A PROBABILISTIC APPROACH TO THE HEAT EQUATION FOR THE $\bar{\partial}$-NEUMANN PROBLEM

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§1. Introduction

In this note, we consider the heat equation for the $\bar{\partial}$-Neumann problem. This is an initial boundary value problem whose boundary consition includes an imaginary directional differentiation. The main purpose of this note is to give a probabilistic expression (2.12) below of the heat kernel for the $\bar{\partial}$-Neumann problem on a strongly pseudoconvex Siegel domain D, which is one of the most fundamental complex manifolds with boundary. In this case, by using the methods of the partial differential equations, Stanton gave an explicit formula of the kernel on the space of (0,q)-forms [8]. We give here an explicit representation of the heat kernel on the space of (p,q)-forms in terms of the Malliavin calculus.

In [6], Malliavin has considered a similar problem in a domain of the complex plane. He constructed the solution of the heat equation for the $\bar{\partial}$-Neumann problem on a half plane by using a method of singular perturbations and pointed out that a method related to the Fourier transform can be applied to this problem. As we will state in §3 below, by combining a method in the Malliavin calculus with the Fourier transform of the Dirac δ-function, we can construct the heat kernel of this problem. To describe the main idea of our method, we first consider the heat equation associated with the $\bar{\partial}_b$-Laplacian on the Heisenberg group H_n:

$$\left(\frac{\partial}{\partial t} + \Box_\alpha\right) F = 0 \tag{1.1}$$

where

$$\Box_\alpha = -\frac{1}{4}\sum_{j=1}^{2n} X_j^2 - i\alpha\frac{\partial}{\partial u}, \qquad -n < \alpha < n,\ \alpha \in \mathbb{Z}.$$

For the notations and notions, see §2 below and Folland-Stein [2]. Let $k_t^\alpha(Y, Y')$, $Y, Y' \in H_n$, be the heat kernel of (1.1). In case of $\alpha = 0$, the kernel $k_t^0(Y, Y')$ can be represented in terms of the generalized expectation in the sense of Watanabe [10]:

$$k_t^0(0,(x,u)) = E\left[\delta_{(x,u)}\left(\frac{x(t)}{\sqrt{2}}, S_0(t)\right)\right] \tag{1.2}$$

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where $x(t)$ is the (2n)-dimensional Brownian motion and $S_0(t)$ is a sum of the Lévy's stochastic areas given by (2.8) below. Since

$$\delta_u(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\ e^{i\lambda(u-v)},$$

as Gaveau [3] has showed, the right hand side of (1.2) can be rewritten in the form

$$k_t^0(0,(x,u)) = \int_{-\infty}^{\infty} d\lambda\ f_t^{0,0}(x,u,\lambda) \tag{1.3}$$

where

$$f_t^{0,0}(x,u,\lambda) = \frac{1}{2\pi}e^{i\lambda u}E\left[e^{-i\lambda S_0(t)}\delta_x\left(\frac{x(t)}{\sqrt{2}}\right)\right]. \tag{1.4}$$

In general, we may formally expect that

$$k_t^\alpha(0,(x,u)) = E\left[\delta_{(x,u)}\left(\frac{x(t)}{\sqrt{2}}, S_0(t) - i\alpha t\right)\right]. \tag{1.5}$$

However, since $\delta_{(x,u)}$ is the Dirac δ function on $\mathbb{R}^{2n+1}$ and $(x(t)/\sqrt{2}, S_0(t) - i\alpha t)$ does not lie on $\mathbb{R}^{2n+1}$, we do not know how to give a correct mathematical sense to the right hand side of (1.5). But, as in the case of Gaveau [3], the function defined by (1.6) below is also a candidate of the kernel $k_t^\alpha(0,(x,u))$:

$$\int_{-\infty}^{\infty} d\lambda\ f_t^{0,\alpha}(x,u,\lambda) \tag{1.6}$$

where

$$\begin{aligned} f_t^{0,\alpha}(x,u,\lambda) &= \frac{1}{2\pi}e^{i\lambda u - t\lambda\alpha}E\left[e^{-i\lambda S_0(t)}\delta_x\left(\frac{x(t)}{\sqrt{2}}\right)\right] \\ &= \frac{1}{2\pi^{n+1}}\left(\frac{\lambda}{\sinh(t\lambda)}\right)^n \exp(i\lambda u - t\lambda\alpha - |x|^2\lambda\coth(t\lambda)). \end{aligned} \tag{1.7}$$

