A proof of the Anderson localization induced by the 2-dimensional white noise

https://www.math.h.kyoto-u.ac.jp/users/ueki/2DWN-Loc1.pdf
https://www.math.h.kyoto-u.ac.jp/users/ueki/presen20250711KansaiProbSem

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Motivation

Study the spectral property of the Schrödinger operator

$$-\Delta + \xi \tag{1}$$

in the case that the potential is the whitenoise on $\mathbb{R}^d$

$\xi = (\xi(x))_{x\in\mathbb{R}^d}$: Gaussian random field, $\mathbb{E}[\xi(x)] = 0$, $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$

Difficulty: "$x \mapsto \xi(x)$" $\in C^{-\epsilon-d/2}$, not a regular function

Dimensions

$$d = 1 \Rightarrow W^{1,2}([a,b]) \ni f, g \mapsto \int_a^b (f'g' + \underset{C_{loc}^{-1/2-\epsilon}}{\xi} fg)dx \ : \ \text{well-defined}$$

$-\Delta + \xi$ is realized as a self-adjoint operator

M. Fukushima and S. Nakao (1977) Spectral asymptotics
(Asymptotics of the Integrated density of states)

N. Minami (1988) (1989) Self-adj. on $\mathbb{R}$, Exp.Loc. at all energies
($\xi \to \partial$(Lévy process))

L. Dumaz and C. Labbé (2020) (2023) Eigenvalues Eigenvectors Statistics

Recently

$d = 2$ or $3 \Rightarrow -\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ are realized as self-adjoint operators

$d \geq 4 \Rightarrow$ No results

Related works on singular SPDEs

M. Hairer (2014) A theory of regularity structures,
M. Gubinelli, P. Imkeller and N. Perkowski (2015) Paracontrolled calculus
A. Kupiainen (2016) Renormalization Group
$\Rightarrow$ Stochastic quatization equation for ϕ_3^4 Euclidean quantum field theory
Generalized continuous parabolic Anderson model
Kardar–Parisi–Zhang type equation
Navier-Stokes equation with very singular forcing
and so on

Eg. Continuous parabolic Anderson model

$$\partial_t u(t,x) = \Delta_x u(t,x) - \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon) u(t,x) \text{ for } t > 0$$

Schrödinger operators on compact spaces

R. Allez and K. Chouk (2015) Paracontrolled calculus based on Fourier Analysis

$-\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ on $\mathbb{R}^2/\mathbb{Z}^2 \Leftarrow$ Self-adjointness, Discrete Spectrum, Asymptotic Distribution

M. Gubinelli, B. Ugurcan and I. Zachhuber (2020) Extension to $\mathbb{R}^3/\mathbb{Z}^3$

$$i\partial_t u(t,x) = \underbrace{(\Delta - \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon) - k_\xi)}_{\geq 0} u(t,x) - (u|u|^2)(t,x)$$

$$\partial_t^2 u(t,x) = (\Delta - \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon) - k_\xi) u(t,x) - (u^3)(t,x)$$

C. Labbé (2019) similar results by regularity structure for

$-\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ on $(-L, L)^{2or3}$ with periodic or Dirichlet conditions

Extensions to noncompact spaces

M. Hairer and C. Labbé, (2015) (2018)
$\partial_t u_\varepsilon = \Delta u_\varepsilon - (\xi_\varepsilon + c_\varepsilon) u_\varepsilon, t > 0, x \in \mathbb{R}^d$, $d = 2, 3$, $u_\varepsilon(0, \cdot) = u_0$
$\exists \lim_{\varepsilon \to 0} u_\varepsilon$ uniformly on compact sets in probability
depends continuously in u_0

Y. Hsu and C. Labbé, (2024)
The generator of the above equation
$\lim_{\varepsilon \to 0}(-\Delta + \xi_\varepsilon + c_\varepsilon)$: self-adjoint Spec$= \mathbb{R}$

Related operators on noncompact spaces

B. Ugurcan, (2022) $-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + \widetilde{c}_\varepsilon(x))$ on $\mathbb{R}^2$ with $\widetilde{c}_\varepsilon(x) \overset{|x|\to\infty}{\longrightarrow} 0$

$$= \underline{-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon^{\uparrow}(x) + \widetilde{c}_\varepsilon(x))} + \underline{\xi_\varepsilon^{\downarrow}(x)},$$

$\uparrow$ An extension of the method for the compact case

$\nwarrow$ By a commutator estimate,

where $\xi = \xi^{\uparrow}(x) + \underbrace{\xi^{\downarrow}(x)}_{\text{Smooth functions in } x}$

$$= \sum_{n=-1}^{\infty} \widetilde{\chi_{[[2^n],2^{n+1}]}}(|x|)\{\underbrace{\widetilde{\chi_{[c2^n,\infty)}}(-\Delta)\xi}_{\text{high energy pert}} + \underbrace{\widetilde{\chi_{[0,c2^n]}}(-\Delta)\xi}_{\text{low energy pert}}\}$$

$\widetilde{c}_\varepsilon(x) = \mathbb{E}[$A resonant product of $\xi_\varepsilon^{\uparrow}(x)$ and $(1-\Delta)^{-1}\xi_\varepsilon^{\uparrow}(x)]$

Heat semigroup approach

Paracontrolled calculus by Heat semigroups
(suitable for noncompact manifolds, graphs and so on)
Paraproducts defined by using Heat Semigroups
(cf. The preceding approach defines paraproducts using Fourier Analysis)

I. Bailleul and F. Bernicot (2016) $\exists \lim_{\varepsilon \to 0} u_\varepsilon$
generalized PAM on 2D manifold (without compactness)

I. Bailleul, F. Bernicot and D. Frey (2018) $\exists \lim_{\varepsilon \to 0} u_\varepsilon$
PAM and multiplicative Burgers eq. on 3D manifold
(without compactness)

A. Mouzard (2022) Self-adjointness, Discrete Spectrum, Its asymptotics of
$\underset{\text{eg. Lap.Belt}}{-\Delta} + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon(x))$ on 2D compact manifold, where $\xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta}\xi)(x)$

$$c_\varepsilon(x) \equiv c_\varepsilon \text{ on } \mathbb{R}^2/\mathbb{Z}^2$$

On $\mathbb{R}^2$

N. Ueki (2025)
A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane, Stochastic Processes and their Applications **186** (2025), 104642

$$\widetilde{H^\xi} = -\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon) \text{ on } \mathbb{R}^2, \text{ where } \xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta} \xi)(x)$$

Self-adjointness
$\mathrm{Spec}(\widetilde{H^\xi}) = \mathbb{R}$

Our topics: By extending the traditional methods, prove
the exponential localization for low energies
i.e. $\exists E_0 > -\infty$ s.t. $(-\infty, E_0] \subset \mathrm{Spec}_{pp}(\widetilde{H^\xi})$
the corresponding eigenfunctions decay exponentially

Products $fg = P_f g + \Pi(f,g) + P_g f + P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g))$

$0 << b \in 2\mathbb{Z}$ fixed $\quad P_t^{(b)} = \sum_{j=0}^{b-1} \frac{(-t\Delta)^j}{j!} e^{t\Delta}$

$$\underbrace{P_0^{(b)}((P_0^{(b)} f)(P_0^{(b)} g))}_{=fg} - P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g)) = -\int_0^1 dt \partial_t \{P_t^{(b)}((P_t^{(b)} f)(P_t^{(b)} g))\}$$
$$= P_f g + \Pi(f,g) + P_g f$$

$P_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$: paraproduct(a well-defind distribution)

$\Pi(f,g) := \sum_\mu c_\mu \int_0^1 \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g))$: resonating term (need regularity)

