Lower Bounds for the Spectra of Schrödinger Operators with Magnetic Fields*

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Abstract: An explicit representation of lower bounds for the spectra of Schrödinger operators with magnetic fields on σ-compact Riemannian manifolds is given, by using the positivity of Pauli Hamiltonian. This representation is applied to show some asymptotic properties of a stochastic oscillatory integral and transverse analyticity of the law of a stochastic line integral.

1. Introduction

In this paper, we concern with the infmum of the spectra of Schrödinger operators with magnetic fields. Our main contribution is to give an explicit representation of lower bounds for the spectra in terms of the curvature and the magnetic field (see Theorems 2.1 below).

On $\mathbb{R}^2$, the Schrödinger operator with a magnetic field db has the following form;

$$L(b) = \frac{1}{2}\sum_{j=1}^{2}(i\frac{\partial}{\partial x^j} + b^j(x))^2 \quad (i = \sqrt{-1}),$$

where $b = \sum_{j=1}^{2} b^j(x)dx^j$. We consider the Dirac operator,

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}(i\frac{\partial}{\partial x^1} + b^1(x)) + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}(i\frac{\partial}{\partial x^2} + b^2(x))$$

acting on $\mathbb{C}^2$-valued functions. Then the Pauli Hamiltonian Q^2 has the following form;

$$Q^2 = \begin{pmatrix} L(b) - db/2 & 0 \\ 0 & L(b) + db/2 \end{pmatrix},$$

where the 2-form db is identified with the function $\partial b^2/\partial x^1 - \partial b^1/\partial x^2$. The positivity of Q^2 leads to

$$\inf \text{spec } L(b) \geq \inf_{x\in\mathbb{R}^2} \frac{|db(x)|}{2},$$

where spec $L(b)$ is the spectrum of $L(b)$.

In this paper, we apply the above method to the operators on a general σ-compact Riemannian manifold. We define a Dirac operator Q (see (4.2) below) and a suitable operator $\mathcal{P}$ (see (4.9) and (4.10) below) on a space of forms, and show the following:

$$\mathcal{P}^*Q^2\mathcal{P} = (L(b) + \mathcal{R})\mathcal{P}_0,$$

where $\mathcal{P}_0$ is a projection onto the space of 0-forms and $\mathcal{R} = \sum_{j=1}^{3} \mathcal{R}_j - \mathcal{B}$ is a function explicitly written as a sum of terms of the curvature and terms of the magnetic field (see (2.3), (2.4) and (3.1) below). Since $\mathcal{P}^* Q^2 \mathcal{P}$ is a positive operator, we obtain

$$\text{inf spec } L(b) \geq \inf_{x \in M} \{-\mathcal{R}(x)\} \tag{1.1}$$

(see Theorem 2.1 below). The operator $\mathcal{P}$ is an extension of the operator used by Shigekawa [20] (cf. also [3]). To define the operator $\mathcal{P}$ for our cases, we introduce an almost complex structure on a subbundle of the tangent bundle (see (2.2) below).

As an application of our estimate mentioned above, we show

$$\varliminf_{\xi \to \infty} \frac{\text{inf spec } L(\xi b)}{\xi} \geq \frac{1}{4} \inf_{x \in M} \|db(x)\|_1 \tag{1.2}$$

under suitable assumptions (A.2) and (A.3) in Section 5 (see Theorem 4.1 below), where $\|\cdot\|_1$ is the trace norm. This type of result was proven in Malliavin [13], [14], though the right hand side of (1.2) is slightly different. Consequently, as Mallivin also discussed, we can deduce results on asymptotics of an oscillatory integral over Wiener space and analyticity of the law of a stochastic line integral (see Corollaries 1 and 2 of Theorem 5.1 below). For asymptotics of oscillatory integrals, see [9] and references therein. The analyticity we prove is known as transverse analyticity, which has been studied by Malliavin [13], [14], [15] and Prat [18].

The organization of this paper is as follows. In Section 2, we state the main theorem, Theorem 2.1. In Section 3, we give several remarks and examples on Theorem 2.1. In Section 4, we prove Theorem 2.1: we define a Dirac operator Q and a suitable operator $\mathcal{P}$ and calculate $\mathcal{P}^* Q^2 \mathcal{P}$. This part is the key of this paper.

In Section 5, we apply the results in Section 2 to show (1.2). Furtheremore, we apply (1.2) to show results on asymptotics of an oscillatory integral and analyticity of the law of a stochastic line integral.

2. The statement of the main theorem

Throughout of this paper, M is a σ-compact C^3 Riemannian manifold, which may have boundary, and b is a real C^1 differential form on M. Let $C_0^2(\overset{\circ}{M})$ be the set of all $\mathbb{C}$-valued C^2 functions on the interior of M with compact support. For any $\varphi \in C_0^2(\overset{\circ}{M})$, we set

$$L(b)\varphi = \frac{1}{2}(id + \mathrm{ext}(b))^*(id + \mathrm{ext}(b))\varphi \tag{2.1}$$

where ext is the exterior multiplication, i.e., $\mathrm{ext}(b)\varphi = b \wedge \varphi$, and $(id + \mathrm{ext}(b))^*$ is the formal adjoint of $id + \mathrm{ext}(b)$. Let $L^2(M)$ be the space of all $\mathbb{C}$-valued square integrable functions with respect to the Riemannian volume. We easily see that this $L(b)$ is a symmetric operator in $L^2(M)$. We take, accordingly, the Friedrichs extension of $L(b)$, which we also denote by $L(b)$: $L(b)$ is the operator with Dirichlet boundary condition when the manifold M has boundary (see Remark 4.1 below). The infmum of the spectrum of this self-adjoint operator $L(b)$ is our object.

Let U, V and W be C^2 Riemannian vector bundles over M satisfying the following:

(2.2)
(i) $TM = U \oplus V \oplus W$ (orthogonal direct sum),

(ii) There is a C^2 isometry J on U such that $J^2 = -(\text{identity})$,

(iii) V is oriented.

As we allow any of these bundles to be zero, this formulation does not make any restrictions. J is an almost complex structure on U for which the Riemannian metric is Hermitian. According to a convention for complex structures, we set $U' = \{v \in U \otimes \mathbb{C};\ Jv = iv\}$ and $U'' = \{v \in U \otimes \mathbb{C};\ Jv = -iv\}$. The orientation of V is needed to define the operator $\mathcal{P}_1$ in Section 4 (see (4.9) below).

Generally, for any complex (resp. real) subbundle E of $TM \otimes \mathbb{C}$ (resp. TM) and any complex (resp. real) subbundle F of E, let $S(E,F)$ be the second fundamental form of F in E (cf. e.g. [7, p-78]). This is a section of the bundle $F^* \otimes T^*M \otimes F^\perp$, where F^* is the dual of F and $F^\perp$ is the orthogonal complement of F in E. Let $\|S(E,F)\|$ be the fibre norm of $S(E,F)$ on $F^* \otimes T^*M \otimes F^\perp$. On the other hand, let $R(\cdot,\cdot)$ be the curvature transformation of TM:

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z \quad \text{for } X, Y, Z \in TM,$$

where ∇ is the Levi-Civita connection of M and $[X,Y] = XY - YX$ is the Lie bracket. Let $\mathcal{R}(\cdot,\cdot;F)$ be a trace on F of curvature tensors as follows;

$$\mathcal{R}(X,Y;F) = \sum_{a=1}^{A} (R(\overline{\delta_a}, X)Y, \delta_a) \quad \text{for } X, Y \in TM \otimes \mathbb{C}$$

where $\{\delta_a\}_{a=1}^A$ is a unitary (resp. an orthonormal) basis of F and the metric $(\cdot,\cdot)$, the curvature transformation $R(\cdot,\cdot)$ are extended to complex vectors as complex linear in every variable. If F equals TM, $\mathcal{R}(\cdot,\cdot;TM)$ is the Ricci tensor. Let $\mathcal{R}(E,F)$ be a trace on $F^\perp$ of $\mathcal{R}(\cdot,\cdot;F)$ as follows:

$$\mathcal{R}(E,F) = \sum_{a'=1}^{A'} \mathcal{R}(\overline{\delta'_{a'}}, \delta'_{a'}; F) = \sum_{a'=1}^{A'} \sum_{a=1}^{A} (R(\overline{\delta_a}, \overline{\delta'_{a'}})\delta'_{a'}, \delta_a)$$

where $\{\delta'_{a'}\}_{a'=1}^{A'}$ is a unitary (resp. an orthonormal) basis of $F^\perp$. We remark that $\|S(E,F)\| = \|S(E,F^\perp)\|$ and $\mathcal{R}(E,F) = \mathcal{R}(E,F^\perp)$.

We define functions $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ on M by

$$\begin{aligned} \mathcal{R}_1 &= \frac{1}{2}\|S(U \otimes \mathbb{C}, U')\|^2 + \frac{1}{2}\mathcal{R}(U \otimes \mathbb{C}, U'), \\ \mathcal{R}_2 &= \frac{1}{4}\|S(TM, U)\|^2 + \frac{1}{4}\mathcal{R}(TM, U), \\ \mathcal{R}_3 &= \frac{1}{2}\|S(V \oplus W, V)\|^2 + \frac{1}{2}\mathcal{R}(V \oplus W, V). \end{aligned} \tag{2.3}$$

For the local expresseion, see Lemma 3.1 below.

We define a function $\mathcal{B}$ by a trace of $db/2i$ on U':

$$\mathcal{B} = \frac{1}{2i} \sum_{\rho=1}^{l} db(\overline{\epsilon_\rho}, \epsilon_\rho), \tag{2.4}$$

where $\{\epsilon_\rho\}_{\rho=1}^{l}$ is a unitary basis of U' and the 2-form $db(\cdot, \cdot)$ are extended to complex vectors as complex linear.

Then our main theorem on lower bounds for the spectra is stated as follows:

Theorem 2.1. *$L(b) + \sum_{j=1}^{3} \mathcal{R}_j - \mathcal{B}$ is a positive operator.*

In particular, we have

$$\text{inf spec } L(b) \geq \inf_{x \in M} \{ - \sum_{j=1}^{3} \mathcal{R}_j(x) + \mathcal{B}(x) \}. \tag{2.5}$$

We prove this theorem at the end of Section 4.

3. Remarks and examples on Theorem 2.1

To use Theorem 2.1, the representation of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{B}$ in terms of local frames is often useful. We summarize the representation:

Lemma 3.1. *Let $2l$ be the fibre dimension of U, m be that of $U \oplus V$ and d be the dimension of M $(2l \leq m \leq d)$. Let $\{e_j\}_{j=1}^{2l}$ be a local orthonormal frame of U such that $\{\epsilon_\rho := (e_{2\rho-1} - ie_{2\rho})/\sqrt{2}\}_{\rho=1}^{l}$ is a unitary frame of U'. Let $\{e_j\}_{j=2l+1}^{m}$ and $\{e_j\}_{j=m+1}^{d}$ be a local orthonormal frame of V and W, respectively.*

Then the functions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{B}$ are represented as

$$
\begin{aligned}
\mathcal{R}_1 &= \frac{1}{2} \sum_{\substack{1 \leq \sigma, \rho \leq l \\ 1 \leq j \leq d}} |(\nabla_{e_j} \epsilon_\sigma, \epsilon_\rho)|^2 + \frac{1}{2} \sum_{1 \leq \sigma, \rho \leq l} (R(\overline{\epsilon_\sigma}, \overline{\epsilon_\rho})\epsilon_\rho, \epsilon_\sigma), \\
\mathcal{R}_2 &= \frac{1}{4} \sum_{\substack{1 \leq p \leq 2l \\ 2l < q \leq d \\ 1 \leq j \leq d}} (\nabla_{e_j} e_q, e_p)^2 + \frac{1}{4} \sum_{\substack{1 \leq p \leq 2l \\ 2l < q \leq d}} (R(e_p, e_q)e_q, e_p), \\
\mathcal{R}_3 &= \frac{1}{2} \sum_{\substack{2l < p \leq m \\ m < q \leq d \\ 1 \leq j \leq d}} (\nabla_{e_j} e_q, e_p)^2 + \frac{1}{2} \sum_{\substack{2l < p \leq m \\ m < q \leq d}} (R(e_p, e_q)e_q, e_p), \\
\mathcal{B} &= \frac{1}{2i} \sum_{\rho=1}^{l} db(\overline{\epsilon_\rho}, \epsilon_\rho).
\end{aligned}
\tag{3.1}
$$

On the other hand, Theorem 2.1 relates to the following, for example:

Remark 3.1. In [10], [12] and [16], the long time asymptotics of the heat semigroup generated by $L(b)$ is studied. Since the leading term of the asymptotics equals $-\inf \operatorname{spec} L(b)$, those works relate to ours.

We now consider various assumptions to use Theorem 2.1 effectively. We first note the following:

Proposition 3.1. *If $\nabla J = 0$, then $\mathcal{R}_1 = 0$.*

This is easily seen from the definition of the function $\mathcal{R}_1$.

