A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane

https://www.math.h.kyoto-u.ac.jp/users/ueki/2DWN-Scr-c1.pdf
https://www.math.h.kyoto-u.ac.jp/users/ueki/presen20231013RORT.pdf

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Motivation

White noise: $\xi = (\xi(x))_{x\in\mathbb{R}^2} \underset{\text{i.i.d.}}{\sim} \mathrm{N}(0, *)$: The most basic but wild random field

ξ: Gaussian random field, $\mathbb{E}[\xi(x)] = 0$, $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$

$\mathbb{P}(\text{"}x \mapsto \xi(x)\text{"} \in C_{loc}^{-1-\epsilon}) = 1$

Claim: Realize $-\Delta + \xi$ as a self-adjoint operator to apply the spectral analysis

Fact: $-\Delta + V$: self-adjoint on $L^2(\mathbb{R}^2)$

closure of the op. on $C_0^\infty(\mathbb{R}^2)$

$\Uparrow$

$-c(|x|^2 + 1) \leq V(x) \in L^2_{loc}(\mathbb{R}^2)$

Gaussian random field potential

For a Gaussian random field V with $\mathbb{C}ov(V(x), V(y)) = \underbrace{C(x-y)}_{\text{usual function}}$

Fischer, Leschke, Müller, J. Statist. Phys.101(2000)
Spectral localization by Gaussian random potentials in multi-dimensional continuous space.

Their results is sufficient for our understanding of the nature.

However, I will extend their theory to the whitenoise in order to a development of the stochastic analysis.

As another merits, some explicit calculations may be possible.
Indeed, in 1D, the density of states is explicitly calculated (cf. Halperin 1965)

Other dimensions

$\xi = (\xi(x))_{x \in \mathbb{R}^d}$: White noise on $\mathbb{R}^d \Rightarrow$ "$x \mapsto \xi(x)$" $\in C_{loc}^{-d/2-\epsilon}$

$d = 1 \Rightarrow W^{1,2}([a,b]) \ni f, g \mapsto \int_a^b (f'g' + \underset{C_{loc}^{-1/2-\epsilon}}{\xi} fg)dx$: well-defined

$-\Delta + \xi$ is realized as a self-adjoint operator

M. Fukushima and S. Nakao (1977) Spectral asymptotics (Asymptotics of the Integrated density of states)

N. Minami (1988) (1989) Self-adj. on $\mathbb{R}$, Exp.Loc ($\xi \to \partial$(Lévy process))

$d \geq 4 \Rightarrow$ No results

$d = 2, 3$ are studied recently

$d = 2$ is easier than $d = 3$

$\xi \in C_{loc}^{-1-\epsilon}$ $\quad\quad$ $\xi \in C_{loc}^{-3/2-\epsilon}$

Related works on singular SPDEs

M. Hairer (2014) A theory of regularity structures,
M. Gubinelli, P. Imkeller and N. Perkowski (2015) Paracontrolled calcuclus
A. Kupiainen (2016) Renormalization Group
$\Rightarrow$ Stochastic quatization equation for ϕ_3^4 Euclidean quantum field theory
Generalized continuous parabolic Anderson model
Kardar–Parisi–Zhang type equation
Navier-Stokes equation with very singular forcing
and so on

Eg. Continuous parabolic Anderson model

$$\partial_t u(t,x) = \partial_x^2 u(t,x) - \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + c_\varepsilon)u(t,x) \text{ for } t>0, x \in \underbrace{\mathbb{R}^2/\mathbb{Z}^2}_{\text{compact}}$$

Related works on Operators

For the Schrödinger operator

R. Allez and K. Chouk (2015) Paracontrolled calcuclus based on Fourier Analysis

$-\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon)$ on $\mathbb{R}^2/\mathbb{Z}^2 \Leftarrow$ Self-adjointness, Discrete Spectrum,

Asymptotic Distribution

M. Gubinelli, B. Ugurcan and I. Zachhuber (2020) Extension to $\mathbb{R}^3/\mathbb{Z}^3$

$$i\partial_t u(t,x) = \underbrace{(\Delta - \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon) - k_\xi)}_{\geq 0} u(t,x) - (u|u|^2)(t,x)$$

$$\partial_t^2 u(t,x) = (\Delta - \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon) - k_\xi) u(t,x) - (u^3)(t,x)$$