$P^\nu, P^\mu \in StGC^{[0,b/2)}, \; Q^{1,\nu}, Q^{2,\nu}, Q^{1,\mu}, Q_t^{2,\mu} \in StGC^{[b/2,2b]}$

For any $I \subset (0,\infty)$, $StGC^I = \{((\sqrt{t}\nabla)^\alpha P_t^{(c)})_{t\in(0,1]} : \alpha \in \mathbb{Z}_+^2, \alpha_1 + \alpha_2 \in I \cap \mathbb{Z}, c \in \mathbb{N} \cap [1,b]\}$

standard families of Gaussian operators with cancellation of orders I

The Besov Spaces

For $p, q \in [1, \infty], \alpha \in (-2b, 2b)$, $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2) = \overline{C_0^{\infty}(\mathbb{R}^2)}^{\|\cdot\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)}}$: the Besov Space

$$\|f\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)} := \|e^{\Delta} f\|_{L^p(\mathbb{R}^2:dx)} + \sum_{Q \in StGC^{(|\alpha|,2b]}} \left\| t^{-\alpha/2} \|Q_t f\|_{L^p(\mathbb{R}^2:dx)} \right\|_{L^q([0,1]:t^{-1}dt)}$$

$\mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2) =: \mathcal{C}^{\alpha}(\mathbb{R}^2)$: the Besov α-Hölder space

$\mathcal{B}^{\alpha}_{2,2}(\mathbb{R}^2) =: \mathcal{H}^{\alpha}(\mathbb{R}^2)$: the Sobolev space with the index α.
$= \mathcal{W}^{\alpha,2}(\mathbb{R}^2)$

$\{\chi_a\}_{a \in \mathbb{Z}^2} \subset C^{\infty}(\mathbb{R}^2 \to [0,1])$ s.t. $\sum_{a \in \mathbb{Z}^2} \chi_a^2 \equiv 1$, $\operatorname{supp} \chi_a \subset \square_2(a) := a + (-1,1)^2$

$\chi_a(\cdot) = \chi_0(\cdot - a)$

The continuity and the exponential decay of paraproducts

(i) For any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$\|\chi_{a_1} P_{\chi_{a_2} g}(\chi_{a_3} f)\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)} \leq \begin{cases} C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^\infty(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^2(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(ii) For any $\alpha \in (-\infty, 0)$ and $\beta \in \mathbb{R}$,

$$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)} \leq \begin{cases} C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{H}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(iii) For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$,

$$\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)} \leq C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)).$$

A modification of the operator

$\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi}(\log(2+|a|))^{1/2}$

$\Delta^{-loc} := -\int_0^1 dt\ e^{t\Delta}$ satisfying $\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^{\Delta}$

$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)} \leq C_{\alpha,\epsilon}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1 - a_2|^2)$

$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)} \leq C_{\alpha,\epsilon}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1 - a_2|^2)$

$\Pi(\Delta^{-loc}\xi, \xi)$ in the formal expression of $H^{\xi} = -\Delta + \xi$ is near to being well-defined, and we can obtain a well-defined operator $\widetilde{H^{\xi}}$ by replacing this by a $\bigcap_{\epsilon>0} C_{loc}^{-\epsilon}(\mathbb{R}^2)$-valued random variable Y_{ξ} s.t.

$\lim_{\varepsilon\to 0} \mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_{\xi})\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$ for any $a \in \mathbb{Z}^2$, $p \in [1,\infty)$ and $\epsilon > 0$, where

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon) - \underline{\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]}$$

$\xi_\varepsilon := e^{\varepsilon^2\Delta}\xi$ is a smooth approximation of ξ $\quad\nwarrow$ diverge as $\varepsilon \to 0$

$\|\chi_a Y_{\xi}\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi}\log(2+|a|)$

Commutators

$$C(f,g,h) := \Pi(\Delta^{-loc}P_f g, h) - f\Pi(\Delta^{-loc}g, h)$$
$$S(f,g,h) := P_h(\Delta^{-loc}P_f g) - {}_f P_h(\Delta^{-loc}g)$$
$$ {}_f P_h g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu h)(Q_t^{2,\nu} g) f)$$

(i) For any $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}, \gamma \in (-\infty, 0)$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$,

$$\|\chi_{a_1} C(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}, \|\chi_{a_1} S(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}$$
$$\leq C_{\epsilon,\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|\chi_{a_4} h\|_{\mathcal{C}^\gamma(\mathbb{R}^2)}$$
$$\times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2 + |a_1 - a_4|^2))$$

(ii) For any $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$\|\chi_{a_1} {}_{\chi_{a_2} h} P_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)}$$
$$\leq C_{\alpha,\beta,\epsilon} \|\chi_{a_3} f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|\chi_{a_4} g\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)}$$
$$\times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))$$

Our operator

$$\Phi_\xi(u) := u - \Delta^{-loc} P_u \xi - \Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi) - \Delta^{-loc} P_u Y_\xi$$

$$\mathrm{Dom}_{+0}(\widetilde{H^\xi}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0 \text{ for any } \epsilon > 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\},$$

$$\begin{aligned}\widetilde{H^\xi} u := & -\Delta\Phi_\xi(u) + P_\xi \Phi_\xi(u) + \Pi(\Phi_\xi(u), \xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi)) \\ & + e^\Delta P_u \xi + e^\Delta{}_u P_\xi(\Delta^{-loc}\xi) + e^\Delta P_u Y_\xi + C(u, \xi, \xi) + S(u, \xi, \xi) \\ & + P_{Y_\xi} u + \Pi(u, Y_\xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_\xi)) \\ & + P_\xi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi)) + \Pi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi), \xi) \\ & + P_\xi(\Delta^{-loc} P_u Y_\xi) + \Pi(\Delta^{-loc} P_u Y_\xi, \xi).\end{aligned}$$

An abstract representation of the operator $\widetilde{H^\xi}$

$$
\begin{aligned}
\widetilde{H^\xi} u \sim -\Delta\Phi_\xi(u) + \sum \int_0^1 \frac{dt}{t} \int dx_1\, O\Big(\frac{1}{t}\exp\Big(-\frac{|x-x_1|^2}{t}\Big)\Big) u(x_1) \\
\times \int dx_2\, O\Big(\frac{1}{t}\exp\Big(-\frac{|x-x_2|^2}{t}\Big)\Big) \begin{matrix}\xi(x_2) \\ Y_\xi(x_2)\end{matrix} \\
\times \Big(\int dx_3\, O\Big(\frac{1}{t}\exp\Big(-\frac{|x-x_3|^2}{t}\Big)\Big)\xi(x_3)\Big)
\end{aligned}
$$

Main Statements

Theorem (Self-adjointness, U(2025))

The operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_{+0}(\widetilde{H^\xi})$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$.

Theorem (Spectrum, U(2025))

The spectral set of the unique self-adjoint extension $\widetilde{H^\xi}$ is $\mathbb{R}$.

