Asymptotics of the Infimum of the Spectrum of Schrödinger Operators with Magnetic Fields*

By

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1. Introduction

Let D be a domain in $\mathbb{R}^d$ given a Riemannian metric and b be a 1-form on D. Let $L(b)$ be the self-adjoint operator corresponding to the closed extension of the form

$$q(b)(\varphi) = \frac{1}{2}\|(id + \mathrm{ext}(b))\varphi\|^2, \qquad i = \sqrt{-1}, \tag{1.1}$$

for any $\varphi \in C_0^\infty(\overset{\circ}{D})$: $L(b)$ is the Schrödinger operator with a magnetic field db and the Dirichlet boundary condition. For the notation, see Section 2 below.

In this paper we give some lower estimates of the asymptotics of the infimum, $\inf \mathrm{spec}\, L(\xi b)$, of the spectrum of the operator $L(\xi b)$ as the real parameter ξ tends to infinity. We intend particularly to its application to the study of the asymptotics of the function

$$I(\xi) := E\left[\exp\Big(-i\xi\int_0^t b(X(s,x)) \circ dX(s,x)\Big)\,\Big|\, X(t,x) = y\right], \tag{1.2}$$

as ξ tends to infinity, where $X(s,x)$ is the absorbing barrier Brownian motion on a domain D, x, y are fixed points in D, and $E[\cdot|\cdot]$ is the conditional expectation. This is called the stochastic oscillatory integral in Malliavin [11], Ikeda and Manabe [7] and so on. The

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connection between the operator $L(\xi b)$ and the function $I(\xi)$ is given by the Feynman-Kac-Itô formula (see (2.17) below). By this formula and our estimate of inf spec $L(\xi b)$, we obtain some upper estimate of the absolute value $|I(\xi)|$ of $I(\xi)$. Accordingly, we obtain some results on the existence and the regularity of the density of the conditional probability with respect to the Lebesgue measure

$$P\left(\int_0^t b(X(s,x)) \circ dX(s,x) \in d\lambda \middle| X(t,x) = y\right) \Big/ d\lambda. \tag{1.3}$$

Moreover we consider the same problem for the operator $L^N(b)$ with the Neumann boundary condition: $L^N(b)$ is the self-adjoint operator corresponding to the closed extension of the form $q(b)(\varphi)$ in (1.1) for any $\varphi \in C_0^\infty(\overline{D})$. For the notation, see Section 4 below. For this operator, we should use the reΩecting barrier Brownian motion $X^N(s,x)$ instead of $X(s,x)$ in (1.2) and (1.3). For this case, we consider only a half space as the domain D. However in both the Dirichlet and Neumann cases, our results are extended to suitable Riemannian manifolds easily, since we can consider the asymptotics locally by the IMS localization (see Lemma 2.1 below).

In particular, a lower bound of the spectrum for the uniform magnetic field and the Neumann boundary condition is obtained (see Theorem 4.1 below). Accordingly, the transverse analyticity, which was proved for the absorbing barrier Brownian motion in [22], is proved also for the reΩecting barrier Brownian motion: the density in (1.3) where $X(s,x)$ is replaced by $X^N(s,x)$ is real analytic in λ when db is nondegenerate at everywhere (see Corollary 2 of Theorem 4.2 below).

The idea of considering the asymptotics of the spectrum to investigate the asymptotics of the function $I(\xi)$ in (1.2) is appeared in Malliavin [12], [13] : he gives a lower bound of the asymptotics of the spectrum when the magnetic field db is non degenerate and the

configulation space D is replaced by a manifold without boundary. For this case, we have more direct study of the function $I(\xi)$ in (1.2) by Ikeda-Manabe [7] and of the density in (1.3) by Malliavin [14] and Plat [18]. The bound of the spectrum by Malliavin is sharpen and is extended to the operator with the Dirichlet boundary condition by the author [22].

For the case that the magnetic field db degenerates finitely on submanifolds, Montogomery [18] and Helffer-Mohamed [4] recently give the following estimate of the spectrum: if

$$U := \{x \in D : db(x) = 0\}$$

is a compact submanifold of D and

$$C_1 d(x, U)^\rho \leq \|db(x)\| \leq C_2 d(x, U)^\rho$$

on a neighborhood of U for some $C_1, C_2, \rho > 0$, then

$$C_3 \xi^{2/(2+\rho)} \leq \inf \mathrm{spec} L(\xi b) \leq C_4 \xi^{2/(2+\rho)} \tag{1.4}$$

for any $\xi \geq 1$ and some $C_3, C_4 > 0$, where $d(x, U)$ is the distance of x from U and $\|\cdot\|$ is a fibre norm on the cotangent bundle (see Remark 2.1 below).

In this paper, we extend the lower estimate in (1.4) to the case that the magnetic field db may degenerate on some finite intersection of compact submanifolds finitely (see Theorem 2.1 below). The main tool is borrowed from Helffer-Mohamed [4]. From this result we will obtain exponential decay of the function $I(\xi)$ in (1.2) and that the density in (1.3) belongs to some Gevrey class (see Corollaries 1 and 2 of Theorem 2.1 below).

On the other hand, we will consider also the case that the magnetic field db degenerates infinitely. For this case, we will use the idea of applying the Malliavin calculus to some problem with infinite degeneracy by Malliavin [10] and Kusuoka and Stroock [8] (see also

[23]). In this case we show

$$\lim_{\xi\uparrow\infty}\frac{\inf \mathrm{spec} L(\xi b)}{\log \xi}=\infty,$$

from which we have

$$|I(\xi)|\leq C_k\xi^k$$

for any $k\in\mathbb{N}$, and the boundedness of all derivatives of the density in (1.3) (see Theorem 3.1, Corollaries 1 and 2 of Theorem 3.1 below).

For the case of the operator $L^N(b)$ with the Neumann boundary condition and the refrecting barrier Brownian motion, our basic method is to consider the double of D to reduce to the case without boundary condition. However, for the neccesary extension of $L^N(b)$ to the double, the corresponding magnetic field db is not continuous. This is the difficulty and the particular point of the Neumann condition. To overcome this difficulty, we use the IMS localization (Lemma 2.1 below). For the operator $L^N(b)$, we discuss only the case that the magnetic field db is nondegenerate. However our results are extended suitably to the case that db may be degenerate.

The organization of this paper is as follows. In Sections 2 and 3, we treat the operator with the Dirichlet boundary condition: in Section 2, we consider finitely degenerate cases and in Section 3, we consider infinitely degenerate cases. In Section 4, we treat the operator with the Neumann boundary condition.

2. The Dirichlet condition (I) finitely degenerate cases

Let D be a domain in $\mathbb{R}^d$ given a metric such that the metric tensors $g_{jk}(x), j, k = 1, 2, \cdots, d$, are C^∞, their derivatives are all bounded and

$$\inf\Big\{\sum_{j,k} g_{jk}(x)\xi_j\xi_k : x \in D,\ \xi = (\xi_1, \xi_2, \cdots, \xi_d) \in \mathbb{R}^d,\ \sum_j \xi_j^2 = 1\Big\} > 0.$$

Let b be an $\mathbb{R}^d$ valued C^∞ function on D, which is identified with the 1 form, and $C_0^\infty(\overset{\circ}{D})$ be the set of all $\mathbb{C}$-valued C^∞ functions on the interior of D with compact support. For any $\varphi \in C_0^\infty(\overset{\circ}{D})$, we set

$$q(b)(\varphi) = \frac{1}{2}\|(id + \mathrm{ext}(b))\varphi\|^2, \tag{2.1}$$

where $\|\cdot\|$ is the L^2 norm with respect to the above metric and ext is the exterior multiplication, i.e., $\mathrm{ext}(b)\varphi = b \wedge \varphi$. Let $\mathrm{Dom}(q(b))$ be the completion of $C_0^\infty(\overset{\circ}{D})$ by the norm $\sqrt{q(b)(\cdot) + \|\cdot\|^2}$. We extend $q(b)(\cdot)$ naturally to $\mathrm{Dom}(q(b))$ so that $(q(b),\mathrm{Dom}(q(b)))$ becomes a closed sesquilinear form. Let $L(b)$ be the associated self-adjoint operator: $L(b)$ is the operator with Dirichlet boundary condition. The 2 form db is regarded as the corresponding magnetic field. We will study the asymptotics of inf spec $L(\xi b)$ as the real parameter ξ tends to infinity, where spec $L(\xi b)$ is the spectral set of the self-adjoint operator $L(\xi b)$.

In this section, we consider the case that the magnetic field db has some finite degeneracy. We first introduce the conditions. For each $x \in D$, let $\|db(x)\|_1$ be the trace norm of $db(x)$:

$$\|db(x)\|_1 = \mathrm{tr}\sqrt{B(x)^t B(x)},$$

where $B(x)$ is the matrix $(db(e_j(x), e_k(x)))_{1\le j,k\le d}$ and $\{e_j(x)\}_{j=1}^d$ is an orthonormal basis of T_xD. Let (G) be the following condition:

$$\sup_{x\in D} \frac{|\partial_\ell db_{jk}(x)|}{\|db(x)\|_1 + 1} < \infty, \qquad \text{for any } 1 \le j, k, \ell \le d, \tag{G}$$

where $\partial_j = \partial/\partial x^j$ and $db_{jk} = db(\partial_j, \partial_k)$.