C. Labbé (2019) similar results by regurarity structure for

$-\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon)$ on $(-L, L)^{2or3}$ with periodic or Dirichlet conditions

K. Chouk and W. van Zuijlen (2021)

Asymptotics of Eigenvalues of the Dirichlet operator on $(-L, L)^2$ as $L \to \infty$

T. Matsuda (2022) Asymptotics of Integrated density of states

Topics on Random Schrödinger operators

Anderson transition:

Point spectrum with exponentially decaying eigenstates
  for strongly random potentials and energies near the edge of the spectrum (Anderson localization)

Absolutely continuous spectrum
  for weakly random potentials and energies far from the edge of the spectrum

This topic is discussed for stationary potentials on noncompact spaces.

Singular SPDEs on noncompact spaces

M. Hairer and C. Labbé, (2015)
M. Hairer and C. Labbé, (2018)
$\partial_t u_\varepsilon = \Delta u_\varepsilon - (\xi_\varepsilon + c_\varepsilon) u_\varepsilon, t > 0, x \in \mathbb{R}^d,\ u_\varepsilon(0, \cdot) = u_0$
Paraboloc Anderson model: ξ space white noise (depends only on x) $d = 2, 3$
Stochastic Heat equation: ξ space-time white noise (depends on (t, x)) $d = 1$
$\exists \lim_{\varepsilon \to 0} u_\varepsilon$ uniformly on compact sets in probability
depends continuously in u_0

W. König, N. Perkowski and W. van Zuijlen, (2022)
Feynman-Kac type representation of the solution of 2DPAM
Asymptotics as $t \to \infty$

Related operators on noncompact spaces

B. Ugurcan, (2022) $-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + \widetilde{c}_\varepsilon(x))$ on $\mathbb{R}^2$ with $\widetilde{c}_\varepsilon(x) \xrightarrow{|x|\to\infty} 0$

$$= \underbrace{-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon^{\uparrow}(x) + \widetilde{c}_\varepsilon(x))}_{\uparrow} + \underbrace{\xi_\varepsilon^{\downarrow}(x)}_{\nwarrow},$$

An extension of the method for the compact case   By a commutator estimate,

where $\xi = \xi^{\uparrow}(x) + \underbrace{\xi^{\downarrow}(x)}_{\text{Smooth functions in } x}$

$$= \sum_{n=-1}^{\infty} \chi_{\widetilde{[[2^n],2^{n+1}]}}(|x|)\{\underbrace{\widetilde{\chi_{[c2^n,\infty)}}(-\Delta)\xi}_{\text{high energy pert}} + \underbrace{\widetilde{\chi_{[0,c2^n]}}(-\Delta)\xi}_{\text{low energy pert}}\}$$

$\widetilde{c}_\varepsilon(x) = \mathbb{E}[$A resonant product of $\xi_\varepsilon^{\uparrow}(x)$ and $(1-\Delta)^{-1}\xi_\varepsilon^{\uparrow}(x)]$

Heat semigroup approach

Paracontrolled calculus by Heat semigroups

(suitable for noncompact manifolds, graphs and so on)

Paraproducts defined by using Heat Semigroups

(cf. Preceding approach defines Paraproducts by using Fourier Analysis)

I. Bailleul and F. Bernicot (2016) $\exists \lim_{\varepsilon \to 0} u_\varepsilon$

generalized PAM on 2D manifold (without compactness)

I. Bailleul, F. Bernicot and D. Frey (2018) $\exists \lim_{\varepsilon \to 0} u_\varepsilon$

PAM and multiplicative Burgers eq. on 3D manifold

(without compactness)

A. Mouzard (2022) Self-adjointness, Discrete Spectrum, Its asymptotics of

$\underset{\text{eg. Lap.Belt}}{-\Delta} + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon(x))$ on 2D Compact manifold, where $\xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta} \xi)(x)$

$$c_\varepsilon(x) \equiv c_\varepsilon \text{ on } \mathbb{R}^2/\mathbb{Z}^2$$

Our Topics

$-\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ on $\mathbb{R}^2$, where $\xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta}\xi)(x)$

