Hypoellipticity for the $\overline{\partial}$-Neumann problem on certain domains with infinite degeneracy

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1. Introduction

The $\overline{\partial}$-Neumann problem has been studied mainly under the condition that the Levi form is positive semidefinite only with finite degeneracy (cf.[**10**],[**11**]). For the case where the Levi form has infinite degeneracy, we have some results on global regularity properties by Boas-Straube [**1**],[**2**] (cf.[**15**]). For the local properties, Doctor Tetsuo Saito proved the local hypoellipticity under some condition in his unpublished notes in the early 1980s where the Levi form degenerates only at isolated points.

On the other hand, the Malliavin calculus has been applied to some problems with infinite degeneracy in the theory of partial differential equations: in [**13**], Malliavin applied his general theory to the Kohn Laplacian, which is also related to the $\overline{\partial}$-Neumann problem, to show that this Laplacian has a smooth heat kernel under certain condition where the Levi form may degenerate infinitely. Furthermore we see that the Kohn Laplacian is also locally hypoelliptic from the work of Kusuoka-Stroock [**12**].

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In this paper we apply the idea of the Malliavin calculus to the $\overline{\partial}$-Neumann problem with an infinite degenerate Levi form. The first difficulty is that we do not have any appropriate expressions of the corresponding heat kernel in the general situation, even if the Levi form does not degenerate. In this paper, we consider on a simple domain $D_f = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \ \mathrm{Im}\, w > f(z)\}$ defined in Section 2 below. This domain is a generalization of the strongly pseudoconvex Siegel domain and also relates to the typical finite type domain studied by Greiner-Stein [**5**] and decoupled domains (see e.g.[**14**]). On this domain, we construct the smooth heat kernels for the $\overline{\partial_b}$-Laplacian (Theorem 1 below) and the $\overline{\partial}$-Laplacian (Theorem 2 below). Furthemore, by estimating the heat kernel following [**12**], we show the local hypoellipticity of the $\overline{\partial}$-Laplacian (Theorem 3 below) and that of the operator $\partial_t + \Box$ (Theorem 4 below), where $\partial_t = \partial/\partial t$, t is an additional parameter and $\Box$ is the $\overline{\partial}$-Laplacian.

Our method is as follows: for the domain D_f, the $\overline{\partial}_b$-Laplacian is a differential operator

$$\Box_b = -\frac{1}{4}\sum_{j=1}^{2n} \widetilde{X_j^2} + iF\widetilde{\partial_u} \tag{1.1}$$

and, on some space of forms, the $\overline{\partial}$-Laplacian is a differential operator

$$\Box = \Box_b - \frac{1}{2}(\widetilde{\partial_u^2} + \widetilde{\partial_r^2}) \tag{1.2}$$

with a boundary condition

$$(\widetilde{\partial_r} - i\widetilde{\partial_u})\varphi \restriction_{r=0} = 0. \tag{1.3}$$

For the notations see Section 2 below. When we study these operators by a probabilistic method, it is difficult to treat the term $iF\partial_u$ in (1.1) and the term

$i\partial_u$ in (1.3). However the metric, the differential structure and the complex structure of the domain D_f are constant in the direction ∂_u. Thus, by the Fourier transform in the variable u, the heat kernel for the $\overline{\partial}_b$-Laplacian is expected to be

$$k(t,(z,u),(z',u')) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\ e^{i\xi(u-u')} q(t,z,z';\xi) \tag{1.4}$$

and that for the $\overline{\partial}$-Laplacian is expected to be

$$h(t,(z,u,r),(z',u',r')) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\ e^{i\xi(u-u')} q(t,z,z';\xi) E(t,r,r',\xi), \tag{1.5}$$

where

$$\begin{aligned} q(t,z,z';\xi) = E\Big[\exp\Big(& -i\xi\int_0^t \nabla f(z+z(s)) * dz(s) \\ & + \xi\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big] \end{aligned} \tag{1.6}$$

with

$$\begin{aligned} & \int_0^t \nabla f(z+z(s)) * dz(s) \\ =& \frac{1}{\sqrt{2}}\sum_{j=1}^{n}\int_0^t \left(\frac{\partial f}{\partial x^{2j}}(z+z(s)) \circ dx^{2j-1}(s) - \frac{\partial f}{\partial x^{2j-1}}(z+z(s)) \circ dx^{2j}(s)\right) \end{aligned}$$

is the heat kernel for the operator

$$\Box_b(\xi) = -\frac{1}{4}\sum_{j=1}^{2n} \widetilde{X_j(\xi)^2} - \xi F \tag{1.7}$$

obtained from $\Box_b$ by the Fourier transform and $E(t,r,r',\xi)$ is an explicit function defined in (2.19) below. This expression was given by Stanton [**16**], [**17**] for the strongly pseudoconvex Siegel domain. She derived the expression by using methods of the partial differential equation theory. For the probabilistic interpretation of the construction, see [**18**]. We will show that the expressions (1.4) and (1.5) give the heat kernels with good regularity properties, and estimate the

kernel given by (1.5) to prove the local hypoellipticity. To do this, we should study the asymptotic behaviour of $q(t,z,z';\xi)$ as $\xi \to \pm\infty$. On this respect, our problem relates to the study of the asymptotics of a stochastic oscillatory integral (see e.g.[**8**]). On the $(0,q)$ forms on the strongly pseudoconvex Siegel domain, $q(t,z,z';\xi)$ can be written as an explicit function and this explicit representation was a base of [**16**],[**17**],[**18**]. However, in our case, we cannot write $q(t,z,z';\xi)$ as an explicit function. Now we note that the operator $\Box_b(\xi)$ is a positive operator. Then we can show that any derivatives of $q(t,z,z';\xi)$ do not diverge exponentially as $\xi \to \pm\infty$ (Proposition 3.1 below). Furtheremore, we impose some condition for the degeneracy of the Levi form (condition $(E,\rho)_\sigma$ in Section 2). Then, by a probabilistic estimate originated from [**12**] (Proposition 3.3 below), we can show that any derivatives of $q(t,z,z';\xi)$ decay faster than any polynomials of ξ as $\xi \to \pm\infty$. The crucial part of our proof of the local hypoellipticity is the proof of Proposition 5.4 below:

$$\sup\left\{\left|\frac{1}{t^k}\partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t,Z,Z')\right| : t\in(0,1], Z,Z'\in K, |u-u'|\geq\varepsilon\right\} < \infty \tag{1.8}$$

for any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n+2}$, $k \in \mathbb{Z}_+$ and compact $K \subset \overline{D_f}$. To show this, we carry out some estimate of the integrand $q(t,z,z';\xi)E(t,r,r',\xi)$ using the Taylor expansion with respect to the variable ξ.

The organization of this paper is as follows. In Section 2, we introduce the domain and the operators. Then we state our theorems. Furthemore we give a few examples. In Section 3, we prepare key estimates to prove the theorems. In Section 4, we prove the theorems on the heat kernels, Theorems 1 and 2. In Section 5, we prove the theorems on the local hypoellipticity, Theorems 3 and 4.

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2. Main theorems

Let f be a real valued smooth function on $\mathbb{C}^n$ with bounded derivatives of all orders ≥ 2. The domain we consider is

$$D_f = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \text{ Im } w > f(z)\}. \tag{2.1}$$

We denote the boundary by bD_f. We give this domain a Hermitian metric for which $\{Z_1, Z_2, \cdots, Z_{n+1}\}$ is a unitary frame of the holomorphic tangent bundle $T^{(1,0)}(D_f)$, where

$$Z_j = \frac{\partial}{\partial z^j} + 2i\frac{\partial f}{\partial z^j}\frac{\partial}{\partial w}, \quad j = 1, 2, \cdots, n, \quad Z_{n+1} = i\sqrt{2}\frac{\partial}{\partial w}. \tag{2.2}$$

The dual frame of $\{Z_1, Z_2, \cdots, Z_{n+1}\}$ is $\{\omega^1, \omega^2, \cdots, \omega^{n+1}\}$, where

$$\begin{aligned} \omega^j = dz^j, \quad j = 1, 2, \cdots, n, \quad \omega^{n+1} &= \sqrt{2}\partial(\text{Im } w - f) \\ &= \frac{1}{i\sqrt{2}}\left(dw - \sum_{j=1}^{n} 2i\frac{\partial f}{\partial z^j}dz^j\right). \end{aligned} \tag{2.3}$$

The Levi form of the domain D_f can be represented as the matrix valued function

$$\left(\frac{\partial^2 f}{\partial z^j \partial \overline{z^k}}\right)_{1 \leq j,k \leq n} \tag{2.4}$$

on the boundary bD_f (cf.[**4**] Proposition (3.2.1)).

Example 1. If $f(z) = |z|^2$, then D_f is the strongly pseudoconvex Siegel domain (cf.[**16**],[**17**],[**18**]). If we set $n = 1$ and $f(z) = |z|^{2k}$, then D_f is the typical finite type domain studied by Greiner-Stein [**5**]. This example is out of our framework because the derivarives of f are not bounded. However the local results as Theorems 3 and 4 below are applicable. If $f(z) = \sum_{j=1}^{n} f_j(z^j)$, then the structure of D_f near the boundary is the same as that of a decoupled domain (see e.g.[**14**]).

We assume that the function f is decomposed as

$$f(z) = f_+(z^+) + f_-(z^-) + f_0(z^0)$$
$$\text{for } z = (z^+, z^-, z^0) \in \mathbb{C}^{n_+} \times \mathbb{C}^{n_-} \times \mathbb{C}^{n_0} \ (n_+ + n_- + n_0 = n). \tag{2.5}$$

For $\sigma = +, -, 0$, let $(z^{\sigma,1}, z^{\sigma,2}, \cdots, z^{\sigma,n_\sigma})$ be the coordinate of $z^\sigma \in \mathbb{C}^{n_\sigma}$ and $\{Z_{\sigma,j}\}_{1\le j\le n_\sigma}$ and $\{\omega^{\sigma,j}\}_{1\le j\le n_\sigma}$ be the corresponding $\{Z_j\}$ and $\{\omega^j\}$, respectively. For $\sigma = +, -$, we assume that there are functions $h_\sigma(\lambda) : \mathbb{R} \to [0, \infty)$ and $\psi_\sigma(z^\sigma) : \mathbb{C}^{n_\sigma} \to \mathbb{R}$ satisfying the following:

(i) h_σ is even, non-decreasing on $[0, \infty)$ and $h_\sigma^{-1}(0) = \{0\}$;

(ii) the derivatives of ψ_σ are all bounded and

$\inf_{z^\sigma \in \mathbb{C}^{n_\sigma}} \{\psi_\sigma(z^\sigma)^2 + |\nabla \psi_\sigma(z^\sigma)|^2\} > 0$;

(iii) $\partial^2 f_\sigma / (\partial z^{\sigma,j} \partial \overline{z^{\sigma,k}}) = 0$ for $j \neq k$ and $\Delta f_\sigma(z^\sigma) \ge h_\sigma(\psi_\sigma(z^\sigma))$ for any $z^\sigma \in \mathbb{C}^{n_\sigma}$, where $\Delta = \sum_{j=1}^{n_\sigma} \partial^2 / (\partial z^{\sigma,j} \partial \overline{z^{\sigma,j}})$.

For $\sigma = +, -$, $\rho > 0$ and $\kappa > 1$, we introduce the following conditions:

$$(E, \rho)_\sigma \qquad \lim_{\lambda \downarrow 0} \lambda^\rho \log \frac{1}{h_\sigma(\lambda)} = 0$$

and

$$(H, \kappa)_\sigma \qquad |\Delta \psi_\sigma(z^\sigma)| \le \eta_1 |\psi_\sigma(z^\sigma)|^\kappa \text{ if } |\psi_\sigma(z^\sigma)| \le \eta_2, \text{ for some } \eta_1, \eta_2 > 0.$$

We use as coordinates on D_f (z, u, r) where $u = \text{Re } w$ and $r = \text{Im } w - |z|^2$. Then we can regard D_f as the product of some domain B_f, which is identified with the boundary bD_f, and $\mathbb{R}_+$. (z, u) is a coordinate of B_f and r is a coordinate of $\mathbb{R}_+$. The metric we gave is the product of a metric on B_f and the standard metric on $\mathbb{R}^+$. In terms of these coordinates,

$$Z_j = \frac{1}{2}(X_j - iX_{n+j}), \ j = 1, 2, \cdots, n, \quad Z_{n+1} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial r} + i \frac{\partial}{\partial u} \right) , \tag{2.6}$$

where

$$X_j = \frac{\partial}{\partial x^j} + \frac{\partial f}{\partial x^{n+j}}\frac{\partial}{\partial u}, \quad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - \frac{\partial f}{\partial x^j}\frac{\partial}{\partial u}$$

and $z^j = x^j + ix^{n+j}$. The volume element of the domain B_f is $2^n dx^1 dx^2 \cdots dx^{2n} du$ and that of the domain D_f is $2^n dx^1 dx^2 \cdots dx^{2n} du dr$. We note that, for any $u_0 \in \mathbb{R}$, the transformation $(z, u, r) \mapsto (z, u + u_0, r)$ acts isometrically and holomorphically on D_f.

Let $\Lambda^{p,q}(B_f)$ be the bundle on B_f spanned by $\omega^J \wedge \overline{\omega^K}$, $J \in \mathcal{I}(p)$, $K \in \mathcal{I}(q)$, where $\mathcal{I}(p) = \{J = (j_1, j_2, \cdots, j_p) : 1 \leq j_1 < j_2 < \cdots < j_p \leq n\}$ and $\omega^J = \omega^{j_1} \wedge \omega^{j_2} \wedge \cdots \wedge \omega^{j_p}$ for $J = (j_1, j_2, \cdots j_p)$ and $\overline{\omega^K} = \overline{\omega^{k_1}} \wedge \overline{\omega^{k_2}} \wedge \cdots \wedge \overline{\omega^{k_q}}$ for $K = (k_1, k_2, \cdots k_q)$. Since $\{\omega^j\}$ is a global frame, we identify $\{\omega^J \wedge \overline{\omega^K}\}_{J \in \mathcal{I}(p), K \in \mathcal{I}(q)}$ with the canonical frame of $\Lambda^{p,q}(\mathbb{C}^n)$, and we identify $\Lambda^{p,q}(B_f)$ with the trivial bundle $B_f \times \Lambda^{p,q}(\mathbb{C}^n)$. For any subbundle U of $\Lambda^{p,q}(B_f)$, let $\mathcal{S}(U)$ be the space of sections on B_f to U whose coefficients relative to $\omega^J \wedge \overline{\omega^K}$ are rapidly decreasing functions of $(x^1, x^2, \cdots, x^{2n}, u) \in \mathbb{R}^{2n+1}$, $L^2(U)$ be that of square integrable sections with respect to the Lebesgue measure $dx^1 dx^2 \cdots dx^{2n} du$ and $\Gamma_0^\infty(U)$ be that of smooth sections with compact supports.

We now introduce the $\overline{\partial_b}$-Laplacian. For any $\varphi = \varphi_{J\overline{K}} \omega^J \wedge \overline{\omega^K} \in \Gamma_0^\infty(\Lambda^{p,q}(B_f))$, let

$$\overline{\partial_b}\varphi = \sum_{j=1}^{n} (\overline{Z_j}\varphi_{J\overline{K}}) \overline{\omega^j} \wedge \omega^J \wedge \overline{\omega^K}. \tag{2.7}$$

$\overline{\partial_b}$ maps $\Gamma_0^\infty(\Lambda^{p,q}(B_f))$ to $\Gamma_0^\infty(\Lambda^{p,q+1}(B_f))$. Let ϑ_b be the formal adjoint of $\overline{\partial_b}$ Then the $\overline{\partial_b}$-Laplacian $\Box_b$ is defined on $\Gamma_0^\infty(\Lambda^{p,q}(B_f))$ by

$$\Box_b = \overline{\partial_b}\vartheta_b + \vartheta_b\overline{\partial_b} \tag{2.8}$$

This operator $\Box_b$ is essentially self-adjoint as an operator on $L^2(\Lambda^{p,q}(B_f))$ (cf.[**3**]). We denote the self-adjoint extension also by $\Box_b$.

To express the operator $\Box_b$ more concretely, we introduce a few notations. For any $\omega \in \Lambda^{p,q}(B_f)$, let $\mathrm{ext}(\omega)$ be the exterior multiplication, i.e.,

$$\mathrm{ext}(\omega)\eta = \omega \wedge \eta \quad \text{for} \quad \eta \in \Lambda^{p,q}(B_f),$$

and $\mathrm{int}(\overline{\omega})$ be the interior multiplication, i.e., the dual operator of $\mathrm{ext}(\omega)$. This $\mathrm{int}(\omega)$ is complex linear in ω. We note that in some literature, e.g., [**4**], $\mathrm{int}(\omega)$ is conjugate linear in ω. For a differential operator A on a space of scalar valued functions, we define $\widetilde{A}$ to be the operator acting only on the coefficients of $\omega^J \wedge \overline{\omega^K}$, i.e.,

$$\widetilde{A}(\sum \varphi_{J\overline{K}} \omega^J \wedge \overline{\omega^K}) = \sum (A\varphi_{J\overline{K}}) \omega^J \wedge \overline{\omega^K}. \tag{2.9}$$

In terms of the above notations, the operator $\Box_b$ is expressed as

$$\Box_b = -\frac{1}{4} \sum_{j=1}^{2n} \widetilde{X_j^2} - Fi\widetilde{\frac{\partial}{\partial u}} \tag{2.10}$$

on the space $\mathcal{S}(\Lambda^{p,q}(B_f))$, where

$$F = \sum_{j=1}^{n} \frac{-1}{4} \frac{\partial^2 f}{\partial z^j \partial \overline{z^j}} [\mathrm{int}(\omega^j), \mathrm{ext}(\overline{\omega^j})] + \sum_{j \neq k} 2 \frac{\partial^2 f}{\partial z^j \partial \overline{z^k}} \mathrm{ext}(\overline{\omega^k}) \mathrm{int}(\omega^j).$$

Remark 2.1. The operator $\Box_b$ is difficult to analyze because of the presence of the term $Fi\partial/\partial u$. Simpler differential operator $1/4 \sum_{j=1}^{2n} X_j^2$ is the Kohn Laplacian studied by Malliavin [**13**]. If the Levi form has only finite degeneracy, the Kohn Laplacian is a prototype of operators considered by Hörmander [**7**].