Theorem (Exponential Localization, U(New))

$\exists E_0 \in (-\infty, 0)$ s. t. for a. a. ξ, $(-\infty, E_0] \subset spec_{pp}(\widetilde{H^\xi})$ and any corresponding eigenfunction ϕ_ξ satisfies

$$\overline{\lim_{|a|\to\infty}} |a|^{-1} \log \|\chi_a \phi_\xi\|_{L^2(\mathbb{R}^2)} < 0.$$

Restrict the white noise to the square $\square_{L,a} := \boldsymbol{a} + (-L/2, L/2)^2$

$\xi_{L-2,\boldsymbol{a}} = \sum_{a \in \mathbb{Z}^2 \cap \square_{L-2,\boldsymbol{a}}} \chi_a^2 \xi \quad \xi_{\varepsilon, L-2,\boldsymbol{a}} = \sum_{a \in \mathbb{Z}^2 \cap \square_{L-2,\boldsymbol{a}}} \chi_a^2 e^{\varepsilon^2 \Delta} \xi$ Omit $\boldsymbol{a}$ if $\boldsymbol{a} = 0$

$\overline{\xi_{L,\boldsymbol{a}}} = \sum_{a \in \mathbb{Z}^2 \cap (\square_{L,\boldsymbol{a}} \setminus \square_{L-2,\boldsymbol{a}})} \chi_a^2 \overline{\xi}_a \quad \overline{\xi} = (\overline{\xi}_a)_{a \in \mathbb{Z}^2}$ i.i.d. bdd, with C_0^∞ density indep. of ξ

$$Y_{\xi, L-2,\boldsymbol{a}} = \lim_{\varepsilon \to 0} (\Pi(\Delta^{-loc} \xi_{\varepsilon, L-2,\boldsymbol{a}}, \xi_{\varepsilon, L-2,\boldsymbol{a}}) - \mathbb{E}[\Pi(\Delta^{-loc} \xi_{\varepsilon, L-2,\boldsymbol{a}}, \xi_{\varepsilon, L-2,\boldsymbol{a}})])$$

$$\underbrace{\Phi_{\xi, L-2,\boldsymbol{a}}}(u) = \underbrace{u} - \Delta^{-loc}\{P_u \xi_{L-2,\boldsymbol{a}} +_u P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc} \xi_{L-2,\boldsymbol{a}}) + P_u Y_{\xi, L-2,\boldsymbol{a}}\} \in \mathcal{H}^2(\mathbb{R}^2)$$

$$H_{L,\boldsymbol{a}}^{\widetilde{\xi}} u = H_{L-2,\boldsymbol{a}}^{\xi} u + \overline{\xi_{L,\boldsymbol{a}}} u$$

$$\|$$

$$-\Delta \Phi_{\xi, L-2,\boldsymbol{a}}(u) + P_{\xi_{L-2,\boldsymbol{a}}} \Phi_{\xi, L-2,\boldsymbol{a}}(u) + \Pi(\Phi_{\xi, L-2,\boldsymbol{a}}(u), \xi_{L-2,\boldsymbol{a}}) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi_{L-2,\boldsymbol{a}}))$$
$$+ e^{\Delta} P_u \xi_{L-2,\boldsymbol{a}} + e^{\Delta}{}_u P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc} \xi_{L-2,\boldsymbol{a}}) + e^{\Delta} P_u Y_{\xi, L-2,\boldsymbol{a}} + C(u, \xi_{L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}})$$
$$+ S(u, \xi_{L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}}) + P_{Y_{\xi, L-2,\boldsymbol{a}}} u + \Pi(u, Y_{\xi, L-2,\boldsymbol{a}}) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_{\xi, L-2,\boldsymbol{a}}))$$
$$+ P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}{}_u P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc} \xi_{L-2,\boldsymbol{a}})) + \Pi(\Delta^{-loc}{}_u P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc} \xi_{L-2,\boldsymbol{a}}), \xi_{L-2,\boldsymbol{a}})$$
$$+ P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc} P_u Y_{\xi, L-2,\boldsymbol{a}}) + \Pi(\Delta^{-loc} P_u Y_{\xi, L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}})$$

Properties of the restriction

$\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}$ is the norm resolvent limit of $\widetilde{H^{\tilde{\xi}_\varepsilon}_{L,\boldsymbol{a}}} := \widetilde{H^{\xi_\varepsilon}_{L-2,\boldsymbol{a}}} + \overline{\xi}_{L,\boldsymbol{a}}$ as $\varepsilon \to 0$

The negative spectra of the operator $\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}$ are discrete.

Lemma (Moments of numbers of negative eigenvalues)

$\forall \lambda > 0$, $\forall p \geq 1$, $\exists c_{\lambda,p,1}, \exists c_{\lambda,p,2} \in (0,\infty)$ *s.t.*

$\mathbb{E}[Tr[1_{(-\infty,-\lambda]}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}})]^p]^{1/p} \leq c_{\lambda,p,1} L^{c_{\lambda,p,2}}$ *for* $\forall L \in 2\mathbb{N}$.

$$\because\ \mathbb{E}[\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H^{\tilde{\xi}}_L})]^p]^{1/p} \leq \varliminf_{\varepsilon\to 0} \mathbb{E}[\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H^{\tilde{\xi}_\varepsilon}_L})]^p]^{1/p}$$

Since $\widetilde{H^{\tilde{\xi}_\varepsilon}_L}$ is a relatively compact perturbation of $-\Delta$,
we apply the Birman-Schwinger principle.

To treat the limit as $\varepsilon \to 0$, we replace $-\Delta$ by $\widehat{H^{\tilde{\xi}_\varepsilon, s}_{L-2}}$ in the next slide:

Operators with the restriction whose lower bound is near to zero

In the definition of $\widetilde{H_L^{\tilde{\xi}}}$, replace $P_f g$ and $\Pi(f,g)$ by

$$P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)), \quad \Pi^s(f,g) := \sum_\mu c_\mu \int_0^s \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g))$$

$$\Phi^s_{\xi,L-2}(u) := u - \Delta^{-loc} P_u^s \xi_{L-2} - \Delta^{-loc}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc}\xi_{L-2}) - \Delta^{-loc} P_u^s Y^s_{\xi,L-2} \in \mathcal{H}^2(\mathbb{R}^2)$$

$$\begin{aligned}
\widehat{H^{\xi,s}_{L-2}} u = & -\Delta\Phi^s_{\xi,L-2}(u) + P^s_{\xi_{L-2}}(\Phi^s_{\xi,L-2}(u)) + \Pi^s(\Phi^s_{\xi,L-2}(u), \xi_{L-2}) \\
& + e^{s\Delta} P_u^s \xi_{L-2} + e^{s\Delta}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc}\xi_{L-2}) + e^{s\Delta} P_u^s Y^s_{\xi,L-2} \\
& + C^s(u, \xi_{L-2}, \xi_{L-2}) + S^s(u, \xi_{L-2}, \xi_{L-2}) \\
& + P^s_{Y^s_{\xi,L-2}} u + \Pi^s(u, Y^s_{\xi,L-2}) + P_s^{(b)}((P_s^{(b)} u)(P_s^{(b)} Y^s_{\xi,L-2})) \\
& + P^s_{\xi_{L-2}}(\Delta^{-loc}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc}\xi_{L-2})) + \Pi^s(\Delta^{-loc}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc}\xi_{L-2}), \xi_{L-2}) \\
& + P^s_{\xi_{L-2}}(\Delta^{-loc} P_u^s Y^s_{\xi,L-2}) + \Pi^s(\Delta^{-loc} P_u^s Y^s_{\xi,L-2}, \xi_{L-2})
\end{aligned}$$

Properties of the operator with the restriction

$$(u, \widehat{H^{\xi,s}_{L-2}}u)_{L^2(\mathbb{R}^2)} \geq -c_1 s^{c_2}(1 + \|\xi_{L-2}\|^2_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{s\in(0,1]} \|Y^s_{\xi,L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)})^{c_3}\|u\|^2_{L^2(\mathbb{R}^2)}$$

$s(\xi,\lambda,L-2) := (\lambda/(4c_1(1+\|\xi_{L-2}\|^2_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{s\in(0,1]}\|Y^s_{\xi,L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)})^{c_3}))^{1/c_2}$

$\Rightarrow \widehat{H^{\xi,s(\xi,\lambda,L-2)}_{L-2}} \geq -\lambda/4$

$\widetilde{H^{\xi}_{L-2}}u = \widehat{H^{\xi,s}_{L-2}}u + P^{(b)}_s((P^{(b)}_s u)(P^{(b)}_s \xi_{L-2})) - \overline{Y^s_{L-2}}u$

with $\overline{Y^s_{L-2}} := \mathbb{E}[(\Pi - \Pi^s)(\Delta^{-loc}\xi_{L-2}, \xi_{L-2})]$