Self-adjointness

Spectral set $= \mathbb{R}$

We should remark $\|\xi\|_{C^{-1-\epsilon}(\{|x|<R\})} \lesssim (\log R)^2$

Tools: Paracontrolled calcuclus by Heat semigroup referring Mouzard (2020) and the partition of unity

Products $fg = P_f g + \Pi(f,g) + P_g f + P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g))$

$0 << b \in 2\mathbb{Z}$ fixed $\quad P_t^{(b)} = \sum_{j=0}^{b-1} \frac{(-t\Delta)^j}{j!} e^{t\Delta}$

$$\underbrace{P_0^{(b)}((P_0^{(b)} f)(P_0^{(b)} g))}_{=fg} - P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g)) = -\int_0^1 dt \partial_t \{P_t^{(b)}((P_t^{(b)} f)(P_t^{(b)} g))\}$$

$$= P_f g + \Pi(f,g) + P_g f$$

$$P_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)): \text{ paraproduct term}$$

$$\Pi(f,g) := \sum_\mu c_\mu \int_0^1 \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g)) : \text{ resonating term}$$

$$P^\nu, P^\mu \in StGC^{[0,b/2)},\ Q^{1,\nu}, Q^{2,\nu}, Q^{1,\mu}, Q_t^{2,\mu} \in StGC^{[b/2,2b]}$$

For any $I \subset (0,\infty)$, $StGC^I = \{((\sqrt{t}\nabla)^\alpha P_t^{(c)})_{t\in(0,1]} : \alpha \in \mathbb{Z}_+^2, \alpha_1 + \alpha_2 \in I \cap \mathbb{Z}, 1 \le c \le b\}$

standard families of Gaussian operators with cancellation of orders I

The Besov Spaces

For $p, q \in [1, \infty], \alpha \in (-2b, 2b)$, $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2) = \overline{C_0^{\infty}(\mathbb{R}^2)}^{\|\cdot\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)}}$: the Besov Space

$$\|f\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)} := \|e^{\Delta} f\|_{L^p(\mathbb{R}^2:dx)} + \sup\{\|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)} : Q \in StGC^{(|\alpha|,2b]}\}$$

$\mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2) =: \mathcal{C}^{\alpha}(\mathbb{R}^2)$: the Besov α-Hölder space

$\mathcal{B}^{\alpha}_{2,2}(\mathbb{R}^2) =: \mathcal{H}^{\alpha}(\mathbb{R}^2)$: the Sobolev space with the index α.
$= \mathcal{W}^{\alpha,2}(\mathbb{R}^2)$

$\{\chi_a\}_{a\in\mathbb{Z}^2} \subset C^{\infty}(\mathbb{R}^2 \to [0,1])$ s.t. $\sum_{a\in\mathbb{Z}^2} \chi_a^2 \equiv 1$, $\operatorname{supp}\chi_a \subset \Lambda_2(a) := a + (-1,1)^2$

$\chi_a(\cdot) = \chi_0(\cdot - a)$

The continuity of paraproducts

(i) For any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$\|\chi_{a_1} P_{\chi_{a_2} g}(\chi_{a_3} f)\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)} \leq \begin{cases} C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^\infty(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^2(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(ii) For any $\alpha \in (-\infty, 0)$ and $\beta \in \mathbb{R}$,

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)} \leq \begin{cases} C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{H}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(iii) For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta>0$,

$$\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)} \leq C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)).$$

1st ansatz for the definition of the operator

$$\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \le C_{\epsilon,\xi}(\log(2+|a|))^2$$

$$u, H^\xi u := -\Delta u + \xi u \in L^2(\mathbb{R}^2) \Rightarrow \Delta u \in \mathcal{H}^{-1-\epsilon}(\mathbb{R}^2) \Rightarrow u \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$$

$$\xi u = \underbrace{P_u \xi}_{\in \mathcal{H}^{-1-\epsilon}_{loc} \uparrow} + \underbrace{P_\xi u}_{\in \mathcal{H}^{-2\epsilon}_{loc}} + \underbrace{\Pi(u,\xi)}_{\text{ill defned}} + \underbrace{P^{(b)}_1((P^{(b)}_1 u)(P^{(b)}_1 \xi))}_{\mathcal{H}^\infty_{loc}}$$

To erase this singularity, $-\Delta u = -P_u \xi + e^\Delta P_u \xi - \Delta \phi_\xi(u)$