Taking $n \in \mathbb{N}$ arbitrarily, we assume the following:

(D, n) There are functions $h_\mu(s) : \mathbb{R} \to [0, \infty)$ and $\psi_\mu(x) : D \to \mathbb{R}$, $\mu = 1, 2, \cdots, n$, satisfying the following:

(i) h_μ is even, non-decreasing on $[0, \infty)$ and $h_\mu^{-1}(0) = \{0\}$;

(ii) ψ_μ is C^∞, the derivatives of ψ_μ are all bounded and

$$\inf_{x \in D} (\psi_\mu(x)^2 + \|d\psi_\mu(x)\|^2) > 0;$$

(iii) for any $x \in D$,

$$\|db(x)\|_1 \geq \prod_{\mu=1}^{n} h_\mu(\psi_\mu(x));$$

(iv) $\inf_{x \notin K} |\psi_\mu(x)| > 0$ for any $\mu = 1, 2, \cdots, n$ and some compact set K in $\overline{D}$.

For $\overline{\rho} = (\rho(1), \rho(2), \cdots, \rho(n)) \in [0, \infty)^n$, let $(P, \overline{\rho})$ be the following condition:

$$(P, \overline{\rho}) \qquad \overline{\lim_{s \downarrow 0}} \frac{s^{\rho(\mu)}}{h_\mu(s)} < \infty, \qquad \text{for any } \mu = 1, 2, \cdots, n.$$

Let (I) be the following condition:

(I) For each $a \in \mathcal{N} := \bigcup_{\mu=1}^n \psi_\mu^{-1}(0)$, the system of the forms $\{d\psi_\mu(a)\}_{\mu \in \Lambda(a)}$ is linear independent, where $\Lambda(a) = \{\mu : \psi_\mu(a) = 0\}$.

Then the main theorem is the following:

Theorem 2.1. *We assume (G), (D, n) and $(P, \overline{\rho})$ for some $\overline{\rho} \in [0, \infty)^n$. Moreover we assume (I) or $\overline{\rho} \in (2\mathbb{Z}_+)^n$. Then we have*

$$\overline{\lim_{\xi \uparrow \infty}} \frac{\inf \mathrm{spec} L(\xi b)}{\xi^{2/(|\overline{\rho}|+2)}} > 0, \tag{2.2}$$

where $|\overline{\rho}| = \sum_{\mu=1}^n \rho(\mu)$.

Remark 2.1. When $n = \overline{\rho} = 1$ and $d = 2$, Theorem 2.1 is a part of the results of Montgomery [17]. When $n = 1$ and $\overline{\rho}$, d are general, Theorem 2.1 is a part of the results of Helffer-Mohamed [4]. In the case of $n = 1$, the condition (I) holds automatically.

We give simple examples:

Example 2.1. On $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ with the Euclidean metric, we define b by $b_1 = (x^1)^{\rho(1)}(x^2)^{\rho(2)+1}$, $b_2 = 0$ and $\overline{\rho} = (\rho(1), \rho(2)) \in (\mathbb{Z}_+)^2$. Then we have $db_{12} = -(\rho(2)+1)(x^1)^{\rho(1)}(x^2)^{\rho(2)}$. This satisfies the conditions $(D, 2)$, $(P, \overline{\rho})$ and (I) with $\psi_\mu = x^\mu$ and $h_\mu = |s|^{\rho(\mu)}$. Thus we obtain the result of Theorem 2.1.

Example 2.2. On the same domain as that of Example 2.1, we take b so that $db_{12} = (x^1)^{\rho(1)}(\psi_2(x))^{\rho(2)}$ for some $\overline{\rho} = (\rho(1), \rho(2)) \in (2\mathbb{Z}_+)^2$, where $\psi_2 = (x^2)^2 + x^1$. This does not satisfy the condition (I), since $d\psi_1 = d\psi_2$ at $x = 0$. However this satisfies the conditions $(D, 2)$, $(P, \overline{\rho})$ and $\overline{\rho} \in (2\mathbb{Z}_+)^n$. Thus we still obtain the result of Theorem 2.1.

For the proof, we prepare several lemmas. The following lemma is one of the fundamental tools to consider the lower bound of the spectrum:

Lemma 2.1 (IMS localization). *(i) Let $\{\chi_m\}_m \subset C^\infty(D)$ satisfying $\sum_m \chi_m^2 \equiv 1$. Then, for any $\varphi \in C_0^\infty(\overset{\circ}{D})$, we have*

$$L(b)\varphi = \sum_m \chi_m L(b)\chi_m\varphi - \sum_m \frac{\|d\chi_m\|^2}{2}\varphi. \tag{2.3}$$

(ii) Let $\{\chi_m\}_m \subset H^{1,2}_{loc}(D)$ satisfying $\sum_m \chi_m^2 \equiv 1$, and b be an $\mathbb{R}^d$ valued continuous function on D. Then, for any $\varphi \in C_0^\infty(\overset{\circ}{D})$, we have $\chi_m\varphi \in \mathrm{Dom}(q(b))$ and

$$q(b)(\varphi) = \sum_m q(b)(\chi_m\varphi) - (\sum_m \frac{\|d\chi_m\|^2}{2}\varphi, \varphi). \tag{2.4}$$

In (ii), $H^{1,2}_{loc}(D) = \{\varphi : D \to \mathbb{C}, \varphi \restriction_{D'} \in H^{1,2}(D') \text{ for any relatively compact } D' \subset D\}$. For the definition of $H^{1,2}(D')$, see [1]. We use (i) in this section and use (ii) in later sections.

Proof. In the proof, we may assume that the domain D is relatively compact. For the proof of (i), see [21] for example. For that of (ii), we use an approximation arguement: for each m, there is a sequence $\{\chi^n_m\}_n \subset C^\infty(D)$ such that $\chi^n_m \to \chi_m$, $d\chi^n_m \to d\chi_m$ in L^2. Then for any $\varphi \in C^\infty_0(\overset{\circ}{D})$, we can show that $\chi^n_m\varphi \to \chi_m\varphi$ in L^2 and $\{\chi^n_m\varphi\}_n$ is a Cauchy sequence in $\mathrm{Dom}(q(b))$. Therefore we have $\chi_m\varphi \in \mathrm{Dom}(q(b))$ and $\chi^n_m\varphi \to \chi_m\varphi$ in $\mathrm{Dom}(q(b))$. For each m and n, we have

$$q(b)(\chi^n_m\varphi) = \frac{1}{2}\|\chi^n_m(id + \mathrm{ext}(b))\varphi\|^2 + \mathrm{Re}(i\varphi\chi^n_m d\chi^n_m, (id + \mathrm{ext}(b))\varphi) + \frac{1}{2}\|\varphi d\chi^n_m\|^2.$$

By taking the limit in n and then taking the sum in m, we obtain (2.4).

The following lemma is due to Helffer-Nourrigat [3], [5], [16]:

Lemma 2.2. *Let $b^0_j(x)$, $j = 1, 2, \cdots, d$, and V_ℓ, $\ell = 1, 2, \cdots, m$, be real polynomials on $\mathbb{R}^d$ of degree $\leq r$. Let*

$$\Psi(x) = \sum_{1\leq j<k\leq d} \sum_{\substack{\alpha\in\mathbb{Z}^d_+ \\ |\alpha|\leq r-1}} |\partial^\alpha db^0_{jk}(x)|^{1/(|\alpha|+2)} + \sum_{\ell=1}^m \sum_{\substack{\beta\in\mathbb{Z}^d_+ \\ |\beta|\leq r}} |\partial^\beta V_\ell(x)|^{1/(|\beta|+1)}. \tag{2.5}$$

Then there exists a constant C depending only on d, m and r such that

$$\int_{\mathbb{R}^d} (\sum_{j=1}^d |(i\partial_j + b^0_j)\varphi|^2 + \sum_{\ell=1}^m |V_\ell\varphi|^2)dx \geq C\int_{\mathbb{R}^d} |\Psi\varphi|^2 dx \tag{2.6}$$

for any $\varphi \in C^\infty_0(\mathbb{R}^d)$.

In [4], the following is deduced from Lemmas 2.1 and 2.2:

Lemma 2.3. *Under the assumption* (G)*, for any* $\varepsilon \in (0,1)$*, there exist constants* C_ε*,* $C'_\varepsilon > 0$ *such that*

$$q(\xi b)(\varphi) \geq C_\varepsilon \int_D \xi \|db\|_1 |\varphi|^2 d\mathrm{vol} - C'_\varepsilon \xi^\varepsilon \|\varphi\|^2 \tag{2.7}$$

for any $\varphi \in C_0^\infty(\overset{\circ}{D})$ *and* $\xi > 1$.

