A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane

https://www.math.h.kyoto-u.ac.jp/users/ueki/2DWN-Scr-c1.pdf
https://www.math.h.kyoto-u.ac.jp/users/ueki/presen20230707TohokuProbSem

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Motivation

White noise: $\xi = (\xi(x))_{x\in\mathbb{R}^2} \underset{\text{i.i.d.}}{\sim}$ N(0, *) : The most basic but wild random field

ξ: Gaussian random field, $\mathbb{E}[\xi(x)] = 0$, $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$

$\mathbb{P}(\text{"}x \mapsto \xi(x)\text{"} \in C_{loc}^{-1-\epsilon}) = 1$

The Schrödinger operator: $-\Delta + V$: $\underset{\text{closure of the op. on } C_0^\infty(\mathbb{R}^2)}{\text{self-adjoint on}}$ $L^2(\mathbb{R}^2) \to$ Spectral Analysis

$$\Uparrow$$

$$-c(|x|^2+1) \le V(x) \in L^2_{loc}(\mathbb{R}^2)$$

Claim: Realize $-\Delta + \xi$ as a self-adjoint operator

Other dimensions

$\xi = (\xi(x))_{x \in \mathbb{R}^d}$: White noise on $\mathbb{R}^d \Rightarrow$ "$x \mapsto \xi(x)$" $\in C_{loc}^{-d/2-\epsilon}$

$d = 1 \Rightarrow W^{1,2}([a,b]) \ni f, g \mapsto \int_a^b (f'g' + \underset{C_{loc}^{-1/2-\epsilon}}{\xi} fg)dx$: well-defined

$-\Delta + \xi$ is realized as a self-adjoint operator

M. Fukushima and S. Nakao (1977) Spectral asymptotics

(Asymptotics of the Integrated density of states)

N. Minami (1988) (1989) Self-adj. on $\mathbb{R}$, Exp.Loc ($\xi \to \partial$(Lévy process))

$d \geq 4 \Rightarrow$ No results

$d = 2, 3$ are studied recently

$d = 2$ is easier than $d = 3$

$\xi \in C_{loc}^{-1-\epsilon}$ $\qquad$ $\xi \in C_{loc}^{-3/2-\epsilon}$

Related works on singular SPDEs

M. Hairer (2014) A theory of regularity structures,
M. Gubinelli, P. Imkeller and N. Perkowski (2015) Paracontrolled calcuclus
A. Kupiainen (2016) Renormalization Group
$\Rightarrow$ Stochastic quatization equation for ϕ^4_3 Euclidean quantum field theory
Generalized continuous parabolic Anderson model
Kardar–Parisi–Zhang type equation
Navier-Stokes equation with very singular forcing
and so on

Eg. Continuous parabolic Anderson model

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) - c_\varepsilon)u(t,x) \text{ for } t>0, x \in \underbrace{\mathbb{R}^2/\mathbb{Z}^2}_{\text{compact}}$$

Related works on Operators

For the Schrödinger operator

R. Allez and K. Chouk (2015) Paracontrolled calcuclus based on Fourier Analysis

$-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) - c_\varepsilon)$ on $\mathbb{R}^2/\mathbb{Z}^2 \Leftarrow$ Self-adjointness, Discrete Spectrum,

Asymptotic Distribution

M. Gubinelli, B. Ugurcan and I. Zachhuber (2020) Extension to $\mathbb{R}^3/\mathbb{Z}^3$

$$i\partial_t u(t,x) = \underbrace{(-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) - c_\varepsilon) + k_\xi)}_{\geq 0} u(t,x) - (u|u|^2)(t,x)$$

$$\partial_t^2 u(t,x) = (-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) - c_\varepsilon) + k_\xi) u(t,x) - (u^3)(t,x)$$

C. Labbé (2019) similar results by regurarity structure for

$-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) - c_\varepsilon)$ on $(-L, L)^{2or3}$ with periodic or Dirichlet conditions

K. Chouk and W. van Zuijlen (2021)

Asymptotics of Eigenvalues of the Dirichlet operaotor on $(-L, L)^2$ as $L \to \infty$

T. Matsuda (2022) Asymptotics of Integrated density of states

Topics on Random Schrödinger operators

Anderson transition:

Point spectrum with exponentially decaying eigenstates
for stronly random potentials and energies near the edge of the spectrum
(Anderson localization)

Absolutely continuous spectrum
for weakly random potentials and energies far from the edge of the spectrum

This topic is discussed for stationary potentials on noncompact spaces.