Ansatz I: $u = \Delta^{-loc} P_u \xi + \phi_\xi(u)^{\nearrow}$ with $\phi_\xi(u) \in \mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)$,

and $\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$, $\|\chi_a \phi_\xi(u)\|_{\mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)}$ decays sufficiently fast as $|a| \to \infty$

where $\Delta^{-loc} := -\int_0^1 dt\ e^{t\Delta}$ satisfying $\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^\Delta$

$$\|\chi_{a_1} \Delta^{-loc} \chi_{a_2} f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \le C_{\alpha,\epsilon} \|\chi_{a_2} f\|_{\mathcal{C}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1 - a_2|^2)$$

$$\|\chi_{a_1} \Delta^{-loc} \chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \le C_{\alpha,\epsilon} \|\chi_{a_2} f\|_{\mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1 - a_2|^2)$$

Commutators

$$H^\xi u = -\underbrace{\Delta\phi_\xi(u)}_{\in\mathcal{H}^{-2\epsilon}} + \underbrace{P_\xi(\Delta^{-loc}P_u\xi)}_{\in\mathcal{H}^{-3\epsilon}} + \underbrace{\Pi(\Delta^{-loc}P_u\xi,\xi)}_{\text{ill defned}} + (L^2)$$

In the 2nd and 3rd terms, move the function u to outer places by the commutators

$C(f,g,h) := \Pi(\Delta^{-loc}P_f g, h) - f\Pi(\Delta^{-loc}g, h)$

$S(f,g,h) := P_h(\Delta^{-loc}P_f g) - {}_fP_h(\Delta^{-loc}g)$

where ${}_fP_h g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu h)(Q_t^{2,\nu}g)f)$

The continuity of commutators and so on

(i) For any $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}, \gamma \in (-\infty, 0)$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$,

$$\left.\begin{array}{l} \|\chi_{a_1} C(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \\ \|\chi_{a_1} S(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)} \end{array}\right\}$$

$$\leq C_{\epsilon,\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|\chi_{a_4} h\|_{\mathcal{C}^{\gamma}(\mathbb{R}^2)} \times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2 + |a_1 - a_4|^2))$$

(ii) For any $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$\|\chi_{a_1 \chi_{a_2} h} P_{\chi_{a_3} f} (\chi_{a_4} g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)}$$

$$\leq C_{\alpha,\beta,\epsilon} \|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))$$

Modification

Then $H^{\xi}u = -\underbrace{\Delta\phi_{\xi}(u)}_{\in\mathcal{H}^{-2\epsilon}} + \underbrace{{}_{u}P_{\xi}(\Delta^{-loc}\xi)}_{\in\mathcal{H}^{-4\epsilon}} + u\underbrace{\Pi(\Delta^{-loc}\xi,\xi)}_{\text{ill defned}} + (L^2)$

Replace the ill defned term by a $\bigcap_{\epsilon>0} C^{-\epsilon}_{loc}(\mathbb{R}^2)$-valued random variable Y_{ξ} s.t. $\lim_{\varepsilon\to 0}\mathbb{E}[\|\chi_a(Y_{\xi_{\varepsilon}} - Y_{\xi})\|^{p}_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$ for any $a\in\mathbb{Z}^2$, $p\in[1,\infty)$ and $\epsilon>0$, where

$$Y_{\xi_{\varepsilon}} := \Pi(\Delta^{-loc}\xi_{\varepsilon},\xi_{\varepsilon}) - \underline{\mathbb{E}[\Pi(\Delta^{-loc}\xi_{\varepsilon},\xi_{\varepsilon})]}$$

$\xi_{\varepsilon} := e^{\varepsilon^2\Delta}\xi$ is a smooth approximation of ξ $\qquad\nwarrow$ diverge as $\varepsilon\to 0$

$\|\chi_a Y_{\xi}\|_{C^{-\epsilon}(\mathbb{R}^2)} \le C_{\epsilon,\xi}\log(2+|a|)$

Here the operator H^{ξ} is replaced by a new operator which we denote as $\widetilde{H^{\xi}}$

2nd ansatz

$$\widetilde{H^\xi} u = -\Delta\phi_\xi(u) + \underbrace{{}_uP_\xi(\Delta^{-loc}\xi)}_{\in\mathcal{H}^{-4\epsilon}} + \underbrace{P_u Y_\xi}_{\in\mathcal{H}^{-2\epsilon}} + (L^2)$$