Proof. Let $\chi^{(1)}$and $\chi^{(2)}$be smooth functions on $\overline{D}$ such that $(\chi^{(1)})^2 + (\chi^{(2)})^2 \equiv 1$, supp $\chi^{(1)}$ is compact, supp $\chi^{(2)} \subset K^c$ and $d\chi^{(2)}$ is bounded, where K is the compact set in the condition (D,n) (iv). By Lemma 2.1, we have

$$q(\xi b)(\varphi) = \sum_{\nu=1}^{2} \{q(\xi b)(\chi^{(\nu)}\varphi) - \frac{1}{2}\|\varphi d\chi^{(\nu)}\|^2\}.$$

Then Theorem 4.5 of [4] leads to the following:

$$q(\xi b)(\chi^{(1)}\varphi) \geq C_\varepsilon \int_D \xi \|db\|_1 |\chi^{(1)}\varphi|^2 d\mathrm{vol} - C'_\varepsilon \xi^\varepsilon \|\chi^{(1)}\varphi\|^2. \tag{2.8}$$

Since $\inf_{x \notin K} \|db(x)\|_1 > 0$, Theorem 3.1 of [4] leads to the following:

$$q(\xi b)(\chi^{(2)}\varphi) \geq C \int_D \xi \|db\|_1 |\chi^{(2)}\varphi|^2 d\mathrm{vol}$$

for some $C > 0$. By these, we obtain (2.7).

The following lemma is well known (see e.g., [6], [20]):

Lemma 2.4 (Diamagnetic estimate). *For any* $\mathbb{R}^d$ *valued continuous function* b *and any real continuous function* V *on* D *which is bounded below, we have*

$$\inf \mathrm{spec}(L(b) + V) \geq \inf \mathrm{spec}(L(0) + V), \tag{2.9}$$

where $L(b) + V$ *is the Friedrichs extension of the corresponding operator on* $C_0^\infty(\overset{\circ}{D})$.

We now prove Theorem 2.1.

Proof of Theorem 2.1.

We take $0 < \varepsilon < 2/(2 + |\overline{\rho}|)$ and fix it. By Lemmas 2.3 and 2.4, and the assumptions of this theorem, we have

$$\inf \mathrm{spec} L(\xi b) \geq C_1 \inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)}) - C_2 \xi^{\varepsilon} \tag{2.10}$$

for some constants C_1 and C_2, where Δ is the Laplace-Beltrami operator with Dirichlet boundary condition.

We first assume the condition (I). Then, for each $a \in \mathcal{N}$, there exists a coordinate neighborhood $(\nu(a), (y^1, y^2, \cdots, y^d))$ of D around a such that $\psi_\mu(y) = y^{n(\mu,a)}$ for some $1 \leq n(\mu, a) \leq d$ and any $\mu \in \Lambda(a)$ and $\psi_\mu(y) \neq 0$ for any $y \in \nu(a)$ and any $\mu \in \Lambda(a)^c$. Let $\nu_0(a)$ be a neighborhood of a so that $\overline{\nu_0(a)} \subset \nu(a)$. Since $\mathcal{N}$ is compact, we can take a finite set $\mathcal{N}_0 \subset \mathcal{N}$ so that

$$\bigcup_{a \in \mathcal{N}_0} \nu_0(a) \supset \mathcal{N}. \tag{2.11}$$

By using Lemma 2.1, we have

$$\begin{aligned} &\inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)}) \\ &\geq \inf_{a \in \mathcal{N}_0} \inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)})_{\nu(a)} \\ &\quad \wedge \inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)})_{\nu} - C_3 \end{aligned} \tag{2.12}$$

for some $C_3 > 0$, where $(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)})_{\nu(a)}$ is the Friedrichs extension of the restriction of the corresponding operator to $C_0^\infty(\nu(a))$ and

$$\nu = \left(\overline{\bigcup_{a \in \mathcal{N}_0} \nu_0(a)} \right)^c. \tag{2.13}$$

We easily see that

$$\inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)})_\nu \geq C_4 \xi \tag{2.14}$$

for some $C_4 > 0$. For each $a \in \mathcal{N}_0$, by using the coordinate and the ellipticity of the metric, we have

$$\begin{aligned} &\inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)})_{\nu(a)} \\ \geq & C_5 \inf \Big\{ \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} |\partial_j \varphi(y)|^2 + \xi \prod_{\mu \in \Lambda(a)} |y^{n(\mu,a)}|^{\rho(\mu)} |\varphi(y)|^2 \Big) dy \\ & \Big| \varphi \in C_0^\infty(\mathbb{R}^d), \int |\varphi(y)|^2 dy = 1 \Big\} \end{aligned} \tag{2.15}$$

for some C_5. By making the change of vasriables ($y \to \xi^{-1/(2+\rho[a])} y$ where $\rho[a] = \sum_{\mu \in \Lambda(a)} \rho(\mu)$), we can rewrite the right hand side of (2.15) as

$$\begin{aligned} & C_5 \xi^{2/(2+\rho[a])} \inf \Big\{ \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} |\partial_j \varphi(y)|^2 + \prod_{\mu \in \Lambda(a)} |y^{n(\mu,a)}|^{\rho(\mu)} |\varphi(y)|^2 \Big) dy \\ & : \varphi \in C_0^\infty(\mathbb{R}^d), \int |\varphi(y)|^2 dy = 1 \Big\}. \end{aligned}$$

Since this is positive, we obtain (2.2).

We next assume $\overline{\rho} \in (2\mathbb{Z}_+)^n$ for any $\mu = 1, 2, \cdots, n$. For each $a \in \mathcal{N}$, let $(\nu(a), (y^1, y^2, \cdots, y^d))$ be a coordinate neighborhood of D around a, such that

$$\sum_{|\alpha| = \rho[a]/2+1} \Big| \partial^\alpha \prod_{\mu \in \Lambda(a)} \psi_\mu(y)^{\rho(\mu)/2} \Big| \leq C_a \sum_{|\alpha| \leq \rho[a]/2} \Big| \partial^\alpha \prod_{\mu \in \Lambda(a)} \psi_\mu(y)^{\rho(\mu)/2} \Big| \neq 0$$

for any $y \in \nu(a)$ and some $C_a > 0$ depending only on a, and $\psi_\mu(y) \neq 0$ for any $y \in \nu(a)$ and $\mu \in \Lambda(a)^c$. Let $\nu_0(a)$ be a neighborhood of a so that $\overline{\nu_0(a)} \subset \nu(a)$, $\mathcal{N}_0 \subset \mathcal{N}$ be a finite set satisfying (2.11) and ν be the set as in (2.13). Then we have (2.12) and (2.14). For

each $a \in \mathcal{N}_0$, by using the ellipticity of the metric, we have

$$
\begin{aligned}
&\inf \operatorname{spec}(-\Delta + \xi \prod_{\mu=1}^{n} |\psi_\mu|^{\rho(\mu)})_{\nu(a)} \\
\geq & C_6 \inf \Big\{ \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} |\partial_j \varphi(y)|^2 + \xi \prod_{\mu \in \Lambda(a)} |\psi_\mu(y)|^{\rho(\mu)} |\varphi(y)|^2 \Big) dy \\
& : \varphi \in C_0^\infty(\nu(a)), \int |\varphi(y)|^2 dy = 1 \Big\}
\end{aligned} \tag{2.16}
$$

By Theorem (1.1) of [16], this is dominated from below by

$$
\begin{aligned}
& C_7 \inf \Big\{ \int_{\mathbb{R}^d} \Big| \sum_{k=0}^{\rho[a]/2} \Big(1 + \sum_{|\alpha|=k} \Big| \xi^{1/2} \partial^\alpha \prod_{\mu \in \Lambda(a)} \psi_\mu^{\rho(\mu)/2} \Big| \Big)^{1/(k+1)} \varphi \Big|^2 dy \\
& \quad : \varphi \in C_0^\infty(\nu(a)), \int |\varphi(y)|^2 dy = 1 \Big\} - C_8 \\
& \geq C_9 \xi^{1/(\rho[a]/2+1)} - C_{10}.
\end{aligned}
$$

By all these, we obtain (2.2).

We next apply Theorem 2.1 to the asymptotics of heat kernels as in Malliavin [12], [13] and Ueki [22]. In the rest of this section, for simplicity, we assume

$$
\sup_{x \in D} \frac{|\nabla^2 b(x)|}{(1+|x|)^k} < \infty \qquad \text{for some } k \in \mathbb{N}. \tag{PG}
$$

Let $e^{-tL(\xi b)}(x,y)$, $(t,x,y) \in (0,\infty) \times D \times D$ be the integral kernel of the semigroup $e^{-tL(\xi b)}$ generated by the operator $L(\xi b)$, which is called the heat kernel: for any $\varphi \in L^2(D)$,

$$
(e^{-tL(\xi b)} \varphi)(x) = < e^{-tL(\xi b)}(x, \cdot), \varphi >,
$$

where $< \cdot, \cdot >$ is the inner product in L^2 which is complex linear in the both variables. The heat kernel has the following representation: let $X(t,x)$, $t > 0$, $x \in D$ be the absorbing barrier Brownian motion on D starting at x: $X(t,x)$ is a diffusion process generated by $\Delta/2$, where Δ is the Laplace-Beltrami operator with Dirichlet boundary condition. Let

$\int_0^t b(X(s,x)) \circ dX(s,x)$ be the stochastic line integral (see e.g. [9] §VI-6) and $e^{t\Delta/2}(x,y)$ be the integral kernel of the semigroup $e^{t\Delta/2}$. Then we have

$$
\begin{aligned}
&e^{-tL(\xi b)}(x,y) \\
&= E\left[\exp\Big(-i\xi \int_0^t b(X(s,x)) \circ dX(s,x) \Big) \middle| X(t,x) = y \right] e^{t\Delta/2}(x,y),
\end{aligned} \tag{2.17}
$$

where $E[\cdot | X(t,x) = y]$ is the conditional expectation with respect to the Brownian motion $X(\cdot, x)$ conditioned by $X(t,x) = y$.

