}$ such that $\Lambda_{L_k}(\boldsymbol{a}')$ is $(m_0/2, K_k, \lambda)$-singular for any $k \geq k'(\xi)$. Then $\Lambda_{L_k}(\boldsymbol{a})$ is $(m_0/2, K_k, \lambda)$-regular for any $\boldsymbol{a} \in A_{k+1}(\boldsymbol{a}') \cap \mathbb{Z}^2$ and $k \geq k'(\boldsymbol{a}', \xi) := k(\boldsymbol{a}', \xi) \vee k'(\xi)$. By Lemma 3.3 (ii) in [21] for $\xi(\cdot + \boldsymbol{a}')$, we have

$$\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi,\boldsymbol{a}'} (\log(2 + |a - \boldsymbol{a}'|))^{1/2}$$

and

$$\|\chi_a Y_\xi\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi,\boldsymbol{a}'} \log(2 + |a - \boldsymbol{a}'|).$$

Moreover, Lemma 4.3 (i) in [21] is generalized as follows

**Lemma 7.1.** *For any $\epsilon \in (0,1)$ and almost all $\xi$, there exist $C_{\epsilon,\xi}, C'_{\epsilon,\xi}, C''_{\epsilon,\xi}, C_\epsilon, C'_\epsilon, C''_\epsilon \in (0, \infty)$ such that*

$$\begin{aligned}\|\chi_a Y_{\xi,L,\boldsymbol{a}}\|_{C^{-\epsilon}(\mathbb{R}^2)} &\leq C_{\epsilon,\xi} (\log(2 + |a| + |\boldsymbol{a}| + L)) \exp(-C_\epsilon d(a, \Lambda_L(\boldsymbol{a}))^2) \\ &\leq C'_{\epsilon,\xi} (\log(2 + |a| + L)) \exp(-C'_\epsilon d(a, \Lambda_L(\boldsymbol{a}))^2)\end{aligned}$$

*and*

$$\|\chi_a (Y_\xi - Y_{\xi,L,\boldsymbol{a}})\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq C''_{\epsilon,\xi} (\log(2 + |a| + |\boldsymbol{a}| + L)) \exp(-C''_\epsilon d(a, \Lambda_L(\boldsymbol{a})^c)^2).$$

*for any $a, \boldsymbol{a} \in \mathbb{Z}^2$ and $L \in \mathbb{N}$.*

By this lemma for $\xi(\cdot + \boldsymbol{a}')$, we have

$$\begin{aligned}\|\chi_a Y_{\xi,L_k-2,\boldsymbol{a}}\|_{C^{-\epsilon}(\mathbb{R}^2)} &\leq C_{\epsilon,\xi,\boldsymbol{a}'} (\log(2 + |a - \boldsymbol{a}'| + |\boldsymbol{a} - \boldsymbol{a}'| + L_k)) \exp(-C_\epsilon d(a - \boldsymbol{a}', \Lambda_{L_k}(\boldsymbol{a}))^2) \\ &\leq C'_{\epsilon,\xi,\boldsymbol{a}'} (\log(2 + |a - \boldsymbol{a}'| + L_k)) \exp(-C'_\epsilon d(a - \boldsymbol{a}', \Lambda_{L_k}(\boldsymbol{a}))^2)\end{aligned}$$

and

$$\|\chi_a(Y_\xi - Y_{\xi,L_k-2,\boldsymbol{a}})\|_{C^{-\epsilon}(\mathbb{R}^2)} \le C''_{\epsilon,\xi,\boldsymbol{a}'}(\log(2+|a-\boldsymbol{a}'|+|\boldsymbol{a}-\boldsymbol{a}'|+L_k))\exp(-C''_\epsilon d(a-\boldsymbol{a}',\Lambda_{L_k}(\boldsymbol{a})^c)^2).$$

Then we have

$$\begin{aligned}\Xi(\boldsymbol{a},L_k-2,\xi) \le& C_{\xi,\boldsymbol{a}'}\log(2+|a-\boldsymbol{a}'|+L_k),\\ \Xi_c(a,\boldsymbol{a},L_k-2,\xi) \le& C'_{\xi,\boldsymbol{a}'}(\log(2+|a-\boldsymbol{a}'|+|\boldsymbol{a}-\boldsymbol{a}'|+L_k))\exp(-C'_\epsilon d(a,\Lambda_{L_k}(\boldsymbol{a}))^2)\end{aligned}$$

and

$$\Xi_c(a,\boldsymbol{a},L_k-2,\xi) \le C'_{\xi,\boldsymbol{a}'}\log(2+|a-\boldsymbol{a}'|).$$