To erase remaining singularities,

Ansatz II: $\phi_\xi(u) = \Delta^{-loc}({}_uP_\xi(\Delta^{-loc}\xi) + P_u Y_\xi) + \Phi_\xi(u)$

with $\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2)$ and $\|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)}$ decays sufficiently fast as $|a| \to \infty$

Then, since

$$-\Delta\phi_\xi(u) = -{}_uP_\xi(\Delta^{-loc}\xi) - P_u Y_\xi + \underbrace{e_u^\Delta P_\xi(\Delta^{-loc}\xi) + e^\Delta P_u Y_\xi - \Delta\Phi_\xi(u)}_{\in L^2},$$

$$\widetilde{H^\xi} u \in L^2$$

Our operator

$$
\begin{aligned}
\widetilde{H^{\xi}}u := & -\Delta\Phi_{\xi}(u) + P_{\xi}\Phi_{\xi}(u) + \Pi(\Phi_{\xi}(u), \xi) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi)) \\
& + e^{\Delta}P_u\xi + e^{\Delta}{}_uP_{\xi}(\Delta^{-loc}\xi) + e^{\Delta}P_uY_{\xi} \\
& + C(u,\xi,\xi) + S(u,\xi,\xi) \\
& + P_{Y_{\xi}}u + \Pi(u, Y_{\xi}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_{\xi})) \\
& + P_{\xi}(\Delta^{-loc}{}_uP_{\xi}(\Delta^{-loc}\xi)) + \Pi(\Delta^{-loc}{}_uP_{\xi}(\Delta^{-loc}\xi), \xi) \\
& + P_{\xi}(\Delta^{-loc}P_uY_{\xi}) + \Pi(\Delta^{-loc}P_uY_{\xi}, \xi),
\end{aligned}
$$

with $\Phi_{\xi}(u) := u - \Delta^{-loc}P_u\xi - \Delta^{-loc}{}_uP_{\xi}(\Delta^{-loc}\xi) - \Delta^{-loc}P_uY_{\xi}$

Main Statements

$$\mathrm{Dom}_0(\widetilde{H^\xi}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0 \text{ for any } \epsilon > 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\}$$

Theorem (Self-adjointness)

The operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_0(\widetilde{H^\xi})$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$.

Theorem (Spectrum)

The spectral set of the closure $\overline{\widetilde{H^\xi}}$ is $\mathbb{R}$.

Characteristic points of our operators

Smooth approximation $\widetilde{H^{\xi_\varepsilon}} = -\Delta + \xi_\varepsilon - \mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]$
is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ since $|\underbrace{\xi_\varepsilon(x)}_{\text{smooth}}| \leq C_{\xi,\varepsilon}(\log(2+|x|))^2$

but $\mathrm{Dom}_0(\widetilde{H^\xi})$ does not include $C_0^\infty(\mathbb{R}^2)$ since $\Phi_\xi(C_0^\infty(\mathbb{R}^2)) \not\subset \mathcal{H}^2(\mathbb{R}^2)$

Our powerful tool is

$$\Phi_\xi^{\boldsymbol{s}}(u) := u - \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s(a)}(\chi_a^2\xi) - \sum_{a,a'\in\mathbb{Z}^2} \Delta^{-loc}{}_u P_{\chi_a^2\xi}^{s_1(a,a')}(\Delta^{-loc}\chi_{a'}^2\xi)$$
$$- \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s_2(a)}(\chi_a^2 Y_\xi)$$

$$P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$$

$${}_h P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)h)$$

The continuity of restricted paraproducts

(i) For any $\beta < \gamma$,

$$\begin{aligned}
&\|\chi_{a_1} P^s_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \\
\le & C_{\beta,\gamma} s^{(\gamma-\beta)/2} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \\
&\times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2))
\end{aligned}$$

(ii) For any $\beta, \gamma_1, \gamma_2 \in \mathbb{R}$ satisfying $\gamma_1 \le 0$ and $\beta < \gamma_1 + \gamma_2$,

$$\begin{aligned}
&\|\chi_{a_1 \chi_{a_2} h} P^s_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^\beta(\mathbb{R}^2)} \\
\le & C_{\beta,\gamma_1,\gamma_2} s^{(\gamma_1+\gamma_2-\beta)/2} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\
&\times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))
\end{aligned}$$