Singular SPDEs on noncompact spaces

M. Hairer and C. Labbé, (2015)
M. Hairer and C. Labbé, (2018)
$\partial_t u_\varepsilon = \Delta u_\varepsilon + (\xi_\varepsilon - c_\varepsilon) u_\varepsilon, t > 0, x \in \mathbb{R}^d,\ u_\varepsilon(0, \cdot) = u_0$
Paraboloc Anderson model: ξ space white noise (depends only on x) $d = 2, 3$
Stochastic Heat equation: ξ space-time white noise (depends on (t, x)) $d = 1$
$\exists \lim_{\varepsilon \to 0} u_\varepsilon$ uniformly on compact sets in probability
depends continuously in u_0

A. Dahlqvist, J. Diehl and B. Driver, (2019)
PAM on 2D closed manifolds by regularity structres,
$c_\varepsilon(x)$ is not constant in x when the manifold is not $\mathbb{R}^2/\mathbb{Z}^2$

W. König, N. Perkowski and W. van Zuijlen, (2022)
Feynman-Kac type representation of the solution
Asymptotics as $t \to \infty$

Related operators on noncompact spaces

B. Ugurcan, (2022) $-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) - \widetilde{c}_\varepsilon(x))$ on $\mathbb{R}^2$ with $\widetilde{c}_\varepsilon(x) \xrightarrow{|x|\to\infty} 0$

$$= \underline{-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon^{\uparrow}(x) - \widetilde{c}_\varepsilon(x))} + \underline{\xi_\varepsilon^{\downarrow}(x)},$$

$\uparrow$ An extension of the method for the compact case $\nwarrow$ By a commutator estimate,

where $\xi = \xi^{\uparrow}(x) + \xi^{\downarrow}(x)$ (Smooth functions in x)

$$= \sum_{n=-1}^{\infty} \widetilde{\chi_{[[2^n],2^{n+1}]}}(|x|)\{\underbrace{\widetilde{\chi_{[c2^n,\infty)}}(-\Delta)\xi}_{\text{high energy pert}} + \underbrace{\widetilde{\chi_{[0,c2^n]}}(-\Delta)\xi}_{\text{low energy pert}}\}$$

$\widetilde{c}_\varepsilon(x) = \mathbb{E}[$A resonant product of $\xi_\varepsilon^{\uparrow}(x)$ and $(1-\Delta)^{-1}\xi_\varepsilon^{\uparrow}(x)]$

Heat semigroup approach

Paracontrolled calcuclus by Heat semigroup
(suitable for noncompact manifolds, graphs and so on)
Paraproducts defined by using Heat Semigroup
(cf. More people define Paraproducts by using Fourier Analysis)

I. Bailleul and F. Bernicot (2016) $\exists \lim_{\varepsilon \to 0} u_\varepsilon$
generalized PAM on 2D manifold (without compactness)

I. Bailleul, F. Bernicot and D. Frey (2018) $\exists \lim_{\varepsilon \to 0} u_\varepsilon$
generalized PAM and multiplicative Burgers eq. on 3D manifold
(without compactness)

A. Mouzard (2022) Self-adjointness, Discrete Spectrum, Its asymptotics of
$\underset{\text{eg. Lap.Belt}}{-\Delta} + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) - c_\varepsilon(x))$ on 2D Compact manifold, where $\xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta} \xi)(x)$

$$c_\varepsilon(x) \equiv c_\varepsilon \text{ on } \mathbb{R}^2/\mathbb{Z}^2$$

Our Topics

$-\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) - c_\varepsilon)$ on $\mathbb{R}^2$, where $\xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta}\xi)(x)$

Self-adjointness

Spectral set $= \mathbb{R}$

Singular SPDEs — Random Schrödinger operators

Strengthen the relation

We should remark $\|\xi\|_{C^{-1-\epsilon}(\{|x|<R\})} \lesssim (\log R)^2$

Tools: Paracontrolled calcuclus by Heat semigroup referring Mouzard (2020) and the partition of unity

The Besov Spaces

$\mathcal{C}^\alpha(\mathbb{R}^2) = \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{R}^2)$: the Besov α-Hölder space
$\mathcal{H}^\alpha(\mathbb{R}^2) = \mathcal{W}^{\alpha,2}(\mathbb{R}^2) = \mathcal{B}^\alpha_{2,2}(\mathbb{R}^2)$: the Sobolev space with the index α.
Generalization by heat semigoup approach: $0 << b \in 2\mathbb{Z}$ fixed
For $p, q \in [1,\infty], \alpha \in (-2b, 2b)$, $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2) = \overline{C_0^\infty(\mathbb{R}^2)}^{\|\cdot\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)}}$

$$\|f\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{R}^2)} := \|e^\Delta f\|_{L^p(\mathbb{R}^2:dx)} \qquad \bigcup_{k\in(|\alpha|,2b]\cap\mathbb{Z}} StGC^k$$

$$\| $$

$$+\sup\{\|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)} : Q \in StGC^{(|\alpha|,2b]}\}$$

$$StGC^k = \{((\sqrt{t}\nabla)^\alpha \underbrace{\sum_{j=0}^{c-1}\frac{(-t\Delta)^j}{j!}e^{t\Delta}}_{=:P_t^{(c)}})_{t\in(0,1]} : \alpha \in \mathbb{Z}^2_+, \alpha_1+\alpha_2 = k, 1 \le c \le b\}$$

Standard families of Gaussian operators with cancellation of order k.