By the Birman-Schwinger principle, we have

$$\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H^{\xi_\varepsilon}_{L-2}} + \overline{\xi}_L)] \leq \mathrm{Tr}[1_{[1,\infty)}(\Gamma^{(\widetilde{\xi},\varepsilon)})] \leq \mathrm{Tr}[(\Gamma^{(\widetilde{\xi},\varepsilon)})^2] = \|\Gamma^{(\widetilde{\xi},\varepsilon)}\|^2_{\mathcal{I}_2},$$

Our Birman-Schwinger kernel

$$\Gamma^{(\widetilde{\xi},\varepsilon)} = -\Gamma_0^{(\xi,\varepsilon)} + \Gamma_1^{(\xi,\varepsilon)} - \Gamma_2^{(\widetilde{\xi},\varepsilon)},$$

where

$$\begin{aligned}
\Gamma_0^{(\xi,\varepsilon)} =& (H_{L-2}^{\widehat{\xi_\varepsilon, s(\xi,\lambda,L}-2)} + \lambda)^{-1/2} (P^{(b)}_{s(\xi,\lambda,L-2)}((P^{(b)}_{s(\xi,\lambda,L-2)}\xi_{\varepsilon,L-2})(P^{(b)}_{s(\xi,\lambda,L-2)}\cdot))) \\
& \times (H_{L-2}^{\widehat{\xi_\varepsilon, s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}, \\
\Gamma_1^{(\xi,\varepsilon)} =& (H_{L-2}^{\widehat{\xi_\varepsilon, s(\xi,\lambda,L}-2)} + \lambda)^{-1/2} \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}} (H_{L-2}^{\widehat{\xi_\varepsilon, s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}, \\
\Gamma_2^{(\widetilde{\xi},\varepsilon)} =& (H_{L-2}^{\widehat{\xi_\varepsilon, s(\xi,\lambda,L}-2)} + \lambda)^{-1/2} \overline{\xi}_L (H_{L-2}^{\widehat{\xi_\varepsilon, s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}.
\end{aligned}$$

$\|\Gamma_1^{(\xi,\varepsilon)}\|_{\mathcal{I}_2} \lesssim \|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2} + \|\Gamma_{1,2}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$

$k_{\xi,L-2}$: a positive polynomial of $\|\xi_{L-2}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}$ and $\|Y_{\xi,L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)}$ s.t.

$$\|u\|^2_{L^2(\mathbb{R}^2)} \leq (u, (\widetilde{H^{\xi_\varepsilon}_{L-2}} + k_{\xi,L-2})u)_{L^2(\mathbb{R}^2)} \text{ for } \forall\varepsilon \in 0,1)$$

By $(\widetilde{H^{\xi_\varepsilon}_{L-2}} + k_{\xi,L-2})^{-1}$

$$= \int_0^T \frac{dt}{2} \exp\left(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}} - \frac{t}{2}k_{\xi,L-2}\right) + \exp\left(-\frac{T}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}} - \frac{T}{2}k_{\xi,L-2}\right)(\widetilde{H^{\xi_\varepsilon}_{L-2}} + k_{\xi,L-2})^{-1},$$

for any $T \in (0,\infty)$, it is enough to estimate $\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$ and $\Gamma_{1,2}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$, where

$$\Gamma_{1,1}^{(\xi,\varepsilon)} = \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}} \int_0^T \frac{dt}{2} \exp\left(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}}\right) \exp\left(-\frac{t}{2}k_{\xi,L-2}\right)$$

$$\Gamma_{1,2}^{(\xi,\varepsilon)} = \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}} \exp\left(-\frac{T}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}}\right) \exp\left(-\frac{T}{2}k_{\xi,L-2}\right)$$

A moment

By applying the Feynman-Kac formula and omitting $k_{\xi,L}$,

$$
\begin{aligned}
&\mathbb{E}^{\xi}[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^2] \\
&\leq \int_{(\mathbb{R}^2)^2} dx \, \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}}(x)^2 \int_{[0,T]^2} \frac{dt d\underline{t}}{8\pi(t+\underline{t})} \\
&\quad \times \mathbb{E}^{\xi,w}\left[\exp\left(-\int_0^{t+\underline{t}} \frac{dt'}{2}(\xi_{\varepsilon,L-2} - \mathbb{E}^{\xi}[\Pi(\Delta^{-loc}\xi_{\varepsilon,L-2}, \xi_{\varepsilon,L-2})])(x + w_0^{t+\underline{t}}(t'))\right)\right],
\end{aligned}
$$

where $w_0^{t+\underline{t}}$ is a 2D Brownian bridge s.t.$w_0^{t+\underline{t}}(0) = w_0^{t+\underline{t}}(t+\underline{t}) = 0$.

ξ-expectation

$$
\begin{aligned}
&\mathbb{E}^{\xi}[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^2] \\
&\leq \int_{(\mathbb{R}^2)^2} dx \ \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}}(x)^2 \int_{[0,T]^2} \frac{dtd\underline{t}}{8\pi(t+\underline{t})} \\
&\quad \times \mathbb{E}^{w}\left[\exp\left(\frac{1}{2}\chi_{\varepsilon}(t+\underline{t},x,L-2,w_0^{t+\underline{t}})\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_{\varepsilon}(t,x,\ell,w) := &\int_0^t ds_1 \int_0^t ds_2 (e^{\varepsilon^2\Delta}\widetilde{\chi}_{\ell}^2 e^{\varepsilon^2\Delta})(x+w(s_1),x+w(s_2)) \\
&+4\int_0^t ds \mathbb{E}^{\xi}[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell})](x+w(s)).
\end{aligned}
$$

and $\widetilde{\chi}_{\ell} = \sum_{a\in\mathbb{Z}^2\cap\square_{\ell}} \chi_a^2$

To a restriction of the renormalized intersection local time

$$\chi_\varepsilon(t,x,\ell,w_0^t) = 2\chi_\varepsilon^0(t,x,\ell,w_0^t) + 4\chi_\varepsilon^{bdd}(t,x,\ell,w_0^t), \text{ where}$$

$$\chi_\varepsilon^0(t,x,\ell,w_0^t) := \iint_{0\le s_1\le s_2\le t} ds_1 ds_2 \int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2(x+w_0^t(s_1)+y) \times \{e^{\varepsilon^2\Delta}(y+w_0^t(s_1)-w_0^t(s_2)) - \mathbb{E}^w[e^{\varepsilon^2\Delta}(y+w_0^t(s_1)-w_0^t(s_2))]\},$$

$$\chi_\varepsilon^{bdd}(t,x,\ell,w_0^t) := \int_0^t ds \int_{[0,(t-s)/2]\setminus[0,1]} dr \int dy e^{\varepsilon^2\Delta}(y)\widetilde{\chi}_\ell^2(x+w_0^t(s)+y)e^{(r+\varepsilon^2)\Delta}(y) + \int_0^t ds \mathbb{E}^\xi[\Pi(\Delta^{-loc}\xi_{\varepsilon,\ell},\xi_{\varepsilon,\ell}) - (\Delta^{-loc}\xi_{\varepsilon,\ell})\xi_{\varepsilon,\ell}](x+w_0^t(s))$$

$$\sup_{t\in[0,1],x\in\mathbb{R}^2,\varepsilon\in(0,1],\ell\in\mathbb{N},w} |\chi_\varepsilon^{bdd}(t,x,\ell,w)| < \infty$$

$$\lim_{\varepsilon\to 0}\chi_\varepsilon^0(t,x,\ell,w_0^t) = \iint_{0\le s_1\le s_2\le t} ds_1 ds_2 \widetilde{\chi}_\ell^2(x+w_0^t(s_1))\{\delta(w_0^t(s_1)-w_0^t(s_2)) - \mathbb{E}^w[\delta(w_0^t(s_1)-w_0^t(s_2))]\}$$ formally