Choice of s

For any $\epsilon \in (0,1)$ and almost all ξ, there exist $s(\epsilon,\xi), s_1(\epsilon,\xi), s_2(\epsilon,\xi) \in (0,1)$ and $M, M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ s.t.

$$\|\chi_a(I - \Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq \delta \sum_{a' \in \mathbb{Z}^2} \exp(-M|a-a'|^2)\|\chi_{a'} u\|_{L^2(\mathbb{R}^2)}$$

for any $\delta \geq 0$, where $\boldsymbol{s}(\epsilon,\xi,\delta) = (s(a;\epsilon,\xi,\delta), s_1(a,a';\epsilon,\xi,\delta), s_2(a;\epsilon,\xi,\delta))_{a\in\mathbb{Z}^2}$ is

$$s(a;\epsilon,\xi,\delta) = s(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2}\right)^{M(\epsilon)},$$

$$s_1(a,a';\epsilon,\xi,\delta) = s_1(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2(\log(2+|a'|))^2}\right)^{M_1(\epsilon)}$$

$$s_2(a;\epsilon,\xi,\delta) = s_2(\epsilon,\xi)\left(\frac{\delta}{\log(2+|a|)}\right)^{M_2(\epsilon)}.$$

Inverse of Φ_ξ^s

$$\|(I-\Phi_\xi^{s(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le C_{\xi,\epsilon}\delta\|u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$$

Thus for $\delta \in (0, 1/C_{\xi,\epsilon})$, there exists the inverse $(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1} = \sum_{n=0}^{\infty}(I-\Phi_\xi^{s(\epsilon,\xi,\delta)})^n$

s.t. $\|(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(v)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le \|v\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}/(1-C_{\xi,\epsilon}\delta)$

$(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(\{v \in \mathcal{H}^2(\mathbb{R}^2) : \text{supp } v \text{ is compact}\}) \subset \mathrm{Dom}_0(\widetilde{H^\xi})$

since $\Phi_\xi - \Phi_\xi^{s(\epsilon,\xi,\delta)}$ is controlled by the following.

The continuity of the difference of paraproducts

$$\|\chi_{a_1}(P_{\chi_{a_2} f}(\chi_{a_3} g) - P^s_{\chi_{a_2} f}(\chi_{a_3} g))\|_{\mathcal{H}^\beta(\mathbb{R}^2)}$$

$$\leq \begin{cases} \dfrac{C_{\beta,\gamma}}{s^{(\beta-\gamma)/2}} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2)) \text{ if } \beta > \gamma, \\ C_{\beta,\gamma} \left(\log \dfrac{1}{s} \right) \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2)) \text{ if } \beta = \gamma, \\ C_{\beta,\gamma} \|\chi_{a_2} f\|_{L^2(\mathbb{R}^2)} \|\chi_{a_3} g\|_{C^\gamma(\mathbb{R}^2)} \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2)) \text{ if } \beta < \gamma \end{cases}$$

$$\|\chi_{a_1}(\chi_{a_2} h P_{\chi_{a_3} f}(\chi_{a_4} g) - \chi_{a_2} h\, P^s_{\chi_{a_3} f}(\chi_{a_4} g))\|_{\mathcal{H}^\beta(\mathbb{R}^2)}$$

$$\leq \begin{cases} \dfrac{C_{\beta,\gamma_1,\gamma_2}}{s^{(\beta-\gamma_1-\gamma_2)/2}} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2)) \text{ if } \beta > \gamma_1 + \gamma_2, \gamma_1 \leq 0, \\ C_{\beta,\gamma_1,\gamma_2} \left(\log \dfrac{1}{s} \right) \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2)) \text{ if } \beta = \gamma_1 + \gamma_2, \gamma_1 \leq 0, \\ C_{\beta,\gamma_1,\gamma_2} \|\chi_{a_3} f\|_{C^{\gamma_1}(\mathbb{R}^2)} \|\chi_{a_4} g\|_{C^{\gamma_2}(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)} \\ \qquad \times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2)) \text{ if } \beta < \gamma_1 + \gamma_2, \gamma_1 \leq 0. \end{cases}$$