We fix $\boldsymbol{a}'$ and $\xi$ such that the above estimates hold. By Proposition 6.1, we have

$$\|\chi_{\boldsymbol{a}}u\|_{L^2(\mathbb{R}^2)} \le \mathcal{K}_k\sum_{a_1\in\mathbb{Z}^2}\|\chi_{a_1}u\|_{L^2(\mathbb{R}^2)}\exp(-c_1|a_1-\boldsymbol{a}|_\infty - c_2L_k)$$

for any $\boldsymbol{a}\in A_{k+1}(\boldsymbol{a}')\cap\mathbb{Z}^2$ with $k\ge k'(\boldsymbol{a}',\xi)$, where

$$\mathcal{K}_k := C_{\xi,\boldsymbol{a}'}K_k(\log L_k)^c(\log(2+|\boldsymbol{a}-\boldsymbol{a}'|)^{(15)/2}(\max|\overline{\xi_0}|+|\lambda|+1).$$

Thus for any $\boldsymbol{a}\in A_{k+1}(\boldsymbol{a}')\cap\mathbb{Z}^2$ with $k\ge k'(\boldsymbol{a}',\xi)$, we have

$$\begin{aligned}&\|\chi_{\boldsymbol{a}}u\|_{L^2(\mathbb{R}^2)}\\ \le&\sum_{a_1\in\mathbb{Z}^2\cap\Lambda_{2L_k+2}(\boldsymbol{a}')}c\|u\|_{\mathcal{H}_-}(1+|\boldsymbol{a}'|+L_k)^{2\nu}\mathcal{K}_k\exp(-c_1|a_1-\boldsymbol{a}|_\infty-c_2L_k)\\ &+\sum_{a_1\in\mathbb{Z}^2\cap A_{k+1}(\boldsymbol{a}')}\|\chi_{a_1}u\|_{L^2(\mathbb{R}^2)}\mathcal{K}_k\exp(-c_1|a_1-\boldsymbol{a}|_\infty-c_2L_k)\\ &+\sum_{k'>k}\sum_{a_1\in\mathbb{Z}^2\cap A_{k'+1}(\boldsymbol{a}')}\sum_{a_2\in\mathbb{Z}^2}c\|u\|_{\mathcal{H}_-}(1+|a_2|)^{2\nu}\mathcal{K}_k\mathcal{K}_{k'}\\ &\times\exp(-c_1(|a_2-a_1|_\infty+|a_1-\boldsymbol{a}|_\infty)-c_2(L_k+L_{k'})).\end{aligned}$$

We repeat this estimate for the second term of the right hand side to obtain

$$\begin{aligned}
&\|\chi_{\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} \\
\leq &\sum_{h=1}^{m} \sum_{a_1,\dots,a_{h-1}\in\mathbb{Z}^2\cap A_{k+1}(\boldsymbol{a}'),a_h\in\Lambda_{2L_k+2}(\boldsymbol{a}')} c\|u\|_{\mathcal{H}_-}(1+|\boldsymbol{a}'|+L_k)^{2\nu}\mathcal{K}_k^h \\
&\times \exp(-c_1(|a_h-a_{h-1}|_\infty+|a_{h-1}-a_{h-2}|_\infty+\cdots+|a_1-\boldsymbol{a}|_\infty)-c_2hL_k) \\
&+ \sum_{a_1,a_2,\dots a_m\in\mathbb{Z}^2\cap A_{k+1}(\boldsymbol{a}')} \|\chi_{a_m}u\|_{L^2(\mathbb{R}^2)}\mathcal{K}_k^m \\
&\times \exp(-c_1(|a_m-a_{m-1}|_\infty+|a_{m-1}-a_{m-2}|_\infty+\cdots+|a_1-\boldsymbol{a}|_\infty)-c_2mL_k) \\
&+ \sum_{h=1}^{m}\sum_{k'>k}\sum_{a_1,\dots,a_h\in\mathbb{Z}^2\cap A_{k+1}(\boldsymbol{a}'),a_h\in\mathbb{Z}^2\cap A_{k'+1}(\boldsymbol{a}'),a_{h+1}\in\mathbb{Z}^2} c\|u\|_{\mathcal{H}_-}(1+|a_{h+1}|)^{2\nu}\mathcal{K}_k^h\mathcal{K}_{k'} \\
&\times \exp(-c_1(|a_{h+1}-a_h|_\infty+|a_h-a_{h-1}|_\infty+\cdots+|a_1-\boldsymbol{a}|_\infty)-c_2(hL_k+L_{k'}))
\end{aligned}$$

for any $m\in\mathbb{N}$. In the first term,

$$\begin{aligned}
&|a_h-a_{h-1}|_\infty+|a_{h-1}-a_{h-2}|_\infty+\cdots+|a_1-\boldsymbol{a}|_\infty \\
\geq &|\boldsymbol{a}-\boldsymbol{a}'|_\infty-L_k-1.
\end{aligned}$$