Paraproducts

$$fg = \underbrace{-\int_0^1 dt\partial_t\{P_t^{(b)}((P_t^{(b)}f)(P_t^{(b)}g))\}}_{=P_f g+\Pi(f,g)+P_g f} + P_1^{(b)}((P_1^{(b)}f)(P_1^{(b)}g)) \because P_0^{(b)} = I$$

$$P_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)) \text{: paraproduct term}$$

$$\Pi(f,g) := \sum_\mu c_\mu \int_0^1 \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g)) \text{ : resonating term}$$

$$P^\nu, P^\mu \in StGC^{[0,b/2)},\ Q^{1,\nu}, Q^{2,\nu}, Q^{1,\mu}, Q_t^{2,\mu} \in StGC^{[b/2,2b]}$$

$$\{\chi_a\}_{a\in\mathbb{Z}^2} \subset C^\infty(\mathbb{R}^2 \to [0,1]) \text{ s.t. } \sum_{a\in\mathbb{Z}^2} \chi_a^2 \equiv 1, \text{ supp}\,\chi_a \subset \Lambda_2(a) := a + (-1,1)^2$$

$$\chi_a(\cdot) = \chi_0(\cdot - a)$$

The continuity of paraproducts

(i) For any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$\|\chi_{a_1} P_{\chi_{a_2} g}(\chi_{a_3} f)\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)} \leq \begin{cases} C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^\infty(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^2(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(ii) For any $\alpha \in (-\infty,0)$ and $\beta \in \mathbb{R}$,

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)} \leq \begin{cases} C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{H}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(iii) For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta>0$,

$$\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)} \leq C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)).$$

1st ansatz for the definition of the operator

$$\|\chi_a\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \le C_{\epsilon,\xi}(\log(2+|a|))^2$$

$$u, H^\xi u := -\Delta u + \xi u \in L^2(\mathbb{R}^2) \Rightarrow \Delta u \in \mathcal{H}^{-1-\epsilon}(\mathbb{R}^2) \Rightarrow u \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$$

$$\mathcal{H}^{1-\epsilon}(\mathbb{R}^2) \ni u \mapsto \xi u$$

$$= \underbrace{P_u\xi}_{\substack{\in\mathcal{H}^{-1-\epsilon}_{loc}\\ \uparrow}} + \underbrace{P_\xi u}_{\in\mathcal{H}_{loc}^{(-1-\epsilon)+(1-\epsilon)=-2\epsilon}} + \underbrace{\Pi(u,\xi)}_{\text{ill defned}} + \underbrace{P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi))}_{\mathcal{H}^\infty_{loc}}$$

To erase this singularity, $\quad -\Delta u = -P_u\xi + e^\Delta P_u\xi - \Delta u^{(\#)}$

Ansatz I: $u = \Delta^{-loc}P_u\xi + u^{(\#)\nearrow}$ with $u^{(\#)} \in \mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)$,

and $\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$, $\|\chi_a u^{(\#)}\|_{\mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)}$ decays sufficiently fast as $|a| \to \infty$

where $\Delta^{-loc} := -\int_0^1 dt\ e^{t\Delta}$ satisfying $\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^\Delta$

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \le C_{\alpha,\epsilon}\|\chi_{a_2}f\|_{\mathcal{C}^{\alpha+\epsilon-2}(\mathbb{R}^2)}\exp(-C|a_1-a_2|^2)$$

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2}f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \le C_{\alpha,\epsilon}\|\chi_{a_2}f\|_{\mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)}\exp(-C|a_1-a_2|^2)$$

Commutators

Then $H^{\xi}u := -\Delta u + \xi u$

$$= -\underbrace{\Delta u^{(\#)}}_{\in \mathcal{H}^{-2\epsilon}} + \underbrace{P_{\xi}(\Delta^{-loc} P_u \xi)}_{\in \mathcal{H}^{(-1-\epsilon)+(1-2\epsilon)=-3\epsilon}} + \underbrace{\Pi(\Delta^{-loc} P_u \xi, \xi)}_{\text{ill defned}} + (L^2)$$

Move the function u to outer places in the 2nd and 3rd terms by the commutators

$$C(f, g, h) := \Pi(\Delta^{-loc} P_f g, h) - f\Pi(\Delta^{-loc} g, h)$$

$$S(f, g, h) := P_h(\Delta^{-loc} P_f g) - {}_f P_h(\Delta^{-loc} g)$$

where ${}_f P_h g := \sum_{\nu} c_{\nu} \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^{\nu} h)(Q_t^{2,\nu} g) f)$

The continuity of commutators and so on

(i) For any $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}, \gamma \in (-\infty, 0)$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$,

$$
\left.\begin{array}{l}
\|\chi_{a_1} C(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\
\|\chi_{a_1} S(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}
\end{array}\right\}
$$

$$
\leq C_{\epsilon,\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|\chi_{a_4} h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \\
\quad \times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2 + |a_1 - a_4|^2))
$$

(ii) For any $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$
\|\chi_{a_1 \chi_{a_2} h} P_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)} \\
\leq C_{\alpha,\beta,\epsilon} \|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\
\quad \times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))
$$

Modification

Then $H^{\xi}u = -\underbrace{\Delta u^{(\#)}}_{\in\mathcal{H}^{-2\epsilon}} + \underbrace{{}_uP_{\xi}(\Delta^{-loc}\xi)}_{\in\mathcal{H}^{(-1-\epsilon)+(1-\epsilon)-2\epsilon=-4\epsilon}} + u\underbrace{\Pi(\Delta^{-loc}\xi,\xi)}_{\text{ill defned}} + (L^2)$