To analyze the operator $\Box_b$, we use the Fourier transform in the variable u. Let $\widetilde{B_f}$ be the quotient of B_f by the equivalence relation $(z, u_1) \sim (z, u_2)$ for any $z \in \mathbb{C}^n$ and $u_1,\ u_2 \in \mathbb{R}$. Then B_f is identified with the product of $\widetilde{B_f}$ and $\mathbb{R}$: z is a coordinate of $\widetilde{B_f}$ and u is a coordinate of $\mathbb{R}$. The forms $\omega^1, \omega^2, \cdots, \omega^n$ are regarded as forms on $\widetilde{B_f}$. Let $\Lambda^{p,q}(\widetilde{B_f})$ be the bundle on $\widetilde{B_f}$ spanned by $\omega^J \wedge \overline{\omega^K}$, $J \in \mathcal{I}(p)$, $K \in \mathcal{I}(q)$. We identify $\Lambda^{p,q}(\widetilde{B_f})$ with $\widetilde{B_f} \times \Lambda^{p,q}(\mathbb{C}^n)$ in the same way for $\Lambda^{p,q}(B_f)$. For any subbundle $\tilde{U}$ of $\Lambda^{p,q}(\widetilde{B_f})$, we define the spaces $\mathcal{S}(\tilde{U})$, $L^2(\tilde{U})$, $\Gamma_0^\infty(\tilde{U})$ of sections on $\widetilde{B_f}$ in the same way for any subbundle U of $\Lambda^{p,q}(B_f)$. For any $\varphi \in \mathcal{S}(\Lambda^{p,q}(B_f))$, we put

$$\widehat{\varphi}(z,\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi u} \varphi(z,u) du. \tag{2.11}$$

Then we have

$$\widehat{(\Box_b \varphi)}(z,\xi) = \Box_b(\xi) \widehat{\varphi}(z,\xi), \tag{2.12}$$

where $\Box_b(\xi)$, $-\infty < \xi < \infty$, are the differential operators on $\mathcal{S}(\Lambda^{p,q}(\widetilde{B_f}))$ defined by

$$\begin{aligned} \Box_b(\xi) &= -\frac{1}{4} \sum_{j=1}^{2n} \widetilde{X_j(\xi)^2} - \xi F, \\ X_j(\xi) &= \frac{\partial}{\partial x^j} - i\xi \frac{\partial f}{\partial x^{n+j}}, \quad X_{n+j}(\xi) = \frac{\partial}{\partial x^{n+j}} + i\xi \frac{\partial f}{\partial x^j}. \end{aligned} \tag{2.13}$$

For each ξ, the operator $\Box_b(\xi)$ is the Schrödinger operator with magnetic field for a spin 1/2 particle and is known to be essentially self-adjoint as an operator on $L^2(\Lambda^{p,q}(\widetilde{B_f}))$ with the domain $\Gamma_0^\infty(\Lambda^{p,q}(\widetilde{B_f}))$ (cf.[**3**]). We denote the self-adjoint extension also by $\Box_b(\xi)$. This operator $\Box_b(\xi)$ has the heat kernel (the integral kernel of the heat semigroup $e^{-t\Box_b(\xi)}$ generated by $\Box_b(\xi)$) with respect to

the Lebesgue measure $dx^1 dx^2 \cdots dx^{2n}$, which has the representation $q(t, z, z'; \xi)$, $t > 0$, $z, z' \in \widetilde{B_f}$, given in (3.2) below. This $q(t, z, z'; \xi)$ is smooth in $t > 0$, $z, z' \in \widetilde{B_f}$ and $-\infty < \xi < \infty$.

Under the decomposition (2.5), the opetrator $\Box_b(\xi)$ is decomposed as

$$\Box_b(\xi) = \Box_b^+(\xi) \oplus \Box_b^-(\xi) \oplus \Box_b^0(\xi) \tag{2.14}$$

where, for each $\sigma = +, -, 0$,

$$\Box_b^\sigma(\xi) = -\frac{1}{4} \sum_{j=1}^{2n_\sigma} \widetilde{X_{\sigma,j}(\xi)^2} - \xi F_\sigma$$

with

$$F_\sigma = \sum_{j=1}^{n_\sigma} \frac{-1}{4} \frac{\partial^2 f_\sigma}{\partial z^{\sigma,j} \partial \overline{z^{\sigma,j}}} [\mathrm{int}(\omega^{\sigma,j}), \mathrm{ext}(\overline{\omega^{\sigma,j}})]$$
$$+ \sum_{j \neq k} 2 \frac{\partial^2 f_\sigma}{\partial z^{\sigma,j} \partial \overline{z^{\sigma,k}}} \mathrm{ext}(\overline{\omega^{\sigma,k}}) \mathrm{int}(\omega^{\sigma,j})$$

is the corresponding Schrödinger operator on $\widetilde{B_{f_\sigma}}$. For $q \geq n_+$, let $\Lambda_r^{p,q}(B_f)$ be the subbundle of $\Lambda^{p,q}(B_f)$ defined by $\Lambda_r^{p,q}(B_f) = \Lambda^{p,0}(B_f) \wedge \Lambda^{0,n_+}(B_{f_+})' \wedge \Lambda_r^{0,q-n_+}(B_{f_0})'$, where $\Lambda^{0,n_+}(B_{f_+})'$ and $\Lambda_r^{0,q-n_+}(B_{f_0})'$ are the bundles on B_f obtained by extending $\Lambda^{0,n_+}(B_{f_+})$ and $\Lambda_r^{0,q-n_+}(B_{f_0})$ trivially. The operator $\Box_b$ preserves the bundle $\Lambda_r^{0,q}(B_f)$. Then we have the following:

THEOREM 1. *For both $\sigma = +$ and $-$, we assume $n_\sigma \geq 1$ and either $(E, \rho)_\sigma$ for some $\rho < 1$ or $(E, \rho)_\sigma$ and $(H, \kappa)_\sigma$ for some $\rho < 2$ and $\kappa > 1$. Then, on the bundle $\Lambda_r^{p,q}(B_f)$, the $\overline{\partial_b}$-Laplacian $\Box_b$ has the heat kernel (the integral kernel of the heat semigroup $e^{-t\Box_b}$ generated by $\Box_b$) with respect to the Lebesgue measure $dx^1 dx^2 \cdots dx^{2n} du$, which has the following representation:*

$$k(t, (z, u), (z', u')) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{-i\xi(u-u')} q(t, z, z'; \xi). \tag{2.15}$$

This $k(t, X, X')$, *where* $X = (z, u)$ *and* $X' = (z', u')$, *is a smooth function of* $(t, X, X') \in (0, \infty) \times B_f \times B_f$ *and is a rapidly decreasing function of* X *(resp.*X'*)* $\in B_f$, *for each fixed* (t, X') *(resp.*(t, X)*)* $\in (0, \infty) \times B_f$. *Derivatives of* $k(t, X, X')$ *can be calculated by differentiating under the integral sign in* (2.15).

We next introduce the $\overline{\partial}$-Laplacian. Let $\Lambda^{p,q}(D_f)$ be the bundle of the (p, q)-forms on D_f. For any subbundle V of $\Lambda^{p,q}(D_f)$, let $\mathcal{S}(V)$ be the space of forms on D_f to V whose coefficients relative to $\omega^J \wedge \overline{\omega^K}$, can be extended to rapidly decreasing functions of $(x^1, x^2, \cdots, x^{2n}, u, r) \in \mathbb{R}^{2n+2}$, $L^2(V)$ be that of square integrable sections with respect to the Lebesgue measure $dx^1 dx^2 \cdots dx^{2n} du dr$ and $\Gamma_0^\infty(V)$ be that of smooth sections with compact supports in D_f. $\overline{\partial}$ maps $\mathcal{S}(\Lambda^{p,q}(D_f))$ to $\mathcal{S}(\Lambda^{p,q+1}(D_f))$ and has the smallest closed extension to $L^2(\Lambda^{p,q}(D_f))$, which we also denote by $\overline{\partial}$. Let $\overline{\partial}^*$ be the adjoint of $\overline{\partial}$ on $L^2(\Lambda^{p,q}(D_f))$. We define the $\overline{\partial}$-Laplacian $\Box$ by

$$\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

$$\text{Dom}\ (\Box) = \{\varphi \in L^2(\Lambda^{p,q}(D_f)) : \varphi \in \text{Dom}\ \overline{\partial} \cap \text{Dom}\ \overline{\partial}^*, \tag{2.16}$$

$$\overline{\partial}\varphi \in \text{Dom}\ \overline{\partial}^*,\ \overline{\partial}^*\varphi \in \text{Dom}\ \overline{\partial}\}.$$

Then the operator $\Box$ is a positive self-adjoint operator on $L^2(\Lambda^{p,q}(D_f))$ (cf.[**4**] Proposition 1.3.8).

For simplicity, we consider the operator $\Box$ on some restricted spaces: let $\Lambda^{(p)(q)}(D_f)$ be the vector bundle over D_f obtained by extending $\Lambda^{p,q}(B_f)$ trivially to D_f. In other words, $\Lambda^{(p)(q)}(D_f)$ is the space of the forms not containing ω^{n+1} and $\overline{\omega^{n+1}}$. On the forms containing ω^{n+1}, we have inessential complexibility, and on the forms containing $\overline{\omega^{n+1}}$, since the boundary condition is the Dirichlet

condition, many properties of the operator $\Box$ are well known. Then we have

$$\begin{aligned}&\mathrm{Dom}(\Box)\cap\mathcal{S}(\Lambda^{(p)(q)}(D_f))\\ &=\{\varphi\in\mathcal{S}(\Lambda^{(p)(q)}(D_f)):\ (\widetilde{\partial}_r-i\widetilde{\partial_u})\varphi=0\text{ on }bD_f\}\end{aligned}\tag{2.17}$$

and

$$\Box=\Box_b-\frac{1}{2}\left(\widetilde{\partial_u^2}+\widetilde{\partial_r^2}\right)\ \text{ on }\ \mathrm{Dom}(\Box)\cap\mathcal{S}(\Lambda^{(p)(q)}(D_f)).\tag{2.18}$$

Furthermore, for $q\geq n_+$, we consider the subbundle $\Lambda_r^{(p)(q)}(D_f)$ obtained by extending $\Lambda_r^{p,q}(B_f)$ trivially to D_f. Then we have the following:

THEOREM 2. *We assume $n_+\geq 1$ and either $(E,\rho)_+$ for some $\rho<1$ or $(E,\rho)_+$ and $(H,\kappa)_+$ for some $\rho<2$ and $\kappa>1$. Then, on the bundle $\Lambda_r^{(p)(q)}(D_f)$, the $\overline{\partial}$-Laplacian $\Box$ has the heat kernel (the integral kernel of the heat semigroup $e^{-t\Box}$ generated by $\Box$) with respect to the Lebesgue measure $dx^1dx^2\cdots dx^{2n}dudr$, which has the following representation:*

$$\begin{aligned}&h(t,(z,u,r),(z',u',r'))\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi e^{-i\xi(u-u')}q(t,z,z';\xi)E(t,r,r',\xi),\end{aligned}\tag{2.19}$$

where

$$E(t,r,r',\xi)=e^{-\xi^2t/2}\{e(t,r-r')+e(t,r+r')\}-\xi\sqrt{\frac{2}{\pi}}\varepsilon(t,r+r',\xi),$$

$$e(t,r)=\frac{e^{-r^2/(2t)}}{\sqrt{2\pi t}},\quad \varepsilon(t,r,\xi)=e^{r\xi}\int_{r/\sqrt{t}+\xi\sqrt{t}}^{\infty}e^{-\mu^2/2}d\mu.$$

This $h(t,Z,Z')$, where $Z=(z,u,r)$ and $Z'=(z',u',r')$, is a smooth function of $(t,Z,Z')\in(0,\infty)\times\overline{D_f}\times\overline{D_f}$ and is the restriction to $\overline{D_f}$ of a rapidly decreasing function of Z (resp.Z'), for each fixed (t,Z') (resp.(t,Z)) $\in(0,\infty)\times\overline{D_f}$. Derivatives of $h(t,Z,Z')$ can be calculated by differentiating under the integral sign in (2.19).

We next state our theorem on the local hypoellipticity for the $\overline{\partial}$-Laplacian:

THEOREM 3. *We assume $n_+ = n \geq 1$, $n_- = n_0 = 0$ and either $(E,\rho)_+$ for some $\rho < 1/3$ or $(E,\rho)_+$ and $(H,\kappa)_+$ for some $\rho < 2/3$ and $\kappa > 1$. Then, on the subbundle $\Lambda_r^{(p)(q)}(D_f)$, the $\overline{\partial}$-Laplacian $\Box$ is locally hypoelliptic in the following sense: if $u \in \mathrm{Dom}(\Box) \cap L^2(\Lambda_r^{(p)(q)}(D_f))$ and $\Box u$ is smooth on an open set W of $\overline{D_f}$, then u is smooth on W.*

We can also consider the hypoellipticity of the operator $\partial_t + \Box$, as in [**12**]. On the subbundle $\Lambda_r^{(p)(q)}(D_f)$ for $n_+ = n \geq 1$ and $n_- = n_0 = 0$, we may regard the $\overline{\partial}$-Laplacian as an operator acting on scalar valued functions, since $F = \Delta f/4$. For any domain Ω, let $\mathcal{D}(\Omega)$ be the space of all test functions, $\mathcal{D}'(\Omega)$ be that of all distributions and $< \cdot, \cdot >$ be the pairing that is complex linear in the first variable and conjugate linear in the second variable. For any $u \in \mathcal{D}'(\mathbb{R} \times \overline{D_f})$, let L_u be the continuous complex linear map from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\overline{D_f})$ defined by

$$< \psi, L_u \phi > = < \overline{\phi} \cdot \psi, u >$$

for any $\phi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\overline{D_f})$. For any $u \in \mathcal{D}'(\mathbb{R} \times \overline{D_f})$ such that $L_u(\mathcal{D}(\mathbb{R})) \subset \mathrm{Dom}(\Box)$, let $\Box u \in \mathcal{D}'(\mathbb{R} \times \overline{D_f})$ be the distribution corresponding to the continuous linear map $\Box L_u$ from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\overline{D_f})$ defined by the composition of L_u and $\Box$ (cf.[**6**] the kernel theorem). Then we have the following:

THEOREM 4. *We assume $n_+ = n \geq 1$, $n_- = n_0 = 0$ and either $(E,\rho)_+$ for some $\rho < 1/3$ or $(E,\rho)_+$ and $(H,\kappa)_+$ for some $\rho < 2/3$ and $\kappa > 1$. Then, on the subbundle $\Lambda_r^{p,q}(\overline{D_f})$, the operator $\partial_t + \Box$ is locally hypoelliptic in the following sense: if $u \in \mathcal{D}'(\mathbb{R} \times \overline{D_f})$ such that $L_u(\mathcal{D}(\mathbb{R})) \subset \mathrm{Dom}(\Box)$ and $\partial_t u + \Box u$ is smooth on an open set W of $\mathbb{R} \times \overline{D_f}$, then u is smooth on W.*

Before closing this section, we give examples.

EXAMPLE 2. For $\rho > 0$, let f_ρ be a smooth function on $\mathbb{C}$ with bounded derivatives of all orders ≥ 2 such that

$$\Delta f_\rho(z) = \exp(-1/|x|^\rho) \quad \text{for} \quad |x| < 1 \tag{2.20}$$

($x = \text{Re } z$). From Theorem 2, we see that the $\overline{\partial}$-Laplacian on the bundle generated by $\overline{\omega^1}$ on D_{f_ρ} has a smooth heat kernel, if $\rho < 2$. From Theorem 3, we see that the $\overline{\partial}$-Laplacian is locally hypoelliptic, if $\rho < 2/3$.

Let $f_\rho^2(z^1, z^2) = f_\rho(z^1) + f_\rho(z^2)$. From Theorem 1, we see that the $\overline{\partial_b}$-Laplacian on the bundle generated by $\overline{\omega^1}$ on $B_{f_\rho^2}$ has a smooth heat kernel, if $\rho < 2$.

EXAMPLE 3. Let $\psi(z) = \log(x^2 + y^2) - 1$ ($z = x + iy$). For $\rho > 0$, let g_ρ be a smooth function on $\mathbb{C}$ with bounded derivatives of all orders ≥ 2 such that

$$\Delta g_\rho(z) = \exp(-1/|\psi(z)|^\rho) \quad \text{for} \quad |\psi(z)| < 1. \tag{2.21}$$

We note that $\Delta\psi = 0$ on $\{z : z \neq 0\}$. The Levi form of the domain D_{g_ρ} degenerates on a tube $\{(z, u, 0) : |z| = \sqrt{e}\}$. From Theorem 2, we see that the $\overline{\partial}$-Laplacian on the bundle generated by $\overline{\omega^1}$ on D_{g_ρ} has a smooth heat kernel, if $\rho < 2$. From Theorem 3, we see that the Laplacian is locally hypoelliptic, if $\rho < 2/3$.

Let $g_\rho^2(z^1, z^2) = g_\rho(z^1) + g_\rho(z^2)$. From Theorem 1, we see that the $\overline{\partial_b}$-Laplacian on the bundle generated by $\overline{\omega^1}$ on $B_{g_\rho^2}$ has a smooth heat kernel, if $\rho < 2$.

3. Key estimates

In this section, we prepare several estimates for $q(t,z,z';\xi)$.

We first give a probabilistic representation of the kernel $q(t,z,z';\xi)$. Let $(x^1(t),x^2(t),\cdots,x^{2n}(t))$, $t>0$, be the $2n$-dimensional standard Brownian motion. Let $z^j(t):=(x^{2j-1}(t)+ix^{2j}(t))/\sqrt{2}$ $j=1,2,\cdots,n$, and $z(t):=(z^1(t),z^2(t),\cdots,z^n(t))$. We set

$$\begin{aligned}&\int_0^t \nabla f(z+z(s))*dz(s)\\ &:=\frac{1}{\sqrt{2}}\sum_{j=1}^n\int_0^t\Big(\frac{\partial f}{\partial x^{2j}}(z+z(s))\circ dx^{2j-1}(s)\\ &\qquad-\frac{\partial f}{\partial x^{2j-1}}(z+z(s))\circ dx^{2j}(s)\Big),\end{aligned}\tag{3.1}$$

where $\circ dx^j(s)$ is the Stratonovich integral. This is also written as

$$\begin{aligned}&\int_0^t \nabla f(z+z(s))*dz(s)\\ &=\frac{1}{\sqrt{2}}\sum_{j=1}^n\int_0^t\Big(\frac{\partial f}{\partial x^{2j}}(z+z(s))\cdot dx^{2j-1}(s)\\ &\qquad-\frac{\partial f}{\partial x^{2j-1}}(z+z(s))\cdot dx^{2j}(s)\Big),\end{aligned}\tag{3.1}'$$

where $\cdot dx^j(s)$ is the Itô integral. Then $q(t,z,z';\xi)$ has the following representation in terms of the generalized Wiener functional theory by S.Watanabe:

$$\begin{aligned}&q(t,z,z';\xi)\\ &=E\Big[\exp\Big(-i\xi\int_0^t\nabla f(z+z(s))*dz(s)\\ &\qquad+\xi\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big],\end{aligned}\tag{3.2}$$

where $\delta_{z'}$ is the Dirac delta function at z' (cf.[**9**],[**19**]). This is also rewritten as the conditional expectation by the relation

$$E[\Phi(z(\cdot))\delta_{z'}(z+z(t))]=E[\Phi(z(\cdot))|z+z(t)=z']\frac{1}{(\pi t)^n}\exp\left(-\frac{|z-z'|^2}{t}\right)\tag{3.3}$$

for any smooth Wiener functional Φ i.e., $\Phi \in \mathbb{D}^\infty$ in [**9**],[**19**].

We use the multi-index notation: $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n}) \in \mathbb{Z}_+^{2n}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{2n}$ and $\partial_z^\alpha = \partial_{x^1}^{\alpha_1} \partial_{x^2}^{\alpha_2} \cdots \partial_{x^{2n}}^{\alpha_{2n}}$. For any endomorphism A on $\Lambda^{p,q}(\mathbb{C}^n)$, let $\|A\|$ be the Hilbert-Schmidt norm:

$$\|A\| = \Big(\sum_{J,K,J',K'} |A_{(J,\overline{K})(J',\overline{K'})}|^2 \Big)^{1/2}$$

for $A = \sum_{J,K,J',K'} A_{(J,\overline{K})(J',\overline{K'})} \operatorname{ext}(\omega^J \wedge \overline{\omega^K}) \operatorname{int}(\omega^{J'} \wedge \overline{\omega^{K'}})$.