Thus if we take $c_1,c_2$ so that $c_1<c_2$, then the first term is dominated by a geometric series

$$c\|u\|_{\mathcal{H}_-}(1+|\boldsymbol{a}'|+L_k)^{2\nu}\exp(-c_1|\boldsymbol{a}-\boldsymbol{a}'|_\infty)\sum_{h=1}^{\infty}(\#(\mathbb{Z}^2\cap A_{k+1}(\boldsymbol{a}'))^{h-1}\mathcal{K}_k^h\exp(-c_2(h-1)L_k)$$

independently of $m$. The common ratio is less than 1 if $L_k\geq c\log\log(2+|\boldsymbol{a}-\boldsymbol{a}'|_\infty)$. Then the series is dominated by

$$c\|u\|_{\mathcal{H}_-}(1+|\boldsymbol{a}'|+L_k)^{2\nu}\exp(-c_1|\boldsymbol{a}-\boldsymbol{a}'|_\infty/2).$$

In the third term,

$$\begin{aligned}
&|a_{h-1}-a_{h-2}|_\infty+|a_{h-2}-a_{h-3}|_\infty+\cdots+|a_1-\boldsymbol{a}|_\infty \\
\geq &|\boldsymbol{a}-\boldsymbol{a}'|_\infty-L_{k+1}.
\end{aligned}$$

Thus if we take $c_1,c_2$ so that $c_1b_1<c_2$, then the third term is dominated by a geometric series

$$\begin{aligned}
&c\|u\|_{\mathcal{H}_-}(1+|\boldsymbol{a}'|+L_{k+1})^{2\nu}(\log(2+|\boldsymbol{a}-\boldsymbol{a}'|_\infty))^{(15)/2}(\max|\overline{\xi_0}|+|\lambda|+1) \\
&\times\exp(-c_1|\boldsymbol{a}-\boldsymbol{a}'|_\infty)\sum_{h=1}^{\infty}\mathcal{K}_k^hL_{k+1}^{2h}\exp(-c_2hL_k)
\end{aligned}$$

independently of $m$. Thus if $L_k\geq c\log\log(2+|\boldsymbol{a}-\boldsymbol{a}'|_\infty)$, then the series is dominated by

$$c\|u\|_{\mathcal{H}_-}(1+|\boldsymbol{a}'|+L_{k+1})^{2\nu}(\log(2+|\boldsymbol{a}-\boldsymbol{a}'|_\infty))^{(15)/2}(\max|\overline{\xi_0}|+|\lambda|+1)\exp(-c_1|\boldsymbol{a}-\boldsymbol{a}'|_\infty/2).$$

In the second term,

$$
\begin{aligned}
&|a_m - a_{m-1}|_\infty + |a_{m-1} - a_{m-2}|_\infty + \cdots + |a_1 - \boldsymbol{a}|_\infty \\
\geq &|\boldsymbol{a} - \boldsymbol{a}'|_\infty - b_1 L_{k+1}.
\end{aligned}
$$

Thus if we take $m$ larger than $(2b_1 L_{k+1})/(c_2 L_k)$, then the second term is dominated by

$$
(\#(\mathbb{Z}^2 \cap A_{k+1}(\boldsymbol{a}'))^m c\|u\|_{\mathcal{H}_-}(1 + |\boldsymbol{a}'| + L_{k+1})^{2\nu} \mathcal{K}_k^m \exp(-c_1|\boldsymbol{a} - \boldsymbol{a}'|_\infty - c_2 m L_k/2).
$$

Moreover we can assume $m < C_0 L_k^{\alpha-1}$ with a finite number $C_0$. Then the second term is dominated by

$$
c\|u\|_{\mathcal{H}_-} \exp(-c_1|\boldsymbol{a} - \boldsymbol{a}'|_\infty - c_2 L_k^\alpha + c_3 L_k^{\alpha-1} \log\log(2 + |\boldsymbol{a} - \boldsymbol{a}'|_\infty)).
$$