Replace the ill defned term by a $\bigcap_{\epsilon>0} C^{-\epsilon}_{loc}(\mathbb{R}^2)$-valued random variable Y_{ξ} s.t. $\lim_{\varepsilon\to 0}\mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_{\xi})\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$ for any $a \in \mathbb{Z}^2$, $p \in [1,\infty)$ and $\epsilon > 0$, where

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon) - \underline{\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]}$$

$\xi_\varepsilon := e^{\varepsilon^2\Delta}\xi$ is a smooth approximation of ξ $\qquad \nwarrow$ diverge as $\varepsilon \to 0$

$\|\chi_a Y_{\xi}\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi}\log(2+|a|)$

Here the operator H^{ξ} is replaced by a new operator which we denote as $\widetilde{H^{\xi}}$

2nd ansatz

$$\widetilde{H^{\xi}}u = -\Delta u^{(\#)} + \underbrace{{}_uP_{\xi}(\Delta^{-loc}\xi)}_{\in \mathcal{H}^{-4\epsilon}} + \underbrace{P_u Y_{\xi}}_{\in \mathcal{H}^{-2\epsilon}} + (L^2)$$

To erase these singularities,

Ansatz II: $u^{(\#)} = \Delta^{-loc}({}_uP_{\xi}(\Delta^{-loc}\xi) + P_u Y_{\xi}) + u^{\#}$

with $u^{\#} \in \mathcal{H}^2(\mathbb{R}^2)$ and $\|\chi_a u^{\#}\|_{\mathcal{H}^2(\mathbb{R}^2)}$ decays sufficiently fast as $|a| \to \infty$

Then, since $-\Delta u^{(\#)} = -{}_uP_{\xi}(\Delta^{-loc}\xi) - P_u Y_{\xi} + \underbrace{e_u^{\Delta} P_{\xi}(\Delta^{-loc}\xi) + e^{\Delta} P_u Y_{\xi} - \Delta u^{\#}}_{\in L^2}$,

$$\widetilde{H^{\xi}}u \in L^2$$

Our operator

$$
\begin{aligned}
\widetilde{H^{\xi}} u := & - \Delta \Phi_{\xi}(u) + P_{\xi} \Phi_{\xi}(u) + \Pi(\Phi_{\xi}(u), \xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi)) \\
& + e^{\Delta} P_u \xi + e^{\Delta}{}_u P_{\xi}(\Delta^{-loc} \xi) + e^{\Delta} P_u Y_{\xi} \\
& + C(u, \xi, \xi) + S(u, \xi, \xi) \\
& + P_{Y_{\xi}} u + \Pi(u, Y_{\xi}) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_{\xi})) \\
& + P_{\xi}(\Delta^{-loc}{}_u P_{\xi}(\Delta^{-loc} \xi)) + \Pi(\Delta^{-loc}{}_u P_{\xi}(\Delta^{-loc} \xi), \xi) \\
& + P_{\xi}(\Delta^{-loc} P_u Y_{\xi}) + \Pi(\Delta^{-loc} P_u Y_{\xi}, \xi),
\end{aligned}
$$

with $\underbrace{\Phi_{\xi}(u)}_{=u^{\#}} := u - \Delta^{-loc} P_u \xi - \Delta^{-loc}{}_u P_{\xi}(\Delta^{-loc} \xi) - \Delta^{-loc} P_u Y_{\xi}$

Main Statements

$$\mathrm{Dom}_0(\widetilde{H^\xi}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0 \text{ for any } \epsilon > 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\}$$

Theorem (Self-adjointness)

The operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_0(\widetilde{H^\xi})$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$.

Theorem (Spectrum)

The spectral set of the closure $\overline{\widetilde{H^\xi}}$ is $\mathbb{R}$.

Characteristic points of our operators

$$\widetilde{H^{\xi_\varepsilon}} = -\Delta + \xi_\varepsilon - \mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]$$

: essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ since $|\underbrace{\xi_\varepsilon(x)}_{\text{smooth}}| \le C_{\xi,\varepsilon}(\log(2+|x|))^2$

However $C_0^\infty(\mathbb{R}^2) \not\subset \mathrm{Dom}_0(\widetilde{H^\xi})$ since $\Phi_\xi(C_0^\infty(\mathbb{R}^2)) \not\subset \mathcal{H}^2(\mathbb{R}^2)$

Our powerful tool is

$$\Phi_\xi^{\mathbf{s}}(u) := u - \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s(a)}(\chi_a^2\xi) - \sum_{a,a'\in\mathbb{Z}^2} \Delta^{-loc}{}_u P_{\chi_a^2\xi}^{s_1(a,a')}(\Delta^{-loc}\chi_{a'}^2\xi)$$

$$- \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s_2(a)}(\chi_a^2 Y_\xi)$$

$$P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$$

$$_hP_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)h)$$

The continuity of restricted paraproducts

(i) For any $\beta < \gamma$,

$$\begin{aligned}
&\|\chi_{a_1} P^s_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \\
\leq &C_{\beta,\gamma} s^{(\gamma-\beta)/2} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \\
&\times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))
\end{aligned}$$

