Spectral Analysis of Schrödinger Operators with Magnetic Fields

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Abstract. The spectra of quadratic Schrödinger operators in general dimensional Euclidean spaces are determined by considering the corresponding classical mechanics and the corresponding Dirac operators. The results are applied to the study on the semiclassical approximation of the eigenvalues of Schrödinger operators with magnetic fields.

1. Introduction

In this paper we will first be concerned with the Schrödinger operator

$$H_{(B,K)} = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - (Bx)_j/2)^2 + \frac{1}{2}\langle x, Kx\rangle, \qquad \partial_j = \frac{\partial}{\partial x_j}, \tag{1.1}$$

defined on the Hilbert space $L^2(\mathbf{R}^d) = L^2(\mathbf{R}^d, dx)$, where B is a real skew-symmetric $d \times d$ matrix, $(Bx)_j$ is the j-th component of $Bx \in \mathbf{R}^d$, K is a real symmetric non-negative definite matrix, $\langle \cdot, \cdot \rangle$ is the inner product of $\mathbf{R}^d$ and dx is the Lebesgue measure. $H_{(B,K)}$ is quadratic in the sense that the corresponding classical Hamiltonian, which is obtained by setting $p_j = \partial_j/\sqrt{-1}$, is a homogeneous quadratic polynomial in $(x, p) \in \mathbf{R}^d \times \mathbf{R}^d$.

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If $B = 0$, $H_{(0,K)}$ is the usual harmonic oscillator and, if $K = 0$, $H_{(B,0)}$ is the Schrödinger operator with a uniform magnetic field. The properties of these particular operators are well known.

For the general case, when $d = 2$ or 3 and K is nondegenerate, Davies [2] and the first author [17] determined the eigenvalues by using the Van Vleck-Pauli formula (cf. [4], [13] and [28]). Moreover, when $d = 2$, the first author [18] has given explicitly the eigenfunctions, which are related to the Hermite and the complex Hermite functions, by some direct calculations and he has also determined the spectral properties when K is degenerate.

In this paper we will study the operator $H_{(B,K)}$ in the above general setting by two systematic methods. We will firstly represent it in terms of the creation and annihilation operators by referring to the corresponding Dirac operator. By this representation, we can give all of the eigenvalues and the eigenfunctions when K is nondegenerate. Secondly we will use the metaplectic representation of the real symplectic group (cf.[5]). We will show that the classical mechanics corresponding to $H_{(B,K)}$ is equivalent to that corresponding to a harmonic oscillator by some symplectic transform. This implies the unitary equivalence of the Schrödinger operators and we can determine the spectra of $H_{(B,K)}$ completely including the case when K is degenerate and the spectrum is not discrete.

We will apply the study of $H_{(B,K)}$ to discuss the semiclassical approximation of the low lying eigenvalues. We will consider the Schrödinger operator with a magnetic field

$$H(\hbar) = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\hbar\partial_j - A_j)^2 + V + \hbar v. \tag{1.2}$$

where $V, v, A_j, j = 1, 2, ..., d$, are the multiplication operators defined by the smooth functions and $\hbar$ is the Planck constant. We will assume that V has nondegenerate or degenerate zeros (see Sections 7, 8 and 11). Then we will show that the asymptotics of the eigenvalues of $H(\hbar)$ below the essential spectrum as $\hbar \downarrow 0$ are determined by the quadratic operators, which is equivalent to the operator of the form $H_{(B,K)}$ and is obtained by ap-

proximating the operator $H(\hbar)$ locally at each zero of V.

When $A \equiv 0$, this type of asymptotics have been studied by several authors and many results are obtained (cf. Helffer [6], Simon [25] and the references therein). In the presence of magnetic fields, Helffer-Sjöstrand [11] have first studied this problem under the assumption that the magnetic field is weak. Moreover the first author [17] discussed this problem when $d = 2$ and V has only finite nondegenerate zeros.

We now describe the contents of this paper.

As is mentioned above, we have two purposes in this paper. Firstly we will study on the quadratic Hamiltonian defined by (1.1) and, secondly, on the semiclassical approximation of the eigenvalues for Schrödinger operator with a magnetic field defined by (1.2).

In Section 2, we will rewrite the quadratic Hamiltonian $H_{(B,K)}$ by using the creation and the annihilation operators which satisfy the canonical commutation relation. For this purpose we will introduce the Dirac operators and use the techniques in the Clifford algebras. In Section 3, we will give the exact forms of the eigenfunctions and determine the spectrum of $H_{(B,K)}$ under the assumption that K is non-degenerate. The results in these two sections will be useful in the study of the semiclassical approximation of the eigenvalues.

Sections 4, 5 and 6 will be devoted to the more detailed study on the quadratic Hamiltonian $H_{(B,K)}$. We obtain a system of orthogonal polynomials from the eigenfunctions for $H_{(B,K)}$ and we will study on them in Section 4. In Section 5, we will calculate an index of the Pauli operator given in Section 2. The calculation will give us an alternative proof for the completeness in $L^2(\mathbf{R}^d)$ of the set of the eigenfunctions given in Section 3.

In Section 6, we will consider the classical mechanics corresponding to $H_{(B,K)}$ and will show that the classical paths are mapped into classical paths for a harmonic oscillator by a symplectic transformation. This enables us to determine completely the spectrum of $H_{(B,K)}$ involving the case when K is degenerate and $H_{(B,K)}$ is not discrete by virtue of

the theory of the metaplectic representations.

At Section 7, we will start the study on the semiclassical approximation of the eigenvalues of $H(\hbar)$ defined by (1.2) and we will consider the problem in that section under the assumption that the Hessian matrices of the scalar potential V at its zeros are all non-degenerate. The case when V has degenerate zeros will be discussed in Section 8. The following two sections, Sections 9 and 10, will be devoted to the proof of the main theorem in Section 8. A similar degenerate asymptotics to that mentioned in Section 8 is seen for the eigenvalues of the Schrödinger operator given by $H_A(\hbar) = (\sqrt{-1}\hbar\nabla - A)^2$ if $H_A(\hbar)$ has some eigenvalues below the essential spectrum. It will be given in the final Section 11.

2. Quadratic Hamiltonians

In this section, for a given quadratic Hamiltonian, we construct concretely the creation and the annihilation operators satisfying the canonical commutation relations. It enables us to identify the spectrum as a set if the spectrum is discrete.

Let $K = (K_{i,j})_{i,j=1,2,\dots,d}$ be a real non-negative definite symmetric $d \times d$ matrix and $B = (B_{i,j})$ be a real skew-symmetric one. We consider the Schrödinger operator $H_{(B,K)}$ defined by

$$H_{(B,K)} = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - (Bx)_j/2)^2 + \frac{1}{2}\sum_{i,j=1}^{d} K_{i,j}x_i x_j, \tag{2.1}$$

where $(Bx)_j$ is the j-th component of $Bx \in \mathbf{R}^d, x \in \mathbf{R}^d$. We denote also by $H_{(B,K)}$ its unique self-adjoint realization on $L^2(\mathbf{R}^d)$.

This operator is quadratic in the sense that the corresponding classical Hamiltonian H_{cl} is given by a homogeneous quadratic polynomial in the position and the momenta

$(x,p) \in \mathbf{R}^d \times \mathbf{R}^d$:

$$H_{cl}(x,p) = \frac{1}{2}\sum_{j=1}^{d}(p_j + (Bx)_j/2)^2 + \frac{1}{2}\sum_{i,j=1}^{d} K_{i,j}x_i x_j. \tag{2.2}$$

The Hamilton equation is written as

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \Omega \begin{pmatrix} x \\ p \end{pmatrix}, \qquad \Omega = \Omega_{(B,K)} = \begin{pmatrix} B/2 & E_d \\ B^2/4 - K & B/2 \end{pmatrix}, \tag{2.3}$$

where E_d is the $d \times d$ unit matrix.

We see that the eigenvalues of Ω are purely imaginary or zero and we denote them by $\pm\sqrt{-1}s_j, s_j \geqq 0, j = 1,2,...,d$. In fact, if we denote by $\sqrt{K}$ the non-negative symmetric square root of K and introduce the skew-symmetric matrix

$$\widetilde{\Omega} = \begin{pmatrix} B & \sqrt{K} \\ -\sqrt{K} & 0 \end{pmatrix},$$

then we can show the following easily :

Lemma 2.1. *For any nonzero eigenvalue α of Ω,*

$$T\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} x \\ -\sqrt{K}x/\alpha \end{pmatrix}$$

is a homomorphism from the eigenspace of Ω onto the eigenspace of $\widetilde{\Omega}$ corresponding to the same eigenvalue α. Its inverse is written as

$$T^{-1}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} x \\ (\alpha - B/2)x \end{pmatrix}.$$

In particular, $\pm\sqrt{-1}s_j, s_j \geqq 0, j = 1,2,...,d$ are also the eigenvalues of $\widetilde{\Omega}$ and we have

$$\prod_{j=1}^{d} s_j^2 = \det \widetilde{\Omega} = \det K. \tag{2.4}$$

In our analysis, the matrix $\widetilde{\Omega}$ plays important roles. This matrix $\widetilde{\Omega}$ appears naturally if we consider the Dirac operator and the Pauli Hamiltonian (see Lemma 2.4 below).

Now we are in the position to mention the first result.

Theorem 2.2. *If K is strictly positive definite, then the spectrum $\sigma(H_{(B,K)})$ of the quadratic operator $H_{(B,K)}$ is given by*

$$\sigma(H_{(B,K)}) = \{\sum_{j=1}^{d}(n_j + 1/2)s_j; n_j \in \mathbf{Z}_+, j = 1, 2, ..., d\}, \tag{2.5}$$

including the multiplicities.

In the rest of this section, we prove that (2.5) holds as a set by rewriting $H_{(B,K)}$ in a canonical way. The proof will be completed if we show the completeness in $L^2(\mathbf{R}^d)$ of the set of the corresponding eigenfunctions in the next section or if we show the non-degeneracy of the ground state in Section 5.

Let $U = (U_{\alpha,\beta})$ be a $2d \times 2d$ orthogonal matrix such that

$$U^t \ \widetilde{\Omega} \ U = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \tag{2.6}$$

where U^t is the transpose of U and Λ is the diagonal matrix whose diagonal elements are $s_j, j = 1, 2, ..., d$. We define the first order differential operators $R_\alpha, \alpha = 1, 2, ..., 2d$, and $A_j, j = 1, 2, ..., d$, by

$$R_\alpha = \sum_{k=1}^{d}\{U_{k,\alpha}(\sqrt{-1}\partial_k - (Bx)_k/2) + U_{d+k,\alpha}(\sqrt{K}x)_k\}$$

and

$$A_j = \frac{1}{\sqrt{2}}R_j + \frac{\sqrt{-1}}{\sqrt{2}}R_{d+j},$$

respectively.

Then, by the following theorem and the standard commutator calculus, we can prove that (2.5) holds as a set. Note that we do not have to assume the non-degeneracy of K.

Theorem 2.3. (i) $[A_j, A_k] = 0, [A_j, A_k^*] = \delta_{j,k} s_j$,

(ii)

$$H_{(B,K)} = \sum_{j=1}^{d} (A_j^* A_j + s_j/2),$$

where A_k^* *is the formal adjoint of* A_k *and* $[\ ,\ ]$ *is the commutator defined by* $[A, A'] = AA' - A'A$.

For the proof we introduce the Dirac matrices $\{\gamma_\alpha\}_{\alpha=1}^{2d}$ and the Dirac operator. γ_α's are the $2^d \times 2^d$ Hermitian matrices satisfying

$$\{\gamma_\alpha, \gamma_\beta\} \equiv \gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha,\beta} E_{2^d} \tag{2.7}$$

and the Dirac operator Q is defined by

$$Q = \frac{1}{\sqrt{2}} \sum_{j=1}^{d} \{\gamma_j (\sqrt{-1}\partial_j - (Bx)_j/2) + \gamma_{d+j} (\sqrt{K} x)_j\}. \tag{2.8}$$

For details of the Dirac matrices and the Dirac operators, we refer to [1].

Then we can prove the following lemmas. The proofs are easy and we omit them.

Lemma 2.4. $$Q^2 = H_{(B,K)} + \frac{\sqrt{-1}}{2} \sum_{\alpha<\beta} \widetilde{\Omega}_{\alpha,\beta} \gamma_\alpha \gamma_\beta.$$

Lemma 2.5. *Setting* $\omega_\beta = \sum_{\alpha=1}^{2d} U_{\alpha,\beta} \gamma_\alpha$, $\{\omega_\beta\}_{\beta=1}^{2d}$ *is also a system of Dirac matrices and*

$$\sum_{\alpha<\beta} \widetilde{\Omega}_{\alpha,\beta} \gamma_\alpha \gamma_\beta = \sum_{j=1}^{d} s_j \omega_j \omega_{d+j}.$$

Lemma 2.6. $$Q = \frac{1}{\sqrt{2}} \sum_{\beta=1}^{2d} R_\beta \omega_\beta.$$

Now we prove Theorem 2.3.