We can take also $L_k$ so that $c \log\log(2 + |\boldsymbol{a} - \boldsymbol{a}'|_\infty) \leq L_k \leq C \log\log(2 + |\boldsymbol{a} - \boldsymbol{a}'|_\infty)$ with a finite number $C$. Then the second term is dominated by

$$
c\|u\|_{\mathcal{H}_-} \exp(-c_1|\boldsymbol{a} - \boldsymbol{a}'|_\infty/2).
$$

Therefore we obtain

$$
\overline{\lim_{|\boldsymbol{a} - \boldsymbol{a}'|_\infty \to \infty}} |\boldsymbol{a} - \boldsymbol{a}'|_\infty^{-1} \log \|\chi_{\boldsymbol{a}} u\|_{L^2(\mathbb{R}^2)} < 0.
$$

This implies $u \in L^2(\mathbb{R}^2)$. Since Theorem 1 in [21] showes the operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_{+0}(\widetilde{H^\xi})$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$, $u$ is an eigenfunction of the self-adjoint extension $\overline{\widetilde{H^\xi}}$ with the eigenvalue $\lambda$. Since the spectrum of $\overline{\widetilde{H^\xi}}$ is $\mathbb{R}$ by Theorem 2, we can comlete the proof.

## References


1. R. Allez and K. Chouk, The continuous Anderson hamiltonian in dimension two, arXiv:1511.02718v2.
2. I. Bailleul, F. Bernicot and D. Frey, Spacetime paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations, Ann. Scient. Éc. Norm. Sup., **51** (2018),1399–1457.
3. R. Carmona and J. Lacroix, Spectral theory of random Schrödinger operators, Probab. Appl.., Birkhäuser Boston Inc., (1990).
4. X. Chen, Random walk intersections: large deviations and related topics, Mathematical Surveys and Monographs, vol. 157, American Mathematical Society, (2010).
5. J. M. Combes and L. Thomas, Asymptotic behavior of eigenfunctions for multiparticle Schroödinger operators, Comm. Math. Phys., **34** (1973), 251–270.
6. J. Fröhlich and T. Spencer, Absence of diffusion in the Anderson tight binding model for large disorder or low energy, *Comm. Math. Phys.*, **88** (1983), 151–184.
7. F. Germinet and A. Klein, Bootstrap multiscale analysis and localization in random media, *Comm. Math. Phys.*, **222** (2001), 415–448.

8. M. Hairer and C. Labbé, A simple construction of the continuum parabolic Anderson model on $\mathbb{R}^2$, Electron. Commun. Probab. **20** (2015), 1–11.

9. M. Hairer and C. Labbé, Multiplicative stochastic heat equations on the whole space, J. Eur. Math. Soc., **20** 4 (2018), 1005–1054.

10. Y. Hsu and C. Labbé, Construction and spectrum of the Anderson Hamiltonian with white noise potential on $\mathbb{R}^2$ and $\mathbb{R}^3$, preprint.

11. N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland, Amsterdam, 1989.

12. A. Klein, A. Koines and M. Seifert, Generalized eigenfunctions for waves in inhomogeneous media. J. Funct. Anal., **190** (2002), 255–291.

13. T. Matsuda, Integrated density of states of the Anderson Hamiltonian with two-dimensional white noise, Stochastic Processes and their Applications **153** (2022), 91-–127

14. A. Mouzard, Weyl law for the Anderson Hamiltonian on a two-dimensional manifold, Ann. Inst. H. Poincaré, Probabilités et Statistiques **58**(3) (2022), 1385–1425.

15. S. Nakao, On the spectral distribution of the Schrödinger operator with random potential, Jpn. J. Math. New Ser. **3** (1) (1977) 111—139.

16. D. Nualart, The Malliavin calculus and related topics, Second edition, Probability and its Applications (New York). Springer-Verlag, Berlin, 2006

17. L. Pastur and A. Figotin, Spectra of Random and Almost-Periodic Operators, Grundlehren Math. Wiss., 297, Springer, (1992).

18. I. Shigekawa, Stochastic analysis, American Mathematical Society, Providence, RI (2004).

19. P. Stollmann, Caught by disorder, Bound states in random media, Progr. Math. Phys., 20, Birkhäuser, (2001).

20. D. W. Stroock and S. R. S. Varadhan, Multidimensional diffusion processes, Springer-Verlag, Berlin, (2006). ISBN = 978-3-540-28998-2; 3-540-28998-4,

21. N. Ueki, A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane, Stochastic Processes and their Applications **186** (2025), 104642.

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