In fact, it is easily seen by using the last line of the above expression that the integral in (1.6) is well defined and gives the desired heat kernel of (1.1) (cf. [8]). Keeping the above observations in mind, we can show that similar methods can be applied to the construction of the heat kernel for the $\bar{\partial}$-Neumann problem on the domain D and so we obtain the representation (2.12) below of the kernel. However, in our case, to show the convergence of integrals appeared in the representation of the heat kernel, we need many steps of discussions including the proof of (2.14) below, which plays a crucial role. Hence, for simplicity, we here give only the proof of the representation of the heat kernel for the $\bar{\partial}$-Neumann problem on a half plane, which Malliavin also considered. The details of the proof of (2.12) will appear in [9].

§2. Results in the strongly pseudoconvex Siegel domain

In this section, we state our results in the strongly pseudoconvex Siegel domain. Let $D = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}; \text{Im } w > |z|^2\}$. This D is called a strongly pseudoconvex Siegel domain. We use the global coordinate $(x^1, x^2, \cdots, x^{2n}, u, r)$ defined by

$$\begin{aligned} & z^j = x^j + ix^{n+j}, \quad j = 1, 2, \cdots, n, \\ & \text{Re } w = u, \\ & \text{Im } w - |z|^2 = r \ (> 0). \end{aligned} \tag{2.1}$$

Then the domain D can be identified with the product $H_n \times \mathbb{R}_+$, and $(x^1, x^2, \cdots, x^{2n}, u)$ and r are coordinates of H_n and $\mathbb{R}_+$, respectively. This H_n is the Heisenberg group whose group law is given by

$$(x', u')(x, u) = (x' + x, u' + u + 2\sum_{j=1}^{n}(x'^{n+j}x^j - x'^j x^{n+j})). \tag{2.2}$$

We define the vector fields $\{X_1, X_2, \cdots, X_{2n}\}$ on D by

$$X_j = \frac{\partial}{\partial x^j} + 2x^{n+j}\frac{\partial}{\partial u}, \qquad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - 2x^j\frac{\partial}{\partial u}, \quad j = 1, 2, \cdots, n. \tag{2.3}$$

We give D the Hermitian metric so that the vector fields $\{X_1/\sqrt{2}, X_2/\sqrt{2}, \cdots, X_{2n}/\sqrt{2}, \partial/\partial u, \partial/\partial r\}$ constitutes an orthonormal frame of the tangent bundle TD. This domain D is one of the most fundamental Hermitian manifolds with boundary.

We define the complex differential 1-forms $\{\omega^1, \omega^2, \cdots, \omega^{n+1}\}$ by

$$\omega^j = dx^j + idx^{n+j}, \quad j = 1, 2, \cdots, n, \qquad \omega^{n+1} = \partial r. \tag{2.4}$$

Then $\{\omega^1, \omega^2, \cdots, \omega^{n+1}\}$ constitutes a unitary frame of the holomorphic cotangent bundle $\Lambda^{1,0}(T^*D)$. We define the projectors P and Q on $\Lambda^{p,q}(T^*D)$ by

$$P = \text{ext}(\bar{\omega}^{n+1})\text{int}(\omega^{n+1}), \qquad Q = \text{int}(\omega^{n+1})\text{ext}(\bar{\omega}^{n+1}). \tag{2.5}$$

Then the heat equation for the $\bar{\partial}$-Neumann problem is the following equation on the space of (p,q)-forms:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \Box\right)F(t, X) = 0, \qquad t > 0, \ X \in D, \\ \lim\limits_{t \searrow 0} F(t, X) = f(X), \\ PF(t, X) = Q\left(\dfrac{\partial}{\partial r} - i\dfrac{\partial}{\partial u}\right)F(t, X) = 0 \qquad \text{on } bD, \end{cases} \tag{2.6}$$

where bD is the boundary of the domain D and $\Box$ is the complex Laplacian acts as follows:

$$\begin{aligned} \Box = & -\frac{1}{4}\sum_{j=1}^{2n} X_j^2 - \frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2}\right) + i(n - 2C)\frac{\partial}{\partial u} \\ & - \sum_{j+1}^{2n}\frac{A_j}{\sqrt{2}}X_j - B. \end{aligned} \tag{2.7}$$