(ii) For any $\beta, \gamma_1, \gamma_2 \in \mathbb{R}$ satisfying $\gamma_1 \leq 0$ and $\beta < \gamma_1 + \gamma_2$,

$$\begin{aligned}
&\|\chi_{a_1 \chi_{a_2} h} P^s_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \\
\leq &C_{\beta,\gamma_1,\gamma_2} s^{(\gamma_1+\gamma_2-\beta)/2} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\
&\times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))
\end{aligned}$$

Choice of $\mathbf{s}$

For any $\epsilon \in (0,1)$ and almost all ξ, there exist $s(\epsilon,\xi), s_1(\epsilon,\xi), s_2(\epsilon,\xi) \in (0,1)$ and $M, M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ s.t.

$$\|\chi_a(I - \Phi_\xi^{\mathbf{s}(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq \delta \sum_{a' \in \mathbb{Z}^2} \exp(-M|a-a'|^2)\|\chi_{a'}u\|_{L^2(\mathbb{R}^2)}$$

for any $\delta \geq 0$, where $\mathbf{s}(\epsilon,\xi,\delta) = (s(a;\epsilon,\xi,\delta), s_1(a,a';\epsilon,\xi,\delta), s_2(a;\epsilon,\xi,\delta))_{a\in\mathbb{Z}^2}$ is

$$s(a;\epsilon,\xi,\delta) = s(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2}\right)^{M(\epsilon)},$$

$$s_1(a,a';\epsilon,\xi,\delta) = s_1(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2(\log(2+|a'|))^2}\right)^{M_1(\epsilon)}$$

$$s_2(a;\epsilon,\xi,\delta) = s_2(\epsilon,\xi)\left(\frac{\delta}{\log(2+|a|)}\right)^{M_2(\epsilon)}.$$

Inverse of $\Phi^{\mathbf{s}}_{\xi}$

$$\|(I - \Phi_{\xi}^{\mathbf{s}(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq C_{\xi,\epsilon}\delta\|u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$$

Thus for $\delta \in (0, 1/C_{\xi,\epsilon})$, there exists the inverse $(\Phi_{\xi}^{\mathbf{s}(\epsilon,\xi,\delta)})^{-1} = \sum_{n=0}^{\infty}(I - \Phi_{\xi}^{\mathbf{s}(\epsilon,\xi,\delta)})^n$

s.t. $\|(\Phi_{\xi}^{\mathbf{s}(\epsilon,\xi,\delta)})^{-1}(v)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq \|v\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}/(1 - C_{\xi,\epsilon}\delta)$

$(\Phi_{\xi}^{\mathbf{s}(\epsilon,\xi,\delta)})^{-1}(\{v \in \mathcal{H}^2(\mathbb{R}^2) : \text{supp } v \text{ is compact}\}) \subset \mathrm{Dom}_0(\widetilde{H^{\xi}})$

since $\Phi_{\xi} - \Phi_{\xi}^{\mathbf{s}(\epsilon,\xi,\delta)}$ is good.

The continuity of the difference of paraproducts

$$\|\chi_{a_1}(P_{\chi_{a_2} f}(\chi_{a_3} g) - P^s_{\chi_{a_2} f}(\chi_{a_3} g))\|_{\mathcal{H}^\beta(\mathbb{R}^2)}$$

$$\leq \begin{cases} \dfrac{C_{\beta,\gamma}}{s^{(\beta-\gamma)/2}} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \text{ if } \beta > \gamma, \\ C_{\beta,\gamma}\left(\log \dfrac{1}{s}\right) \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \text{ if } \beta = \gamma, \\ C_{\beta,\gamma} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \text{ if } \beta < \gamma \end{cases}$$

$$\|\chi_{a_1}(\chi_{a_2} h P_{\chi_{a_3} f}(\chi_{a_4} g) - \chi_{a_2} h\, P^s_{\chi_{a_3} f}(\chi_{a_4} g))\|_{\mathcal{H}^\beta(\mathbb{R}^2)}$$

$$\leq \begin{cases} \dfrac{C_{\beta,\gamma_1,\gamma_2}}{s^{(\beta-\gamma_1-\gamma_2)/2}} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times \exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2)) \text{ if } \beta > \gamma_1+\gamma_2, \gamma_1 \leq 0, \\ C_{\beta,\gamma_1,\gamma_2}\left(\log \dfrac{1}{s}\right) \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times \exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2)) \text{ if } \beta = \gamma_1+\gamma_2, \gamma_1 \leq 0, \\ C_{\beta,\gamma_1,\gamma_2} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times \exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2)) \text{ if } \beta < \gamma_1+\gamma_2, \gamma_1 \leq 0. \end{cases}$$

For the self-adjointness

$\widetilde{H^\xi}$: essentially self-adjoint $\Leftarrow \overline{\mathrm{Ran}(\widetilde{H^\xi} + \underset{\uparrow}{i})} = L^2(\mathbb{R}^2)$

we can take as a real number for operators bounded below

Thus we first restrict the randomness to consider an operator $\widetilde{H^\xi_R}$ bounded below. This is the limit as $\varepsilon \to 0$ of

$$\widetilde{H^{\xi_\varepsilon}_R} = -\Delta + \xi_{\varepsilon,R} - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\varepsilon,R}, \xi_{\varepsilon,R})],$$

where $\xi_R = \sum_{a\in\mathbb{Z}^2\cap\Lambda_R} \chi_a^2\xi,\ \xi_{\varepsilon,R} = \sum_{a\in\mathbb{Z}^2\cap\Lambda_R} \chi_a^2 e^{\varepsilon^2\Delta}\xi.$

The exact definition of $\widetilde{H^\xi_R}$ is in the next slide.