We next assume the following:

(A.1) $$\text{For any } v \in U, \quad \frac{1}{i} db(\overline{v}, v) \geq 0.$$

Under this assumption, we have the following:

Proposition 3.2. *Under the assumption (A.1), we have*

$$\mathcal{B}(x) = \frac{1}{4} \| db(x)|_U \|_1, \tag{3.2}$$

where $db(x)|_U$ *is the matrix* $(db(e_j(x), e_k(x)))_{1 \leq j,k \leq 2l}$ *for an orthonormal basis* $\{e_j(x)\}_{j=1}^{2l}$ *of* U_x *and, for any matrix* A, $\|A\|_1$ *is its trace norm:*

$$\|A\|_1 = \text{tr } \sqrt{{}^t A \cdot A}.$$

In particular, if $db(v, w) = 0$ *for any* $v \in TM$ *and any* $w \in V \oplus W$, *we have*

$$\mathcal{B}(x) = \frac{1}{4} \| db(x) \|_1. \tag{3.3}$$

Proof. For any local orthonormal frame $\{e_j\}_{j=1}^d$ taken as in Lemma 3.1, $(db(\overline{\epsilon_\rho}, \epsilon_\sigma))_{1 \leq \rho, \sigma \leq l}$ is a skew-Hermitian matrix. Therefore, we can take such a frame $\{e_j\}_{j=1}^d$ satisfying

$$(db(\overline{\epsilon_\rho}, \epsilon_\sigma))_{1 \leq \rho, \sigma \leq l} = \begin{pmatrix} i\beta_1 & & & 0 \\ & i\beta_2 & & \\ & & \ddots & \\ 0 & & & i\beta_l \end{pmatrix}.$$

The assumption (A.1) implies $\beta_\sigma \geq 0$, $\sigma = 1, 2, \cdots, l$. Then, by easy calculations, we have

$$\mathcal{B} = \frac{1}{2} \sum_{\sigma=1}^{l} \beta_\sigma = \frac{1}{4} \| db(x)|_U \|_1. \qquad \blacksquare$$

About the assumption (A.1), we first note the following (cf. Darboux's theorem [1, Theorem 3.2.2]):

Proposition 3.3. *If the matrix $db(x)|_U$ in Proposition 3.2 is nonsingular at any point $x \in M$, then there exists J satisfying (2.2) and (A.1).*

Proof. Under the assumption, we can choose a local orthonormal frame $\{e_j\}_{j=1}^{2l}$ such that

$$(db(e_j, e_k))_{1\le j,k\le 2l} = \begin{pmatrix} 0 & -\beta_1 & & & & & \\ \beta_1 & 0 & & & & & \\ & & 0 & -\beta_2 & & & \\ & & \beta_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -\beta_l \\ & & & & & \beta_l & 0 \end{pmatrix}.$$

where β_σ, $\sigma = 1, 2, \cdots, l$ are positive functions defined locally. Then we can define J by

$$Je_{2\sigma-1} = e_{2\sigma} \text{ and } Je_{2\sigma} = -e_{2\sigma-1}$$

for any $\sigma = 1, 2, \cdots, l$. ∎

If the dimension of the manifold M is 2, to consider the effect of the magnetic field to the lower bound, we should assume that $TM = U$ in (2.2). This implies $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = 0$ in (2.3). Then, in using Theorem 2.1, we may assume that the matrix $db(x)$ is nonsingular at any point $x \in M$, since $L(b)$ is positive. Therefore we have the following:

Proposition 3.4. *If the dimension d of M is 2, we always have*

$$\text{inf spec } L(b) \ge \inf_{x\in M} \frac{1}{4}\|db(x)\|_1. \tag{3.4}$$

However, if the dimension d of M is greater than 3, there are examples the assumption (A.1) is not satisfied:

Example 1. On $\mathbb{R}^d$, $d \geq 3$, we consider the magnetic field db having the following form:

$$db(x) = \begin{pmatrix} 0 & -x^1 & 0 & \dots & 0 \\ x^1 & 0 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & * & \\ 0 & 0 & & & \end{pmatrix}.$$

Then, for the assumption (A.1), we should take J satisfying the following: on $\{x^1 > 0\}$, ${}^t(1, -i, 0, \cdots, 0) \in U'$ and ${}^t(1, +i, 0, \cdots, 0) \in U''$; on $\{x^1 < 0\}$, ${}^t(1, +i, 0, \cdots, 0) \in U'$ and ${}^t(1, -i, 0, \cdots, 0) \in U''$. However we can not take such a J continuously.

For other remarks, see the end of the next section.

We next apply our theorem on Euclidean space:

Example 2. On $\mathbb{R}^d$ with the standard metric, the easiest way of applying Theorem 2.1 is the following: we put $U =< \partial_1, \partial_2, \cdots, \partial_{2l} >$, $W =< \partial_{2l+1}, \partial_{2l+2}, \cdots, \partial_d >$, $0 \leq 2l \leq d$, and $J\partial_{2\sigma-1} = \partial_{2\sigma}$, $\sigma = 1, 2, \cdots, l$, where $\partial_j = \partial/\partial x^j$. Then we have $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = 0$ and

$$\mathcal{B} = \sum_{\sigma=1}^{l} (db)_{2\sigma, 2\sigma-1}(x). \tag{3.5}$$

Therefore, from Theorem 2.1, we have

$$\text{inf spec } L(b) \geq \inf_{x \in M} \sum_{\sigma=1}^{l} (db)_{2\sigma, 2\sigma-1}(x). \tag{3.6}$$

However, we can obtain better estimates from Theorem 2.1. Let us suppose that $d = 2l$ and db is written as follows:

(3.7)

$$db(x) = e(x) \begin{pmatrix} 0 & -\beta_1(x) & & & & & \\ \beta_1(x) & 0 & & & & & \\ & & 0 & -\beta_2(x) & & & \\ & & \beta_2(x) & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -\beta_l(x) \\ & & & & & \beta_l(x) & 0 \end{pmatrix} {}^t e(x),$$

where $\mathbb{R}^d \ni x \mapsto e(x) = (e_{jk}(x))_{1\le j,k\le d}$ is an orthogonal matrix valued C^2 function and β_σ, $\sigma = 1, 2, \cdots, l$, are nonnegative continuous functions. We define a global orthonormal frame $\{e_j(x)\}_{j=1}^d$ by

$$e_j(x) = \sum_{k=1}^d e_{kj}(x)\partial_k$$

and we put $T\mathbb{R}^d = U$ and $Je_{2\sigma-1} = e_{2\sigma}$, $\sigma = 1, 2, \cdots, l$. Then we have

$$\begin{aligned}
\mathcal{R}_1 =& \frac{1}{32} \sum_{\substack{1\le\sigma,\rho\le l\\ 1\le j\le d}} \left| \sum_{k=1}^d \partial_j (e_{k,2\sigma-1} - ie_{k,2\sigma})^2 \right|^2, \\
\mathcal{R}_2 =& \mathcal{R}_3 = 0, \\
\mathcal{B} =& \frac{\|db(x)\|_1}{4}.
\end{aligned}$$

Therefore, from Theorem 2.1, we have

$$\text{(3.8)}\quad \inf \operatorname{spec} L(b) \ge \inf_{x\in M} \left(\frac{\|db(x)\|_1}{4} - \frac{1}{32} \sum_{\substack{1\le\sigma,\rho\le l\\ 1\le j\le d}} \left| \sum_{k=1}^d \partial_j (e_{k,2\sigma-1} - ie_{k,2\sigma})^2 \right|^2 \right).$$

On the other hand, we are interested in our estimates on spaces of negative constant curvature, since sectional curvatures appear with positive coefficients in (2.3). We now consider on the hyperbolic space in the half-space (cf. [2], [4, Section 5.7]).

Example 3. Let $\mathbb{H}^{n+1} = \{x = (x^1, x^2, \cdots, x^{n+1}) \in \mathbb{R}^{n+1}; 0 < x^{n+1} < \infty\}$. We give $\mathbb{H}^{n+1}$ the metric $(\partial_j, \partial_k) = \delta_{j,k}/y^2$, $j, k = 1, 2, \cdots, n+1$, where $\partial_j = \partial/\partial x^j$, $y = x^{j+1}$ and $\delta_{j,k}$ is Kronecker's delta. Then the vector fields $e_j = y\partial_j$, $j = 1, 2, \cdots, n+1$, form a global orthonormal frame of $T\mathbb{H}^{n+1}$. For this frame, we have

$$(\nabla_{e_j} e_k, e_h) = \begin{cases} \delta_{j,k}, & \text{if } h = n+1, \\ -\delta_{j,h}, & \text{if } k = n+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$(R(e_j, e_k)e_q, e_p) = \delta_{j,q}\delta_{k,p} - \delta_{j,p}\delta_{k,q}.$$

If we put $U =< e_1, e_2, \cdots, e_{2l} >$, $V =< e_{2l+1}, e_{2l+2}, \cdots, e_m >$,

$W =< e_{m+1}, e_{m+2}, \cdots, e_{n+1} >$, $0 \leq 2l \leq m \leq n$, and $Je_{2\sigma-1} = e_{2\sigma}$, $\sigma = 1, 2, \cdots, l$, then we have

$$\mathcal{R}_1 = -\frac{l(l-1)}{2}, \quad \mathcal{R}_2 = -\frac{l(n-2l)}{4}, \quad \mathcal{R}_3 = -\frac{(m-2l)(n-m)}{2}. \tag{3.9}$$

If n is even, $\sum_{j=1}^{3} \mathcal{R}_j$ takes on its minimum with respect to m at $m = l + n/2$ and its value is

$$\sum_{j=1}^{3} \mathcal{R}_j = -\frac{n^2}{8} + \frac{l}{2}. \tag{3.10}$$

This value increases as l increases. Therefore, the effect of the term $\sum_{j=1}^{3} \mathcal{R}_j$ becomes worse as direction we estimate the magnetic field becomes wider. When $l = 0$ and $m = n/2$, we get $\sum_{j=1}^{3} \mathcal{R}_j = -n^2/8$. Therefore we obtain

$$\text{inf spec } \left(-\frac{\Delta}{2}\right) \geq \frac{n^2}{8}, \tag{3.11}$$

where Δ is the Laplace-Beltrami operator. This is the best estimate, since it is known that inf spec $(-\Delta/2) = n^2/8$ for any n [4, Theorem 5.7.1].

If n is odd, $\sum_{j=1}^{3} \mathcal{R}_j$ takes on its minimum with respect to m at $m = l + (n \pm 1)/2$ and its value is

$$\sum_{j=1}^{3} \mathcal{R}_j = -\frac{n^2}{8} + \frac{l}{2} + \frac{1}{8} \geq -\frac{n^2}{8} + \frac{1}{8}. \tag{3.12}$$

We next put $U =< e'_1, e'_2, \cdots, e'_{2l} >$, $V =< e'_{2l+1}, e'_{2l+2}, \cdots, e'_m >$,

$W =< e'_{m+1}, e'_{m+2}, \cdots, e'_{n+1} >$, $2 \leq 2l \leq m \leq n+1$, and $Je'_{2\sigma-1} = e'_{2\sigma}$, $\sigma = 1, 2, \cdots, l$, where $e'_j = e_{n+1-j}$: we consider the case $e_{n+1} \in U$. If n is odd, $\sum_{j=1}^{3} \mathcal{R}_j$ takes on its minimum with respect to m at $m = l + (n+1)/2$ and its

value is

$$\sum_{j=1}^{3} \mathcal{R}_j = -\frac{n^2}{8} + \frac{l}{2} - \frac{3}{8} \geq -\frac{n^2}{8} + \frac{1}{8}, \quad l \geq 1. \tag{3.13}$$

If n is even, $\sum_{j=1}^{3} \mathcal{R}_j$ takes on its minimum with respect to m at $m = l+(n+1\pm1)/2$ and its value is

$$\sum_{j=1}^{3} \mathcal{R}_j = -\frac{n^2}{8} + \frac{l}{2} - \frac{1}{4} \geq -\frac{n^2}{8} + \frac{1}{4}, \quad l \geq 1. \tag{3.14}$$

4. Proof of Theorem 2.1

To prove Theorem 2.1 in the previous section, we define a Dirac operator Q and an operator $\mathcal{P}$ and calculate $\mathcal{P}^*Q^2\mathcal{P}$. This calculation is the most important part of this paper. The result is summarized in Theorem 4.1 below.

In the following, we do local calculation for the most part. Let $\{e_j\}_{j=1}^d$ be a local orthonormal frame of TM taken as in Lemma 3.1.

We first prepare the following formula for $L(b)$, which is easily shown:

Lemma 4.1. *$L(b)$ is locally represented as*

$$L(b) = -\frac{1}{2}\sum_{j=1}^{d}(e_je_j - \sum_{k=1}^{d}\Gamma_{jjk}e_k) + i\sum_{j=1}^{d}b(e_j)e_j - \frac{id^*b}{2} + \frac{\|b\|^2}{2} \tag{4.1}$$

where

$$\Gamma_{jkh} = (\nabla_{e_j}e_k, e_h), \quad j,k,h = 1,2,\cdots,d,$$

and $\|b\|$ is the usual norm on cotangent space of b.