From Theorem 2.1, we obtain the following:

Corollary 1. *Under the assumption of Theorem 2.1 and (PG), it holds that*

$$
\overline{\lim_{\xi \to \pm\infty}} \sup_{(t,x,y) \in [t_0,\infty) \times D \times D} \frac{1}{t|\xi|^{2/(|\overline{\rho}|+2)}} \log |e^{-tL(\xi b)}(x,y)| < 0 \tag{2.18}
$$

for any $t_0 > 0$.

Corollary 1 is an upper estimate of asymptotics of a stochastic oscillatory integral for which the phase function is a stochastic line integral (see e.g. [7] and [11]). Moreover, since $e^{-tL(\xi b)}(x,y)$ is the Fourier transform of the distribution of $\int_0^t b(X(s,x)) \circ dX(s,x)$ under the conditional probability $P(\cdot | X(t,x) = y)$, we have the following:

Corollary 2. *Under the assumption of Theorem 2.1 and (PG), for any* $t > 0$, $x, y \in D$, *the distribution of* $\int_0^t b(X(s,x)) \circ dX(s,x)$ *under the conditional probability* $P(\cdot | X(t,x) = y)$ *has a smooth density function belonging to the Gevrey class of order* $(|\overline{\rho}| + 2)/2$*: there is a function* $p(t,x,y;\lambda)$, $t > 0$, $x, y \in D$, $\lambda \in \mathbb{R}$ *such that*

$$
P\left(\int_0^t b(X(s,x)) \circ dX(s,x) \in A \middle| X(t,x) = y \right) = \int_A p(t,x,y;\lambda) d\lambda \tag{2.19}
$$

for any Borel set A *in* $\mathbb{R}$, $p(t,x,y;\lambda)$ *is smooth in* λ, *and*

$$
\left| \frac{\partial^k}{\partial \lambda^k} p(t,x,y;\lambda) \right| \leq \frac{1}{\pi \alpha C^{(k+1)/\alpha}} \Gamma\left(\frac{n+1}{\alpha} \right) \tag{2.20}
$$

for any $\lambda \in \mathbb{R}$, $k \in \mathbb{Z}_+$ *and some* $C > 0$, *where* $\alpha = 2/(|\overline{\rho}| + 2)$ *and* $\Gamma(\cdot)$ *is the gamma function.*

3. The Dirichlet condition (II) infinitely degenerate cases

In this section, we consider the case that the magnetic field db has some infinite degeneracy: for $\rho > 0$, let (E, ρ) be the following condition:

$$(E,\rho) \qquad \lim_{s\downarrow 0} s^{\rho} \log \frac{1}{h_{\mu}(s)} = 0 \quad \text{for any} \quad \mu = 1, 2, \cdots, n.$$

Let (H) be the following condition:

$$(H) \qquad \Delta\psi_{\mu} = 0 \text{ and } d\Delta\psi_{\mu} = 0 \text{ on } \psi_{\mu}^{-1}(0) \text{ for any } \mu = 1, 2, \cdots, n,$$

where Δ is the Laplace-Beltrami operator.

Then the main theorem is the following:

Theorem 3.1. *We assume (D, n) and (E, ρ) for some $\rho < 2$. Moreover we assume (H) or $\rho < 1$. Then we have*

$$\lim_{\xi\uparrow\infty} \frac{\inf \mathrm{spec} L(\xi b)}{\log \xi} = \infty. \tag{3.1}$$

We give simple examples:

Example 3.1. On $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ with the Euclidean metric, we define b by

$$b_1 = \exp(-|x^1|^{-\rho}) \int_0^{x^2} \exp(-|t|^{-\rho}) dt,$$

$b_2 = 0$ and $\rho \in (0, 2)$. Then we have $db_{12} = -\exp(-|x^1|^{-\rho} - |x^2|^{-\rho})$. This satisfies the conditions $(D, 2)$, (H) and (E, ρ') for any $\rho' \in (0, 2)$ with $\psi_{\mu} = x^{\mu}$ and $h_{\mu} = \exp(-|s|^{-\rho})$. Thus we obtain the result of Theorem 3.1.

Example 3.2. On $D = \{x \in \mathbb{R}^2 : |x - e| < 1\}$ with the Euclidean metric, we take b so that $db_{12} = \exp(-|\psi_1|^{-\rho})$ for some $\rho \in (0, 2)$, where $\psi_1 = \log|x| - 1$. This db vanishes on the arc $\{|x| = e\}$. However, since $\Delta\psi_1 = 0$, the condition (H) is satisfied. Moreover

the conditions $(D,1)$ and (E,ρ') for any $\rho' \in (0,2)$ are satisfied. Thus we still obtain the result of Theorem 3.1.

For the proof, we prepare several lemmas. We first show the following:

Lemma 3.1. *Under the assumption (D,n), there exist constants C_1 and C_2 independent of ξ such that*

$$q(\xi b)(\varphi) \geq C_1 \int_{\mathbb{R}^d} \xi \Big(\prod_{\mu=1}^{n} |\psi_\mu| h_\mu \circ \psi_\mu\Big) |\varphi|^2 dx - C_2 \|\varphi\|^2 \tag{3.2}$$

for any $\varphi \in C_0^\infty(\overset{\circ}{D})$ and $\xi \geq 0$.

Proof. For each $a \in D$, there exist a neighborhood $\nu(a)$ of a and an orthonormal frame $e_1, e_2, \cdots, e_d$ on $\nu(a)$ such that

$$\frac{\|db(x)\|_1}{4} = |\mathcal{B}(x)|,$$

where

$$\mathcal{B}(x) = \frac{1}{2} \sum_{\sigma=1}^{[d/2]} db(e_{2\sigma-1}(x), e_{2\sigma}(x)).$$

By Theorem 2.1 of [22], we have

$$q(\xi b)(\varphi) \geq ((\pm \xi \mathcal{B} - \mathcal{R})\varphi, \varphi)$$

for any $\varphi \in C_0^\infty(\nu(a))$, where $\mathcal{R}$ is some bounded function independent of ξ.

For each $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(n)) \in \{0,1\}^n$, let $\{\nu(a,\sigma,\iota) : \iota = 1, 2, \cdots, n(a,\sigma)\}$ be the arcwise connected components of the set $\{x \in \nu(a) : (-1)^{\sigma(\mu)} \psi_\mu(x) > 0 \text{ for } \mu = 1, 2, \cdots, n\}$. Then for each σ and ι, there exists $\varepsilon(a,\sigma,\iota) \in \{0,1\}$ such that $\nu(a,\sigma,\iota) \subset \{x \in \nu(a) : (-1)^{\varepsilon(a,\sigma,\iota)} \mathcal{B}(x) > 0\}$. We now introduce a function $\eta(x)$ on $\nu(a)$ by

$$\eta(x) = \begin{cases} (-1)^{\varepsilon(a,\sigma,\iota)} \Big| \prod_{\mu=1}^{n} \psi_\mu(x) \Big| & \text{if } x \in \nu(a,\sigma,\iota), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta_\pm$ be smooth functions on $\mathbb{R}$ satisfying $\zeta_\pm(s) = \sqrt{(1 \pm s)/2}$ for $s \in [-1/2, 1/2]$, $0 \le \zeta_\pm \le 1$ and $\zeta_+^2 + \zeta_-^2 \equiv 1$. We set $\chi_\pm(x) := \zeta_\pm(\eta(x))$. These belong to $H^{1,2}_{loc}(\nu(a))$. Therefore by Lemma 2.1 (ii), we have

$$\begin{aligned} q(\xi b)(\varphi) =& q(\xi b)(\chi_+\varphi) + q(\xi b)(\chi_-\varphi) \\ & - \frac{1}{2}((|d\chi_+|^2 + |d\chi_-|^2)\varphi, \varphi) \\ \ge& \xi((\chi_+^2 - \chi_-^2)\mathcal{B}\varphi, \varphi) - (\mathcal{R}_1\varphi, \varphi) \end{aligned}$$

for any $\varphi \in C_0^\infty(\nu(a))$ and some bounded function $\mathcal{R}_1$. It is easy to see that

$$(\chi_+^2 - \chi_-^2)\mathcal{B} \ge C \prod_{\mu=1}^{n} |\psi_\mu| h_\mu \circ \psi_\mu$$

for some $C > 0$. By the IMS localization (Lemma 2.1), we obtain (3.2).

Lemma 3.2. *Let $L(\xi)$, $\xi > 0$ be a family of self-adjoint operators on a Hilbert space. Then the following (i) and (ii) are equivalent:*

(i) $$\lim_{\xi \uparrow \infty} \frac{\inf \mathrm{spec} L(\xi)}{\log \xi} = \infty.$$

(ii) $$\lim_{\xi \uparrow \infty} \xi^k \| e^{-tL(\xi)} \|_o = 0 \text{ for any } k \in \mathbb{N} \text{ and } t > 0,$$

where $e^{-tL(\xi)}$, $t > 0$, is the semigroup generated by $L(\xi)$ and $\| \cdot \|_o$ is the operator norm on the Hilbert space.