Proof of Theorem 2.3. By Lemmas 2.4 and 2.5, we have

$$Q^2 = H_{(B,K)} + \frac{\sqrt{-1}}{2} \sum_{j=1}^{d} s_j \omega_j \omega_{d+j}. \tag{2.9}$$

On the other hand, Lemma 2.6 implies

$$Q^2 = \frac{1}{2}\sum_{\alpha=1}^{2d} R_\alpha^2 + \frac{1}{2}\sum_{\alpha<\beta}[R_\alpha, R_\beta]\omega_\alpha\omega_\beta.$$

Therefore, by the fundamental fact on the Dirac matrices that $E_{2^d}, \omega_\alpha\omega_\beta, \alpha < \beta$, are linearly independent, we obtain

$$H_{(B,K)} = \frac{1}{2}\sum_{\alpha=1}^{2d} R_\alpha^2, \tag{2.10}$$

$$[R_j, R_k] = 0,\ [R_{d+j}, R_{d+k}] = 0 \tag{2.11}$$

and

$$[R_j, R_{d+k}] = \sqrt{-1}\delta_{j,k}s_j \tag{2.12}$$

for $j, k = 1, 2, ..., d$.

Note that $R_\alpha^* = R_\alpha$ by definition. Then we can prove the assertion (i) easily by (2.11) and (2.12). Moreover, by using (2.10) and (2.12), we get

$$\sum_{j=1}^{d} A_j^* A_j = \frac{1}{2}\sum_{\alpha=1}^{2d} R_\alpha^2 + \frac{\sqrt{-1}}{2}\sum_{j=1}^{d}[R_j, R_{d+j}] = H_{(B,K)} - \frac{1}{2}\sum_{j=1}^{d} s_j. \qquad \square$$

We end this section by giving a remark on the relations between the matrices Ω and $\widetilde{\Omega}$. When the symmetric matrix K is non-degenerate, we have

$$T\ \Omega\ T^{-1} = \widetilde{\Omega}, \qquad T = \begin{pmatrix} B/2 & E_d \\ -\sqrt{K} & 0 \end{pmatrix}.$$

When K is degenerate, Ω and $\widetilde{\Omega}$ are not equivalent in general because $\mathrm{rank}(\Omega) = d + \mathrm{rank}(K)$, which follows from

$$\begin{pmatrix} E_d & 0 \\ -B/2 & E_d \end{pmatrix}\ \Omega\ \begin{pmatrix} E_d & 0 \\ -B/2 & E_d \end{pmatrix} = \begin{pmatrix} 0 & E_d \\ -K & 0 \end{pmatrix}.$$

This causes the various properties of the spectrum of $H_{(B,K)}$. For details, see Section 6 below.

3. The Eigenfunctions of $H_{(B,K)}$

We have obtained the eigenvalues of the quadratic operator $H_{(B,K)}$ in the previous section. In this section we give the corresponding eigenfunctions and complete the proof of Theorem 2.2. We use the same notations as those in the previous section and assume that the symmetric matrix K is non-degenerate.

First of all we look for an eigenfunction corresponding to the least eigenvalue $\mu_0 = \sum s_j/2 = \|\widetilde{\Omega}\|_1/4$, where $\|\widetilde{\Omega}\|_1$ is the trace norm of $\widetilde{\Omega}$. For this purpose we let $X = (X_{i,j})$ be a complex symmetric (not Hermitian) matrix and set

$$\phi_0(x) = \exp(-\langle x, Xx\rangle/2),$$

where $\langle \xi, \eta\rangle = \sum_{j=1}^d \xi_j\eta_j$ for $\xi, \eta \in \mathbf{C}^d$.

Then, since $X, \sqrt{K}$ are symmetric and B is skew-symmetric,

$$\phi_0(x)^{-1}A_j\phi_0(x) = 2^{-1/2}\sum_{i,k=1}^{d}\{(X_{i,k} + \sqrt{-1}B_{i,k}/2)(-\sqrt{-1}U_{k,j} + U_{k,d+j}) + \sqrt{K}_{i,k}(U_{d+k,j} + \sqrt{-1}U_{d+k,d+j})\}x_i.$$

Now we set

$$U = \begin{pmatrix} U_{(11)} & U_{(12)} \\ U_{(21)} & U_{(22)} \end{pmatrix}, \quad U_1 = \sqrt{-1}U_{(11)} - U_{(12)}, \quad U_2 = U_{(21)} + \sqrt{-1}U_{(22)},$$

where $U_{(ij)}$'s are $d \times d$ matrices. Then $A_j\phi_0 = 0$ if

$$XU_1 = \sqrt{K}U_2 - \sqrt{-1}BU_1/2. \tag{3.1}$$

We show that (3.1) has a solution X which is symmetric and whose real part is strictly positive definite. For this purpose we show the following.

Lemma 3.1. (i) $\sqrt{K}U_1U_2^* = U_2\Lambda U_2^*$, *where* $U_2^* = \overline{U_2^t}$ *is the adjoint matrix of* U_2.

(ii) U_1 *and* U_2 *are non-degenerate.*

Proof. By the definition of U, we have

$$\begin{aligned} BU_{(11)} + \sqrt{K}U_{(21)} &= -U_{(12)}\Lambda, \qquad & BU_{(12)} + \sqrt{K}U_{(22)} &= U_{(11)}\Lambda, \\ \sqrt{K}U_{(11)} &= U_{(22)}\Lambda, & -\sqrt{K}U_{(12)} &= U_{(21)}\Lambda. \end{aligned} \tag{3.2}$$

and

$$\sqrt{K}U_1 = \sqrt{-1}\sqrt{K}(U_{(11)} + \sqrt{-1}U_{(12)}) = \sqrt{-1}(U_{(22)}\Lambda - \sqrt{-1}U_{(21)}\Lambda) = U_2\Lambda.$$

For the proof of (ii), it is sufficient to show that the Hermitian matrix $U_2\Lambda U_2^*$ is strictly positive definite. By (3.2) and the unitarity of U, we get

$$\begin{aligned} U_2\Lambda U_2^* &= U_{(21)}\Lambda U_{(21)}^t + U_{(22)}\Lambda U_{(22)}^t + \sqrt{-1}\sqrt{K}(U_{(11)}U_{(21)}^t + U_{(12)}U_{(22)}^t) \\ &= U_{(21)}\Lambda U_{(21)}^t + U_{(22)}\Lambda U_{(22)}^t. \end{aligned}$$

Therefore $U_2\Lambda U_2^t$ is real, symmetric and non-negative definite. Moreover, since the diagonal matrix Λ is strictly positive definite by the assumption and the $2d$ column vectors of $U_{(21)}$ and $U_{(22)}$ span $\mathbf{R}^d$, $U_2\Lambda U_2^*$ is strictly positive definite. $\square$

Now we give the ground state for the quadratic operator $H_{(B,K)}$.

Theorem 3.2. *Let X be the $d \times d$ matrix defined by*

$$X = -\sqrt{-1}B/2 + \sqrt{K}U_2U_1^{-1}.$$

Then X is symmetric, $\mathrm{Re}(X)$ is strictly positive definite and $\phi_0(x) = \exp(-\langle x, Xx\rangle/2)$ is an eigenfunction of $H_{(B,K)}$ corresponding to the least eigenvalue μ_0.

Proof. The symmetry of X can be proved from the definitions and we omit the details.

To prove the strictly positive definiteness of $\mathrm{Re}(X)$, we note

$$U_1U_2^* = U_{(11)}U_{(22)}^t - U_{(12)}U_{(21)}^t \quad \text{and} \quad \mathrm{Re}(U_2U_2^*) = E_d,$$

which can be shown by the unitarity of U. Then we get

$$\mathrm{Re}(X)U_1U_2^* = \mathrm{Re}((X + \sqrt{-1}B/2)U_1U_2^*) = \mathrm{Re}(\sqrt{K}U_2U_2^*) = \sqrt{K}.$$

Moreover, by using Lemma 3.1, we obtain

$$(U_1U_2^*)^* \operatorname{Re}(X) U_1U_2^* = U_2U_1^* \sqrt{K} = U_2 \Lambda U_2^*$$

and

$$\operatorname{Re}(X) = (U_1^*)^{-1} \Lambda U_1^{-1}.$$

Therefore $\operatorname{Re}(X)$ is strictly positive definite.

By the second assertion proved above, $\phi_0 \in L^2(\mathbf{R}^d)$. Moreover, since X solves (3.1), $A_j\phi_0 = 0, j = 1, 2, ..., d$.

The proof of the theorem is now completed. $\square$

We proceed to the study on the general eigenfunctions of $H_{(B,K)}$. We set

$$\phi_{\mathbf{n}} = (A_1^*)^{n_1}(A_2^*)^{n_2} \cdots (A_d^*)^{n_d}\phi_0 \quad \text{and} \quad \mu_{\mathbf{n}} = \sum_{j=1}^{d} (n_j + 1/2)s_j$$

for $\mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbf{Z}_+^d$, where A_j^*'s are the creation operators defined in the previous section.

Then the following is proved by the standard commutator calculus.

Proposition 3.3. (i) *$\phi_{\mathbf{n}}$ is an eigenfunction of $H_{(B,K)}$ corresponding to the eigenvalue $\mu_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{Z}_+^d$.*

(ii) $A_j\phi_{(n_1,...,n_j+1,...,n_d)} = (n_j + 1)s_j\phi_{(n_1,...,n_j,...,n_d)}$.

(iii) *Denoting by $(\cdot, \cdot)$ and $\| \cdot \|$ the inner product and the norm in $L^2(\mathbf{R}^d)$,*

$$(\phi_{\mathbf{n}}, \phi_{\mathbf{n}'}) = \|\phi_0\|^2 \prod_{j=1}^{d} \delta_{n_j,n_j'} n_j! s_j^{n_j}.$$

In order to prove that the spectrum of $H_{(B,K)}$ coincides with $\{\mu_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{Z}_+^d}$ including the multiplicities, we show the completeness of the set $\{\phi_{\mathbf{n}}\}$ in $L^2(\mathbf{R}^d)$. For this purpose we consider the generating function $F(\xi, x), \xi \in \mathbf{C}^d$:

$$F(\xi, x) = \sum_{\mathbf{n}\in\mathbf{Z}_+^d} \frac{\xi^{\mathbf{n}}}{\mathbf{n}!}\phi_{\mathbf{n}}(x) = \sum_{\mathbf{n}\in\mathbf{Z}_+^d} \left(\prod_{j=1}^{d} \frac{1}{n_j!} (\xi_j A_j^*)^{n_j} \right) \phi_0(x).$$

Lemma 3.4. *$F(\xi, x) = \phi_0(x) \exp(g(\xi, x))$, where*

$$g(\xi, x) = \sqrt{2}\langle (\mathrm{Re}\, X)\overline{U_1}\xi, x\rangle - \frac{1}{2}\langle (\mathrm{Re}\, X)\overline{U_1}\xi, \overline{U_1}\xi\rangle. \tag{3.3}$$

Proof. We set $f(t, x, \xi) = F(t\xi, x)$. Then f is the unique solution of the following first order partial differential equation :

$$\frac{\partial f}{\partial t} - \sum_{j=1}^{d} \xi_j A_j^* f = 0$$

or

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{d} a_j \frac{\partial f}{\partial x_j} = b(x) f, \tag{3.4}$$

satisfying $f|_{t=0} = \phi_0$, where $a_j = (\overline{U_1}\xi)_j/\sqrt{2}$ and $b(x) = \langle \overline{XU_1}\xi, x\rangle/\sqrt{2}$.

(3.4) can be solved easily and we get

$$f(t, \xi, x) = \phi_0(x) \exp(g(t, \xi, x)),$$
$$g(t, \xi, x) = \sqrt{2}\langle (\mathrm{Re}\, X)\overline{U_1}\xi, x\rangle t - \langle (\mathrm{Re}\, X)\overline{U_1}\xi, \overline{U_1}\xi\rangle t^2/2.$$

Setting $t = 1$, we get (3.3). □

By this lemma and the non-degeneracy of the matrices $\mathrm{Re}\, X$ and U_1, we have the following theorem, which proves that the spectrum of $H_{(B,K)}$ is given by $\{\mu_{\mathbf{n}}\}$ including the multiplicities and completes the proof of Theorem 2.2.

Theorem 3.5. *$\{\phi_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{Z}_+^d}$ forms a complete orthogonal basis of $L^2(\mathbf{R}^d)$ consisting of the eigenfunctions of $H_{(B,K)}$.*

4. Generalized Hermite polynomials

We have given the explicit expressions of the eigenfunctions $\{\phi_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{Z}_+^d}$ of the quadratic operator $H_{(K,B)}$ defined by (2.1). By definition, $\phi_{\mathbf{n}}$ is of the form $G_{\mathbf{n}}\phi_0$ for some polynomial $G_{\mathbf{n}}$ in $x = (x_1, x_2, ..., x_d)$ of order $|\mathbf{n}| = \sum n_j$.