We define a Dirac operator Q as follows: let $\Gamma_0^1(\overset{\circ}{M};\Lambda(T^*M)\otimes\mathbb{C})$ be the set of all C^1 forms on the interior of M with compact support. Let Q be an operator defined on $\Gamma_0^1(\overset{\circ}{M};\Lambda(T^*M)\otimes\mathbb{C})$ by

$$Q = \frac{1}{\sqrt{2}}\{(id + \mathrm{ext}(b)) + (id + \mathrm{ext}(b))^*\}. \tag{4.2}$$

This Q is locally represented as

$$Q = \frac{1}{\sqrt{2}}\sum_{j=1}^{d}\gamma^j(i\nabla_{e_j} + b(e_j)) \tag{4.3}$$

(cf. [6, p-149]), where γ^j, $j = 1,2,\cdots,d$, are local sections to the bundle of Hermitian matrices of $\Lambda(T^*M)\otimes\mathbb{C}$ defined by

$$\gamma^j = \mathrm{ext}(e^j) + \mathrm{int}(e^j). \tag{4.4}$$

We remark that $\{\gamma^j\}_{j=1}^d$ satisfies the following commutation relation:

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 2\delta_{j,k}, \quad j,k = 1,2,\cdots,d. \tag{4.5}$$

When $b = 0$, the corresponding Pauli Hamiltonian Q^2 becomes the de Rham-Hodge-Kodaira Laplacian.

The Pauli Hamiltonian Q^2 has the following representation:

Lemma 4.2. *Q^2 is locally represented as*

$$\begin{aligned} Q^2 =& \underline{L(b)} + \sum_{1\le j,p,q\le d} \Gamma_{jqp}\mathrm{ext}(e^q)\mathrm{int}(e^p)\underline{e_j} \\ &+ \sum_{1\le j,p,q\le d} \{\frac{1}{2}(e_j\Gamma_{jqp}) - \frac{1}{2}\Gamma_{jjk}\Gamma_{kqp} - ib(e_j)\Gamma_{jqp}\}\mathrm{ext}(e^q)\mathrm{int}(e^p) \\ &- \frac{1}{2}\sum_{1\le j,p,q,r,s\le d} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p) \\ &+ \frac{1}{2}\sum_{\substack{1\le j<k\le d\\ 1\le p,q\le d}} R_{jkqp}\gamma^j\gamma^k\mathrm{ext}(e^q)\mathrm{int}(e^p) \\ &+ \frac{1}{2}\sum_{1\le j<k\le d} i\gamma^j\gamma^k db(e_j,e_k) \end{aligned} \tag{4.6}$$

where

$$R_{jkqp} = (R(e_j,e_k)e_q,e_p), \quad j,k,q,p = 1,2,\cdots,d,$$

and, for any differential operator A on space of functions, $\underline{A}$ is the operator acting only on the coefficients of $\{e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_p}\}_{1\le j_1<j_2<\cdots<j_p\le d}$, i.e.,

$$\begin{aligned} &\underline{A}\sum_{1\le j_1<j_2<\cdots<j_p\le d} \varphi_{j_1j_2\cdots j_p}e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_p} \\ &= \sum_{1\le j_1<j_2<\cdots<j_p\le d} \varphi_{j_1j_2\cdots j_p}(A\varphi_{j_1j_2\cdots j_p})e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_p}. \end{aligned}$$

Proof. We take any $x_0 \in M$ and fix it. We take $\{e_j\}$ such that $\nabla e_j = 0$ at x_0. At

x_0, we can calculate as follows:

$$\begin{aligned}
Q^2 =&\frac{1}{2}\sum_{j=1}^{d}\gamma^j\gamma^j(i\nabla_{e_j}+b(e_j))(i\nabla_{e_j}+b(e_j))\\
&+\frac{1}{2}\sum_{j<k}\gamma^j\gamma^k[i\nabla_{e_j}+b(e_j),i\nabla_{e_k}+b(e_k)]\\
=&\frac{1}{2}\sum_{j=1}^{d}(-\nabla_{e_j}\nabla_{e_j}+i(e_jb(e_j))+2ib(e_j)\nabla_{e_j}+b(e_j)^2)\\
&+\frac{1}{2}\sum_{j<k}\gamma^j\gamma^k\{-[\nabla_{e_j},\nabla_{e_k}]+i(e_jb(e_k)-e_kb(e_j))\}.
\end{aligned}$$

We introduce the curvature operator $R^{T^*M}(\cdot,\cdot)$ on space of forms by

$$R^{T^*M}(X,Y)\omega=(\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]})\omega,\quad X,Y\in TM,\ \omega\in T^*M,$$

and we rewrite the above equation as follows:

$$\begin{aligned}
Q^2 =&-\frac{1}{2}\sum_{j=1}^{d}\nabla^2_{e_je_j}+i\sum_{j=1}^{d}b(e_j)\nabla_{e_j}-\frac{id^*b}{2}+\frac{\|b\|^2}{2}\\
&-\frac{1}{2}\sum_{j<k}\gamma^j\gamma^kR^{T^*M}(e_j,e_k)+\frac{1}{2}\sum_{j<k}i\gamma^j\gamma^kdb(e_j,e_k).
\end{aligned}\tag{4.7}$$

Both sides of (4.7) are independent of the choice of $\{e_j\}$ and (4.7) holds at any point. Furthermore, we use Lemma 4.1 and the following to obtain (4.6):

$$\begin{aligned}
\nabla_{e_j} =&\ \underline{e_j}-\sum_{p,q}\Gamma_{jqp}\text{ext }(e^q)\text{int }(e^p),\\
\nabla^2_{e_je_j} =&\ \underline{e_je_j}-\sum_{p,q}(e_j\Gamma_{jqp})\text{ext }(e^q)\text{int }(e^p)\\
&-2\sum_{p,q}\Gamma_{jqp}\text{ext }(e^q)\text{int }(e^p)\underline{e_j}\\
&+\sum_{p,q,r,s}\Gamma_{jsr}\Gamma_{jqp}\text{ext }(e^s)\text{int }(e^r)\text{ext }(e^q)\text{int }(e^p)\\
&-\sum_k\Gamma_{jjk}\underline{e_k}+\sum_{k,p,q}\Gamma_{jjk}\Gamma_{kqp}\text{ext }(e^q)\text{int }(e^p),\\
R^{T^*M}(e_j,e_k) =&-\sum_{p,q}R_{jkqp}\text{ext }(e^q)\text{int }(e^p).\blacksquare
\end{aligned}\tag{4.8}$$

We introduce the following operators:

$$
\begin{aligned}
&\mathcal{P}_0 := \mathrm{int}(e^d)\cdots\mathrm{int}(e^2)\mathrm{int}(e^1)\mathrm{ext}(e^1)\mathrm{ext}(e^2)\cdots\mathrm{ext}(e^d),\\
&\mathcal{P}_1 := \mathrm{ext}(e^{2l+1})\mathrm{ext}(e^{2l+2})\cdots\mathrm{ext}(e^m),\\
&\mathcal{P}_2 := \prod_{\sigma=1}^{l}\frac{1+i\gamma^{2\sigma-1}\gamma^{2\sigma}}{2} = \prod_{\sigma=1}^{l}\frac{\gamma^{2\sigma-1}-i\gamma^{2\sigma}}{2}\cdot\frac{\gamma^{2\sigma-1}+i\gamma^{2\sigma}}{2}.
\end{aligned}
\tag{4.9}
$$

To define $\mathcal{P}_1$ globally, in the following, we assume that the frame $\{e_j\}_{j=2l+1}^m$ is positively oriented. Then $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are globally defined elements of $\Gamma^2(M;\mathrm{End}(\Lambda(T^*M)\otimes\mathbb{C}))$. $\mathcal{P}_0$ is the orthogonal projection onto the space of zero forms. $\mathcal{P}_2$ is the orthogonal projection onto the space $\{\omega\in\Lambda(T^*M)\otimes\mathbb{C}; i\gamma^{2\sigma-1}\gamma^{2\sigma}\omega=\omega,\ \sigma=1,2,\cdots,l\}$. We use the operator $\mathcal{P}$ defined by

$$
\mathcal{P} = 2^{\frac{l}{2}}\mathcal{P}_2\mathcal{P}_1\mathcal{P}_0. \tag{4.10}
$$

This operator also can be writen as follows:

$$
\mathcal{P} = 2^{-\frac{l}{2}}\mathrm{ext}\left(\exp\left(\sum_{\sigma=1}^{l}e^{2\sigma-1}\wedge e^{2\sigma}\right)\right)\mathcal{P}_1\mathcal{P}_0. \tag{4.11}
$$

The form $\sum_{\sigma=1}^{l}e^{2\sigma-1}\wedge e^{2\sigma}$ is analogous to symplectic form, and $\mathcal{P}_1$ is analogous to the Hodge $*$-operator on V.

Then we can state the following important formula:

Theorem 4.1.

$$
\mathcal{P}^*Q^2\mathcal{P} = \left\{L(b)+\sum_{j=1}^{3}\mathcal{R}_j-\mathcal{B}\right\}\mathcal{P}_0 \tag{4.12}
$$

where $\mathcal{R}_j$, $j=1,2,3$ and $\mathcal{B}$ are given in (2.3) and (2.4).

Proof. We consider each term of the right hand side of (4.6) separately. We devide the calculation into several steps. In Steps 0-4, we use the following

$$
\mathcal{P}1 = 2^{-\frac{l}{2}}\bigwedge_{\sigma=1}^{l}(1+ie^{2\sigma-1}\wedge e^{2\sigma})\wedge e^{2l+1}\wedge e^{2l+2}\wedge\cdots\wedge e^m, \tag{4.13}
$$

$$(1 + i\gamma^{2\sigma-1}\gamma^{2\sigma})1 = 1 + ie^{2\sigma-1} \wedge e^{2\sigma}, \tag{4.14}$$

$$(1 + i\gamma^{2\sigma-1}\gamma^{2\sigma})e^{2\sigma-1} = e^{2\sigma-1} - ie^{2\sigma}, \tag{4.15}$$

$$(1 + i\gamma^{2\sigma-1}\gamma^{2\sigma})e^{2\sigma} = i(e^{2\sigma-1} - ie^{2\sigma}), \tag{4.16}$$

$$(1 + i\gamma^{2\sigma-1}\gamma^{2\sigma})(e^{2\sigma-1} \wedge e^{2\sigma}) = -i(1 + ie^{2\sigma-1} \wedge e^{2\sigma}), \tag{4.17}$$

$$\mathcal{P}^* = 2^{-\frac{l}{2}}\mathcal{P}_0 \mathrm{int}(e^m) \cdots \mathrm{int}(e^{2l+2})\mathrm{int}(e^{2l+1})\mathcal{P}_2, \tag{4.18}$$

$$\mathcal{P}^*\mathcal{P} = \mathcal{P}_0. \tag{4.19}$$

Step 0. Calculation of $\mathcal{P}^*\underline{L(b)}\mathcal{P}$.

From (4.19), we have

$$\mathcal{P}^*\underline{L(b)}\mathcal{P} = L(b)\mathcal{P}_0. \tag{4.20}$$

Step 1. Calculation of $\mathcal{P}^*\Gamma_{jqp}\mathrm{ext}(e^q)\mathrm{int}(e^p)\underline{e_j}\mathcal{P}$

and $\mathcal{P}^*\{\frac{1}{2}\Gamma_{jjk}\Gamma_{kqp} - ib(e_j)\Gamma_{jqp}\}\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}$.

When $p = 2\rho - 1$ $(\rho \leq l)$, we have

$$\mathrm{int}(e^p)\mathcal{P}1 = 2^{-\frac{l}{2}}ie^{2\rho} \bigwedge_{\sigma\neq\rho} (1 + ie^{2\sigma-1} \wedge e^{2\sigma}) \wedge e^{2l+1} \wedge e^{2l+2} \wedge \cdots \wedge e^m.$$

From (4.16) and (4.18), we know that $q = p$ is necessary for $\mathcal{P}^*\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}1 \neq 0$. Similarly, for $p = 2\rho$ $(\rho \leq l)$, we know the same thing, from (4.15) instead of (4.16). For $p > 2l$, it is quite easy to see the same thing, from (4.13) and (4.18). On the other hand, $\Gamma_{jqp} = 0$ for $q = p$. Therefore we conclude that

$$\begin{aligned} &\mathcal{P}^*\Gamma_{jqp}\mathrm{ext}(e^q)\mathrm{int}(e^p)\underline{e_j}\mathcal{P} \\ &= \mathcal{P}^*\{\frac{1}{2}\Gamma_{jjk}\Gamma_{kqp} - ib(e_j)\Gamma_{jqp}\}\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\ &= 0. \end{aligned} \tag{4.21}$$

Step 2. Calculation of $\mathcal{P}^* \sum_{j<k} i\gamma^j\gamma^k db(e_j, e_k)\mathcal{P}$.

When $k = 2\rho - 1$ $(\rho \le l)$, we have

$$\gamma^k\mathcal{P}1 = 2^{-\frac{l}{2}}(e^{2\rho-1} + ie^{2\rho}) \wedge \bigwedge_{\sigma\neq\rho}(1 + ie^{2\sigma-1} \wedge e^{2\sigma}) \wedge e^{2l+1} \wedge e^{2l+2} \wedge \cdots \wedge e^m.$$

From (4.15), (4.16) and (4.18), we know that $\mathcal{P}^*\gamma^j\gamma^k\mathcal{P} = 0$ for any $j < k$.