The operator with the restricted white noise $\xi_R = \sum_{a\in\mathbb{Z}^2\cap\Lambda_R} \chi_a^2 \xi$

$$
\begin{aligned}
\widetilde{H_R^{\xi}} u = & - \Delta\Phi_{\xi,R}(u) + P_{\xi_R}\Phi_{\xi,R}(u) + \Pi(\Phi_{\xi,R}(u),\xi_R) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_R)) \\
& + e^{\Delta}P_u\xi_R + e^{\Delta}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R) + e^{\Delta}P_u Y_{\xi,R} + C(u,\xi_R,\xi_R) \\
& + S(u,\xi_R,\xi_R) + P_{Y_{\xi,R}}u + \Pi(u, Y_{\xi,R}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_{\xi,R})) \\
& + P_{\xi_R}(\Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R)) + \Pi(\Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R),\xi_R) \\
& + P_{\xi_R}(\Delta^{-loc}P_u Y_{\xi,R}) + \Pi(\Delta^{-loc}P_u Y_{\xi,R},\xi_R),
\end{aligned}
$$

$$
\Phi_{\xi,R}(u) = u - \Delta^{-loc}P_u\xi_R - \Delta^{-loc}{}_u P_{\xi_R}(\Delta^{-loc}\xi_R) - \Delta^{-loc}P_u Y_{\xi,R}
$$

$$
Y_{\xi,R} = \lim_{\varepsilon\to 0}\Big(\Pi(\Delta^{-loc}\xi_{\varepsilon,R},\xi_{\varepsilon,R}) - \underbrace{\mathbb{E}[\Pi(\Delta^{-loc}\xi_{\varepsilon,R},\xi_{\varepsilon,R})]}_{\neq\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Rightarrow\widetilde{H_R^{\xi}}\neq\widetilde{H^{\xi_R}}}\Big) \quad (\xi_{\varepsilon,R} = \textstyle\sum_{a\in\mathbb{Z}^2\cap\Lambda_R}\chi_a^2 e^{\varepsilon^2\Delta}\xi)
$$

$$
\mathrm{Dom}(\widetilde{H_R^{\xi}}) := \left\{ u \in \bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,R}(u) \in \mathcal{H}^2(\mathbb{R}^2) \right\}
$$

Properties of the operator with the restricted whitenoise

$$\|\nabla \Phi_{\xi,R}(u)\|^2_{L^2(\mathbb{R}^2)} \leq (u, (\widetilde{H^\xi_R} + k(\xi, R))u)_{L^2(\mathbb{R}^2)}$$

We can show that $\mathrm{Ran}(\widetilde{H^\xi_R} + k(\xi, R)) = L^2(\mathbb{R}^2)$

Lemma (Self-adjointness of the operator with the restricted whitenoise)

The operator $\widetilde{H^\xi_R}$ with the domain $\mathrm{Dom}(\widetilde{H^\xi_R})$ is self-adjoint on $L^2(\mathbb{R}^2)$.

Proof of the self-adjointness

For $\forall f \in \mathrm{Ran}(\widetilde{H^\xi} + i)^\perp$,

$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \widetilde{\chi_R} f)_{L^2(\mathbb{R}^2)}$,

where $\widetilde{\chi_R} \in C_0^\infty(\Lambda_R \to [0,1])$ s.t. $\widetilde{\chi_R} = 1$ on Λ_{R-1}.

$\varphi_{R,L} := (H^\xi_{R+L} + i)^{-1} \widetilde{\chi_R} f \in \mathrm{Dom}(H^\xi_{R+L})$

We can show that $\widetilde{\varphi_{R,L}} := (\Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1} (\Phi_{\xi,R+L}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(\varphi_{R,L})) \in \mathrm{Dom}_0(\widetilde{H^\xi})$

Since $\widetilde{\chi_R} f = (H^\xi_{R+L} + i)\varphi_{R,L}$ and $(\widetilde{H^\xi} + i)\widetilde{\varphi_{R,L}} \in \mathrm{Ran}(\widetilde{H^\xi} + i)$, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \underbrace{(H^\xi_{R+L} + i)\varphi_{R,L} - (\widetilde{H^\xi} + i)\widetilde{\varphi_{R,L}}}_{\substack{\downarrow \text{ as } L\to\infty \\ 0}})_{L^2(\mathbb{R}^2)} = 0$$

$$\therefore \overline{\mathrm{Ran}(\widetilde{H^\xi} + i)} = L^2(\mathbb{R}^2)$$