Moreover, by Proposition 3.3 (iii) and Theorem 3.5, $\{G_{\mathbf{n}}\}$ forms a system of the complete orthogonal polynomials in $L^2(\mathbf{R}^d, |\phi_0|^2 dx)$.

In general the orthogonal polynomials satisfy some fundamental properties, e.g., some recurrence formulae. See, e.g., Szegö [26]. We will show some of these properties in this section. See also [18] for the case of $d = 2$.

At first we define an isometry T from $L^2(\mathbf{R}^d) = L^2(\mathbf{R}^d, dx)$ onto $L^2(\mathbf{R}^d, |\phi_0|^2 dx)$ by

$$T\varphi = \phi_0^{-1}\varphi, \quad \varphi \in L^2(\mathbf{R}^d),$$

and set

$$L = TH_{(B,K)}T^{-1} \quad \text{and} \quad A_j^0 = TA_jT^{-1}.$$

L is so-called the ground state transformation of the Schrödinger operator $H_{(B,K)}$.

Then the following is easily verified.

Proposition 4.1. (i) *The system of the polynomials* $\{G_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{Z}_+^d}$ *forms a complete orthogonal basis of* $L^2(\mathbf{R}^d, |\phi_0|^2 dx)$ *satisfying* $LG_{\mathbf{n}} = \mu_{\mathbf{n}} G_{\mathbf{n}}$.

(ii) $A_j^0 G_{(n_1,\dots,n_j+1,\dots,n_d)} = (n_j + 1)s_j G_{(n_1,\dots,n_j,\dots,n_d)}$.

(iii) $(A_j^0)^* G_{(n_1,\dots,n_j-1,\dots,n_d)} = G_{(n_1,\dots,n_j,\dots,n_d)}$, *where* $(A_j^0)^*$ *is the formal adjoint of* A_j^0 *with respect to the measure* $|\phi_0|^2 dx$.

Secondly we show a recurrence formula for $\{G_{\mathbf{n}}\}$.

Proposition 4.2. *Let* U_1 *and* X *be the* $d \times d$ *matrices introduced in the previous section, then*

$$G_{(n_1,\dots,n_j+1,\dots,n_d)}(x) = \sqrt{2}(U_1^*(\operatorname{Re} X)x)_j G_{(n_1,\dots,n_d)}(x) - \sum_{k=1}^{d} n_k (U_1^*(\operatorname{Re} X)\overline{U_1})_{j,k} G_{(n_1,\dots,n_k-1,\dots,n_d)}(x).$$

Proof. By Lemma 3.4, we have

$$\sum_{\mathbf{n}\in\mathbf{Z}_+^d} \frac{\xi^{\mathbf{n}}}{\mathbf{n}!} G_{\mathbf{n}}(x) = \exp(\sqrt{2}\langle (\operatorname{Re} X)\overline{U_1}\xi, x\rangle - \langle (\operatorname{Re} X)\overline{U_1}\xi, \overline{U_1}\xi\rangle/2).$$

By differentiating both hand sides with respect to ξ_j and by using the symmetry of X, we get the assertion of the proposition. $\square$

5. An index of the Pauli Hamiltonian

In this section we calculate an index of the Pauli Hamiltonian $P = Q^2$, where Q is the Dirac operator defined by (2.8). We will see that the index is equal to one and will get an alternative proof of Theorem 2.2. We will use the same notations as those in the previous sections and assume that the symmetric matrix K is non-degenerate.

First of all we define an Hermitian matrix Γ by

$$\Gamma = \prod_{j=1}^{d} \sqrt{-1}\gamma_{d+j}\gamma_j.$$

Then the following is easily established by using the anti-commutativity (2.7) of the Dirac matrices $\{\gamma_\alpha\}_{\alpha=1}^{2d}$.

Proposition 5.1. *(P, Γ, Q) has the supersymmetry, that is,* (i) $P = Q^2 \geqq 0$, (ii) Γ^2 *is equal to the identity and* (iii) $\{Q, \Gamma\} = 0$.

Since Γ is Hermitian and Γ^2 is equal to the identity, it only has the eigenvalues ± 1. We denote the corresponding eigenspaces by $V_\pm$ and the restriction of P on $L^2(\mathbf{R}^d) \otimes V_\pm$ by $P_\pm$. Then the supersymmetric index (the Witten index) of P is defined in the usual manner ([1]).

Definition. $\quad \mathrm{Ind}(P) \equiv \dim \mathrm{Ker}(P_+) - \dim \mathrm{Ker}(P_-).$

Before proceeding to the calculations of the index, we notice the following. The proof is easy and is omitted.

Lemma 5.2. (*Berezin's formula*) (i) $\{\sqrt{-1}\gamma_{d+j}\gamma_j\}_{j=1}^{d}$ *is a commutative system of Hermitian matrices and* $(\sqrt{-1}\gamma_{d+j}\gamma_j)^2$ *is equal to the identity,* $j = 1, 2, ..., d$.

(ii) *For any* $\ell = 0, 1, ..., d-1$, *and any* $j_1, j_2, ..., j_\ell$ *with* $1 \leqq j_1 < \cdots < j_\ell \leqq d$,

$$\mathrm{Tr}(\Gamma \prod_{k=1}^{\ell} \sqrt{-1}\gamma_{d+j_k}\gamma_{j_k}) = 0.$$

(iii) *Setting* $\omega_\beta = \sum_{\alpha=1}^{2d} U_{\alpha,\beta}\gamma_\alpha$ *for an orthogonal matrix* $(U_{\alpha,\beta})$,

$$\mathrm{Tr}(\Gamma \prod_{j=1}^{d} \sqrt{-1}\omega_{d+j}\omega_j) = 2^d.$$

We first prove :

Proposition 5.3. $\mathrm{Ind}(P) = 1.$

Proof. We note the following McKean-Singer identity ([1]) : for $t > 0$,

$$\begin{aligned}\mathrm{Ind}(P) &= \int_{\mathbf{R}^d} \mathrm{Tr}(\Gamma e^{-tP}(x,x))dx \\ &= \int_{\mathbf{R}^d} \mathrm{Tr}(\Gamma e^{-tP}(t^{-1/2}x, t^{-1/2}x))t^{-d/2}dx, \end{aligned} \tag{5.1}$$

where $\exp(-tP)(x,y), t > 0, x, y \in \mathbf{R}^d$, is the integral kernel of the semigroup generated by P.

For the integral kernel, we have the Feynman-Kac-Itô formula ([24]). We denote by $E_0[\cdot|A]$ the standard conditional expectation under the condition A with respect to the d-dimensional Wiener measure on the path space starting from 0. Then we have, evaluating the integral kernel on the diagonal set,

$$\begin{aligned}e^{-tP}(x,x) = (2\pi t)^{-d/2} E_0[\exp\{\sqrt{-1}S(t,x,w) - F(t,x,w)\}|w_t = 0] \\ \times \exp(-\sqrt{-1}t \sum_{\alpha<\beta} \widetilde{\Omega}_{\alpha,\beta}\gamma_\alpha\gamma_\beta/2),\end{aligned} \tag{5.2}$$

where S and F are the quadratic Wiener functionals defined by

$$S(t,x,w) = \frac{1}{2}\int_0^t \langle B(x+w_s), \circ dw_s\rangle$$

and

$$F(t,x,w) = \frac{1}{2}\int_0^t \langle x+w_s, K(x+w_s)\rangle ds,$$

respectively. $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbf{R}^d$ and $\circ dw_s$ is the Stratonovich stochastic integral. S is so-called the stochastic area of P. Lévy. We refer to [14] and [24] for the notions on the probability theory.

Let us consider the asymptotics of the integrand of (5.1). Then, by (5.2) and by using the self-similarity of the Wiener measure, we get

$$\begin{aligned}
& t^{-d/2} \operatorname{Tr}(\Gamma e^{-tP}(t^{-1/2}x, t^{-1/2}x)) \\
&= (2\pi t^2)^{-d/2} e^{-\langle x, Kx \rangle/2} \operatorname{Tr}(\Gamma \exp(t \sum_{j=1}^{d} s_j \sqrt{-1} \omega_{d+j} \omega_j / 2)) \cdot (1 + o(1)) \\
&= (2\pi)^{-d/2} e^{-\langle x, Kx \rangle/2} \prod_{j=1}^{d} s_j \cdot (1 + o(1))
\end{aligned}$$

as $t \downarrow 0$ for each $x \in \mathbf{R}^d$. We have used Lemmas 2.5 and 5.2 at the second and the third lines, respectively.

The estimate above is pointwise and depends on x. However, since the magnetic field is constant and the scalar potential grows quadratically at infinity, the magnetic field does not contribute to the leading asymptotics of the integrals over $\mathbf{R}^d$ as $t \downarrow 0$ (cf. [16]). That is, we can show

$$\begin{aligned}
\operatorname{Ind}(P) &= \lim_{t \downarrow 0} \int_{\mathbf{R}^d} \operatorname{Tr}(\Gamma e^{-tP}(x, x)) dx \\
&= \int_{\mathbf{R}^d} \lim_{t \downarrow 0} t^{-d/2} \operatorname{Tr}(\Gamma e^{-tP}(t^{-1/2}x, t^{-1/2}x)) dx \\
&= \int_{\mathbf{R}^d} (2\pi)^{-d/2} e^{-\langle x, Kx \rangle/2} dx \prod_{j=1}^{d} s_j \\
&= (\det K)^{-1/2} \prod_{j=1}^{d} s_j.
\end{aligned}$$

By (2.4), we have proved that the index is equal to one and completed the proof of the proposition. $\square$

Secondly we show the following proposition, which proves Theorem 3.5 if it is combined with Proposition 5.3.

Proposition 5.4. $\mathrm{Ind}(P) = \dim \mathrm{Ker}(H_{(B,K)} - \mu_0)$.

Proof. Let $\{\omega_\beta\}_{\beta=1}^{2d}$ be the system of the Dirac matrices introduced in the statement of Lemma 2.5. Since $\sqrt{-1}\omega_{d+j}\omega_j, j = 1, 2, ..., d$, are Hermitian and $(\sqrt{-1}\omega_{d+j}\omega_j)^2$ is equal to the identity, their eigenvalues are only ± 1. Letting ε_j be 1 or -1, we put

$$V(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d) = \{v \in \mathbf{C}^{2^d}; \sqrt{-1}\omega_{d+j}\omega_j v = \varepsilon_j v, j = 1, 2, ..., d\}.$$

Then, since $\{\sqrt{-1}\omega_{d+j}\omega_j\}$ are commutative, the eigenspace $V_\pm$ of Γ are decomposed by

$$V_\pm = \bigoplus_{\varepsilon_1\varepsilon_2\cdots\varepsilon_d=\pm 1} V(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d) \qquad \text{(direct sum)}.$$

Therefore, letting $P(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d)$ be the restriction of P on $L^2(\mathbf{R}^d) \otimes V(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d)$, we get

$$\mathrm{Ker}(P_\pm) = \bigoplus_{\varepsilon_1\varepsilon_2\cdots\varepsilon_d=\pm 1} \mathrm{Ker}(P(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d)).$$

On the other hand, we have

$$P = H_{(B,K)} - \sum_{j=1}^{d} s_j\sqrt{-1}\omega_{d+j}\omega_j/2 \geqq H_{(B,K)} - \mu_0$$

by (2.9) because $\omega_{d+j}\omega_j = -\omega_j\omega_{d+j}$. Since μ_0 is the least eigenvalue of $H_{(B,K)}$, we get

$$\mathrm{Ker}(P(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d)) = \begin{cases} \mathrm{Ker}(H_{(B,K)} - \mu_0) & \text{if } \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_d = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\dim \mathrm{Ker}(P_+) = \dim \mathrm{Ker}(P(1, 1, ..., 1)) = \dim \mathrm{Ker}(H_{(B,K)} - \mu_0)$$

and

$$\dim \mathrm{Ker}\,(P_-) = 0.$$

The proof is completed. □

6. Quadratic Hamiltonians and harmonic oscillators

In this section, we determine the spectrum of the quadratic operator defined by (2.1) without assuming that the matrix K is strictly positive. For this we first show that the corresponding classical mechanics is transformed to that of a harmonic oscillator by a symplectic transform and then we use the theory of the metaplectic representation (cf.[5]).

Let $Sp(d;\mathbf{R})$ be the group of linear symplectic transforms in $\mathbf{R}^{2d}$ and $\mathfrak{sp}(d;\mathbf{R})$ be its Lie algebra. The classical mechanics corresponding to (2.1) is given by (2.2) and (2.3). Note that $\Omega \in \mathfrak{sp}(d;\mathbf{R})$.