We can prove this easily by using the relation

(3.3) $$\| e^{-tL(\xi)} \|_o = \exp(-t \inf \mathrm{spec} L(\xi)).$$

We omit the detail.

Lemma 3.3. *Let V be a nonnegative continuous function on D. Then, for any $k \in \mathbb{N}$, there is a constant $C_k > 0$ such that*

$$\begin{aligned} &\xi^k \Big\langle \exp\Big\{ -t\Big(-\frac{\Delta}{2} + \xi V\Big)\Big\}\Big\rangle_o \\ &\leq C_k \sup_{x\in D} E\Big[\Big(\int_0^t V(X(s,x))ds\Big)^{-2k} : \tau > t\Big], \end{aligned} \tag{3.4}$$

where $\tau = \inf\{t > 0 : X(t,x) \in \partial D\}$.

Proof. For any $\varphi \in C_0^\infty(\overset{\circ}{D})$, we have

$$\begin{aligned} &\Big|\xi^k\Big[\exp\Big\{ -t\Big(-\frac{\Delta}{2} + \xi V\Big)\Big\}\varphi\Big](x)\Big| \\ =&\Big|E\Big[\xi^k \exp\Big(-\xi\int_0^t V(X(s,x))ds\Big)\varphi(X(t,x)) : \tau > t\Big]\Big| \\ \leq& C_k E\Big[\Big(\int_0^t V(X(s,x))ds\Big)^{-2k} : \tau > t\Big]^{1/2} E[|\varphi(X(t,x))|^2 : \tau > t]^{1/2} \end{aligned}$$

and

$$\begin{aligned} &\xi^k \Big\langle \exp\Big\{ -t\Big(-\frac{\Delta}{2} + \xi V\Big)\Big\}\Big\rangle_o \\ \leq& C_k \sup_{x\in D} E\Big[\Big(\int_0^t V(X(s,x))ds\Big)^{-2k} : \tau > t\Big]^{1/2} \|e^{t\Delta/2}\|_{(1)}^{1/2}, \end{aligned}$$

where $\|\cdot\|_{(1)}$ is the operator norm on the L^1 space. Since

$$\|e^{t\Delta/2}\|_{(1)} \leq 1,$$

we have (3.4)

Lemma 3.4. *Under the assumption (H), there exist constants C_1 and C_2 such that $|\psi_\mu(x)| \leq C_1$ implies $|\Delta\psi_\mu(x)| \leq C_2|\psi_\mu(x)|^2$.*

Proof. By virtue of the condition (D,n) (ii), for each $a \in \psi_\mu^{-1}(0)$, there exists a coordinate neighborhood $(\nu(a), (y^1, y^2, \cdots, y^d))$ of D around a such that $y^1 = \psi_\mu(y)$ and $(y^2, \cdots, y^d)$ is a coordinate of the submanifold $\psi_\mu^{-1}(0)$. By the Taylor theorem, we estimate as

$$|\Delta\psi_\mu(y^1, y^2, \cdots, y^d) - \Delta\psi_\mu(0, y^2, \cdots, y^d)| \leq C_a|y^1|^2$$

for some constant C_a depending on a. This implies

$$|\Delta\psi_\mu(y)| \le C_a|\psi_\mu(y)|^2$$

on $\nu(a)$. Since $\psi_\mu^{-1}(0)$ is compact, we can take constants C_1 and C_2 satisfying the statement of this lemma.

The following is the key estimate to prove Theorem 3.1 (cf. Kusuoka-Stroock [8] and Proposition 3.2 in [23]).

Proposition 3.1. *Under the assumptions of Theorem 3.1, we have*

$$\sup_{x\in D} E\left[\left(\int_0^t \prod_{\mu=1}^n h_\mu(\psi_\mu(X(s,x)))ds\right)^{-p} : \tau > t\right] < \infty \tag{3.5}$$

for any $p \ge 1$.

Proof. We give the proof only under the condition (H). The proof under the condition $\rho < 1$ can be given by referring the following proof and the proof of Proposition 3.3 (ii) in [23]. Let $(w^1(t), w^2(t), \cdots, w^d(t))$ be the d-dimensional Brownian motion, $\sigma(x) = (\sigma^{jk}(x))_{1\le j,k\le d}$ be the square root of $(g^{jk}(x))_{1\le j,k\le d}$ which is the inverse of the metric tensor $(g_{jk}(x))_{1\le j,k\le d}$, $\beta(x) = (\beta^j(x))_{1\le j\le d} = ((1/2g)\sum_{k=1}^d \partial(gg^{jk})/\partial x^k)_{1\le j\le d}$ and $g(x) = \sqrt{\det(g_{jk}(x))}$. Then we can regard the Brownian motion $X(t,x)$ as the solution of the following stochastic differential equation:

$$\begin{cases} X(0) = x, \\ dX^j(t) = \displaystyle\sum_{k=1}^d \sigma^{jk}(X(t))dw^k(t) + \beta^j(X(t))dt \text{ for } \tau > t, \\ X(t) \text{ is killed at } t = \tau, \end{cases}$$

where $\tau = \inf\{t > 0 : X(t) \in \partial D\}$. In particular we have

$$\begin{aligned} &\psi_\mu(X(t\wedge\tau, x, w)) - \psi_\mu(x) \\ &= \sum_{j,k=1}^d \int_0^{t\wedge\tau} (\sigma^{jk}\partial_k\psi_\mu)(X(s,x,w))dw^j(s) + \int_0^{t\wedge\tau} \frac{1}{2}(\Delta\psi_\mu)(X(s,x,w))ds \end{aligned} \tag{3.6}$$

for any $\mu = 1, 2, \cdots, n$. As in [23], we estimate the probability

$$\begin{aligned}
&P\left(\int_0^t \prod_{\mu=1}^n h_\mu(\psi_\mu(X(s,x)))ds < \frac{1}{R}, \tau > t\right) \\
=&P\left(\int_0^t \prod_{\mu=1}^n h_\mu(\psi_\mu(X(s,x)))ds < \frac{1}{R}, \tau_\lambda(x) \le \frac{t}{2}, \tau > t\right) \\
&+ P\left(\int_0^t \prod_{\mu=1}^n h_\mu(\psi_\mu(X(s,x)))ds < \frac{1}{R}, \tau_\lambda(x) > \frac{t}{2}, \tau > t\right) \\
=:&I_1 + I_2,
\end{aligned}$$

where $\lambda \in (0, 1]$ is an additional variable and $\tau_\lambda(x) = \inf\{s \ge 0 : |\psi_\mu(X(s,x))| \ge \lambda$ for any $\mu = 1, 2, \cdots, n\}$.

By using the strong Markov property as in [23], we have

$$\begin{aligned}
I_1 \le &\sup\left\{P\left(\frac{t}{2} \wedge \sigma_{\lambda/2}(x) < \frac{1}{R\prod_{\mu=1}^n h_\mu\left(\frac{\lambda}{2}\right)}\right)\right. \\
&\left. : x \in D, |\psi_\mu(x)| > \lambda \text{ for any } \mu = 1, 2, \cdots, n\right\},
\end{aligned}$$

where $\sigma_{\lambda/2}(x) = \inf\{s \ge 0 : |\psi_\mu(X(s,x))| < \lambda/2$ for some $\mu \in \{1, 2, \cdots, n\}\}$. If $R\prod_{\mu=1}^n h_\mu(\lambda/2) \ge 2/t$, then we have

$$\begin{aligned}
I_1 \le &\sup_{x\in D} P\Bigg(\sup_{0<s<1/\{R\prod_{\mu=1}^n h_\mu(\lambda/2)\}} |\psi_\nu(X(s,x)) - \psi_\nu(x)| > \frac{\lambda}{2} \\
&\qquad \text{for some } \nu \in \{1, 2, \cdots, n\}\Bigg) \\
\le &\sum_{\nu=1}^n \sup_{x\in D} P\Bigg(\sup_{0<s<1/\{R\prod_{\mu=1}^n h_\mu(\lambda/2)\}} \left|\sum_{j,k=1}^d \int_0^s (\sigma^{jk}\partial_k\psi_\mu)(X(u,x,w))dw^j(u)\right| > \frac{\lambda}{4}, \\
&\qquad \tau > \frac{1}{R\prod_{\mu=1}^n h_\mu(\lambda/2)}\Bigg) \\
&+ \sum_{\nu=1}^n \sup_{x\in D} P\Bigg(\sup_{0<s<1/\{R\prod_{\mu=1}^n h_\mu(\lambda/2)\}} \left|\int_0^s \frac{\Delta\psi_\nu}{2}(X(u,x))du\right| > \frac{\lambda}{4}, \\
&\qquad \tau > \frac{1}{R\prod_{\mu=1}^n h_\mu(\lambda/2)}\Bigg) \\
=:&I_{11} + I_{12}.
\end{aligned}$$

By Theorem II-7.2' of [9], for each $\nu = 1, 2, \cdots, n$ and each $x \in D$, there exists a 1-dimensional Brownian motion $B(t)$ such that

$$\begin{aligned}(3.7)\qquad &\sum_{j,k=1}^{d} \int_0^s (\sigma^{jk}\partial_k \psi_\nu)(X(u,x,w))dw^j(u) \\ =&B\left(\int_0^s \frac{1}{2}\|d\psi_\nu(X(u,x,w))\|^2 du\right).\end{aligned}$$

Since $\|d\psi_\nu\|$ is bounded, we have

$$\begin{aligned}I_{11} \leq& nP\left(\sup_{0<s<C_1/\{R\prod_{\mu=1}^n h_\mu(\lambda/2)\}} |B(s)| > \frac{\lambda}{4}\right) \\ \leq& C_2 \exp\Big(-C_3\lambda^2 R \prod_{\mu=1}^n h_\mu\Big(\frac{\lambda}{2}\Big)\Big)\end{aligned}$$

from (8.29) in [8]. On the other hand, since $\Delta\psi_\nu$ is bounded, there is some $C_4 > 0$ such that $I_{12} = 0$ if $\lambda R \prod_{\mu=1}^n h_\mu(\lambda/2) \geq C_4$.