By a standard argument in the Malliavin calculus, we have the following:

LEMMA 3.1. *For any α, $\alpha' \in \mathbb{Z}_+^{2n}$, $\beta \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$, there is a polynomial $\mathcal{P}(\xi, t, 1/t)$ such that*

$$\|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi)\| \le \mathcal{P}\Big(\xi, t, \frac{1}{t}\Big) \exp\Big(|\xi| t \sup \|F\| - \frac{|z - z'|^2}{2t} \Big) \tag{3.4}$$

for all $z \in K$ and $z' \in \widetilde{B_f}$.

PROOF. In the formula (3.2), we exchange the order of differentiation and integration and use the formula of the integration by parts on the Wiener space (see e.g., Section V-9 of [**9**], (2.21) of [**19**]) to obtain

$$\begin{aligned} &\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi) \\ =&E\Big[\Big\{ \partial_\xi^\beta \Big(\partial_z - \frac{2}{t} D - z(t) \Big)^\alpha \Big(\frac{2}{t} D + z(t) \Big)^{\alpha'} \\ &\quad \times \exp\Big(- i\xi \int_0^t \nabla f(z + z(s)) * dz(s) + \xi \int_0^t F(z + z(s)) ds \Big) \Big\} \\ &\quad \times \delta_{z_0}(z + z(t)) \Big], \end{aligned} \tag{3.5}$$

where

$$\Big(\frac{2}{t} D + z(t) \Big)^\alpha = \Big(\frac{2}{t} D_1 + \frac{x^1(t)}{\sqrt{2}} \Big)^{\alpha_1} \Big(\frac{2}{t} D_2 + \frac{x^2(t)}{\sqrt{2}} \Big)^{\alpha_2} \cdots \Big(\frac{2}{t} D_{2n} + \frac{x^{2n}(t)}{\sqrt{2}} \Big)^{\alpha_{2n}}$$

and D_j is the H-differentiation in the direction of $h_j(s) = (0, \cdots, \overset{j-\text{th}}{s \wedge t}, \cdots, 0)$, i.e.,

$$D_j \Phi(z(\cdot)) = \lim_{\eta \downarrow 0} \frac{\Phi(z(\cdot) + \eta h_j(\cdot)) - \Phi(z(\cdot))}{\eta}.$$

Then we have

$$\begin{aligned} &\|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi)\| \\ \leq & E[\Pi(\xi, t, z, z(\cdot)) | z + z(t) = z'] \frac{1}{(\pi t)^n} \exp\left(|\xi| t \sup \|F\| - \frac{|z - z'|^2}{t} \right), \end{aligned} \tag{3.6}$$

where $\Pi(\xi, t, z, z(\cdot))$ is a polynomial of ξ, t, $1/t$, $z(t)$ and finite number of elements of

$$\left\{ \left| \int_0^t \nabla \partial_z^\Omega f(z + z(s)) s^\delta * dz(s) \right|, \left| \int_0^t \partial_z^\Omega f(z + z(s)) s^\delta ds \right| : \Omega \in \mathbb{Z}_+^{2n}, \delta \in \mathbb{Z}_+ \right\}$$

For any $\Omega \in \mathbb{Z}_+^{2n}$, $\delta \in \mathbb{Z}_+$ and $p \geq 1$, by standard arguments in the Malliavin calculus (cf.[**9**],[**19**]), we can give the following estimate:

$$\begin{aligned} & E\left[\left\langle\!\!\left\langle \int_0^t \nabla \partial^\Omega f(z + z(s)) s^\delta * dz(s) \right\rangle\!\!\right\rangle^p \middle| z + z(t) = z' \right] \\ & \qquad \times \frac{1}{(\pi t)^{n/2}} \exp\left(- \frac{|z - z_0|^2}{2t} \right) \\ & \leq \left\{ E\left[\left\langle\!\!\left\langle \int_0^t \nabla \partial^\Omega f(z + z(s)) s^\delta * dz(s) \right\rangle\!\!\right\rangle^{2p} \delta_{z_0}(z + z(t)) \right] \right\}^{1/2} \\ & \leq C \left\langle\!\!\left\langle \int_0^t \nabla \partial^\Omega f(z + z(s)) s^\delta * dz(s) \right\rangle\!\!\right\rangle_{4p,k}^p \left\langle\!\!\left\langle \delta_{z'}(z + z(t)) \right\rangle\!\!\right\rangle_{2,-k}^{1/2} \\ & \leq \mathcal{P}\left(t, \frac{1}{t} \right), \end{aligned} \tag{3.7}$$

for all $z \in K$ and some polynomial $\mathcal{P}(t, 1/t)$, where k is a positive number depending only on the dimension n and $\| \cdot \|_{2,k}$ and $\| \cdot \|_{2,-k}$ are the norms of the Sobolev spaces $\mathbb{D}_{2,k}$ and $\mathbb{D}_{2,-k}$, respectively, in the Malliavin calculus. Similarly, for any $\Omega \in \mathbb{Z}_+^{2n}$, $\delta \in \mathbb{Z}_+$ and $p \geq 1$, we have

$$\begin{aligned} & E\left[\left\langle\!\!\left\langle \int_0^t \partial_z^\Omega f(z + z(s)) s^\delta ds \right\rangle\!\!\right\rangle^p \middle| z + z(t) = z_0 \right] \frac{1}{(\pi t)^{n/2}} \exp\left(- \frac{|z - z_0|^2}{2t} \right) \\ & \leq \mathcal{P}\left(t, \frac{1}{t} \right), \end{aligned} \tag{3.8}$$

for all $z \in K$ and some polynomial $\mathcal{P}(t, 1/t)$. Therefore we obtain (3.4). $\square$

We use this lemma and the positivity of the operator $\Box_b(\xi)$ to obtain the following:

PROPOSITION 3.1. *For any α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, $\Omega \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$, there is a polynomial $\mathcal{P}(\xi, t, 1/t)$ such that*

$$\sup\{|z - z'|^{\Omega} \|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi)\| : z \in K, z' \in \widetilde{B_f}\} \leq \mathcal{P}\left(\xi, t, \frac{1}{t}\right) \tag{3.9}$$

for all $0 < t \leq 1$ and $\xi \in \mathbb{R}$.

PROOF. Let $\varepsilon = 1/((t|\xi|) \vee 3)$. We expand as follows:

$$\begin{aligned} &q(t, z, z'; \xi) \\ &= \int_{\widetilde{B_f}} dz_0 q(\varepsilon t, z, z_0; \xi) \int_{\widetilde{B_f}} dz_1 q((1 - 2\varepsilon)t, z_0, z_1; \xi) q(\varepsilon t, z_1, z'; \xi), \end{aligned} \tag{3.10}$$

where $dz = dx^1 dx^2 \cdots dx^{2n}$. Since

$$|z - z'|^{\Omega} \leq C_\Omega \sum_{\Omega_1 + \Omega_2 + \Omega_3 = \Omega} |z - z_0|^{\Omega_1} |z_0 - z_1|^{\Omega_2} |z_1 - z'|^{\Omega_3}, \tag{3.11}$$

it is sufficient to show that, for any β_1, β_2, β_3, Ω_1, Ω_2, $\Omega_3 \in \mathbb{Z}_+$, there is a polynomial $\mathcal{P}(\xi, t, 1/t)$ such that

$$
\begin{aligned}
I :=& \sup\Big\{ \int_{\widetilde{B_f}} dz_0 \int_{\widetilde{B_f}} dz_1 |z-z_0|^{\Omega_1} \|\partial_\xi^{\beta_1}\partial_z^\alpha q(\varepsilon t, z, z_0;\xi)\| \\
&\qquad \times |z_0-z_1|^{\Omega_2}\|\partial_\xi^{\beta_2} q((1-2\varepsilon)t, z_0, z_1;\xi)\| \\
&\qquad \times |z_1-z'|^{\Omega_3}\|\partial_\xi^{\beta_3}\partial_{z'}^{\alpha'} q(\varepsilon t, z_1, z';\xi)\| : z\in K, z'\in \widetilde{B_f}\Big\}, \\
\le& \mathcal{P}\Big(\xi, t, \frac{1}{t}\Big).
\end{aligned} \tag{3.12}
$$

From Lemma 3.1, we have

$$
\|\partial_\xi^{\beta_1}\partial_z^\alpha q(\varepsilon t, z, z_0;\xi)\| \le \mathcal{P}\left(\xi, \varepsilon t, \frac{1}{\varepsilon t}\right) \exp\left(|\xi|\varepsilon t \sup\|F\| - \frac{|z-z_0|^2}{2\varepsilon t}\right).
$$

Since $\varepsilon t = (1/|\xi|)\wedge(t/3)$, for any $\Omega_1' \in \mathbb{Z}_+$, we have

$$
\begin{aligned}
&|z-z_0|^{\Omega_1'}\|\partial_\xi^{\beta_1}\partial_z^\alpha q(\varepsilon t, z, z_0;\xi)\| \\
&\le \mathcal{P}\left(\xi, t, \frac{1}{t}\right) \exp\left(-\Big(|\xi|\vee\frac{3}{t}\Big)\frac{|z-z_0|^2}{2}\right)
\end{aligned} \tag{3.13}
$$

for some polynomial $\mathcal{P}(\xi, t, 1/t)$. Similarly, we have

$$
\begin{aligned}
&|z_1-z'|^{\Omega_3}\|\partial_\xi^{\beta_3}\partial_{z'}^{\alpha'} q(\varepsilon t, z_1, z';\xi)\| \\
&\le \mathcal{P}\left(\xi, t, \frac{1}{t}, |z-z_1|\right) \exp\left(-\Big(|\xi|\vee\frac{3}{t}\Big)\frac{|z_1-z'|^2}{2}\right).
\end{aligned} \tag{3.14}
$$

Therefore, we obtain

$$
\begin{aligned}
I \le& \mathcal{P}\left(\xi, t, \frac{1}{t}\right) \sum_{\Omega_2'\in S} \sup\Big\{ \int_{\widetilde{B_f}} dz_0 \exp\left(-\Big(|\xi|\vee\frac{3}{t}\Big)\frac{|z-z_0|^2}{2}\right) \\
&\qquad \times |z_0-z'|^{\Omega_2'}\|\partial_\xi^{\beta_2} q((1-2\varepsilon)t, z_0, z';\xi)\| : z\in K, z'\in\widetilde{B_f}\Big\},
\end{aligned} \tag{3.15}
$$

for some finite set S of $\mathbb{Z}_+$. Accordingly, it is sufficient to show

$$
\begin{aligned}
&\sup\Big\{ \int_{\widetilde{B_f}} dz_0 \exp\left(-\Big(|\xi|\vee\frac{3}{t}\Big)\frac{|z-z_0|^2}{2}\right) \\
&\qquad \times |z_0-z'|^{\Omega}\|\partial_\xi^{\beta} q((1-2\varepsilon)t, z_0, z';\xi)\| : z\in K, z'\in\widetilde{B_f}\Big\} \\
\le& \mathcal{P}\Big(\xi, t, \frac{1}{t}\Big)
\end{aligned} \tag{3.16}
$$

for some polynomial $\mathcal{P}(\xi, t, 1/t)$.

We now introduce an additional variable $\mu > 0$. Let $B(\mu) = \{z \in \widetilde{B_f} : \ |z| \leq \mu\}$, $q(t, z_0, z'; \xi, \mu)$ be the heat kernel for the operator $\Box_b(\xi)$ on $B(\mu)$ with the Dirichlet boundary condition, and

$$q^d(t, z_0, z'; \xi, \mu) := \begin{cases} q(t, z_0, z'; \xi) - q(t, z_0, z'; \xi, \mu), & \text{if } z, z' \in B(\mu), \\ q(t, z_0, z'; \xi), & \text{otherwise.} \end{cases}$$

These has the following representations:

$$\begin{aligned} q(t, z, z'; \xi, \mu) =& E[\exp(\xi \mathcal{F}(t, z, z(\cdot))) \\ & |z + z(s) \in B(\mu) \text{ for any } s \in [0, t], z + z(t) = z'] \\ & \times e(t, z, z'; \mu), \end{aligned} \tag{3.17}$$

$$\begin{aligned} q^d(t, z, z'; \xi, \mu) =& E[\exp(\xi \mathcal{F}(t, z, z(\cdot))) \\ & : z + z(s) \notin B(\mu) \text{ for some } s \in [0, t] | z + z(t) = z'] \\ & \times \frac{1}{(\pi t)^n} \exp\Big(- \frac{|z - z'|^2}{t} \Big), \end{aligned} \tag{3.18}$$

where

$$\mathcal{F}(t, z, z(\cdot)) = -i \int_0^t \nabla f(z + z(s)) * dz(s) + \int_0^t F(z + z(s)) ds$$

and $e(t, z, z'; \mu)$ is the heat kernel for the operator $\Delta/4$ on $B(\mu)$ with the Dirichlet boundary condition. Then we see that $q(t, z_0, z'; \xi, \mu)$ and $q^d(t, z_0, z'; \xi, \mu)$ are differentiable in ξ. Thus we dominate the left hand side of (3.16) by $I_1 + I_2$, where

$$\begin{aligned} I_1 := \sup \Big\{ & \int_{B(\mu)} dz_0 \exp\left(- \Big(|\xi| \vee \frac{3}{t} \Big) \frac{|z - z_0|^2}{2} \right) \\ & \times |z_0 - z'|^{\Omega} \| \partial_\xi^\beta q((1 - 2\varepsilon)t, z_0, z'; \xi, \mu) \| : z \in K, z' \in B(\mu) \Big\} \end{aligned} \tag{3.19}$$

and

$$
\begin{aligned}
I_2 := \sup\Big\{ & \int_{\widetilde{B}_f} dz_0 \exp\left(-\Big(|\xi| \vee \frac{3}{t}\Big)\frac{|z-z_0|^2}{2}\right) \\
& \times |z_0 - z'|^{\Omega} \|\partial_\xi^\beta q^d((1-2\varepsilon)t, z_0, z'; \xi, \mu)\| : z \in K, z' \in \widetilde{B}_f\Big\}.
\end{aligned} \tag{3.20}
$$

For any $m \in \mathbb{N}$, we use the semigroup property of the heat kernel to expand

$$
\begin{aligned}
& q((1-2\varepsilon)t, z_0, z'; \xi, \mu) \\
& = \int_{B(\mu)} dz_1 q((1-2\varepsilon)t/m, z_0, z_1; \xi, \mu) \\
& \quad \times \int_{B(\mu)} dz_2 q((1-2\varepsilon)t/m, z_1, z_2; \xi, \mu) \\
& \quad \times \cdots \times \int_{B(\mu)} dz_{m-1} q((1-2\varepsilon)t/m, z_{m-2}, z_{m-1}; \xi, \mu) \\
& \quad \times q((1-2\varepsilon)t/m, z_{m-1}, z'; \xi, \mu).
\end{aligned} \tag{3.21}
$$

Then we have

$$
\begin{aligned}
& \|\partial_\xi^\beta q((1-2\varepsilon)t, z_0, z'; \xi, \mu)\| \\
& \le \sum_{\beta_1+\beta_2+\cdots+\beta_m=\beta} \frac{\beta!}{\beta_1!\beta_2!\cdots\beta_m!} \|\partial_\xi^{\beta_1} q((1-2\varepsilon)t/m, z_0, \cdot; \xi, \mu)\|_{2,B(\mu)} \\
& \quad \times \prod_{\nu=2}^{m-1} |||\partial_\xi^{\beta_\nu} \exp(-(1-2\varepsilon)t\Box_b(\xi)_{B(\mu)}/m)|||_{2,B(\mu)} \\
& \quad \times \|\partial_\xi^{\beta_m} q((1-2\varepsilon)t/m, \cdot, z'; \xi, \mu)\|_{2,B(\mu)},
\end{aligned} \tag{3.22}
$$

where $\|\cdot\|_{2,B(\mu)}$ is the L^2-norm on the restriction to $B(\mu)$ of $\Lambda(\widetilde{B}_f)$, $|||\cdot|||_{2,B(\mu)}$ is the operator norm and $\partial_\xi^{\beta_\nu} \exp(-(1-2\varepsilon)t\Box_b(\xi)_{B(\mu)}/m)$ is the operator with the integral kernel $\partial_\xi^{\beta_\nu} q((1-2\varepsilon)t/m, z, z'; \xi, \mu)$, $z, z' \in B(\mu)$.

Since

$$
\begin{aligned}
& \partial_\xi^{\beta_1} q((1-2\varepsilon)t/m, z_0, z_1; \xi, \mu) \\
& = E[\mathcal{F}((1-2\varepsilon)t/m, z_0, z(\cdot))^{\beta_1} \exp(\xi\mathcal{F}((1-2\varepsilon)t/m, z_0, z(\cdot))) \\
& \qquad |z_0 + z(s) \in B(\mu) \text{ for any } s \in [0, (1-2\varepsilon)t/m], \\
& \qquad z_0 + z((1-2\varepsilon)t/m) = z_1] e((1-2\varepsilon)t/m, z, z_1; \mu)
\end{aligned}
$$

and

$$\sup_{z\in B(\mu)} |\nabla f(z)| \le C'(1+\mu),$$

we obtain

$$\begin{aligned} &\|\partial_\xi^{\beta_1} q((1-2\varepsilon)t/m, z_0, \cdot; \xi, \mu)\|^2_{2,B(\mu)} \\ &\le C(1+\mu)^{2\beta_1} \exp\Big(\frac{2|\xi|t}{m} \sup \|F\|\Big) \frac{m^n}{t^n}, \end{aligned} \tag{3.23}$$

by the Schwarz inequality. Similarly, we obtain

$$\begin{aligned} &\|\partial_\xi^{\beta_m} q((1-2\varepsilon)t/m, \cdot, z'; \xi, \mu)\|^2_{2,B(\mu)} \\ &\le C(1+\mu)^{2\beta_m} \exp\Big(\frac{2|\xi|t}{m} \sup \|F\|\Big) \frac{m^n}{t^n} \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &|||\partial_\xi^{\beta_\nu} \exp(-(1-2\varepsilon)t\Box_b(\xi)_{B(\mu)}/m)|||^2_{2,B(\mu)} \\ &\le C(1+\mu)^{2\beta_\nu} \exp\Big(\frac{2|\xi|t}{m} \sup \|F\|\Big). \end{aligned} \tag{3.25}$$

When $\beta_\nu = 0$, by using the positivity of $\Box_b(\xi)_{B(\mu)}$, we have

$$|||\exp(-(1-2\varepsilon)t\Box_b(\xi)_{B(\mu)}/m)|||_{2,B(\mu)} \le 1. \tag{3.26}$$

Therefore, by taking $m = [|\xi|t] + 1$, we have

$$\|\partial_\xi^\beta q((1-2\varepsilon)t, z_0, z'; \xi, \mu)\| \le \mathcal{P}(\xi, t)(1+\mu)^{\beta+2} \frac{1}{t^n} \tag{3.27}$$

for some polynomial $\mathcal{P}(\xi, t)$. Therefore we obtain

$$I_1 \le \mathcal{P}\left(\xi, \mu, t, \frac{1}{t}\right) \tag{3.28}$$

for some polynomial $\mathcal{P}(\xi, \mu, t, 1/t)$.