We set $r = \mathrm{rank}(K)$ and $\ell = \mathrm{rank}(\widetilde{\Omega})/2$. Then, it is easy to see

$$d + r \geqq 2\ell.$$

Taking Lemma 2.1 into account, we denote the non-zero eigenvalues of Ω by $\pm\sqrt{-1}s_j, s_j > 0, j = 1, 2, ..., \ell$, including the multiplicities. For $\ell \leqq m \leqq d$, we define $T_{m,\ell} \in \mathfrak{sp}(d;\mathbf{R})$ by

$$T_{m.\ell} = \begin{pmatrix} 0 & E'_m \\ -(\Lambda'_\ell)^2 & 0 \end{pmatrix},$$

where

$$E'_m = \begin{pmatrix} E_m & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda'_\ell = \begin{pmatrix} \Lambda_\ell & 0 \\ 0 & 0 \end{pmatrix},$$

E_m is the $m \times m$ unit matrix and Λ_ℓ is the $\ell \times \ell$ diagonal matrix whose nonzero elements are $s_1, ..., s_\ell$. It is easy to show that $T_{m,\ell}$ is diagonalizable if and only if $m = \ell$.

Then we first show :

Theorem 6.1. *There exists an $S \in Sp(d;\mathbf{R})$ such that*

$$S^{-1}\ \Omega\ S = T_{m,\ell}, \qquad m = d + r - \ell. \tag{6.1}$$

Proof. We only give the matrix S. Let us first consider the case of $d + r = 2\ell$. We let $\mathbf{U}_i = \begin{pmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{pmatrix}, \mathbf{a}_i, \mathbf{b}_i \in \mathbf{C}^d, i = 1, 2, ..., \ell$, be the eigenvectors of Ω corresponding the the

eigenvalue $\sqrt{-1}s_i$ so that

$$((K+s_i^2)\mathbf{a}_i, \mathbf{a}_i) = 2s_i^2$$

and, if $s_i = s_j$ for $i \neq j$,

$$((K+s_i^2)\mathbf{a}_i, \mathbf{a}_j) = 0$$

and we set $\mathbf{X}_i = \mathrm{Re}(\mathbf{U}_i)$ and $\mathbf{Y}_i = \mathrm{Im}(\mathbf{U}_i/s_i)$, where $(\cdot,\cdot)$ is the usual inner product in $\mathbf{C}^d$, $(\mathbf{a},\mathbf{b}) = \sum_{i=1}^d \overline{a_i} b_i$ for $\mathbf{a},\ \mathbf{b} \in \mathbf{C}^d$. Moreover we let $\mathbf{Z}_i = \begin{pmatrix} \mathbf{c}_i \\ \mathbf{d}_i \end{pmatrix}, \mathbf{c}_i, \mathbf{d}_i \in \mathbf{R}^d, i = 1,2,...,2n, n = d-\ell$, be the eigenvectors of Ω corresponding to the zero eigenvalue so that

$$(\mathbf{c}_i, B\mathbf{c}_j) = (\mathbf{c}_{n+i}, B\mathbf{c}_{n+j}) = 0 \qquad \text{and} \qquad (\mathbf{c}_i, B\mathbf{c}_{n+j}) = -\delta_{i,j}$$

for $i,j = 1,...,n$. Note that $K\mathbf{c}_i = 0$. Then we can show that the matrix

$$S = (\mathbf{X}_1 \cdots \mathbf{X}_l \mathbf{Z}_1 \cdots \mathbf{Z}_n \mathbf{Y}_1 \cdots \mathbf{Y}_l \mathbf{Z}_{n+1} \cdots \mathbf{Z}_{2n})$$

is symplectic and satisfies (6.1).

We next consider the case of $d + r > 2\ell$. We change the coordinate so that

$$K = \begin{pmatrix} K^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B^{(1)} & B^{(2)} & B^{(3)} \\ -(B^{(2)})^t & 0 & 0 \\ -(B^{(3)})^t & 0 & B^{(0)} \end{pmatrix},$$

where $K^{(1)}$ is the $r \times r$ nondegenerate diagonal matrix, $B^{(1)}$ is an $r \times r$ matrix, and

$$B^{(0)} = \begin{pmatrix} B_1^{(0)} & & \\ & \ddots & \\ & & B_s^{(0)} \end{pmatrix}, \quad B_i^{(0)} = \begin{pmatrix} 0 & -\beta_i \\ \beta_i & 0 \end{pmatrix},$$

$\beta_i > 0, s = \ell - r$. We set $h = m - \ell$. Then the size of $B^{(2)}$ is $r \times h$.

We take $\mathbf{X}_i, \mathbf{Y}_i, i = 1,2,...,\ell$ as in the last case. We take the eigenvectors $\mathbf{Z}_i = \begin{pmatrix} \mathbf{c}_i \\ \mathbf{d}_i \end{pmatrix}, i = 1,2,...,2s$, of Ω corresponding to the eigenvalue zero as follows: for $i =$

$1, 2, ..., s$,

$$\mathbf{c}_i = (0 \cdots \overset{(r+h+2i-1)-\text{th}}{1} \cdots 0)^t/\sqrt{\beta_i}, \qquad \mathbf{c}_{s+i} = (0 \cdots \overset{(r+h+2i)-\text{th}}{1} \cdots 0)^t/\sqrt{\beta_i}.$$

For the $h \times h$ positive definite symmetric matrix

$$\mathcal{A} = I + (B^{(2)})^t (K^{(1)})^{-1} B^{(2)},$$

let $\lambda_1, ..., \lambda_h$ be its eigenvalues and $\{\mathbf{n}_1, ..., \mathbf{n}_h\}$ be the corresponding orthonormal eigenvectors. For $i = 1, 2, ..., h$, put $\mathbf{q}_i^{(1)} = 0 \in \mathbf{R}^r, \mathbf{q}_i^{(2)} = \mathbf{n}_i/\sqrt{\lambda_i} \in \mathbf{R}^h, \mathbf{q}_i^{(3)} = 0 \in \mathbf{R}^{2s}, \mathbf{q}_i = \begin{pmatrix} \mathbf{q}_i^{(1)} \\ \mathbf{q}_i^{(2)} \\ \mathbf{q}_i^{(3)} \end{pmatrix} \in \mathbf{R}^d, \mathbf{r}_i = -B\mathbf{q}_i/2 \in \mathbf{R}^d$ and $\mathbf{W}_i = \begin{pmatrix} \mathbf{q}_i \\ \mathbf{r}_i \end{pmatrix} \in \mathbf{R}^{2d}$. We see that $\{\mathbf{Z}_1, ..., \mathbf{Z}_{2s}, \mathbf{W}_1, ..., \mathbf{W}_h\}$ is a basis of the eigenspace of the eigenvalue zero.

For $i = 1, 2, ..., h$, set $\mathbf{v}_i^{(1)} = (K^{(1)})^{-1} B^{(2)} \mathbf{q}_i^{(2)} \in \mathbf{R}^r, \mathbf{v}_i^{(3)} = (B^{(0)})^{-1} (B^{(3)})^t \mathbf{v}_i^{(1)} \in \mathbf{R}^{2s}$ and $\mathbf{v}_i^{(2)} \in \mathbf{R}^h$ as follows :

$$(\mathbf{v}_1^{(2)} \cdots \mathbf{v}_h^{(2)}) = -\mathcal{B}\mathcal{C}^{-1}\mathcal{D}^t\mathcal{E}\mathcal{D}$$

where $\mathcal{B} = (\mathbf{n}_1 \cdots \mathbf{n}_h)$, $\mathcal{C}$ is the $h \times h$ non-degenerate diagonal matrix whose nonzero element is $\sqrt{\lambda_1}, ..., \sqrt{\lambda_h}$, $\mathcal{D}$ is the $(r + 2s) \times h$ matrix defined by

$$\mathcal{D} = \begin{pmatrix} \mathbf{v}_1^{(1)} & \cdots & \mathbf{v}_h^{(1)} \\ \mathbf{v}_1^{(3)} & \cdots & \mathbf{v}_h^{(3)} \end{pmatrix},$$

and $\mathcal{E}$ is the $(r + 2s) \times (r + 2s)$ matrix defined by

$$\mathcal{E} = \frac{1}{2} \begin{pmatrix} B^{(1)} & B^{(3)} \\ -(B^{(3)})^t & B^{(0)} \end{pmatrix}.$$

Moreover, we set $\mathbf{v}_i = \begin{pmatrix} \mathbf{v}_i^{(1)} \\ \mathbf{v}_i^{(2)} \\ \mathbf{v}_i^{(3)} \end{pmatrix} \in \mathbf{R}^d, \mathbf{w}_i = \mathbf{q}_i - B\mathbf{v}_i/2 \in \mathbf{R}^d$ and $\mathbf{V}_i = \begin{pmatrix} \mathbf{v}_i \\ \mathbf{w}_i \end{pmatrix} \in \mathbf{R}^{2d}$ for $i = 1, 2, ..., h$. It is easy to see $\Omega \mathbf{V}_i = \mathbf{W}_i$.

Then we can show that the matrix

$$S = (\mathbf{X}_1 \cdots \mathbf{X}_\ell \mathbf{W}_1 \cdots \mathbf{W}_h \mathbf{Z}_1 \cdots \mathbf{Z}_s \mathbf{Y}_1 \cdots \mathbf{Y}_\ell \mathbf{V}_1 \cdots \mathbf{V}_h \mathbf{Z}_{s+1} \cdots \mathbf{Z}_{2s})$$

is symplectic and satisfies (6.1). $\square$

By (6.1), we get the following. Define the classical Hamiltonian $H_{cl}^{m,\ell}$ by

$$H_{cl}^{m,\ell}(x,p) = \frac{1}{2}\sum_{j=1}^{m} p_j^2 + \frac{1}{2}\sum_{j=1}^{\ell} s_j^2 x_j^2$$

and set $\begin{pmatrix} x' \\ p' \end{pmatrix} = S^{-1} \begin{pmatrix} x \\ p \end{pmatrix}$. Then the following identity holds :

$$H_{cl}^{m,\ell}(x',p') = H_{cl}(x,p), \tag{6.2}$$

where H_{cl} is the classical Hamiltonian defined by (2.2). This identity holds also when we put $p = -\sqrt{-1}\nabla$.

Now, by using the metaplectic representation (cf.[5]), we get the following:

Theorem 6.2. *$H_{(B,K)}$ is unitary equivalent to the (possibly degenerate) harmonic oscillator*

$$H_{osc}^{m,\ell} = \frac{1}{2}\sum_{j=1}^{m} \partial_j^2 + \frac{1}{2}\sum_{j=1}^{\ell} s_j^2 x_j^2, \quad m = d + r - \ell,$$

defined on $L^2(\mathbf{R}^d)$.

Proof. Let G be the subgroup of the unitary transforms on $L^2(\mathbf{R}^d)$ defined in pp.71 of [5] and $\mathfrak{G}$ be its Lie algebra. In [5], a two to one homomorphism ρ from G onto $Sp(d;\mathbf{R})$ is constructed and it is shown that its differentiation $d\rho$ is an isomorphism from $\mathfrak{G}$ to $\mathfrak{sp}(d;\mathbf{R})$ such that

$$\sqrt{-1}H_{(B,K)} \in \mathfrak{G}, \quad d\rho(\sqrt{-1}H_{(B,K)}) = \Omega.$$

Therefore, for the matrix S in Theorem 6.1, there is a unitary transform $U \in G$ such that $\rho(U) = S$ and it satisfies $U^{-1}H_{(B,K)}U = H_{osc}^{m,\ell}$. $\square$

Theorem 6.2 allows us to determine the spectra of $H_{(B,K)}$:

Theorem 6.3. (i) *If K is non-degenerate, then $\sigma(H_{(B,K)})$ consists only of the eigenvalues with finite multiplicities and is given by* (2.5) *including the multiplicities.*
(ii) *If K is degenerate and $d + \mathrm{rank}(K) = \mathrm{rank}(\widetilde{\Omega}) = 2\ell$, then $\sigma(H_{(B,K)})$ consists only of the eigenvalues with infinite multiplicities and is given by*

$$\{\sum_{j=1}^{\ell}(n_j + 1/2)s_j; n_j \in \mathbf{Z}_+, j = 1, 2, ..., \ell\}.$$

(iii) *If $d + \mathrm{rank}(K) > \mathrm{rank}(\widetilde{\Omega}) = 2\ell$, then $\sigma(H_{(B,K)})$ is equal to $[\mu_0, \infty)$ and is absolutely continuous, where $\mu_0 = \sum_{j=1}^{\ell} s_j/2$.*

Remark. When $d = 2$, this result has been proved in [18] by a different method.

It should be noted that $d+\mathrm{rank}(K) = \mathrm{rank}(\widetilde{\Omega})$ if and only if the matrix Ω in the classical Hamilton equation is diagonalizable and that, in this case, the spectrum of $H_{(B,K)}$ is pure point.

7. Semiclassical Asymptotics of Eigenvalues

In the following sections, we discuss the asymptotics of the spectrum of $H(\hbar)$ defined by (1.2) as $\hbar \downarrow 0$. We are interested in the leading asymptotics of the low lying eigenvalues.

Setting $\lambda = 1/\hbar$, we consider the Schrödinger operator $H(\lambda)$ defined by the following instead of $H(\hbar)$:

$$H(\lambda) = H_{(A,V,v)}(\lambda) = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - \lambda A_j)^2 + \lambda^2 V + \lambda v, \quad \partial_j = \partial/\partial x_j. \tag{7.1}$$

We use the same notation because there is no fear of confusion.