Restricted randomness

$$\widetilde{H_R^\xi} u = -\Delta\Phi_{\xi,R}(u) + P_{\xi_R}\Phi_{\xi,R}(u) + \Pi(\Phi_{\xi,R}(u),\xi_R) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_R))$$
$$+ e^\Delta P_u\xi_R + e^\Delta{}_u P_{\xi_R}(\Delta^{-loc}\xi_R) + e^\Delta P_u Y_{\xi,R} + C(u,\xi_R,\xi_R)$$
$$+ S(u,\xi_R,\xi_R) + P_{Y_{\xi,R}}u + \Pi(u, Y_{\xi,R}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_{\xi,R}))$$
$$+ P_{\xi_R}(\Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R)) + \Pi(\Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R),\xi_R)$$
$$+ P_{\xi_R}(\Delta^{-loc}P_u Y_{\xi,R}) + \Pi(\Delta^{-loc}P_u Y_{\xi,R},\xi_R),$$

$$\Phi_{\xi,R}(u) = u - \Delta^{-loc}P_u\xi_R - \Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R) - \Delta^{-loc}P_u Y_{\xi,R}$$
$$Y_{\xi,R} = \lim_{\varepsilon\to 0}(\Pi(\Delta^{-loc}\xi_{\varepsilon,R},\xi_{\varepsilon,R}) - \underbrace{\mathbb{E}[\Pi(\Delta^{-loc}\xi_{\varepsilon,R},\xi_{\varepsilon,R})]}_{\neq\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Rightarrow\widetilde{H_R^\xi}\neq\widetilde{H^{\xi_R}}})$$

$$\mathrm{Dom}(\widetilde{H_R^\xi}) := \left\{u\in\bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,R}(u)\in\mathcal{H}^2(\mathbb{R}^2)\right\}$$

Properties of the operator with the restricted randomness

$$\|\nabla\Phi_{\xi,R}(u)\|^2_{L^2(\mathbb{R}^2)} \leq (u, (\widetilde{H_R^\xi} + k(\xi, R))u)_{L^2(\mathbb{R}^2)}$$

We can show that $\mathrm{Ran}(\widetilde{H_R^\xi} + k(\xi, R)) = L^2(\mathbb{R}^2)$

Lemma (Self-adjointness of the restricted randomness)

The operator $\widetilde{H_R^\xi}$ with the domain $\mathrm{Dom}(\widetilde{H_R^\xi})$ is self-adjoint on $L^2(\mathbb{R}^2)$.

Proof of the self-adjointness

For $\forall f \in \mathrm{Ran}(\widetilde{H^\xi} + i)^\perp$,
$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \widetilde{\chi_R} f)_{L^2(\mathbb{R}^2)}$,
where $\widetilde{\chi_R} \in C_0^\infty(\Lambda_R \to [0,1])$ s.t. $\widetilde{\chi_R} = 1$ on Λ_{R-1}.
$\varphi_{R,L} := (H^\xi_{R+L} + i)^{-1}\widetilde{\chi_R} f \in \mathrm{Dom}(H^\xi_{R+L})$
We can show that $\widetilde{\varphi_{R,L}} := (\Phi_\xi^{\mathbf{s}(\epsilon,\xi,\delta)})^{-1}(\Phi_{\xi,R+L}^{\mathbf{s}(\epsilon,\xi,\delta)}(\varphi_{R,L})) \in \mathrm{Dom}_0(\widetilde{H^\xi})$
Since $\widetilde{\chi_R} f = (H^\xi_{R+L} + i)\varphi_{R,L}$ and $(\widetilde{H^\xi} + i)\widetilde{\varphi_{R,L}} \in \mathrm{Ran}(\widetilde{H^\xi} + i)$, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \underbrace{(H^\xi_{R+L} + i)\varphi_{R,L} - (\widetilde{H^\xi} + i)\widetilde{\varphi_{R,L}}}_{\downarrow \text{ as } L\to\infty \atop 0})_{L^2(\mathbb{R}^2)} = 0$$

$$\therefore \overline{\mathrm{Ran}(\widetilde{H^\xi} + i)} = L^2(\mathbb{R}^2)$$

Resolvent convergence

For $\widetilde{H^\xi}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\sup_{\|v\|_{L^2(\mathbb{T}^2)}=1} \|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^\xi} + z)^{-1}v\|_{L^2(\mathbb{T}^2)} \xrightarrow{\varepsilon\to 0} 0 \text{ for a large } z \in \mathbb{R}$$

(Mouzard Prop.2.14)

$\lambda_n(\widetilde{H^{\xi_\varepsilon}}) \xrightarrow{\varepsilon\to 0} \lambda_n(\widetilde{H^\xi})$ (Mouzard Cor.2.15)

For $\widetilde{H^\xi}$ on $\mathbb{R}^2$,

$\|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^\xi} + z)^{-1}v\|_{L^2(\mathbb{R}^2)} \xrightarrow{\varepsilon\to 0} 0$ for each $v \in L^2(\mathbb{R}^2)$ and $z \in \mathbb{C} \setminus \mathbb{R}$

However the estimate with $\sup\limits_{\|v\|_{L^2(\mathbb{R}^2)}=1}$ may be difficult.