On the other hand, by the Schwarz inequality, we have

$$\begin{aligned} &\|\partial_\xi^\beta q^d((1-2\varepsilon)t, z_0, z', \xi, \mu)\| \\ &\le E[\|\mathcal{F}((1-2\varepsilon)t/m, z_0, z(\cdot))\|^{2\beta} | z_0 + z((1-2\varepsilon)t/m) = z_1]^{1/2} \\ &\quad \times P(z_0 + z(s) \notin B(\mu) \text{ for some } s \in [0, (1-2\varepsilon)t] \\ &\qquad | z_0 + z((1-2\varepsilon)t) = z')^{1/2} \\ &\quad \times \exp\Big(|\xi|t \sup \|F\| - \frac{|z_0 - z'|^2}{(1-2\varepsilon)t}\Big) \frac{1}{(\pi t)^n}. \end{aligned} \tag{3.29}$$

Noting that ∇f may be of linear growth, we estimate as in the proof Lemma 3.1 to obtain

$$\begin{aligned} &E[\|\mathcal{F}((1-2\varepsilon)t/m, z_0, z(\cdot))\|^{2\beta}|z_0 + z((1-2\varepsilon)t/m) = z_1]^{1/2} \\ &\times \exp\Big(-\frac{|z_0 - z'|^2}{2(1-2\varepsilon)t}\Big)\frac{1}{(\pi t)^{n/2}} \\ \le &(1+|z-z_0|^2)^{\beta}\mathcal{P}\Big(t, \frac{1}{t}\Big) \end{aligned} \tag{3.30}$$

for some polynomial $\mathcal{P}(t, 1/t)$. Therefore we have

$$\begin{aligned} I_2 \le &\mathcal{P}\Big(t, \frac{1}{t}\Big)\exp(|\xi|t\sup\|F\|)\sup\Big\{\int_{\widetilde{B_f}} dz_0(1+|z-z_0|^2)^{\beta} \\ &\times |z_0 - z'|^{\Omega}\exp\Big(-\Big(|\xi| \vee \frac{3}{t}\Big)\frac{|z-z_0|^2}{2} - \frac{|z_0-z'|^2}{2(1-2\varepsilon)t}\Big) \\ &\times P(z_0 + z(s) \notin B(\mu) \text{ for some } s \in [0,(1-2\varepsilon)t] \\ &\qquad |z_0 + z((1-2\varepsilon)t) = z')^{1/2} \\ &: z \in K, z' \in \widetilde{B_f}\Big\}. \end{aligned} \tag{3.31}$$

We now take $\mu > 0$ large enough so that $K \subset B(\mu/4)$. In (3.31), if $z_0 \notin B(\mu/2)$, then $|z - z_0| \ge \mu/4$. If $z_0 \in B(\mu/2)$ and $z' \notin B(3\mu/4)$, then $|z_0 - z'| \ge \mu/4$. If $z_0 \in B(\mu/2)$ and $z' \in B(3\mu/4)$, then, by standard arguments in the probability theorey, we have

$$\begin{aligned} &P(z_0 + z(s) \notin B(\mu) \text{ for some } s \in [0,(1-2\varepsilon)t]|z_0 + z((1-2\varepsilon)t) = z') \\ &\le C'\exp\Big(-C\frac{\mu^2}{t}\Big)\ , \end{aligned} \tag{3.32}$$

for some $C, C' > 0$ (cf.[**9**] Lemma V-10.5). Therefore, by using also the fact that

$$\int_{\widetilde{B_f}} dz_0(1+|z-z_0|^2)^{\beta}|z_0 - z'|^{\Omega}\exp\Big(-\Big(|\xi| \vee \frac{3}{t}\Big)\frac{|z-z_0|^2}{4} - \frac{|z_0-z'|^2}{4(1-2\varepsilon)t}\Big)$$

is dominated by a polynomial of t and $1/t$, we have

$$I_2 \le \mathcal{P}\Big(t, \frac{1}{t}\Big)\exp\Big(|\xi|t\sup\|F\| - C\frac{\mu^2}{t}\Big) \tag{3.33}$$

for some $C > 0$ and some polynomial $\mathcal{P}(t, 1/t)$. By taking μ appropriately, we can complete the proof. $\square$

PROPOSITION 3.2. *We assume that* $n_+ = n \geq 1$, $n_- = n_0 = 0$. *Then we have the following:*

(i) For any $\alpha, \alpha' \in \mathbb{Z}_+^{2n}$, $\beta \in \mathbb{Z}_+$ *and compact* $K \subset \widetilde{B_f}$, *there is a polynomial* $\mathcal{P}(\xi, t, 1/t)$ *such that*

$$
\begin{aligned}
&|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi)| \\
&\leq \mathcal{P}\Big(\xi, t, \frac{1}{t}\Big) E\bigg[\exp\Big(3\xi \int_0^{t/2} F(z + z(s))ds\Big)\bigg]^{1/3} \exp\Big(-\frac{|z - z'|^2}{3t}\Big),
\end{aligned} \tag{3.34}
$$

for all $t > 0$, $z \in K$, $z' \in \widetilde{B_f}$ *and* $\xi \leq 0$.

(ii) We further assume that, for any $p \geq 1$ *and compact* $K \subset \widetilde{B_f}$,

$$
B(p, t; K) := \sup_{z \in K} E\bigg[\Big(\int_0^t \Delta f(z + z(s))ds\Big)^{-p}\bigg] < \infty. \tag{3.35}
$$

Then, for any $\alpha, \alpha' \in \mathbb{Z}_+^{2n}$, $\beta, \Omega \in \mathbb{Z}_+$ *and compact* $K \subset \widetilde{B_f}$, *there is a polynomial* $\mathcal{P}(t, 1/t)$ *and a finite subset* S *of* $(1, \infty)$ *such that*

$$
\begin{aligned}
&\sup\Big\{|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi)||\xi|^\Omega \exp\Big(\frac{|z - z'|^2}{3t}\Big) : z \in K, z' \in \widetilde{B_f}, \xi \leq 0\Big\} \\
&\leq \mathcal{P}\left(t, \frac{1}{t}\right) \left\{\sum_{p \in S} B\Big(p, \frac{t}{2}; K\Big)\right\}^{1/3}
\end{aligned} \tag{3.36}
$$

for all $t > 0$,

PROOF. Since (ii) is easily obtained from (i), we prove only (i). Under the assumption of this proposition, $F = \Delta f/4$ is a scalar valued function. Thus, in (3.5), we calculate as

$$
\begin{aligned}
&\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi) \\
&= E\bigg[\Pi(\xi, t, z, z(\cdot)) \exp\Big(-i\xi \int_0^t \nabla f(z + z(s)) * dz(s) \\
&\qquad + \xi \int_0^t F(z + z(s))ds\Big) \delta_{z'}(z + z(t))\bigg],
\end{aligned} \tag{3.37}
$$

where $\Pi(\xi, t, z, z(\cdot))$ is a polynomial of ξ, t, $1/t$, $z(t)$ and finite number of elements of

$$\Big\{ \int_0^t \nabla \partial_z^\Omega f(z+z(s)) s^\delta * dz(s), \int_0^t \partial_z^\Omega f(z+z(s)) s^\delta ds : \Omega \in \mathbb{Z}_+^{2n}, \delta \in \mathbb{Z}_+ \Big\}$$

By the Hölder inequality, we have

$$\begin{aligned}
&|\partial_\xi^\beta \partial_z^\alpha \partial_{z'}^{\alpha'} q(t, z, z'; \xi)| \\
\le &\Big\{ E[|\Pi(\xi, t, z, z(\cdot))|^3 | z + z(t) = z'] \frac{1}{(\pi t)^n} \exp\Big(- \frac{|z-z'|^2}{t} \Big) \Big\}^{1/3} \\
&\times \Big\{ E\Big[\exp\Big(3\xi \int_0^t F(z+z(s)) ds \Big) \Big| z + z(t) = z' \Big] \\
&\times \frac{1}{(\pi t)^n} \exp\Big(- \frac{|z-z'|^2}{t} \Big) \Big\}^{1/3} \Big\{ \frac{1}{(\pi t)^n} \exp\Big(- \frac{|z-z'|^2}{t} \Big) \Big\}^{1/3}.
\end{aligned}$$

The first factor is estimated as in the proof of Lemma 3.1. For the second factor, noting F is positive, we use the Markov property to estimate as follows:

$$\begin{aligned}
&E\Big[\exp\Big(2\xi \int_0^t F(z+z(s)) ds \Big) \Big| z + z(t) = z' \Big] \frac{1}{(\pi t)^n} \exp\Big(- \frac{|z-z'|^2}{t} \Big) \\
\le &E\Big[\exp\Big(2\xi \int_0^{t/2} F(z+z(s)) ds \Big) \delta_{z'}(z+z(t)) \Big] \\
\le &E\Big[\exp\Big(2\xi \int_0^{t/2} F(z+z(s)) ds \Big) \Big] \sup\{ E[\delta_{z'}(z+z(t/2))] : z, z' \in \widetilde{B_f} \} \\
\le &\frac{C}{t^n} E\Big[\exp\Big(2\xi \int_0^{t/2} F(z+z(s)) ds \Big) \Big].
\end{aligned} \tag{3.38}$$

Therefore we obtain (3.34). □

To estimate $B(p, t; K)$ in (3.35), we prepare the following:

PROPOSITION 3.3. *We assume $n_+ = n \ge 1$, $n_- = n_0 = 0$ and $(E, \rho)_+$ for some $\rho > 0$. Let $g(z) = h_+(\psi_+(z))$ for $z \in \widetilde{B_f}$.*

(i) If $\rho < 2$ and $(H, \kappa)_+$ is satisfied for some $\kappa > 1$, then, for any $1 < p < \infty$,

there are constants C, C' and $C''>0$ such that

$$
\begin{aligned}
&E\Big[\Big(\int_0^t g(z+z(s))ds\Big)^{-p}\Big] \\
\le& C\Big\{\exp(C't^{\rho/(\kappa-1)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{\rho/(2-\rho)}\Big)\Big\}
\end{aligned}
\tag{3.39}
$$

for all $0<t<\infty$.

(ii) If $\rho<1$, we take some $\nu\in(1/\rho, 2/\rho-1)$. Then, for any $1<p<\infty$, there are constants C, C' and $C''>0$ such that

$$
\begin{aligned}
&E\Big[\Big(\int_0^t g(z+z(s))ds\Big)^{-p}\Big] \\
\le& C\Big\{\exp(C't^{\rho/(\nu\rho-1)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{\rho/(2-\nu\rho-\rho)}\Big)\Big\}
\end{aligned}
\tag{3.40}
$$

for all $0<t<\infty$.

REMARK 3.1. This type of estimate is given in Kusuoka-Stroock [**12**] to estimate the Malliavin covariance. We make their estimate more precise so that the dependence on ρ of the bound becomes clear.

PROOF. We will drop the subscript + during the course of the proof.

(i) As in [**12**], we first estimate the probability,

$$
\begin{aligned}
&P\left(\int_0^t g(z+z(s))ds<\frac{1}{R}\right) \\
=&P\left(\int_0^t g(z+z(s))ds<\frac{1}{R},\ \tau_\lambda(z)\le\frac{t}{2}\right) \\
&+P\left(\int_0^t g(z+z(s))ds<\frac{1}{R},\ \tau_\lambda(z)>\frac{t}{2}\right) \\
=:&I_1+I_2,
\end{aligned}
\tag{3.41}
$$

where $\lambda\in(0,1]$ is an additional variable and

$\tau_\lambda(z)=\inf\{s\ge 0:\ |\psi(z+z(s))|>\lambda\}$.

By using the strong Markov property, we have

$$
\begin{aligned}
I_1 &\le P\left(\int_{\tau_\lambda(z)}^{\tau_\lambda(z)+t/2} g(z+z(s))ds < \frac{1}{R},\ \tau_\lambda(z) \le \frac{t}{2}\right) \\
&\le \sup_{z:|\psi(z)|>\lambda} P\left(\int_0^{t/2} g(z+z(s))ds < \frac{1}{R}\right) \\
&\le \sup_{z:|\psi(z)|>\lambda} P\left(\int_0^{(t/2)\wedge\sigma_{\lambda/2}(z)} g(z+z(s))ds < \frac{1}{R}\right) \\
&\le \sup_{z:|\psi(z)|>\lambda} P\left(\frac{t}{2} \wedge \sigma_{\lambda/2}(z) \le \frac{1}{Rh(\lambda/2)}\right)\ ,
\end{aligned} \tag{3.42}
$$

where $\sigma_{\lambda/2}(z) = \inf\{s \ge 0 :\ |\psi(z+z(s))| < \lambda/2\}$. If $Rh(\lambda/2) \ge 2/t$, then we have

$$
\begin{aligned}
I_1 &\le \sup_z P\left(\sigma_{\lambda/2}(z) \le \sup_z \frac{1}{Rh(\lambda/2)}\right) \\
&\le P\left(\sup_{0<s\le 1/(Rh(\lambda/2))} |\psi(z+z(s)) - \psi(z)| > \frac{\lambda}{2}\right) \\
&\le P\left(\sup_{0<s\le 1/(Rh(\lambda/2))} |z(s)| > C\lambda\right) \\
&\le C' \exp\left(-C''\lambda^2 Rh\Big(\frac{\lambda}{2}\Big)\right)
\end{aligned} \tag{3.43}
$$

from (8.29) in [**12**].

On the other hand, we have

$$
\begin{aligned}
I_2 &\le P\left(\tau_\lambda(z) > \frac{t}{2}\right) \\
&= P\left(\sup_{0<s<t/2} |\psi(z+z(s))| \le \lambda\right) \\
&\le \sup_{z:|\psi(z)|\le\lambda} P\Big(\sup_{0<s<t/2} |\psi(z+z(s)) - \psi(z)| \le 2\lambda, \\
&\qquad\qquad \sup_{0<s<t/2} |\psi(z+z(s))| \le \lambda\Big).
\end{aligned} \tag{3.44}
$$

By using the Itô formula, we have

$$
\begin{aligned}
I_2 \le & \sup_{z:|\psi(z)|\le\lambda} P\Bigg(\sup_{0<s<t/2} \left| \sum_{j=1}^{2n} \int_0^s \frac{\partial_j \psi}{\sqrt{2}}(z+z(r))dx^j(r) \right| \le 4\lambda, \\
& \qquad \sup_{0<s<t/2} |\psi(z+z(s))| \le \lambda \Bigg) \\
& + \sup_{z:|\psi(z)|\le\lambda} P\Bigg(\sup_{0<s<t/2} \left| \int_0^s \frac{\Delta\psi}{4}(z+z(r))dr \right| \ge 2\lambda, \\
& \qquad \sup_{0<s<t/2} |\psi(z+z(s))| \le \lambda \Bigg) \\
=: & I_{21} + I_{22}.
\end{aligned} \tag{3.45}
$$

By Theorem II-7.2′ of [**9**], there exists a 1-dimensional Brownian motion $B(t)$ such that

$$
\sum_{j=1}^{2n} \int_0^s \frac{\partial_j \psi}{\sqrt{2}}(z+z(r))dx^j(r) = B\left(\int_0^s \frac{1}{2} \sum_{j=1}^{2n} |\partial_j \psi(z+z(r))|^2 dr \right).
$$

Since $\inf_{z\in\widetilde{B_f}} \{\psi(z)^2 + |\nabla\psi(z)|^2\} > 0$, we have

$$
\frac{1}{2} \sum_{j=1}^{2n} |\partial_j \psi(z+z(r))|^2 \ge C - C'\lambda^2
$$

for some C, $C' > 0$. Therefore, from (8.27) in [**12**], we obtain

$$
I_{21} \le C'' \exp\left(-\frac{t(C - C'\lambda^2)}{\lambda^2} \right). \tag{3.46}
$$

On the other hand, remarking $|\Delta\psi(z)| \le \eta_1 |\psi(z)|^\kappa$ for $z \in \widetilde{B_f}$ such that $|\psi(z)| \le \eta_2$, we see that

$$
I_{22} = 0 \ \text{ if } \lambda < C\left(1 \wedge \frac{1}{t} \right)^{1/(\kappa-1)}, \tag{3.47}
$$

for some $C > 0$.

We now note that $\lim_{\lambda\downarrow 0} \lambda^\rho \log(1/h(\lambda)) = 0$. For an arbitrary fixed $0 < \varepsilon < 1$, we take $\lambda_\varepsilon > 0$ such that $\lambda^\rho \log(1/h(\lambda)) < \varepsilon$, for any $0 < \lambda < \lambda_\varepsilon$. If we set

$\lambda = 2(\log R)^{-1/\rho}$, then we see that $\lambda^2 Rh(\lambda/2) > R^{1/2}$ and $t(C - C'\lambda)/\lambda^2 > C''t(\log R)^{2/\rho}$ for large enough R. Thus there are $C_1, C_2, \cdots, C_6 > 0$ such that

$$
\begin{aligned}
&P\left(\int_0^t g(z+z(s))ds < \frac{1}{R}\right) \\
&\le C_1 \exp\left(-C_2 R^{1/2}\right) + C_3 \exp\left(-C_4 t(\log R)^{2/\rho}\right)
\end{aligned} \tag{3.48}
$$

for any $R \ge \exp(C_5 t^{\rho/(\kappa-1)}) \vee (2/t)^{1/(1-\varepsilon)} \vee C_6$.

Therefore, noting $\rho < 2$, we obtain

$$
\begin{aligned}
&E\left[\left(\int_0^t g(z+z(s))ds\right)^{-p}\right] \\
&= \int_0^\infty pR^{p-1} P\left(\int_0^t g(z+z(s))ds < \frac{1}{R}\right) dR \\
&\le \exp(pC_5 t^{\rho/(\kappa-1)}) \vee \left(\frac{2}{t}\right)^{p/(1-\varepsilon)} \vee C_6^p \vee 1 \\
&\quad + \int_0^\infty pR^{p-1} C_1 \exp(-C_2 R^{1/2}) dR \\
&\quad + \int_1^\infty pR^{p-1} C_3 \exp\left(-C_4 t(\log R)^{2/\rho}\right) dR \\
&\le \exp(pC_5 t^{\rho/(\kappa-1)}) \vee \left(\frac{2}{t}\right)^{p/(1-\varepsilon)} \vee C_6^p \vee 1 + C_7 \\
&\quad + \frac{pC_8}{t^{1/\alpha}} \int_0^\infty \exp\left(-2S^\alpha + \frac{C_9 pS}{t^{1/\alpha}}\right) dS
\end{aligned} \tag{3.49}
$$

for some C_7, C_8, $C_9 > 0$, where $\alpha := 2/\rho$. Let $g_t(S) = S^\alpha - C_9 pS/t^{1/\alpha}$. Then there exists C_{10} such that $g_t(S) \ge -C_{10}(1/t)^{1/(\alpha-1)}$ for any $S \ge 0$ and $t > 0$. Therefore, we have

$$
\int_0^\infty \exp\left(-2S^\alpha + \frac{C_9 pS}{t^{1/\alpha}}\right) dS \le C_{11} \exp\left(C_{10}\left(\frac{1}{t}\right)^{1/(\alpha-1)}\right). \tag{3.50}
$$

(ii) In stead of (3.41), we decompose

$$
\begin{aligned}
&P\left(\int_0^t g(z+z(s))ds < \frac{1}{R}\right) \\
=&P\left(\int_0^t g(z+z(s))ds < \frac{1}{R},\ \tau_\lambda(z) \le \frac{t}{(\log R)^\nu}\right) \\
&+ P\left(\int_0^t g(z+z(s))ds < \frac{1}{R},\ \tau_\lambda(z) > \frac{t}{(\log R)^\nu}\right) \\
=:& I_1' + I_2',
\end{aligned} \tag{3.51}
$$