On the other hand, for each $\nu = 1, 2, \cdots, n$, we set

$$\begin{aligned}\sigma^1_{\lambda,\nu}(x) :=& \inf\{s \geq 0 : |\psi_\nu(X(s,x))| < \lambda\}, \\ \tau^1_{\lambda,\nu}(x) :=& \inf\{s \geq \sigma^1_{\lambda,\nu}(x) : |\psi_\nu(X(s,x))| > \lambda\}, \\ &\vdots \\ \sigma^n_{\lambda,\nu}(x) :=& \inf\{s \geq \tau^{n-1}_{\lambda,\nu}(x) : |\psi_\nu(X(s,x))| < \lambda\}, \\ \tau^n_{\lambda,\nu}(x) :=& \inf\{s \geq \sigma^n_{\lambda,\nu}(x) : |\psi_\nu(X(s,x))| > \lambda\}, \\ &\vdots\end{aligned}$$

Then we have

$$I_2 \leq P\Big(\sum_{m=1}^{\infty} \Big|\Big(\tau^m_{\lambda,\nu}(x) \wedge \frac{t}{2}\Big) - \Big(\sigma^m_{\lambda,\nu}(x) \wedge \frac{t}{2}\Big)\Big| > \frac{t}{2n} \text{ for some } \nu \in \{1, 2, \cdots, n\}\Big)$$

and, for each $\nu = 1, 2, \cdots, n$,

$$(3.8)\qquad \sup_{0\leq s\leq t/2} \Big|\sum_{m=1}^{\infty} (\psi_\nu(X(\tau^m_{\lambda,\nu}(x) \wedge s)) - \psi_\nu(X(\sigma^m_{\lambda,\nu}(x) \wedge s)))\Big| \leq 2\lambda.$$

By using (3.6), we have either

$$\sup_{0\le s\le t/2}\left|\sum_{m=1}^{\infty}\sum_{j,k=1}^{d}\int_{\sigma^m_{\lambda,\nu}(x)\wedge s}^{\tau^m_{\lambda,\nu}(x)\wedge s}(\sigma^{jk}\partial_k\psi_\nu)(X(u,x,w))dw^j(u)\right|\le 4\lambda$$

or

$$\sup_{0\le s\le t/2}\left|\sum_{m=1}^{\infty}\int_{\sigma^m_{\lambda,\nu}(x)\wedge s}^{\tau^m_{\lambda,\nu}(x)\wedge s}\frac{\Delta\psi_\nu}{2}(X(u,x,w))du\right|\ge 2\lambda$$

from (3.8). Accordingly, we estimate as

$$\begin{aligned}
I_2 \le& \sum_{\nu=1}^{n} P\Bigg(\sum_{m=1}^{\infty}\left|\left(\tau^m_{\lambda,\nu}(x)\wedge\frac{t}{2}\right)-\left(\sigma^m_{\lambda,\nu}(x)\wedge\frac{t}{2}\right)\right|>\frac{t}{2n},\\
&\sup_{0\le s\le t/2}\left|\sum_{m=1}^{\infty}\sum_{j,k=1}^{d}\int_{\sigma^m_{\lambda,\nu}(x)\wedge s}^{\tau^m_{\lambda,\nu}(x)\wedge s}(\sigma^{jk}\partial_k\psi_\nu)(X(u,x,w))dw^j(u)\right|\le 4\lambda,\tau>\frac{t}{2}\Bigg)\\
&+\sum_{\nu=1}^{n} P\Bigg(\sup_{0\le s\le t/2}\left|\sum_{m=1}^{\infty}\int_{\sigma^m_{\lambda,\nu}(x)\wedge s}^{\tau^m_{\lambda,\nu}(x)\wedge s}\frac{\Delta\psi_\nu}{2}(X(u,x,w))du\right|\ge 2\lambda,\tau>\frac{t}{2}\Bigg)\\
=:& I_{21}+I_{22}.
\end{aligned}$$

As in (3.7), there exists a 1-dimensional Brownian motion $B(t)$ such that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{j,k=1}^{d}\int_{\sigma^m_{\lambda,\nu}(x)\wedge s}^{\tau^m_{\lambda,\nu}(x)\wedge s}(\sigma^{jk}\partial_k\psi_\nu)(X(u,x,w))dw^j(u)\\
=&B\left(\sum_{m=1}^{\infty}\int_{\sigma^m_{\lambda,\nu}(x)\wedge s}^{\tau^m_{\lambda,\nu}(x)\wedge s}\|d\psi_\nu(X(u.x,w))\|^2du\right).
\end{aligned}$$

By the condition (D,n) (ii), we have

$$\inf\{\|d\psi_\nu(x)\|:x\in D,|\psi_\nu(x)|\le\lambda\}\ge C-C'\lambda^2$$

for some $C,C'>0$. Therefore, from (8.27) in [8], we obtain

$$I_{21}\le C''\exp\left(-\frac{t(C-C'\lambda^2)}{\lambda^2}\right).$$

On the other hand, by Lemma 3.4, we see that

$$I_{22}=0\qquad\text{if}\qquad\lambda<C\left(1\wedge\frac{1}{t}\right)$$

for some $C > 0$.

As in [23], we now note the condition (E, ρ). For an arbitrary fixed $0 < \varepsilon < 1/n$, we take $\lambda_\varepsilon > 0$ such that $\lambda^\rho \log(1/h_\mu(\lambda)) < \varepsilon/2$ for any $0 < \lambda < \lambda_\varepsilon$ and any $\mu = 1, 2, \cdots, n$. If we set $\lambda = 2(\log R)^{-1/\rho}$, then we see that $\lambda^2 R \prod_{\mu=1}^n h_\mu(\lambda/2) > R^{1/2}$ and $t(C - C'\lambda^2)/\lambda^2 > C''t(\log R)^{2/\rho}$ for large enough R. Thus there are $C_1, C_2, \cdots, C_6 > 0$ such that

$$\begin{aligned} &P\left(\int_0^t \prod_{\mu=1}^n h_\mu(\psi_\mu(X(s,x))ds < \frac{1}{R}, \tau > t\right) \\ \leq &C_1 \exp(-C_2 R^{1/2}) + C_3 \exp(-C_4 t(\log R)^{2/\rho}) \end{aligned}$$

for any $R \geq \exp(C_5 t^\rho) \vee (2/t)^2 \vee C_6$. Then as in (3.49) of [23], we obtain (3.5) under the condition (E, ρ) for some $\rho < 2$.

Proof of Theorem 3.1. By Lemma 3.1 and the diamagnetic estimate (Lemma 2.4), we have

$$\inf \mathrm{spec} L(\xi b) \geq C_1 \inf \mathrm{spec}(-\Delta + \xi \prod_{\mu=1}^n |\psi_\mu| h_\mu \circ \psi_\mu) - C_2$$

for some C_1, $C_2 > 0$. Then, by Lemmas 3.2 and 3.3, it is enough to show that

$$\sup_{x \in D} E\left[\left(\int_0^t \prod_{\mu=1}^n (|\psi_\mu| h_\mu \circ \psi_\mu)(X(s,x))ds\right)^{-p} : \tau > t\right] < \infty$$

for any $p \geq 1$. This follows from Proposition 3.1.

From Theorem 3.1, we obtain the following as in Section 2:

Corollary 1. *Under the assumption of Theorem 3.1 and (PG), it holds that*

$$\sup_{\substack{\xi \in \mathbb{R} \\ x, y \in D}} |\xi|^k |e^{-tL(\xi b)}(x, y)| < \infty$$

for any $k \in \mathbb{N}$ and $t > 0$.