Fourier series representation

For the identification of the spectrum, we take a positive probability event that $\xi \fallingdotseq$ const on Λ_L with a large L.

For this, identify Λ_L with $\mathbb{R}^2/(L\mathbb{Z})^2$ and represent the white noise ξ^L on Λ_L as

$$\xi^L(x) = \sum_{\mathbf{n}\in\mathbb{Z}^2} X_{\mathbf{n}}(\xi^L)\varphi^L_{\mathbf{n}}(x),$$

where $\{X_{\mathbf{n}}(\xi^L)\}_{\mathbf{n}\in\mathbb{Z}^2} \underset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$ and $\{\varphi^L_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^2}$ is ONB of $L^2(\mathbb{R}^2/(L\mathbb{Z})^2)$ $\varphi^L_0 \equiv 1/L$

defined by $\varphi^L_{(n_1,n_2)}(x_1,x_2) = \phi^L_{n_1}(x_1)\phi^L_{n_2}(x_2)$

$$\text{and } \phi^L_{n_1}(x_1) = \begin{cases} \sqrt{2/L}\cos(2\pi n_1 x_1/L) & \text{for } 0 < n_1 \in \mathbb{Z}, \\ \sqrt{1/L} & \text{for } n_1 = 0, \\ \sqrt{2/L}\sin(2\pi n_1 x_1/L) & \text{for } 0 > n_1 \in \mathbb{Z}. \end{cases}$$

Decomposition by the Fourier series

Write $\xi = \widetilde{\chi_L \xi^L_{N\geq}} + \widetilde{\xi^L_{N<}}$, where

$$\xi^L_{N\geq}(x) = \sum_{\mathbf{n}\in\mathbb{Z}^2\cap\Lambda_N} X_{\mathbf{n}}(\xi^L)\varphi^L_{\mathbf{n}}(x),$$

$\widetilde{\xi^L_{N<}}$ is a Gaussian random field independent of $\xi^L_{N\geq}(x)$
Let $Y_{\xi,L,N<}$ be a $\bigcap_{\epsilon>0} C^{-\epsilon}_{loc}(\mathbb{R}^2)$-valued random variable such that

$$\lim_{\varepsilon\to 0} \mathbb{E}[\|\chi_a(Y_{\xi,\varepsilon,L,N<} - Y_{\xi,L,N<})\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0 \text{ for any } p \in [1,\infty), a \in \mathbb{Z}^2, \text{ where}$$

$$Y_{\xi,\varepsilon,L,N<} := \Pi(\Delta^{-loc} e^{\varepsilon^2\Delta}\widetilde{\xi^L_{N<}}, e^{\varepsilon^2\Delta}\widetilde{\xi^L_{N<}}) - \mathbb{E}[\Pi(\Delta^{-loc} e^{\varepsilon^2\Delta}\widetilde{\xi^L_{N<}}, e^{\varepsilon^2\Delta}\widetilde{\xi^L_{N<}})_\varepsilon]).$$

Operator depending only on $\widetilde{\xi^L_{N<}}$

$$\widetilde{H^{\xi,L,N<}}u = -\Delta\Phi_{\xi,L,N<}(u) + P_{\widetilde{\xi^L_{N<}}}\Phi_{\xi,L,N<}(u) + \Pi(\Phi_{\xi,L,N<}(u), \widetilde{\xi^L_{N<}}) + P^{(b)}_1((P^{(b)}_1 u)(P^{(b)}_1 \widetilde{\xi^L_{N<}}))$$

$$+ e^{\Delta}P_u\widetilde{\xi^L_{N<}} + e^{\Delta}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}) + e^{\Delta}P_u Y_{\xi,L,N<}$$

$$+ C(u, \widetilde{\xi^L_{N<}}, \widetilde{\xi^L_{N<}}) + S(u, \widetilde{\xi^L_{N<}}, \widetilde{\xi^L_{N<}})$$

$$+ P_{Y_{\xi,L,N<}}u + \Pi(u, Y_{\xi,L,N<}) + P^{(b)}_1((P^{(b)}_1 u)(P^{(b)}_1 Y_{\xi,L,N<}))$$

$$+ P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}) + \Pi(\Delta^{-loc}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}), \widetilde{\xi^L_{N<}})$$

$$+ P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}P_u Y_{\xi,L,N<})) + \Pi(\Delta^{-loc}P_u Y_{\xi,L,N<}, \widetilde{\xi^L_{N<}}),$$

$$\Phi_{\xi,L,N<}(u) := u - \Delta^{-loc}P_u\widetilde{\xi^L_{N<}} - \Delta^{-loc}{}_u P_{\widetilde{\xi^L_{N<}}}(\Delta^{-loc}\widetilde{\xi^L_{N<}}) - \Delta^{-loc}P_u Y_{\xi,L,N<}$$

$$\mathrm{Dom}_0(\widetilde{H^{\xi,L,N<}}) := \Big\{ u \in \bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0,$$

$$\Phi_{\xi,L,N<}(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty}\frac{1}{|a|}\log\|\chi_a\Phi_{\xi,L,N<}(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0\Big\}$$