We take R large enough so that $(\log R)^{-\nu} < 1/2$. Then, as in (3.42) and (3.43), we have

$$I_1' \le C' \exp\left(-C''\lambda^2 Rh\Big(\frac{\lambda}{2}\Big)\right) \tag{3.52}$$

if $Rh(\lambda/2) \ge 2/t$. As in (3.44) and (3.45), we have

$$\begin{aligned} I_2' \le & \sup_{z:|\psi(z)|\le\lambda} P\Bigg(\sup_{0<s<t/(\log R)^\nu} \Bigg| \sum_{j=1}^{2n} \int_0^s \frac{\partial_j\psi}{\sqrt{2}}(z+z(r))dx^j(r) \Bigg| \le 4\lambda, \\ & \qquad \sup_{0<s<t/(\log R)^\nu} |\psi(z+z(s))| \le \lambda \Bigg) \\ & + \sup_{z:|\psi(z)|\le\lambda} P\Bigg(\sup_{0<s<t/(\log R)^\nu} \Bigg| \int_0^s \frac{\Delta\psi}{4}(z+z(r))dr \Bigg| \ge 2\lambda, \\ & \qquad \sup_{0<s<t/(\log R)^\nu} |\psi(z+z(s))| \le \lambda \Bigg) \\ =: & I_{21}' + I_{22}'. \end{aligned} \tag{3.53}$$

As in (3.46), we have

$$I_{21}' \le C'' \exp\left(-\frac{t(C-C'\lambda)}{\lambda^2(\log R)^\nu}\right). \tag{3.54}$$

We easily see that

$$I_{22}' = 0 \ \text{ if } \lambda > \frac{Ct}{(\log R)^\nu}, \tag{3.55}$$

for some $C > 0$. As in the proof of (i), we set $\lambda = 2(\log R)^{-1/\rho}$. Then, noting $\nu > 1/\rho$, we see that there are $C_1, C_2, \cdots, C_6 > 0$ such that

$$\begin{aligned} & P\left(\int_0^t g(z+z(s))ds < \frac{1}{R}\right) \\ & \le C_1 \exp\left(-C_2 R^{1/2}\right) + C_3 \exp\left(-C_4 t(\log R)^{2/\rho-\nu}\right) \end{aligned} \tag{3.56}$$

for any $R \ge \exp(C_5 t^{\rho/(\nu\rho-1)}) \vee (2/t)^{1/(1-\varepsilon)} \vee C_6$. As in (3.49), noting $2/\rho - \nu > 1$, we obtain

$$\begin{aligned} & E\left[\Big(\int_0^t g(z+z(s))ds\Big)^{-p}\right] \\ & \le \exp(pC_5 t^{\rho/(\nu\rho-1)}) \vee \left(\frac{2}{t}\right)^{p/(1-\varepsilon)} \vee C_6^p \vee 1 + C_7 \\ & \quad + \frac{pC_8}{t^{1/\beta}} \int_0^\infty \exp\left(-2S^\beta + \frac{C_9 pS}{t^{1/\beta}}\right) dS \end{aligned} \tag{3.57}$$

for some C_7, C_8, $C_9 > 0$, where $\beta := 2/\rho - \nu$. $\square$

From Propositions 3.2 and 3.3, we obtain the following:

COROLLARY 3.1. *We assume $n_\pm \geq 1$ and either $(E,\rho)_\pm$ for some $\rho < 1$ or $(E,\rho)_\pm$ and $(H,\kappa)_\pm$ for some $\rho < 2$ and $\kappa > 1$. We choose $(A(\pm), B(\pm))$ from the set*

$$\Big\{\Big(\frac{\rho}{\kappa-1}, \frac{\rho}{2-\rho}\Big) : \kappa \text{ is the number } (H,\kappa)_\pm \text{ is satisfied}\Big\}$$
$$\cup \Big\{\Big(\frac{\rho}{\nu\rho-1}, \frac{\rho}{2-\nu\rho-\rho}\Big) : \nu \text{ is the number satisfying } \frac{1}{\rho} < \nu < \frac{2}{\rho} - 1\Big\}.$$

Then, for any α, $\alpha' \in \mathbb{Z}_+^{2n_\pm}$, β, $\Omega \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_{f_\pm}}$, there are constants C, C', $C'' > 0$ such that

$$\begin{aligned} &\sup\{|\partial_\xi^\beta \partial_{z^\pm}^\alpha \partial_{z'^\pm}^{\alpha'} q_\pm(t, z^\pm, z'^\pm; \xi)||\xi|^\Omega \exp\Big(\frac{|z^\pm - z'^\pm|^2}{3t}\Big) \\ &\qquad : z^\pm \in K, z'^\pm \in \widetilde{B_{f_\pm}}, \mp\xi \geq 0\} \\ &\leq C\Big\{\exp(C' t^{A(\pm)}) + \exp\Big(C''\Big(\frac{1}{t}\Big)^{B(\pm)}\Big)\Big\} \end{aligned} \tag{3.58}$$

for all $t > 0$, where $q_\pm(t, z^\pm, z'^\pm; \xi)$ is the heat kernel for $\Box_b^\pm(\xi)$.

4. Proof of Theorems 1 and 2

We first show the following:

PROPOSITION 4.1. *(i) Under the same assumption as for Theorem 1, the function* $k(t, X, X')$ *defined in* (2.15) *is a smooth function of* $(X, X') \in B_f \times B_f$ *with the following estimate: for any* α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, β', k, $l \in \mathbb{Z}_+$ *and compact* $K \subset \widetilde{B_f}$, *there are constants* C, C', C'' *such that*

$$\sup\{|z-z'|^k|u-u'|^l\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_u^\beta\partial_{u'}^{\beta'}k(t,(z,u),(z',u'))\|$$

$$: z \in K, z' \in \widetilde{B_f}, u, u' \in \mathbb{R}\} \tag{4.1}$$

$$\leq C\Big\{\exp(C't^A) + \exp\Big(C''\Big(\frac{1}{t}\Big)^B\Big)\Big\}$$

and

$$\sup\{|z-z'|^k|u-u'|^l\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_u^\beta\partial_{u'}^{\beta'}k(t,(z,u),(z',u'))\|$$

$$: z \in \widetilde{B_f}, z' \in K, u, u' \in \mathbb{R}\} \tag{4.1}'$$

$$\leq C\Big\{\exp(C't^A) + \exp\Big(C''\Big(\frac{1}{t}\Big)^B\Big)\Big\}$$

for all $t > 0$, *where* $A = A(+) \vee A(-)$, $B = B(+) \vee B(-)$, *and* $(A(\pm), B(\pm))$ *are choosen as in Corollary 3.1. Derivatives of* $k(t, X, X')$ *can be calculated by differentiating under the integral sign in* (2.15).

(ii) Under the same assumption as for Theorem 2, the function $h(t, Z, Z')$ *defined in* (2.19) *is a smooth function of* $(Z, Z') \in \overline{D_f} \times \overline{D_f}$ *with the following estimate: for any* α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, β', Ω, Ω', k, l, $m \in \mathbb{Z}_+$ *and compact* $K \subset \widetilde{B_f}$, *there are constants* C, C', C'' *such that*

$$\sup\{|z-z'|^k|u-u'|^l|r-r'|^m\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_u^\beta\partial_{u'}^{\beta'}\partial_r^\delta\partial_{r'}^{\delta'}h(t,(z,u,r),(z',u',r'))\|$$

$$: z \in K, z' \in \widetilde{B_f}, u, u' \in \mathbb{R}, r, r' \geq 0\} \tag{4.2}$$

$$\leq C\Big\{\exp(C't^{A(+)}) + \exp\Big(C''\Big(\frac{1}{t}\Big)^{B(+)}\Big)\Big\}$$

and

$$\sup\{|z-z'|^k|u-u'|^l|r-r'|^m\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_u^\beta\partial_{u'}^{\beta'}\partial_r^\delta\partial_{r'}^{\delta'}h(t,(z,u,r),(z',u',r'))\|$$

$$(4.2)' \qquad : z\in\widetilde{B_f}, z'\in K, u,u'\in\mathbb{R}, r,r'\geq 0\}$$

$$\leq C\Big\{\exp(C't^{A(+)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{B(+)}\Big)\Big\}$$

for all $t>0$*, where* $(A(\pm),B(\pm))$ *are choosen as in Corollary 3.1. Derivatives of* $h(t,Z,Z')$ *can be calculated by differentiating under the integral sign in* (2.19).

Proof. (i) Under the decomposition (2.5), on the bundle $\Lambda_r^{p,q}(B_f)$, the heat kernel $q(t,z,z';\xi)$ is decomposed as

$$(4.3) \qquad q(t,z,z';\xi)=q_+(t,z^+,z'^+;\xi)q_-(t,z^-,z'^-;\xi)q_0(t,z^0,z'^0;\xi),$$

where, for each $\sigma=+,-,0$, $q_\sigma(t,z^\sigma,z'^\sigma;\xi)$ is the heat kernel for $\Box_b^\sigma(\xi)$ (cf.(2.14)). We note that $q_+(t,z^+,z'^+;\xi)$ and $q_-(t,z^-,z'^-;\xi)$ are scalar valued functions. For $\xi\geq 0$, we apply Proposition 3.1 to $q_+(t,z^+,z'^+;\xi)q_0(t,z^0,z'^0;\xi)$ and Corollary 3.1 to $q_-(t,z^-,z'^-;\xi)$. Then we see that

$$\sup\{|z-z'|^\Omega\|\partial_\xi^\beta\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)\|\xi^\delta : z\in K, z'\in\widetilde{B_f}, \xi\geq 0\}$$

$$(4.4) \qquad \leq C\Big\{\exp(C't^{A(-)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{B(-)}\Big)\Big\}$$

for some C, C', $C''\geq 0$. For $\xi\leq 0$, we obtain the similar estimate by applying Proposition 3.1 to $q_-(t,z^-,z'^-;\xi)q_0(t,z^0,z'^0;\xi)$ and Corollary 3.1 to $q_+(t,z^+,z'^+;\xi)$. Therefore, we have

$$\sup\{|z-z'|^\Omega\|\partial_\xi^\beta\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)\|\xi^\delta : z\in K, z'\in\widetilde{B_f}, \xi\in\mathbb{R}\}$$

$$(4.5) \qquad \leq C\Big\{\exp(C't^{A})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{B}\Big)\Big\}$$

for some C, C', $C''\geq 0$.

By the symmetry of the operator $\Box_b(\xi)$, we also have

$$
\begin{aligned}
&\sup\{|z-z'|^{\Omega}\|\partial_\xi^\beta\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)\|\xi^\delta : z\in\widetilde{B_f}, z'\in K, \xi\in\mathbb{R}\} \\
&\le C\Big\{\exp(C't^A)+\exp\Big(C''\Big(\frac{1}{t}\Big)^B\Big)\Big\}.
\end{aligned}
\tag{4.5}'
$$

Thus we see the statements of (i) hold.

(ii) As in the proof of (i), we show

$$
\begin{aligned}
&\sup\{|z-z'|^k|r-r'|^m\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_\xi^\beta q(t,z,z';\xi)\partial_r^{\Omega}\partial_{r'}^{\Omega'}E(t,r,r',\xi)\||\xi|^l \\
&\qquad : z\in K, z'\in\widetilde{B_f}, \xi\in\mathbb{R}, r,r'\ge 0\} \\
&\le C\Big\{\exp(C't^{A(+)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{B(+)}\Big)\Big\}
\end{aligned}
\tag{4.6}
$$

for any $t>0$. From Lemma 4.1 below, it is sufficient to show that for any β', $l'\in\mathbb{Z}_+$, there are a polynomial $\mathcal{P}(t,1/t)$ and constants C, C', C'' such that

$$
\begin{aligned}
&\sup\Big\{|z-z'|^k\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_\xi^{\beta'}q(t,z,z';\xi)\||\xi|^{l'}\exp\Big(-\frac{\xi^2t}{2}\Big) \\
&\qquad : z\in K, z'\in\widetilde{B_f}, \xi\in\mathbb{R}\Big\} \\
&\le\mathcal{P}\Big(t,\frac{1}{t}\Big)
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
&\sup\Big\{|z-z'|^k\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_\xi^{\beta'}q(t,z,z';\xi)\||\xi|^{l'} : z\in K, z'\in\widetilde{B_f}, \xi\le 0\Big\} \\
&\le C\Big\{\exp(C't^{A(+)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{B(+)}\Big)\Big\}
\end{aligned}
\tag{4.8}
$$

for any $t>0$. (4.7) is easily shown by using Lemma 3.1, and (4.8) is shown as in (4.4).

By the symmetry of the operator $\Box_b(\xi)$, we also have

$$
\begin{aligned}
&\sup\{|z-z'|^k|r-r'|^m\|\partial_z^\alpha\partial_{z'}^{\alpha'}\partial_\xi^\beta q(t,z,z';\xi)\partial_r^{\Omega}\partial_{r'}^{\Omega'}E(t,r,r',\xi)\||\xi|^l \\
&\qquad : z\in\widetilde{B_f}, z'\in K, \xi\in\mathbb{R}, r,r'\ge 0\} \\
&\le C\Big\{\exp(C't^{A(+)})+\exp\Big(C''\Big(\frac{1}{t}\Big)^{B(+)}\Big)\Big\}. \quad \Box
\end{aligned}
\tag{4.6}'
$$

LEMMA 4.1. *For any β, Ω, $\Omega' \in \mathbb{Z}_+$, there are polynomials $\mathcal{P}(t, 1/t)$ and $\mathcal{P}(\xi, t, 1/t)$ such that*

$$
\begin{aligned}
&|\partial_\xi^\beta \partial_r^\Omega \partial_{r'}^{\Omega'} E(t, r, r', \xi)| \\
&\leq \begin{cases} \mathcal{P}\Big(t, \dfrac{1}{t}\Big) \exp\Big(- \dfrac{|r - r'|^2}{2t} - \dfrac{\xi^2 t}{2}\Big) & \text{if } \xi \geq -\dfrac{r + r'}{t}, \\ \mathcal{P}\Big(\xi, t, \dfrac{1}{t}\Big) \exp\Big(- \dfrac{|r - r'|^2}{2t}\Big) & \text{if } \xi \leq -\dfrac{r + r'}{t} \end{cases}
\end{aligned} \tag{4.9}
$$

for all $t > 0$, r, $r' \geq 0$.

PROOF. For any Ω, $l \in \mathbb{Z}_+$, there is a polynomial $\mathcal{P}(t, 1/t)$ such that

$$
|r^l \partial_r^\Omega e(t, r)| \leq \mathcal{P}\Big(t, \frac{1}{t}\Big) \exp\Big(- \frac{r^2}{2t}\Big) . \tag{4.10}
$$

On the other hand, we have

$$
\partial_r^\Omega \varepsilon(t, r, \xi) = \sum_{\Omega_1 + \Omega_2 = \Omega} \frac{\Omega!}{\Omega_1! \Omega_2!} \xi^{\Omega_1} e^{r\xi} \partial_r^{\Omega_2} \int_{r/\sqrt{t} + \xi\sqrt{t}}^\infty e^{-\mu^2/2} d\mu. \tag{4.11}
$$

For $\Omega_2 > 0$, we have

$$
e^{r\xi} \partial_r^{\Omega_2} \int_{r/\sqrt{t} + \xi\sqrt{t}}^\infty e^{-\mu^2/2} d\mu = \mathcal{P}\Big(r, \sqrt{t}, \frac{1}{\sqrt{t}}\Big) \exp\Big(- \frac{r^2}{2t} - \frac{\xi^2 t}{2}\Big), \tag{4.12}
$$

where $\mathcal{P}(r, \sqrt{t}, 1/\sqrt{t})$ is a polynomial. For $\Omega_2 = 0$, if $r/\sqrt{t} + \xi\sqrt{t} \geq 0$, we estimate as

$$
\begin{aligned}
&e^{r\xi} \int_{r/\sqrt{t} + \xi\sqrt{t}}^\infty e^{-\mu^2/2} d\mu \\
&\leq e^{r\xi} \int_0^\infty \exp\Big(- \frac{\mu^2}{2} - \frac{1}{2}\Big(\frac{r}{\sqrt{t}} + \xi\sqrt{t}\Big)^2\Big) d\mu \\
&= \sqrt{\frac{\pi}{2}} \exp\Big(- \frac{r^2}{2t} - \frac{\xi^2 t}{2}\Big).
\end{aligned} \tag{4.13}
$$

If $r/\sqrt{t} + \xi\sqrt{t} \leq 0$, we estimate as follows:

$$
e^{r\xi} \int_{r/\sqrt{t} + \xi\sqrt{t}}^\infty e^{-\mu^2/2} d\mu \leq e^{r\xi} \sqrt{\frac{\pi}{2}} \leq \sqrt{\frac{\pi}{2}} e^{-r^2/t}. \tag{4.14}
$$

By all this we obtain (4.9). □

We now note the following (cf.Stanton [**16**] Proposition 3.1):

PROPOSITION 4.2. *(i) We assume that K_t, $t > 0$, is a one parameter family of bounded operators on the space $L^2(\Lambda_r^{p,q}(B_f))$ such that, for any $\varphi \in \Gamma_0^\infty(\Lambda_r^{p,q}(B_f))$,*

(i-i) for any $T > 0$, there is a constant C depending on φ and T such that $\|K_t\varphi\|_2 \leq C$ for all $t \in [0,T]$, where $\|\cdot\|_2$ is the L^2-norm;

(i-ii) $K_t\varphi$ is differentiable in t;

(i-iii) $K_t\varphi \in \mathrm{Dom}(\Box_b)$;

(i-iv) $(\partial_t + \Box_b)K_t\varphi = 0$;

(i-v) $K_t\varphi \to \varphi$ in L^2 as $t \to 0$;

(i-vi) $\Box_b K_t\varphi = K_t\Box_b\varphi$.

Then K_t is a unique heat semigroup $e^{-t\Box_b}$ generated by $-\Box_b$.

(ii) We assume that H_t, $t > 0$, is a one parameter family of bounded operators on the space $L^2(\Lambda_r^{(p)(q)}(D_f))$ such that, for any $\Phi \in \Gamma_0^\infty(\Lambda_r^{(p)(q)}(D_f))$,

(ii-i) for any $T > 0$, there is a constant C depending on Φ and T such that $\|H_t\Phi\|_2 \leq C$ for all $t \in [0,T]$, where $\|\cdot\|_2$ is the L^2-norm;

(ii-ii) $H_t\Phi$ is differentiable in t;

(ii-iii) $H_t\Phi \in \mathrm{Dom}(\Box)$;

(ii-iv) $(\partial_t + \Box)H_t\Phi = 0$;

(ii-v) $H_t\Phi \to \Phi$ in L^2 as $t \to 0$;

(ii-vi) $\Box H_t\Phi = H_t\Box\Phi$.

Then H_t is a unique heat semigroup $e^{-t\Box}$ generated by $-\Box$.

Stanton proved (ii) of this proposition on the strongly pseudoconvex Siegel domain [**16**]. Her proof is applicable to our framework. To complete the proof of Theorems 1 and 2, it is sufficient to prove the following:

PROPOSITION 4.3. *(i) For any* $\varphi \in \Gamma_0^\infty(\Lambda_r^{p,q}(B_f))$, *we put*

$$(K_t\varphi)(X) := \int_{B_f} k(t, X, X')\varphi(X')dX', \tag{4.15}$$

where $dX' = dx'^1 dx'^2 \cdots dx'^{2n} du'$. *Then* K_t *is extended to a bounded operator on* $L^2(\Lambda_r^{p,q}(B_f))$ *and satisfies the conditions (i-i)–(i-vi) in Proposition 4.2. Therefore we have* $K_t = e^{-t\Box_b}$.

(ii) For any $\Phi \in \Gamma_0^\infty(\Lambda_r^{(p)(q)}(D_f))$, *we put*

$$(H_t\Phi)(Z) := \int_{D_f} h(t, Z, Z')\Phi(Z')dZ', \tag{4.16}$$

where $dZ' = dx'^1 dx'^2 \cdots dx'^{2n} du' dr'$. *Then,* H_t *is extended to a bounded operator on* $L^2(\Lambda_r^{(p)(q)}(D_f))$ *and satisfies the conditions (ii-i)–(ii-vi) in Proposition 4.2. Therefore we have* $H_t = e^{-t\Box}$.