Moreover Corollary 1 leads to the following:

Corollary 2. *Under the assumption of Theorem 3.1 and* (PG)*, for any* $t > 0$*,* $x, y \in D$*, the distribution of* $\int_0^t b(X(s,x)) \circ dX(s,x)$ *under the conditional probability* $P(\cdot | X(t,x) = y)$ *has a smooth density function whose derivatives are all bounded: there is a function* $p(t,x,y;\lambda)$*,* $t > 0$*,* $x, y \in D$*,* $\lambda \in \mathbb{R}$ *such that* (2.19) *holds,* $p(t,x,y;\lambda)$ *is smooth in* λ*, and* $\partial^k p(t,x,y;\lambda)/\partial\lambda^k$ *are all bounded in* λ *for any* $k \in \mathbb{Z}_+$*.*

4. The Neumann condition

In this section, we consider the Neumann boundary condition. We consider only nondegenerate magnetic fields and we take only a half space as the domain D : let $D = \{x = (x^1, x^2, \cdots, x^d) \in \mathbb{R}^d : x^1 > 0\}$ given a metric as before, and b be a real C^∞differential form as before. Let $C_0^\infty(\overline{D})$ be the set of all $\mathbb{C}$-valued functions on D which can be extended to C^∞ functions with compact supports on $\mathbb{R}^d$. For any $\varphi \in C_0^\infty(\overline{D})$, we set $q(b)(\varphi)$ as in (2.1) and $\mathrm{Dom}(q^N(b))$ as the completion of $C_0^\infty(\overline{D})$ by the norm $\sqrt{q(b)(\cdot) + \|\cdot\|^2}$. We extend $q(b)(\cdot)$ naturally to $\mathrm{Dom}(q^N(b))$ so that $(q(b),\mathrm{Dom}(q^N(b)))$ becomes a closed sesquilinear form. Let $L^N(b)$ be the associated self-adjoint operator: $L^N(b)$ is the operator with the Neumann boundary condition. We will study the asymptotics of inf spec $L^N(\xi b)$ as ξ tends to infinity, where spec $L^N(\xi b)$ is the spectral set of the self adjoint operator $L^N(\xi b)$.

The main theorems are the following:

Theorem 4.1. *We assume that the metric on D is standard and that the magnetic field is uniform: $g_{jk}(x) \equiv \delta_{jk}$ and $db_{jk}(x), j, k = 1, 2, \cdots, d$ are independent of x. Then there exists a universal positive constant $\mathcal{K}$ such that*

$$\inf \mathrm{spec} L^N(b) \geq \mathcal{K}\|db\|_\infty, \tag{4.1}$$

where $\|db\|_\infty$ is the maximum norm of db as

$$\|db\|_\infty = \max_{1 \leq j,k \leq d} |db_{jk}|.$$

Theorem 4.2. *Under the condition (G), it holds that*

$$\varliminf_{\xi \uparrow \infty} \frac{\inf \mathrm{spec} L^N(\xi b)}{\xi} \geq \inf_{x \in D} \mathcal{K}\|db(x)\|_\infty, \tag{4.2}$$

where $\mathcal{K}$ is the constant in Theorem 4.1.

Remark 4.1. (i) In the circumstances of Theorem 4.1, by Weyl's criterion (see e.g.[19]), we can show

$$\text{ess spec}L^N(b) \supset \text{spec}\mathcal{L}(b),$$

where ess spec$L^N(b)$ is the essential spectrum of the operator $L^N(b)$ and $\mathcal{L}(b)$ is the operator with the uniform magnetic field on $\mathbb{R}^d$. If we consider the Dirichlet condition instead of the Neumann condition, then, by the same criterion, we can show

$$\text{spec}L(b) = \text{ess spec}L(b) = \text{spec}\mathcal{L}(b).$$

In particular we have

$$\inf \text{spec}L(b) = \frac{1}{4}\|db\|_1. \tag{4.1)'}$$

(ii) In Theorem 4.2, if we consider the Dirichlet condition instead of the Neumann condition, then we have

$$\lim_{\xi\uparrow\infty} \frac{\inf \text{spec}L(\xi b)}{\xi} = \inf_{x\in D} \frac{1}{4}\|db(x)\|_1 \tag{4.2)'}$$

(cf. Theorem 11.1 in [15], Theorem 4.1 in [22]).

For the proof, we first prepare a fundamental lemma. For any $x = (x^1, x^2, \cdots, x^d) \in \mathbb{R}^d$, we write $\overline{x} = (x^2, \cdots, x^d)$. For any function φ on D, let $\widehat{\varphi}$ and $\widetilde{\varphi}$ be functions on $\mathbb{R}^d$, defined by

$$\widehat{\varphi}(x) = \text{sgn}(x^1)\varphi(|x^1|, \overline{x}),$$

$$\widetilde{\varphi}(x) = \varphi(|x^1|, \overline{x}).$$

We extend the metric to $\mathbb{R}^d$ by

$$(\check{g}_{jk}(x))_{1\le j,k\le d} = \begin{pmatrix} \widetilde{g_{11}}(x) & \widehat{g_{12}}(x) & \cdots & \widehat{g_{1d}}(x) \\ \widehat{g_{21}}(x) & \widetilde{g_{22}}(x) & \cdots & \widetilde{g_{2d}}(x) \\ \vdots & \vdots & & \vdots \\ \widehat{g_{d1}}(x) & \widetilde{g_{d2}}(x) & \cdots & \widetilde{g_{dd}}(x) \end{pmatrix}$$

and extend the form b to $\mathbb{R}^d$ by

$$\check{b}(x) = \widehat{b_1}(x)dx^1 + \widetilde{b_2}(x)dx^2 + \cdots + \widetilde{b_d}(x)dx^d.$$

In the following, for simplicity, we take b so that $b_1 = 0$ on $\{x^1 = 0\}$. This is always possible by the gauge invariance. Then $\check{b}$ is continuous on $\mathbb{R}^d$. For any $\psi \in C_0^\infty(\mathbb{R}^d)$, we define $Q(\check{b})(\psi)$ as in (2.1) where D is replaced by $\mathbb{R}^d$. Then we have the following:

Lemma 4.1.

$$\inf\{q(b)(\varphi) : \varphi \in C_0^\infty(\overline{D}), \|\varphi\| = 1\} = \inf\{Q(\check{b})(\psi) : \psi \in C_0^\infty(\mathbb{R}^d), \|\psi\| = 1\}. \tag{4.3}$$

Proof. It is easy to see that

$$\begin{aligned} &\int_D \sum_{j,k=1}^d g^{jk}(x)\{(i\partial_j + b_j(x))\varphi(x)\}\overline{\{(i\partial_k + b_k(x))\varphi(x)\}}g(x)dx \\ &= \int_{D^*} \sum_{j,k=1}^d \check{g}^{jk}(x)\{(i\partial_j + \check{b}_j(x))\widetilde{\varphi}(x)\}\overline{\{(i\partial_k + \check{b}_k(x))\widetilde{\varphi}(x)\}}\check{g}(x)dx \end{aligned} \tag{4.4}$$

for any $\varphi \in C_0^\infty(\overline{D})$, where $D^* = (-\infty, 0) \times \mathbb{R}^{d-1}$ and $\check{g}(x) = \sqrt{\det(\check{g}_{jk}(x))}$. From this, we have

$$q(b)(\varphi) = \frac{1}{2}Q(\check{b})(\widetilde{\varphi}).$$

Since $\widetilde{\varphi}$can be approximated by elements of $C_0^\infty(\mathbb{R}^d)$ in $\mathrm{Dom}(Q(\check{b}))$, we have

$$\inf\{q(b)(\varphi) : \varphi \in C_0^\infty(\overline{D}), \|\varphi\| = 1\} \geq \inf\{Q(\check{b})(\psi) : \psi \in C_0^\infty(\mathbb{R}^d), \|\psi\| = 1\}.$$

On the other hand, from (4.4), we have

$$Q(\check{b})(\psi) = q(b)(\psi|_D) + q(b)(\psi(-x^1, \overline{x})|_D)$$

for any $\psi \in C_0^\infty(\mathbb{R}^d)$. From this, we have

$$\inf\{Q(\check{b})(\psi) : \psi \in C_0^\infty(\mathbb{R}^d), \|\psi\| = 1\} \geq \inf\{q(b)(\varphi) : \varphi \in C_0^\infty(\overline{D}), \|\varphi\| = 1\}.$$

Proof of Theorem 4.1.

We may assume that the dimension d is 2. In fact, for the general dimensional case, we choose $1 \leq j, k \leq d$ so that

$$db_{jk} = \|db\|_\infty,$$

and estimate as

$$q(b)(\varphi) \geq \frac{1}{2}(\|(i\partial_j + b_j(x))\varphi(x)\|^2 + \|(i\partial_k + b_k(x))\varphi(x)\|^2).$$

The right hand side is estimated by the result of the 2 dimensional case.