When $k = 2\rho$ $(\rho \le l)$, we have

$$\gamma^k\mathcal{P}1 = 2^{-\frac{l}{2}}\frac{e^{2\rho-1} + ie^{2\rho}}{i} \wedge \bigwedge_{\sigma\neq\rho}(1 + ie^{2\sigma-1} \wedge e^{2\sigma}) \wedge e^{2l+1} \wedge \cdots \wedge e^m.$$

From (4.15), (4.16) and (4.18), we know that $\mathcal{P}^*\gamma^j\gamma^k\mathcal{P} = 0$ for any $j < 2\rho - 1$ and

$$\begin{aligned} &\mathcal{P}^*\gamma^{2\rho-1}\gamma^{2\rho}\mathcal{P}1 \\ &= \mathcal{P}^*2^{-\frac{l}{2}}\frac{1 + ie^{2\rho-1} \wedge e^{2\rho}}{i} \wedge \bigwedge_{\sigma\neq\rho}(1 + ie^{2\sigma-1} \wedge e^{2\sigma}) \wedge e^{2l+1} \wedge \cdots \wedge e^m \\ &= \frac{1}{i}. \end{aligned} \tag{4.22}$$

If $2l < k \le m$, then

$$\gamma^k\mathcal{P}1 = (-1)^{k-1}2^{-\frac{l}{2}} \bigwedge_{\sigma=1}^{l}(1 + ie^{2\sigma-1} \wedge e^{2\sigma}) \wedge e^{2l+1} \wedge \cdots \wedge e^{k-1} \wedge e^{k+1} \wedge \cdots \wedge e^m$$

and so $\mathcal{P}^*\gamma^j\gamma^k\mathcal{P} = 0$ for any $j < k$.

If $m < k$, then $\gamma^k\mathcal{P}1 = e^k \wedge \mathcal{P}1$ and so $\mathcal{P}^*\gamma^j\gamma^k\mathcal{P} = 0$ for any $j < k$.

Therefore we have

$$\begin{aligned} \mathcal{P}^* \sum_{j<k} i\gamma^j\gamma^k db(e_j, e_k)\mathcal{P} &= \sum_{\sigma=1}^{l} db(e_{2\sigma-1}, e_{2\sigma}) \\ &= -\sum_{\sigma=1}^{l} \frac{db(\overline{\epsilon_\sigma}, \epsilon_\sigma)}{i}. \end{aligned} \tag{4.23}$$

Step 3. Calculation of $\mathcal{P}^*\Gamma_{jqp}\Gamma_{jrs}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}$ *(I).*

If the number of elements of the set $\{p, q, r, s\} \cap \{1, 2, \cdots, 2l\}$ is odd, then

$$\mathcal{P}^*\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} = 0, \tag{4.24}$$

from (4.15), (4.16) and (4.18). Therefore, we decompose as follows:

$$
\begin{aligned}
&\mathcal{P}^* \sum_{1\le p,q,r,s\le d} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&= \mathcal{P}^* \sum_{p,q,r,s\le 2l} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{p,q\le 2l<r,s} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{p,r\le 2l<q,s} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{p,s\le 2l<q,r} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{q,r\le 2l<p,s} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{q,s\le 2l<p,r} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{r,s\le 2l<p,q} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&\quad + \mathcal{P}^* \sum_{2l<p,q,r,s} \Gamma_{jqp}\Gamma_{jsr}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{4.25}
$$

In I_3, the suffices for the exterior multiplication are greater than $2l$ but the suffices for the interior multiplication are less than or equal to $2l$. Therefore $I_3 = 0$.

Similarly, in I_6, the suffices for the interior multiplication are greater than $2l$ but the suffices for the exterior multiplication are less than or equal to $2l$. Therefore $I_6 = 0$.

In I_2 and I_7, we may consider the operations of $\mathrm{ext}(e^s)\mathrm{int}(e^r)$ and $\mathrm{ext}(e^q)\mathrm{int}(e^p)$ separately. Therefore, by the same reason in Step 1, we have $I_2 = I_7 = 0$.

In I_4, from (4.13) and (4.18), we have

$$
I_4 = \mathcal{P}^* \sum_{\substack{p,s\le 2l \\ q>m}} \Gamma_{jqp}\Gamma_{jsq}\mathrm{ext}(e^s)\mathrm{int}(e^p)\mathcal{P}.
$$

The same reason in Step 1 leads to

$$I_4 = \mathcal{P}^* \sum_{\substack{p\le 2l\\ q>m}} \Gamma_{jqp}\Gamma_{jpq}\mathrm{ext}(e^p)\mathrm{int}(e^p)\mathcal{P}.$$

From (4.13), (4.17) and (4.18), we see that

$$\mathcal{P}^*\mathrm{ext}(e^p)\mathrm{int}(e^p)\mathcal{P} = \frac{1}{2} \tag{4.26}$$

for $p \le 2l$. Therefore, we obtain

$$I_4 = \frac{1}{2}\sum_{\substack{p\le 2l\\ q>m}} \Gamma_{jqp}\Gamma_{jpq} = -\frac{1}{2}\sum_{\substack{p\le 2l\\ q>m}} (\Gamma_{jqp})^2. \tag{4.27}$$

In I_5, from (4.13) and (4.18), we have

$$I_5 = \mathcal{P}^* \sum_{\substack{q,r\le 2l\\ 2l<p\le m}} \Gamma_{jqp}\Gamma_{jpr}\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathcal{P}.$$

As in Step 1, we have

$$I_5 = \mathcal{P}^* \sum_{\substack{q\le 2l\\ 2l<p\le m}} \Gamma_{jqp}\Gamma_{jpq}\mathrm{int}(e^q)\mathrm{ext}(e^q)\mathcal{P}.$$

From (4.13), (4.14) and (4.18), we see that

$$\mathcal{P}^*\mathrm{int}(e^q)\mathrm{ext}(e^q)\mathcal{P} = \frac{1}{2} \tag{4.28}$$

for $p \le 2l$. Therefore, we obtain

$$I_5 = \frac{1}{2}\sum_{\substack{q\le 2l\\ m<p\le m}} \Gamma_{jqp}\Gamma_{jpq} = -\frac{1}{2}\sum_{\substack{p\le 2l\\ 2l<q\le m}} (\Gamma_{jqp})^2. \tag{4.29}$$

In I_8, by using (4.13), (4.18) and the fact that $\Gamma_{jqp} = 0$ for $q = p$, we have

$$I_8 = \sum_{\substack{2l<p\le m\\ q>m}} \Gamma_{jqp}\Gamma_{jpq} = -\sum_{\substack{2l<p\le m\\ q>m}} (\Gamma_{jqp})^2. \tag{4.30}$$

We consider I_1 in Step 6 below.

Step 4. Calculation of $\mathcal{P}^* R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}$ *(I).*

If the number of elements of the set $\{j,k,q,p\} \cap \{1,2,\cdots,2l\}$ is odd, then

$$\mathcal{P}^*\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P} = 0, \tag{4.31}$$

from (4.15), (4.16) and (4.18). Therefore, we decompose as follows:

$$\begin{aligned}
&\mathcal{P}^* \sum_{\substack{1\le j<k\le d\\ 1\le p,q\le d}} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&= \mathcal{P}^* \sum_{\substack{j,k,q,p\le 2l\\ j<k}} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&\quad + \mathcal{P}^* \sum_{j<k\le 2l<q,p} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&\quad + \mathcal{P}^* \sum_{j,q\le 2l<k,p} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&\quad + \mathcal{P}^* \sum_{j,p\le 2l<k,q} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&\quad + \mathcal{P}^* \sum_{p,q\le 2l<j<k} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&\quad + \mathcal{P}^* \sum_{\substack{2l<j,k,q,p\\ j<k}} R_{jkqp}\gamma^j\gamma^k \mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}\\
&=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned} \tag{4.31}$$

In J_2 and J_5, we may consider the operations of $\gamma^j\gamma^k$ and $\mathrm{ext}(e^q)\mathrm{int}(e^p)$ separately. Therefore, by the same reason in Step 1, we have $J_2 = J_5 = 0$.

In J_3, from (4.13) and (4.18), we have

$$J_3 = -\mathcal{P}^* \sum_{\substack{j,q\le 2l\\ 2l<p\le m}} R_{jpqp}\gamma^j \mathrm{ext}(e^q)\mathcal{P}. \tag{4.33}$$

For $q = 2\rho - 1$ $(\rho \le l)$, we have

$$\mathcal{P}^*\gamma^j \mathrm{ext}(e^{2\rho-1})\mathcal{P}1 = \begin{cases} \frac{1}{2}, & \text{if } j = 2\rho-1, \quad \text{from (4.14) and (4.18)},\\ \frac{i}{2}, & \text{if } j = 2\rho, \quad \text{from (4.17) and (4.18)},\\ 0, & \text{otherwise}, \quad \text{from (4.15), (4.16) and (4.18)}. \end{cases} \tag{4.34}$$

For $q = 2\rho$ $(\rho \leq l)$, we have

(4.35)
$$\mathcal{P}^*\gamma^j \mathrm{ext}(e^{2\rho})\mathcal{P}1 = \begin{cases} -\frac{i}{2}, & \text{if } j = 2\rho - 1, \quad \text{from (4.17) and (4.18)}, \\ \frac{1}{2}, & \text{if } j = 2\rho, \quad \text{from (4.14) and (4.18)}, \\ 0, & \text{otherwise}, \quad \text{from (4.15), (4.16) and (4.18)}. \end{cases}$$

Therefore, we obtain

(4.36)
$$\begin{aligned} J_3 =& -\sum_{\substack{\rho \leq l \\ 2l < p \leq m}} \left(\frac{1}{2} R_{2\rho-1,p,2\rho-1,p} + \frac{i}{2} R_{2\rho,p,2\rho-1,p} - \frac{i}{2} R_{2\rho-1,p,2\rho,p} + \frac{1}{2} R_{2\rho,p,2\rho,p} \right) \\ =& \frac{1}{2} \sum_{\substack{p \leq 2l \\ 2l < q \leq m}} R_{pqqp}, \end{aligned}$$

since $R_{2\rho,p,2\rho-1,p} = R_{2\rho-1,p,2\rho,p}$.

In J_4, from (4.13) and (4.18), we have

$$J_4 = \mathcal{P}^* \sum_{\substack{j,p \leq 2l \\ q > m}} R_{jqqp} \gamma^j \mathrm{int}(e^p)\mathcal{P}.$$

For $p = 2\rho - 1$ $(\rho \leq l)$, we have

(4.37)
$$\mathcal{P}^*\gamma^j \mathrm{int}(e^{2\rho-1})\mathcal{P}1 = \begin{cases} \frac{1}{2}, & \text{if } j = 2\rho - 1, \quad \text{from (4.17) and (4.18)}, \\ \frac{i}{2}, & \text{if } j = 2\rho, \quad \text{from (4.14) and (4.18)}, \\ 0, & \text{otherwise}, \quad \text{from (4.15), (4.16) and (4.18)}. \end{cases}$$

For $p = 2\rho$ $(\rho \leq l)$, we have

(4.38)
$$\mathcal{P}^*\gamma^j \mathrm{int}(e^{2\rho})\mathcal{P}1 = \begin{cases} -\frac{i}{2}, & \text{if } j = 2\rho - 1, \quad \text{from (4.14) and (4.18)}, \\ \frac{1}{2}, & \text{if } j = 2\rho, \quad \text{from (4.17) and (4.18)}, \\ 0, & \text{otherwise}, \quad \text{from (4.15), (4.16) and (4.18)}. \end{cases}$$

Therefore, we obtain

(4.39)
$$\begin{aligned} J_4 =& \sum_{\substack{\rho \leq l \\ q > m}} \left(\frac{1}{2} R_{2\rho-1,q,q,2\rho-1} + \frac{i}{2} R_{2\rho,q,q,2\rho-1} - \frac{i}{2} R_{2\rho-1,q,q,2\rho} + \frac{1}{2} R_{2\rho,q,q,2\rho} \right) \\ =& \frac{1}{2} \sum_{\substack{p \leq 2l \\ q > m}} R_{pqqp}, \end{aligned}$$

since $R_{2\rho,q,q,2\rho-1} = R_{2\rho-1,q,q,2\rho}$.

In J_6, by using (4.13), (4.18) and the fact that $R_{jkqp} = 0$ for $q = p$, we have

$$J_6 = \sum_{\substack{2l<p\leq m \\ q>m}} R_{pqqp}. \tag{4.40}$$

We consider J_1 in Step 7 below.

Step 5. Complex expression.

From now on, we may assume that $d = 2l$.

To calculate I_1 and J_1 in Steps 3 and 4, the unitary frame $\{\epsilon_\sigma = (e_{2\sigma-1} - ie_{2\sigma})/\sqrt{2}\}_{\sigma=1}^{l}$ is useful. Let $\{\epsilon^\sigma\}$ and $\{\overline{\epsilon^\sigma}\}$ be the dual and conjugate dual basis, respectively. In this step, we express $\Gamma_{jqp}\text{ext}(e^q)\text{int}(e^p)$ in terms of $\{\epsilon_\sigma\}$ and give fundamental formulas with respect to the exterior and interior multiplication of $\{\epsilon^\sigma\}$ and $\{\overline{\epsilon^\sigma}\}$. These provide us with useful tools in the next two steps.