Resolvent convergence

For $\widetilde{H^\xi}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\sup_{\|v\|_{L^2(\mathbb{T}^2)}=1} \|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^\xi} + z)^{-1}v\|_{L^2(\mathbb{T}^2)} \xrightarrow{\varepsilon\to 0} 0 \text{ for a large } z \in \mathbb{R}$$

(Mouzard Prop.2.14)

$\lambda_n(\widetilde{H^{\xi_\varepsilon}}) \xrightarrow{\varepsilon\to 0} \lambda_n(\widetilde{H^\xi})$ (Mouzard Cor.2.15)

For $\widetilde{H^\xi}$ on $\mathbb{R}^2$,

$\|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^\xi} + z)^{-1}v\|_{L^2(\mathbb{R}^2)} \xrightarrow{\varepsilon\to 0} 0$ for each $v \in L^2(\mathbb{R}^2)$ and $z \in \mathbb{C} \setminus \mathbb{R}$

However the estimate with $\sup_{\|v\|_{L^2(\mathbb{R}^2)}=1}$ may be difficult.

For the identification of the spectrum, we use the method used for stationary random operators.

Fourier series representation

$$\xi = \widetilde{\chi_L} \sum_{\boldsymbol{n} \in \mathbb{Z}^2 \cap \Lambda_{L^{10}}} X_{\boldsymbol{n}}(\xi^L) \varphi_{\boldsymbol{n}}^L(x) + \widetilde{\xi^L}$$

Partial Fourier sum independent remaining terms

$\{X_{\boldsymbol{n}}(\xi^L)\} \underset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$

$\{\varphi_{\boldsymbol{n}}^L\}$ is ONS of $L^2(\Lambda_L)$

$\varphi_0^L \equiv 1/L$

$Y(\widetilde{\xi^L})$ be a $\bigcap_{\epsilon>0} C_{loc}^{-\epsilon}(\mathbb{R}^2)$-valued random variable obtained by replacing ξ by $\widetilde{\xi^L}$ in the definition of Y_ξ

The event s.t. $\xi \underset{r}{\doteqdot} \text{const.}$ and $Y(\widetilde{\xi^L}) \doteqdot 0$ on $\Lambda_{L/2}$

$$E(\varepsilon, r, L) = \Bigg\{ \xi : \ |X_{\mathbf{0}}(\xi^L) - r|, |X_{\boldsymbol{n}}(\xi^L)| \leq \varepsilon / L^{20} \text{ for } \boldsymbol{n} \in \mathbb{Z}^2 \cap \Lambda_{L^{10}} \setminus \{\mathbf{0}\},$$

$$\left. \begin{matrix} \|\chi_a \widetilde{\xi^L}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}, \\ \|\chi_a Y(\widetilde{\xi^L})\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \end{matrix} \right\} \leq \left\{ \begin{matrix} 1/L \ (a \in \mathbb{Z}^2 \cap \Lambda_L) \\ |a| \ (a \in \mathbb{Z}^2 \setminus \Lambda_L) \end{matrix} \right\},$$

which satisfies $\mathbb{P}(E(\varepsilon, r, L)) > 0$.
$\forall \lambda \in \mathbb{R} \ \exists r, c(\lambda, L) \in \mathbb{R} \ \forall \xi \in E(\varepsilon, r, L) \ \exists \varphi \in C_0^\infty(\Lambda_{L/2})$
s.t. $\|(\widetilde{H^\xi} - \lambda)\varphi\|_{L^2(\mathbb{R}^2)} < c\varepsilon, \ \|\varphi\| = 1$

Probability One

Let $E(x_0, \varepsilon, r, L) := \{\xi : \xi(\cdot - x_0) \in E(\varepsilon, r, L)\}$
Then $\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, r, L)$ is $\mathbb{Z}^2$-invariant.
By the ergodicity of the white noise, we have

$$\mathbb{P}\Big(\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, r, L)\Big) = 1.$$

By the shift, we can take a Weyl sequence with probability 1.
Thus $\lambda \in \mathrm{Spec}(\overline{\widetilde{H^\xi}})$.