From Lemma 4.1, we have

$$\inf \mathrm{spec} L^N(b) = \inf\{Q(\check{b})(\psi) : \psi \in C_0^\infty(\mathbb{R}^2), \|\psi\| = 1\}.$$

We can show that

$$Q(\check{b})(\psi) \geq \pm\frac{|db|}{2}((\mathrm{sgn}\ x^1)\psi, \psi), \tag{4.5}$$

where db is identified with the component db_{12}. In fact, we have

$$\begin{aligned} 0 \leq & \frac{1}{2}\int_{\mathbb{R}^2} dx \left|\{(i\partial_1 + \check{b}_1) \pm (i\partial_2 + \check{b}_2)\}\psi\right|^2 \\ = & Q(\check{b})(\psi) \pm \frac{1}{2}\int_{\mathbb{R}^2} dx\ \mathrm{sgn}(x^1)(db)|\psi|^2. \end{aligned}$$

For an arbitrarily fixed $0 < \alpha < 1$, we set $\eta(t) = (t \wedge \alpha) \vee (-\alpha)$ for $t \in \mathbb{R}$ and

$$\chi_\pm(x) = \sqrt{\frac{1 \pm \eta(x^1)}{2}}.$$

It is easy to see that $\chi_\pm \in H^{1,2}_{loc}(\mathbb{R}^2)$. Then, by the IMS localization (Lemma 2.1), we have

$$Q(\check{b})(\psi) \geq \left(\left(\frac{|db|}{2}|\eta(x^1)| - \sum_{\sigma\in\{+,-\}} \frac{|d\chi_\sigma|^2}{2}\right)\psi, \psi\right).$$

By an easy calculation, we have

$$\text{ess sup} \sum_{\sigma\in\{+,-\}} |d\chi_\sigma|^2 = \frac{1}{4(1-\alpha^2)}.$$

Moreover, by using also the diamagnetic estimate (Lemma 2.4), we have

$$\inf \text{spec} L^N(b) \geq \inf \text{spec} \left(-\frac{\Delta}{4} + \frac{|db|}{4}|\eta(x^1)| \right) - \frac{1}{16(1-\alpha^2)},$$

where the operator on the right hand side is the Friedrichs extension of the corresponding operator on $C_0^\infty(\mathbb{R}^2)$. However, by the min-max principle, we easily see that

$$\inf \text{spec} \left(-\frac{\Delta}{4} + \frac{|db|}{4}\eta(x^1) \right) \geq \frac{1}{4} \inf \text{spec} \left(-\frac{d^2}{dt^2} + |db|t \right)^N_{[0,\alpha]},$$

where the operator on the right hand side is the operator on the interval $[0,\alpha]$ with the Neumann boundary condition.

On the other hand, by using the min-max principle and changing the variables, we easily see that

$$\inf \text{spec} L^N(b) = \frac{\inf \text{spec} L^N(\xi b)}{\xi} \tag{4.7}$$

for any $\xi > 0$. Therefore we have

$$\inf \text{spec} L^N(b) \geq \frac{1}{4\xi} \inf \text{spec} \left(-\frac{d^2}{dt^2} + \xi|db|t \right)^N_{[0,\alpha]} - \frac{1}{16\xi(1-\alpha^2)}.$$

By using the min-max principle and changing the variables, we have

$$\frac{1}{\xi} \inf \text{spec} \left(-\frac{d^2}{dt^2} + |db|t \right)^N_{[0,\alpha]} = \frac{|db|^{2/3}}{\xi^{1/3}} N((\xi|db|)^{1/3}),$$

where

$$N(R) = \inf \text{spec} \left(-\frac{d^2}{dt^2} + t \right)^N_{[0,R]}$$

for $R > 0$. By the min-max principle, we see that $N(R)$ is an increasing function in R. Therefore for arbitrarily fixed $R > 0$, we have

$$\inf \mathrm{spec} L^N(b) \geq \frac{|db|^{2/3}}{4\xi^{1/3}} N(R) - \frac{1}{16\xi(1-\alpha^2)}$$

for any $\xi \geq R^3/|db|$. If we regard the right hand side of this inequality as a function of all $\xi > 0$, then the maximum is

$$\frac{|db| N(R)^{3/2} \sqrt{1-\alpha^2}}{3^{3/2}} =: \mathcal{A}$$

and this is attained at

$$\left\{ \frac{3}{4(1-\alpha^2)N(R)} \right\}^{3/2} \frac{1}{|db|} =: \xi_0.$$

Therefore, if $\xi_0 \geq R^3/|db|$, then we have $\inf \mathrm{spec} L^N(b) \geq \mathcal{A}$. This condition is rewritten as

$$\frac{3}{4} \geq N(R) R^2 (1-\alpha^2).$$

Now we take α and R so that this holds as an equality. Then we have

$$\mathcal{A} = \frac{|db| N(R)}{6R}.$$

Therefore we have

$$\inf \mathrm{spec} L^N(b) \geq \frac{|db|}{6} \sup_{R>0} \frac{N(R)}{R}, \tag{4.8}$$

which completes the proof.

The proof of Theorem 4.2 is reduced to that of the following modification of Theorem 3.1 of [4]:

Lemma 4.2. *Under the condition (G), there exist constants C_1, C_2 such that*

$$\left(1+\frac{C_1}{\xi^{1/4}}\right)q(\xi b)(\varphi)\geq \xi\int_D\left(\mathcal{K}\|db(x)\|_\infty-\frac{C_2}{\xi^{1/4}}\right)|\varphi(x)|^2 d\mathrm{vol}(x) \tag{4.9}$$

for any $\varphi\in C_0^\infty(\overline{D})$ and large enough $\xi>0$.

The proof is almost the same with that of Theorem 3.1 of [4] except for using Theorem 4.1.

We next consider the asymptotics of the heat kernel as in Sections 2 and 3. We assume

(BO) $g_{1j}(x)\equiv\delta_{1j}$.

We first give the heat kernel $e^{-tL^N(\xi b)}(x,y)$: for any $\varphi\in L^2(D)$,

$$(e^{-tL^N(\xi b)}\varphi)(x)=<e^{-tL^N(\xi b)}(x,\cdot),\varphi>,$$

where $<\cdot,\cdot>$ is the inner product in L^2 which is complex linear in the both variables. Then we have the following representation:

Proposition 4.1. *We assume the conditions (BO) and (PG). Let $X^N(t,x)$, $t>0$, $x\in\overline{D}$, be the reΩecting barrier Brownian motion on $\overline{D}$, starting at x: $X^N(t,x)$ is a diffusion process generated by $\Delta^N/2$, where Δ^N is the Laplace-Beltrami operator with the Neumann boundary condition. Let $e^{t\Delta^N/2}(x,y)$ be the integral kernel of the semigroup $e^{t\Delta^N/2}$. Then we have*

$$\begin{aligned}&e^{-tL^N(b)}(x,y)\\ =&E\left[\exp\left(-i\int_0^t b(X^N(s,x))\circ dX^N(s,x)\right)\middle| X^N(t,x)=y\right]e^{t\Delta^N/2}(x,y),\end{aligned} \tag{4.10}$$

for $(t,x,y)\in(0,\infty)\times\overline{D}\times\overline{D}$.

Proof. If $b_1=0$ on ∂D, then the proof is the same with that for the case $b=0$ (cf.[2], [9]). For the general case, we choose $f\in C_b^\infty(\overline{D})$ so that $\partial f/\partial x^1=b_1$ on ∂D and set

$\overset{\circ}{b} := b - df$, which satisfies $\overset{\circ}{b}_1 = 0$ on ∂D. Then, by the gauge invariance, we can show (4.10) as follows:

$$\begin{aligned}
&E\left[\exp\Big(-i\int_0^t b(X^N(s,x))\circ dX^N(s,x)\Big)\,\Big|\,X^N(t,x)=y\right]e^{t\Delta^N/2}(x,y)\\
=&e^{if(x)}E\left[\exp\Big(-i\int_0^t \overset{\circ}{b}(X^N(s,x))\circ dX^N(s,x)\Big)\,\Big|\,X^N(t,x)=y\right]\\
&\qquad\times e^{t\Delta^N/2}(x,y)e^{-if(y)}\\
=&e^{if(x)}e^{-tL^N(\overset{\circ}{b})}(x,y)e^{-if(y)}\\
=&e^{-tL^N(b)}(x,y).
\end{aligned}$$

From Theorem 4.2, we obtain the following as in [22]:

Corollary 1. *Under the conditions (PG) and (BO), it holds that*

$$\overline{\lim_{\xi\to\pm\infty}}\sup_{(t,x,y)\in[t_0,\infty)\times\overline{D}\times\overline{D}}\frac{1}{t|\xi|}\log|e^{-tL^N(\xi b)}(x,y)| \le -\inf_{x\in D}\mathcal{K}\|db(x)\|_\infty \tag{4.11}$$

for any $t_0 > 0$.

Moreover Corollary 1 leads to the following:

Corollary 2. *We assume the conditions (PG), (BO) and*

$$\inf_{x\in D}\|db(x)\|_\infty > 0.$$

Then for any $t > 0$, $x, y \in \overline{D}$, the distribution of $\int_0^t b(X^N(s,x))\circ dX^N(s,x)$ under the conditional probability $P(\cdot|X^N(t,x)=y)$ has a real analytic density function: there is a function $p^N(t,x,y;\lambda)$, $t>0$, $x,y\in\overline{D}$, $\lambda\in\mathbb{R}$ such that $p^N(t,x,y;\lambda)$ is real analytic in λ and

$$P\left(\int_0^t b(X^N(s,x))\circ dX^N(s,x)\in A\,\Big|\,X^N(t,x)=y\right)=\int_A p^N(t,x,y;\lambda)d\lambda \tag{4.12}$$

for any Borel set A in $\mathbb{R}$. The radius of convergence of its Taylor series around any point is greater than or equal to $t\inf_{x\in D}\ \mathcal{K}\|db(x)\|_\infty$.

Remark 4.2. Corollary 2 is known as transverse analyticity. This is proven in Malliavin [12], [13], [14] and Prat [18] for the case without boundary and in Ueki [22] for the absorbing barrier Brownian motion.

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