Throughout the following sections, we assume the following.

(A.1) (i) V is a non-negative C^∞ function and the set of the zeros (sometimes called the wells) of V is not empty ;

(ii) v is a real C^∞ function bounded from below ;

(iii) $A = (A_1, A_2, ..., A_d)$ is a $\mathbf{R}^d$-valued C^∞ function.

Under this assumption, $H(\lambda)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$ ([22]). We will denote the unique self-adjoint realization on $L^2(\mathbf{R}^d)$ by the same notation $H(\lambda)$.

Moreover we identify the vector potential A with the differential 1-form $\sum A_j dx_j$ as usual and we denote it also by A. Then the differential 2-form dA represents the corresponding magnetic field.

In this section, we assume the non-degeneracy of the Hessian matrices of V at all of the wells. The main theorem in this section is typical and fundamental in the study of the semiclassical approximation of the eigenvalues. We will discuss a more general case in the following sections.

First of all we give a fundamental proposition. We assume also the following.

(A.2) $\qquad \liminf_{|x|\to\infty} V(x) > 0.$

Proposition 7.1. *Assume* (A. 1) *and* (A. 2). *Then, for any* $m \in \mathbf{N}$, *there exists a constant* λ_m *such that* $H(\lambda)$ *has more than* m *discrete spectrum* $\{\mu_n(\lambda)\}, \mu_1(\lambda) \leqq \mu_2(\lambda) \leqq \cdots$, *below the essential spectrum for any vector potential* A *and for any* $\lambda > \lambda_m$.

The proof is omitted here because, when we give an upper bound for $\mu_n(\lambda)$ by using the Rayleigh-Ritz technique (cf. [25]), this proposition will be proved simultaneously. For details, see Section 9.

To proceed further, we make one more assumption.

(A.3) $V^{-1}(0)$ is a finite subset $\{p^a\}_{a=1}^N$ of $\mathbf{R}^d$ and the Hessian matrix $K^a = (\partial_i \partial_j V(p^a))$ at each p^a is non-degenerate.

Setting $B_{i,j}(x) = \partial_j A_i(x) - \partial_j A_i(x)$, $i, j = 1, 2, ..., d$, $B(x) = (B_{i,j}(x))$ and $B^a = B(p^a)$,

$a = 1, 2, ..., N$, we consider the quadratic operator $H^a = H_{(B^a, K^a)}$ defined by (2.1) and let $\{\mu^a_{n^a}\}_{n^a=1}^{\infty}$ be the increasing sequence of the eigenvalues of H^a, counting the multiplicities.

Moreover we define the increasing sequence $\{\mu_n\}_{n=1}^{\infty}$ so that

$$\{\mu_n\}_{n=1}^{\infty} = \bigcup_{a=1}^{N} \{v(p^a) + \mu^a_{n^a}\}_{n^a=1}^{\infty}$$

holds including the multiplicities.

Then the main theorem in this section is the following.

Theorem 7.2. ([17]) *Under the assumptions* (A. 1)– (A. 3),

$$\lim_{\lambda \to \infty} \frac{1}{\lambda}\, \mu_n(\lambda) = \mu_n \qquad \textit{for any} \quad n \in \mathbf{N}. \tag{7.2}$$

We end this section by giving some remarks on Theorem 7.2.

Essentially we have only to consider $H(\lambda)$ in the neighborhoods of the wells of V and we can approximate it by the quadratic operator $H_{(\lambda B^a, \lambda^2 K^a)} + \lambda v(p^a)$ near each well p^a.

In order to illustrate this folklore, it might be useful to refer to the cases when we can consider the trace $\mathrm{Tr}(\exp(-tH(\lambda))), t > 0$, of the semigroup generated by $H(\lambda)$. For example, if we assume moreover that there exists a constant $\delta > 0$ such that $V(x) \geqq |x|^{\delta}$ holds for sufficiently large $|x|$, then we can prove

$$\lim_{\lambda \to \infty} \mathrm{Tr}(\exp(-tH(\lambda))) = \sum_{a=1}^{N} \mathrm{Tr}(\exp(-tH^a)) = \sum_{n=1}^{\infty} \exp(-t\mu_n)$$

and (7.2) by virtue of the continuity theorem of the Laplace transformation.

If we consider the problem on a compact Riemannian manifold, the additional assumptions on V are not necessary and we can prove (7.2) under the assumptions corresponding to (A.1) and (A.3). See [17] and [29].

In [17], it is assumed that the wells of V are also stationary points of the magnetic field $(B_{i,j}(x))$ by a technical reason. We found that such an assumption is not necessary to prove (7.2). For details, see Section 10.

8. Degenerate Wells

In this section, continued from the previous section, we discuss the asymptotic behavior of the eigenvalues of the Schrödinger operator $H(\lambda)$ defined by (7.1). We admit that the Hessian matrices of V at the wells are degenerate and that the set of the wells contains a compact submanifold of $\mathbf{R}^d$. For the related topics when $A \equiv 0$, see the works of Helffer-Sjöstrand [9] and [10].

Besides the assumptions (A.1) and (A.2), we assume the following :

(A.4)(i) $V^{-1}(0)$ is decomposed into the disjoint union as

$$V^{-1}(0) = M_1 \bigcup M_2 \bigcup M,$$

where $M_1 = \{p^a\}_{a\in\Lambda_1}, M_2 = \{q^b\}_{b\in\Lambda_2}$ are finite subsets of $\mathbf{R}^d$ and M is a compact submanifold of $\mathbf{R}^d$;

(ii) The Hessian matrix $K^a = K(p^a)$ of V at p^a, $a \in \Lambda_1$, is non-degenerate ;

(iii) The Hessian matrix $K(q^b)$ of V at q^b, $b \in \Lambda_2$, is degenerate ;

(iv) The rank of the Hessian matrix of V at $q \in M$ is equal to the codimension of M.

Denoting the spectrum of a self-adjoint operator L on $L^2(\mathbf{R}^d)$ by $\sigma(L)$, we define the increasing sequence $\{\mu_n\}_{n=1}^{\infty}$ in the same way as in the previous section such that

$$\{\mu_n\}_{n=1}^{\infty} = \bigcup_{a\in\Lambda_1} \{\sigma(H^a) + v(p^a)\}$$

holds including the multiplicities, where $H^a = H_{(B^a,K^a)}$ is the quadratic operator defined by (2.1).

For $q \in M_2 \cup M$ also, we consider the quadratic operator $H^q = H_{(B(q),K(q))}$. Note that, since $K(q)$ is degenerate for $q \in M_2 \cup M$, $\sigma(H^q)$ consists only of the eigenvalues with infinite multiplicities or it is absolutely continuous by Theorem 6.3.

We set

$$\nu = \min\{\inf \sigma(H^q); q \in M_2 \bigcup M\}. \tag{8.1}$$

When V has only the finite non-degenerate wells, we define $\nu = \infty$.

Then we can mention the main result in this section.

Theorem 8.1. *Assume* (A. 1), (A. 2) *and* (A. 4). *Then, for every eigenvalue* $\mu_n(\lambda)$ *of* $H(\lambda)$ *below the essential spectrum, it holds that*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda}\ \mu_n(\lambda) = \min\{\mu_n, \nu\}. \tag{8.2}$$

The proof of the theorem will be given in the following Sections 9 and 10.

The degenerate convergence in (8.2) is intuitively clear. If we admit the one-to-one correspondence of the eigenvalues $\{\mu_n(\lambda)\}$ of $H(\lambda)$ and those of the direct sum of the quadratic operators, which we have exactly in the previous section, ν in the essential spectrum can correspond to any number of $\mu_n(\lambda)$'s.

9. Upper Bounds

The purpose of this section is to prove the following :

Proposition 9.1. *Under the assumptions of Theorem* 8.1, $H(\lambda)$ *has any number of the discrete spectrum* $\mu_n(\lambda)$ *below the essential spectrum if* λ *is sufficiently large and*

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda}\ \mu_n(\lambda) \leqq \min\{\mu_n, \nu\}. \tag{9.1}$$

The method of the proof, essentially borrowed from Simon [25], is based on the Rayleigh-Ritz principle. The eigenfunctions for the quadratic operators are important in order to construct the test functions. See also [17].

Before proceeding to the proof, we prepare some lemmas. For $p \in V^{-1}(0)$, we consider the following quadratic operator :

$$H_p^0(\lambda) = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - \lambda A_{0,j}^p)^2 + \frac{\lambda^2}{2}\sum_{i,j=1}^{d} K_{i,j}^p x_i x_j + \lambda v(p)$$

and

$$H_p^\varepsilon(\lambda) = H_p^0(\lambda) + \varepsilon^2\lambda^2|x|^2,$$

where $\varepsilon > 0$ and $A_{0,j}^p(x) = \sum_{k=1}^{d} \partial_k A_j(p) x_k$. Moreover we set $H_p^\varepsilon = H_p^\varepsilon(1)$. Then we have the following.

Lemma 9.2. *For each $p \in V^{-1}(0)$, there is an increasing sequence $\{\mu_n^p(\varepsilon)\}_{n=1}^\infty$ of eigenvalues and the complete orthonormal basis $\{\phi_n^{p,\varepsilon}\}_{n=1}^\infty$ of $L^2(\mathbf{R}^d)$ consisting of the corresponding eigenfunctions of H_p^ε. Moreover $\phi_n^{p,\varepsilon}$ decays exponentially at infinity and the order of the decay depends on $\varepsilon > 0$.*

Proof. Let $\theta_0^p(x) = B(p)x/2, B(p)_{i,j} = \partial_j A_i(p) - \partial_i A_j(p)$. Since $d(A_0^p - \theta_0^p) = 0$ if we identify the vector potentials with the differential 1-forms, there exists a homogeneous quadratic polynomial g^p such that $A_0^p - \theta_0^p = dg^p$. Then the gauge invariance says

$$H_p^\varepsilon(\lambda) = (G_\lambda^p)^*\ H_p^{s,\varepsilon}(\lambda)\ G_\lambda^p,$$

where G_λ^p is the multiplication operator defined by $\exp(\sqrt{-1}\lambda g^p)$ and

$$H_p^{s,\varepsilon}(\lambda) = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - \lambda\theta_{0,j}^p)^2 + \frac{\lambda^2}{2}\sum_{i,j=1}^{d} K_{i,j}^p x_i x_j + \varepsilon^2\lambda^2|x|^2 + \lambda v(p).$$

For the operator $H_p^{s,\varepsilon}(\lambda)$, the corresponding eigenvalues and eigenfunctions are given by Theorem 2.2 and Proposition 3.3. Combining these results, we can complete the proof. □

We consider also the following quadratic operators : for $p \in V^{-1}(0)$,

$$P_p^0(\lambda) = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - \lambda A_j^p)^2 + \frac{\lambda^2}{2}\sum_{i,j=1}^{d} K_{i,j}^p (x_i - p_i)(x_j - p_j) + \lambda v(p) \qquad (9.2)$$

and

$$P_p^\varepsilon(\lambda) = P_p^0(\lambda) + \varepsilon^2\lambda^2|x-p|^2,$$

where $A_j^p(x) = \sum_{k=1}^d \partial_k A_j(p)(x_k - p_k)$.

Then, defining a unitary operator U_λ^p in $L^2(\mathbf{R}^d)$ by

$$(U_\lambda^p f)(x) = \lambda^{d/4} f(\lambda^{1/2}(x-p)), \qquad f \in L^2(\mathbf{R}^d),$$

it is easy to verify the following.

Lemma 9.3. $\quad P_p^0(\lambda) = \lambda U_\lambda^p H_p^0 (U_\lambda^p)^{-1}$.

We let h be a C^∞ function such that $0 \leqq h \leqq 1$, $h(x) = 1$ if $|x| \leqq 1$ and $\mathrm{supp}(h) \subseteq \{|x| \leqq 2\}$ and set

$$h_\lambda^p(x) = h(\lambda^{2/5}(x-p)).$$

Moreover, setting

$$f^p(x) = \sum_{j=1}^d A_j(p)(x_j - p_j) \qquad \text{and} \qquad F_\lambda^p(x) = \exp(\sqrt{-1}\lambda f^p(x)),$$

we define a function $\varphi_{p,n}^{\varepsilon,\lambda}$ by

$$\varphi_{p,n}^{\varepsilon,\lambda}(x) = \overline{F_\lambda^p(x)} h_\lambda^p(x)(U_\lambda^p \phi_n^{p,\varepsilon})(x).$$

Lemma 9.4. *There exists a constant C_ε, dependent on ε and independent of λ, such that*

$$(\varphi_{p,k}^{\varepsilon,\lambda}, \varphi_{p,\ell}^{\varepsilon,\lambda}) = \delta_{k,\ell} + O(e^{-C_\varepsilon \lambda^{1/5}}) \tag{9.3}$$

as $\lambda \uparrow \infty$, where $(\cdot,\cdot)$ is the inner product in $L^2(\mathbf{R}^d)$ and the large O depends on k and ℓ.