Since $\{e_j\}$ is orthonormal, $\Gamma_{jqp} = 0$ for $q = p$. For any $j \in \{1, 2, \cdots, d\}$ and $A, B \in \{1, 2, \cdots, l, \bar{1}, \bar{2}, \cdots, \bar{l}\}$, let $\Gamma^c_{jAB} = (\nabla_{e_j}\epsilon_A, \epsilon_B)$, where $\epsilon_{\bar{\sigma}} := \overline{\epsilon_\sigma}$, $\sigma = 1, 2, \cdots, d$. Since

$$e_{2\sigma-1} = \frac{\epsilon_\sigma + \overline{\epsilon_\sigma}}{\sqrt{2}} \quad \text{and} \quad e_{2\sigma} = i\frac{\epsilon_\sigma - \overline{\epsilon_\sigma}}{\sqrt{2}},$$

we have

$$\begin{aligned}
\Gamma_{j,2\sigma,2\sigma-1} =& i\Gamma^c_{j\sigma\bar{\sigma}}, \\
\Gamma_{j,2\sigma-1,2\sigma} =& -i\Gamma^c_{j\sigma\bar{\sigma}}, \\
\Gamma_{j,2\rho-1,2\sigma-1} =& \frac{1}{2}(\Gamma^c_{j\rho\sigma} + \Gamma^c_{j\bar{\rho}\sigma} + \Gamma^c_{j\rho\bar{\sigma}} + \Gamma^c_{j\bar{\rho}\bar{\sigma}}), \\
\Gamma_{j,2\rho,2\sigma} =& \frac{1}{2}(-\Gamma^c_{j\rho\sigma} + \Gamma^c_{j\bar{\rho}\sigma} + \Gamma^c_{j\rho\bar{\sigma}} - \Gamma^c_{j\bar{\rho}\bar{\sigma}}), \\
\Gamma_{j,2\rho,2\sigma-1} =& \frac{i}{2}(\Gamma^c_{j\rho\sigma} - \Gamma^c_{j\bar{\rho}\sigma} + \Gamma^c_{j\rho\bar{\sigma}} - \Gamma^c_{j\bar{\rho}\bar{\sigma}}), \\
\Gamma_{j,2\rho-1,2\sigma} =& \frac{i}{2}(\Gamma^c_{j\rho\sigma} + \Gamma^c_{j\bar{\rho}\sigma} - \Gamma^c_{j\rho\bar{\sigma}} - \Gamma^c_{j\bar{\rho}\bar{\sigma}}),
\end{aligned} \tag{4.41}$$

for $\sigma \neq \rho$. Therefore, we have

$$\begin{aligned}
&\sum_{p,q} \Gamma_{jqp}\mathrm{ext}(e^q)\mathrm{int}(e^p) \\
&= \sum_{1\leq\sigma\leq l} \Gamma^c_{j\sigma\bar{\sigma}} i(\mathrm{ext}(e^{2\sigma})\mathrm{int}(e^{2\sigma-1}) - \mathrm{ext}(e^{2\sigma-1})\mathrm{int}(e^{2\sigma})) \\
&\quad + \sum_{\substack{1\leq\sigma,\rho\leq l \\ \sigma\neq\rho}} \{\frac{1}{2}\Gamma^c_{j\rho\sigma}(\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma-1}) - \mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma}) \\
&\qquad\qquad + i\mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma-1}) + i\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma})) \\
&\qquad + \frac{1}{2}\Gamma^c_{j\bar{\rho}\sigma}(\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma-1}) + \mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma}) \\
&\qquad\qquad - i\mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma-1}) + i\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma})) \\
&\qquad + \frac{1}{2}\Gamma^c_{j\rho\bar{\sigma}}(\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma-1}) + \mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma}) \\
&\qquad\qquad + i\mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma-1}) - i\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma})) \\
&\qquad + \frac{1}{2}\Gamma^c_{j\bar{\rho}\bar{\sigma}}(\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma-1}) - \mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma}) \\
&\qquad\qquad - i\mathrm{ext}(e^{2\rho})\mathrm{int}(e^{2\sigma-1}) - i\mathrm{ext}(e^{2\rho-1})\mathrm{int}(e^{2\sigma}))\}.
\end{aligned}$$

Since

$$e^{2\sigma-1} = \frac{\epsilon^\sigma + \overline{\epsilon^\sigma}}{\sqrt{2}} \quad \text{and} \quad e^{2\sigma} = \frac{\epsilon^\sigma - \overline{\epsilon^\sigma}}{\sqrt{2}i},$$

we have

$$\begin{aligned}
&\sum_{p,q} \Gamma_{jqp}\mathrm{ext}(e^q)\mathrm{int}(e^p) \\
&= \sum_{1\leq\sigma\leq l} \Gamma^c_{j\sigma\bar{\sigma}}(\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\sigma}) - \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\sigma)) \\
&\quad + \sum_{\substack{1\leq\sigma,\rho\leq l \\ \sigma\neq\rho}} \sum_{\substack{A\in\{\sigma,\bar{\sigma}\} \\ B\in\{\rho,\bar{\rho}\}}} \Gamma^c_{jBA}\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A),
\end{aligned} \tag{4.42}$$

where

$$\mathrm{ext}(\omega + i\eta) = \mathrm{ext}(\omega) + i\ \mathrm{ext}(\eta)$$

$$\mathrm{int}(\omega + i\eta) = \mathrm{int}(\omega) + i\ \mathrm{int}(\eta)$$

for $\omega, \eta \in T^*M$.

In the next two steps, the following is fundamental:

$$\mathcal{P} = \prod_{\sigma=1}^{l} \frac{\mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{ext}(\epsilon^\sigma) + 1}{\sqrt{2}} \mathcal{P}_0 \tag{4.43}$$

and

$$\mathcal{P}^* = \mathcal{P}_0 \prod_{\sigma=1}^{l} \frac{\mathrm{int}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\sigma) + 1}{\sqrt{2}}. \tag{4.44}$$

Furtermore, the operation of $\mathrm{int}(\epsilon^\sigma)$ and $\mathrm{int}(\overline{\epsilon^\sigma})$ is as follows:

$$\begin{aligned} &\mathrm{int}(\epsilon^\sigma)\epsilon^\rho = 0, \qquad &&\mathrm{int}(\epsilon^\sigma)\overline{\epsilon^\rho} = \delta_{\sigma\rho}, \\ &\mathrm{int}(\overline{\epsilon^\sigma})\epsilon^\rho = \delta_{\sigma\rho}, \qquad &&\mathrm{int}(\overline{\epsilon^\sigma})\overline{\epsilon^\rho} = 0. \end{aligned} \tag{4.45}$$

Step 6. Calculation of $\mathcal{P}^*\Gamma_{jqp}\Gamma_{jrs}\mathrm{ext}(e^s)\mathrm{int}(e^r)\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}$ *(II).*

We consider I_1 in Step 3. I_1 is written as

$$I_1 = \mathcal{P}^* \left(\sum_{p,q} \Gamma_{jqp}\mathrm{ext}(e^q)\mathrm{int}(e^p) \right)^2 \mathcal{P}. \tag{4.46}$$

Since we have

$$\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\sigma})\mathcal{P}1 = \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\sigma)\mathcal{P}1$$

from (4.43), we obtain

$$(\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\sigma}) - \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\sigma))\mathcal{P} = 0 \tag{4.47}$$

and

$$\mathcal{P}^*(\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\sigma}) - \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\sigma)) = 0. \tag{4.48}$$

Therefore we have

$$I_1 = \mathcal{P}^* \left(\sum_{\substack{1 \le \sigma,\rho \le l \\ \sigma \ne \rho}} \sum_{\substack{A \in \{\sigma,\bar{\sigma}\} \\ B \in \{\rho,\bar{\rho}\}}} \Gamma^c_{jBA}\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A) \right)^2 \mathcal{P}, \tag{4.49}$$

from (4.42).

By using (4.43)-(4.45), we see that, if $\{\sigma,\rho\} \neq \{\sigma',\rho'\}$,

$$\mathcal{P}^*\mathrm{ext}(\epsilon^D)\mathrm{int}(\epsilon^C)\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A)\mathcal{P} = 0 \tag{4.50}$$

for $A \in \{\sigma,\bar{\sigma}\}$, $B \in \{\rho,\bar{\rho}\}$, $C \in \{\sigma',\bar{\sigma'}\}$, $D \in \{\rho',\bar{\rho'}\}$. From this, we have

$$I_1 = \mathcal{P}^* \sum_{1\le\sigma<\rho\le l} \left\{ \sum_{\substack{A\in\{\sigma,\bar{\sigma}\}\\ B\in\{\rho,\bar{\rho}\}}} \Gamma^c_{jBA}(\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A) - \mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A)) \right\}^2 \mathcal{P}. \tag{4.51}$$

Since we have

$$\begin{aligned}
&\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma)\mathcal{P}1 = -\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\epsilon^\rho)\mathcal{P}1,\\
&\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma})\mathcal{P}1 = -\mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\overline{\epsilon^\rho})\mathcal{P}1,\\
&\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\epsilon^\sigma)\mathcal{P}1 = \mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\rho})\mathcal{P}1,\\
&\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\overline{\epsilon^\sigma})\mathcal{P}1 = \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\rho)\mathcal{P}1,
\end{aligned}$$

from (4.43)-(4.45), we obtain

$$\begin{aligned}
&\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma)\mathcal{P} = -\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\epsilon^\rho)\mathcal{P},\\
&\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma})\mathcal{P} = -\mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\overline{\epsilon^\rho})\mathcal{P},\\
&\mathcal{P}^*\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma) = -\mathcal{P}^*\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\epsilon^\rho),\\
&\mathcal{P}^*\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma}) = -\mathcal{P}^*\mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\overline{\epsilon^\rho}),\\
&(\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\epsilon^\sigma) - \mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\rho}))\mathcal{P} = 0,\\
&(\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\overline{\epsilon^\sigma}) - \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\rho))\mathcal{P} = 0,\\
&\mathcal{P}^*(\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\epsilon^\sigma) - \mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\rho})) = 0,\\
&\mathcal{P}^*(\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\overline{\epsilon^\sigma}) - \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\rho)) = 0.
\end{aligned} \tag{4.52}$$

Therefore, we have

$$I_1 = \mathcal{P}^* 4 \sum_{1\le\sigma<\rho\le l} \{\Gamma^c_{j\rho\sigma}\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma) + \Gamma^c_{j\bar{\rho}\bar{\sigma}}\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma})\}^2\mathcal{P}. \tag{4.53}$$

Since

$$(\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma))^2 = (\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma}))^2 = 0,$$

we have

$$\begin{aligned} I_1 =& \mathcal{P}^* 4 \sum_{1\le\sigma<\rho\le l} |\Gamma^c_{j\rho\sigma}|^2 (\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma})\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma) \\ &+ \mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma)\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma}))\mathcal{P}. \end{aligned} \tag{4.54}$$

Finally, since we have

$$\begin{aligned} &\mathcal{P}^*\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma})\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma)\mathcal{P} \\ &= \mathcal{P}^*\mathrm{ext}(\epsilon^\rho)\mathrm{int}(\epsilon^\sigma)\mathrm{ext}(\overline{\epsilon^\rho})\mathrm{int}(\overline{\epsilon^\sigma})\mathcal{P} \\ &= -\frac{1}{4} \end{aligned} \tag{4.55}$$

from (4.43)-(4.45), we obtain

$$I_1 = -2 \sum_{1\le\sigma<\rho\le l} |\Gamma^c_{j\rho\sigma}|^2. \tag{4.56}$$

Step 7. Calculation of $\mathcal{P}^* R_{jkqp}\gamma^j\gamma^k\mathrm{ext}(e^q)\mathrm{int}(e^p)\mathcal{P}$ *(II).*

We consider J_1 in Step 4. By the same procedure in Step 5, we have

$$\begin{aligned} &\sum_{\substack{j<k\le 2l \\ q,p\le l}} R_{jkqp}\gamma^j\gamma^k\mathrm{ext}(e^q)\mathrm{int}(e^p) \\ &= \sum_{\substack{j<k\le 2l \\ 1\le\sigma\le l}} (R(e_j,e_k)\epsilon_\sigma,\overline{\epsilon_\sigma})\gamma^j\gamma^k(\mathrm{ext}(\epsilon^\sigma)\mathrm{int}(\overline{\epsilon^\sigma}) - \mathrm{ext}(\overline{\epsilon^\sigma})\mathrm{int}(\epsilon^\sigma)) \\ &\quad + \sum_{\substack{j<k\le 2l \\ 1\le\sigma,\rho\le l \\ \sigma\ne\rho}} \sum_{\substack{A\in\{\sigma,\bar\sigma\} \\ B\in\{\rho,\bar\rho\}}} (R(e_j,e_k)\epsilon_B,\epsilon_A)\gamma^j\gamma^k\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A). \end{aligned} \tag{4.57}$$

From (4.47), we have

$$\begin{aligned} J_1 &= \mathcal{P}^* \sum_{\substack{j<k\le 2l \\ 1\le\sigma,\rho\le l \\ \sigma\ne\rho}} \sum_{\substack{A\in\{\sigma,\bar\sigma\} \\ B\in\{\rho,\bar\rho\}}} (R(e_j,e_k)\epsilon_B,\epsilon_A)\gamma^j\gamma^k\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A)\mathcal{P} \\ &= \mathcal{P}^* \frac{1}{2} \sum_{\substack{j,k\le 2l \\ 1\le\sigma,\rho\le l \\ \sigma\ne\rho}} \sum_{\substack{A\in\{\sigma,\bar\sigma\} \\ B\in\{\rho,\bar\rho\}}} (R(e_j,e_k)\epsilon_B,\epsilon_A)\gamma^j\gamma^k\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A)\mathcal{P}. \end{aligned} \tag{4.58}$$

We now define δ^σ and $\overline{\delta^\sigma}$ by

$$\delta^\sigma = \mathrm{ext}(\epsilon^\sigma) + \mathrm{int}(\epsilon^\sigma) = \frac{\gamma^{2\sigma-1} - i\gamma^{2\sigma}}{\sqrt{2}},$$
$$\overline{\delta^\sigma} = \mathrm{ext}(\overline{\epsilon^\sigma}) + \mathrm{int}(\overline{\epsilon^\sigma}) = \frac{\gamma^{2\sigma-1} + i\gamma^{2\sigma}}{\sqrt{2}}.$$

For any $A, B, C, D \in \{1, 2, \cdots, l, \bar{1}, \bar{2}, \cdots, \bar{l}\}$, let $R^c_{ABCD} = (R(\epsilon_A, \epsilon_B)\epsilon_C, \epsilon_D)$.