Intersection local time for 2D Brownian motion

$$\chi_0^0(t,x,\infty,w_0^t) = \iint_{0\le s_1\le s_2\le t} ds_1\, ds_2 \{\delta(w_0^t(s_1)-w_0^t(s_2)) - \mathbb{E}^w[\delta(w_0^t(s_1)-w_0^t(s_2))]\}$$

X. Chen, Random walk intersections: large deviations and related topics, Mathematical Surveys and Monographs, vol. 157, AMS (2010)

$\sup_\varepsilon \mathbb{E}^w[\exp(\chi_\varepsilon^0(t,x,\infty,w_0))] < \infty$ for small enough $t > 0$,

where $\chi_\varepsilon^0(t,x,\infty,w_0)$ is the function obtained by replacing the Brownian bridge w_0^t by the Brownian motion w_0 starting at 0 in the definition of $\chi_\varepsilon^0(t,x,\infty,w_0^t)$

T. Matsuda, Integrated density of states of the Anderson Hamiltonian with two-dimensional white noise, Stochastic Processes and their Applications **153** (2022), 91-–127

$\sup_\varepsilon \mathbb{E}^w[\exp(\chi_\varepsilon^0(t,x,\infty,w_0^t))] < \infty$ for small enough $t > 0$

The end of our estimates of the numbers of negative eigenvalues

By the same method, we have
$\sup_{\varepsilon} \mathbb{E}^{w}\left[\exp\left(\frac{1}{2}\chi_{\varepsilon}(t,x,\ell,w_0^t)\right)\right] < \infty$ for small enough $t > 0$
We also use

$$|\overline{Y_{L-2}^{s(\xi,\lambda,L-2)}}| \leq c_1 \exp(-c_2 d(x,\Lambda_L)^2/s(\xi,\lambda,L-2)) \log(1/s(\xi,\lambda,L-2))$$

and take $T > 0$ sufficiently small to obtain
$\mathbb{E}^{\xi}[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^2] < \infty$
Similarly we obtain
$\mathbb{E}^{\xi}[\|\Gamma_{j}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}] < \infty$ for $j \in \{0,1,2\}$ and $p \in \mathbb{N}$.
Thus we can complete the proof of the lemma on the moments of numbers on negative eigenvalues.

Outline of a traditional proof of the localization

I. Initial estimate : for some $L_0 \in \mathbb{N}$

$\|\chi_x(\widetilde{H^{\widetilde{\xi}}_{L_0}} - E)^{-1}\chi_y\| \lesssim \exp(-m|x-y|)$

by Combes-Thomas estimate

II(Multi Scale Analysis).

Similar estimate for $L_1 > L_0$

Similar estimate for $L_2 > L_1 \cdots$

by Geometric resolvent inequality

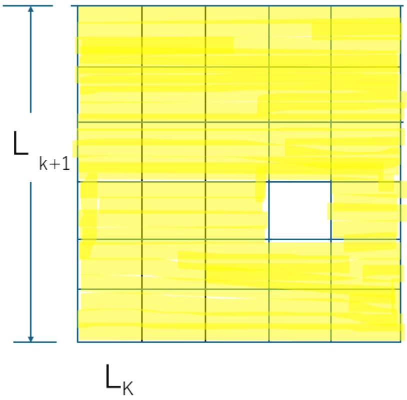
and Wegner estimate

III. Apply the preceding estimates

to generalized eigenfunctions

(from a generalized eigenfunction expansion)

by eigenfunction decay inequality



A definition of regular squares

$\square_{L,\boldsymbol{a}}$: (m, E, K)-regular for $\widetilde{\xi} = (\xi, \overline{\xi})$

$\overset{def}{\Leftrightarrow}$

$E \notin \mathrm{spec}(\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}})$ and

$$\|\chi_{a_1}(\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1}\chi_{a_2}\|_{op} \leq K \exp(-m(|a_1 - a_2|_\infty \wedge d_\infty(a_2, \partial\square_{L,\boldsymbol{a}}) + d(a_1, \square_{L,\boldsymbol{a}})))$$

for any $a_1 \in \mathbb{Z}^2$ and $a_2 \in \mathbb{Z}^2 \cap \square_{L/3,\boldsymbol{a}}$

Combes-Thomas type estimate

$$
\begin{aligned}
&\|\chi_{a_1}(\widetilde{H_L^{\tilde{\xi}}} - E)^{-1}\chi_{a_2}\|_{op} \\
&\leq \frac{3}{d(E, \mathrm{spec}\widetilde{H_L^{\tilde{\xi}}})} \\
&\times \exp\left(\frac{-(|a_1 - a_2| - 2\sqrt{2})_+ \color{red}{d(E, \mathrm{spec}\widetilde{H_L^{\tilde{\xi}}})}}{2\sqrt{d(E, \mathrm{spec}\widetilde{H_L^{\tilde{\xi}}}) + c(1 + \sup_{a \in \mathbb{Z}^2 \cap \square_{L-2}} \|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{a \in \mathbb{Z}^2} \|\chi_a Y_{\xi, L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)})}} \right)
\end{aligned}
$$

$$
\because \color{blue}{\|e^{-v\cdot x}(\widetilde{H_L^{\tilde{\xi}}} - E)^{-1} e^{v\cdot x}\|_{op} \text{ for } v \in \mathbb{R}^2} : |v| < \sqrt{d(E, \mathrm{spec}\widetilde{H_L^{\tilde{\xi}}})}
$$

$$
\leq \|(\widetilde{H_L^{\tilde{\xi}}} - |v|^2 - E)^{-1}\|_{op} \sum_{n=0}^{\infty} \|(\widetilde{H_L^{\tilde{\xi}}} - |v|^2 - E)^{-1/2} 2v \cdot \nabla (\widetilde{H_L^{\tilde{\xi}}} - |v|^2 - E)^{-1/2}\|_{op}
$$

Initial estimate

$$\mathbb{P}(\|\chi_a\xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \leq \Xi_1\sqrt{\log(2+|a|)} \text{ and }$$
$$\|\chi_a Y_{\xi,L_0}\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq \Xi_2 \log(2+|a|)\exp(-\widetilde{c}_* d(a,\square_{L_0})) \text{ for any } a\in\mathbb{Z}^2) \geq 1-1/L_0^{p/2}$$

for some $\Xi_1, \Xi_2 \in (0,\infty)$.

Under this event, we have $\inf \mathrm{spec} \widetilde{H_{L_0}^{\widetilde{\xi}}} \geq -\Xi_3(\log L_0)^{\Xi_4}$ for some $\Xi_3, \Xi_4 \in (0,\infty)$.
For sufficiently small $m_0 > 0$
$\square_{L_0}$: (m_0, E, K_0)-regular for any $E \in [E_1, E_0]$ if
$E_0 < -\Xi_5(\log L_0)^{\Xi_6}$ for some $\Xi_5, \Xi_6 \in (0,\infty)$,
where K_0 is a number depend on E_0, E_1, L_0.
Since the bound of the energy tends to $-\infty$ as $L_0 \to \infty$,
we need more techniques like Multi Scale Analysis.