C, A_j and B are endomorphisms on the bundle $\Lambda^{p,q}(T^*D)$ whose coefficients of the frame $\{\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_q}\}$ are independent of the point D. Especially, C is expressed as

$$C = qQ + (q-1)P.$$

In (2.7), we assume that the differential operators act only on the coefficients of the frame $\{\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\wedge\bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_q}\}$. The equation (2.6) has the similar form to that of the heat equation for the d-Neumann problem ([1],[5],[7]) except for *the imaginary directional differentiation* in the boundary condition. This imaginary directional differentiation is the particular point of the equation (2.6). Our purpose is to construct the fundamental solution (heat kernel) $h_t(X,X')$ of the equation (2.6).

Now, let (W_0^{2n},P) be the (2n)-dimensional classical Wiener space and $x(t) = (x^1(t),x^2(t),\cdots,x^{2n}(t))$ be the element of W_0^{2n}. We define a process $S_0(t)$ by

$$S_0(t) = \sum_{j=1}^{n}\int_0^t (x^{n+j}(s)dx^j(s) - x^j(s)dx^{n+j}(s)) \tag{2.8}$$

and an $\mathrm{End}(\Lambda^{p,q}(T^*D))$-valued process $M(t)$ by the solution of the following stochastic integral equation:

$$M(t) = I + \int_0^t M(s)\left\{\sum_{j=1}^{2n} A_j dx^j(s) + Bds\right\}. \tag{2.9}$$

Keeping the obserbation mentioned in §1 in mind, we set, for any $\alpha\in\mathbb{Z}$ and $(t,x,u,\lambda)\in[0,\infty)\times H_n\times\mathbb{R}$,

$$\mathbf{f}_t^\alpha(x,u,\lambda) = \frac{1}{2\pi}e^{i\lambda u - t\lambda\alpha}E\left[e^{-i\lambda S_0(t)}M(t)\delta_x\left(\frac{x(t)}{\sqrt{2}}\right)\right]. \tag{2.10}$$

where $E[\,\cdot\,]$ in the right hand side means the generalized Wiener functional expectation (cf. [10]). Next, we set

$$\begin{aligned}\mathbf{p}_t^\alpha(x,u) &= \int_{-\infty}^{\infty} d\lambda\ \mathbf{f}_t^\alpha(x,u,\lambda)\exp\left(-\frac{t}{2}\lambda^2\right),\\ \mathbf{q}_t^\alpha(x,u,r) &= -\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\lambda\ \lambda\mathbf{f}_t^\alpha(x,u,\lambda)e^{r\lambda}\int_{\sqrt{\frac{t}{2}}\left(\frac{r}{t}+\lambda\right)}^{\infty} d\mu\ e^{-\mu^2}\end{aligned} \tag{2.11}$$

and

$$e_t^{\pm}(r,r') = \frac{1}{\sqrt{2\pi t}}\left\{\exp\left(-\frac{(r-r')^2}{2t}\right)\pm\exp\left(-\frac{(r+r')^2}{2t}\right)\right\}.$$

Then our results are stated as follows:

Main Theorem. *The heat kernel $h_t(X, X')$ for the $\bar{\partial}$-Neumann problem on the (p,q)-forms is expressed as follows:*

$$
\begin{aligned}
&h_t((x,u,r),(x',u',r')) \\
&= \mathbf{p}_t^{n-2q+2}((x,u)^{-1}(x',u'))e_t^-(r,r')P \\
&\quad + \mathbf{p}_t^{n-2q}((x,u)^{-1}(x',u'))e_t^+(r,r')Q \\
&\quad + \mathbf{q}_t^{n-2q}((x,u)^{-1}(x',u'),r+r')Q.
\end{aligned} \tag{2.12}
$$

This $h_t((x,u,r),(x',u',r'))$ is the restriction to $\bar{D}$ of a rapidly decreasing function of (x,u,r) (resp. (x',u',r')) for each fixed $(t,x',u',r') \in (0,\infty)\times D$ (resp. $(t,x,u,r) \in (0,\infty)\times D$) uniformly with respect to $t \in [t_0,t_1]$ for any $0 < t_0 \leqq t_1$.