PROOF. Since the proof of (i) is similar to that of (ii), we prove only (ii). Let $\Phi \in \Gamma_0^\infty(\Lambda_r^{(p)(q)}(D_f))$. From $(4.2)'$, we see that $H_t\Phi$ is a well defined element of $\mathcal{S}(\Lambda_r^{(p)(q)}(D_f))$. We set

$$\widehat{\Phi}(z, \xi, r) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi u}\Phi(z, u, r)du \tag{4.17}$$

and

$$(\widehat{H}_t\widehat{\Phi})(z, \xi, r) := \left\{ e^{-t\Box_b(\xi)} \int_0^\infty dr' E(t, r, r', \xi)\widehat{\Phi}(\cdot, \xi, r') \right\}(z). \tag{4.18}$$

Then, by the Parseval equality, we have

$$\|\Phi\|_2 = \|\widehat{\Phi}\|_{2,(z,\xi,r)} \qquad \text{and} \qquad \|H_t\Phi\|_2 = \|\widehat{H}_t\widehat{\Phi}\|_{2,(z,\xi,r)},$$

where $\|\cdot\|_{2,(z,\xi,r)}$ is the L^2 norm with respect to the variable (z,ξ,r). Since $\Box_b(\xi)$ is a positive operator, $e^{-t\Box_b(\xi)}$ is a contraction operator. Thus we have

$$\begin{aligned}\|\widehat{H}_t\widehat{\Phi}\|_2 &\le \Big\langle\!\Big\langle \int_0^\infty dr' E(t,r,r',\xi)\widehat{\Phi}(z,\xi,r') \Big\rangle\!\Big\rangle_{2,(z,\xi,r)} \\ &\le \|\widehat{\Phi}\|_{2,(z,\xi,r)} \sup\Big\{ \int_0^\infty dr' |E(t,r,r',\xi)| : r \ge 0\Big\}.\end{aligned}$$

We easily see that

$$\sup\Big\{ \int_0^\infty dr' e(t, r \pm r') : t > 0, r \ge 0\Big\} < \infty.$$

From (4.13), we see that

$$\sup\Big\{ \int_0^\infty dr' \xi\varepsilon(t, r + r', \xi) : t > 0, r \ge 0, \xi \ge 0\Big\} < \infty.$$

For $\xi < 0$, we estimate as follows:

$$\int_0^\infty dr(-\xi)\varepsilon(t,r,\xi) \le \int_0^\infty dr(-\xi)e^{r\xi}\sqrt{2\pi} = \sqrt{2\pi}.$$

Thus we have

$$\sup\Big\{ \int_0^\infty dr' |E(t,r,r',\xi)| : t > 0, r \ge 0\Big\} < \infty.$$

Consequently, we have

$$\|\widehat{H}_t\widehat{\Phi}\|_{2,(z,\xi,r)} \le C\|\widehat{\Phi}\|_{2,(z,\xi,r)} \tag{4.19}$$

where C is independent of t, from which we have

$$\|H_t\Phi\|_2 \le C\|\Phi\|_2. \tag{4.20}$$

Thus H_t is extended to a bounded operator on $L^2(\Lambda_r^{(p)(q)}(D_f))$.

By an easy calculation, we see

$$(\widetilde{\partial_r} - i\widetilde{\partial_u})H_t\Phi(z,u,r)\restriction_{r=0} = 0. \tag{4.21}$$

From this and (2.17), we have (ii-iii). By an easy calculation, we also see

$$\begin{aligned}\partial_t(\widehat{H}_t\widehat{\Phi})(z,\xi,r) &= \left(-\Box_b(\xi) - \frac{\xi^2}{2} + \frac{1}{2}\widetilde{\partial_r^2}\right)(\widehat{H}_t\widehat{\Phi})(z,\xi,r) \\ &= -(\widehat{H_t(\Box\Phi)})(z,\xi,r).\end{aligned} \tag{4.22}$$

Thus we see that H_t satisfies (ii-ii), (ii-iv) and (ii-vi). Furthermore, since

$$\lim_{t\to 0}(\widehat{H}_t\widehat{\Phi})(z,\xi,r)) = \widehat{\Phi}(z,\xi,r)$$

for any $(z,\xi,r) \in \widetilde{B_f} \times \mathbb{R} \times \mathbb{R}_+$, we have

$$\begin{aligned}&(H_t\Phi)(z,u,r) - \Phi(z,u,r) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi u}\{(\widehat{H}_t\widehat{\Phi})(z,\xi,r) - \widehat{\Phi}(z,\xi,r)\}d\xi \\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi u}\int_0^t (\widehat{H_t(\Box\Phi)})(z,\xi,r)dsd\xi.\end{aligned} \tag{4.23}$$

From this and (4.19), we have

$$\begin{aligned}&\|H_t\Phi - \Phi\|_2 \\ &= \left(\int_{\widetilde{B_f}\times\mathbb{R}\times\mathbb{R}_+}\left|\int_0^t (\widehat{H_t(\Box\Phi)})(z,\xi,r)ds\right|^2 dzd\xi dr\right)^{1/2} \\ &\le \int_0^t ds\left(\int_{\widetilde{B_f}\times\mathbb{R}\times\mathbb{R}_+} dzd\xi dr|(\widehat{H_t(\Box\Phi)})(z,\xi,r)|^2\right)^{1/2} \\ &\le C\int_0^t ds\left(\int_{\widetilde{B_f}\times\mathbb{R}\times\mathbb{R}_+} dzd\xi dr|\widehat{(\Box\Phi)}(z,\xi,r)|^2\right)^{1/2} \\ &= Ct\|\Box\Phi\|_2.\end{aligned} \tag{4.24}$$

From this, we see that

$$\|H_t\Phi\|_2 \le \|\Phi\|_2 + Ct\|\Box\Phi\|_2 \tag{4.25}$$

and

$$H_t\Phi \longrightarrow \Phi \ \text{ in } L^2 \ \text{ as } \ t \downarrow 0. \tag{4.26}$$

Therefore, we obtain (ii-i) and (ii-v). $\square$

5. Proof of Theorems 3 and 4

In this section, we prove the local hypoellipticity of the $\overline{\partial}$-Laplacian. We assume that the conclusion of Theorem 2 is satisfied. We assume $n_+ = n \geq 1$ and $n_- = n_0 = 0$, and consider on the subbundle $\Lambda_r^{(p)(q)}(D_f)$. Thus we regard the $\overline{\partial}$-Laplacian as an operator acting on scalar valued functions. A sufficient condition for the local hypoellipticity of differential operators without boundary conditon is given by Kusuoka-Stroock [**12**]. Following their argument, we obtain the following:

PROPOSITION 5.1. *(i) We assume*

$$\sup_{t\in(0,1]} \|e^{-t\Box}\phi\|_{2,m,K} < \infty \tag{5.1}$$

for any $\phi \in C_0^\infty(\overline{D_f})$, $m \in \mathbb{Z}_+$ *and compact* $K \subset \overline{D_f}$, *where*

$$\|\phi\|_{2,m,K} = \sum_{\alpha\in\mathbb{Z}_+^{2n+2}:|\alpha|\leq m} \|\partial_Z^\alpha \phi\|_{2,K}$$

and $\|\cdot\|_{2,K}$ *is the* L^2*-norm on* K. *We also assume that*

$$\sup\{|\partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t,Z,Z')| : t \in (0,1], Z, Z' \in K, |Z-Z'| \geq \varepsilon\} < \infty \tag{5.2}$$

for any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n+2}$ *such that* $|\alpha'| \leq 1$ *and compact* $K \subset \overline{D_f}$, *where, for* $Z = (z,u,r)$, $|Z| = \sqrt{|z|^2 + u^2 + r^2}$ *is the Euclidean distance. Then* $\Box$ *is locally hypoelliptic in the sense of Theorem 3.*

(ii) For any $m \in \mathbb{Z}_+$ *and compact* $K \subset \overline{D_f}$, *we assume there are* $m' \in \mathbb{Z}_+$ *and* $C > 0$ *such that*

$$\sup_{t\in(0,1]} \|e^{-t\Box}\phi\|_{2,m,K} \leq C\|\phi\|_{2,m'} \tag{5.3}$$

for all $\phi \in C_0^\infty(\overline{D_f})$, *where* $\|\cdot\|_{2,m'} = \|\cdot\|_{2,m',\overline{D_f}}$. *We also assume that*

$$(5.4) \quad \sup\left\{\left|\frac{1}{t^k}\partial_t^m \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t,Z,Z')\right| : t \in (0,1],\, Z, Z' \in K, |Z-Z'| \geq \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n+2}$ k, $m \in \mathbb{Z}_+$ *and compact* $K \subset \overline{D_f}$. *Then* $\partial_t + \Box$ *is locally hypoelliptic in the sense of Theorem 4.*

PROOF. (i) Let $u \in \mathrm{Dom}(\Box)$ such that $\Box u$ is smooth on an open set W in $\overline{D_f}$. For any $Z_0 \in W$, we take $\varepsilon > 0$ satisfying $B(Z_0, 3\varepsilon) \subset W$, where $B(Z_0,\varepsilon) = \{Z \in \overline{D_f} : |Z - Z_0| < \varepsilon\}$. We will show that u is smooth on $B(Z_0,\varepsilon)$. We choose $\eta \in C_0^\infty(B(Z_0, 3\varepsilon))$ so that $\eta \equiv 1$ on $B(Z_0, 2\varepsilon)$ and set $\tilde{u}(Z) := \eta(Z)u(Z) \in L^2(D_f)$. Since $e^{-t\Box}$ is a C^0-semigroup, we have

$$e^{-t\Box}\tilde{u} \longrightarrow \tilde{u} \quad \text{in } L^2 \text{ as } t \downarrow 0.$$

Then it is sufficient to show

$$\sup_{t\in(0,1]} \|\partial_Z^\alpha (e^{-t\Box}\tilde{u})(Z)\|_{2,(\varepsilon)} < \infty$$

for any $\alpha \in \mathbb{Z}_+^{2n+2}$, where $\|\cdot\|_{2,(\varepsilon)}$ is the L^2-norm on $B(Z_0,\varepsilon)$. Since

$$e^{-\Box_\sim}\tilde{u} - e^{-t\Box}\tilde{u} = \int_t^1 \partial_\tau e^{-\tau\Box}\tilde{u}\, d\tau$$

for any $t \in (0,1]$ and

$$\|\partial_Z^\alpha e^{-\Box_\sim}\tilde{u}\|_{2,(\varepsilon)} < \infty$$

by (5.1), it is sufficient to show that

$$\sup_{t\in(0,1]} \|\partial_Z^\alpha (e^{-t\Box}\tilde{u})(Z)\|_{2,(\varepsilon)} < \infty.$$

We show this by decomposing as follows:

$$\begin{aligned}
&\partial_t e^{-t\Box}\tilde{u}(Z)\\
&= -\int_{D_f}\{\eta(Z')\overline{\Box_{Z'}}h(t,Z,Z')\}u(Z')dZ'\\
&= -\int_{D_f}\{\overline{\Box_{Z'}}\eta(Z')h(t,Z,Z')\}u(Z')dZ'\\
&\quad+\int_{D_f}\{[\overline{\Box_{Z'}},\eta(Z')]h(t,Z,Z')\}u(Z')dZ'\\
&=: -\tilde{u}_t(Z)+v_t(Z),
\end{aligned}$$

where $\overline{\Box_{Z'}}$ is the complex conjugate of $\Box$ regarded as a differential operator (1.2) without boundary condition acting on the variable Z' and $[\cdot,\cdot]$ is the commutator. Since u belongs to $\mathrm{Dom}(\Box)$, we have

$$\tilde{u}_t(Z)=(e^{-t\Box}\eta\Box u)(Z).$$

Since $\eta\Box u$ is smooth, we have

$$\sup_{t\in(0,1]}\|\partial_Z^\alpha\tilde{u}_t(Z)\|_{2,(\varepsilon)}<\infty$$

from (5.1). On the other hand, we remark that

$$[\overline{\Box_Z},\eta(Z)]=a_0(Z)+\sum_{j=1}^{2n+2}a_j(Z)\partial_{x^j},\quad\text{where}\quad\mathrm{supp}\,a_j\subset B(Z_0,3\varepsilon)-\overline{B(Z_0,2\varepsilon)},$$

and $x^{2n+1}=u$, $x^{2n+2}=r$. Then, from (5.2), we have

$$\sup_{t\in(0,1]}\|\partial_Z^\alpha v_t(Z)\|_{2,(\varepsilon)}<\infty.$$

(ii) Let $u\in\mathcal{D}'(\mathbb{R}\times\overline{D_f})$ such that $L_u(\mathcal{D}(\mathbb{R}))\subset\mathrm{Dom}(\Box)$ and $\partial_t u+\Box u$ is smooth on an open set W in $\mathbb{R}\times\overline{D_f}$. For any $(s_0,Z_0)\in\mathbb{R}\times\overline{D_f}$, we take $\varepsilon\in(0,1/2]$ satisfying $(s_0-5\varepsilon,s_0+5\varepsilon)\times B(Z_0,5\varepsilon)\subset W$. We will show that u is smooth on

$(s_0-\varepsilon, s_0+\varepsilon)\times B(Z_0,\varepsilon)$. We choose $\eta \in C_0^\infty((s_0-4\varepsilon, s_0+4\varepsilon)\times B(Z_0,4\varepsilon))$ so that $\eta \equiv 1$ on $[s_0-3\varepsilon, s_0+3\varepsilon]\times\overline{B(Z_0,3\varepsilon)}$ and set $\tilde{u}(s,Z) := \eta(s,Z)u(s,Z) \in \mathcal{E}'(\mathbb{R}\times\overline{D_f})$, the space of all distributions with compact support. For any $\tau \in (0,1/2)$, we set

$$\tilde{u}_\tau(s,Z) :=< \rho_\tau(s-s')h(s-s',Z,Z'), \overline{\tilde{u}(s',Z')} >_{(s',Z')}$$

where $< \cdot,\cdot >_{(s',Z')}$ is the pairing with respect to the variable (s',Z'), $\rho_\tau(t) = \rho(t/\tau)/\tau$ and ρ is a function on $\mathbb{R}$ such that $\operatorname{supp}\rho \subset (1,2)$ and $\int_{\mathbb{R}}\rho(t)dt = 1$. Then we see that $\tilde{u}_\tau \in \mathcal{S}(\mathbb{R}\times\overline{D_f})$.

We see that $\tilde{u}_\tau \to \tilde{u}$ in $\mathcal{D}'(\mathbb{R}\times\overline{D_f})$ as $\tau \downarrow 0$. In fact, for any $\phi \in \mathcal{D}(\mathbb{R}\times\overline{D_f})$, we have

$$\int_{\mathbb{R}\times D_f} \phi(s,Z)\overline{\tilde{u}_\tau(s,Z)}dsdZ =< \phi_\tau, \tilde{u} >,$$

where

$$\phi_\tau(s',Z') = \int_{\mathbb{R}\times D_f} \rho_\tau(s-s')\phi(s,Z)\overline{h(s-s',Z,Z')}dsdZ.$$

From (5.3), we see that $\{\phi_\tau : 0 < \tau \le 1/2\}$ is relatively compact in $\mathcal{E}(\mathbb{R}\times\overline{D_f})$. For any $\psi \in \mathcal{D}(\mathbb{R}\times\overline{D_f})$, we have

$$(\phi_\tau,\psi)_2 = \int_{\mathbb{R}^2} \rho(s')(\phi(s,\cdot), \{e^{-\tau s'\Box}\psi(s+\tau s',\cdot)\})_2 ds'ds.$$

Since $e^{-s\Box}$ is a C_0-semigroup, we have

$$\lim_{\tau\downarrow 0}(\phi_\tau,\psi)_2 = (\phi,\psi)_2,$$

from which we have $\phi_\tau \to \phi$ in $\mathcal{D}'(\mathbb{R}\times\overline{D_f})$ as $\tau\downarrow 0$ and so in $\mathcal{E}(\mathbb{R}\times\overline{D_f})$. Thus we have

$$\lim_{\tau\downarrow 0}\int_{\mathbb{R}\times D_f} \phi(s,Z)\overline{\tilde{u}_\tau(s,Z)}dsdZ =< \phi, \tilde{u} > .$$

Then it is sufficient to show that

$$\sup_{\tau\in(0,T]} \|\partial^{\alpha}_{(s,Z)}\tilde{u}_{\tau}(s,Z)\|_{2,(\varepsilon)} < \infty$$

for any $\alpha \in \mathbb{Z}_{+}^{2n+3}$ and some $T > 0$, where $\|\cdot\|_{2,(\varepsilon)}$ is the L^2-norm on $(s_0-\varepsilon, s_0+\varepsilon)\times B(Z_0,\varepsilon)$. Since $\tilde{u}_{5\varepsilon}(s,Z) = 0$ for $s \in (s_0-\varepsilon, s_0+\varepsilon)$, we have

$$\tilde{u}_{\tau}(s,Z) = -\int_{\tau}^{5\varepsilon} d\sigma \partial_{\sigma}\tilde{u}_{\sigma}(s,Z)$$

for $0 < \tau \leq 5\varepsilon$ and $s \in (s_0-\varepsilon, s_0+\varepsilon)$. Thus it is sufficient to show that

$$\sup_{0<\tau\leq 5\varepsilon} \|\partial^{\alpha}_{(s,Z)}\partial_{\tau}\tilde{u}_{\tau}(s,Z)\|_{2,(\varepsilon)} < \infty.$$

We show this by decomposing as follows:

$$\begin{aligned}
&\partial_{\tau}\tilde{u}_{\tau}(s,Z)\\
&= < (\partial_{s'} - \overline{\Box_{Z'}})\eta(s',Z')\tilde{\rho}_{\tau}(s-s')h(s-s',Z,Z'), \overline{u(s',Z')} >_{(s',Z')}\\
&\quad + < [-\partial_{s'} + \overline{\Box_{Z'}}, \eta(s',Z')]\tilde{\rho}_{\tau}(s-s')h(s-s',Z,Z'), \overline{u(s',Z')} >_{(s',Z')}\\
&=: f_{\tau}(s,Z) + v_{\tau}(s,Z),
\end{aligned}$$

where $\tilde{\rho}_{\tau}(s) = (s/\tau^2)\rho(s/\tau)$.