Proof. By definitions, we have

$$(\varphi_{p,k}^{\varepsilon,\lambda}, \varphi_{p,\ell}^{\varepsilon,\lambda}) = \delta_{k,\ell} + \int_{\mathbf{R}^d} (h(\lambda^{-1/10}y)^2 - 1)\overline{\phi_k^{p,\varepsilon}(y)}\phi_\ell^{p,\varepsilon}(y)dy$$

and

$$|\phi_n^{p,\varepsilon}(y)| \leqq |G_n^{p,\varepsilon}(y)| \exp(-C_\varepsilon|x|^2) \tag{9.4}$$

for some polynomial $G_n^{p,\varepsilon}$ and some positive constant C_ε, both of which depends on ε in general.

The rest of the proof is easy. $\square$

Moreover we show the following.

Lemma 9.5. *It holds that*

$$(\varphi_{p,k}^{\varepsilon,\lambda}, H(\lambda)\varphi_{p,\ell}^{\varepsilon,\lambda}) = \lambda(\phi_k^{p,\varepsilon}, H_p^0\phi_\ell^{p,\varepsilon}) + O(\lambda^{4/5})$$

as $\lambda \uparrow \infty$, where the large O depends on k, ℓ and ε.

Proof. We set $\widetilde{H}^p(\lambda) = F_\lambda^p H(\lambda)(F_\lambda^p)^*$. Then we have

$$(\varphi_{p,k}^{\varepsilon,\lambda}, H(\lambda)\varphi_{p,\ell}^{\varepsilon,\lambda}) = (h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, \widetilde{H}^p(\lambda) h_\lambda^p U_\lambda^p \phi_\ell^{\varepsilon,\lambda}).$$

Setting

$$I_1 = (h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, P_p^0(\lambda) h_\lambda^p U_\lambda^p \phi_\ell^{p,\varepsilon})$$

and

$$I_2 = (h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, (\widetilde{H}^p(\lambda) - P_p^0(\lambda)) h_\lambda^p U_\lambda^p \phi_\ell^{p,\varepsilon}),$$

we show

$$I_1 = \lambda(\phi_k^{p,\varepsilon}, H_p^0\phi_\ell^{p,\varepsilon}) + O(e^{-C_\varepsilon \lambda^{1/5}}) \quad \text{and} \quad I_2 = O(\lambda^{4/5}).$$

For this purpose, we write

$$I_1 = I_{11} + I_{12} + I_{13} + I_{14},$$

where

$$I_{11} = (h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, h_\lambda^p P_p^0(\lambda) U_\lambda^p \phi_\ell^{p,\varepsilon}), \qquad I_{12} = -(h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, (\Delta h_\lambda^p) U_\lambda^p \phi_\ell^{p,\varepsilon})/2,$$

$$I_{13} = -(h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, \langle \nabla h_\lambda^p, \nabla U_\lambda^p \rangle \phi_\ell^{p,\varepsilon}), \quad I_{14} = -\lambda(h_\lambda^p U_\lambda^p \phi_k^{p,\varepsilon}, \sqrt{-1}\langle A^p, \nabla h_\lambda^p \rangle U_\lambda^p \phi_\ell^{p,\varepsilon}).$$

By the same way as in the proof of Lemma 9.4, we get

$$I_{11} = (U_\lambda^p \phi_k^{p,\varepsilon}, P_p^0(\lambda) U_\lambda^p \phi_\ell^{p,\varepsilon}) + O(e^{-C_\varepsilon \lambda^{1/5}}).$$

Therefore Lemma 9.3 allows us to conclude

$$I_{11} = \lambda(\phi_k^{p,\varepsilon}, H_p^0 \phi_\ell^{p,\varepsilon}) + O(e^{-C_\varepsilon \lambda^{1/5}}).$$

Since the supports of Δh_λ^p and ∇h_ϕ^p are included in the set $\{x; \lambda^{-2/5} \leqq |x-p| \leqq 2\lambda^{-2/5}\}$, we have only to consider the integral over this set for I_{12}, I_{13}, I_{14} and it is easy to get

$$|I_{1i}| = O(e^{-C_\varepsilon \lambda^{1/5}}), \qquad i = 2, 3, 4.$$

By expanding $\widetilde{H}(\lambda) - P_p^0(\lambda)$, we can prove $I_2 = O(\lambda^{4/5})$ by the same way as in [17] and [25]. The details are omitted. $\square$

Now we are in the position to give a proof of Proposition 9.1.

Proof of Proposition 9.1. By the assumptions (A.1) and (A.2), there exists $C > 0$ such that

$$\inf \sigma_{ess}(H(\lambda)) \geqq C\lambda^2$$

for sufficiently large λ, where $\sigma_{ess}(H(\lambda))$ is the essential spectrum of $H(\lambda)$. Therefore the first assertion follows from Lemmas 9.4, 9.5 and the Rayleigh-Ritz principle.

We set $\Psi_{k,\ell}^{p,\varepsilon} = (\phi_k^{p,\varepsilon}, H_p^0 \phi_\ell^{p,\varepsilon})$ and let $\nu_m^p(\varepsilon)$ be the maximum eigenvalue of the $m \times m$ symmetric matrix $(\Psi_{k,\ell}^{p,\varepsilon})_{k,\ell=1,2,\ldots,m}$. Moreover we set

$$\phi_\xi^{p,\varepsilon} = \sum_{k=1}^m \xi_k \phi_k^{p,\varepsilon} \qquad \text{for} \quad \xi \in \mathbf{C}^m.$$

Then we have

$$\nu_m^p(\varepsilon) = \sup\{(\phi_\xi^{p,\varepsilon}, H_p^0 \phi_\xi^{p,\varepsilon}); |\xi| = 1\} \leqq \sup\{(\phi_\xi^{p,\varepsilon}, H_p^\varepsilon \phi_\xi^{p,\varepsilon}); |\xi| = 1\} = \mu_m^p(\varepsilon).$$

Now we define $n_0 \in \mathbf{N}$ by

$$n_0 = \sup\{n; \mu_n \leqq \nu\}.$$

For $n = 1, 2, \ldots, n_0$, there exists $p_n \in M_1$ and $m(n) \in \mathbf{N}$ such that

$$\lim_{\varepsilon \downarrow 0} \nu_{m(n)}^{p_n}(\varepsilon) = \lim_{\varepsilon \downarrow 0} \mu_{m(n)}^{p_n}(\varepsilon) = \mu_n.$$

Therefore we may assume that $\{\nu^{p_k}_{m(k)}(\varepsilon)\}_{k=1}^{n_0}$ is an increasing sequence.

Then, by the Rayleigh-Ritz principle, we have

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \leqq \nu^{p_n}_{m(n)}(\varepsilon)$$

and, letting ε tend to 0, we get

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \leqq \mu_n$$

for $n = 1, 2, ..., n_0$.

Next we choose any $q \in M_2 \cup M$ such that

$$\nu = \inf \sigma(P^0_q(1)),$$

where $P^0_q(\lambda)$ is the quadratic operator defined by (9.2). Then we get, by Lemmas 9.4, 9.5 and the Rayleigh-Ritz principle again,

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \leqq \nu^q_{n-n_0}(\varepsilon)$$

for every $n > n_0$. Since $\nu^q_k(\varepsilon)$ tends to ν as $\varepsilon \downarrow 0$ for every $k = 1, 2, ...$, we get

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \leqq \nu \qquad \text{for every} \quad n > n_0.$$

We have completed the proof of Proposition 9.1. $\square$

10. Lower Bounds

The purpose of this section is to prove

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \geqq \min\{\mu_n, \nu\} \tag{10.1}$$

and complete the proof of Theorem 8.1.

We begin with a general proposition.

Proposition 10.1. *Letting $W_1, W_2, ..., W_d$ be C^∞ functions and θ be a differential 1-form on $\mathbf{R}^d$, we set*

$$\sigma(x) = (\sigma_{\alpha,\beta}(x)) = \begin{pmatrix} -d\theta(x) & \nabla W(x) \\ -(\nabla W(x))^t & 0 \end{pmatrix},$$

where $d\theta$ is identified with the $d \times d$ skew-symmetric matrix $(\partial_i \theta_j - \partial_j \theta_i)$ and $\nabla W = (\partial_i W_j)$. Then, for any $\xi \in \bigwedge T^*(\mathbf{R}^d) \otimes \mathbf{C}$ with $(\xi, \xi) = 1$ and for any system of Dirac matrices $\{\gamma_\alpha\}_{\alpha=1}^{2d}$, the operator P_ξ defined by

$$P_\xi = \frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - \theta_j)^2 + \frac{1}{2}\sum_{j=1}^{d} W_j^2 + \frac{1}{4}(\xi, \sum_{\alpha,\beta=1}^{2d} \sigma_{\alpha,\beta}\sqrt{-1}\gamma_\alpha\gamma_\beta\xi)$$

is a positive operator, where $(\cdot,\cdot)$ is the natural inner product in $\bigwedge T^*(\mathbf{R}^d) \otimes \mathbf{C}$.

Proof. Consider the Dirac operator

$$Q = \frac{1}{\sqrt{2}}\sum_{j=1}^{d}\{\gamma_j(\sqrt{-1}\partial_j - \theta_j) + \gamma_{d+j}W_j\}.$$

Then, noting that we have adopted the inner product which is complex linear in the second variable, we get

$$(\varphi, P_\xi\varphi)_{L^2} = \|Q(\varphi\xi)\|_{L^2}^2$$

for any $\varphi \in C_0^\infty(\mathbf{R}^d)$, where $(\cdot,\cdot)_{L^2}$ is the inner product in $L^2(\mathbf{R}^d)$ and the norm in the right hand side is that of $L^2(\bigwedge T^*(\mathbf{R}^d) \otimes \mathbf{C})$. $\square$

Remark. By Proposition 10.1, we can show the following extension of the results in [27]: if the matrix valued function $\sigma(x)$ is uniformly continuous, then

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\inf \sigma(L(\lambda)) \geqq \inf_{x\in\mathbf{R}^d} \frac{\|\sigma(x)\|_1}{4},$$

where $L(\lambda)$ is the Friedrichs extension of the symmetric operator

$$\frac{1}{2}\sum_{j=1}^{d}(\sqrt{-1}\partial_j - \lambda\theta_j)^2 + \frac{1}{2}\sum_{j=1}^{d}(\lambda W_j)^2$$

on $C_0^\infty(\mathbf{R}^d)$ and $\|\cdot\|_1$ is the trace norm. For the related results, see also [8] and [19].

Before proceeding to the proof of (10.1), we show the following.

Lemma 10.2. *There exists a differential 1-form* $\alpha = \sum_{i=1}^{d} \alpha_i dx_i$ *such that* (i) $d\alpha = dA$, (ii) $\alpha_k(p^a) = 0$ *and* (iii) $\partial_j \alpha_k(p^a) = \partial_j A_k(p^a)$ *for every* $j, k = 1, 2, ..., d$ *and every* $a \in \Lambda_1$.

Proof. Let $j_a, a \in \Lambda_1$, be C^∞ functions with compact disjoint supports such that j_a is equal to one near p^a and the supports do not intersect with $M_2 \cup M$. Moreover we set

$$g_a(x) = \sum_{i=1}^{d} A_i(p^a)(x_i - p_i^a)$$

Then, setting

$$f(x) = \sum_{a \in \Lambda_1} j_a(x) g_a(x) \qquad \text{and} \qquad \alpha_k(x) = A_k(x) - \partial_k f(x),$$

we can easily show that α satisfies the assertion of the lemma. □

Note that, denoting by F_λ the multiplication operator defined by $\exp(\sqrt{-1}\lambda f)$, we have

$$\widetilde{H}(\lambda) \equiv H_{(\alpha,V,v)}(\lambda) = F_\lambda H_{(A,V,v)}(\lambda) F_\lambda^*. \tag{10.2}$$

For the proof of (10.1), we introduce the IMS localization formula ([1] and [25]) in a convenient form for our purpose.

Fix arbitrary $\varepsilon > 0$. Then, since M is assumed to be compact, there exists a finite subset $\{r^c\}_{c \in \Lambda_3}$ of M such that

$$\bigcup_{c \in \Lambda_3} B(r^c, \varepsilon) \supset M,$$

where $B(r^c, \varepsilon)$ is the ε-ball with center r^c. We set $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ and $\varepsilon_c = \lambda^{-2/5}$ for $c \in \Lambda_1$ and $\varepsilon_c = \varepsilon$ for $c \in \Lambda_2 \cup \Lambda_3$, respectively. Thus, denoting p^c and q^c also by r^c for $c \in \Lambda_1 \cup \Lambda_2$, we get a finite covering $\{B(r^c, \varepsilon_c)\}_{c \in \Lambda}$ of $V^{-1}(0)$.