An idea of Geometric resolvent inequality

$\square_{\ell,\boldsymbol{a}} \subset \square_L$, $a \in \square_{\ell-8,\boldsymbol{a}}$, $a_* \notin \square_{\ell,\boldsymbol{a}}$

$\phi \in C_0^\infty(\square_{\ell-2,\boldsymbol{a}} \to [0,1])$ $\phi = 1$ on $\square_{\ell-6,\boldsymbol{a}}$

$$(-\Delta + V1_{\square_L} - z)^{-1}\phi - \phi(-\Delta + V1_{\square_{\ell,\boldsymbol{a}}} - z)^{-1}$$
$$= (-\Delta + V1_{\square_L} - z)^{-1}((2\nabla\phi)\cdot\nabla + (\Delta\phi))(-\Delta + V1_{\square_{\ell,\boldsymbol{a}}} - z)^{-1}$$

$$\|\chi_{a_*}(-\Delta + V1_{\square_L} - z)^{-1}\chi_a\|_{op}$$
$$= \|\chi_{a_*}((-\Delta + V1_{\square_L} - z)^{-1}\phi - \phi(-\Delta + V1_{\square_{\ell,\boldsymbol{a}}} - z)^{-1})\chi_a\|_{op}$$
$$\leq c_V \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{\ell-2,\boldsymbol{a}} \setminus \square_{\ell-6,\boldsymbol{a}}}} \|\chi_{a_*}(-\Delta + V1_{\square_L} - z)^{-1}\chi_{a_1}\|_{op}\|\chi_{a_1}(-\Delta + V1_{\square_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{op}$$
$$\leq c_V^2 \sum_{\substack{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{\ell-2,\boldsymbol{a}} \setminus \square_{\ell-6,\boldsymbol{a}}} \\ a_2 \in \mathbb{Z}^2 \cap \overline{\square_{\ell-2,\boldsymbol{a}_1} \setminus \square_{\ell-6,\boldsymbol{a}_1}}}} \|\chi_{a_*}(-\Delta + V1_{\square_{L_{k+1}}} - z)^{-1}\chi_{a_2}\|_{op}\|\chi_{a_2}(-\Delta + V1_{\square_{\ell,\boldsymbol{a}_1}} - z)^{-1}\chi_{a_1}\|_{op}$$
$$\times \|\chi_{a_1}(-\Delta + V1_{\square_{\ell,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{op}$$

Iterate until $a_n \doteqdot a_*$

Toward our Geometric resolvent inequality

$$(\widetilde{H_L^{\tilde{\xi}}} - z)^{-1}\phi - \phi(\widetilde{H_{\ell,\boldsymbol{a}}^{\tilde{\xi}}} - z)^{-1}$$
$$= (\widetilde{H_L^{\tilde{\xi}}} - z)^{-1}((2\nabla\phi)\cdot\nabla + (\Delta\phi)$$

$\nwarrow$ Need the paracontrolled calculus

$$+\underline{\mathbb{E}[\phi(\Pi(\Delta^{-loc}(\xi_{L-2} - \xi_{\ell-2,\boldsymbol{a}}), \xi_{L-2}) + \Pi(\Delta^{-loc}\xi_{\ell-2,\boldsymbol{a}}, \xi_{L-2} - \xi_{\ell-2,\boldsymbol{a}}))]]}(\widetilde{H_{\ell,\boldsymbol{a}}^{\tilde{\xi}}} - z)^{-1}$$

supp($''$) $\not\subset$ supp $\nabla\phi$

supp($''$) $\subset$ supp ϕ, $\lesssim \exp(-cd(\cdot, \mathbb{R}^2 \setminus \square_{\ell-2,\boldsymbol{a}}))$

Our inequality becomes complicated as you will see in the next slide but new terms decay exponentially.

Our Geometric resolvent inequality

$$\|\chi_{a_*}(\widetilde{H_L^{\widetilde{\xi}}} - z)^{-1}\chi_a\|_{op} \quad \Xi(\boldsymbol{a},\ell,\xi) := \sup_{a_0\in\mathbb{Z}^2}\begin{pmatrix}\sum_{j=1}^2\left(\sum_{a_1\in\mathbb{Z}^2\cap\square_{\ell,\boldsymbol{a}}}\|\chi_{a_1}^2\xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)}\exp(-c_*|a_0-a_1|^2)\right)^j \\ +\sum_{a_1\in\mathbb{Z}^2}\|\chi_{a_1}^2 Y_{\xi,\ell,\boldsymbol{a}}\|_{C^{-\varepsilon}(\mathbb{R}^2)}\exp(-c_*|a_0-a_1|^2)\end{pmatrix}$$

$$\leq \sum_{a_1\in\mathbb{Z}^2\cap\overline{\square_{\ell-2,\boldsymbol{a}}\setminus\square_{\ell-6.\boldsymbol{a}}}}\|\chi_{a_*}(\widetilde{H_L^{\widetilde{\xi}}}-z)^{-1}\chi_{a_1}\|_{op}c_1\exp(-c_*|a_1-a|)$$

$$+\sum_{a_1\in\mathbb{Z}^2\cap\overline{\square_{\ell-2,\boldsymbol{a}}\setminus\square_{\ell-6,\boldsymbol{a}}},a_2\in\mathbb{Z}^2}\|\chi_{a_2}(\widetilde{H_{\ell,\boldsymbol{a}}^{\widetilde{\xi}}}-z)^{-1}\ \chi_a\|_{op}c_2\exp(-c_*(|a_1-a_*|+|a_1-a_2|^2))\Xi(\boldsymbol{a},\ell-2,\xi)^{1/2}$$

$$+\sum_{a_1\in\mathbb{Z}^2\cap\overline{\square_{\ell-2,\boldsymbol{a}}\setminus\square_{\ell-6,\boldsymbol{a}}},a_2\in\mathbb{Z}^2}\|\chi_{a_*}(\widetilde{H_L^{\widetilde{\xi}}}-z)^{-1}\chi_{a_1}\|_{op}\|\chi_{a_2}(\widetilde{H_{\ell,\boldsymbol{a}}^{\widetilde{\xi}}}-z)^{-1}\chi_a\|_{op}\begin{matrix}c_3\exp(-c_*|a_1-a_2|)\\ \times(|z|+\Xi(\boldsymbol{a},\ell-2,\xi))^{c_4}\end{matrix}$$

$$+\sum_{a_1\in\mathbb{Z}^2\cap\overline{\square_{\ell-2,\boldsymbol{a}}\setminus\square_{\ell-6,\boldsymbol{a}}},a_2,a_3\in\mathbb{Z}^2}\|\chi_{a_*}(\widetilde{H_L^{\widetilde{\xi}}}-z)^{-1}\chi_{a_2}\ \|_{op}\|\chi_{a_3}(\widetilde{H_{\ell,\boldsymbol{a}}^{\widetilde{\xi}}}-z)^{-1}\chi_a\|_{op}$$

$$\times\ c_5\exp(-c_*(|a_1-a_2|+|a_1-a_3|^2))(|z|+\Xi(\boldsymbol{0},L-2,\xi))^{c_6},$$

A Wegner type estimate

There exist finite positive constants c_0, c_1, c_2, c_3 such that

$$\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\tilde{\xi}}})]] \leq c_1 \eta L^{c_2}$$

for any $E \leq -c_3$, $0 < \eta \leq 1 \wedge (-E/2)$ and $L \in 2\mathbb{N}$.
($c_2 = 2 \Rightarrow$ An upper bound of the density of states
Wegner (1981) discrete Anderson model)
The present estimate is enough for MSA since Fröhlich-Spencer (1983)

Idea of the proof of Wegner type estimates I

(Variation of energies)$\underset{\sim}{\to}$(Variation of random variables)

$\lambda_0 \in [E-\eta, E+\eta]$, $H_L^{\widetilde{\xi}} \varphi_0 = \lambda_0 \varphi_0$, $\|\varphi_0\|_{L^2(\mathbb{R}^2)} = 1$
$\chi_{out} \in \underset{\sim}{C^\infty}(\mathbb{R}^2 \setminus \Box_{L-1})$, $\chi_{in} \in \underset{\sim}{C^\infty}(\Box_{L-1/2})$ s.t.. $\chi_{out}^2 + \chi_{in}^2 \equiv 1$.