Relations with the original operator

$$\mathrm{Dom}_0(\widetilde{H^\xi}) = \mathrm{Dom}_0(\widetilde{H^{\xi,L,N<}})$$

$$\widetilde{H^\xi} = \widetilde{H^{\xi,L,N<}} + \widetilde{\chi_L}\xi^L_{N\geq} - Y^{L,N\geq},$$

where

$$Y^{L,N\geq} := \sum_{\mathbf{n}\in\mathbb{Z}^2\cap\Lambda_N} \Pi(\Delta^{-loc}\widetilde{\chi_L}\varphi^L_{\mathbf{n}}, \widetilde{\chi_L}\varphi^L_{\mathbf{n}}).$$

$Y^{L,N\geq}$ may diverge as $N \to \infty$. However we have

$$\sup_{x\in\mathbb{R}^2} |Y^{L,N\geq}(x)| \leq c_\eta (N/L)^\eta \text{ for } \forall \eta.$$

$\varphi_{\varepsilon,R}$

For $1 < R < L/2$ and large N, we can take an event s.t. $\widetilde{H^{\xi,L,N<}} \fallingdotseq -\Delta$ on Λ_R.
For $\forall \varepsilon > 0$,
take $\varphi_{\varepsilon,R} \in C_0^\infty(\Lambda_R)$ s.t. $\|\varphi_{\varepsilon,R}\|_{L^2(\mathbb{R})} = 1$ and

$$\|((-\Delta - Y^{L,N\geq}) - \inf\operatorname{spec}(-\Delta - Y^{L,N\geq})^{\text{Dirichlet}}_{\Lambda_R})\varphi_{\varepsilon,R}\|_{L^2(\mathbb{R}^2)} < \varepsilon.$$

For $\forall \lambda \in \mathbb{R}$, $\qquad ||$

$$\|(-\Delta + r(\lambda, L, N, R)\varphi_{\mathbf{0}}^{L} - Y^{L,N\geq}) - \lambda)\varphi_{\varepsilon,R}\|_{L^2(\mathbb{R}^2)} \text{ with } r(\lambda, L, N, R) \in \mathbb{R}$$

$\dot{||}.$

$$\|(\widetilde{H^{\xi}} - \lambda)\widetilde{\varphi_{\varepsilon,R}}\|_{L^2(\mathbb{R}^2)} \text{ for } \xi \in E(\varepsilon, \epsilon, \lambda, L) \ \& \ \widetilde{\varphi_{\varepsilon,L}} := (\Phi^{\mathbf{s}(\epsilon,\xi,L,\delta)}_{\xi,L,N<})^{-1}(\varphi_{\varepsilon,R})$$

The event we consider

$$
\begin{aligned}
& E(\varepsilon, \epsilon, \lambda, L) \\
=& \Big\{ \xi : \text{ In the representation } \xi = \widetilde{\chi_L \xi^L_{N\geq}} + \widetilde{\xi^L_{N<}} \text{ with } N = L^{10}, \text{ it holds that} \\
& |X_{\mathbf{0}}(\xi^L) - r(\lambda, L, N, R)|, |X_{\mathbf{n}}(\xi^L)| \leq \varepsilon/N^2 \text{ for any } \mathbf{n} \in \mathbb{Z}^2 \cap \Lambda_N \setminus \{\mathbf{0}\}, \\
& \|\chi_a \widetilde{\xi^L_{N<}}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}, \|\chi_a Y_{\xi,L,N<}\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \leq 1_{\Lambda_{L/2}}(a) L^{-\epsilon} + 1_{\Lambda^c_{L/2}}(a)|a|^{\epsilon} \\
& \text{ for any } a \in \mathbb{Z}^2. \Big\},
\end{aligned}
$$

which satisfies $\mathbb{P}(E(\varepsilon, \epsilon, \lambda, L)) > 0$.

Probability One

Let $E(x_0, \varepsilon, \epsilon, \lambda, L) := \{\xi : \xi(\cdot - x_0) \in E(\varepsilon, \epsilon, \lambda, L)\}$
Then $\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, \epsilon, \lambda, L)$ is $\mathbb{Z}^2$-invariant.
By the ergodicity of the white noise, we have

$$\mathbb{P}\Big(\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, \epsilon, \lambda, L) \Big) = 1.$$

By the shift, we can take a Weyl sequence with probability 1.
Thus $\lambda \in \mathrm{Spec}(\overline{\widetilde{H^\xi}})$.