This theorem has been proved in the (0,q)-form case by Stanton [8]. In the (0,q)-form case, $M(t)$ becomes identity. So by using the Lévy's formula, as we stated in §1, we can represent $\mathbf{f}_t^\alpha$ in (2.10) more explicitly:

$$
\begin{aligned}
\mathbf{f}_t^\alpha(x,u,\lambda) &= f_t^{0,\alpha}(x,u,\lambda) \\
&= \frac{1}{2\pi^{n+1}}\left(\frac{\lambda}{\sinh(t\lambda)}\right)^n \exp(i\lambda u - t\lambda\alpha - |x|^2\lambda\coth(t\lambda)).
\end{aligned} \tag{2.13}
$$

In the proof of the Main Theorem, the following lemma plays a crucial role:

Key Lemma. *For any $\alpha,\beta \in \mathbb{Z}_+^{2n}$, $k \in \mathbb{Z}_+$ and $0 < t_0 < t_1$, the following estimate holds;*

$$
\begin{aligned}
&\overline{\lim_{\lambda\to\pm\infty}} \sup_{t_0\leqq t\leqq t_1} \frac{1}{t|\lambda|}\log \sup_{x\in\mathbb{R}^{2n}} \Bigg\| x^\beta \partial_x^\alpha E\Bigg[S_0(t)^k e^{-i\lambda S_0(t)} \\
&\qquad \times M(t)\delta_x\left(\frac{x(t)}{\sqrt{2}}\right)\Bigg]\Bigg\| \\
&\leqq -n,
\end{aligned} \tag{2.14}
$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm in $\mathrm{End}(\Lambda^{p,q}(T^*D))$.

By using this lemma, we can prove that the expression (2.12) is well-defined. Next, by using the Feynman-Kac formula, we see the following:

$$
\frac{\partial}{\partial t}\mathbf{f}_t^\alpha((x,u)^{-1}(x',u'),\lambda) = -\Box_\lambda^\alpha \mathbf{f}_\lambda^\alpha((x,u)^{-1}(x',u'),\lambda) \tag{2.15}
$$

where

$$
\begin{aligned}
\Box_\lambda^\alpha = &-\frac{1}{4}\sum_{j=1}^n \left\{\left(\frac{\partial}{\partial x^j} - 2i\lambda x^{n+j}\right)^2 + \left(\frac{\partial}{\partial x^{n+j}} + 2i\lambda x^j\right)^2\right\} \\
&-\sum_{j=1}^n \left\{\frac{A_j}{\sqrt{2}}\left(\frac{\partial}{\partial x^j} - 2i\lambda x^{n+j}\right) + \frac{A_{n+j}}{\sqrt{2}}\left(\frac{\partial}{\partial x^{n+j}} + 2i\lambda x^j\right)\right\} \\
&+ B + \lambda\alpha.
\end{aligned}
$$

Furthermere, by refering the Key Lemma, we obtain

$$\left(\frac{\partial}{\partial t}+\Box\right)h_t((x,u,r),(x',u',r'))=0. \tag{2.16}$$

By repeating straightforward calculations, we can show that the expression (2.12) satisfies the boundary condition. For the initial condition, we need further devices. The detail of the matters in this section will appear in [9].

§3. The case of a half plane

In this section, we apply our methods to the $\bar{\partial}$-Neumann problem on a half plane, which Malliavin also considered in [6]. Let $M=\{z=(x,y)\in\mathbb{C}:\ x\geqq 0\}$ with the canonical metric. On the space of (0,0)-forms, the heat equation for the $\bar{\partial}$-Neumann problem is as follows:

$$\begin{cases} \dfrac{\partial F(t,z)}{\partial t}=\dfrac{1}{2}\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)F(t,z), & t>0,\ z\in M,\\ \lim\limits_{t\searrow 0}F(t,z)=f(z), & \\ \left(\dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y}\right)F(t,z)=0 & \text{on } bM. \end{cases} \tag{3.1}$$

This is exactly the heat equation with the Oblique boundary condition except for *the imaginary directional differentiation.*

Let $X(t,x)$ be a reflecting Brownian motion starting at a point $x\geqq 0$ and $\phi(t,x)$ be the local time at the point 0. Then as same as in the case of the Oblique boundary condition, if $Y(t,y)$ is a Brownian motion starting at a point y independent of $X(t,x)$, the heat kernel $h_t((x,y),(x',y'))$ is expected to be

$$h_t((x,y),(x',y'))=E\left[\delta_{(x',y')}(X(t,x),Y(t,y)+i\phi(t,x))\right]. \tag{3.2}$$

However, since $\delta_{(x',y')}$ is the Dirac δ function on $\mathbb{R}^2$ and the $(X(t,x),Y(t,y)+i\phi(t,x))$ does not lie on $\mathbb{R}^2$, we do not know how to give a correct mathematical sense to the right hand side of (3.2).