IMS localization: $H_L^{\widetilde{\xi}} = \chi_{out} H_L^{\widetilde{\xi}} \chi_{out} + \chi_{in} H_L^{\widetilde{\xi}} \chi_{in} - |\nabla \chi_{out}|^2 - |\nabla \chi_{in}|^2$
(Ismagilov-Morgan-Simon, Ref. Sigal(1982))

$$\chi_{out} H_L^{\widetilde{\xi}} \chi_{out} \doteqdot \chi_{out}\Big(-\Delta + \sum_{a \in \mathbb{Z}^2 \cap \Box_L \setminus \Box_{L-2}} \chi_a^2 \overline{\xi_a} - \mathbb{E}[\Pi(\Delta^{-loc} \xi_{L-2}, \xi_{L-2})]\Big)\chi_{out} \geq -c_2 \chi_{out}^2$$

$\|\chi_{in}\varphi_0\|_{L^2(\mathbb{R}^2)}^2 \geq (-\lambda_0 - c_1)/B(\xi)$,
$(B(\xi) := c_4(1 + \|\xi_{L-2}\|_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} + \|\xi_{L-2}\|^2_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} + \|Y_{\xi,L-2}\|_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)})^{c_3})$
We assume $E \leq -c_1 - 2$ and $\lambda_0 \leq E + 1$. Then $\|\chi_{in}\varphi_0\|_{L^2(\mathbb{R}^2)}^2 \geq 1/B(\xi)$.

Idea of the proof of Wegner type estimates II

If $\lambda_0 \in [E-\eta, E+\eta]$ and $t \geq 2\eta B(\xi)/c_5$, then we have

$$(\varphi_0, \underbrace{(\widetilde{H_L^{\widetilde{\xi}}} + t \sum_{a\in\mathbb{Z}^2\cap\Box_L} \chi_a^2)}_{=\widetilde{H_L^{\widetilde{\xi}+t}}}\varphi_0)_{L^2(\mathbb{R}^2)} \geq E+\eta, \text{ where } \underbrace{\widetilde{\xi}+t}_{\text{shift}} = ((\xi(x)+t)_{x\in\mathbb{R}^2}, (\overline{\xi_a}+t)_{a\in\mathbb{Z}^2})$$

$$(\varphi_0, (\widetilde{H_L^{\widetilde{\xi}-t}} - t \sum_{a\in\mathbb{Z}^2\cap\Box_L} \chi_a^2)\varphi_0)_{L^2(\mathbb{R}^2)} \leq E-\eta.$$

$$\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\widetilde{\xi}}})]] = \sum_{n=1}^{\infty} \mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\widetilde{\xi}}})] : B(\xi) \in [n-1, n]]$$

$$\leq \sum_{n=1}^{\infty} \mathbb{E}\left[\mathrm{Tr}\left[\widetilde{1_{(-\infty,0]}}\left(\frac{1}{\eta}\left(H_L^{\widetilde{\xi}-2\eta n/c_5} - E\right)\right) - \widetilde{1_{(-\infty,0]}}\left(\frac{1}{\eta}\left(H_L^{\widetilde{\xi}+2\eta n/c_5} - E\right)\right)\right]\widetilde{\chi_{[n-1,n]}}(B(\xi))\right.$$

Idea of the proof of Wegner type estimates III

By the Cameron-Martin theorem,

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \mathbb{E}\Big[\Big(\prod_{a\in\mathbb{Z}^2\cap(\Box_L\setminus\Box_{L-2})} \int_{\mathbb{R}} d\overline{\xi}_a g\Big(\overline{\xi}_a + \frac{2\eta n}{c_5}\Big)\Big) \mathrm{Tr}[\widetilde{1_{(-\infty,0]}}((\widetilde{H_L^{\tilde{\xi}}} - E)/\eta)] \\
&\times \exp\Big(-\frac{2\eta n}{c_5}\int_{\Box_{L-2}} \xi(x)dx - 2\Big(\frac{\eta n(L-2)}{c_5}\Big)^2\Big) \widetilde{\chi_{[n-1,n]}}\Big(B\Big(\xi + \frac{2\eta n}{c_5}\Big)\Big) \\
&- \Big(\prod_{a\in\mathbb{Z}^2\cap(\Box_L\setminus\Box_{L-2})} \int_{\mathbb{R}} d\overline{\xi}_a g\Big(\overline{\xi}_a - \frac{2\eta n}{c_5}\Big)\Big) \mathrm{Tr}[\widetilde{1_{(-\infty,0]}}((\widetilde{H_L^{\tilde{\xi}}} - E)/\eta)] \\
&\times \exp\Big(\frac{2\eta n}{c_5}\int_{\Box_{L-2}} \xi(x)dx - 2\Big(\frac{\eta n(L-2)}{c_5}\Big)^2\Big) \widetilde{\chi_{[n-1,n]}}\Big(B\Big(\xi - \frac{2\eta n}{c_5}\Big)\Big)\Big],
\end{aligned}
$$

where g is the probability density of the random variable $\overline{\xi_0}$

Multiscale Analysis

For $1 \le \forall p < \infty$, $1 < \forall \alpha < 1 + p/4$, $0 < \forall m_0$ sufficiently small,
$\forall E_1 < \forall E_0 < 0$ sufficiently small,
$\exists L_0 \in 6\mathbb{N}$, $\exists c_1, c_2 \in (0, \infty)$ s.t.
$\mathbb{P}(\text{for } E_1 \le \forall E \le E_0, \square_{L_k,\boldsymbol{a}} \text{ or } \square_{L_k,\boldsymbol{a}'} \text{ is } (m_0, E, K_k)\text{-regular for } \widetilde{\xi}) > 1 - L_k^{-p}$
for any $k \in \mathbb{N}$ and any $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{Z}^2$ satisfying $|\boldsymbol{a} - \boldsymbol{a}'|_\infty > L_k + 2$, where $\{L_k\}_{k \in \mathbb{N}}$ is a sequence defined by $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}} := \max\{(-\infty, L_k^\alpha] \cap (6\mathbb{N})\}$, and
$K_k = c_2 \exp(c_1 L_k^{1/\alpha})$.
–Variable energy type von Dreifus and Klein (1989) Germinet and Klein (2001)

cf. simpler MSA (Fixed energy type) Fröhlich and Spencer (1983)
Spectral averaging methods are used for the proof of the localization

Generalized Eigenfunction Expansion

$\nu > 1/4$ fix, $\langle x \rangle := (1 + |x|^2)^{1/2}$ (Ref. Klein, Koines, Seifert, JFA(2002))

$\mathbb{E}[\mathrm{Tr}[\langle x \rangle^{-\nu} E(I : \widetilde{H^\xi}) \langle x \rangle^{-\nu}]^p] < \infty$ for any bounded interval I and $p > 0$ –(SGEE)

$\mu^\xi(I) := \mathrm{Tr}[\langle x \rangle^{-\nu} E(I : \widetilde{H^\xi}) \langle x \rangle^{-\nu}] \to$ Borel measure

By extending the Radon-Nikodym theorem,

$\mathbb{R} \ni \lambda \mapsto \exists Q^\xi(\lambda) \in \mathcal{I}_1(L^2(\mathbb{R}^2, dx))$: μ^ξ-locally integrable s.t. $Q^\xi(\lambda) \geq 0$

$\nwarrow$ Banach space of the trace class operators

$\langle x \rangle^{-\nu} E(I : \widetilde{H^\xi}) \langle x \rangle^{-\nu} = \int_I Q^\xi(\lambda) \mu^\xi(d\lambda)$ in the sense of the Banach integral