Then, by the same procedure in Step 5, we have

$$\begin{aligned} J_1 =& \mathcal{P}^* \frac{1}{2} \sum_{\substack{\sigma,\rho,\sigma' \le l \\ \sigma \ne \rho}} \sum_{\substack{A \in \{\sigma,\bar{\sigma}\} \\ B \in \{\rho,\bar{\rho}\}}} R^c_{\sigma'\sigma' BA} (\delta^{\sigma'} \overline{\delta^{\sigma'}} - \overline{\delta^{\sigma'}} \delta^{\sigma'}) \mathrm{ext}(\epsilon^B) \mathrm{int}(\epsilon^A) \mathcal{P} \\ &+ \mathcal{P}^* \frac{1}{2} \sum_{\substack{\sigma,\rho,\sigma',\rho' \le l \\ \sigma \ne \rho \\ \sigma' \ne \rho'}} \sum_{\substack{A \in \{\sigma,\bar{\sigma}\} \\ B \in \{\rho,\bar{\rho}\} \\ C \in \{\sigma',\bar{\sigma}'\} \\ D \in \{\rho',\bar{\rho}'\}}} R^c_{DCBA} \delta^D \delta^C \mathrm{ext}(\epsilon^B) \mathrm{int}(\epsilon^A) \mathcal{P} \\ =:& J_{11} + J_{12}. \end{aligned} \tag{4.59}$$

By using (4.43) and (4.45), we have

$$\overline{\delta^\sigma} \delta^\sigma \mathcal{P} = 2\mathcal{P}, \qquad \overline{\delta^\sigma} \mathcal{P} = 0, \tag{4.60}$$

and so

$$\mathcal{P}^* \overline{\delta^\sigma} \delta^\sigma = 2\mathcal{P}^*, \qquad \mathcal{P}^* \delta^\sigma = 0. \tag{4.61}$$

From this, we have

$$J_{11} = -\mathcal{P}^* \sum_{\sigma,\rho,\sigma' \le l} \sum_{\substack{A \in \{\sigma,\bar{\sigma}\} \\ B \in \{\rho,\bar{\rho}\}}} R^c_{\sigma'\bar{\sigma}' BA} \mathrm{ext}(\epsilon^B) \mathrm{int}(\epsilon^A) \mathcal{P}. \tag{4.62}$$

Then, as in Step 1, we see that

$$J_{11} = 0. \tag{4.63}$$

By using (4.61) and the fact that $\overline{\delta^\sigma} \delta^\rho = -\delta^\rho \overline{\delta^\sigma}$ for $\sigma \ne \rho$, we have

$$J_{12} = \mathcal{P}^* \frac{1}{2} \sum_{\substack{\sigma,\rho,\sigma',\rho' \le l \\ \sigma \ne \rho \\ \sigma' \ne \rho'}} \sum_{\substack{A \in \{\sigma,\bar{\sigma}\} \\ B \in \{\rho,\bar{\rho}\}}} R^c_{\bar{\rho}'\bar{\sigma}' BA} \overline{\delta^{\rho'}}\ \overline{\delta^{\sigma'}} \mathrm{ext}(\epsilon^B) \mathrm{int}(\epsilon^A) \mathcal{P}. \tag{4.64}$$

As in (4.50), we see that, if $\{\sigma,\rho\} \neq \{\sigma',\rho'\}$ as sets,

$$\mathcal{P}^*\delta^D\delta^C \mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A)\mathcal{P} = 0 \tag{4.65}$$

for $A \in \{\sigma,\bar{\sigma}\}$, $B \in \{\rho,\bar{\rho}\}$, $C \in \{\sigma',\bar{\sigma'}\}$ and $D \in \{\rho',\bar{\rho'}\}$. From this and symmetricity of $R(\overline{\epsilon_{\sigma'}},\overline{\epsilon_{\rho'}})\overline{\delta^{\rho'}}\ \overline{\delta^{\sigma'}}$ in (ρ',σ'), we have

$$J_{12} = \mathcal{P}^* \sum_{\substack{1\leq\sigma,\rho\leq l \\ \sigma\neq\rho}} \sum_{\substack{A\in\{\sigma,\bar{\sigma}\} \\ B\in\{\rho,\bar{\rho}\}}} R^c_{\bar{\sigma}\bar{\rho}BA}\overline{\delta^{\sigma}}\ \overline{\delta^{\rho}}\mathrm{ext}(\epsilon^B)\mathrm{int}(\epsilon^A)\mathcal{P}. \tag{4.66}$$

By using antisymmetricity of curvature tensors, this is rewritten as

$$\begin{aligned} J_{12} =& \mathcal{P}^* 2 \sum_{1\leq\sigma,\rho\leq l} \overline{\delta^{\sigma}}\ \overline{\delta^{\rho}}\{R^c_{\bar{\sigma}\bar{\rho}\rho\sigma}(\mathrm{ext}(\epsilon^{\rho})\mathrm{int}(\epsilon^{\sigma}) - \mathrm{ext}(\epsilon^{\sigma})\mathrm{int}(\epsilon^{\rho})) \\ &+ R^c_{\bar{\sigma}\bar{\rho}\bar{\rho}\bar{\sigma}}(\mathrm{ext}(\overline{\epsilon^{\rho}})\mathrm{int}(\overline{\epsilon^{\sigma}}) - \mathrm{ext}(\overline{\epsilon^{\sigma}})\mathrm{int}(\overline{\epsilon^{\rho}})) \\ &+ R^c_{\bar{\sigma}\bar{\rho}\bar{\rho}\sigma}(\mathrm{ext}(\overline{\epsilon^{\rho}})\mathrm{int}(\epsilon^{\sigma}) - \mathrm{ext}(\epsilon^{\sigma})\mathrm{int}(\overline{\epsilon^{\rho}})) \\ &+ R^c_{\bar{\sigma}\bar{\rho}\rho\bar{\sigma}}(\mathrm{ext}(\epsilon^{\rho})\mathrm{int}(\overline{\epsilon^{\sigma}}) - \mathrm{ext}(\overline{\epsilon^{\sigma}})\mathrm{int}(\epsilon^{\rho}))\}\mathcal{P}. \end{aligned} \tag{4.67}$$

By using (4.52), we have

$$\begin{aligned} J_{12} =& \mathcal{P}^* 4 \sum_{1\leq\sigma,\rho\leq l} \overline{\delta^{\sigma}}\ \overline{\delta^{\rho}}\{R^c_{\bar{\sigma}\bar{\rho}\rho\sigma}\mathrm{ext}(\epsilon^{\rho})\mathrm{int}(\epsilon^{\sigma}) \\ &+ R^c_{\bar{\sigma}\bar{\rho}\bar{\rho}\bar{\sigma}}\mathrm{ext}(\overline{\epsilon^{\rho}})\mathrm{int}(\overline{\epsilon^{\sigma}})\}\mathcal{P}. \end{aligned} \tag{4.68}$$

Finally, by using (4.43)-(4.45), we see that

$$\begin{aligned} &\mathcal{P}^*\overline{\delta^{\sigma}}\ \overline{\delta^{\rho}}\mathrm{ext}(\epsilon^{\rho})\mathrm{int}(\epsilon^{\sigma})\mathcal{P} = 1, \\ &\mathcal{P}^*\overline{\delta^{\sigma}}\ \overline{\delta^{\rho}}\mathrm{ext}(\overline{\epsilon^{\rho}})\mathrm{int}(\overline{\epsilon^{\sigma}})\mathcal{P} = 0. \end{aligned} \tag{4.69}$$

Therefore we obtain

$$J_{12} = 4 \sum_{1\leq\sigma,\rho\leq l} R^c_{\bar{\sigma}\bar{\rho}\rho\sigma}. \tag{4.70}$$

∎

Finally, we prove Theorem 2.1 by using Theorem 4.1.

Proof of Theorem 2.1. We set $\mathcal{R} = \sum_{j=1}^{3} \mathcal{R}_j - \mathcal{B}$. Let $(\cdot,\cdot)_{L^2(M)}$ and $(\cdot,\cdot)_{L^2(M;\Lambda(T^*M)\otimes\mathbb{C})}$ be the inner product of $L^2(M)$ and $L^2(M;\Lambda(T^*M)\otimes\mathbb{C})$, respectively, where $L^2(M;\Lambda(T^*M)\otimes\mathbb{C})$ is the space of all complex coefficient square integrable forms on M. For any $\varphi \in C_0^2(\overset{\circ}{M})$, we have

$$
\begin{aligned}
&((L(b)+\mathcal{R})\varphi,\varphi)_{L^2(M)} \\
&= ((L(b)+\mathcal{R})\varphi,\varphi)_{L^2(M;\Lambda(T^*M)\otimes\mathbb{C})} \\
&= (\mathcal{P}^*Q^2\mathcal{P}\varphi,\varphi)_{L^2(M;\Lambda(T^*M)\otimes\mathbb{C})} \qquad (4.71)\\
&= \|Q\mathcal{P}\varphi\|^2_{L^2(M;\Lambda(T^*M)\otimes\mathbb{C})} \\
&\geq 0,
\end{aligned}
$$

where $\|\cdot\|^2_{L^2(M;\Lambda(T^*M)\otimes\mathbb{C})}$ is the norm of $L^2(M;\Lambda(T^*M)\otimes\mathbb{C})$. Since $C_0^2(\overset{\circ}{M})$ is dence in $L^2(M)$, we see that $L(b)+\mathcal{R}$ is a positive operator. ∎

Remark 4.1. In (4.71), it is essential that the support of φ does not intersect the boundary of M. Therefore our proof works only for the operator with Dirichlet boundary condition.

Remark 4.2. In the proof of Theorem 2.1, we use only the positivity of the Pauli Hamiltonian Q^2. However, if $\inf \operatorname{spec} Q^2$ is greater than zero, then Theorem 2.1 is improved as follows: $L(b) + \sum_{j=1}^{3} \mathcal{R}_j - \mathcal{B} - \inf \operatorname{spec} Q^2$ is a positive operator. In particular, we have

$$
\inf \operatorname{spec} L(b) \geq \inf_{x\in M}\{-\sum_{j=1}^{3} \mathcal{R}_j(x) + \mathcal{B}(x)\} + \inf \operatorname{spec} Q^2. \qquad (4.72)
$$

5. Applications to asymptotics of a stochastic oscillatory integral and transverse analyticity

In this section, we apply Theorem 2.1 to studies of asymptotic behavior of an oscillatory integral and transverse analyticity.

For any $x \in M$, there is an orthonormal basis

$[e_{1,x}, e_{2,x}, \cdots, e_{d,x}]$ of T_xM such that

$$(db(e_{j,x}, e_{k,x}))_{1\le j,k\le 2l} = \begin{pmatrix} \begin{matrix} 0 & -\beta_1(x) \\ \beta_1(x) & 0 \end{matrix} & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & -\beta_{l(x)}(x) \\ \beta_{l(x)}(x) & 0 \end{matrix} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \tag{5.1}$$

where $\beta_\sigma(x) > 0$ for $1 \le \sigma \le l(x)$. We define a homomorphism $\mathcal{J}_x : T_xM \to T_xM$ by

$$\begin{aligned} \mathcal{J}_x e_{2\sigma-1,x} &= e_{2\sigma,x}, && \sigma \le l(x), \\ \mathcal{J}_x e_{2\sigma,x} &= -e_{2\sigma-1,x}, && \sigma \le l(x), \\ \mathcal{J}_x e_{j,x} &= 0, && j \ge l(x). \end{aligned} \tag{5.2}$$

This definition of $\mathcal{J}_x$ is independent of the choice of the basis $\{e_{j,x}\}$ as above. We need not consider the continuity of $\mathcal{J}_x$ in $x \in M$.