If we take the Fourier transformation of the both hand sides of (3.2) and exchange the order of integrations formally, then we have

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dy'e^{-i\lambda y'}h_t((x,y),(x',y))\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dy'e^{-i\lambda y'}E\left[\delta_{(x',y')}(X(t,x),Y(t,y)+i\phi(t,x))\right]\\ &=\frac{1}{\sqrt{2\pi}}E\left[e^{-i\lambda Y(t,y)}e^{\lambda\phi(t,x)}\middle|X(t,x)=x'\right]P(X(t,x)=x'). \end{aligned} \tag{3.3}$$

The right hand side of (3.3) is well-defined regardless the value of $Y(t,y)+i\phi(t,x)$. By using the Fourier inversion theorem, we have

$$\begin{aligned} &h_t((x,y),(x',y'))\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda e^{-i\lambda y'}E\left[e^{-i\lambda Y(t,y)}\right]E\left[e^{\lambda\phi(t,x)}\middle|X(t,x)=x'\right]P(X(t,x)=x'). \end{aligned} \tag{3.4}$$

Hence the problem has reduced to show that the integral of the right hand side of (3.3) is well defined. The term $E[e^{\lambda\phi(t)}|X(t) = x']$ diverges exponentially as λ tends to $\pm\infty$. But the term $E[e^{-i\lambda Y(t)}] = \exp(-i\lambda y - \frac{\lambda^2}{2}t)$ decays very well as λ tends to $\pm\infty$. After all, if $x + x' > 0$, the right hand side of (3.4) converges and gives the desired heat kernel. In fact, the right hand side of (3.4) can be rewritten in the form

$$\begin{aligned} &\frac{1}{2\pi t}\left\{ \exp\left(-\frac{(x-x')^2}{2t}\right) + \exp\left(-\frac{(x+x)^2}{2t}\right)\right\} \exp\left(-\frac{(y-y')^2}{2t}\right) \\ &+ \frac{i}{\pi}\frac{\partial}{\partial y}\left\{ \frac{1}{x+x'+i(y-y')} \exp\left(-\frac{(x+x')^2+(y-y')^2}{2t}\right)\right\}. \end{aligned} \tag{3.5}$$

We can easily check that this satisfies the heat equation (3.1).

In our discussion, the important point is that we can dissolve the problem of the imaginary value by only using the simple Fourier transformation in one valuable. This is due to the fact that the metric, differential and complex structures are *constant in the direction that we must consider the imaginary value.* So this fact is a key point in our discussion. The strongly pseudoconvex Siegel domain in the last section is one of the most fundamental complex manifolds with boundary satisfying this condition.

On the other hand, we can also consider the right hand side of (3.2) as follows: in Gelfand-Silov [4], the following series expansion in a space of complex generalized functions is discussed:

$$\delta(z+ib) = \sum_{j=0}^{\infty} \frac{(ib)^j}{j!}\delta^{(j)}(z), \quad z \in \mathbb{C},\ b \in \mathbb{R}. \tag{3.6}$$

As was pointed out by Ikeda, if we formally apply this expansion to the right hand side of (3.2), we obtain the following expression:

$$\begin{aligned} &E\left[\delta_{(x',y')}(X(t,x), Y(t,y) + i\phi(t,x))\right] \\ &= \int_0^{\infty} db \sum_{j=0}^{\infty} E\left[\frac{(ib)^j}{j!}\frac{\partial^j}{\partial y^j}\delta_{y'}(Y(t,y))\right] P(X(t,x) = x',\ \phi(t,x) = b). \end{aligned} \tag{3.7}$$

The each term of the right hand side of (3.7) can be given a correct mathematical sense (see [7], [10]). In the above simple case, we easily see that the value of (3.4) and (3.7) coincide.

Acknowledgement

The author thanks Professor P. Malliavin and Professor G. Ben Arous for helpful advices at the conference. This work was supported by Grant-in-Aid for Scientific Research, The Ministry of Education, Science and Culture.

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