We consider a partition of unity $\{h_c\}_{c \in \Lambda'}, \Lambda' = \Lambda \cup \{0\}$ subordinate to $\{B(r^c, \varepsilon_c)\}_{c \in \Lambda}$,

that is, $\{h_c\}$ is a system of non-negative functions such that

$$\mathrm{supp}(h_c) \subset B(r^c, 2\varepsilon_c), \qquad \mathrm{supp}(h_0) \subset \overline{\bigcap_{c\in\Lambda} \{\mathbf{R}^d \setminus B(r^c, \varepsilon_c)\}} \equiv B(\lambda, \varepsilon),$$

$$\sum_{c\in\Lambda'} h_c(x)^2 \equiv 1, \qquad \sum_{c\in\Lambda'} |\nabla h_c(x)|^2 < \infty.$$

Then we have the following identity, called the IMS localization formula :

$$H(\lambda) = \sum_{c\in\Lambda'} h_c H(\lambda) h_c - \frac{1}{2} \sum_{c\in\Lambda'} |\nabla h_c|^2. \tag{10.3}$$

This formula is well known when $A = 0$ and it is easy to show that it holds without any change in the presence of magnetic fields. We may assume that

$$\sum_{c\in\Lambda'} |\nabla h_c(x)|^2 \leqq C_1(\varepsilon^{-2} + \lambda^{4/5}) \tag{10.4}$$

for some positive constant C_1. We will denote the constants independent of λ and ε by C_i's.

Next let $H_c(\lambda, \varepsilon)$ be the operator $H(\lambda)$ defined on $L^2(B(r^c, 2\varepsilon_c))$ and $L^2(B(\lambda, \varepsilon))$ for $c \in \Lambda$ and $c = 0$, respectively, with the Dirichlet boundary conditions. Then, by (10.3) and (10.4), we have

$$\inf \sigma(H(\lambda)) \geqq \inf_{c\in\Lambda'} \{\inf \sigma(H_c(\lambda, \varepsilon))\} - C_1(\varepsilon^2 + \lambda^{4/5}). \tag{10.5}$$

As the first step to the proof of (10.1), we give a lower bound for $\sigma(H_c(\lambda, \varepsilon))$.

Since $V(x) \geqq \min\{C_\varepsilon, C_2\lambda^{-4/5}\}$, $x \in B(\lambda, \varepsilon)$, for some $C_\varepsilon, C_2 > 0$, we have

$$\inf \sigma(H_0(\lambda, \varepsilon)) \geqq \min\{C_\varepsilon \lambda^2, C_2 \lambda^{6/5}\}. \tag{10.6}$$

For $c \in \Lambda$, we consider the quadratic operator $H_c = H_{(B^c, K^c)}$, where $B^c_{i,j} = \partial_i A_j(r^c) - \partial_j A_i(r^c)$ and $K^c_{i,j} = \partial^2 V(r^c)/\partial x_i \partial x_j$. We set

$$n(r^c) = \inf \sigma(H_c) = \sum_{j=1}^{d} s^c_j/2,$$

where $s^c_j \geqq 0$ and $\pm\sqrt{-1}s^c_j$, $j = 1, 2, ..., d$, are the eigenvalues of the $2d \times 2d$ matrix $\Omega_{(B^c, K^c)}$ defined by (2.3). Note that at least one of s^c_j's is zero for $c \in \Lambda_2 \cup \Lambda_3$ since K^c is degenerate.

Then we show :

Lemma 10.3. *Under the assumptions* (A. 1), (A. 2) *and* (A. 4), *there exists a positive constant* C_3, *independent of* λ *and* ε, *such that*

$$\inf \sigma(H_c(\lambda, \varepsilon)) \geqq \lambda\{n(r^c) + v(r^c) - C_3\varepsilon\} \tag{10.7}$$

holds for every $c \in \Lambda$ *and any sufficiently large* λ.

Proof. We give the proof for $c \in \Lambda_3$. The proof of the case $c \in \Lambda_1 \cup \Lambda_2$ is similar and easier.

Fix $c \in \Lambda_3$ and choose a coordinate system $(y_1, y_2, ..., y_d)$ around r^c such that $\check{y} = (y_{\ell+1}, ..., y_d)$ is a coordinate in $V^{-1}(0) \cap B(r^c, 2\varepsilon)$, where $d - \ell = \dim M$. Moreover we set $\hat{y} = (y_1, ..., y_\ell)$ and

$$\hat{\nabla}^2 V(y) = (\partial^2 V / \partial y_i \partial y_j)_{i,j=1,...,\ell}.$$

Then, by the assumption (A.4), there exist $C_4 > 0$ and $C_5 > 0$ such that

$$V(y) \geqq \frac{1}{2}(1 - C_4\varepsilon)\langle \hat{y}, \hat{\nabla}^2 V(r^c)\hat{y}\rangle_\ell$$

and

$$v(y) \geqq v(r^c) - C_5\varepsilon$$

hold for any $y \in B(r^c, 2\varepsilon)$, where $\langle \cdot, \cdot \rangle_\ell$ is the usual inner product in $\mathbf{R}^\ell$.

Now we apply Proposition 10.1 by setting

$$W_j(y) = \lambda(\sqrt{\hat{\nabla}^2 V(r^c)}\hat{y})_j, \quad j = 1, 2, ..., \ell,$$

$$W_{\ell+1} = \cdots = W_d = 0,$$

where $\sqrt{\hat{\nabla}^2 V}$ is the positive symmetric root of $\hat{\nabla}^2 V$. Then we get

$$\inf \sigma(H_c(\lambda, \varepsilon)) \geqq \lambda(L^c_\xi/2 + v(r^c) - C_5\varepsilon), \tag{10.8}$$

where

$$L^c_\xi = -(\xi, \sum_{\alpha<\beta} \sigma^c_{\alpha,\beta}\sqrt{-1}\gamma_\alpha\gamma_\beta\xi),$$

$$\sigma^c = (\sigma^c_{\alpha,\beta}) = \begin{pmatrix} -dA(r^c) & \sqrt{\nabla^2_y V(r^c)} \\ -\sqrt{\nabla^2_y V(r^c)} & 0 \end{pmatrix}.$$

As in Section 2, we choose a $2d \times 2d$ orthogonal matrix $U = (U_{\alpha,\beta})$ such that

$$U^t\ \sigma^c\ U = \begin{pmatrix} 0 & \Lambda^c \\ -\Lambda^c & 0 \end{pmatrix},$$

Λ^c is a diagonal matrix with diagonal elements s^c_j, and define $\omega_\beta, \beta = 1, 2, ..., 2d$, by

$$\omega_\beta = \sum_{\alpha=1}^{2d} U_{\alpha,\beta}\gamma_\alpha.$$

Then $\{\omega_\beta\}_{\beta=1}^{2d}$ is also a system of Dirac matrices and we get

$$L^c_\xi = (\xi, \sum_{j=1}^{d} s^c_j\sqrt{-1}\omega_{d+j}\omega_j\xi).$$

Letting ξ be an element in $V(1, 1, ..., 1)$ defined in the proof of Proposition 5.4, we obtain

$$L^c_\xi = \sum_{j=1}^{d} s^c_j.$$

Combining this with (10.8), we get (10.7). $\square$

Now we are in the position to complete the proof of (10.1). The method in the first part is essentially borrowed from [25]. We refer also to [17].

Proof of (10.1). We set $n_0 = \sup\{n; \mu_n \leqq \nu\}$.

For $n = 1$, (10.5) and (10.7) imply

$$\begin{aligned} \liminf_{\lambda\to\infty} \frac{1}{\lambda}\ \mu_1(\lambda) &= \liminf_{\lambda\to\infty} \frac{1}{\lambda}\ \inf \sigma(H(\lambda)) \\ &\geqq \min_{c\in\Lambda}\{n(r^c) + v(r^c) - C_3\varepsilon\}. \end{aligned}$$

Since $\mu_1 = \min\{n(r^c) + v(r^c); c \in \Lambda\}$ and ε is arbitrary, we get

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\ \mu_1(\lambda) \geqq \mu_1.$$

If μ_1 has the multiplicity $n_1 \geqq 2$,

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\, \mu_{n_1}(\lambda) \geqq \liminf_{\lambda\to\infty} \frac{1}{\lambda}\, \mu_1(\lambda) = \mu_1 = \mu_{n_1}.$$

Next we let n be such that $\mu_n < \mu_{n+1} \leqq \mu_{n_0}$ and choose μ with $\mu_n < \mu < \mu_{n+1}$.

We use the IMS localization formula. At first we note that

$$(\phi, h_c H(\lambda) h_c \phi) \geqq \nu\lambda \|h_c\phi\|^2 \tag{10.9}$$

holds for every $c \in \Lambda_2 \cup \Lambda_3$ and every $\phi \in C_0^\infty(\mathbf{R}^d)$, where $\|\ \cdot\ \|$ and $(\cdot,\cdot)$ are the norm and the inner product in $L^2(\mathbf{R}^d)$, respectively. Moreover, by (10.6), we have

$$(\phi, h_0 H(\lambda) h_0 \phi) \geqq \min\{C_\varepsilon \lambda^2, C_2 \lambda^{6/5}\} \|h_0\phi\|^2 \geqq \mu\lambda \|h_0\phi\|^2 \tag{10.10}$$

if $\varepsilon > 0$ is fixed and λ is sufficiently large.

For $a \in \Lambda_1$, using the operators $P_p^0(\lambda)$ and $\widetilde{H}(\lambda)$ defined by (9.2) and (10.2), we write

$$(\phi, h_a H(\lambda) h_a \phi) = I_{1,a}^\lambda(\phi) + I_{2,a}^\lambda(\phi),$$

where

$$I_{1,a}^\lambda(\phi) = (\phi, h_a F_\lambda^* P_{p^a}^0(\lambda) F_\lambda h_a \phi)$$

and

$$I_{2,a}^\lambda(\phi) = (\phi, h_a F_\lambda^* (\widetilde{H}(\lambda) - P_{p^a}^0(\lambda)) F_\lambda h_a \phi).$$

Now let $L_\lambda^a, a \in \Lambda_1$, be the orthogonal projections onto the eigenspace of $P_{p^a}^0(\lambda)$ corresponding to the eigenvalues less than $\lambda\mu$.

Then we have

$$\begin{aligned} I_{1,a}^\lambda(\phi) &= (F_\lambda h_a\phi, P_{p^a}^0(\lambda)(1 - L_\lambda^a) F_\lambda h_a \phi) + (F_\lambda h_a \phi, P_{p^a}^0(\lambda) L_\lambda^a F_\lambda h_a \phi) \\ &= ((1 - L_\lambda^a) F_\lambda h_a \phi, P_{p^a}^0(\lambda)(1 - L_\lambda^a) F_\lambda h_a \phi) + (F_\lambda h_a \phi, P_{p^a}^0(\lambda) L_\lambda^a F_\lambda h_a \phi) \\ &\geqq \lambda\mu \|F_\lambda h_a \phi\|^2 + (\phi, h_a F_\lambda^* L_\lambda^a P_{p^a}^0(\lambda) F_\lambda h_a \phi). \end{aligned} \tag{10.11}$$

For the estimate for $I^{\lambda}_{2,a}(\phi)$, we write

$$\widetilde{H}(\lambda) - P^0_{p^a}(\lambda) = J^a_1(\lambda) + J^a_2(\lambda),$$

where $J^a_1(\lambda)$ is the first order differential operator defined by

$$J^a_1(\lambda) = -\sqrt{-1}\lambda \sum_{j=1}^{d} \{\alpha_j(x) - \sum_{k=1}^{d} \partial_k \alpha_j(p^a)(x_k - p^a_k)\} \frac{\partial}{\partial x_j}$$

and $J^a_2(\lambda)$ is the multiplication operator defined by the function

$$\frac{-\sqrt{-1}\lambda}{2}(\operatorname{div}\alpha(x) - \operatorname{div}\alpha(p^a)) + \frac{\lambda^2}{2}\{|\alpha(x)|^2 - \sum_{j=1}^{d}(\sum_{k=1}^{d}\partial_k\alpha_j(p^a)(x_k - p^a_k))^2\}$$
$$+ \lambda^2(V(x) - \frac{1}{2}\sum_{i,j=1}^{d} K^a_{i,j}(x_i - p^a_i)(x_j - p^a_j)) + \lambda(v(x) - v(p^a)).$$

For the term related $J^a_1(\lambda)$, we estimate as follows:

$$|(\phi, h_a F^*_\lambda J^a_1(\lambda) F_\lambda h_a \phi)| \leqq (\lambda^{4/5}\|h_a\phi\|^2 + \lambda^{-4/5}\|J^a_1(\lambda)F_\lambda h_a\phi\|^2)/2.$$

Since the coefficients of $J^a_1(\lambda)$ is of $O(|x-p^a|^2)$ as $x \to p^a$, we have

$$\|J^a_1(\lambda)F_\lambda h_a\phi\|^2 \leqq C_6\lambda^{2/5}\|\nabla(F_\lambda h_a\phi)\|^2.$$