In this section, we assume the following:

(A.2) (i) $\displaystyle\sup_{r=(x,e)\in O(M)} (R(e_1, e_2)e_2, e_1) < \infty,$

(ii) $\displaystyle\sup_{r\in O(M)} (R(\mathcal{J}_x e_1, e_2)\mathcal{J}_x e_2, e_1) < \infty,$

(A.3) $O(M) \ni r \longmapsto db(e_1, e_2) \in \mathbb{R}$ is uniformly continuous,

where $O(M)$ is the orthonormal frame bundle of TM, i.e.,

$$O(M) = \{r = (x, e); e = [e_1, e_2, \cdots, e_d] \text{ is an orthonormal basis of } T_xM\}.$$

We can give $O(M)$ a natural Riemannian structure.

Then we obtain the following:

Theorem 5.1. *Under (A.2) and (A.3), it holds that*

$$\varliminf_{\xi\to\infty} \frac{\inf \operatorname{spec} L(\xi b)}{\xi} \geq \frac{1}{4} \inf_{x\in M} \|db(x)\|_1. \tag{5.3}$$

This type of theorem is proven by Malliavin ([13] for compact manifolds, [14] for Euclidean spaces. cf. also [5]). In [13] and [14], the right hand side of (5.3) is slightly different: the constant and the norm are different. However, the estimate (5.3) is best possible when we consider a uniform magnetic field on $\mathbb{R}^d$.

The conclusion (5.3) of this theorem is weaker than the conclusions of Section 2 in some cases. In fact, in the circumstances of Section 2, we assume that $\sup_{x\in M} \mathcal{R}_j(x) < \infty$ for $j = 1, 2, 3$. Then we easily obtain

$$\varliminf_{\xi\to\infty} \frac{\inf \ \operatorname{spec} L(\xi b)}{\xi} \geq \inf_{x\in M} \mathcal{B}(x) \tag{5.4}$$

from Theorem 2.1, without (A.3).

Proof of Theorem 5.1. It is enough to show that

$$\varliminf_{\mathbb{N}\ni n\nearrow\infty} \frac{\inf \ \operatorname{spec} L(nb)}{n} \geq \frac{1}{4} \inf_{x\in M} \|db(x)\|_1. \tag{5.5}$$

For any bounded operator A on any Banach space B, we denote the operator norm by $\|A\|_{\mathrm{End}(B)}$. Then we have

$$\inf \ \operatorname{spec} L(nb) = -\log \|e^{-L(nb)}\|_{\mathrm{End}(L^2(M))}, \tag{5.6}$$

where $e^{-tL(nb)}$, $t > 0$, is the semigroup generated by $L(nb)$. We put $L^n(b) := L(nb)/n$. Then we have

$$\frac{1}{n} \log \|e^{-L(nb)}\|_{\mathrm{End}(L^2(M))} \leq \log \|e^{-L^n(b)}\|_{\mathrm{End}(L^2(M))}. \tag{5.7}$$

Therefore the proof of Thorem 5.1 is reduced to Lemma 5.1 below.∎

Lemma 5.1. *Under (A.2) and (A.3), it holds that*

$$\overline{\lim_{\mathbb{N}\ni n\nearrow\infty}} \log \|e^{-L^n(b)}\|_{\mathrm{End}(L^2(M))} \leq -\frac{1}{4} \inf_{x\in M} \|db(x)\|_1. \tag{5.8}$$

Proof. Since M is σ-compact, the set $C_0(M)$ of all continuous functions with compact support is dence in $L^2(M)$. Therefore, we have

$$\begin{aligned} &\overline{\lim_{n\nearrow\infty}} \log \|e^{-L^n(b)}\|_{\mathrm{End}(L^2(M))} \\ &= \overline{\lim_{n\nearrow\infty}} \log \inf_{\substack{\varphi\in C_0(M) \\ \|\varphi\|_{L^2(M)}=1}} \|e^{-L^n(b)}\varphi\|_{L^2(M)} \\ &\leq \inf_{\substack{\varphi\in C_0(M) \\ \|\varphi\|_{L^2(M)}=1}} \log \overline{\lim_{n\nearrow\infty}} \|e^{-L^n(b)}\varphi\|_{L^2(M)}. \end{aligned} \tag{5.9}$$

Let $X(t,x)$, $t>0$, $x\in M$ be the absorbing barrier Brownian motion on M, starting at x: $X(t,x)$ is a diffusion process generated by $\Delta/2$, where Δ is the Laplace-Beltrami operator with Dirichlet boundary condition. It is well known that $e^{-L^n(b)}\varphi$ has the folowing representation:

$$(e^{-L^n(b)}\varphi)(x) = E\left[\exp\left(-in\int_0^{\frac{t}{n}} b(X(s,x))\circ dX(s,x)\right)\varphi\Big(X\Big(\frac{t}{n},x\Big)\Big)\right], \tag{5.10}$$

where $E[\cdot]$ is the expectation with respect to $X(\cdot,x)$ and $\int_0^{\frac{t}{n}} b(X(s,x))\circ dX(s,x)$ is a stochastic line integral of the 1-form b (see e.g. [11] and [19]).

For any $\delta>0$ and $x\in M$, there is an open set $U_\delta(x)$ of M satisfying the following:

(i) there is an orthonormal frame $[e_{1,x}(y), e_{2,x}(y), \cdots, e_{d,x}(y)]_{y\in U_\delta(x)}$ of TM on $U_\delta(x)$ such that

$$\begin{aligned} &[e_{1,x}(x), e_{2,x}(x), \cdots, e_{d,x}(x)] = [e_{1,x}, e_{2,x}, \cdots, e_{d,x}], \\ &|(R(e_{p,x}(y), e_{q,x}(y))e_{r,x}(y), e_{s,x}(y)) - (R(e_{p,x}, e_{q,x})e_{r,x}, e_{s,x})| < 1, \\ &|(\nabla_{e_{p,x}(y)} e_{q,x}(y), e_{r,x}(y))| < 1, \end{aligned} \tag{5.11}$$

for any $1 \leq p, q, r, s \leq d$ and $y \in U_\delta(x)$, where $[e_{1,x}, e_{2,x}, \cdots, e_{d,x}]$ is the base taken at (5.1).

(ii) $d(x, y) < \delta$ for any $y \in U_\delta(x)$, where $d(x, y)$ is the Riemannian distance between x and y.

(iii) $U_\delta(x)$ is a domain surrounded by finite number of C^2 hypersurfaces.

Since M is σ-compact, there is a sequence $\{x_\iota\}_{\iota=1}^\infty$ of points of M such that $\bigcup_{\iota=1}^\infty U_\delta(x_\iota) = M$. For any $\iota \in \mathbb{N}$, we set

$$V_{\delta,\iota} := U_\delta(x_\iota) - \bigcup_{\kappa=1}^{\iota} U_\delta(x_\kappa). \tag{5.12}$$

Then $V_{\delta,\iota}$ is a finite union of sets surrounded by finite number of C^2 hypersurfaces and M is a disjoint union of $V_{\delta,\iota}$, $\iota \in \mathbb{N}$. We devide the integral (5.10) as follows: when $x \in V_{\delta,\iota}$,

$$\begin{aligned}
&(e^{-L^n(b)}\varphi)(x) \\
&= E\left[\exp\left(-in \int_0^{\frac{t}{n}} b(X(s,x)) \circ dX(s,x) \right) \varphi\Big(X\Big(\frac{t}{n}, x\Big)\Big) \right. \\
&\qquad\qquad \left. ; X(s,x) \in V_{\delta,\iota} \text{ for any } 0 \leq s \leq \frac{t}{n} \right] \\
&\quad + E\left[\exp\left(-in \int_0^{\frac{t}{n}} b(X(s,x)) \circ dX(s,x) \right) \varphi\Big(X\Big(\frac{t}{n}, x\Big)\Big) \right. \\
&\qquad\qquad \left. ; X(s,x) \notin V_{\delta,\iota} \text{ for some } 0 \leq s \leq \frac{t}{n} \right].
\end{aligned} \tag{5.13}$$

The first term is a representation of

$$(e^{-L^n(b)_{V_{\delta,\iota}}}\varphi)(x),$$

where $L^n(b)_{V_{\delta,\iota}}$ is the Friedrichs extension of the restriction of $L^n(b)$ to the set $C_0^2(\overset{\circ}{V}_{\delta,\iota})$ of all C^2 functions on the interior $\overset{\circ}{V}_{\delta,\iota}$ with compact support. The second

term is dominated by

$$E\left[\left|\varphi\Big(X\Big(\frac{t}{n},x\Big)\Big)\right|; X(s,x)\notin V_{\delta,\iota} \text{ for some } 0\le s\le \frac{t}{n}\right].$$

Then we have

$$\begin{aligned}
&\overline{\lim_{n\nearrow\infty}}\ \|e^{-L^n(b)}\varphi\|_{L^2(M)}\\
&=\overline{\lim_{n\nearrow\infty}}\left(\sum_{\iota=1}^{\infty}\|e^{-L^n(b)_{V_{\delta,\iota}}}\varphi\|^2_{L^2(V_{\delta,\iota})}\right)^{1/2}\\
&\quad+\left(\overline{\lim_{n\nearrow\infty}}\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}}E\left[\left|\varphi\Big(X\Big(\frac{t}{n},x\Big)\Big)\right|\right.\right.\\
&\qquad\left.\left.;X(s,x)\notin V_{\delta,\iota} \text{ for some } 0\le s\le\frac{t}{n}\right]^2\mathrm{dvol}(x)\right)^{1/2},
\end{aligned}\tag{5.14}$$

where $\mathrm{dvol}(x)$ is the Riemannian volume.

We now consider the second term of the right hand side of (5.14). By using Hölder's inequality,

$$\begin{aligned}
&\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}}E\left[\left|\varphi\Big(X\Big(\frac{t}{n},x\Big)\Big)\right|\right.\\
&\qquad\left.;X(s,x)\notin V_{\delta,\iota} \text{ for some } 0\le s\le\frac{t}{n}\right]^2\mathrm{dvol}(x)\\
&\le\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}}E\left[\left|\varphi\Big(X\Big(\frac{t}{n},x\Big)\Big)\right|^2\right]\\
&\qquad\times P\left(X(s,x)\notin V_{\delta,\iota} \text{ for some } 0\le s\le\frac{t}{n}\right)\mathrm{dvol}(x)\\
&\le\|\varphi\|^2_{\infty}\left\{\int_M(e^{\frac{t\Delta}{2n}}\chi_{\mathrm{supp}\ \varphi})(x)\mathrm{dvol}(x)\right.\\
&\qquad\left.-\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}}(e^{\frac{t\Delta_{V_{\delta,\iota}}}{2n}}\chi_{\mathrm{supp}\ \varphi|V_{\delta,\iota}})(x)\mathrm{dvol}(x)\right\},
\end{aligned}\tag{5.15}$$

where $P(\cdot)$ is the probability with respect to $X(\cdot,x)$, $\|\varphi\|_\infty=\sup_{x\in M}|\varphi(x)|$, $\chi_{\mathrm{supp}\ \varphi}$ is the defining function of the support $\mathrm{supp}\ \varphi$ of the function φ, and $\Delta_{V_{\delta,\iota}}=L(0)_{V_{\delta,\iota}}$.