For any $\phi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\overline{D_f})$, we have

$$< \Box(\phi\psi), u > = < \phi\psi, \Box u >,$$

since $L_u(\phi) \in \mathrm{Dom}(\Box)$. Since the algebraic tensor product $\mathcal{D}(\mathbb{R})\otimes\mathcal{D}(\overline{D_f})$ is dense in $\mathcal{D}(\mathbb{R}\times\overline{D_f})$, we have

$$< \Box\varphi, u > = < \varphi, \Box u >$$

for any $\varphi \in \mathcal{D}(\mathbb{R}\times\overline{D_f})$. Thus we have

$$f_{\tau}(s,Z) = - < \tilde{\rho}_{\tau}(s-s')h(s-s',Z,Z'), \overline{\eta(\partial_{s'} + \Box_{Z'})u(s',Z')} >_{(s',Z')} .$$

Since $\eta(\partial_s + \Box)u$ is smooth, we have

$$\sup_{\tau\in(0,5\varepsilon]} \|\partial^{\alpha}_{(s,Z)} f_\tau(s,Z)\|_{2,(\varepsilon)} < \infty$$

from (5.3). On the other hand, we remark that

$$[-\partial_s + \overline{\Box_Z}, \eta(s,Z)] = a_0(s,Z) + \sum_{j=1}^{2n+2} a_j(s,Z)\partial_{x^j},$$

where

$$\operatorname{supp} a_j \subset (s_0-4\varepsilon, s_0+4\varepsilon)\times B(Z_0,4\varepsilon) - [s_0-3\varepsilon, s_0+3\varepsilon]\times\overline{B(Z_0,2\varepsilon)}.$$

Then to prove

$$\sup_{\tau\in(0,5\varepsilon]} \|\partial^{\alpha}_{(s,Z)} v_\tau(s,Z)\|_{2,(\varepsilon)} < \infty,$$

it is sufficient to show that

$$\begin{aligned}
\sup\Big\{\Big\langle\Big\langle \frac{1}{(s-s')^k}\partial_s^\beta\partial_Z^\alpha\partial_{Z'}^{\alpha'} h(s-s',Z,Z')\Big\rangle\Big\rangle \\
: 0<\tau\le 5\varepsilon, \tau < s-s' < 2\tau, (s,Z)\in(s_0-\varepsilon,s_0+\varepsilon)\times B(Z_0,\varepsilon), \\
(s',Z')\in(s_0-4\varepsilon,s_0+4\varepsilon)\times B(Z_0,4\varepsilon) \\
- [s_0-3\varepsilon,s_0+3\varepsilon]\times\overline{B(Z_0,3\varepsilon)}\Big\} \\
<\infty
\end{aligned}$$

for any $\alpha,\ \alpha' \in \mathbb{Z}_+^{2n},\ \beta,\ k \in \mathbb{Z}_+$. Under the condition in the above supremum,

$\tau \le \varepsilon$ implies $s' \in [s_0 - 3\varepsilon, s_0 + 3\varepsilon]$. Thus, from (5.4), we have

$$\begin{aligned}
&\sup\Big\{\Big\langle\!\Big\langle \frac{1}{(s-s')^k}\partial_s^\beta \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(s-s', Z, Z')\Big\rangle\!\Big\rangle \\
&\qquad : 0 < \tau \le \varepsilon, \tau < s - s' < 2\tau, (s, Z) \in (s_0 - \varepsilon, s_0 + \varepsilon) \times B(Z_0, \varepsilon), \\
&\qquad\quad (s', Z') \in (s_0 - 4\varepsilon, s_0 + 4\varepsilon) \times B(Z_0, 4\varepsilon) \\
&\qquad\qquad - [s_0 - 3\varepsilon, s_0 + 3\varepsilon] \times \overline{B(Z_0, 3\varepsilon)}\Big\} \\
\le &\sup\Big\{\Big\langle\!\Big\langle \frac{1}{t^k}\partial_t^\beta \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(s-s', Z, Z')\Big\rangle\!\Big\rangle \\
&\qquad : 0 < t \le 2\varepsilon, Z \in B(Z_0, \varepsilon), Z' \in B(Z_0, 4\varepsilon) - \overline{B(Z_0, 3\varepsilon)}\Big\} \\
< &\infty.
\end{aligned}$$

On the other hand, since $h(t, Z, Z')$ is smooth on $(0, \infty) \times \overline{D_f} \times \overline{D_f}$, we have

$$\begin{aligned}
&\sup\Big\{\Big\langle\!\Big\langle \frac{1}{(s-s')^k}\partial_s^\beta \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(s-s', Z, Z')\Big\rangle\!\Big\rangle \\
&\qquad : \varepsilon \le \tau \le 5\varepsilon, \tau < s - s' < 2\tau, (s, Z) \in (s_0 - \varepsilon, s_0 + \varepsilon) \times B(Z_0, \varepsilon), \\
&\qquad\quad (s', Z') \in (s_0 - 4\varepsilon, s_0 + 4\varepsilon) \times B(Z_0, 4\varepsilon) \\
&\qquad\qquad - [s_0 - 3\varepsilon, s_0 + 3\varepsilon] \times \overline{B(Z_0, 3\varepsilon)}\Big\} \\
\le &C_\varepsilon \sup\Big\{\Big\langle\!\Big\langle \frac{1}{t^k}\partial_t^\beta \partial_Z^\alpha \partial_{Z'}^{\alpha'} h(s-s', Z, Z')\Big\rangle\!\Big\rangle \\
&\qquad : \varepsilon < t \le 5\varepsilon, Z \in B(Z_0, \varepsilon), Z' \in B(Z_0, 4\varepsilon)\Big\} \\
< &\infty. \quad \square
\end{aligned}$$

Remark 5.1. (i) As we show in Proposition 5.3 below, we can choose $m' = m$ in (5.3). Therefore, from our proof of Proposition 5.1 (i), we have the following: for any m, $k \in \mathbb{Z}_+$, there is $C > 0$ such that

$$\|\phi\|_{2,m,B(Z_0,\varepsilon)} \le C(\|\eta \Box \phi\|_{2,m} + \|\phi\|_{2,-k})$$

for any $\phi \in C_0^\infty(\overline{D_f})$, where

$$\|\phi\|_{2,-k} = \sup\{|(\phi, \psi)_2| : \psi \in C_0^\infty(\overline{D_f}), \|\psi\|_{2,k} \le 1\}$$

(cf.[**4**] Proposition (3.1.1), [**10**] Theorem 5.14, [**1**], [**2**]).

(ii) In our proof of (ii), the definition of $\tilde{u}_\tau$ is different from that of [**12**], because the proof of [**12**] corresponds to a proof for $\partial_t - \Box$. This difference is not important because the hypoellipticity of $\partial_t - \Box$ is equivalent to that of $\partial_t + \Box$.

We now consider the proof of Theorems 3 and 4. Since (5.1) and (5.2) are included in (5.3) and (5.4), respectively, it is sufficient to show (5.3) and (5.4). Since

$$\partial_t^m h(t, Z, Z') = (-\Box)^m h(t, Z, Z'),$$

it is sufficient to let $m = 0$ in (5.4). To show (5.4) with $m = 0$, it is sufficient to show

$$\sup\left\{\left|\frac{1}{t^k}\partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t, Z, Z')\right| : t \in (0, 1], Z, Z' \in K, |z - z'| \geq \varepsilon\right\} < \infty, \tag{5.5}$$

$$\sup\left\{\left|\frac{1}{t^k}\partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t, Z, Z')\right| : t \in (0, 1], Z, Z' \in K, |u - u'| \geq \varepsilon\right\} < \infty, \tag{5.6}$$

and

$$\sup\left\{\left|\frac{1}{t^k}\partial_Z^\alpha \partial_{Z'}^{\alpha'} h(t, Z, Z')\right| : t \in (0, 1], Z, Z' \in K, |r - r'| \geq \varepsilon\right\} < \infty, \tag{5.7}$$

for any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n+2}$, $k \in \mathbb{Z}_+$ and compact $K \subset \overline{D_f}$. Therefore we show (5.3), (5.5), (5.6) and (5.7).

For (5.5) and (5.7), we have the following:

PROPOSITION 5.2. *We assume that for any $p \geq 1$ and compact $K \subset \widetilde{B_f}$, there are constants $C > 0$ and $\lambda < 1$ such that*

$$B(p, t; K) < C \exp\left(\frac{1}{t^\lambda}\right) \tag{5.8}$$

for all $t > 0$, where $B(p,t;K)$ is the quantity defined in (3.35). *Then, we have the following:*

(i) For any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, β', Ω, Ω', $k \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$, we have

$$\begin{aligned} &\sup\Big\{\Big|\frac{1}{t^k}\partial_z^\alpha \partial_u^\beta \partial_r^\Omega \partial_{z'}^{\alpha'} \partial_u^{\beta'} \partial_r^{\Omega'} h(t,(z,u,r),(z',u',r'))\Big| \\ &\qquad : t \in (0,1], z, z' \in K, |z-z'| \geq \varepsilon, u, u' \in \mathbb{R}, r, r' \geq 0\Big\} \\ &< \infty. \end{aligned} \tag{5.9}$$

(ii) For any $\kappa > \varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, β', Ω, Ω', $k \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$, we have

$$\begin{aligned} &\sup\Big\{\Big|\frac{1}{t^k}\partial_z^\alpha \partial_u^\beta \partial_r^\Omega \partial_{z'}^{\alpha'} \partial_u^{\beta'} \partial_r^{\Omega'} h(t,(z,u,r),(z',u',r'))\Big| \\ &\qquad : t \in (0,1], z, z' \in K, u, u' \in \mathbb{R}, \varepsilon \leq |r-r'| \leq \kappa\Big\} \\ &< \infty. \end{aligned} \tag{5.10}$$

Remark 5.2. The assumption of this proposition is satisfied if we assume either $(E,\rho)_+$ for some $\rho < 1/2$ or $(E,\rho)_+$ and $(H,\kappa)_+$ for some $\rho < 1$ and $\kappa > 1$ (cf. Proposition 3.3).

Proof of Proposition 5.2. (i) It is sufficient to show that

$$\begin{aligned} &\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha \partial_{z'}^{\alpha'} q(t,z,z';\xi)||\partial_r^\Omega \partial_{r'}^{\Omega'} E(t,r,r',\xi)| \\ &\qquad : t \in (0,1], z, z' \in K, |z-z'| \geq \varepsilon, \xi \in \mathbb{R}, r, r' \geq 0\Big\} \\ &< \infty \end{aligned} \tag{5.11}$$

for any α, $\alpha' \in \mathbb{Z}_+^{2n}$ and β, Ω, Ω', $k \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$. To show this, by

Lemma 4.1, it is sufficient to show that

$$\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)|\exp\Big(-\frac{\xi^2 t}{2}\Big) : t\in(0,1], z,z'\in K, |z-z'|\ge\varepsilon, \xi\in\mathbb{R}\Big\} \tag{5.12}$$

$$<\infty$$

and

$$\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)| : t\in(0,1], z,z'\in K, |z-z'|\ge\varepsilon, \xi\le 0\Big\} \tag{5.13}$$

$$<\infty$$

for any α, $\alpha'\in\mathbb{Z}_+^{2n}$, β, $k\in\mathbb{Z}_+$ and compact $K\subset\widetilde{B_f}$. (5.12) is easily shown by using Lemma 3.1. For (5.13), we use Proposition 3.2 (ii) to have

$$\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)| : t\in(0,1], z,z'\in K, |z-z'|\ge\varepsilon, \xi\le 0\Big\} \tag{5.14}$$

$$\le\sup\Big\{\mathcal{P}\Big(\frac{1}{t}\Big)\Big(\sum_{p\in S}B\Big(p,\frac{t}{2};K\Big)\Big)^{1/3}e^{-\varepsilon^2/(3t)} : t\in(0,1]\Big\}.$$

From the assumption of this proposition, we see this is finite.

(ii) It is sufficient to show that

$$\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)||\partial_r^\Omega\partial_{r'}^{\Omega'}E(t,r,r',\xi)| : t\in(0,1], z,z'\in K, \xi\in\mathbb{R}, \varepsilon\le|r-r'|\le\kappa\Big\} \tag{5.15}$$

$$<\infty$$

for any α, $\alpha'\in\mathbb{Z}_+^{2n}$and β, Ω, Ω', $k\in\mathbb{Z}_+$ and compact $K\subset\widetilde{B_f}$. To do this, by Lemma 4.1, it is sufficient to show that

$$\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)|\exp\Big(-\frac{\varepsilon^2}{2t}-\frac{\xi^2 t}{2}\Big) : t\in(0,1], z,z'\in K, \xi\in\mathbb{R}\Big\} \tag{5.16}$$

$$<\infty$$

and

$$\sup\Big\{\frac{|\xi|^\beta}{t^k}|\partial_z^\alpha\partial_{z'}^{\alpha'}q(t,z,z';\xi)|\exp\Big(-\frac{\varepsilon^2}{2t}\Big) \tag{5.17}$$
$$: t\in(0,1], z,z'\in K, |z-z'|\ge\varepsilon, \xi\le-\frac{\varepsilon}{t}\Big\}$$
$$<\infty$$

for any α, $\alpha'\in\mathbb{Z}_+^{2n}$, β, $k\in\mathbb{Z}_+$ and compact $K\subset\widetilde{B_f}$. Then the rest of the proof is the same with that of (i). $\square$

For (5.3), we have the following:

PROPOSITION 5.3. *Under the same assumption as for Proposition 5.2, for any $m\in\mathbb{Z}_+$ and compact $K\subset\overline{D_f}$, there is a constant $C>0$ such that*

$$\|e^{-t\Box}\phi\|_{2,m,K}\le C\|\phi\|_{2,m} \tag{5.18}$$

for all $t>0$ and $\phi\in C_0^\infty(\overline{D_f})$.

PROOF. We take $\eta\in C_0^\infty(\overline{D_f})$ such that $\eta\equiv1$ on $\{z\in\widetilde{B_f}:|z-z'|\le1$ for some $(z',u',r')\in K\}$. We take k, l, $m\in\mathbb{Z}_+$ and $\phi\in C_0^\infty(\overline{D_f})$ arbitrarily. By using Lemma 5.1 below, we have

$$\begin{aligned}
&\|(1-\Delta_z)^k\eta(z)(1-\partial_u^2)^q(1-\partial_r^2)^m(e^{-t\Box}\phi)(z,u,r)\|_2\\
=&\|(1-\Delta_z)^k\eta(z)(1+\xi^2)^q(1-\partial_r^2)^m(\widehat{H}_t\widehat{\phi})(z,\xi,r)\|_{2,(z,\xi,r)}\\
\le&C\|(1+\xi^2+\Box_b(\xi))^k\eta(z)(1+\xi^2)^q(1-\partial_r^2)^m(\widehat{H}_t\widehat{\phi})(z,\xi,r)\|_{2,(z,\xi,r)},
\end{aligned}$$

where $\Delta_z=\sum_{j=1}^{2n}\partial_{x^j}^2$. We divide the right hand side as

$$\begin{aligned}
&\|(1+\xi^2+\Box_b(\xi))^k\eta(z)(1+\xi^2)^q(1-\partial_r^2)^m(\widehat{H}_t\widehat{\phi})(z,\xi,r)\|_{2,(z,\xi,r)}\\
\le&\|\eta(z)(1+\xi^2+\Box_b(\xi))^k(1+\xi^2)^q(1-\partial_r^2)^m(\widehat{H}_t\widehat{\phi})(z,\xi,r)\|_{2,(z,\xi,r)}\\
&+\|[(1+\xi^2+\Box_b(\xi))^k,\eta(z)](1+\xi^2)^q(1-\partial_r^2)^m(\widehat{H}_t\widehat{\phi})(z,\xi,r)\|_{2,(z,\xi,r)}\\
=:&I_1+I_2.
\end{aligned}$$

By the commutativity, we have

$$(1+\xi^2+\Box_b(\xi))^k(1+\xi^2)^q(1-\partial_r^2)^m(\widehat{H}_t\widehat{\phi})(z,\xi,r)=(\widehat{H}_t\widehat{\phi_*})(z,\xi,r),$$

where

$$\phi_*:=(1-\partial_u^2+\Box_b)^k(1-\partial_u^2)^q(1-\partial_r^2)^m\phi(z,u,r).$$

Then, as in the proof of Proposition 4.3, we have

$$I_1\le C\|\phi_*\|_2.$$

On the other hand, we use (5.11) to obtain

$$I_2\le C\|(1-\partial_u^2)^q(1-\partial_r^2)^m\phi\|_2. \quad \Box$$

LEMMA 5.1. *For any $k\in\mathbb{Z}_+$ and compact $K\subset\widetilde{B_f}$, there is a constant C such that*

$$\|(1-\Delta_z)^k\phi(z)\|_2\le C\|(1+\xi^2+\Box_b(\xi))^k\phi(z)\|_{2,z} \tag{5.19}$$

for all $\xi\in\mathbb{R}$ and $\phi\in C_0^\infty(\widetilde{B_f})$ satisfying $\operatorname{supp}\phi\subset K$.

PROOF. We prove by the induction. Since $\Box_b(\xi)$ is written as

$$\Box_b(\xi)=-\frac{1}{4}\Delta_z+\sum_{j=1}^{2n}i\xi A_j(z)\partial_{x^j}+\xi B(z)+\xi^2C(z)$$

with some functions $A_j(z)$, $B(z)$ and $C(z)$, there is a constant C such that

$$\|(1-\Delta_z)\phi(z)\|_2\le C(\|(1+|\xi|^2+\Box_b(\xi))\phi(z)\|_{2,z}+|\xi|\|\nabla\phi(z)\|_{2,z}).$$

By using the Fourier transform, we have

$$\|\nabla\phi(z)\|_2\le\varepsilon\|(1-\Delta_z)\phi(z)\|_2+\frac{1}{4\varepsilon}\|\phi\|_2,$$

for any $\varepsilon > 0$. By taking ε appropriately, we have

$$\|(1-\Delta_z)\phi(z)\|_2 \le C\|(1+|\xi|^2+\Box_b(\xi))\phi(z)\|_{2,z}.$$

Let $k \ge 2$. We assume that

$$\|(1-\Delta_z)^{k'}\phi(z)\|_2 \le C\|(1+|\xi|^2+\Box_b(\xi))^{k'}\phi(z)\|_{2,z},$$

for any $k' < k$. Then we have

$$\begin{aligned}\|(1-\Delta_z)^k\phi(z)\|_2 \le& C\|(1+|\xi|^2+\Box_b(\xi))^{k-1}(1-\Delta_z)\phi(z)\|_{2,z}\\ \le& C\|(1-\Delta_z)(1+|\xi|^2+\Box_b(\xi))^{k-1}\phi(z)\|_{2,z}\\ &+C\|[\Delta_z,(1+|\xi|^2+\Box_b(\xi))^{k-1}]\phi(z)\|_{2,z}.\end{aligned}$$

The first term is dominated by $\|(1+|\xi|^2+\Box_b(\xi))^k\phi(z)\|_{2,z}$. Since $1+|\xi|^2+\Box_b(\xi)$ has the form

$$1+|\xi|^2+\Box_b(\xi) = \sum_{|\alpha|+l\le 2} A'_{\alpha,l}(z)\xi^l\partial_z^\alpha,$$

$[\Delta_z,(1+|\xi|^2+\Box_b(\xi))^{k-1}]$ has the form

$$[\Delta_z,(1+|\xi|^2+\Box_b(\xi))^{k-1}] = \sum_{|\alpha|+l\le 2k-1} A_{\alpha,l}(z)\xi^l\partial_z^\alpha.$$

Thus we have

$$\begin{aligned}&\|[\Delta_z,(1+|\xi|^2+\Box_b(\xi))^{k-1}]\phi(z)\|_{2,z}\\ \le& C\sum_{l+m\le 2k-1}|\xi|^l\|\nabla^m\phi(z)\|_2.\end{aligned}$$

By using the hypothesis of the induction, we have

$$\sum_{\substack{l+m\le 2k-1\\ m\ne 2k-1}}|\xi|^l\|\nabla^m\phi(z)\|_2 \le C\|(1+|\xi|^2+\Box_b(\xi))^k\phi(z)\|_{2,z}.$$

On the other hand, as in the case of $k=1$, we have

$$\begin{aligned}\|\nabla^{2k-1}\phi(z)\|_2 \le& \varepsilon\|(1-\Delta_z)\nabla^{2k-2}\phi(z)\|_2+\frac{1}{4\varepsilon}\|\nabla^{2k-2}\phi\|_2\\ \le& C(\varepsilon\|(1-\Delta_z)^k\phi(z)\|_2+\frac{1}{4\varepsilon}\|(1-\Delta_z)^{k-1}\phi\|_2\end{aligned}$$

for any $\varepsilon > 0$. By taking ε appropriately and using the hypothesis of the induction, we obtain (5.19) for k. $\square$

For (5.6), we have the following:

PROPOSITION 5.4. *We assume that, for any $p \geq 1$ and compact $K \subset \widetilde{B_f}$, there are constants C and $\lambda < 1/2$ such that*

$$B(p,t;K) < C \exp\left(\frac{1}{t^\lambda}\right) \tag{5.20}$$

for all $t > 0$, where $B(p,t;K)$ is the quantity defined in (3.35). *Then we have*

$$\begin{aligned} &\sup\Big\{\Big|\frac{1}{t^k}\partial_z^\alpha \partial_u^\beta \partial_r^\Omega \partial_{z'}^{\alpha'} \partial_u^{\beta'} \partial_r^{\Omega'} h(t,(z,u,r),(z',u',r'))\Big| \\ &\qquad : t \in (0,1], z, z' \in K, |u-u'| \geq \varepsilon, r, r' \geq 0\Big\} \\ &< \infty. \end{aligned} \tag{5.21}$$

for any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, β', Ω, Ω', $k \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$.