Moreover, since α is of $O(|x-p^a|)$ as $x \to p^a$, we have

$$\|\nabla(F_\lambda h_a\phi)\|^2 \leqq 2(\|(\sqrt{-1}\nabla - \lambda\alpha)(F_\lambda h_a\phi)\|^2 + \|\lambda\alpha h_a\phi\|^2)$$
$$\leqq 2\|(\sqrt{-1}\nabla - \lambda\alpha)(F_\lambda h_a\phi)\|^2 + C_7\lambda^{6/5}\|h_a\phi\|^2.$$

Combining these, we obtain

$$\|(\phi, h_a F^*_\lambda J^a_1(\lambda) F_\lambda h_a\phi)| \leqq C_8\lambda^{-2/5}(h_a\phi, H(\lambda)h_a\phi) + C_9\lambda^{4/5}\|h_a\phi\|^2. \tag{10.12}$$

For the term related to $J^a_2(\lambda)$, it is easy to prove (cf. [17])

$$|(\phi, h_a F^*_\lambda J^a_2(\lambda) F_\lambda h_a\phi)| \leqq C_{10}\lambda^{4/5}\|h_a\phi\|^2. \tag{10.13}$$

By (10.11), (10.12) and (10.13), we get

$$\sum_{a\in\Lambda_1}(\phi, h_a H(\lambda)h_a\phi) \geqq \lambda\mu\sum_{a\in\Lambda_1}\|h_a\phi\|^2 + (\phi, L_n(\lambda)\phi)$$
$$- C_{11}\lambda^{4/5}\|\phi\|^2 - C_{12}\lambda^{-2/5}(\phi, H(\lambda)\phi), \tag{10.14}$$

where $L_n(\lambda)$ is the operator of rank at most n given by

$$L_n(\lambda) = \sum_{a \in \Lambda_1} h_a F_\lambda^* L_\lambda^a P_{p^a}^0(\lambda) F_\lambda h_a.$$

Combining this with (10.9) and (10.10), we obtain

$$(1 + C_{12}\lambda^{-2/5})(\phi, H(\lambda)\phi) \geqq \lambda\mu\|\phi\|^2 + (\phi, L_n(\lambda)\phi) - C_{11}\lambda^{4/5}\|\phi\|^2$$

and

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\, \mu_{n+1}(\lambda) \geqq \mu, \qquad \mu_n < \mu < \mu_{n+1}.$$

Since μ is arbitrary, we have proved

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\, \mu_{n+1}(\lambda) \geqq \mu_{n+1}.$$

Finally we consider $n > n_0$. We have $\nu < \mu_n$. Then, choosing $\mu = \nu$, we can prove (10.10) and (10.14) by the same way. Therefore, combining these with (10.9) and summing up, we get

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \geqq \nu.$$

The proof of (10.1) and Theorem 8.1 is now completed. $\square$

11. Magnetic Well

In this section, we consider the Schrödinger operator $H_A(\lambda)$ defined by

$$H_A(\lambda) = \frac{1}{2}\sum_{j=1}^{d} (\sqrt{-1}\partial_j - \lambda A_j)^2$$

on $L^2(\mathbf{R}^d)$, where $A_j, j = 1, 2, ..., d$ are real smooth functions. We set as before $B_{i,j}(x) = \partial_j A_i(x) - \partial_i A_j(x)$ and $B(x) = (B_{i,j}(x))$.

Denoting by $s(x)$ the trace norm $\|B(x)\|_1$ of $B(x)$, we assume the following :

(A.5) (i) $\liminf_{|x|\to\infty} s(x) \equiv s_\infty^- > s_0 \equiv \inf\{s(x); x \in \mathbf{R}^d\}$;

(ii) $\sup_{x\in\mathbf{R}^d} \dfrac{|(\nabla B_{j,k})(x)|}{1 + s(x)} < \infty$, for any $j, k = 1, 2, ..., d$.

Following [7], we call each connected component of the set $\{x \in \mathbf{R}^d; s(x) = s_0\}$ a magnetic well.

The purpose of this section is to show the following :

Theorem 11.1. *Under the assumption* (A. 5), $H_A(\lambda)$ *has any number of the discrete spectrum* $\{\mu_n(\lambda)\}$ *below its essential spectrum if* λ *is sufficiently large and it holds that*

$$\lim_{\lambda\uparrow\infty} \frac{1}{\lambda}\ \mu_n(\lambda) = s_0/4$$

for every $n \in \mathbf{N}$.

For the related results, see [3], [15], [20] and [27].

Proof. Modifying the proof of the Persson's theorem (cf. [1] and [20]), we can prove

$$\inf \sigma_{ess}(H_A(\lambda)) \geqq s_\infty^- \lambda/4 - C\lambda^{3/4} \tag{11.1}$$

where $\sigma_{ess}(H_A(\lambda))$ is the essential spectrum of $H_A(\lambda)$ and C is a constant. It is a consequence of the inequality

$$(H_A(\lambda)\phi, \phi) \geqq \|\phi\|^2 \inf\{(\lambda s(x)/4 - C\lambda^{3/4}); x \in \text{supp } \phi\} \tag{11.2}$$

for $\phi \in C_0^\infty(\mathbf{R}^d)$, which dues to Theorem 3.1 of [7].

Next we choose any $p \in \mathbf{R}^d$ such that $s(p) = s_0$. Changing the coordinate if necessary, we may assume that $B(p)$ is of the form

$$B(p) = \begin{pmatrix} B'(p) & 0 \\ 0 & 0 \end{pmatrix},$$

where $B'(p)$ is a skew-symmetric non-degenerate matrix.

Now we consider the quadratic operator

$$H_p(\lambda) = \frac{1}{2}\sum_{j=1}^{r}(\sqrt{-1}\partial_j - \lambda(B(p)x)_j/2)^2 - \frac{1}{2}\sum_{j=r+1}^{d}\partial_j^2,$$

$$H_p^r(\lambda) = \frac{1}{2}\sum_{j=1}^{r}(\sqrt{-1}\partial_j - \lambda(B'(p)\hat{x})_j/2)^2,$$

where r is the half of the rank of the matrix $B(p)$ and $\hat{x} = (x_1, ..., x_{2r})$. The spectrum of $H_p^r(\lambda)$ on $L^2(\mathbf{R}^r)$ consists only of the eigenvalues with infinite multiplicities and the least eigenvalue is $\lambda s_0/4$. We denote by $\{\phi_k^\lambda(\hat{x})\}_{k=1}^\infty$ an orthonormal basis of the eigenspace of $H_p^r(\lambda)$ corresponding to the least eigenvalue $\lambda s_0/4$.

Moreover we set

$$\varphi_k^{\lambda,\varepsilon}(x) = \phi_k^\lambda(\hat{x}) \prod_{j=r+1}^{d} \frac{1}{\varepsilon\sqrt{2\pi}} \exp(-x_j^2/2\varepsilon^2)$$

for $\varepsilon > 0$. Then, by the same way as in Section 9, we can prove

$$(\varphi_k^{\lambda,\varepsilon}, H_A(\lambda)\varphi_\ell^{\lambda,\varepsilon}) = \lambda\delta_{k,\ell}s_0/4 + O(\lambda^{4/5})$$

as $\lambda \uparrow \infty$, where the large O depends on ε. Therefore, by (11.1), the Rayleigh-Ritz principle enables us to prove that $H_A(\lambda)$ has any number of the discrete spectrum $\{\mu_n(\lambda)\}$ below $\sigma_{ess}(H_A(\lambda))$ and that

$$\limsup_{\lambda\uparrow\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \leqq s_0/4 \qquad \text{for every} \quad n.$$

On the other hand, (11.2) implies $\inf \sigma(H_A(\lambda)) \geqq \lambda s_0/4 - C\lambda^{3/4}$ and

$$\liminf_{\lambda\uparrow\infty} \frac{1}{\lambda}\, \mu_n(\lambda) \geqq s_0/4.$$

We have completed the proof of Theorem 11.1. $\square$

Remark. For the operator $H_A(\lambda)$, recently Helffer-Mohamed [7] and Montgomery [20] have discussed the asymptotic behavior of the eigenvalues $\{\mu_n(\lambda)\}$, in particular, of the least eigenvalue in detail when $s_0 = 0$. Moreover the results of Hempel-Herbst [12] on the convergence of the resolvents show the zeros of A and dA are important for this problem. See also Nakamura [21].

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References

1. H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, "Schrödinger Operators with Application to Quantum Mechanics and Global Geometry," Texts and Monographs in Physics, Springer-Verlag, Berlin/Heiderberg/New York, 1987.
2. I. M. Davies, The propagator for a charged particle in a constant magnetic field and with a quadratic potential, *J. Phys. A: Math. Gen.*, **18** (1985), 2737–2741.
3. J-P. Demailly, Champs magnétiques et inégaliés de Morse pour la d''-cohomologie, *Ann. Inst. Fourier Grenoble*, **35** (1985), 189–229.
4. C. Dewitt-Morette, The semiclassical expansion, *Ann. Phys.*, **97** (1976), 367–399.
5. V. Guillemin and S. Sternberg, "Symplectic Techniques in Physics," Cambridge Univ. Press, Cambridge, 1984.
6. B. Helffer, "Semi-classical Analysis for the Schrödinger Operators and Applications," Lec. Notes in Math., **1336**, Springer-Verlag, Berlin/Heiderberg/New York, 1988.
7. B. Helffer and A. Mohamed, Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells, to appear in *J. Func. Anal.*
8. B. Helffer and J. Nourrigat, "Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs,"Birkhäuser, Boston, 1985.
9. B. Helffer and J. Sjöstrand, Puits multiples en limite semiclassique V (étude des minipuits), *Current Topics in Partial Differential Equations* (*Volume in honor of S. Mizohata*), Kinokuniya, Tokyo, 133–186, (1986).
10. B. Helffer and J. Sjöstrand, Puits multiples en mécanique semi-classique VI (Cas des puits sous-variétés), *Ann. Inst. Henri Poincaré, Physique Theorique*, **46** (1987), 353–372.
11. B. Helffer and J. Sjöstrand, Effect tunnel pour l'equation de Schrödinger avec champ magnétique, *Ann. Scuola Norm. Sup. Pisa*, **14** (1987), 625–657.

12. R. Hempel and I. Herbst, Strong magnetic fields, Dirichlet boundaries, and spectral gaps, *Comm. Math. Phys.*, **169** (1995), 237-259.

13. N. Ikeda, S. Kusuoka and S. Manabe, Lévy's stochastic area formula and related problem, in *Stochastic Analysis*, *Proc. Symp. Pure Math.* **57**, *ed. by M.C. Cranston and M.A. Pinsky*, 281–305, Amer. Math. Soc., 1995.

14. N. Ikeda and S. Watanabe, "Stochastic Differential Equations and Diffusion Processes," 2nd. ed., Kodansha/North-Holland, Tokyo/Amsterdam, 1989.

15. P. Malliavin, Analyticité transverse d'opérateurs hypoelliptiques C^3 sur des fibrés principaux. Spectre équivariant et courbure, *C. R. Acad. Sci.*, **301** (1985), 767–770.

16. H. Matsumoto, The short time asymptotics of the traces of the heat kernels for the magnetic Schrödinger operators, *J. Math. Soc. Japan*, **42** (1990), 677–689.

17. H. Matsumoto, Semiclassical asymptotics of eigenvalues for Schrödinger operators with magnetic fields, *J. Func. Anal.*, **129** (1995), 168–190.

18. H. Matsumoto, Quadratic Hamiltonians and associated orthogonal polynomials, to appear in *J. Func. Anal.*

19. P. A. Mohamed and J. Nourrigat, Encadrement du $N(\lambda)$ pour un opérateur de Schrödinger avec un champ magnétique et un potentiel électrique, *J. Math. Pure Appl.*, **70** (1991), 87–99.

20. R. Montgomery, Hearing the zero locus of a magnetic field, *Comm. Math. Phys.*, **168** (1995), 651–675.

21. S. Nakamura, Band spectrum for Schrödinger operators with strong magnetic fields, *preprint.*

22. M. Reed and B. Simon, "Methods of Modern Mathematical Physics, II : Fourier Analysis and Self-Adjointness," Academic Press, New York, 1975.

23. M. Reed and B. Simon, "Methods of Modern Mathematical Physics, IV : Analysis of Operators," Academic Press, New York, 1978.

24. B. Simon, "Functional Integration and Quantum Physics," Academic Press, London, 1979.

25. B. Simon, Semi-classical analysis of low lying eigenvalues I, Non degenerate minima: Asymptotic expansions, *Ann. Inst. H. Poncaré, Physique Theorique*, **38** (1983), 295–307.

26. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc., 1939.

27. N. Ueki, Lower bounds for the spectra of Schrödinger operators with magnetic fields, *J. Func. Anal.*, **120** (1994), 344–379.

28. J. H. Van Vleck, The correspondence principle in the statistical interpretation of quantum mechanics,

Proc. Nat. Acad. Sci. U.S.A., **14** (1928), 178–188.

29. S. Watanabe, Generalized Wiener functionals and their applications, *Proc. Fifth Japan-USSR Symp. Prob. Theory and Math. Stat.*, Lec. Notes in Math., **1299**, 541-548, Springer-Verlag, Berlin/Heiderberg/New York, 1988.