The second factor of the right hand side of (5.15) is rewritten as follows:

$$
\begin{aligned}
&\int_M (e^{\frac{t\Delta}{2n}}\chi_{\text{supp }\varphi})(x)\text{dvol}(x) \\
&\qquad -\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}} (e^{\frac{t\Delta_{V_{\delta,\iota}}}{2n}}\chi_{\text{supp }\varphi|_{V_{\delta,\iota}}})(x)\text{dvol}(x) \\
&=\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}\cap\text{ supp }\varphi} P\Big(X(s,x)\notin V_{\delta,\iota} \text{ for some } 0\le s\le \frac{t}{n} \\
&\qquad\qquad \text{and } X\Big(\frac{t}{n},x\Big)\in M\Big)\text{dvol}(x).
\end{aligned}
\tag{5.16}
$$

In fact, since $V_{\delta,\iota}$ is relatively compact, we have

$$
\begin{aligned}
&\int_{V_{\delta,\iota}} (e^{\frac{t\Delta_{V_{\delta,\iota}}}{2n}}\chi_{\text{supp }\varphi|_{V_{\delta,\iota}}})(x)\text{dvol}(x) \\
&=\int_{V_{\delta,\iota}} \chi_{\text{supp }\varphi}(x)(e^{\frac{t\Delta_{V_{\delta,\iota}}}{2n}}1)(x)\text{dvol}(x) \\
&=\int_{V_{\delta,\iota}\cap\text{ supp }\varphi} P\Big(X(s,x)\in V_{\delta,\iota} \text{ for any } 0\le s\le \frac{t}{n}\Big)\text{dvol}(x).
\end{aligned}
\tag{5.17}
$$

On the other hand, though M is not assumed to be compact, we take an increasing sequence $\{K_j\}_{j=1}^{\infty}$ of compact sets of M such that $\bigcup_{j=1}^{\infty}K_j=M$. Then we use the fact that $\chi_{K_j}\in L^2(M)$ and the monotone converjence theorem to obtain

$$
\begin{aligned}
&\int_M (e^{\frac{t\Delta}{2n}}\chi_{\text{supp }\varphi})(x)\text{dvol}(x) \\
&=\lim_{j\to\infty}\int_M (e^{\frac{t\Delta}{2n}}\chi_{\text{supp }\varphi})(x)\chi_{K_j}(x)\text{dvol}(x) \\
&=\lim_{j\to\infty}\int_M \chi_{\text{supp }\varphi}(x)(e^{\frac{t\Delta}{2n}}\chi_{K_j})(x)\text{dvol}(x) \\
&=\lim_{j\to\infty}\int_{\text{supp }\varphi} P\Big(X\Big(\frac{t}{n},x\Big)\in K_j\Big)\text{dvol}(x) \\
&=\int_{\text{supp }\varphi} P\Big(X\Big(\frac{t}{n},x\Big)\in M\Big)\text{dvol}(x).
\end{aligned}
\tag{5.18}
$$

Form (5.17) and (5.18), we have (5.16).

For the right hand side of (5.16), we easily have

$$
\begin{aligned}
&\overline{\lim_{n\to\infty}}\sum_{\iota=1}^{\infty}\int_{V_{\delta,\iota}\cap\text{ supp }\varphi} P\Big(X(s,x)\notin V_{\delta,\iota} \text{ for some } 0\le s\le \frac{t}{n} \\
&\qquad\qquad \text{and } X\Big(\frac{t}{n},x\Big)\in M\Big)\text{dvol}(x) \\
&=0,
\end{aligned}
\tag{5.19}
$$

by using bounded convergence theorem, since supp φ is compact. Therefore we conclude that

$$\begin{aligned}
&\overline{\lim_{n\nearrow\infty}} \sum_{\iota=1}^{\infty} \int_{V_{\delta,\iota}} E\Big[\Big|\varphi\Big(X\Big(\frac{t}{n},x\Big)\Big)\Big| \\
&\qquad\qquad ; X(s,x) \notin V_{\delta,\iota} \text{ for some } 0 \le s \le \frac{t}{n}\Big]^2 \mathrm{dvol}(x) \\
&= 0.
\end{aligned} \tag{5.20}$$

On the other hand, for any $\varphi \in C_0(M)$ such that $\|\varphi\|_{L^2(M)} = 1$, we have

$$\begin{aligned}
&\left(\sum_{\iota=1}^{\infty} \|e^{-L^n(b)_{V_{\delta,\iota}}}\varphi\|^2_{L^2(V_{\delta,\iota})}\right)^{1/2} \\
&\le \left(\sum_{\iota=1}^{\infty} \exp\Big(-2\inf\ \mathrm{spec}\ L^n(b)_{V_{\delta,\iota}}\Big)\|\varphi\|^2_{L^2(V_{\delta,\iota})}\right)^{1/2} \\
&\le \exp\Big(\inf_{1\le\iota<\infty} \inf\ \mathrm{spec}\ L^n(b)_{V_{\delta,\iota}}\Big) \\
&\le \exp\Big(\inf_{1\le\iota<\infty} \inf\ \mathrm{spec}\ L^n(b)_{U_\delta(x_\iota)}\Big).
\end{aligned} \tag{5.21}$$

On $U_\delta(x_\iota)$, let U_ι (resp. W_ι) be the Riemannian vector bundle such that $[e_{1,x_\iota}(y), e_{2,x_\iota}(y), \cdots, e_{2l,x_\iota}(y)]$ (resp. $[e_{2l+1,x_\iota}(y), e_{2l+2,x_\iota}(y), \cdots, e_{d,x_\iota}(y)]$) is the global orthonormal frame. We define $J_\iota : U_\iota \to U_\iota$ by $J_\iota e_{2\sigma-1,x} = e_{2\sigma,x}$, $J_\iota e_{2\sigma,x} = -e_{2\sigma-1,x}$, $1 \le \sigma \le l$. In this situation, we apply Theorem 2.1 and we have

$$\begin{aligned}
&\inf\ \mathrm{spec}\ L^n(b)_{U_\delta(x_\iota)} \\
&\ge \frac{1}{n} \inf_{y\in U_\iota} \left(-\sum_{j=1}^{2} \mathcal{R}^\iota_j(y) + n\mathcal{B}^\iota(y)\right) \\
&\ge \frac{-1}{n} \sup_{y\in U_\iota} \sum_{j=1}^{2} \mathcal{R}^\iota_j(y) + \inf_{y\in U_\iota} \mathcal{B}^\iota(y),
\end{aligned} \tag{5.22}$$

where $\mathcal{R}^\iota_j$ and $\mathcal{B}^\iota$ are the corresponding $\mathcal{R}_j$ and $\mathcal{B}$. From the definition of U_ι, we have

$$\sup_{1\le\iota<\infty} \sup_{y\in U_\iota} \sum_{j=1}^{2} \mathcal{R}^\iota_j(y) < \infty. \tag{5.23}$$

Therefore, for any $\varphi \in C_0(M)$ such that $\|\varphi\|_{L^2(M)} = 1$, we have

$$\overline{\lim_{n\nearrow\infty}} \left(\sum_{\iota=1}^{\infty} \|e^{-L^n(b)_{V_{\delta,\iota}}} \varphi\|^2_{L^2(V_{\delta,\iota})} \right)^{1/2} \leq \exp\left(- \inf_{1\leq\iota<\infty} \inf_{y\in U_\delta(x_\iota)} \mathcal{B}^\iota(y) \right). \tag{5.24}$$

From (5.9), (5.14), (5.20) and (5.24), we obtain

$$\overline{\lim_{n\nearrow\infty}} \log \|e^{-L^n(b)}\|_{\mathrm{End}(L^2(M))} \leq - \inf_{1\leq\iota<\infty} \inf_{y\in U_\delta(x_\iota)} \mathcal{B}^\iota(y). \tag{5.25}$$

Since $\delta > 0$ is arbitrary, by using (A.3) and the fact $\mathcal{B}^\iota(x_\iota) = \|db(x_\iota)\|_1/4$, we obtain (5.8). ∎

Let $e^{-tL(\xi b)}(x,y)$, $(t,x,y) \in (0,\infty)\times M \times M$ be the integral kernel of the semigroup $e^{-tL(\xi b)}$, which is called the heat kernel: for any $\varphi \in L^2(M)$,

$$(e^{-tL(\xi b)}\varphi)(x) = \int_M e^{-tL(\xi b)}(x,y)\varphi(y)\mathrm{dvol}(y).$$

From Theorem 5.1, we obtain a result on asymptotic behavior of $e^{-tL(\xi b)}(x,y)$:

Corollary 1. *Under (A.2) and (A.3), it holds that*

$$\overline{\lim_{\xi\nearrow\infty}} \frac{1}{\xi} \log \left| e^{-tL(\xi b)}(x,y) \right| \leq -\frac{t}{4} \inf_{x\in M} \|db(x)\|_1 \tag{5.26}$$

for any $x, y \in M$.

As in (5.10), the heat kernel has the following representation:

$$\begin{aligned} &e^{-tL(\xi b)}(x,y) \\ &= E\left[\exp\left(-i\xi \int_0^t b(X(s,x)) \circ dX(s,x) \right) \middle| X(t,x) = y \right] e^{\frac{t\Delta}{2}}(x,y), \end{aligned} \tag{5.27}$$

where $e^{\frac{t\Delta}{2}}(x,y)$ is the integral kernel of the semigroup generated by $\Delta/2$. Therefore Corollary 1 is a general statement on the upper estimate of an asymptotic behaviour of a stochastic oscillatory integral for which the phase function is a stochastic line integral (see e.g. [9]).

Proof of Corollary 1. We take $\delta > 0$ sufficiently small. By using the semigroup property, we have

$$
\begin{aligned}
&e^{-tL(\xi b)}(x,y) \\
&= \int_M \int_M e^{-\delta L(\xi b)}(x,x') e^{-(t-2\delta)L(\xi b)}(x',y') \\
&\qquad \times e^{-\delta L(\xi b)}(y',y)\mathrm{dvol}(x')\mathrm{dvol}(y') \\
&= \left(e^{-\delta L(\xi b)}(x,\cdot), \overline{e^{-(t-2\delta)L(\xi b)}\left(e^{-\delta L(\xi b)}(\cdot,y)\right)(\cdot)} \right)_{L^2(M)}.
\end{aligned} \tag{5.28}
$$

This is estimated as follows:

$$
\begin{aligned}
&\left| e^{-tL(\xi b)}(x,y) \right| \\
&\le \left\| e^{-\delta L(\xi b)}(x,\cdot) \right\|_{L^2(M)} \\
&\qquad \times \left\| e^{-(t-2\delta)L(\xi b)}\left(e^{-\delta L(\xi b)}(\cdot,y)\right)(\cdot) \right\|_{L^2(M)} \\
&\le \left\| e^{-\delta L(\xi b)}(x,\cdot) \right\|_{L^2(M)} \left\| e^{-(t-2\delta)L(\xi b)} \right\|_{\mathrm{End}(L^2(M))} \\
&\qquad \times \left\| e^{-\delta L(\xi b)}(\cdot,y) \right\|_{L^2(M)}.
\end{aligned} \tag{5.29}
$$

By using (5.27) and the symmetricity and the semigroup property of $e^{\frac{t\Delta}{2}}(x,y)$, we have

$$
\begin{aligned}
&\left\| e^{-\delta L(\xi b)}(x,\cdot) \right\|^2_{L^2(M)} = \left\| e^{-\delta L(\xi b)}(\cdot,x) \right\|^2_{L^2(M)} \\
&\le \int_M e^{\frac{\delta\Delta}{2}}(x,x')^2 \mathrm{dvol}(x') \\
&= e^{\delta\Delta}(x,x).
\end{aligned} \tag{5.30}
$$

Using also

$$
\left\| e^{-(t-2\delta)L(\xi b)} \right\|_{\mathrm{End}(L^2(M))} = \exp\left(-(t-2\delta)\inf\ \mathrm{spec}\ L(\xi b)\right),
$$

we have

$$
\begin{aligned}
&\log \left| e^{-tL(\xi b)}(x,y) \right| \\
&\le \frac{1}{2}\log e^{\delta\Delta}(x,x) + \frac{1}{2}\log e^{\delta\Delta}(y,y) \\
&\qquad - (t-2\delta)\inf\ \mathrm{spec}\ L(\xi b).
\end{aligned} \tag{5.31}
$$

If we take $\delta = 1/\xi$, we have

$$
\begin{aligned}
&\overline{\lim_{\xi\nearrow\infty}} \frac{1}{\xi} \log \left| e^{-tL(\xi b)}(x,y) \right| \\
&\leq \frac{1}{2} \overline{\lim_{\xi\nearrow\infty}} \frac{1}{\xi} \log e^{\frac{\Delta}{\xi}}(x,x) + \frac{1}{2} \overline{\lim_{\xi\nearrow\infty}} \frac{1}{\xi} \log e^{\frac{\Delta}{\xi}}(y,y) \\
&\quad - \underline{\lim_{\xi\nearrow\infty}} \frac{1}{\xi} \left(t - \frac{2}{\xi} \right) \inf \ \text{spec}\ L(\xi b).
\end{aligned} \tag{5.32}
$$

It is well known that the first and the second terms are zero (cf.e.g. [17]). Therefore, we can conclude the proof by using Theorem 5.1. ∎

The formula (5.27) says that $e^{-tL(\xi b)}(x,y)$ is the Fourier transform of the distribution of $\int_0^t b(X(s,x)) \circ dX(s,x)$ under the conditional probability $P(\cdot | X(t,x) = y)$. Consequently, Corollary 1 leads the following:

Corollary 2. *We assume that (A.2) and (A.3) holds and*

$$\inf_{x \in M} \|db(x)\|_1 > 0.$$

Then, for any $t > 0$, $x, y \in M$, the distribution of $\int_0^t b(X(s,x)) \circ dX(s,x)$ under the conditional probability $P(\cdot | X(t,x) = y)$ has a real analytic density function: there is a function $p(t,x,y;\lambda)$, $t > 0$, $x, y \in M$, $\lambda \in \mathbb{R}$ such that $p(t,x,y;\lambda)$ is real analytic in λ and

$$P\left(\int_0^t b(X(s,x)) \circ dX(s,x) \in A \middle| X(t,x) = y \right) = \int_A p(t,x,y;\lambda) d\lambda$$

for any Borel set A in $\mathbb{R}$.

The radius of convergence of its Taylor series around any point is greater than or equal to $t \inf_{x \in M} \|db(x)\|_1 / 4$.

This type of result is known as transverse analyticity, which has been studied by Malliavin [13],[14],[15], and Prat [18].

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