REMARK 5.3. The assumption of this proposition is satisfied if we assume either $(E,\rho)_+$ for some $\rho < 1/3$ or $(E,\rho)_+$ and $(H,\kappa)_+$ for some $\rho < 2/3$ and $\kappa > 1$ (cf. Proposition 3.3).

PROOF. For large enough $R > 0$, we choose a smooth function $\tilde{f} : \widetilde{B_f} \to \mathbb{R}^{2n}$ with bounded derivatives of all orders such that $\tilde{f} = \nabla f$ on $\{z \in \widetilde{B_f} : |z| \leq R\}$.

We divide $q(t,z,z';\xi)$ as follows:

$$
\begin{aligned}
&q(t,z,z';\xi)\\
=&E\Big[\exp\Big(-i\xi\int_0^t \tilde f(z+z(s))*dz(s)\\
&\qquad +\xi\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big]\\
&+E\Big[\Big\{\exp\Big(-i\xi\int_0^t \nabla f(z+z(s))*dz(s)\Big)\\
&\qquad -\exp\Big(-i\xi\int_0^t \tilde f(z+z(s))*dz(s)\Big)\Big\}\\
&\qquad \times\exp\Big(\xi\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big]\\
=:&\tilde q(t,z,z';\xi)+q^d(t,z,z',\xi)
\end{aligned}
\tag{5.22}
$$

and, accordingly,

$$
\begin{aligned}
&h(t,(z,u,r),(z',u',r'))\\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\tilde q(t,z,z';\xi)E(t,r,r',\xi)\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi q^d(t,z,z',\xi)E(t,r,r',\xi)\\
=:&\tilde h(t,(z,u,r),(z',u',r'))+h^d(t,(z,u,r),(z',u',r')).
\end{aligned}
\tag{5.23}
$$

By a proof similar to that of Proposition 5.2 (ii), we have

$$
\begin{aligned}
&\sup\Big\{\Big|\frac{1}{t^k}\partial_z^\alpha\partial_u^\beta\partial_r^\Omega\partial_{z'}^{\alpha'}\partial_u^{\beta'}\partial_r^{\Omega'}h^d(t,(z,u,r),(z',u',r'))\Big|\\
&\qquad : t\in(0,1],z,z'\in K,u,u'\in\mathbb{R},r,r'\ge 0\Big\}\\
<&\infty
\end{aligned}
\tag{5.24}
$$

for any $\varepsilon>0$, α, $\alpha'\in\mathbb{Z}_+^{2n}$, β, β', Ω, Ω', $k\in\mathbb{Z}_+$ and compact $K\subset\widetilde{B_f}$. We estimate $\tilde h(t,Z,Z')$.

We may assume that $u-u'\ge\varepsilon$. We make the change of the variable ($\xi\to\xi/\sqrt{t}$):

$$
\begin{aligned}
&\tilde h(t,(z,u,r),(z',u',r'))\\
=&C\int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{t}}\exp\Big(-i\xi\frac{u-u'}{\sqrt{t}}\Big)\tilde q(t,z,z',\xi/\sqrt{t})E(t,r,r',\xi/\sqrt{t}).
\end{aligned}
$$

By using the Taylor expansion

$$\exp\Big(\frac{u-u'}{\sqrt{t}}\Big)=\sum_{N=0}^{\infty}\frac{1}{N!}\Big(\frac{u-u'}{\sqrt{t}}\Big)^N,$$

we have

$$\begin{aligned}
&\tilde{h}(t,(z,u,r),(z',u',r'))\\
&=\exp\Big(-\frac{u-u'}{\sqrt{t}}\Big)\sum_{N=0}^{\infty}\frac{1}{N!}C\int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{t}}\Big\{(i\partial_\xi)^N\exp\Big(-i\xi\frac{u-u'}{\sqrt{t}}\Big)\Big\}\\
&\qquad\times\tilde{q}(t,z,z',\xi/\sqrt{t})E\Big(t,r,r',\frac{\xi}{\sqrt{t}}\Big)\\
&=\exp\Big(-\frac{u-u'}{\sqrt{t}}\Big)\sum_{N=0}^{\infty}\frac{1}{N!}C\int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{t}}\exp\Big(-i\xi\frac{u-u'}{\sqrt{t}}\Big)\\
&\qquad\times(-i\partial_\xi)^N\Big\{\tilde{q}(t,z,z',\xi/\sqrt{t})E\Big(t,r,r',\frac{\xi}{\sqrt{t}}\Big)\Big\}.
\end{aligned}$$

Thus it is sufficient to show that

$$\begin{aligned}
&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\int_{-\infty}^{\infty}d\xi\sum_{N=0}^{\infty}\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z',\xi/\sqrt{t})\\
&\qquad\times e^{-\xi^2/2}\partial_r^\Omega e(t,r)\Big\}\Big|:t\in(0,1],z,z'\in K,r\geq 0\Big\}\\
&<\infty
\end{aligned}\tag{5.25}$$

and

$$\begin{aligned}
&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\int_{-\infty}^{\infty}d\xi\sum_{N=0}^{\infty}\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z',\xi/\sqrt{t})\\
&\qquad\times\partial_r^\Omega\varepsilon(t,r,\xi/\sqrt{t})\Big\}\Big|:t\in(0,1],z,z'\in K,r\geq 0\Big\}\\
&<\infty
\end{aligned}\tag{5.26}$$

for any $\varepsilon>0$, α, $\alpha'\in\mathbb{Z}_+^{2n}$, β, Ω, $k\in\mathbb{Z}_+$ and compact $K\subset\widetilde{B_f}$.

By the Leibniz formula, we have

$$\begin{aligned}
&\sum_{N=0}^{\infty}\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z',\xi/\sqrt{t})e^{-\xi^2/2}\partial_r^\Omega e(t,r)\Big\}\Big|\\
&\leq\Big(\sum_{N=0}^{\infty}\frac{1}{N!}|\partial_\xi^N\xi^\beta|\Big)\Big(\sum_{N=0}^{\infty}\frac{1}{N!}|\partial_\xi^N\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z',\xi/\sqrt{t})|\Big)\\
&\quad\times\Big(\sum_{N=0}^{\infty}\frac{1}{N!}|\partial_\xi^N e^{-\xi^2/2}|\Big)\Big(\frac{1}{t^k}|\partial_r^\Omega e(t,r)|\Big).
\end{aligned}$$

Then we have

$$\sum_{N=0}^{\infty}\frac{1}{N!}|\partial_\xi^N\xi^\beta|\le(1+|\xi|)^\beta$$

and

$$\partial_\xi^N e^{-\xi^2/2}=H_N(\xi)e^{-\xi^2/2},$$

where $H_N(\xi)$ is the N-th Hermite polynomial.

As in (3.37), we have

$$\begin{aligned}&\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde q(t,z,z',\xi/\sqrt t)\\&=E\Big[\Pi(\xi,t,z,z(\cdot))\exp\Big(-i\xi\int_0^1\tilde f(z+\sqrt t z(s))*dz(s)\\&\qquad+\xi\sqrt t\int_0^1F(z+\sqrt t z(s))ds\Big)\delta_{z_0}(z+\sqrt t z(1))\Big],\end{aligned}$$

where $\Pi(\xi,t,z,z(\cdot))$ is a polynomial of ξ, $\sqrt t$, $1/\sqrt t$, $Z(1)$ and finite number of elements of

$$\Big\{\int_0^1\partial_z^\Omega\tilde f(z+\sqrt t z(s))s^\delta*dz(s),\int_0^1\partial_z^\Omega f(z+\sqrt t z(s))s^\delta ds:\Omega\in\mathbb{Z}_+^{2n},\delta\in\mathbb{Z}_+\Big\}.$$

Thus we have

$$\begin{aligned}&\sum_{N=0}^{\infty}\frac{1}{N!}|\partial_\xi^N\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde q(t,z,z',\xi/\sqrt t)|\\&\le\sum_{N_1,N_2=0}^{\infty}\frac{1}{N_1!N_2!}\Big|E\Big[\partial_\xi^{N_1}\Pi(\xi,t,z,z(\cdot))\\&\qquad\times\Big(-i\int_0^1\tilde f(z+\sqrt t z(s))*dz(s)+\sqrt t\int_0^1F(z+\sqrt t z(s))ds\Big)^{N_2}\\&\qquad\times\exp\Big(-i\xi\int_0^1\tilde f(z+\sqrt t z(s))*dz(s)\\&\qquad\quad+\xi\sqrt t\int_0^1F(z+\sqrt t z(s))ds\Big)\delta_{z'}(z+\sqrt t z(1))\Big]\Big|\\&\le E\Big[|\Pi'(\xi,t,z,z(\cdot))|\exp\Big(\Big|\int_0^1\tilde f(z+\sqrt t z(s))*dz(s)\Big|\\&\qquad+\sqrt t\Big|\int_0^1F(z+\sqrt t z(s))ds\Big|\Big)\Big|z+\sqrt t z(1)=z'\Big]\\&\qquad\times\frac{1}{t^n}\exp\Big(|\xi|t\sup|F|-\frac{|z-z'|^2}{t}\Big),\end{aligned}\tag{5.27}$$

where $\Pi'(\xi,t,z,z(\cdot))$ is the same kind of polynomial as $\Pi(\xi,t,z,z(\cdot))$. Since $\tilde{f}$is bounded, the right hand side of (5.27) is dominated by

$$\frac{1}{t^n}\exp\Big(\sup\|\tilde{f}\|\frac{|z-z'|}{\sqrt{t}}+|\xi|t\sup|F|-\frac{|z-z'|^2}{t}\Big).$$

Thus we have

$$\sum_{N=0}^{\infty}\frac{1}{N!}|\partial_\xi^N\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}(t,z,z',\xi/\sqrt{t})|\le\frac{C}{t^n}\exp\Big(|\xi|C'\Big). \tag{5.28}$$

Therefore we have

$$\begin{aligned}
&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\int_{-\infty}^{\infty}d\xi\sum_{N=0}^{\infty}\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}(t,z,z',\xi/\sqrt{t})\\
&\qquad\times e^{-\xi^2/2}\partial_r^\Omega e(t,r)\Big\}\Big|:t\in(0,1],z,z'\in K,r\ge0\Big\}\\
\le&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\frac{C}{t^{k+n}}\sum_{N=0}^{\infty}\frac{1}{N!}\int_{-\infty}^{\infty}d\xi|H_N(\xi)|\exp\Big(|\xi|C'-\frac{\xi^2}{2}\Big)\\
&\qquad:t\in(0,1]\Big\}.
\end{aligned} \tag{5.29}$$

By using the Hölder inequality, we have

$$\begin{aligned}
&\sum_{N=0}^{\infty}\frac{1}{N!}\int_{-\infty}^{\infty}d\xi|H_N(\xi)|\exp\Big(|\xi|C'-\frac{\xi^2}{2}\Big)\\
\le&\sum_{N=0}^{\infty}\frac{1}{N!}\sqrt{\int_{-\infty}^{\infty}d\xi H_N(\xi)^2e^{-\xi^2/2}}\sqrt{\int_{-\infty}^{\infty}d\xi\exp\Big(2|\xi|C'-\frac{\xi^2}{2}\Big)}\\
\le&C\sum_{N=0}^{\infty}\frac{1}{\sqrt{N!}}\\
<&\infty.
\end{aligned}$$

Thus we obtain (5.25).

To show (5.26), by (4.10), it is sufficient to show that

$$\begin{aligned}
&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\int_{-\infty}^{\infty}d\xi\sum_{N=0}^{\infty}\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}(t,z,z',\xi/\sqrt{t})\\
&\qquad\times e^{r\xi/\sqrt{t}}\partial_r^\Omega\int_{r/\sqrt{t}+\xi}^{\infty}d\mu e^{-\mu^2/2}\Big\}\Big|:t\in(0,1],z,z'\in K,r\ge0\Big\}\\
<&\infty
\end{aligned}$$

for any $\varepsilon > 0$, α, $\alpha' \in \mathbb{Z}_+^{2n}$, β, Ω, $k \in \mathbb{Z}_+$ and compact $K \subset \widetilde{B_f}$. When $\Omega > 0$, the proof is almost the same as that of (5.25), since

$$e^{r\xi/\sqrt{t}}\partial_r^\Omega \int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2} = \mathcal{P}\Big(\frac{1}{\sqrt{t}}, r, \xi\Big) \exp\Big(-\frac{r^2}{2t} - \frac{\xi^2}{2}\Big)$$

for some polynomial $\mathcal{P}(1/\sqrt{t}, r, \xi)$.

Let $\Omega = 0$. By the Leibniz formula, we have

$$\begin{aligned}
&\sum_{N=0}^\infty \frac{1}{N!}\Big|\partial_\xi^N \Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha \partial_{z'}^{\alpha'} \tilde{q}(t, z, z'; \xi/\sqrt{t}) e^{r\xi/\sqrt{t}} \int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2}\Big\}\Big| \\
\le &\frac{1}{t^k}\left(\sum_{N=0}^\infty \frac{1}{N!}|\partial_\xi^N \xi^\beta|\right)\left(\sum_{N=0}^\infty \frac{1}{N!}|\partial_\xi^N \partial_z^\alpha \partial_{z'}^{\alpha'} \tilde{q}(t, z, z'; \xi/\sqrt{t})|\right) \\
&\times \left(\sum_{N=0}^\infty \frac{1}{N!}\Big|\partial_\xi^N e^{r\xi/\sqrt{t}} \int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2}\Big|\right).
\end{aligned}$$

For the last factor, we estimate as follows:

$$\begin{aligned}
&\Big|\partial_\xi^N e^{r\xi/\sqrt{t}} \int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2}\Big| \\
=&\Big|\partial_\xi^N \int_0^\infty d\mu \exp\Big(-\frac{(\mu+\xi)^2}{2} - \mu r/\sqrt{t} - \frac{r^2}{2t}\Big)\Big| \\
=&\Big|\int_0^\infty d\mu H_N(\mu+\xi) \exp\Big(-\frac{(\mu+\xi)^2}{2} - \mu r/\sqrt{t} - \frac{r^2}{2t}\Big)\Big| \\
\le &\int_\xi^\infty d\mu |H_N(\mu)| e^{-\mu^2/2} \\
\le &\sqrt{\int_\xi^\infty d\mu H_N(\mu)^2 e^{-\mu^2/2}}\sqrt{\int_\xi^\infty d\mu e^{-\mu^2/2}} \\
\le &\sqrt{\sqrt{2\pi} N!}\sqrt{\int_\xi^\infty d\mu e^{-\mu^2/2}}.
\end{aligned}$$

Therefore we have

$$\sum_{N=0}^\infty \frac{1}{N!}\Big|\partial_\xi^N e^{r\xi/\sqrt{t}} \int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2}\Big| \le \begin{cases} C & \text{if } \xi < 0, \\ Ce^{-\xi^2/8} & \text{if } \xi > 0. \end{cases}$$

For $\xi > 0$, we use (5.28) to obtain

$$\begin{aligned}
&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\int_0^\infty d\xi\sum_{N=0}^\infty\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z';\xi/\sqrt{t})\\
&\qquad\times e^{r\xi/\sqrt{t}}\int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2}\Big\}\Big| : t\in(0,1], z,z'\in K, r\geq 0\Big\}\\
\leq& C\sup\Big\{\frac{e^{-\varepsilon/\sqrt{t}}}{t^{n+k}}\int_0^\infty d\xi|\xi|^\beta e^{|\xi|C'-\xi^2/8} : t\in(0,1]\Big\}\\
<&\infty.
\end{aligned}$$

For $\xi < 0$, in (5.27), we estimate as follows:

$$\begin{aligned}
&\sum_{N=0}^\infty\frac{1}{N!}|\partial_\xi^N\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z';\xi/\sqrt{t})|\\
\leq&\frac{1}{t^n}\Big(E[|\Pi'(\xi,t,z(\cdot))|^3|z+\sqrt{t}z(1)=z']\exp\Big(-\frac{|z-z'|^2}{t}\Big)\Big)^{1/3}\\
&\times\Big(E\Big[\exp\Big(3\Big|\int_0^1\tilde{f}(z+\sqrt{t}z(s))*dz(s)\Big|+3\sqrt{t}\Big|\int_0^1 F(z+\sqrt{t}z(s))ds\Big|\Big)\\
&\qquad\Big|z+\sqrt{t}z(1)=z'\Big]\exp\Big(-\frac{|z-z'|^2}{t}\Big)\Big)^{1/3}\\
&\times\Big(E\Big[\exp\Big(3\xi\sqrt{t}\int_0^1 F(z+\sqrt{t}z(s))ds\Big)\\
&\qquad\Big|z+\sqrt{t}z(1)=z'\Big]\exp\Big(-\frac{|z-z'|^2}{t}\Big)\Big)^{1/3}\\
\leq&\frac{C}{t^{2n/3}}E\Big[\exp\Big(\frac{3\xi}{\sqrt{t}}\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big]^{1/3}.
\end{aligned}$$

As in (3.38), we have

$$\begin{aligned}
&E\Big[\exp\Big(\frac{3\xi}{\sqrt{t}}\int_0^t F(z+z(s))ds\Big)\delta_{z'}(z+z(t))\Big]\\
\leq&\frac{C}{t^n}E\Big[\exp\Big(\frac{3\xi}{\sqrt{t}}\int_0^{t/2}F(z+z(s))ds\Big)\Big].
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\sup\Big\{e^{-\varepsilon/\sqrt{t}}\int_{-\infty}^0 d\xi\sum_{N=0}^\infty\frac{1}{N!}\Big|\partial_\xi^N\Big\{\frac{\xi^\beta}{t^k}\partial_z^\alpha\partial_{z'}^{\alpha'}\tilde{q}_1(t,z,z';\xi/\sqrt{t})\\
&\qquad\times e^{r\xi/\sqrt{t}}\int_{r/\sqrt{t}+\xi}^\infty d\mu e^{-\mu^2/2}\Big\}\Big| : t\in(0,1], z,z'\in K, r\geq 0\Big\}\\
\leq& C\sup\Big\{\frac{e^{-\varepsilon/\sqrt{t}}}{t^{k+n}}\sum_{p\in S}B\Big(p,\frac{t}{2},K\Big)^{1/3}t^{p/3} : t\in(0,1]\Big\}
\end{aligned}$$

for some finite set S of $[1, \infty)$. By the assumption of this proposition, this is dominated by

$$\sup \Big\{ \exp \Big(- \frac{\varepsilon}{\sqrt{t}} + \frac{C}{t^{\nu}} \Big) \ : t \in (0, 1] \Big\}.$$

This is finite because $\nu < 1/2$. $\square$

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