$P^\xi(\lambda) := \langle x \rangle^{\nu} Q^\xi(\lambda) \langle x \rangle^{\nu}$

$E(I : \widetilde{H^\xi}) = \int_I P^\xi(\lambda) \mu^\xi(d\lambda)$ in $\mathcal{I}_1(L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx), L^2(\mathbb{R}^2, \langle x \rangle^{-2\nu} dx))$

$:= \{\langle x \rangle^{\nu} A \langle x \rangle^{\nu} : A \in \mathcal{I}_1(L^2(\mathbb{R}^2, dx))\}$

with the norm $\|\langle x \rangle^{-\nu}(\cdot)\langle x \rangle^{-\nu}\|_{\mathcal{I}_1(L^2(\mathbb{R}^2, dx))}$

Generalized Eigenfunction

$\psi \in L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx) \Rightarrow \Psi := P^\xi(\lambda)\psi \in L^2(\mathbb{R}^2, \langle x \rangle^{-2\nu} dx)$

For $\forall \varphi \in \mathrm{Dom}_{+0}(\widetilde{H^\xi}) \subset L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx)$, we have $\widetilde{H^\xi}\varphi \in L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx)$ and

$$\int_{\mathbb{R}^2} dx \Psi(x)(\widetilde{H^\xi}\varphi)(x) = \lambda \int_{\mathbb{R}^2} dx \Psi(x)\varphi(x)$$

$\Psi \neq 0 \Rightarrow \Psi$: a generalized eigenfunction of $\widetilde{H^\xi}$ with a generalized eigenvalue λ

As in the geometric resolvent inequality, we have

Eigenfunction Decay Inequality

$$
\begin{aligned}
&\|\chi_{\boldsymbol{a}}\Psi\|_{L^2(\mathbb{R}^2)} \\
&\le c_3 \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)} (1\vee \Xi(\boldsymbol{a},L-2,\xi))^{3/2} \Xi_c(a_1,\boldsymbol{a},L-2,\xi)^{1/2} \\
&\times (1\vee \Xi_c(a_1,\boldsymbol{a},L-2,\xi))^{1/2} \exp(-c_1|a_1-a_2|) \\
&+c_3 \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)} (1\vee \Xi(\boldsymbol{a},L-2,\xi))^2 \exp(-c_1|a_1-a_2| - c_1 d(a_1,\square_{2L/3}(\boldsymbol{a})\setminus\square_{L/3}(\boldsymbol{a}))) \\
&+c_3 \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)} \sum_{a_3\in\mathbb{Z}^2\cap\overline{\square_2(\boldsymbol{a})}} \|\chi_{a_2}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}-E)^{-1}\chi_{a_3}^2\|_{op}(\log L)^{c_2} \\
&\times (1\vee \Xi(\boldsymbol{a},L-2,\xi))^2 (1\vee \Xi_c(a_2,\boldsymbol{a},L-2,\xi))^{3/2} \Xi_c(a_2,\boldsymbol{a},L-2,\xi)^{1/2} (1\vee \Xi_c(a_2,\xi))^{1/2} \\
&\times (\max|\overline{\xi_0}| + |E| + \Xi(\boldsymbol{a},L-2,\xi)) \exp(-c_1|a_1-a_2|) \\
&+c_3 \sum_{a_1,a_2\in\mathbb{Z}^2} \|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)} \sum_{a_3\in\mathbb{Z}^2\cap\overline{\square_2(\boldsymbol{a})}} \|\chi_{a_2}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}-E)^{-1}\chi_{a_3}^2\|_{op}(\log L)^{c_2} \\
&\times (1\vee \Xi(\boldsymbol{a},L-2,\xi))^{5/2} (1\vee \Xi_c(a_2,\boldsymbol{a},L-2,\xi)) (\max|\overline{\xi_0}| + |E| + \Xi(\boldsymbol{a},L-2,\xi)) \\
&\times \exp(-c_1|a_1-a_2| - c_1 d(a_1,\square_{2L/3}(\boldsymbol{a})\setminus\square_{L/3}(\boldsymbol{a})))
\end{aligned}
$$

Notations in the Eigenfunction Decay Inequality

$$\Xi_c(a_1, \boldsymbol{a}, L-2, \xi) := \sum_{a_2 \in \mathbb{Z}^2 \setminus \square_{L-2}(\boldsymbol{a})} \|\chi_{a_2}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|)$$

$$+ \sum_{(a_2,a_3) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \setminus \square_{L-2}(\boldsymbol{a})^2} \prod_{j=2}^{3} \|\chi_{a_j}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_j|)$$

$$+ \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 (Y_\xi - Y_{\xi, L-2, \boldsymbol{a}})\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|),$$

$$\Xi_c(a_1, \xi) := \sum_{j=1}^{2} \left(\sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|) \right)^j$$

$$+ \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 Y_\xi\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|)$$

Exponential localization

From the eigenfunction inequality

and the estimates of $\{\|\chi_{a_1}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1}\chi_{a_2}^2\|_{op}\}_{a_1,a_2,L}$,

we obtain

$$\|\chi_{\boldsymbol{a}}\Psi\|_{L^2(\mathbb{R}^2)} \leq c_1 \exp(-c_2|\boldsymbol{a}|_\infty)$$

$\because$ of (SGEE) $\mathbb{E}[\mathrm{Tr}[\langle x\rangle^{-\nu}E(I:\widetilde{H^{\xi}})\langle x\rangle^{-\nu}]^p] < \infty$

It is enough to show $\exists t > 0$ s.t.

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \exp\Big(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}}\Big)(x,x)\Big)^m\Big] < \infty \text{ for } \forall m \in \mathbb{N} \text{——} (\mathrm{HM})_m$$

For $m = 1$,

$$\mathbb{E}\Big[\int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \exp\Big(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}}\Big)(x,x)\Big]$$
$$= \int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \mathbb{E}\Big[\exp\Big(-\int_0^t \frac{ds}{2}\xi_\varepsilon(x+w_0^t(s))\Big)\Big] \exp\Big(\frac{t}{2}\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Big)\frac{1}{2\pi t}$$
$$= \int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \mathbb{E}\Big[\exp\Big(\frac{1}{8}\int_0^t ds_1 \int_0^t ds_2 e^{2\varepsilon^2\Delta}(w_0^t(s_1), w_0^t(s_2))\Big)\Big]$$
$$\times \exp\Big(\frac{t}{2}\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]\Big)\frac{1}{2\pi t}.$$

Use the integrability of the intersection local time

By Matsuda (2022),

$$\sup_{\varepsilon} \mathbb{E}\left[\exp\left(\iint_{0 \le s_1, s_2 \le t} ds_1 ds_2 (e^{\varepsilon^2 \Delta}(w_0^t(s_1), w_0^t(s_2)) - \mathbb{E}[e^{\varepsilon^2 \Delta}(w_0^t(s_1), w_0^t(s_2))]) \right) \right] < \infty$$

for sufficiently small $t > 0$ and

$$\int_0^t ds_1 \int_0^t ds_2 \mathbb{E}[e^{\varepsilon^2 \Delta}(w_0^t(s_1), w_0^t(s_2))] = \frac{t}{\pi} \log \frac{t}{2\varepsilon^2} + o(1) \text{ as } \varepsilon \to 0.$$

Moreover by

$$\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)] = \frac{-1}{4\pi} \log \frac{1}{2\varepsilon^2} + o(1) \text{ as } \varepsilon \to 0,$$

we have $(\mathrm{HM})_1$ in the last slide.

Similarly we have $(\mathrm{HM})_m$ for $\forall m \in \mathbb{N}$.